

thm\_2Epred\_\_set\_2EPROD\_\_SET\_\_IMAGE\_\_REDUCTION  
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Zoc5VNGwMZSAHnmQj74dpVfJgG6b3GxQi)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A.27a : \iota.(\lambda V0x \in A.27a.(\lambda V1f \in (2^{A-27a}).(ap V1f V0x)))$

**Definition 4** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 5** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A.27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A-27a})) (\lambda V1x \in 2.V1x)) (\lambda V2x \in 2.V2x)))$

**Definition 6** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow \forall A.27b.nonempty A.27b \Rightarrow c\_2Epair\_2EABS\_prod A.27a A.27b \in ((ty\_2Epair\_2Eprod A.27a A.27b)^{(2^{A-27b})^{A-27a}}) \tag{2}$$

**Definition 7** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A.27a : \iota.\lambda A.27b : \iota.\lambda V0x \in A.27a.\lambda V1y \in A.27b.(ap (c\_2Epair\_2EABS\_prod A.27a A.27b) (V0x V1y))$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow \forall A.27b.nonempty A.27b \Rightarrow c\_2Epred\_set\_2EGSPEC A.27a A.27b \in ((2^{A-27a})^{(ty\_2Epair\_2Eprod A.27a 2)^{A-27b}}) \tag{3}$$

**Definition 8** We define  $c\_2Epred\_set\_2EIMAGE$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0f \in (A\_27b^{A\_27a}). \lambda V1s \in ($

**Definition 9** We define  $c\_2Ebool\_2E2E$  to be  $(ap (c\_2Ebool\_2E\_21) 2) (\lambda V0t \in 2.V0t)$ .

**Definition 10** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota. (\lambda V0x \in A\_27a.c\_2Ebool\_2E2E)$ .

**Definition 11** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c\_2Ebool\_2E\_21) 2) (\lambda V2t \in$

**Definition 12** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota. \lambda V0x \in A\_27a. \lambda V1s \in (2^{A\_27a}). (ap (c\_2E$

**Definition 13** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2. (ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E2E$

**Definition 14** We define  $c\_2Epred\_set\_2E2EDIFF$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). \lambda V1t \in (2^{A\_27a}). (ap (c\_2E$

**Definition 15** We define  $c\_2Epred\_set\_2E2EDELETE$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). \lambda V1x \in A\_27a. (ap (ap$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{4}$$

Let  $c\_2Earithmetic\_2E\_2A : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2A \in ((ty\_2Enum\_2Enum)^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum} \tag{5}$$

**Definition 16** We define  $c\_2Epred\_set\_2E2EFINITE$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). (ap (c\_2Ebool\_2E\_21) 2)$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \tag{6}$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum)^{\omega} \tag{7}$$

**Definition 17** We define  $c\_2Enum\_2E0$  to be  $(ap c\_2Enum\_2EABS\_num c\_2Enum\_2EZERO\_REP)$ .

**Definition 18** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega)^{ty\_2Enum\_2Enum} \tag{8}$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega)^{\omega} \tag{9}$$

**Definition 19** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. (ap c\_2Enum\_2EABS\_num$

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum)^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum} \tag{10}$$

**Definition 20** We define  $c\_2\text{Earithmetic\_2EBIT1}$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap (ap c\_2Earithmetic$

**Definition 21** We define  $c\_2\text{Earithmetic\_2ENUMERAL}$  to be  $\lambda V0x \in ty\_2Enum\_2Enum.V0x$ .

**Definition 22** We define  $c\_2\text{Ecombin\_2EK}$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0x \in A\_27a.(\lambda V1y \in A\_27b.V0x)$

**Definition 23** We define  $c\_2\text{Ecombin\_2ES}$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.(\lambda V0f \in ((A\_27c^{A\_27b})^{A\_27a})$

**Definition 24** We define  $c\_2\text{Ecombin\_2EI}$  to be  $\lambda A\_27a : \iota.(ap (ap (c\_2Ecombin\_2ES A\_27a (A\_27a^{A\_27a}) A$

Let  $c\_2\text{Epred\_set\_2EITSET} : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.\text{nonempty } A\_27a \Rightarrow \forall A\_27b.\text{nonempty } A\_27b \Rightarrow c\_2\text{Epred\_set\_2EITSET} \\ A\_27a A\_27b \in (((A\_27b^{A\_27b})^{(2^{A\_27a})})^{((A\_27b^{A\_27b})^{A\_27a})}) \end{aligned} \quad (11)$$

**Definition 25** We define  $c\_2\text{Epred\_set\_2EPROD\_IMAGE}$  to be  $\lambda A\_27a : \iota.\lambda V0f \in (ty\_2Enum\_2Enum^{A\_27a})$

**Definition 26** We define  $c\_2\text{Epred\_set\_2EPROD\_SET}$  to be  $(ap (c\_2Epred\_set\_2EPROD\_IMAGE ty\_2Enum$

Assume the following.

$$True \quad (12)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (( \\ (p V0t) \Rightarrow False) \Leftrightarrow (\neg (p V0t)))))) \end{aligned} \quad (13)$$

Assume the following.

$$\begin{aligned} ((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ ((\neg False) \Leftrightarrow True))) \end{aligned} \quad (14)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\ p V0t)))))) \end{aligned} \quad (15)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in ( \\ 2^{A\_27a}).(((p V0P) \wedge (\forall V2x \in A\_27a.(p (ap V1Q V2x)))) \Leftrightarrow (\forall V3x \in \\ A\_27a.((p V0P) \wedge (p (ap V1Q V3x)))))) \end{aligned} \quad (16)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1s \in \\ 2^{A\_27a}).((\neg(p (ap (ap (c\_2Ebool\_2EIN A\_27a) V0x) V1s))) \Leftrightarrow ((ap \\ (ap (c\_2Epred\_set\_2EDELETE A\_27a) V1s) V0x) = V1s)))) \end{aligned} \quad (17)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow ( \\ & \quad \forall V0f \in (A.27b^{A.27a}). (\forall V1x \in A.27a. (\forall V2s \in ( \\ & \quad 2^{A.27a}). ((ap\ (ap\ (c.2Epred\_set\_2EIMAGE\ A.27a\ A.27b)\ V0f)\ (ap \\ & \quad (ap\ (c.2Epred\_set\_2EINSERT\ A.27a)\ V1x)\ V2s)) = (ap\ (ap\ (c.2Epred\_set\_2EINSERT \\ & \quad A.27b)\ (ap\ V0f\ V1x))\ (ap\ (ap\ (c.2Epred\_set\_2EIMAGE\ A.27a\ A.27b) \\ & \quad V0f)\ V2s)))))) \\ & \hspace{15em} (18) \end{aligned}$$

Assume the following.

$$\begin{aligned} & (((ap\ c.2Epred\_set\_2EPROD\_SET\ (c.2Epred\_set\_2EEMPTY\ ty\_2Enum\_2Enum)) = \\ & \quad (ap\ c.2Earithmetic\_2ENUMERAL\ (ap\ c.2Earithmetic\_2EBIT1\ c.2Earithmetic\_2EZERO))) \wedge \\ & \quad (\forall V0x \in ty\_2Enum\_2Enum. (\forall V1s \in (2^{ty\_2Enum\_2Enum}). \\ & \quad ((p\ (ap\ (c.2Epred\_set\_2EFINITE\ ty\_2Enum\_2Enum)\ V1s)) \Rightarrow ((ap\ c.2Epred\_set\_2EPROD\_SET \\ & \quad (ap\ (ap\ (c.2Epred\_set\_2EINSERT\ ty\_2Enum\_2Enum)\ V0x)\ V1s)) = ( \\ & \quad ap\ (ap\ c.2Earithmetic\_2E\_2A\ V0x)\ (ap\ c.2Epred\_set\_2EPROD\_SET \\ & \quad (ap\ (ap\ (c.2Epred\_set\_2EDELETE\ ty\_2Enum\_2Enum)\ V1s)\ V0x))))))))) \\ & \hspace{15em} (19) \end{aligned}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (20)$$

Assume the following.

$$(\forall V0A \in 2. ((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (21)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & \quad (((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) \quad (22) \end{aligned}$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & \quad ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) \quad (23) \end{aligned}$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p\ V0p) \Leftrightarrow ( \\ & \quad (p\ V1q) \Leftrightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (p\ V2r))) \wedge (((p\ V0p) \vee ((\neg \\ & \quad p\ V2r)) \vee (\neg(p\ V1q)))) \wedge (((p\ V1q) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V0p)))) \wedge ((p\ V2r) \vee \\ & \quad ((\neg(p\ V1q)) \vee (\neg(p\ V0p)))))))))) \quad (25) \end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \wedge (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee (\neg(p \ V1q)) \vee (\neg(p \ V2r)))) \wedge (((p \ V1q) \vee \\
& (\neg(p \ V0p))) \wedge ((p \ V2r) \vee (\neg(p \ V0p))))))))))
\end{aligned} \tag{26}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \vee (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee (\neg(p \ V1q))) \wedge (((p \ V0p) \vee (\neg(p \ V2r))) \wedge \\
& ((p \ V1q) \vee ((p \ V2r) \vee (\neg(p \ V0p))))))))))
\end{aligned} \tag{27}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \Rightarrow (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee (p \ V1q)) \wedge (((p \ V0p) \vee (\neg(p \ V2r))) \wedge ( \\
& \neg(p \ V1q)) \vee ((p \ V2r) \vee (\neg(p \ V0p))))))))))
\end{aligned} \tag{28}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (((p \ V0p) \Leftrightarrow (\neg(p \ V1q))) \Leftrightarrow (((p \ V0p) \vee \\
& (p \ V1q)) \wedge ((\neg(p \ V1q)) \vee (\neg(p \ V0p))))))
\end{aligned} \tag{29}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \ V0p) \Rightarrow (p \ V1q))) \Rightarrow (p \ V0p))) \tag{30}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \ V0p) \Rightarrow (p \ V1q))) \Rightarrow (\neg(p \ V1q)))) \tag{31}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \ V0p) \vee (p \ V1q))) \Rightarrow (\neg(p \ V0p)))) \tag{32}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \ V0p) \vee (p \ V1q))) \Rightarrow (\neg(p \ V1q)))) \tag{33}$$

Assume the following.

$$(\forall V0p \in 2. ((\neg(\neg(p \ V0p))) \Rightarrow (p \ V0p))) \tag{34}$$

**Theorem 1**

$$\begin{aligned}
& \forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0f \in (ty\_2Enum\_2Enum^{A-27a}). \\
& (\forall V1s \in (2^{A-27a}). (\forall V2x \in A.27a. (((p \ (ap \ (c.2Epred\_set\_2EFINITE \\
& ty\_2Enum\_2Enum) \ (ap \ (ap \ (c.2Epred\_set\_2EIMAGE \ A.27a \ ty\_2Enum\_2Enum) \\
& V0f) \ V1s))) \wedge (\neg(p \ (ap \ (ap \ (c.2Ebool\_2EIN \ ty\_2Enum\_2Enum) \ (ap \ V0f \\
& V2x)) \ (ap \ (ap \ (c.2Epred\_set\_2EIMAGE \ A.27a \ ty\_2Enum\_2Enum) \ V0f) \\
& V1s)))))) \Rightarrow ((ap \ c.2Epred\_set\_2EPROD\_SET \ (ap \ (ap \ (c.2Epred\_set\_2EIMAGE \\
& A.27a \ ty\_2Enum\_2Enum) \ V0f) \ (ap \ (ap \ (c.2Epred\_set\_2EINSERT \ A.27a) \\
& V2x) \ V1s))) = (ap \ (ap \ c.2Earithmic\_2E\_2A \ (ap \ V0f \ V2x)) \ (ap \ c.2Epred\_set\_2EPROD\_SET \\
& (ap \ (ap \ (c.2Epred\_set\_2EIMAGE \ A.27a \ ty\_2Enum\_2Enum) \ V0f) \ V1s))))))
\end{aligned}$$