

thm\_2Epred\_\_set\_2EINJ\_\_CARD\_\_IMAGE  
(TMU1vd1rMwG6wRYyhSHv1JqvxJfjWtBQMbp)

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**Definition 1** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 2** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 4** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 5** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t)))$

**Definition 6** We define  $c\_2Ebool\_2E\_2F$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 7** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge p (ap P x)))$  of type  $\iota \Rightarrow \iota$ .

**Definition 8** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap V0P (ap (c\_2Emin\_2E\_40 A\_27a P)))$

**Definition 9** We define  $c\_2Ebool\_2E\_2IN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap V1f V0x)))$

**Definition 10** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t)))$

**Definition 11** We define  $c\_2Epred\_set\_2EINJ$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (A\_27b^{A\_27a}).\lambda V1s \in (2^{A\_27b})$

**Definition 12** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_2F))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EABS\_prod A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a A\_27b)^{(2^{A\_27b})^{A\_27a}}) \tag{2}$$

**Definition 13** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0x \in A\_27a. \lambda V1y \in A\_27b. (ap (c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota)$  be given. Assume the following.

$$\forall A\_27a. nonempty A\_27a \Rightarrow \forall A\_27b. nonempty A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC A\_27a A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod A\_27a 2)^{A\_27b}})$$
 (3)

**Definition 14** We define  $c\_2Epred\_set\_2EIMAGE$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0f \in (A\_27b^{A\_27a}). \lambda V1s \in$  Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega$$
 (4)

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty ty\_2Enum\_2Enum$$
 (5)

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega})$$
 (6)

**Definition 15** We define  $c\_2Enum\_2E0$  to be  $(ap c\_2Enum\_2EABS\_num c\_2Enum\_2EZERO\_REP)$ .

**Definition 16** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota. (\lambda V0x \in A\_27a. c\_2Ebool\_2EF)$ .

**Definition 17** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A\_27a. (\lambda V2t2 \in A\_27a. ($

**Definition 18** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota. \lambda V0x \in A\_27a. \lambda V1s \in (2^{A\_27a}). (ap (c\_2Epred\_set\_2ECARD : \iota \Rightarrow \iota)$  be given. Assume the following.

$$\forall A\_27a. nonempty A\_27a \Rightarrow c\_2Epred\_set\_2ECARD A\_27a \in (ty\_2Enum\_2Enum^{(2^{A\_27a})})$$
 (7)

**Definition 19** We define  $c\_2Epred\_set\_2EFINITE$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). (ap (c\_2Ebool\_2E\_21 (2$

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum})$$
 (8)

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega})$$
 (9)

**Definition 20** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. (ap c\_2Enum\_2EABS\_num$

Assume the following.

$$True$$
 (10)

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2))))))$$
 (11)

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p V0t))) \quad (12)$$

Assume the following.

$$(\forall V0t \in 2.((p V0t) \vee (\neg(p V0t)))) \quad (13)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\ & (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (14)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (( \\ & (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \end{aligned} \quad (15)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ & ((\neg False) \Leftrightarrow True)))) \end{aligned} \quad (16)$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (17)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\ & p V0t)))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0t1 \in A\_27a.(\forall V1t2 \in \\ & A\_27a.(((ap (ap (ap (c\_2Ebool\_2ECOND A\_27a) c\_2Ebool\_2ET) V0t1) \\ & V1t2) = V0t1) \wedge ((ap (ap (ap (c\_2Ebool\_2ECOND A\_27a) c\_2Ebool\_2EF \\ & V0t1) V1t2) = V1t2)))))) \end{aligned} \quad (19)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty \ A\_27a \Rightarrow \forall A\_27b.nonempty \ A\_27b \Rightarrow ( \\ & \forall V0y \in A\_27b.(\forall V1s \in (2^{A\_27a}).(\forall V2f \in (A\_27b^{A\_27a}). \\ & ((p (ap (ap (c\_2Ebool\_2EIN A\_27b) V0y) (ap (ap (c\_2Epred\_set\_2EIMAGE \\ & A\_27a \ A\_27b) V2f) V1s))) \Leftrightarrow (\exists V3x \in A\_27a.((V0y = (ap V2f V3x)) \wedge \\ & (p (ap (ap (c\_2Ebool\_2EIN A\_27a) V3x) V1s)))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0f \in (A\_27b^{A\_27a}).((ap\ (ap\ (c\_2Epred\_set\_2EIMAGE\ A\_27a \\ & \quad A\_27b)\ V0f)\ (c\_2Epred\_set\_2EEMPTY\ A\_27a)) = (c\_2Epred\_set\_2EEMPTY \\ & \quad A\_27b))) \end{aligned} \tag{21}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0f \in (A\_27b^{A\_27a}).(\forall V1x \in A\_27a.(\forall V2s \in ( \\ & \quad 2^{A\_27a}).((ap\ (ap\ (c\_2Epred\_set\_2EIMAGE\ A\_27a\ A\_27b)\ V0f)\ (ap \\ & \quad (ap\ (c\_2Epred\_set\_2EINSERT\ A\_27a)\ V1x)\ V2s)) = (ap\ (ap\ (c\_2Epred\_set\_2EINSERT \\ & \quad A\_27b)\ (ap\ V0f\ V1x))\ (ap\ (ap\ (c\_2Epred\_set\_2EIMAGE\ A\_27a\ A\_27b) \\ & \quad V0f)\ V2s)))))) \end{aligned} \tag{22}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0f \in (A\_27b^{A\_27a}).(\forall V1x \in A\_27a.(\forall V2s \in ( \\ & \quad 2^{A\_27a}).(\forall V3t \in (2^{A\_27b}).((p\ (ap\ (ap\ (ap\ (c\_2Epred\_set\_2EINJ \\ & \quad A\_27a\ A\_27b)\ V0f)\ (ap\ (ap\ (c\_2Epred\_set\_2EINSERT\ A\_27a)\ V1x)\ V2s)) \\ & \quad V3t)) \Leftrightarrow ((p\ (ap\ (ap\ (ap\ (c\_2Epred\_set\_2EINJ\ A\_27a\ A\_27b)\ V0f)\ V2s) \\ & \quad V3t)) \wedge ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27b)\ (ap\ V0f\ V1x))\ V3t)) \wedge (\forall V4y \in \\ & \quad A\_27a.(((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V4y)\ V2s)) \wedge ((ap\ V0f\ V1x) = \\ & \quad (ap\ V0f\ V4y)))) \Rightarrow (V1x = V4y)))))))))) \end{aligned} \tag{23}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{(2^{A\_27a})}).(( \\ & \quad (p\ (ap\ V0P\ (c\_2Epred\_set\_2EEMPTY\ A\_27a))) \wedge (\forall V1s \in (2^{A\_27a}). \\ & \quad ((p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ V1s)) \wedge (p\ (ap\ V0P\ V1s)))) \Rightarrow \\ & \quad (\forall V2e \in A\_27a.((\neg(p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2e)\ V1s))) \Rightarrow \\ & \quad (p\ (ap\ V0P\ (ap\ (ap\ (c\_2Epred\_set\_2EINSERT\ A\_27a)\ V2e)\ V1s)))))) \Rightarrow \\ & \quad (\forall V3s \in (2^{A\_27a}).((p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a) \\ & \quad V3s)) \Rightarrow (p\ (ap\ V0P\ V3s)))))) \end{aligned} \tag{24}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0s \in (2^{A\_27a}).((p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a) \\ & \quad V0s)) \Rightarrow (\forall V1f \in (A\_27b^{A\_27a}).(p\ (ap\ (c\_2Epred\_set\_2EFINITE \\ & \quad A\_27b)\ (ap\ (ap\ (c\_2Epred\_set\_2EIMAGE\ A\_27a\ A\_27b)\ V1f)\ V0s)))))) \end{aligned} \tag{25}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow ((ap\ (c\_2Epred\_set\_2ECARD\ A\_27a) \\ & \quad (c\_2Epred\_set\_2EEMPTY\ A\_27a)) = c\_2Enum\_2E0) \end{aligned} \tag{26}$$

Assume the following.

$$\begin{aligned}
& \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow (\forall V0s \in (2^{A_{.27a}}).((p\ (ap \\
& (c_{.2}Epred\_set_{.2}EFINITE\ A_{.27a})\ V0s)) \Rightarrow (\forall V1x \in A_{.27a}.(( \\
& ap\ (c_{.2}Epred\_set_{.2}ECARD\ A_{.27a})\ (ap\ (ap\ (c_{.2}Epred\_set_{.2}EINSERT \\
& A_{.27a})\ V1x)\ V0s)) = (ap\ (ap\ (ap\ (c_{.2}Ebool_{.2}ECOND\ ty_{.2}Enum_{.2}Enum) \\
& (ap\ (ap\ (c_{.2}Ebool_{.2}EIN\ A_{.27a})\ V1x)\ V0s))\ (ap\ (c_{.2}Epred\_set_{.2}ECARD \\
& A_{.27a})\ V0s))\ (ap\ c_{.2}Enum_{.2}ESUC\ (ap\ (c_{.2}Epred\_set_{.2}ECARD\ A_{.27a}) \\
& V0s)))))))))
\end{aligned} \tag{27}$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty_{.2}Enum_{.2}Enum.(\forall V1n \in ty_{.2}Enum_{.2}Enum.( \\
& ((ap\ c_{.2}Enum_{.2}ESUC\ V0m) = (ap\ c_{.2}Enum_{.2}ESUC\ V1n)) \Leftrightarrow (V0m = V1n))))
\end{aligned} \tag{28}$$

**Theorem 1**

$$\begin{aligned}
& \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow \forall A_{.27b}.nonempty\ A_{.27b} \Rightarrow ( \\
& \forall V0f \in (A_{.27b}^{A_{.27a}}).(\forall V1t \in (2^{A_{.27b}}).(\forall V2s \in \\
& (2^{A_{.27a}}).((p\ (ap\ (c_{.2}Epred\_set_{.2}EFINITE\ A_{.27a})\ V2s)) \Rightarrow ((p\ ( \\
& ap\ (ap\ (ap\ (c_{.2}Epred\_set_{.2}EINJ\ A_{.27a}\ A_{.27b})\ V0f)\ V2s)\ V1t)) \Rightarrow (( \\
& ap\ (c_{.2}Epred\_set_{.2}ECARD\ A_{.27b})\ (ap\ (ap\ (c_{.2}Epred\_set_{.2}EIMAGE \\
& A_{.27a}\ A_{.27b})\ V0f)\ V2s)) = (ap\ (c_{.2}Epred\_set_{.2}ECARD\ A_{.27a})\ V2s)))))))))
\end{aligned}$$