thm_2Erat_2ERAT__LEQ__CALCULATE (TMNS-DNT2mFXbM8hmip4RYXVg4mAPL9dJLcW)

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Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o \ (x = y)$
of type $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Ebool_2ET *to be* $(ap \ (ap \ ($c_2Emin_2E_3D$ \ (2^2)) \ (\lambda V0x \in 2.V0x)) \ (\lambda V1x \in 2.V1$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty \ ty\_2Enum\_2Enum \tag{1}$$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty \ A0 \Rightarrow \forall A1.nonempty \ A1 \Rightarrow nonempty \ (ty\_2Epair\_2Eprod$$
$$A0 \ A1) \tag{2}$$

Let $ty\_2Einteger\_2Eint : \iota$ be given. Assume the following.

$$nonempty \ ty\_2Einteger\_2Eint \tag{3}$$

Let $c\_2Einteger\_2Eint\_\_REP\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_\_REP\_\_CLASS \in ((2^{(ty\_2Epair\_2Eprod \ ty\_2Enum\_2Enum \ ty\_2Enum\_2Enum)})^{ty\_2Einteger\_2} \tag{4}$$

Definition 3 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.\textbf{if} \ (\exists x \in A.p \ (ap \ P \ x)) \ \textbf{then} \ (the \ (\lambda x.x \in A \wedge p$
of type $\iota \Rightarrow \iota$.

Definition 4 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap \ (ap \ ($c_2Emin_2E_3D$ \ (2^{A_27}$

Definition 5 *We define* c_2Einteger_2Eint__REP *to be* $\lambda V0a \in ty\_2Einteger\_2Eint.(ap \ ($c_2Emin_2E_40$ \ (ty.$

Let $c\_2Einteger\_2Etint\_\_lt : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_lt \in ((2^{(ty\_2Epair\_2Eprod \ ty\_2Enum\_2Enum \ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod \ ty\_2Enum\_}} \tag{5}$$

Definition 6 *We define* c_2Einteger_2Eint_lt *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_2$

Definition 7 *We define* c_2Ebool_2EF *to be* $(ap\ (\text{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 8 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_o\ (p\ P \Rightarrow p\ Q)$
of type ι.

Definition 9 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \text{c\_2Emin\_2E\_3D\_3D\_3E}\ V0t)\ \text{c\_2Ebool\_2EF}$

Definition 10 *We define* c_2Einteger_2Eint_le *to be* $\lambda V0x \in ty\_2Einteger\_2Eint.\lambda V1y \in ty\_2Einteger\_2E$

Definition 11 *We define* c_2Ebool_2E_5C_2F *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\text{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in$

Let $ty\_2Erat\_2Erat : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Erat\_2Erat \qquad (6)$$

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \qquad (7)$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \qquad (8)$$

Definition 12 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Definition 13 *We define* c_2Earithmetic_2EZERO *to be* c_2Enum_2E0.

Let $c\_2Enum\_2EREP\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EREP\_\_num \in (omega^{ty\_2Enum\_2Enum}) \qquad (9)$$

Let $c\_2Enum\_2ESUC\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2ESUC\_\_REP \in (omega^{omega}) \qquad (10)$$

Definition 14 *We define* c_2Enum_2ESUC *to be* $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_\_num$

Let $c\_2Earithmetic\_2E\_2B : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum})$$
$$\qquad (11)$$

Definition 15 *We define* c_2Earithmetic_2EBIT1 *to be* $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

Definition 16 *We define* c_2Earithmetic_2ENUMERAL *to be* $\lambda V0x \in ty\_2Enum\_2Enum.V0x$.

Let $c\_2Einteger\_2Eint\_of\_num : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_num \in (ty\_2Einteger\_2Eint^{ty\_2Enum\_2Enum}) \quad (12)$$

Let $ty\_2Efrac\_2Efrac : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Efrac\_2Efrac \quad (13)$$

Let $c\_2Erat\_2Erep\_rat\_CLASS : \iota$ be given. Assume the following.

$$c\_2Erat\_2Erep\_rat\_CLASS \in ((2^{ty\_2Efrac\_2Efrac})^{ty\_2Erat\_2Erat}) \quad (14)$$

Definition 17 *We define* c_2Erat_2Erep_rat *to be* $\lambda V0a \in ty\_2Erat\_2Erat.(ap$ (c_2Emin_2E_40 $ty\_2Efrac\_$

Let $c\_2Efrac\_2Erep\_frac : \iota$ be given. Assume the following.

$$c\_2Efrac\_2Erep\_frac \in ((ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint$$
$$ty\_2Einteger\_2Eint)^{ty\_2Efrac\_2Efrac}) \quad (15)$$

Let $c\_2Epair\_2ESND : \iota{\Rightarrow}\iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}\forall A\_27b.nonempty\ A\_27b{\Rightarrow}c\_2Epair\_2ESND$$
$$A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$$
$$(16)$$

Definition 18 *We define* c_2Efrac_2Efrac_dnm *to be* $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap$ (c_2Epair_2ESND t

Let $c\_2Epair\_2EFST : \iota{\Rightarrow}\iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}\forall A\_27b.nonempty\ A\_27b{\Rightarrow}c\_2Epair\_2EFST$$
$$A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$$
$$(17)$$

Definition 19 *We define* c_2Efrac_2Efrac_nmr *to be* $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap$ (c_2Epair_2EFST ty

Let $c\_2Einteger\_2Etint\_neg : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_neg \in ((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum$$
$$ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})$$
$$(18)$$

Let $c\_2Einteger\_2Etint\_eq : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum}$$
$$(19)$$

Let $c\_2Einteger\_2Eint\_ABS\_CLASS : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_ABS\_CLASS \in (ty\_2Einteger\_2Eint^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}}$$
$$(20)$$

Definition 20 *We define* c_2Einteger_2Eint_ABS *to be* $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_$

Definition 21 *We define* c_2Einteger_2Eint__neg *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.(ap$ c_2Einteger_2Eint_

Definition 22 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap$ (c_2Ebool_2E_21 2) $(\lambda V2t \in$

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod$$
$$A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})})$$
$$(21)$$

Definition 23 *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2$

Let $c\_2Efrac\_2Eabs\_\_frac : \iota$ be given. Assume the following.

$$c\_2Efrac\_2Eabs\_\_frac \in (ty\_2Efrac\_2Efrac^{(ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)})$$
$$(22)$$

Definition 24 *We define* c_2Efrac_2Efrac__ainv *to be* $\lambda V0f1 \in ty\_2Efrac\_2Efrac.(ap\ c\_2Efrac\_2Eabs\_\_fr$

Let $c\_2Einteger\_2Etint\_\_mul : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum$$
$$ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty}$$
$$(23)$$

Definition 25 *We define* c_2Einteger_2Eint__mul *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteg$

Let $c\_2Einteger\_2Etint\_\_add : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum$$
$$ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty}$$
$$(24)$$

Definition 26 *We define* c_2Einteger_2Eint__add *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteg$

Definition 27 *We define* c_2Efrac_2Efrac__add *to be* $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Ef$

Definition 28 *We define* c_2Efrac_2Efrac__sub *to be* $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Ef$

Definition 29 *We define* c_2Erat_2Erat__equiv *to be* $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Ef$

Let $c\_2Erat\_2Eabs\_\_rat\_CLASS : \iota$ be given. Assume the following.

$$c\_2Erat\_2Eabs\_\_rat\_CLASS \in (ty\_2Erat\_2Erat^{(2^{ty\_2Efrac\_2Efrac})})$$
$$(25)$$

Definition 30 *We define* c_2Erat_2Eabs__rat *to be* $\lambda V0r \in ty\_2Efrac\_2Efrac.(ap\ c\_2Erat\_2Eabs\_\_rat\_\_C1$

Definition 31 *We define* c_2Erat_2Erat__sub *to be* $\lambda V0r1 \in ty\_2Erat\_2Erat.\lambda V1r2 \in ty\_2Erat\_2Erat.(ap$

Definition 32 *We define* c_2Ebool_2ECOND *to be* $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

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Definition 33 *We define* c_2EintExtension_2ESGN *to be* $\lambda V0x \in ty\_2Einteger\_2Eint.(ap\ (ap\ (ap\ ($c_2Eboo...

Definition 34 *We define* c_2Efrac_2Efrac__sgn *to be* $\lambda V0f1 \in ty\_2Efrac\_2Efrac.(ap$ c_2EintExtension_2ES...

Definition 35 *We define* c_2Erat_2Erat__sgn *to be* $\lambda V0r \in ty\_2Erat\_2Erat.(ap$ c_2Efrac_2Efrac__sgn $(ap$ c_2...

Definition 36 *We define* c_2Erat_2Erat__les *to be* $\lambda V0r1 \in ty\_2Erat\_2Erat.\lambda V1r2 \in ty\_2Erat\_2Erat.(ap\ ($

Definition 37 *We define* c_2Erat_2Erat__leq *to be* $\lambda V0r1 \in ty\_2Erat\_2Erat.\lambda V1r2 \in ty\_2Erat\_2Erat.(ap\ ($

Assume the following.
$$True \tag{26}$$

Assume the following.
$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \tag{27}$$

Assume the following.
$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.$$
$$((p\ (ap\ (ap\ \text{c\_2Einteger\_2Eint\_\_le}\ V0x)\ V1y)) \Leftrightarrow ((p\ (ap\ (ap\ \text{c\_2Einteger\_2Eint\_\_lt}$$
$$V0x)\ V1y)) \vee (V0x = V1y)))))$$
$$\tag{28}$$

Assume the following.
$$(\forall V0f1 \in ty\_2Efrac\_2Efrac.(\forall V1f2 \in ty\_2Efrac\_2Efrac.$$
$$(((ap\ \text{c\_2Erat\_2Eabs\_\_rat}\ V0f1) = (ap\ \text{c\_2Erat\_2Eabs\_\_rat}\ V1f2)) \Leftrightarrow$$
$$((ap\ (ap\ \text{c\_2Einteger\_2Eint\_\_mul}\ (ap\ \text{c\_2Efrac\_2Efrac\_\_nmr}\ V0f1))$$
$$(ap\ \text{c\_2Efrac\_2Efrac\_\_dnm}\ V1f2)) = (ap\ (ap\ \text{c\_2Einteger\_2Eint\_\_mul}$$
$$(ap\ \text{c\_2Efrac\_2Efrac\_\_nmr}\ V1f2))\ (ap\ \text{c\_2Efrac\_2Efrac\_\_dnm}\ V0f1)))))) \tag{29}$$

Assume the following.
$$(\forall V0f1 \in ty\_2Efrac\_2Efrac.(\forall V1f2 \in ty\_2Efrac\_2Efrac.$$
$$((p\ (ap\ (ap\ \text{c\_2Erat\_2Erat\_\_les}\ (ap\ \text{c\_2Erat\_2Eabs\_\_rat}\ V0f1))\ ($$
$$ap\ \text{c\_2Erat\_2Eabs\_\_rat}\ V1f2))) \Leftrightarrow (p\ (ap\ (ap\ \text{c\_2Einteger\_2Eint\_\_lt}$$
$$(ap\ (ap\ \text{c\_2Einteger\_2Eint\_\_mul}\ (ap\ \text{c\_2Efrac\_2Efrac\_\_nmr}\ V0f1))$$
$$(ap\ \text{c\_2Efrac\_2Efrac\_\_dnm}\ V1f2)))\ (ap\ (ap\ \text{c\_2Einteger\_2Eint\_\_mul}$$
$$(ap\ \text{c\_2Efrac\_2Efrac\_\_nmr}\ V1f2))\ (ap\ \text{c\_2Efrac\_2Efrac\_\_dnm}\ V0f1)))))))$$
$$\tag{30}$$

Theorem 1
$$(\forall V0f1 \in ty\_2Efrac\_2Efrac.(\forall V1f2 \in ty\_2Efrac\_2Efrac.$$
$$((p\ (ap\ (ap\ \text{c\_2Erat\_2Erat\_\_leq}\ (ap\ \text{c\_2Erat\_2Eabs\_\_rat}\ V0f1))\ ($$
$$ap\ \text{c\_2Erat\_2Eabs\_\_rat}\ V1f2))) \Leftrightarrow (p\ (ap\ (ap\ \text{c\_2Einteger\_2Eint\_\_le}$$
$$(ap\ (ap\ \text{c\_2Einteger\_2Eint\_\_mul}\ (ap\ \text{c\_2Efrac\_2Efrac\_\_nmr}\ V0f1))$$
$$(ap\ \text{c\_2Efrac\_2Efrac\_\_dnm}\ V1f2)))\ (ap\ (ap\ \text{c\_2Einteger\_2Eint\_\_mul}$$
$$(ap\ \text{c\_2Efrac\_2Efrac\_\_nmr}\ V1f2))\ (ap\ \text{c\_2Efrac\_2Efrac\_\_dnm}\ V0f1)))))))$$