

# thm\_2Epoly\_2EPOLY\_\_ROOTS\_\_FINITE (TMZnETNbEwnNdnnQBh- SrZDiC8X579ucwQuC)

October 26, 2020

**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ecombin\_2EK$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0x \in A\_27a.(\lambda V1y \in A\_27b.V0x))$

**Definition 4** We define  $c\_2Ecombin\_2ES$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.(\lambda V0f \in ((A\_27c^{A\_27b})^{A\_27a}))$

**Definition 5** We define  $c\_2Ecombin\_2EI$  to be  $\lambda A\_27a : \iota.(ap (ap (c\_2Ecombin\_2ES A\_27a (A\_27a^{A\_27a})) A\_27a))$

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \tag{1}$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{2}$$

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erealax\_2Ereal \tag{3}$$

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{ty\_2Erealax\_2Ereal}) \tag{4}$$

**Definition 6** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.if\ (\exists x \in A.p\ (ap\ P\ x))\ \mathbf{then}\ (the\ (\lambda x.x \in A \wedge p\ x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 7** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$



Let  $c\_Ereal\_Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_Ereal\_Ereal\_of\_num \in (ty\_Erealax\_Ereal^{ty\_Eenum\_Eenum}) \quad (14)$$

Let  $c\_Elist\_ENIL : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_Elist\_ENIL\ A\_27a \in (ty\_Elist\_Elist\ A\_27a) \quad (15)$$

Let  $c\_Epoly\_Epoly : \iota$  be given. Assume the following.

$$c\_Epoly\_Epoly \in ((ty\_Erealax\_Ereal^{ty\_Erealax\_Ereal})(ty\_Elist\_Elist\ ty\_Erealax\_Ereal)) \quad (16)$$

**Definition 13** We define  $c\_Ebool\_2E$  to be  $(ap\ (c\_Ebool\_2E\_21\ 2)\ (\lambda V0t \in 2.V0t))$ .

**Definition 14** We define  $c\_Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o\ (p\ P \Rightarrow p\ Q)$  of type  $\iota$ .

**Definition 15** We define  $c\_Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap\ (ap\ c\_Emin\_2E\_3D\_3D\_3E\ V0t)\ c\_Ebool\_2E\_7E))$ .

**Definition 16** We define  $c\_Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2.V2t))))$ .

Let  $c\_Eenum\_EREP\_num : \iota$  be given. Assume the following.

$$c\_Eenum\_EREP\_num \in (omega^{ty\_Eenum\_Eenum}) \quad (17)$$

Let  $c\_Eenum\_ESUC\_REP : \iota$  be given. Assume the following.

$$c\_Eenum\_ESUC\_REP \in (omega^{omega}) \quad (18)$$

**Definition 17** We define  $c\_Eenum\_ESUC$  to be  $\lambda V0m \in ty\_Eenum\_Eenum.(ap\ c\_Eenum\_EABS\_num\ V0m)$ .

**Definition 18** We define  $c\_Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_Emin\_2E\_40\ 40)\ V0P))))$ .

**Definition 19** We define  $c\_Eprim\_rec\_2E\_3C$  to be  $\lambda V0m \in ty\_Eenum\_Eenum.\lambda V1n \in ty\_Eenum\_Eenum.(ap\ c\_Eprim\_rec\_2E\_3C\ V0m\ V1n)$ .

Let  $c\_Erealax\_Etreal\_lt : \iota$  be given. Assume the following.

$$c\_Erealax\_Etreal\_lt \in ((2^{(ty\_Epair\_Eprod\ ty\_Ehreal\_Ehreal\ ty\_Ehreal\_Ehreal)})(ty\_Epair\_Eprod\ ty\_Ehreal\_Ehreal)) \quad (19)$$

**Definition 20** We define  $c\_Erealax\_Ereal\_lt$  to be  $\lambda V0T1 \in ty\_Erealax\_Ereal.\lambda V1T2 \in ty\_Erealax\_Ereal.(ap\ c\_Erealax\_Etreal\_lt\ V0T1\ V1T2)$ .

**Definition 21** We define  $c\_Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2.V2t))))$ .

Assume the following.

$$True \quad (20)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (21)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \quad (22)$$

Assume the following.

$$(\forall V0t \in 2.(((p\ V0t) \Rightarrow False) \Rightarrow (\neg(p\ V0t)))) \quad (23)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(p\ V0t)) \Rightarrow ((p\ V0t) \Rightarrow False))) \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (( \\ & (p\ V0t) \Rightarrow False) \Leftrightarrow (\neg(p\ V0t)))))) \end{aligned} \quad (25)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \quad (26)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (27)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (28)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \forall V0f \in (A\_27b^{A\_27a}).(\forall V1g \in (A\_27b^{A\_27a}).((\forall V2x \in \\ & A\_27a.((ap\ V0f\ V2x) = (ap\ V1g\ V2x)) \Rightarrow (V0f = V1g)))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg(p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\ & p\ V0t)))))) \end{aligned} \quad (30)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).((\neg(\forall V1x \in A\_27a.(p\ (ap\ V0P\ V1x)))) \Leftrightarrow (\exists V2x \in A\_27a.(\neg(p\ (ap\ V0P\ V2x)))))) \quad (31)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).((\neg(\exists V1x \in A.27a.(p \ (ap \ V0P \ V1x)))) \Leftrightarrow (\forall V2x \in A.27a.(\neg(p \ (ap \ V0P \ V2x)))))) \quad (32)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in (2^{A.27a}).((p \ V0P) \wedge (\forall V2x \in A.27a.(p \ (ap \ V1Q \ V2x)))) \Leftrightarrow (\forall V3x \in A.27a.((p \ V0P) \wedge (p \ (ap \ V1Q \ V3x)))))) \quad (33)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).(\forall V1Q \in (2^{A.27a}).((\exists V2x \in A.27a.((p \ (ap \ V0P \ V2x)) \vee (p \ (ap \ V1Q \ V2x)))) \Leftrightarrow ((\exists V3x \in A.27a.(p \ (ap \ V0P \ V3x))) \vee (\exists V4x \in A.27a.(p \ (ap \ V1Q \ V4x))))))) \quad (34)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in (2^{A.27a}).((p \ V0P) \vee (\exists V2x \in A.27a.(p \ (ap \ V1Q \ V2x)))) \Leftrightarrow (\exists V3x \in A.27a.((p \ V0P) \vee (p \ (ap \ V1Q \ V3x)))))) \quad (35)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).(\forall V1Q \in 2.((\exists V2x \in A.27a.((p \ (ap \ V0P \ V2x)) \wedge (p \ V1Q))) \Leftrightarrow ((\exists V3x \in A.27a.(p \ (ap \ V0P \ V3x)) \wedge (p \ V1Q)))))) \quad (36)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in (2^{A.27a}).((\exists V2x \in A.27a.((p \ V0P) \wedge (p \ (ap \ V1Q \ V2x)))) \Leftrightarrow ((p \ V0P) \wedge (\exists V3x \in A.27a.(p \ (ap \ V1Q \ V3x)))))) \quad (37)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0Q \in 2.(\forall V1P \in (2^{A.27a}).((\forall V2x \in A.27a.((p \ (ap \ V1P \ V2x)) \vee (p \ V0Q))) \Leftrightarrow ((\forall V3x \in A.27a.(p \ (ap \ V1P \ V3x)) \vee (p \ V0Q)))))) \quad (38)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in (2^{A.27a}).((\forall V2x \in A.27a.((p \ V0P) \vee (p \ (ap \ V1Q \ V2x)))) \Leftrightarrow ((p \ V0P) \vee (\forall V3x \in A.27a.(p \ (ap \ V1Q \ V3x)))))) \quad (39)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p \ V0A) \vee (p \ V1B) \vee (p \ V2C))) \Leftrightarrow (((p \ V0A) \vee (p \ V1B)) \vee (p \ V2C))))) \quad (40)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((p \ V0A) \vee (p \ V1B)) \Leftrightarrow ((p \ V1B) \vee (p \ V0A)))))) \quad (41)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p \ V0A) \wedge (p \ V1B))) \Leftrightarrow ((\neg(p \ V0A)) \vee (\neg(p \ V1B)))))) \wedge (((\neg((p \ V0A) \vee (p \ V1B))) \Leftrightarrow ((\neg(p \ V0A)) \wedge (\neg(p \ V1B))))))) \quad (42)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty \ A.27a \Rightarrow \forall A.27b.nonempty \ A.27b \Rightarrow ( \\ & \quad \forall V0P \in ((2^{A.27b})^{A.27a}).((\forall V1x \in A.27a.(\exists V2y \in \\ & \quad A.27b.(p \ (ap \ (ap \ V0P \ V1x) \ V2y)))) \Leftrightarrow (\exists V3f \in (A.27b^{A.27a}).( \\ & \quad \forall V4x \in A.27a.(p \ (ap \ (ap \ V0P \ V4x) \ (ap \ V3f \ V4x))))))) \end{aligned} \quad (43)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.((ap \ (c.2Ecombin.2El \ A.27a) \ V0x) = V0x)) \quad (44)$$

Assume the following.

$$\begin{aligned} & ((\forall V0x \in ty.2Erealax.2Ereal.((ap \ (ap \ c.2Epoly.2Epoly \ ( \\ & \quad c.2Elist.2ENIL \ ty.2Erealax.2Ereal)) \ V0x) = (ap \ c.2Ereal.2Ereal\_of\_num \\ & \quad c.2Enum.2E0))) \wedge (\forall V1h \in ty.2Erealax.2Ereal.(\forall V2t \in \\ & \quad (ty.2Elist.2Elist \ ty.2Erealax.2Ereal).(\forall V3x \in ty.2Erealax.2Ereal. \\ & \quad ((ap \ (ap \ c.2Epoly.2Epoly \ (ap \ (ap \ (c.2Elist.2ECONS \ ty.2Erealax.2Ereal \\ & \quad V1h) \ V2t)) \ V3x) = (ap \ (ap \ c.2Erealax.2Ereal\_add \ V1h) \ (ap \ (ap \ c.2Erealax.2Ereal\_mul \\ & \quad V3x) \ (ap \ (ap \ c.2Epoly.2Epoly \ V2t) \ V3x))))))) \end{aligned} \quad (45)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in (ty.2Elist.2Elist \ ty.2Erealax.2Ereal).((\neg((ap \\ & \quad c.2Epoly.2Epoly \ V0p) = (ap \ c.2Epoly.2Epoly \ (c.2Elist.2ENIL \ ty.2Erealax.2Ereal)))) \Rightarrow \\ & \quad (\exists V1N \in ty.2Enum.2Enum.(\exists V2i \in (ty.2Erealax.2Ereal^{ty.2Enum.2Enum}). \\ & \quad (\forall V3x \in ty.2Erealax.2Ereal.(((ap \ (ap \ c.2Epoly.2Epoly \ V0p) \\ & \quad V3x) = (ap \ c.2Ereal.2Ereal\_of\_num \ c.2Enum.2E0)) \Rightarrow (\exists V4n \in \\ & \quad ty.2Enum.2Enum.((p \ (ap \ (ap \ c.2Eprim\_rec.2E.3C \ V4n) \ V1N)) \wedge (V3x = \\ & \quad (ap \ V2i \ V4n)))))))))) \end{aligned} \quad (46)$$

Assume the following.

$$\begin{aligned} & (\forall V0i \in (ty.2Erealax.2Ereal^{ty.2Enum.2Enum}).(\forall V1N \in \\ & \quad ty.2Enum.2Enum.(\forall V2P \in (2^{ty.2Erealax.2Ereal}).((\forall V3x \in \\ & \quad ty.2Erealax.2Ereal.((p \ (ap \ V2P \ V3x)) \Rightarrow (\exists V4n \in ty.2Enum.2Enum. \\ & \quad ((p \ (ap \ (ap \ c.2Eprim\_rec.2E.3C \ V4n) \ V1N)) \wedge (V3x = (ap \ V0i \ V4n)))))) \Rightarrow \\ & \quad (\exists V5a \in ty.2Erealax.2Ereal.(\forall V6x \in ty.2Erealax.2Ereal. \\ & \quad ((p \ (ap \ V2P \ V6x)) \Rightarrow (p \ (ap \ (ap \ c.2Erealax.2Ereal\_lt \ V6x) \ V5a))))))) \end{aligned} \quad (47)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\neg(p (ap (ap c\_2Erealax\_2Ereal\_lt V0x) V0x)))) \quad (48)$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \quad (49)$$

Assume the following.

$$(\forall V0A \in 2. ((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \quad (50)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow ((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))))) \quad (51)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))))) \quad (52)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (53)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg(p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee ((\neg(p V1q)) \vee (\neg(p V0p)))))))))) \quad (54)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \wedge (p V2r)) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \vee (\neg(p V2r)))) \wedge (((p V1q) \vee ((\neg(p V0p)) \wedge ((p V2r) \vee (\neg(p V0p)))))))))) \quad (55)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \vee (p V2r)) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((p V1q) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \quad (56)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \Rightarrow (p V2r)) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((\neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \quad (57)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(((p \ V0p) \Leftrightarrow (\neg(p \ V1q))) \Leftrightarrow (((p \ V0p) \vee (p \ V1q)) \wedge ((\neg(p \ V1q)) \vee (\neg(p \ V0p))))))) \quad (58)$$

**Theorem 1**

$$\begin{aligned} & (\forall V0p \in (ty\_2Elist\_2Elist \ ty\_2Erealax\_2Ereal).(\neg((ap \\ c\_2Epoly\_2Epoly \ V0p) = (ap \ c\_2Epoly\_2Epoly \ (c\_2Elist\_2ENIL \ ty\_2Erealax\_2Ereal)))) \Leftrightarrow \\ & (\exists V1N \in ty\_2Enum\_2Enum.(\exists V2i \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}). \\ & (\forall V3x \in ty\_2Erealax\_2Ereal.(((ap \ (ap \ c\_2Epoly\_2Epoly \ V0p) \\ V3x) = (ap \ c\_2Ereal\_2Ereal\_of\_num \ c\_2Enum\_2E0)) \Rightarrow (\exists V4n \in \\ ty\_2Enum\_2Enum.(p \ (ap \ (ap \ c\_2Eprim\_rec\_2E\_3C \ V4n) \ V1N)) \wedge (V3x = \\ (ap \ V2i \ V4n)))))))))) \end{aligned}$$