

thm\_2Epred\_\_set\_2ECARD\_\_SUBSET  
 (TMdZJQqBmSZA-  
 ZDsL9kL6t6JZSMYfnC4TZxo)

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**Definition 1** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.$ if  $(\exists x \in A.p (ap P x))$  then (the  $(\lambda x.x \in A \wedge p)$  of type  $\iota \Rightarrow \iota$ ).

**Definition 2** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap V0P (ap (c\_2Emin\_2E\_40 A$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

**Definition 4** We define  $c\_2Ebool\_2E\_T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 5** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A-27a}$

**Definition 6** We define  $c\_2Ebool\_2E\_F$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 7** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 8** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_F$

**Definition 9** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \tag{2}$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \tag{3}$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \tag{4}$$

**Definition 10** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_num$

**Definition 11** We define  $c\_2Eprim\_rec\_2E\_3C$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

**Definition 12** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in$

**Definition 13** We define  $c\_2Earithmetic\_2E\_3C\_3D$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

**Definition 14** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap\ V1f\ V0x))$

**Definition 15** We define  $c\_2Epred\_set\_2ESUBSET$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap\ ($

**Definition 16** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2EF).$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \quad (5)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (6)$$

**Definition 17** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}}) \quad (7)$$

**Definition 18** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap\ (c\_2$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (8)$$

**Definition 19** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP).$

**Definition 20** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (9)$$

**Definition 21** We define  $c\_2Earithmetic\_2EBIT1$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

**Definition 22** We define  $c\_2Earithmetic\_2ENUMERAL$  to be  $\lambda V0x \in ty\_2Enum\_2Enum.V0x$ .

Let  $c\_2Earithmetic\_2E\_2D : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2D \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})ty\_2Enum\_2Enum) \quad (10)$$

**Definition 23** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A\_27a. (\lambda V2t2 \in A\_27a. ($

**Definition 24** We define  $c\_2Epred\_set\_2EDIFF$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). \lambda V1t \in (2^{A\_27a}). (ap (c\_2E$

**Definition 25** We define  $c\_2Epred\_set\_2EDELETE$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). \lambda V1x \in A\_27a. (ap (a$

Let  $c\_2Epred\_set\_2ECARD : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a. nonempty A\_27a \Rightarrow c\_2Epred\_set\_2ECARD A\_27a \in (ty\_2Enum\_2Enum^{(2^{A\_27a})}) \quad (11)$$

**Definition 26** We define  $c\_2Epred\_set\_2EFINITE$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). (ap (c\_2Ebool\_2E\_21 (2$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum. ((V0m = c\_2Enum\_2E0) \vee (\exists V1n \in ty\_2Enum\_2Enum. (V0m = (ap c\_2Enum\_2ESUC V1n)))))) \quad (12)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. (p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C (ap c\_2Enum\_2ESUC V0m)) (ap c\_2Enum\_2ESUC V1n))) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C V0m) V1n)))))) \quad (13)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum. ((ap (ap c\_2Earithmetic\_2E\_2D (ap c\_2Enum\_2ESUC V0m)) (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO))) = V0m)) \quad (14)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum. (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D V0m) V0m))) \quad (15)$$

Assume the following.

$$True \quad (16)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (17)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p V0t))) \quad (18)$$

Assume the following.

$$(\forall V0t \in 2.((p V0t) \vee (\neg(p V0t)))) \quad (19)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \vee (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \vee True) \Leftrightarrow True) \wedge \\ & (((False \vee (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee False) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee \\ & (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (( \\ & (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \end{aligned} \quad (21)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (22)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\ & p V0t)))))) \end{aligned} \quad (23)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t1 \in A\_27a.(\forall V1t2 \in \\ & A\_27a.(((ap (ap (ap (c\_2Ebool\_2ECOND A\_27a) c\_2Ebool\_2ET) V0t1) \\ & V1t2) = V0t1) \wedge ((ap (ap (ap (c\_2Ebool\_2ECOND A\_27a) c\_2Ebool\_2EF \\ & V0t1) V1t2) = V1t2)))))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}).((p (ap \\ & (ap (c\_2Epred\_set\_2ESUBSET A\_27a) V0s) (c\_2Epred\_set\_2EEMPTY \\ & A\_27a))) \Leftrightarrow (V0s = (c\_2Epred\_set\_2EEMPTY A\_27a)))) \end{aligned} \quad (25)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1s \in \\ & (2^{A\_27a}).((\neg(p (ap (ap (c\_2Ebool\_2EIN A\_27a) V0x) V1s))) \Leftrightarrow ((ap \\ & (ap (c\_2Epred\_set\_2EDELETE A\_27a) V1s) V0x) = V1s)))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1s \in \\ & (2^{A\_27a}).(\forall V2t \in (2^{A\_27a}).((p (ap (ap (c\_2Epred\_set\_2ESUBSET \\ & A\_27a) V1s) (ap (ap (c\_2Epred\_set\_2EINSERT A\_27a) V0x) V2t))) \Leftrightarrow \\ & (p (ap (ap (c\_2Epred\_set\_2ESUBSET A\_27a) (ap (ap (c\_2Epred\_set\_2EDELETE \\ & A\_27a) V1s) V0x)) V2t)))))) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0x \in A.27a. (\forall V1s \in \\ (2^{A.27a}). ((p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27a)\ V0x)\ V1s)) \Rightarrow ((ap\ (ap \\ (c.2Epred\_set.2EINSERT\ A.27a)\ V0x)\ (ap\ (ap\ (c.2Epred\_set.2EDELETE \\ A.27a)\ V1s)\ V0x)) = V1s)))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0P \in (2^{(2^{A.27a})}). (( \\ (p\ (ap\ V0P\ (c.2Epred\_set.2EEMPTY\ A.27a))) \wedge (\forall V1s \in (2^{A.27a}). \\ (((p\ (ap\ (c.2Epred\_set.2EFINITE\ A.27a)\ V1s)) \wedge (p\ (ap\ V0P\ V1s))) \Rightarrow \\ (\forall V2e \in A.27a. ((\neg(p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27a)\ V2e)\ V1s))) \Rightarrow \\ (p\ (ap\ V0P\ (ap\ (ap\ (c.2Epred\_set.2EINSERT\ A.27a)\ V2e)\ V1s)))))) \Rightarrow \\ (\forall V3s \in (2^{A.27a}). ((p\ (ap\ (c.2Epred\_set.2EFINITE\ A.27a)\ \\ V3s)) \Rightarrow (p\ (ap\ V0P\ V3s)))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0x \in A.27a. (\forall V1s \in \\ (2^{A.27a}). ((p\ (ap\ (c.2Epred\_set.2EFINITE\ A.27a)\ (ap\ (ap\ (c.2Epred\_set.2EINSERT \\ A.27a)\ V0x)\ V1s))) \Leftrightarrow (p\ (ap\ (c.2Epred\_set.2EFINITE\ A.27a)\ V1s)))) \end{aligned} \quad (30)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0s \in (2^{A.27a}). ((p\ (ap \\ (c.2Epred\_set.2EFINITE\ A.27a)\ V0s)) \Rightarrow (\forall V1t \in (2^{A.27a}). \\ ((p\ (ap\ (ap\ (c.2Epred\_set.2ESUBSET\ A.27a)\ V1t)\ V0s)) \Rightarrow (p\ (ap\ (c.2Epred\_set.2EFINITE \\ A.27a)\ V1t)))))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (((ap\ (c.2Epred\_set.2ECARD\ A.27a) \\ (c.2Epred\_set.2EEMPTY\ A.27a)) = c.2Enum.2E0) \wedge (\forall V0s \in \\ (2^{A.27a}). ((p\ (ap\ (c.2Epred\_set.2EFINITE\ A.27a)\ V0s)) \Rightarrow (\forall V1x \in \\ A.27a. ((ap\ (c.2Epred\_set.2ECARD\ A.27a)\ (ap\ (ap\ (c.2Epred\_set.2EINSERT \\ A.27a)\ V1x)\ V0s)) = (ap\ (ap\ (ap\ (c.2Ebool.2ECOND\ ty.2Enum.2Enum) \\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27a)\ V1x)\ V0s))\ (ap\ (c.2Epred\_set.2ECARD \\ A.27a)\ V0s))\ (ap\ c.2Enum.2ESUC\ (ap\ (c.2Epred\_set.2ECARD\ A.27a) \\ V0s))))))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow ((ap\ (c.2Epred\_set.2ECARD\ A.27a) \\ (c.2Epred\_set.2EEMPTY\ A.27a)) = c.2Enum.2E0) \end{aligned} \quad (33)$$

Assume the following.

$$\begin{aligned} & \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow (\forall V0s \in (2^{A_{27a}}). ((p \text{ (ap} \\ & \text{(c\_2Epred\_set\_2EFINITE } A_{27a}) V0s)) \Rightarrow (\forall V1x \in A_{27a}. (( \\ & \text{ap (c\_2Epred\_set\_2ECARD } A_{27a}) \text{ (ap (ap (c\_2Epred\_set\_2EINSERT} \\ & A_{27a}) V1x) V0s)) = (\text{ap (ap (ap (c\_2Ebool\_2ECOND } ty\_2Enum\_2Enum) \\ & (\text{ap (ap (c\_2Ebool\_2EIN } A_{27a}) V1x) V0s)) \text{ (ap (c\_2Epred\_set\_2ECARD} \\ & A_{27a}) V0s)) \text{ (ap c\_2Enum\_2ESUC (ap (c\_2Epred\_set\_2ECARD } A_{27a}) \\ & V0s))))))))) \end{aligned} \quad (34)$$

Assume the following.

$$\begin{aligned} & \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow (\forall V0s \in (2^{A_{27a}}). ((p \text{ (ap} \\ & \text{(c\_2Epred\_set\_2EFINITE } A_{27a}) V0s)) \Rightarrow (\forall V1x \in A_{27a}. (( \\ & \text{ap (c\_2Epred\_set\_2ECARD } A_{27a}) \text{ (ap (ap (c\_2Epred\_set\_2EDELETE} \\ & A_{27a}) V0s) V1x)) = (\text{ap (ap (ap (c\_2Ebool\_2ECOND } ty\_2Enum\_2Enum) \\ & (\text{ap (ap (c\_2Ebool\_2EIN } A_{27a}) V1x) V0s)) \text{ (ap (ap c\_2Earithmetic\_2E\_2D} \\ & (\text{ap (c\_2Epred\_set\_2ECARD } A_{27a}) V0s)) \text{ (ap c\_2Earithmetic\_2ENUMERAL} \\ & (\text{ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO)))) \text{ (ap (c\_2Epred\_set\_2ECARD} \\ & A_{27a}) V0s))))))))) \end{aligned} \quad (35)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & ((\text{ap c\_2Enum\_2ESUC } V0m) = (\text{ap c\_2Enum\_2ESUC } V1n)) \Leftrightarrow (V0m = V1n)))) \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned} & (\forall V0n \in ty\_2Enum\_2Enum. (p \text{ (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Enum\_2E0) \\ & (\text{ap c\_2Enum\_2ESUC } V0n)))) \end{aligned} \quad (37)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & (p \text{ (ap (ap c\_2Eprim\_rec\_2E\_3C } V0m) \text{ (ap c\_2Enum\_2ESUC } V1n)) \Leftrightarrow ( \\ & (V0m = V1n) \vee (p \text{ (ap (ap c\_2Eprim\_rec\_2E\_3C } V0m) V1n)))))) \end{aligned} \quad (38)$$

### Theorem 1

$$\begin{aligned} & \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow (\forall V0s \in (2^{A_{27a}}). ((p \text{ (ap} \\ & \text{(c\_2Epred\_set\_2EFINITE } A_{27a}) V0s)) \Rightarrow (\forall V1t \in (2^{A_{27a}}). \\ & ((p \text{ (ap (ap (c\_2Epred\_set\_2ESUBSET } A_{27a}) V1t) V0s)) \Rightarrow (p \text{ (ap (ap} \\ & \text{c\_2Earithmetic\_2E\_3C\_3D (ap (c\_2Epred\_set\_2ECARD } A_{27a}) V1t)) \\ & (\text{ap (c\_2Epred\_set\_2ECARD } A_{27a}) V0s))))))))) \end{aligned}$$