

# thm\_2Erat\_2ERAT\_\_MUL\_\_ASSOC (TMZ6Q7oQ7K2PSpmki61ke9oiTx2z2nTQ38Q)

October 26, 2020

**Definition 1** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Ebool_2ET` to be  $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V 0x \in 2.V 0x)) (\lambda V 1x \in 2.V 1x))$

Let `ty_2Efrac_2Efrac` :  $\iota$  be given. Assume the following.

$$nonempty\ ty\_2Efrac\_2Efrac \tag{1}$$

Let `ty_2Erat_2Erat` :  $\iota$  be given. Assume the following.

$$nonempty\ ty\_2Erat\_2Erat \tag{2}$$

Let `c_2Erat_2Erep__rat__CLASS` :  $\iota$  be given. Assume the following.

$$c\_2Erat\_2Erep\_rat\_CLASS \in ((2^{ty\_2Efrac\_2Efrac})^{ty\_2Erat\_2Erat}) \tag{3}$$

**Definition 3** We define `c_2Emin_2E_40` to be  $\lambda A. \lambda P \in 2^A. \mathbf{if} (\exists x \in A. p (ap P x)) \mathbf{then} (the (\lambda x. x \in A \wedge p x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 4** We define `c_2Ebool_2E_21` to be  $\lambda A. 27a : \iota. (\lambda V 0P \in (2^{A-27a}). (ap (ap (c_2Emin_2E_3D (2^{A-27a})) (\lambda V 1x \in 2.V 1x)) (\lambda V 1x \in 2.V 1x)))$

**Definition 5** We define `c_2Erat_2Erep__rat` to be  $\lambda V 0a \in ty\_2Erat\_2Erat. (ap (c_2Emin_2E_40 ty\_2Efrac\_2Efrac))$

Let `ty_2Einteger_2Eint` :  $\iota$  be given. Assume the following.

$$nonempty\ ty\_2Einteger\_2Eint \tag{4}$$

Let `ty_2Epair_2Eprod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. nonempty\ A0 \Rightarrow \forall A1. nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{5}$$

Let `c_2Efrac_2Erep__frac` :  $\iota$  be given. Assume the following.

$$c\_2Efrac\_2Erep\_frac \in ((ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)^{ty\_2Efrac\_2Efrac}) \tag{6}$$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow c.2Epair\_2ESND \\ A.27a\ A.27b \in (A.27b)^{(ty\_2Epair\_2Eprod\ A.27a\ A.27b)} \end{aligned} \quad (7)$$

**Definition 6** We define  $c\_2Efrac\_2Efrac\_dnm$  to be  $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap\ (c\_2Epair\_2ESND\ ty$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (8)$$

Let  $c\_2Einteger\_2Eint\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Einteger\_2Eint}) \quad (9)$$

**Definition 7** We define  $c\_2Einteger\_2Eint\_REP$  to be  $\lambda V0a \in ty\_2Einteger\_2Eint.(ap\ (c\_2Emin\_2E.40\ (ty$

Let  $c\_2Einteger\_2Etint\_mul : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (10)$$

Let  $c\_2Einteger\_2Etint\_eq : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (11)$$

Let  $c\_2Einteger\_2Eint\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_ABS\_CLASS \in (ty\_2Einteger\_2Eint)^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})} \quad (12)$$

**Definition 8** We define  $c\_2Einteger\_2Eint\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)$

**Definition 9** We define  $c\_2Einteger\_2Eint\_mul$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow c.2Epair\_2EFST \\ A.27a\ A.27b \in (A.27a)^{(ty\_2Epair\_2Eprod\ A.27a\ A.27b)} \end{aligned} \quad (13)$$

**Definition 10** We define  $c\_2Efrac\_2Efrac\_nmr$  to be  $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap\ (c\_2Epair\_2EFST\ ty$

**Definition 11** We define  $c\_2Emin\_2E.3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o\ (p\ P \Rightarrow p\ Q)$  of type  $\iota$ .

**Definition 12** We define  $c\_2Ebool\_2E.2F.5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E.21\ 2)\ (\lambda V2t \in$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod \\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \end{aligned} \quad (14)$$

**Definition 13** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2Efrac\_2Eabs\_frac : \iota$  be given. Assume the following.

$$c\_2Efrac\_2Eabs\_frac \in (ty\_2Efrac\_2Efrac^{(ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)}) \quad (15)$$

**Definition 14** We define  $c\_2Efrac\_2Efrac\_mul$  to be  $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Efrac$

**Definition 15** We define  $c\_2Erat\_2Erat\_equiv$  to be  $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Efrac$

Let  $c\_2Erat\_2Eabs\_rat\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erat\_2Eabs\_rat\_CLASS \in (ty\_2Erat\_2Erat^{(2^{ty\_2Efrac\_2Efrac})}) \quad (16)$$

**Definition 16** We define  $c\_2Erat\_2Eabs\_rat$  to be  $\lambda V0r \in ty\_2Efrac\_2Efrac.(ap\ c\_2Erat\_2Eabs\_rat\_CLASS$

**Definition 17** We define  $c\_2Erat\_2Erat\_mul$  to be  $\lambda V0r1 \in ty\_2Erat\_2Erat.\lambda V1r2 \in ty\_2Erat\_2Erat.(ap$

Assume the following.

$$True \quad (17)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in \\ A\_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow \\ True)) \end{aligned} \quad (19)$$

Assume the following.

$$\begin{aligned} (\forall V0a \in ty\_2Efrac\_2Efrac.(\forall V1b \in ty\_2Efrac\_2Efrac. \\ (\forall V2c \in ty\_2Efrac\_2Efrac.((ap\ (ap\ c\_2Efrac\_2Efrac\_mul \\ V0a)\ (ap\ (ap\ c\_2Efrac\_2Efrac\_mul\ V1b)\ V2c)) = (ap\ (ap\ c\_2Efrac\_2Efrac\_mul \\ (ap\ (ap\ c\_2Efrac\_2Efrac\_mul\ V0a)\ V1b))\ V2c)))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned} ((\forall V0x \in ty\_2Efrac\_2Efrac.(\forall V1y \in ty\_2Efrac\_2Efrac. \\ ((ap\ c\_2Erat\_2Eabs\_rat\ (ap\ (ap\ c\_2Efrac\_2Efrac\_mul\ (ap\ c\_2Erat\_2Erep\_rat \\ (ap\ c\_2Erat\_2Eabs\_rat\ V0x)))\ V1y)) = (ap\ c\_2Erat\_2Eabs\_rat\ ( \\ ap\ (ap\ c\_2Efrac\_2Efrac\_mul\ V0x)\ V1y)))))) \wedge (\forall V2x \in ty\_2Efrac\_2Efrac. \\ (\forall V3y \in ty\_2Efrac\_2Efrac.((ap\ c\_2Erat\_2Eabs\_rat\ (ap\ ( \\ ap\ c\_2Efrac\_2Efrac\_mul\ V2x)\ (ap\ c\_2Erat\_2Erep\_rat\ (ap\ c\_2Erat\_2Eabs\_rat \\ V3y)))) = (ap\ c\_2Erat\_2Eabs\_rat\ (ap\ (ap\ c\_2Efrac\_2Efrac\_mul \\ V2x)\ V3y)))))) \end{aligned} \quad (21)$$

**Theorem 1**

$$\begin{aligned} & (\forall V0a \in ty\_2Erat\_2Erat. (\forall V1b \in ty\_2Erat\_2Erat. ( \\ & \quad \forall V2c \in ty\_2Erat\_2Erat. ((ap (ap c\_2Erat\_2Erat\_mul V0a) \\ & (ap (ap c\_2Erat\_2Erat\_mul V1b) V2c)) = (ap (ap c\_2Erat\_2Erat\_mul \\ & \quad (ap (ap c\_2Erat\_2Erat\_mul V0a) V1b)) V2c)))))) \end{aligned}$$