

thm_2Erat_2ERAT__LES__RMUL__NEG
(TMLKvyQ6nzcjsjT3EsAUC6phAYTp8z9ZFA)

October 26, 2020

Let $ty_2Enum_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Enum_2Enum \tag{1}$$

Let $c_2Earithmetic_2EEVEN : \iota$ be given. Assume the following.

$$c_2Earithmetic_2EEVEN \in (2^{ty_2Enum_2Enum}) \tag{2}$$

Let $c_2Earithmetic_2EODD : \iota$ be given. Assume the following.

$$c_2Earithmetic_2EODD \in (2^{ty_2Enum_2Enum}) \tag{3}$$

Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_T$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c_2Emin_2E_3D (2^{A-27a}))$

Definition 4 We define $c_2Ebool_2E_F$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 5 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p \Rightarrow p \Rightarrow Q)$ of type ι .

Definition 6 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_F$

Definition 7 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t$

Let $c_2Enum_2EREP_num : \iota$ be given. Assume the following.

$$c_2Enum_2EREP_num \in (omega^{ty_2Enum_2Enum}) \tag{4}$$

Let $c_2Enum_2ESUC_REP : \iota$ be given. Assume the following.

$$c_2Enum_2ESUC_REP \in (omega^{omega}) \tag{5}$$

Let $c_2Enum_2EABS_num : \iota$ be given. Assume the following.

$$c_2Enum_2EABS_num \in (ty_2Enum_2Enum^{omega}) \tag{6}$$

Definition 8 We define c_2Enum_2ESUC to be $\lambda V0m \in ty_2Enum_2Enum.(ap\ c_2Enum_2EABS_num$

Definition 9 We define $c_2Emin_2E_40$ to be $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p\ (ap\ P\ x))$ **then** (the $(\lambda x.x \in A \wedge p$
of type $\iota \Rightarrow \iota$).

Definition 10 We define $c_2Ebool_2E_3F$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap\ V0P\ (ap\ (c_2Emin_2E_40$

Definition 11 We define $c_2Eprim_rec_2E_3C$ to be $\lambda V0m \in ty_2Enum_2Enum.\lambda V1n \in ty_2Enum_2Enum$

Definition 12 We define $c_2Earithmetic_2E_3E$ to be $\lambda V0m \in ty_2Enum_2Enum.\lambda V1n \in ty_2Enum_2Enum$

Definition 13 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V2t \in$

Definition 14 We define $c_2Earithmetic_2E_3E_3D$ to be $\lambda V0m \in ty_2Enum_2Enum.\lambda V1n \in ty_2Enum_2Enum$

Definition 15 We define $c_2Earithmetic_2E_3C_3D$ to be $\lambda V0m \in ty_2Enum_2Enum.\lambda V1n \in ty_2Enum_2Enum$

Let $c_2Enum_2EZERO_REP : \iota$ be given. Assume the following.

$$c_2Enum_2EZERO_REP \in \text{omega} \quad (7)$$

Definition 16 We define c_2Enum_2E0 to be $(ap\ c_2Enum_2EABS_num\ c_2Enum_2EZERO_REP)$.

Definition 17 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A_27a.(\lambda V2t2 \in A_27a.($

Definition 18 We define $c_2Eprim_rec_2EPRE$ to be $\lambda V0m \in ty_2Enum_2Enum.(ap\ (ap\ (ap\ (c_2Ebool_2E$

Let $c_2Earithmetic_2EEXP : \iota$ be given. Assume the following.

$$c_2Earithmetic_2EEXP \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \quad (8)$$

Let $c_2Earithmetic_2E_2D : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2D \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \quad (9)$$

Let $c_2Earithmetic_2E_2A : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2A \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \quad (10)$$

Definition 19 We define $c_2Enumeral_2EiZ$ to be $\lambda V0x \in ty_2Enum_2Enum.V0x$.

Let $c_2Earithmetic_2E_2B : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2B \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \quad (11)$$

Definition 20 We define $c_2Earithmetic_2EBIT2$ to be $\lambda V0n \in ty_2Enum_2Enum.(ap\ (ap\ c_2Earithmetic$

Let $ty_2Efrac_2Efrac : \iota$ be given. Assume the following.

$$nonempty\ ty_2Efrac_2Efrac \quad (12)$$

Let $ty_2Erat_2Erat : \iota$ be given. Assume the following.

$$nonempty\ ty_2Erat_2Erat \quad (13)$$

Let $c_2Erat_2Erep_rat_CLASS : \iota$ be given. Assume the following.

$$c_2Erat_2Erep_rat_CLASS \in ((2^{ty_2Efrac_2Efrac})^{ty_2Erat_2Erat}) \quad (14)$$

Definition 21 We define $c_2Erat_2Erep_rat$ to be $\lambda V0a \in ty_2Erat_2Erat.(ap\ (c_2Emin_2E_40\ ty_2Efrac)$

Let $ty_2Einteger_2Eint : \iota$ be given. Assume the following.

$$nonempty\ ty_2Einteger_2Eint \quad (15)$$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \quad (16)$$

Let $c_2Efrac_2Erep_frac : \iota$ be given. Assume the following.

$$c_2Efrac_2Erep_frac \in ((ty_2Epair_2Eprod\ ty_2Einteger_2Eint\ ty_2Einteger_2Eint)^{ty_2Efrac_2Efrac}) \quad (17)$$

Let $c_2Epair_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2ESND\ A_27a\ A_27b \in (A_27b^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \quad (18)$$

Definition 22 We define $c_2Efrac_2Efrac_dnm$ to be $\lambda V0f \in ty_2Efrac_2Efrac.(ap\ (c_2Epair_2ESND\ t$

Let $c_2Epair_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EFST\ A_27a\ A_27b \in (A_27a^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \quad (19)$$

Definition 23 We define $c_2Efrac_2Efrac_nmr$ to be $\lambda V0f \in ty_2Efrac_2Efrac.(ap\ (c_2Epair_2EFST\ t$

Let $c_2Einteger_2Eint_REP_CLASS : \iota$ be given. Assume the following.

$$c_2Einteger_2Eint_REP_CLASS \in ((2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})^{ty_2Einteger_2Eint}) \quad (20)$$

Definition 24 We define $c_2Einteger_2Eint_REP$ to be $\lambda V0a \in ty_2Einteger_2Eint.(ap\ (c_2Emin_2E_40\ t$

Let $c_2Einteger_2Etint_neg : \iota$ be given. Assume the following.

$$\begin{aligned} c_2Einteger_2Etint_neg \in & ((ty_2Epair_2Eprod\ ty_2Enum_2Enum \\ & ty_2Enum_2Enum)^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)}) \end{aligned} \quad (21)$$

Let $c_2Einteger_2Etint_eq : \iota$ be given. Assume the following.

$$c_2Einteger_2Etint_eq \in ((2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum)}) \quad (22)$$

Let $c_2Einteger_2Eint_ABS_CLASS : \iota$ be given. Assume the following.

$$c_2Einteger_2Eint_ABS_CLASS \in (ty_2Einteger_2Eint)^{(2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})} \quad (23)$$

Definition 25 We define $c_2Einteger_2Eint_ABS$ to be $\lambda V0r \in (ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)$

Definition 26 We define $c_2Einteger_2Eint_neg$ to be $\lambda V0T1 \in ty_2Einteger_2Eint.(ap\ c_2Einteger_2Eint)$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow & c_2Epair_2EABS_prod \\ A_27a\ A_27b \in & ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b} A_27a)}) \end{aligned} \quad (24)$$

Definition 27 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap\ (c_2Epair_2EABS_prod\ A_27a\ A_27b))$

Let $c_2Efrac_2Eabs_frac : \iota$ be given. Assume the following.

$$c_2Efrac_2Eabs_frac \in (ty_2Efrac_2Efrac)^{(ty_2Epair_2Eprod\ ty_2Einteger_2Eint\ ty_2Einteger_2Eint)} \quad (25)$$

Definition 28 We define $c_2Efrac_2Efrac_ainv$ to be $\lambda V0f1 \in ty_2Efrac_2Efrac.(ap\ c_2Efrac_2Eabs_frac\ f1)$

Let $c_2Einteger_2Etint_mul : \iota$ be given. Assume the following.

$$\begin{aligned} c_2Einteger_2Etint_mul \in & (((ty_2Epair_2Eprod\ ty_2Enum_2Enum \\ & ty_2Enum_2Enum)^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)}) \end{aligned} \quad (26)$$

Definition 29 We define $c_2Einteger_2Eint_mul$ to be $\lambda V0T1 \in ty_2Einteger_2Eint.\lambda V1T2 \in ty_2Einteger_2Eint.(c_2Einteger_2Etint_mul\ T1\ T2)$

Let $c_2Einteger_2Etint_add : \iota$ be given. Assume the following.

$$\begin{aligned} c_2Einteger_2Etint_add \in & (((ty_2Epair_2Eprod\ ty_2Enum_2Enum \\ & ty_2Enum_2Enum)^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)}) \end{aligned} \quad (27)$$

Definition 30 We define $c_2Einteger_2Eint_add$ to be $\lambda V0T1 \in ty_2Einteger_2Eint.\lambda V1T2 \in ty_2Einteger_2Eint.(c_2Einteger_2Etint_add\ T1\ T2)$

Definition 31 We define $c_2Efrac_2Efrac_add$ to be $\lambda V0f1 \in ty_2Efrac_2Efrac.\lambda V1f2 \in ty_2Efrac_2Efrac.(c_2Efrac_2Eabs_frac\ f1\ f2)$

Definition 32 We define $c_Efrac_Efrac_sub$ to be $\lambda V0f1 \in ty_2Efrac_2Efrac.\lambda V1f2 \in ty_2Efrac_2Efrac$

Definition 33 We define $c_Erat_2Erat_equiv$ to be $\lambda V0f1 \in ty_2Efrac_2Efrac.\lambda V1f2 \in ty_2Efrac_2Efrac$

Let $c_2Erat_2Eabs_rat_CLASS : \iota$ be given. Assume the following.

$$c_2Erat_2Eabs_rat_CLASS \in (ty_2Erat_2Erat^{(2^{ty_2Efrac_2Efrac})}) \quad (28)$$

Definition 34 We define $c_2Erat_2Eabs_rat$ to be $\lambda V0r \in ty_2Efrac_2Efrac.(ap\ c_2Erat_2Eabs_rat_CLASS$

Definition 35 We define $c_2Erat_2Erat_sub$ to be $\lambda V0r1 \in ty_2Erat_2Erat.\lambda V1r2 \in ty_2Erat_2Erat.(ap$

Definition 36 We define $c_2Erat_2Erat_add$ to be $\lambda V0r1 \in ty_2Erat_2Erat.\lambda V1r2 \in ty_2Erat_2Erat.(ap$

Definition 37 We define $c_2Earithmetic_2EZERO$ to be c_2Enum_2E0 .

Definition 38 We define $c_2Earithmetic_2EBIT1$ to be $\lambda V0n \in ty_2Enum_2Enum.(ap\ (ap\ c_2Earithmetic$

Definition 39 We define $c_2Earithmetic_2ENUMERAL$ to be $\lambda V0x \in ty_2Enum_2Enum.V0x$.

Let $c_2Einteger_2Eint_of_num : \iota$ be given. Assume the following.

$$c_2Einteger_2Eint_of_num \in (ty_2Einteger_2Eint^{ty_2Enum_2Enum}) \quad (29)$$

Let $c_2Einteger_2Etint_lt : \iota$ be given. Assume the following.

$$c_2Einteger_2Etint_lt \in ((2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum)}) \quad (30)$$

Definition 40 We define $c_2Einteger_2Eint_lt$ to be $\lambda V0T1 \in ty_2Einteger_2Eint.\lambda V1T2 \in ty_2Einteger$

Definition 41 We define $c_2EintExtension_2ESGN$ to be $\lambda V0x \in ty_2Einteger_2Eint.(ap\ (ap\ (ap\ (c_2Eboo$

Definition 42 We define $c_2Efrac_2Efrac_sgn$ to be $\lambda V0f1 \in ty_2Efrac_2Efrac.(ap\ c_2EintExtension_2ES$

Definition 43 We define $c_2Erat_2Erat_sgn$ to be $\lambda V0r \in ty_2Erat_2Erat.(ap\ c_2Efrac_2Efrac_sgn\ (ap\ c$

Definition 44 We define $c_2Erat_2Erat_les$ to be $\lambda V0r1 \in ty_2Erat_2Erat.\lambda V1r2 \in ty_2Erat_2Erat.(ap\ ($

Definition 45 We define $c_2Erat_2Erat_gre$ to be $\lambda V0r1 \in ty_2Erat_2Erat.\lambda V1r2 \in ty_2Erat_2Erat.(ap\ ($

Definition 46 We define $c_2Erat_2Erat_ainv$ to be $\lambda V0r1 \in ty_2Erat_2Erat.(ap\ c_2Erat_2Eabs_rat\ (ap\ c$

Definition 47 We define $c_2Efrac_2Efrac_mul$ to be $\lambda V0f1 \in ty_2Efrac_2Efrac.\lambda V1f2 \in ty_2Efrac_2Efrac$

Definition 48 We define $c_2Erat_2Erat_mul$ to be $\lambda V0r1 \in ty_2Erat_2Erat.\lambda V1r2 \in ty_2Erat_2Erat.(ap$

Definition 49 We define $c_2Einteger_2EABS$ to be $\lambda V0n \in ty_2Einteger_2Eint.(ap\ (ap\ (ap\ (c_2Eboo$

Definition 50 We define $c_2Efrac_2Efrac_minv$ to be $\lambda V0f1 \in ty_2Efrac_2Efrac.(ap\ c_2Efrac_2Eabs_f$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \wedge (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \wedge True) \Leftrightarrow \\
& (p \ V0t)) \wedge (((False \wedge (p \ V0t)) \Leftrightarrow False) \wedge (((p \ V0t) \wedge False) \Leftrightarrow False) \wedge \\
& (((p \ V0t) \wedge (p \ V0t)) \Leftrightarrow (p \ V0t)))))) \quad (35)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& ((\forall V0t \in 2.((\neg(\neg(p \ V0t))) \Leftrightarrow (p \ V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge \\
& ((\neg False) \Leftrightarrow True))) \quad (36)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a.((V0x = V0x) \Leftrightarrow \\
& True)) \quad (37)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a.(\forall V1y \in \\
& A_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (38)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Leftrightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow True) \Leftrightarrow \\
& (p \ V0t)) \wedge (((False \Leftrightarrow (p \ V0t)) \Leftrightarrow (\neg(p \ V0t))) \wedge (((p \ V0t) \Leftrightarrow False) \Leftrightarrow (\neg \\
& p \ V0t)))))) \quad (39)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty_2Einteger_2Eint.((ap \ (ap \ c_2Einteger_2Eint_mul \\
& (ap \ c_2Einteger_2Eint_of_num \ (ap \ c_2Earithmetic_2ENUMERAL \\
& (ap \ c_2Earithmetic_2EBIT1 \ c_2Earithmetic_2EZERO)))) \ V0x) = V0x)) \quad (40)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty_2Einteger_2Eint.(\forall V1y \in ty_2Einteger_2Eint. \\
& (((ap \ c_2Einteger_2Eint_neg \ V0x) = V1y) \Leftrightarrow (V0x = (ap \ c_2Einteger_2Eint_neg \\
& V1y)))))) \quad (41)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty_2Enum_2Enum.(\forall V1n \in ty_2Enum_2Enum.(\\
& ((ap \ c_2Einteger_2Eint_of_num \ V0m) = (ap \ c_2Einteger_2Eint_of_num \\
& V1n)) \Leftrightarrow (V0m = V1n)))) \quad (42)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty_2Einteger_2Eint.(\forall V1y \in ty_2Einteger_2Eint. \\
& (((ap \ c_2Einteger_2Eint_neg \ V0x) = (ap \ c_2Einteger_2Eint_neg \\
& V1y)) \Leftrightarrow (V0x = V1y)))) \quad (43)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& ((\forall V0m \in ty_2Enum_2Enum. (\forall V1n \in ty_2Enum_2Enum. \\
& (((ap\ c_2Einteger_2Eint_of_num\ V0m) = (ap\ c_2Einteger_2Eint_of_num \\
& \quad V1n)) \Leftrightarrow (V0m = V1n)))) \wedge ((\forall V2x \in ty_2Einteger_2Eint. (\forall V3y \in \\
& ty_2Einteger_2Eint. (((ap\ c_2Einteger_2Eint_neg\ V2x) = (ap\ c_2Einteger_2Eint_neg \\
& \quad V3y)) \Leftrightarrow (V2x = V3y)))) \wedge (\forall V4n \in ty_2Enum_2Enum. (\forall V5m \in \\
& ty_2Enum_2Enum. (((ap\ c_2Einteger_2Eint_of_num\ V4n) = (ap \\
& \quad c_2Einteger_2Eint_neg\ (ap\ c_2Einteger_2Eint_of_num\ V5m)) \Leftrightarrow \\
& ((V4n = c_2Enum_2E0) \wedge (V5m = c_2Enum_2E0))) \wedge (((ap\ c_2Einteger_2Eint_neg \\
& \quad (ap\ c_2Einteger_2Eint_of_num\ V4n)) = (ap\ c_2Einteger_2Eint_of_num \\
& \quad V5m)) \Leftrightarrow ((V4n = c_2Enum_2E0) \wedge (V5m = c_2Enum_2E0))))))))) \\
& \hspace{15em} (44)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& ((\forall V0n \in ty_2Enum_2Enum.((ap (ap c_2Earithmetic_2E_2B \\
& \quad c_2Enum_2E0) V0n) = V0n)) \wedge ((\forall V1n \in ty_2Enum_2Enum.((ap \\
& \quad (ap c_2Earithmetic_2E_2B V1n) c_2Enum_2E0) = V1n)) \wedge ((\forall V2n \in \\
& \quad ty_2Enum_2Enum.(\forall V3m \in ty_2Enum_2Enum.((ap (ap c_2Earithmetic_2E_2B \\
& \quad (ap c_2Earithmetic_2ENUMERAL V2n)) (ap c_2Earithmetic_2ENUMERAL \\
& \quad V3m)) = (ap c_2Earithmetic_2ENUMERAL (ap c_2Enumeral_2EiZ (ap \\
& \quad (ap c_2Earithmetic_2E_2B V2n) V3m)))))) \wedge ((\forall V4n \in ty_2Enum_2Enum. \\
& \quad ((ap (ap c_2Earithmetic_2E_2A c_2Enum_2E0) V4n) = c_2Enum_2E0)) \wedge \\
& \quad ((\forall V5n \in ty_2Enum_2Enum.((ap (ap c_2Earithmetic_2E_2A \\
& \quad V5n) c_2Enum_2E0) = c_2Enum_2E0)) \wedge ((\forall V6n \in ty_2Enum_2Enum. \\
& \quad (\forall V7m \in ty_2Enum_2Enum.((ap (ap c_2Earithmetic_2E_2A (\\
& \quad ap c_2Earithmetic_2ENUMERAL V6n)) (ap c_2Earithmetic_2ENUMERAL \\
& \quad V7m)) = (ap c_2Earithmetic_2ENUMERAL (ap (ap c_2Earithmetic_2E_2A \\
& \quad V6n) V7m)))))) \wedge ((\forall V8n \in ty_2Enum_2Enum.((ap (ap c_2Earithmetic_2E_2D \\
& \quad c_2Enum_2E0) V8n) = c_2Enum_2E0)) \wedge ((\forall V9n \in ty_2Enum_2Enum. \\
& \quad ((ap (ap c_2Earithmetic_2E_2D V9n) c_2Enum_2E0) = V9n)) \wedge ((\forall V10n \in \\
& \quad ty_2Enum_2Enum.(\forall V11m \in ty_2Enum_2Enum.((ap (ap c_2Earithmetic_2E_2D \\
& \quad (ap c_2Earithmetic_2ENUMERAL V10n)) (ap c_2Earithmetic_2ENUMERAL \\
& \quad V11m)) = (ap c_2Earithmetic_2ENUMERAL (ap (ap c_2Earithmetic_2E_2D \\
& \quad V10n) V11m)))))) \wedge ((\forall V12n \in ty_2Enum_2Enum.((ap (ap c_2Earithmetic_2EEXP \\
& \quad c_2Enum_2E0) (ap c_2Earithmetic_2ENUMERAL (ap c_2Earithmetic_2EBIT1 \\
& \quad V12n))) = c_2Enum_2E0)) \wedge ((\forall V13n \in ty_2Enum_2Enum.((ap \\
& \quad (ap c_2Earithmetic_2EEXP c_2Enum_2E0) (ap c_2Earithmetic_2ENUMERAL \\
& \quad (ap c_2Earithmetic_2EBIT2 V13n))) = c_2Enum_2E0)) \wedge ((\forall V14n \in \\
& \quad ty_2Enum_2Enum.((ap (ap c_2Earithmetic_2EEXP V14n) c_2Enum_2E0) = \\
& \quad (ap c_2Earithmetic_2ENUMERAL (ap c_2Earithmetic_2EBIT1 c_2Earithmetic_2EZERO)))))) \wedge \\
& \quad ((\forall V15n \in ty_2Enum_2Enum.(\forall V16m \in ty_2Enum_2Enum. \\
& \quad ((ap (ap c_2Earithmetic_2EEXP (ap c_2Earithmetic_2ENUMERAL V15n)) \\
& \quad (ap c_2Earithmetic_2ENUMERAL V16m)) = (ap c_2Earithmetic_2ENUMERAL \\
& \quad (ap (ap c_2Earithmetic_2EEXP V15n) V16m)))))) \wedge ((ap c_2Enum_2ESUC \\
& \quad c_2Enum_2E0) = (ap c_2Earithmetic_2ENUMERAL (ap c_2Earithmetic_2EBIT1 \\
& \quad c_2Earithmetic_2EZERO))) \wedge ((\forall V17n \in ty_2Enum_2Enum. (\\
& \quad (ap c_2Enum_2ESUC (ap c_2Earithmetic_2ENUMERAL V17n)) = (ap c_2Earithmetic_2ENUMERAL \\
& \quad (ap c_2Enum_2ESUC V17n)))) \wedge ((ap c_2Eprim_rec_2EPRE c_2Enum_2E0) = \\
& \quad c_2Enum_2E0) \wedge ((\forall V18n \in ty_2Enum_2Enum.((ap c_2Eprim_rec_2EPRE \\
& \quad (ap c_2Earithmetic_2ENUMERAL V18n)) = (ap c_2Earithmetic_2ENUMERAL \\
& \quad (ap c_2Eprim_rec_2EPRE V18n)))) \wedge ((\forall V19n \in ty_2Enum_2Enum. \\
& \quad (((ap c_2Earithmetic_2ENUMERAL V19n) = c_2Enum_2E0) \Leftrightarrow (V19n = c_2Earithmetic_2EZERO))) \wedge \\
& \quad ((\forall V20n \in ty_2Enum_2Enum.((c_2Enum_2E0 = (ap c_2Earithmetic_2ENUMERAL \\
& \quad V20n)) \Leftrightarrow (V20n = c_2Earithmetic_2EZERO))) \wedge ((\forall V21n \in ty_2Enum_2Enum. \\
& \quad (\forall V22m \in ty_2Enum_2Enum.(((ap c_2Earithmetic_2ENUMERAL \\
& \quad V21n) = (ap c_2Earithmetic_2ENUMERAL V22m)) \Leftrightarrow (V21n = V22m)))) \wedge \\
& \quad ((\forall V23n \in ty_2Enum_2Enum.((p (ap (ap c_2Eprim_rec_2E_3C \\
& \quad V23n) c_2Enum_2E0)) \Leftrightarrow False)) \wedge ((\forall V24n \in ty_2Enum_2Enum. \\
& \quad ((p (ap (ap c_2Eprim_rec_2E_3C c_2Enum_2E0) (ap c_2Earithmetic_2ENUMERAL \\
& \quad V24n))) \Leftrightarrow (p (ap (ap c_2Eprim_rec_2E_3C c_2Earithmetic_2EZERO) \\
& \quad V24n)))) \wedge ((\forall V25n \in ty_2Enum_2Enum.(\forall V26m \in ty_2Enum_2Enum. \\
& \quad ((p (ap (ap c_2Eprim_rec_2E_3C (ap c_2Earithmetic_2ENUMERAL \\
& \quad V25n)) (ap c_2Earithmetic_2ENUMERAL V26m))) \Leftrightarrow (p (ap (ap c_2Eprim_rec_2E_3C \\
& \quad V25n) V26m)))))) \wedge ((\forall V27n \in ty_2Enum_2Enum.((p (ap (ap c_2Earithmetic_2E_3E \\
& \quad c_2Enum_2E0) V27n)) \Leftrightarrow False)) \wedge ((\forall V28n \in ty_2Enum_2Enum. \\
& \quad ((p (ap (ap c_2Earithmetic_2E_3E (ap c_2Earithmetic_2ENUMERAL \\
& \quad V28n)) c_2Enum_2E0)) \Leftrightarrow (p (ap (ap c_2Eprim_rec_2E_3C c_2Earithmetic_2EZERO) \\
& \quad V28n)))) \wedge ((\forall V29n \in ty_2Enum_2Enum.(\forall V30m \in ty_2Enum_2Enum. \\
& \quad ((p (ap (ap c_2Earithmetic_2E_3E (ap c_2Earithmetic_2ENUMERAL \\
& \quad V29n)) (ap c_2Earithmetic_2ENUMERAL V30m))) \Leftrightarrow (p (ap (ap c_2Eprim_rec_2E_3C \\
& \quad V30m) V29n)))))) \wedge ((\forall V31n \in ty_2Enum_2Enum.((p (ap (ap c_2Earithmetic_2E_3C_3D \\
& \quad c_2Enum_2E0) V31n)) \Leftrightarrow True)) \wedge ((\forall V32n \in ty_2Enum_2Enum. \\
& \quad ((p (ap (ap c_2Earithmetic_2E_3C_3D (ap c_2Earithmetic_2ENUMERAL
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty_2Enum_2Enum. (\forall V1m \in ty_2Enum_2Enum. (\\
& ((c_2Earithmetic_2EZERO = (ap\ c_2Earithmetic_2EBIT1\ V0n)) \Leftrightarrow False) \wedge \\
& (((ap\ c_2Earithmetic_2EBIT1\ V0n) = c_2Earithmetic_2EZERO) \Leftrightarrow \\
& False) \wedge (((c_2Earithmetic_2EZERO = (ap\ c_2Earithmetic_2EBIT2 \\
& V0n)) \Leftrightarrow False) \wedge (((ap\ c_2Earithmetic_2EBIT2\ V0n) = c_2Earithmetic_2EZERO) \Leftrightarrow \\
& False) \wedge (((ap\ c_2Earithmetic_2EBIT1\ V0n) = (ap\ c_2Earithmetic_2EBIT2 \\
& V1m)) \Leftrightarrow False) \wedge (((ap\ c_2Earithmetic_2EBIT2\ V0n) = (ap\ c_2Earithmetic_2EBIT1 \\
& V1m)) \Leftrightarrow False) \wedge (((ap\ c_2Earithmetic_2EBIT1\ V0n) = (ap\ c_2Earithmetic_2EBIT1 \\
& V1m)) \Leftrightarrow (V0n = V1m)) \wedge (((ap\ c_2Earithmetic_2EBIT2\ V0n) = (ap\ c_2Earithmetic_2EBIT2 \\
& V1m)) \Leftrightarrow (V0n = V1m)))))))))
\end{aligned} \tag{46}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty_2Erat_2Erat. (\forall V1b \in ty_2Erat_2Erat. (\\
& \forall V2c \in ty_2Erat_2Erat. ((ap\ (ap\ c_2Erat_2Erat_mul\ V0a) \\
& (ap\ (ap\ c_2Erat_2Erat_mul\ V1b)\ V2c)) = (ap\ (ap\ c_2Erat_2Erat_mul \\
& (ap\ (ap\ c_2Erat_2Erat_mul\ V0a)\ V1b))\ V2c))))))
\end{aligned} \tag{47}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty_2Erat_2Erat. (\forall V1b \in ty_2Erat_2Erat. (\\
& (ap\ (ap\ c_2Erat_2Erat_add\ V0a)\ V1b) = (ap\ (ap\ c_2Erat_2Erat_add \\
& V1b)\ V0a))))
\end{aligned} \tag{48}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty_2Erat_2Erat. ((ap\ (ap\ c_2Erat_2Erat_add\ (ap \\
& c_2Erat_2Erat_of_num\ c_2Enum_2E0))\ V0a) = V0a))
\end{aligned} \tag{49}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty_2Erat_2Erat. ((ap\ (ap\ c_2Erat_2Erat_mul\ V0a) \\
& (ap\ c_2Erat_2Erat_of_num\ (ap\ c_2Earithmetic_2ENUMERAL\ (ap \\
& c_2Earithmetic_2EBIT1\ c_2Earithmetic_2EZERO)))) = V0a))
\end{aligned} \tag{50}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty_2Erat_2Erat. ((\neg(V0a = (ap\ c_2Erat_2Erat_of_num \\
& c_2Enum_2E0))) \Rightarrow ((ap\ (ap\ c_2Erat_2Erat_mul\ V0a)\ (ap\ c_2Erat_2Erat_minv \\
& V0a)) = (ap\ c_2Erat_2Erat_of_num\ (ap\ c_2Earithmetic_2ENUMERAL \\
& (ap\ c_2Earithmetic_2EBIT1\ c_2Earithmetic_2EZERO))))))
\end{aligned} \tag{51}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty_2Erat_2Erat. (\forall V1b \in ty_2Erat_2Erat. (\\
& \forall V2c \in ty_2Erat_2Erat. ((ap\ (ap\ c_2Erat_2Erat_mul\ (ap\ (\\
& ap\ c_2Erat_2Erat_add\ V0a)\ V1b))\ V2c) = (ap\ (ap\ c_2Erat_2Erat_add \\
& (ap\ (ap\ c_2Erat_2Erat_mul\ V0a)\ V2c))\ (ap\ (ap\ c_2Erat_2Erat_mul \\
& V1b)\ V2c))))))
\end{aligned} \tag{52}$$

Assume the following.

$$(\forall V0r1 \in ty_2Erat_2Erat. (\forall V1r2 \in ty_2Erat_2Erat. ((ap (ap c_2Erat_2Erat_sub V0r1) V1r2) = (ap (ap c_2Erat_2Erat_add V0r1) (ap c_2Erat_2Erat_ainv V1r2)))))) \quad (53)$$

Assume the following.

$$(\forall V0r1 \in ty_2Erat_2Erat. (\forall V1r2 \in ty_2Erat_2Erat. ((ap c_2Erat_2Erat_ainv (ap (ap c_2Erat_2Erat_add V0r1) V1r2)) = (ap (ap c_2Erat_2Erat_add (ap c_2Erat_2Erat_ainv V0r1)) (ap c_2Erat_2Erat_ainv V1r2)))))) \quad (54)$$

Assume the following.

$$(\forall V0r1 \in ty_2Erat_2Erat. (\forall V1r2 \in ty_2Erat_2Erat. ((ap c_2Erat_2Erat_ainv (ap (ap c_2Erat_2Erat_mul V0r1) V1r2)) = (ap (ap c_2Erat_2Erat_mul (ap c_2Erat_2Erat_ainv V0r1)) V1r2)))) \quad (55)$$

Assume the following.

$$(\forall V0r1 \in ty_2Erat_2Erat. (\forall V1r2 \in ty_2Erat_2Erat. (((ap c_2Erat_2Erat_ainv V0r1) = V1r2) \Leftrightarrow (V0r1 = (ap c_2Erat_2Erat_ainv V1r2)))))) \quad (56)$$

Assume the following.

$$\begin{aligned} & (\forall V0r1 \in ty_2Erat_2Erat. (((ap c_2Erat_2Erat_sgn V0r1) = \\ & (ap c_2Einteger_2Eint_neg (ap c_2Einteger_2Eint_of_num (\\ & ap c_2Earithmetic_2ENUMERAL (ap c_2Earithmetic_2EBIT1 c_2Earithmetic_2EZERO)))))) \Leftrightarrow \\ & (p (ap (ap c_2Erat_2Erat_les V0r1) (ap c_2Erat_2Erat_of_num \\ & c_2Enum_2E0)))) \wedge (((ap c_2Erat_2Erat_sgn V0r1) = (ap c_2Einteger_2Eint_of_num \\ & c_2Enum_2E0)) \Leftrightarrow (V0r1 = (ap c_2Erat_2Erat_of_num c_2Enum_2E0))) \wedge \\ & (((ap c_2Erat_2Erat_sgn V0r1) = (ap c_2Einteger_2Eint_of_num \\ & (ap c_2Earithmetic_2ENUMERAL (ap c_2Earithmetic_2EBIT1 c_2Earithmetic_2EZERO)))) \Leftrightarrow \\ & (p (ap (ap c_2Erat_2Erat_gre V0r1) (ap c_2Erat_2Erat_of_num \\ & c_2Enum_2E0)))))) \end{aligned} \quad (57)$$

Assume the following.

$$(\forall V0r1 \in ty_2Erat_2Erat. ((ap c_2Einteger_2Eint_neg (ap c_2Erat_2Erat_sgn (ap c_2Erat_2Erat_ainv V0r1))) = (ap c_2Erat_2Erat_sgn V0r1))) \quad (58)$$

Assume the following.

$$(\forall V0r1 \in ty_2Erat_2Erat. (\forall V1r2 \in ty_2Erat_2Erat. ((ap c_2Erat_2Erat_sgn (ap (ap c_2Erat_2Erat_mul V0r1) V1r2)) = (ap (ap c_2Einteger_2Eint_mul (ap c_2Erat_2Erat_sgn V0r1) (ap c_2Erat_2Erat_sgn V1r2)))))) \quad (59)$$

Assume the following.

$$\begin{aligned}
& (\forall V0r1 \in ty_2Erat_2Erat. ((\neg(V0r1 = (ap\ c_2Erat_2Erat_of_num \\
& c_2Enum_2E0))) \Rightarrow ((ap\ c_2Erat_2Erat_sgn\ (ap\ c_2Erat_2Erat_minv \\
& V0r1)) = (ap\ c_2Erat_2Erat_sgn\ V0r1)))) \quad (60)
\end{aligned}$$

Theorem 1

$$\begin{aligned}
& (\forall V0r1 \in ty_2Erat_2Erat. (\forall V1r2 \in ty_2Erat_2Erat. \\
& (\forall V2r3 \in ty_2Erat_2Erat. ((p\ (ap\ (ap\ c_2Erat_2Erat_les \\
V2r3)\ (ap\ c_2Erat_2Erat_of_num\ c_2Enum_2E0))) \Rightarrow ((p\ (ap\ (ap\ c_2Erat_2Erat_les \\
& (ap\ (ap\ c_2Erat_2Erat_mul\ V1r2)\ V2r3))\ (ap\ (ap\ c_2Erat_2Erat_mul \\
& V0r1)\ V2r3))) \Leftrightarrow (p\ (ap\ (ap\ c_2Erat_2Erat_les\ V0r1)\ V1r2))))))
\end{aligned}$$