

# thm\_2Erat\_2ERAT\_\_1 (TMc- CWgo8KUiM1C3wxHKDtRrg7wG9e8Vds6y)

October 26, 2020

**Definition 1** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. \text{inj\_o } (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Ebool_2E_2T` to be  $(\text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define `c_2Ebool_2E_21` to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (\text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^{A-27a}))))$

**Definition 4** We define `c_2Ebool_2E_2F` to be  $(\text{ap } (\text{c\_2Ebool\_2E\_21 } 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define `c_2Emin_2E_3D_3D_3E` to be  $\lambda P \in 2. \lambda Q \in 2. \text{inj\_o } (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 6** We define `c_2Ebool_2E_2F_5C` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (\text{ap } (\text{c\_2Ebool\_2E\_21 } 2) (\lambda V2t \in 2.V2t))))$

**Definition 7** We define `c_2Emin_2E_40` to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p (\text{ap } P x)) \text{ then } (\text{the } (\lambda x. x \in A \wedge p x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 8** We define `c_2Ebool_2ECOND` to be  $\lambda A. 27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A. 27a. (\lambda V2t2 \in A. 27a. (\text{ap } (\text{c\_2Emin\_2E\_40 } (2^{A-27a}))))))$

Let `ty_2Enum_2Enum` :  $\iota$  be given. Assume the following.

$$\text{nonempty } \text{ty\_2Enum\_2Enum} \tag{1}$$

Let `ty_2Epair_2Eprod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. \text{nonempty } A0 \Rightarrow \forall A1. \text{nonempty } A1 \Rightarrow \text{nonempty } (\text{ty\_2Epair\_2Eprod } A0 A1) \tag{2}$$

Let `ty_2Einteger_2Eint` :  $\iota$  be given. Assume the following.

$$\text{nonempty } \text{ty\_2Einteger\_2Eint} \tag{3}$$

Let `c_2Einteger_2Eint__REP__CLASS` :  $\iota$  be given. Assume the following.

$$\text{c\_2Einteger\_2Eint\_REP\_CLASS} \in ((2^{(\text{ty\_2Epair\_2Eprod } \text{ty\_2Enum\_2Enum } \text{ty\_2Enum\_2Enum}) \text{ty\_2Einteger\_2Eint}})) \tag{4}$$

**Definition 9** We define  $c\_2Einteger\_2Eint\_REP$  to be  $\lambda V0a \in ty\_2Einteger\_2Eint.(ap (c\_2Emin\_2E\_40 (ty$

Let  $c\_2Einteger\_2Etint\_neg : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_neg \in ((ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum) (ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)) \quad (5)$$

Let  $c\_2Einteger\_2Etint\_eq : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_eq \in ((2^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)}) (ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)) \quad (6)$$

Let  $c\_2Einteger\_2Eint\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_ABS\_CLASS \in (ty\_2Einteger\_2Eint)^{(2^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})} \quad (7)$$

**Definition 10** We define  $c\_2Einteger\_2Eint\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)$

**Definition 11** We define  $c\_2Einteger\_2Eint\_neg$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.(ap c\_2Einteger\_2Eint$

Let  $c\_2Einteger\_2Eint\_of\_num : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_num \in (ty\_2Einteger\_2Eint)^{ty\_2Enum\_2Enum} \quad (8)$$

Let  $c\_2Einteger\_2Etint\_add : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_add \in (((ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum) (ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)) (ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)) \quad (9)$$

**Definition 12** We define  $c\_2Einteger\_2Eint\_add$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger$

Let  $c\_2Earithmetic\_2EEVEN : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2EEVEN \in (2^{ty\_2Enum\_2Enum}) \quad (10)$$

Let  $c\_2Earithmetic\_2EODD : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2EODD \in (2^{ty\_2Enum\_2Enum}) \quad (11)$$

**Definition 13** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E$

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (12)$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \quad (13)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum)^{\omega} \quad (14)$$

**Definition 14** We define  $c\_Enum\_ESUC$  to be  $\lambda V0m \in ty\_Enum\_Enum.(ap\ c\_Enum\_EABS\_num$

**Definition 15** We define  $c\_Ebool\_E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_Emin\_E\_40$

**Definition 16** We define  $c\_Eprim\_rec\_E\_3C$  to be  $\lambda V0m \in ty\_Enum\_Enum.\lambda V1n \in ty\_Enum\_Enum$

**Definition 17** We define  $c\_Earithmetic\_E\_3E$  to be  $\lambda V0m \in ty\_Enum\_Enum.\lambda V1n \in ty\_Enum\_Enum$

**Definition 18** We define  $c\_Ebool\_E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_Ebool\_E\_21\ 2)\ (\lambda V2t \in$

**Definition 19** We define  $c\_Earithmetic\_E\_3E\_3D$  to be  $\lambda V0m \in ty\_Enum\_Enum.\lambda V1n \in ty\_Enum\_Enum$

Let  $c\_Enum\_EZERO\_REP : \iota$  be given. Assume the following.

$$c\_Enum\_EZERO\_REP \in \omega \tag{15}$$

**Definition 20** We define  $c\_Enum\_E0$  to be  $(ap\ c\_Enum\_EABS\_num\ c\_Enum\_EZERO\_REP)$ .

**Definition 21** We define  $c\_Eprim\_rec\_EPRE$  to be  $\lambda V0m \in ty\_Enum\_Enum.(ap\ (ap\ (ap\ (c\_Ebool\_E$

Let  $c\_Earithmetic\_EEXP : \iota$  be given. Assume the following.

$$c\_Earithmetic\_EEXP \in ((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum}) \tag{16}$$

Let  $c\_Earithmetic\_E\_2D : \iota$  be given. Assume the following.

$$c\_Earithmetic\_E\_2D \in ((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum}) \tag{17}$$

Let  $c\_Earithmetic\_E\_2A : \iota$  be given. Assume the following.

$$c\_Earithmetic\_E\_2A \in ((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum}) \tag{18}$$

**Definition 22** We define  $c\_Enumeral\_EiZ$  to be  $\lambda V0x \in ty\_Enum\_Enum.V0x$ .

**Definition 23** We define  $c\_Earithmetic\_ENUMERAL$  to be  $\lambda V0x \in ty\_Enum\_Enum.V0x$ .

Let  $c\_Earithmetic\_E\_2B : \iota$  be given. Assume the following.

$$c\_Earithmetic\_E\_2B \in ((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum}) \tag{19}$$

**Definition 24** We define  $c\_Earithmetic\_EBIT2$  to be  $\lambda V0n \in ty\_Enum\_Enum.(ap\ (ap\ c\_Earithmetic$

**Definition 25** We define  $c\_Earithmetic\_EBIT1$  to be  $\lambda V0n \in ty\_Enum\_Enum.(ap\ (ap\ c\_Earithmetic$

**Definition 26** We define  $c\_Earithmetic\_EZERO$  to be  $c\_Enum\_E0$ .

**Definition 27** We define  $c\_Earithmetic\_E\_3C\_3D$  to be  $\lambda V0m \in ty\_Enum\_Enum.\lambda V1n \in ty\_Enum\_Enum$

Let  $ty\_2Efrac\_2Efrac : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Efrac\_2Efrac \quad (20)$$

Let  $ty\_2Erat\_2Erat : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erat\_2Erat \quad (21)$$

Let  $c\_2Erat\_2Erep\_rat\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erat\_2Erep\_rat\_CLASS \in ((2^{ty\_2Efrac\_2Efrac})^{ty\_2Erat\_2Erat}) \quad (22)$$

**Definition 28** We define  $c\_2Erat\_2Erep\_rat$  to be  $\lambda V0a \in ty\_2Erat\_2Erat.(ap\ (c\_2Emin\_2E40\ ty\_2Efrac)$

Let  $c\_2Efrac\_2Erep\_frac : \iota$  be given. Assume the following.

$$c\_2Efrac\_2Erep\_frac \in ((ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)^{ty\_2Efrac\_2Efrac}) \quad (23)$$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND \\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \end{aligned} \quad (24)$$

**Definition 29** We define  $c\_2Efrac\_2Efrac\_dnm$  to be  $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap\ (c\_2Epair\_2ESND\ t$

Let  $c\_2Einteger\_2Etint\_mul : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (25)$$

**Definition 30** We define  $c\_2Einteger\_2Eint\_mul$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST \\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \end{aligned} \quad (26)$$

**Definition 31** We define  $c\_2Efrac\_2Efrac\_nmr$  to be  $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap\ (c\_2Epair\_2EFST\ ty$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod \\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \end{aligned} \quad (27)$$

**Definition 32** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2$

Let  $c\_2Efrac\_2Eabs\_frac : \iota$  be given. Assume the following.

$$c\_2Efrac\_2Eabs\_frac \in (ty\_2Efrac\_2Efrac^{(ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)}) \quad (28)$$

**Definition 33** We define  $c\_2Efrac\_2Efrac\_add$  to be  $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Efrac$ .

**Definition 34** We define  $c\_2Erat\_2Erat\_equiv$  to be  $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Efrac$ .

Let  $c\_2Erat\_2Eabs\_rat\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erat\_2Eabs\_rat\_CLASS \in (ty\_2Erat\_2Erat^{(2^{ty\_2Efrac\_2Efrac})}) \quad (29)$$

**Definition 35** We define  $c\_2Erat\_2Eabs\_rat$  to be  $\lambda V0r \in ty\_2Efrac\_2Efrac.(ap\ c\_2Erat\_2Eabs\_rat\_CLASS)$ .

**Definition 36** We define  $c\_2Erat\_2Erat\_add$  to be  $\lambda V0r1 \in ty\_2Erat\_2Erat.\lambda V1r2 \in ty\_2Erat\_2Erat.(ap\ c\_2Erat\_2Erat\_add)$ .

**Definition 37** We define  $c\_2Efrac\_2Efrac\_1$  to be  $(ap\ c\_2Efrac\_2Eabs\_frac\ (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Efrac\_2Efrac))))$ .

**Definition 38** We define  $c\_2Erat\_2Erat\_1$  to be  $(ap\ c\_2Erat\_2Eabs\_rat\ c\_2Efrac\_2Efrac\_1)$ .

**Definition 39** We define  $c\_2Efrac\_2Efrac\_0$  to be  $(ap\ c\_2Efrac\_2Eabs\_frac\ (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Efrac\_2Efrac))))$ .

**Definition 40** We define  $c\_2Erat\_2Erat\_0$  to be  $(ap\ c\_2Erat\_2Eabs\_rat\ c\_2Efrac\_2Efrac\_0)$ .

**Definition 41** We define  $c\_2Ecombin\_2EK$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0x \in A\_27a.(\lambda V1y \in A\_27b.V0x))$ .

**Definition 42** We define  $c\_2Ecombin\_2ES$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.(\lambda V0f \in ((A\_27c^{A\_27b})^{A\_27a}))$ .

**Definition 43** We define  $c\_2Ecombin\_2EI$  to be  $\lambda A\_27a : \iota.(ap\ (ap\ (c\_2Ecombin\_2ES\ A\_27a\ (A\_27a^{A\_27a})))$ .

Let  $c\_2Earithmetic\_2Enum\_CASE : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Earithmetic\_2Enum\_CASE\ A\_27a \in \left( ((A\_27a^{(A\_27a^{ty\_2Enum\_2Enum})})_{A\_27a})^{ty\_2Enum\_2Enum} \right) \quad (30)$$

**Definition 44** We define  $c\_2Erelation\_2EWF$  to be  $\lambda A\_27a : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).(ap\ (c\_2Ebool\_2E\_2I))$ .

Let  $c\_2Ebool\_2EARB : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ebool\_2EARB\ A\_27a \in A\_27a \quad (31)$$

**Definition 45** We define  $c\_2Erelation\_2ERESTRICT$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (A\_27b^{A\_27a}).\lambda V1R \in (A\_27b^{A\_27a})$ .

**Definition 46** We define  $c\_2Erelation\_2ETC$  to be  $\lambda A\_27a : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1a \in A\_27a.\lambda V2b \in A\_27a$ .

**Definition 47** We define  $c\_2Erelation\_2Eapprox$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1M \in (A\_27a^{A\_27a})$ .

**Definition 48** We define  $c\_2Erelation\_2Ethe\_fun$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1M \in (A\_27a^{A\_27a})$ .

**Definition 49** We define  $c\_Erelation\_2EWFREC$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1M$

**Definition 50** We define  $c\_Erat\_2Erat\_of\_num$  to be  $(ap (ap (c\_Erelation\_2EWFREC ty\_2Enum\_2Enum$

**Definition 51** We define  $c\_Erelation\_2Einv\_image$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((2^{A\_27b})^{A\_27b}).\lambda V1M$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow ((\forall V0v \in A\_27a. (\forall V1f \in \\ & (A\_27a^{ty\_2Enum\_2Enum}). ((ap (ap (ap (c\_2Earithmetic\_2Enum\_CASE \\ & A\_27a) c\_2Enum\_2E0) V0v) V1f) = V0v))) \wedge (\forall V2n \in ty\_2Enum\_2Enum. \\ & (\forall V3v \in A\_27a. (\forall V4f \in (A\_27a^{ty\_2Enum\_2Enum}). ((ap \\ & (ap (ap (c\_2Earithmetic\_2Enum\_CASE A\_27a) (ap c\_2Enum\_2ESUC \\ & V2n)) V3v) V4f) = (ap V4f V2n)))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & ((ap (ap c\_2Earithmetic\_2E\_2B c\_2Enum\_2E0) V0m) = V0m) \wedge ((ap ( \\ & ap c\_2Earithmetic\_2E\_2B V0m) c\_2Enum\_2E0) = V0m) \wedge ((ap (ap c\_2Earithmetic\_2E\_2B \\ & (ap c\_2Enum\_2ESUC V0m)) V1n) = (ap c\_2Enum\_2ESUC (ap (ap c\_2Earithmetic\_2E\_2B \\ & V0m) V1n))) \wedge ((ap (ap c\_2Earithmetic\_2E\_2B V0m) (ap c\_2Enum\_2ESUC \\ & V1n)) = (ap c\_2Enum\_2ESUC (ap (ap c\_2Earithmetic\_2E\_2B V0m) V1n)))))) \end{aligned} \quad (33)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & (ap (ap c\_2Earithmetic\_2E\_2B V0m) V1n) = (ap (ap c\_2Earithmetic\_2E\_2B \\ & V1n) V0m)))) \end{aligned} \quad (34)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & (p (ap (ap c\_2Eprim\_rec\_2E\_3C (ap c\_2Enum\_2ESUC V0m)) (ap c\_2Enum\_2ESUC \\ & V1n))) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C V0m) V1n)))) \end{aligned} \quad (35)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & (\neg (p (ap (ap c\_2Eprim\_rec\_2E\_3C V0m) V1n))) \Leftrightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\ & V1n) V0m)))) \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\
& ((ap (ap c\_2Earithmetic\_2E\_2A c\_2Enum\_2E0) V0m) = c\_2Enum\_2E0) \wedge \\
& (((ap (ap c\_2Earithmetic\_2E\_2A V0m) c\_2Enum\_2E0) = c\_2Enum\_2E0) \wedge \\
& (((ap (ap c\_2Earithmetic\_2E\_2A (ap c\_2Earithmetic\_2ENUMERAL \\
& (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO))) V0m) = V0m) \wedge \\
& (((ap (ap c\_2Earithmetic\_2E\_2A V0m) (ap c\_2Earithmetic\_2ENUMERAL \\
& (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO))) = V0m) \wedge ( \\
& ((ap (ap c\_2Earithmetic\_2E\_2A (ap c\_2Enum\_2ESUC V0m)) V1n) = (ap \\
& (ap c\_2Earithmetic\_2E\_2B (ap (ap c\_2Earithmetic\_2E\_2A V0m) V1n)) \\
& V1n)) \wedge ((ap (ap c\_2Earithmetic\_2E\_2A V0m) (ap c\_2Enum\_2ESUC V1n)) = \\
& (ap (ap c\_2Earithmetic\_2E\_2B V0m) (ap (ap c\_2Earithmetic\_2E\_2A \\
& V0m) V1n))))))))))
\end{aligned} \tag{37}$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\
& \forall V2p \in ty\_2Enum\_2Enum. ((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& (ap (ap c\_2Earithmetic\_2E\_2B V0m) V1n)) (ap (ap c\_2Earithmetic\_2E\_2B \\
& V0m) V2p))) \Leftrightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D V1n) V2p))))))
\end{aligned} \tag{38}$$

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum. ((ap c\_2Enum\_2ESUC V0n) = (ap (ap \\
& c\_2Earithmetic\_2E\_2B (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 \\
& c\_2Earithmetic\_2EZERO))) V0n)))
\end{aligned} \tag{39}$$

Assume the following.

$$True \tag{40}$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in \\
A\_27a.(p V0t)) \Leftrightarrow (p V0t))) \tag{41}$$

Assume the following.

$$\begin{aligned}
& ((\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge \\
& ((\neg False) \Leftrightarrow True)))
\end{aligned} \tag{42}$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a. ((V0x = V0x) \Leftrightarrow \\
True)) \tag{43}$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in \\
A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \tag{44}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\
& (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow \neg(p V0t)) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow \neg( \\
& p V0t))))))
\end{aligned} \tag{45}$$

Assume the following.

$$(\forall V0t \in 2.(((p V0t) \Rightarrow False) \Leftrightarrow ((p V0t) \Leftrightarrow False))) \tag{46}$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((ap (c\_2Ecombin\_2EI \\
A\_27a) V0x) = V0x)) \tag{47}$$

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum.((ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Enum\_2ESUC V0n)) = (ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_of\_num \\
& V0n)) (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO))))))
\end{aligned} \tag{48}$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.( \\
& ((ap c\_2Einteger\_2Eint\_of\_num V0m) = (ap c\_2Einteger\_2Eint\_of\_num \\
& V1n)) \Leftrightarrow (V0m = V1n))))
\end{aligned} \tag{49}$$



Assume the following.

$$\begin{aligned}
& (\forall V0p \in ty\_2Einteger\_2Eint. (\forall V1n \in ty\_2Enum\_2Enum. \\
& (\forall V2m \in ty\_2Enum\_2Enum. (((ap (ap c\_2Einteger\_2Eint\_add \\
& (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) V0p) = V0p) \wedge ((( \\
& ap (ap c\_2Einteger\_2Eint\_add V0p) (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) = V0p) \wedge (((ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) = (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) \wedge \\
& (((ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_neg V0p)) = \\
& V0p) \wedge (((ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL V1n))) (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL V2m))) = (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Enumeral\_2EiZ (ap (ap c\_2Earithmetic\_2E\_2B \\
& V1n) V2m)))))) \wedge (((ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL V1n))) (ap c\_2Einteger\_2Eint\_neg \\
& (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& V2m)))) = (ap (ap (ap (c\_2Ebool\_2ECOND ty\_2Einteger\_2Eint) (ap \\
& (ap c\_2Earithmetic\_2E\_3C\_3D V2m) V1n)) (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2D V1n) \\
& V2m)))) (ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2D V2m) \\
& V1n)))))) \wedge (((ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_neg \\
& (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& V1n))) (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& V2m))) = (ap (ap (ap (c\_2Ebool\_2ECOND ty\_2Einteger\_2Eint) (ap ( \\
& ap c\_2Earithmetic\_2E\_3C\_3D V1n) V2m)) (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2D V2m) \\
& V1n)))) (ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2D V1n) \\
& V2m)))))) \wedge (((ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_neg \\
& (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& V1n))) (ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL V2m)))) = (ap c\_2Einteger\_2Eint\_neg \\
& (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& (ap c\_2Enumeral\_2EiZ (ap (ap c\_2Earithmetic\_2E\_2B V1n) V2m))))))))))))))
\end{aligned}$$

(50)

Assume the following.

$$\begin{aligned}
& ((\forall V0n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2B \\
& \quad c\_2Enum\_2E0) V0n) = V0n)) \wedge ((\forall V1n \in ty\_2Enum\_2Enum.((ap \\
& \quad (ap c\_2Earithmetic\_2E\_2B V1n) c\_2Enum\_2E0) = V1n)) \wedge ((\forall V2n \in \\
& \quad ty\_2Enum\_2Enum.(\forall V3m \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2B \\
& \quad (ap c\_2Earithmetic\_2ENUMERAL V2n)) (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad V3m)) = (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Enumeral\_2EiZ (ap \\
& \quad (ap c\_2Earithmetic\_2E\_2B V2n) V3m)))))) \wedge ((\forall V4n \in ty\_2Enum\_2Enum. \\
& \quad ((ap (ap c\_2Earithmetic\_2E\_2A c\_2Enum\_2E0) V4n) = c\_2Enum\_2E0)) \wedge \\
& \quad ((\forall V5n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2A \\
& \quad V5n) c\_2Enum\_2E0) = c\_2Enum\_2E0)) \wedge ((\forall V6n \in ty\_2Enum\_2Enum. \\
& \quad (\forall V7m \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2A ( \\
& \quad ap c\_2Earithmetic\_2ENUMERAL V6n)) (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad V7m)) = (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2A \\
& \quad V6n) V7m)))))) \wedge ((\forall V8n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2D \\
& \quad c\_2Enum\_2E0) V8n) = c\_2Enum\_2E0)) \wedge ((\forall V9n \in ty\_2Enum\_2Enum. \\
& \quad ((ap (ap c\_2Earithmetic\_2E\_2D V9n) c\_2Enum\_2E0) = V9n)) \wedge ((\forall V10n \in \\
& \quad ty\_2Enum\_2Enum.(\forall V11m \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2D \\
& \quad (ap c\_2Earithmetic\_2ENUMERAL V10n)) (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad V11m)) = (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2D \\
& \quad V10n) V11m)))))) \wedge ((\forall V12n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2EEXP \\
& \quad c\_2Enum\_2E0) (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 \\
& \quad V12n))) = c\_2Enum\_2E0)) \wedge ((\forall V13n \in ty\_2Enum\_2Enum.((ap \\
& \quad (ap c\_2Earithmetic\_2EEXP c\_2Enum\_2E0) (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad (ap c\_2Earithmetic\_2EBIT2 V13n))) = c\_2Enum\_2E0)) \wedge ((\forall V14n \in \\
& \quad ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2EEXP V14n) c\_2Enum\_2E0) = \\
& \quad (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO)))))) \wedge \\
& \quad ((\forall V15n \in ty\_2Enum\_2Enum.(\forall V16m \in ty\_2Enum\_2Enum. \\
& \quad ((ap (ap c\_2Earithmetic\_2EEXP (ap c\_2Earithmetic\_2ENUMERAL V15n)) \\
& \quad (ap c\_2Earithmetic\_2ENUMERAL V16m)) = (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad (ap (ap c\_2Earithmetic\_2EEXP V15n) V16m)))))) \wedge ((ap c\_2Enum\_2ESUC \\
& \quad c\_2Enum\_2E0) = (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 \\
& \quad c\_2Earithmetic\_2EZERO))) \wedge ((\forall V17n \in ty\_2Enum\_2Enum. ( \\
& \quad (ap c\_2Enum\_2ESUC (ap c\_2Earithmetic\_2ENUMERAL V17n)) = (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad (ap c\_2Enum\_2ESUC V17n)))))) \wedge ((ap c\_2Eprim\_rec\_2EPRE c\_2Enum\_2E0) = \\
& \quad c\_2Enum\_2E0) \wedge ((\forall V18n \in ty\_2Enum\_2Enum.((ap c\_2Eprim\_rec\_2EPRE \\
& \quad (ap c\_2Earithmetic\_2ENUMERAL V18n)) = (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad (ap c\_2Eprim\_rec\_2EPRE V18n)))))) \wedge ((\forall V19n \in ty\_2Enum\_2Enum. \\
& \quad (((ap c\_2Earithmetic\_2ENUMERAL V19n) = c\_2Enum\_2E0) \Leftrightarrow (V19n = c\_2Earithmetic\_2EZERO))) \wedge \\
& \quad ((\forall V20n \in ty\_2Enum\_2Enum.((c\_2Enum\_2E0 = (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad V20n)) \Leftrightarrow (V20n = c\_2Earithmetic\_2EZERO))) \wedge ((\forall V21n \in ty\_2Enum\_2Enum. \\
& \quad (\forall V22m \in ty\_2Enum\_2Enum.(((ap c\_2Earithmetic\_2ENUMERAL \\
& \quad V21n) = (ap c\_2Earithmetic\_2ENUMERAL V22m)) \Leftrightarrow (V21n = V22m)))))) \wedge \\
& \quad ((\forall V23n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Eprim\_rec\_2E\_3C \\
& \quad V23n) c\_2Enum\_2E0)) \Leftrightarrow False)) \wedge ((\forall V24n \in ty\_2Enum\_2Enum. \\
& \quad ((p (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Enum\_2E0) (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad V24n))) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Earithmetic\_2EZERO) \\
& \quad V24n)))))) \wedge ((\forall V25n \in ty\_2Enum\_2Enum.(\forall V26m \in ty\_2Enum\_2Enum. \\
& \quad ((p (ap (ap c\_2Eprim\_rec\_2E\_3C (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad V25n)) (ap c\_2Earithmetic\_2ENUMERAL V26m))) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C \\
& \quad V25n) V26m)))))) \wedge ((\forall V27n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmetic\_2E\_3E \\
& \quad c\_2Enum\_2E0) V27n)) \Leftrightarrow False)) \wedge ((\forall V28n \in ty\_2Enum\_2Enum. \\
& \quad ((p (ap (ap c\_2Earithmetic\_2E\_3E (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad V28n)) c\_2Enum\_2E0)) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Earithmetic\_2EZERO) \\
& \quad V28n)))))) \wedge ((\forall V29n \in ty\_2Enum\_2Enum.(\forall V30m \in ty\_2Enum\_2Enum. \\
& \quad ((p (ap (ap c\_2Earithmetic\_2E\_3E (ap c\_2Earithmetic\_2ENUMERAL \\
& \quad V29n)) (ap c\_2Earithmetic\_2ENUMERAL V30m))) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C \\
& \quad V30m) V29n)))))) \wedge ((\forall V31n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& \quad c\_2Enum\_2E0) V31n)) \Leftrightarrow True)) \wedge ((\forall V32n \in ty\_2Enum\_2Enum. \\
& \quad ((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2ENUMERAL
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum. (\forall V1m \in ty\_2Enum\_2Enum. ( \\
& ((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D c\_2Earithmetic\_2EZERO) V0n)) \Leftrightarrow \\
& True) \wedge (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2EBIT1 \\
& V0n)) c\_2Earithmetic\_2EZERO)) \Leftrightarrow False) \wedge (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& (ap c\_2Earithmetic\_2EBIT2 V0n)) c\_2Earithmetic\_2EZERO)) \Leftrightarrow False) \wedge \\
& (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2EBIT1 \\
& V0n)) (ap c\_2Earithmetic\_2EBIT1 V1m))) \Leftrightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& V0n) V1m))) \wedge (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2EBIT1 \\
& V0n)) (ap c\_2Earithmetic\_2EBIT2 V1m))) \Leftrightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& V0n) V1m))) \wedge (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2EBIT2 \\
& V0n)) (ap c\_2Earithmetic\_2EBIT1 V1m))) \Leftrightarrow (\neg (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& V1m) V0n)))) \wedge (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2EBIT2 \\
& V0n)) (ap c\_2Earithmetic\_2EBIT2 V1m))) \Leftrightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& V0n) V1m)))))))))
\end{aligned} \tag{52}$$

Assume the following.

$$(p (ap (c\_2Erelation\_2EWF ty\_2Enum\_2Enum) c\_2Eprim\_rec\_2E\_3C)) \tag{53}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow ( \\
& \forall V0R \in ((2^{A\_27b} A\_27b)^{A\_27b}). (\forall V1f \in (A\_27b^{A\_27a}). ( \\
& p (ap (c\_2Erelation\_2EWF A\_27b) V0R)) \Rightarrow (p (ap (c\_2Erelation\_2EWF \\
& A\_27a) (ap (ap (c\_2Erelation\_2Einv\_image A\_27a A\_27b) V0R) V1f))))))
\end{aligned} \tag{54}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow ( \\
& \forall V0f \in (A\_27b^{A\_27a}). (\forall V1R \in ((2^{A\_27a} A\_27a)^{A\_27a}). (\forall V2y \in \\
& A\_27a. (\forall V3z \in A\_27a. ((p (ap (ap V1R V2y) V3z)) \Rightarrow ((ap (ap (ap \\
& (ap (c\_2Erelation\_2ERESTRICT A\_27a A\_27b) V0f) V1R) V3z) V2y) = \\
& (ap V0f V2y))))))
\end{aligned} \tag{55}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow ( \\
& \forall V0M \in ((A\_27b^{A\_27a})^{(A\_27b^{A\_27a})}). (\forall V1R \in ((2^{A\_27a} A\_27a)^{A\_27a}). \\
& (\forall V2f \in (A\_27b^{A\_27a}). ((V2f = (ap (ap (c\_2Erelation\_2EWFREC \\
& A\_27a A\_27b) V1R) V0M)) \Rightarrow ((p (ap (c\_2Erelation\_2EWF A\_27a) V1R)) \Rightarrow \\
& (\forall V3x \in A\_27a. ((ap V2f V3x) = (ap (ap V0M (ap (ap (ap (c\_2Erelation\_2ERESTRICT \\
& A\_27a A\_27b) V2f) V1R) V3x)) V3x))))))
\end{aligned} \tag{56}$$

### Theorem 1

$$(c\_2Erat\_2Erat\_1 = (ap c\_2Erat\_2Erat\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
(ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO))))$$