

# thm\_2Eproduct\_2EPRODUCT\_\_SING\_\_NUMSEG (TMYD5yb6Wz4xZYpYJeqVEecj5X9S7jdxKmr)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 4** We define  $c\_2Ebool\_2E\_EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p \Rightarrow q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_EF$

**Definition 7** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t$

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \tag{2}$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \tag{3}$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \tag{4}$$

**Definition 8** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap c\_2Enum\_2EABS\_num$

**Definition 9** We define `c_2Emin_2E_40` to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p \text{ (ap } P \ x)) \text{ then (the } (\lambda x. x \in A \wedge p \text{ of type } \iota \Rightarrow \iota).$

**Definition 10** We define `c_2Ebool_2E_3F` to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (\text{ap } V0P \text{ (ap (c_2Emin_2E_40$

**Definition 11** We define `c_2Eprim__rec_2E_3C` to be  $\lambda V0m \in \text{ty\_2Enum\_2Enum}. \lambda V1n \in \text{ty\_2Enum\_2Enum}$

**Definition 12** We define `c_2Ebool_2E_5C_2F` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (\text{ap (c_2Ebool_2E_21 } 2) (\lambda V2t \in$

**Definition 13** We define `c_2Earithmetic_2E_3C_3D` to be  $\lambda V0m \in \text{ty\_2Enum\_2Enum}. \lambda V1n \in \text{ty\_2Enum\_2Enum}$

Let `ty_2Epair_2Eprod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. \text{nonempty } A0 \Rightarrow \forall A1. \text{nonempty } A1 \Rightarrow \text{nonempty (ty\_2Epair\_2Eprod } A0 \ A1) \quad (5)$$

Let `c_2Epair_2EABS__prod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. 27a. \text{nonempty } A. 27a \Rightarrow \forall A. 27b. \text{nonempty } A. 27b \Rightarrow \text{c\_2Epair\_2EABS\_prod } A. 27a \ A. 27b \in ((\text{ty\_2Epair\_2Eprod } A. 27a \ A. 27b))^{((2^{A-27b})^{A-27a})} \quad (6)$$

**Definition 14** We define `c_2Epair_2E_2C` to be  $\lambda A. 27a : \iota. \lambda A. 27b : \iota. \lambda V0x \in A. 27a. \lambda V1y \in A. 27b. (\text{ap (c_2E$

Let `c_2Epred__set_2EGSPEC` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. 27a. \text{nonempty } A. 27a \Rightarrow \forall A. 27b. \text{nonempty } A. 27b \Rightarrow \text{c\_2Epred\_set\_2EGSPEC } A. 27a \ A. 27b \in ((2^{A-27a})^{((\text{ty\_2Epair\_2Eprod } A. 27a \ 2)^{A-27b})}) \quad (7)$$

**Definition 15** We define `c_2Eiterate_2E_2E_2E` to be  $\lambda V0m \in \text{ty\_2Enum\_2Enum}. \lambda V1n \in \text{ty\_2Enum\_2Enum}$

**Definition 16** We define `c_2Epred__set_2EEMPTY` to be  $\lambda A. 27a : \iota. (\lambda V0x \in A. 27a. \text{c\_2Ebool\_2EF}).$

**Definition 17** We define `c_2Ebool_2EIN` to be  $\lambda A. 27a : \iota. (\lambda V0x \in A. 27a. (\lambda V1f \in (2^{A-27a}). (\text{ap } V1f \ V0x))$

**Definition 18** We define `c_2Epred__set_2EINSERT` to be  $\lambda A. 27a : \iota. \lambda V0x \in A. 27a. \lambda V1s \in (2^{A-27a}). (\text{ap (c_2E$

Let `ty_2Ehreal_2Ehreal` :  $\iota$  be given. Assume the following.

$$\text{nonempty ty\_2Ehreal\_2Ehreal} \quad (8)$$

Let `ty_2Erealax_2Ereal` :  $\iota$  be given. Assume the following.

$$\text{nonempty ty\_2Erealax\_2Ereal} \quad (9)$$

Let `c_2Erealax_2Ereal__REP__CLASS` :  $\iota$  be given. Assume the following.

$$\text{c\_2Erealax\_2Ereal\_REP\_CLASS} \in ((2^{(\text{ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal}) \text{ty\_2Erealax\_2Ereal}})) \quad (10)$$

**Definition 19** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_2Emin\_2E.40 (t$   
Let  $c\_2Erealax\_2Etrealmul : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealmul \in (((ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)))(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal) \quad (11)$$

Let  $c\_2Erealax\_2Etrealeq : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealeq \in ((2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)})(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)) \quad (12)$$

Let  $c\_2Erealax\_2Ereal\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_ABS\_CLASS \in (ty\_2Erealax\_2Ereal)^{(2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)})(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)} \quad (13)$$

**Definition 20** We define  $c\_2Erealax\_2Ereal\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)$

**Definition 21** We define  $c\_2Erealax\_2Ereal\_mul$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$

**Definition 22** We define  $c\_2Eiterate\_2Eneutral$  to be  $\lambda A\_27a : \iota.\lambda V0op \in ((A\_27a^{A\_27a})^{A\_27a}).(ap (c\_2Emin\_2E.40 (t$

**Definition 23** We define  $c\_2Eiterate\_2Esupport$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0op \in ((A\_27b^{A\_27b})^{A\_27b}).\lambda V$

**Definition 24** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

**Definition 25** We define  $c\_2Epred\_set\_2EFINITE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).(ap (c\_2Ebool\_2E.21 (2$

**Definition 26** We define  $c\_2Eiterate\_2EITSET$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in ((A\_27a^{A\_27a})^{A\_27b}).\lambda V$

**Definition 27** We define  $c\_2Eiterate\_2Eiterate$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0op \in ((A\_27b^{A\_27b})^{A\_27b}).\lambda V$

**Definition 28** We define  $c\_2Eproduct\_2Eproduct$  to be  $\lambda A\_27a : \iota.(ap (c\_2Eiterate\_2Eiterate A\_27a ty\_2Erealax\_2Ereal$

Assume the following.

$$True \quad (14)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p V0t)) \Leftrightarrow (p V0t))) \quad (15)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (16)$$

Assume the following.

$$(\forall V0n \in ty\_2Enum\_2Enum.((ap (ap c\_2Eiterate\_2E.2E.2E V0n) V0n) = (ap (ap (c\_2Epred\_set\_2EINSERT ty\_2Enum\_2Enum) V0n) (c\_2Epred\_set\_2EEMPTY ty\_2Enum\_2Enum)))) \quad (17)$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0f \in (ty\_2Erealax\_2Ereal^{A\_27a}). \\
& \quad (\forall V1x \in A\_27a. ((ap (ap (c\_2Eproduct\_2Eproduct A\_27a) (ap \\
& (ap (c\_2Epred\_set\_2EINSERT A\_27a) V1x) (c\_2Epred\_set\_2EEMPTY \\
& \quad A\_27a))) V0f) = (ap V0f V1x)))) \quad (18)
\end{aligned}$$

**Theorem 1**

$$\begin{aligned}
& (\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}). (\forall V1n \in \\
& ty\_2Enum\_2Enum. ((ap (ap (c\_2Eproduct\_2Eproduct ty\_2Enum\_2Enum) \\
& (ap (ap c\_2Eiterate\_2E\_2E V1n) V1n)) V0f) = (ap V0f V1n))))
\end{aligned}$$