

# thm\_2Erat\_2ERAT\_\_MUL\_\_LZERO (TMZPkSUXPN9kuqoKYrVvKKbuMJZUoBqxxN8)

October 26, 2020

**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A-27a})) (\lambda V1P \in 2.V1P)) (\lambda V2P \in 2.V2P))$

**Definition 4** We define  $c\_2Ebool\_2E\_2F$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_27E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_2F))$

**Definition 7** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \tag{2}$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \tag{3}$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \tag{4}$$

**Definition 8** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap c\_2Enum\_2EABS\_num (ap c\_2Enum\_2EREP\_num (ap c\_2Enum\_2ESUC\_REP m)))$

**Definition 9** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.\mathbf{if} (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge P x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 10** We define  $c\_Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota. (\lambda V0P \in (2^{A\_27a}). (ap\ V0P\ (ap\ (c\_Emin\_2E\_40$

**Definition 11** We define  $c\_Eprim\_rec\_2E\_3C$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. \lambda V1n \in ty\_2Enum\_2Enum$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. nonempty\ A0 \Rightarrow \forall A1. nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \quad (5)$$

Let  $ty\_2Einteger\_2Eint : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Einteger\_2Eint \quad (6)$$

Let  $c\_2Einteger\_2Eint\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})\ ty\_2Einteger\_2Eint) \quad (7)$$

**Definition 12** We define  $c\_2Einteger\_2Eint\_REP$  to be  $\lambda V0a \in ty\_2Einteger\_2Eint. (ap\ (c\_Emin\_2E\_40\ (t$

Let  $c\_2Einteger\_2Eint\_neg : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_neg \in ((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (8)$$

Let  $c\_2Einteger\_2Eint\_eq : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum)) \quad (9)$$

Let  $c\_2Einteger\_2Eint\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_ABS\_CLASS \in (ty\_2Einteger\_2Eint)^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})} \quad (10)$$

**Definition 13** We define  $c\_2Einteger\_2Eint\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)$

**Definition 14** We define  $c\_2Einteger\_2Eint\_neg$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint. (ap\ c\_2Einteger\_2Eint$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (11)$$

**Definition 15** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum)^{ty\_2Enum\_2Enum})\ ty\_2Enum\_2Enum \quad (12)$$

**Definition 16** We define  $c\_2Earithmetic\_2EBIT2$  to be  $\lambda V0n \in ty\_2Enum\_2Enum. (ap\ (ap\ c\_2Earithmetic$

Let  $c\_2Einteger\_2Etint\_lt : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum)) \quad (13)$$

**Definition 17** We define  $c\_2Einteger\_2Eint\_lt$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger$ .

Let  $ty\_2Efrac\_2Efrac : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Efrac\_2Efrac \quad (14)$$

Let  $ty\_2Erat\_2Erat : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erat\_2Erat \quad (15)$$

Let  $c\_2Erat\_2Erep\_rat\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erat\_2Erep\_rat\_CLASS \in ((2^{ty\_2Efrac\_2Efrac})^{ty\_2Erat\_2Erat}) \quad (16)$$

**Definition 18** We define  $c\_2Erat\_2Erep\_rat$  to be  $\lambda V0a \in ty\_2Erat\_2Erat.(ap\ (c\_2Emin\_2E40\ ty\_2Efrac$

Let  $c\_2Efrac\_2Erep\_frac : \iota$  be given. Assume the following.

$$c\_2Efrac\_2Erep\_frac \in ((ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)^{ty\_2Efrac\_2Efrac}) \quad (17)$$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow c\_2Epair\_2ESND\ A.27a\ A.27b \in (A.27b)^{(ty\_2Epair\_2Eprod\ A.27a\ A.27b)} \quad (18)$$

**Definition 19** We define  $c\_2Efrac\_2Efrac\_dnm$  to be  $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap\ (c\_2Epair\_2ESND\ t$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow c\_2Epair\_2EFST\ A.27a\ A.27b \in (A.27a)^{(ty\_2Epair\_2Eprod\ A.27a\ A.27b)} \quad (19)$$

**Definition 20** We define  $c\_2Efrac\_2Efrac\_nmr$  to be  $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap\ (c\_2Epair\_2EFST\ ty$

Let  $c\_2Einteger\_2Etint\_mul : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)) \quad (20)$$

**Definition 21** We define  $c\_2Einteger\_2Eint\_mul$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod \\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \end{aligned} \quad (21)$$

**Definition 22** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2E$

Let  $c\_2Efrac\_2Eabs\_frac : \iota$  be given. Assume the following.

$$c\_2Efrac\_2Eabs\_frac \in (ty\_2Efrac\_2Efrac^{(ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)}) \quad (22)$$

**Definition 23** We define  $c\_2Efrac\_2Efrac\_mul$  to be  $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Efrac$

**Definition 24** We define  $c\_2Erat\_2Erat\_equiv$  to be  $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Efrac$

Let  $c\_2Erat\_2Eabs\_rat\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erat\_2Eabs\_rat\_CLASS \in (ty\_2Erat\_2Erat^{(2^{ty\_2Efrac\_2Efrac})}) \quad (23)$$

**Definition 25** We define  $c\_2Erat\_2Eabs\_rat$  to be  $\lambda V0r \in ty\_2Efrac\_2Efrac.(ap\ c\_2Erat\_2Eabs\_rat\_CLASS$

**Definition 26** We define  $c\_2Erat\_2Erat\_mul$  to be  $\lambda V0r1 \in ty\_2Erat\_2Erat.\lambda V1r2 \in ty\_2Erat\_2Erat.(ap$

**Definition 27** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

**Definition 28** We define  $c\_2Earithmetic\_2EBIT1$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

**Definition 29** We define  $c\_2Earithmetic\_2ENUMERAL$  to be  $\lambda V0x \in ty\_2Enum\_2Enum.V0x$ .

Let  $c\_2Einteger\_2Eint\_of\_num : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_num \in (ty\_2Einteger\_2Eint^{ty\_2Enum\_2Enum}) \quad (24)$$

**Definition 30** We define  $c\_2Efrac\_2Efrac\_1$  to be  $(ap\ c\_2Efrac\_2Eabs\_frac\ (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2E$

**Definition 31** We define  $c\_2Erat\_2Erat\_1$  to be  $(ap\ c\_2Erat\_2Eabs\_rat\ c\_2Efrac\_2Efrac\_1)$ .

Let  $c\_2Einteger\_2Eint\_add : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (25)$$

**Definition 32** We define  $c\_2Einteger\_2Eint\_add$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger$

**Definition 33** We define  $c\_2Efrac\_2Efrac\_add$  to be  $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Efrac$

**Definition 34** We define  $c\_2Erat\_2Erat\_add$  to be  $\lambda V0r1 \in ty\_2Erat\_2Erat.\lambda V1r2 \in ty\_2Erat\_2Erat.(ap$

**Definition 35** We define  $c\_2Ecombin\_2EK$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0x \in A\_27a.(\lambda V1y \in A\_27b.V0x))$

**Definition 36** We define  $c\_2Ecombin\_2ES$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.(\lambda V0f \in ((A\_27c^{A\_27b})^{A\_27a}))$

**Definition 37** We define  $c\_2Ecombin\_2EI$  to be  $\lambda A\_27a : \iota.(ap (ap (c\_2Ecombin\_2ES A\_27a (A\_27a^{A\_27a})) A\_27a))$

Let  $c\_2Earithmetic\_2Enum\_CASE : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Earithmetic\_2Enum\_CASE A\_27a \in \left( (A\_27a^{(A\_27a^{ty\_2Enum\_2Enum})})^{A\_27a} \right)^{ty\_2Enum\_2Enum} \quad (26)$$

**Definition 38** We define  $c\_2Efrac\_2Efrac\_0$  to be  $(ap c\_2Efrac\_2Eabs\_frac (ap (ap (c\_2Epair\_2E\_2C ty\_2Enum\_2Enum))))$

**Definition 39** We define  $c\_2Erat\_2Erat\_0$  to be  $(ap c\_2Erat\_2Eabs\_rat c\_2Efrac\_2Efrac\_0)$ .

**Definition 40** We define  $c\_2Erelation\_2EWF$  to be  $\lambda A\_27a : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).(ap (c\_2Ebool\_2E\_21))$

Let  $c\_2Ebool\_2EARB : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Ebool\_2EARB A\_27a \in A\_27a \quad (27)$$

**Definition 41** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.))$

**Definition 42** We define  $c\_2Erelation\_2ERESTRICT$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (A\_27b^{A\_27a}).\lambda V1f \in (A\_27b^{A\_27a}).$

**Definition 43** We define  $c\_2Erelation\_2ETC$  to be  $\lambda A\_27a : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1a \in A\_27a.\lambda V2b \in A\_27a.$

**Definition 44** We define  $c\_2Erelation\_2Eapprox$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1M \in (A\_27a^{A\_27a}).$

**Definition 45** We define  $c\_2Erelation\_2Ethe\_fun$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1M \in (A\_27a^{A\_27a}).$

**Definition 46** We define  $c\_2Erelation\_2EWFREC$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1M \in (A\_27a^{A\_27a}).$

**Definition 47** We define  $c\_2Erat\_2Erat\_of\_num$  to be  $(ap (ap (c\_2Erelation\_2EWFREC ty\_2Enum\_2Enum)))$

**Definition 48** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.))$

Assume the following.

$$True \quad (28)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (29)$$

Assume the following.

$$(\forall V0t \in 2.(((p V0t) \Rightarrow False) \Rightarrow (\neg (p V0t)))) \quad (30)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(p V0t)) \Rightarrow ((p V0t) \Rightarrow False))) \quad (31)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (( \\ & (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge \\ & ((\neg False) \Leftrightarrow True))) \end{aligned} \quad (33)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (34)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (35)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\ & p V0t)))))) \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned} & (\forall V0f \in ty\_2Efrac\_2Efrac.((ap c\_2Efrac\_2Eabs\_frac (ap \\ & (ap (c\_2Epair\_2E\_2C ty\_2Einteger\_2Eint ty\_2Einteger\_2Eint) \\ & (ap c\_2Efrac\_2Efrac\_nmr V0f)) (ap c\_2Efrac\_2Efrac\_dnm V0f))) = \\ & V0f)) \end{aligned} \quad (37)$$

Assume the following.

$$\begin{aligned} & (\forall V0f \in ty\_2Efrac\_2Efrac.(p (ap (ap c\_2Einteger\_2Eint\_lt \\ & (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) (ap c\_2Efrac\_2Efrac\_dnm \\ & V0f)))) \end{aligned} \quad (38)$$

Assume the following.

$$\begin{aligned} & (\forall V0a \in ty\_2Einteger\_2Eint.(\forall V1b \in ty\_2Einteger\_2Eint. \\ & ((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\ & c\_2Enum\_2E0)) V1b)) \Rightarrow ((ap c\_2Efrac\_2Efrac\_nmr (ap c\_2Efrac\_2Eabs\_frac \\ & (ap (ap (c\_2Epair\_2E\_2C ty\_2Einteger\_2Eint ty\_2Einteger\_2Eint) \\ & V0a) V1b))) = V0a)))) \end{aligned} \quad (39)$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty\_2Einteger\_2Eint. (\forall V1b \in ty\_2Einteger\_2Eint. \\
& ((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) V1b)) \Rightarrow ((ap c\_2Efrac\_2Efrac\_dnm (ap c\_2Efrac\_2Eabs\_frac \\
& (ap (ap (c\_2Epair\_2E\_2C ty\_2Einteger\_2Eint ty\_2Einteger\_2Eint) \\
& V0a) V1b))) = V1b))))))
\end{aligned} \tag{40}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a1 \in ty\_2Einteger\_2Eint. (\forall V1b1 \in ty\_2Einteger\_2Eint. \\
& (\forall V2a2 \in ty\_2Einteger\_2Eint. (\forall V3b2 \in ty\_2Einteger\_2Eint. \\
& ((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) V1b1)) \Rightarrow ((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) V3b2)) \Rightarrow ((ap (ap c\_2Efrac\_2Efrac\_mul (ap c\_2Efrac\_2Eabs\_frac \\
& (ap (ap (c\_2Epair\_2E\_2C ty\_2Einteger\_2Eint ty\_2Einteger\_2Eint) \\
& V0a1) V1b1))) (ap c\_2Efrac\_2Eabs\_frac (ap (ap (c\_2Epair\_2E\_2C \\
& ty\_2Einteger\_2Eint ty\_2Einteger\_2Eint) V2a2) V3b2))) = (ap c\_2Efrac\_2Eabs\_frac \\
& (ap (ap (c\_2Epair\_2E\_2C ty\_2Einteger\_2Eint ty\_2Einteger\_2Eint) \\
& (ap (ap c\_2Einteger\_2Eint\_mul V0a1) V2a2)) (ap (ap c\_2Einteger\_2Eint\_mul \\
& V1b1) V3b2))))))))))
\end{aligned} \tag{41}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty\_2Einteger\_2Eint. (\forall V1b \in ty\_2Einteger\_2Eint. \\
& ((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) V0a)) \Rightarrow ((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) V1b)) \Rightarrow (p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) (ap (ap c\_2Einteger\_2Eint\_mul V0a) V1b))))))
\end{aligned} \tag{42}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_mul \\
& V0x) (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& (ap c\_2Earithmic\_2EBIT1 c\_2Earithmic\_2EZERO)))) = V0x))
\end{aligned} \tag{43}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_mul \\
& (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) V0x) = (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)))
\end{aligned} \tag{44}$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\
& ((ap c\_2Einteger\_2Eint\_of\_num V0m) = (ap c\_2Einteger\_2Eint\_of\_num \\
& V1n))) \Leftrightarrow (V0m = V1n)))
\end{aligned} \tag{45}$$

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum. (\forall V1m \in ty\_2Enum\_2Enum. ( \\
& \quad ((p (ap (ap c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_of\_num \\
& \quad c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& \quad (ap c\_2Arithmetic\_2EBIT1 V0n)))))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt \\
& (ap c\_2Integer\_2Eint\_of\_num c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_of\_num \\
& (ap c\_2Arithmetic\_2ENUMERAL (ap c\_2Arithmetic\_2EBIT2 V0n)))))) \Leftrightarrow \\
& \quad True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_of\_num c\_2Enum\_2E0))) \Leftrightarrow \\
& \quad False) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num \\
& (ap c\_2Arithmetic\_2ENUMERAL V0n)))))) \Leftrightarrow False) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt \\
& (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& V0n)) (ap c\_2Integer\_2Eint\_of\_num c\_2Enum\_2E0))) \Leftrightarrow False) \wedge \\
& \quad (((p (ap (ap c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_neg \\
& (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& (ap c\_2Arithmetic\_2EBIT1 V0n)))))) (ap c\_2Integer\_2Eint\_of\_num \\
& c\_2Enum\_2E0))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt (ap \\
& c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& (ap c\_2Arithmetic\_2EBIT2 V0n)))))) (ap c\_2Integer\_2Eint\_of\_num \\
& c\_2Enum\_2E0))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt (ap \\
& c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL V0n))) \\
& (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& V1m)))) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C V0n) V1m))) \wedge (((p (ap (ap \\
& c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num \\
& (ap c\_2Arithmetic\_2ENUMERAL (ap c\_2Arithmetic\_2EBIT1 V0n)))))) \\
& (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& V1m)))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_neg \\
& (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& (ap c\_2Arithmetic\_2EBIT2 V0n)))))) (ap c\_2Integer\_2Eint\_of\_num \\
& (ap c\_2Arithmetic\_2ENUMERAL V1m)))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt \\
& (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& V0n)) (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num \\
& (ap c\_2Arithmetic\_2ENUMERAL V1m)))))) \Leftrightarrow False) \wedge ((p (ap (ap c\_2Integer\_2Eint\_lt \\
& (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num ( \\
& ap c\_2Arithmetic\_2ENUMERAL V0n)))) (ap c\_2Integer\_2Eint\_neg \\
& (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& V1m)))))) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C V1m) V0n))))))))) \\
& \hspace{15em} (46)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0r \in ty\_2Erat\_2Erat. ((ap c\_2Erat\_2Eabs\_rat (ap c\_2Erat\_2Erep\_rat \\
& \quad V0r)) = V0r)) \\
& \hspace{15em} (47)
\end{aligned}$$



Assume the following.

$$\begin{aligned}
& (\forall V0f1 \in ty\_2Efrac\_2Efrac. (\forall V1f2 \in ty\_2Efrac\_2Efrac. \\
& ((ap\ c\_2Erat\_2Eabs\_rat\ V0f1) = (ap\ c\_2Erat\_2Eabs\_rat\ V1f2)) \Leftrightarrow \\
& ((ap\ (ap\ c\_2Einteger\_2Eint\_mul\ (ap\ c\_2Efrac\_2Efrac\_nmr\ V0f1)) \\
& (ap\ c\_2Efrac\_2Efrac\_dnm\ V1f2)) = (ap\ (ap\ c\_2Einteger\_2Eint\_mul \\
& (ap\ c\_2Efrac\_2Efrac\_nmr\ V1f2))\ (ap\ c\_2Efrac\_2Efrac\_dnm\ V0f1))))))
\end{aligned} \tag{48}$$

Assume the following.

$$\begin{aligned}
& (\forall V0f1 \in ty\_2Efrac\_2Efrac. (\forall V1f2 \in ty\_2Efrac\_2Efrac. \\
& ((ap\ (ap\ c\_2Erat\_2Erat\_mul\ (ap\ c\_2Erat\_2Eabs\_rat\ V0f1))\ (ap \\
& c\_2Erat\_2Eabs\_rat\ V1f2)) = (ap\ c\_2Erat\_2Eabs\_rat\ (ap\ (ap\ c\_2Efrac\_2Efrac\_mul \\
& V0f1)\ V1f2))))))
\end{aligned} \tag{49}$$

Assume the following.

$$\begin{aligned}
& (\forall V0n1 \in ty\_2Enum\_2Enum. ((ap\ c\_2Erat\_2Erat\_of\_num\ V0n1) = \\
& (ap\ c\_2Erat\_2Eabs\_rat\ (ap\ c\_2Efrac\_2Eabs\_frac\ (ap\ (ap\ (c\_2Epair\_2E\_2C \\
& ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)\ (ap\ c\_2Einteger\_2Eint\_of\_num \\
& V0n1))\ (ap\ c\_2Einteger\_2Eint\_of\_num\ (ap\ c\_2Earithmetic\_2ENUMERAL \\
& (ap\ c\_2Earithmetic\_2EBIT1\ c\_2Earithmetic\_2EZERO))))))
\end{aligned} \tag{50}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \tag{51}$$

Assume the following.

$$(\forall V0A \in 2. ((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \tag{52}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\
& (((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False))))))
\end{aligned} \tag{53}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\
& ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False))))))
\end{aligned} \tag{54}$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \tag{55}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p\ V0p) \Leftrightarrow ( \\
& (p\ V1q) \Leftrightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (p\ V2r))) \wedge (((p\ V0p) \vee ((\neg \\
& p\ V2r)) \vee (\neg(p\ V1q)))) \wedge (((p\ V1q) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V0p)))) \wedge ((p\ V2r) \vee \\
& ((\neg(p\ V1q)) \vee (\neg(p\ V0p))))))))))
\end{aligned} \tag{56}$$

Assume the following.

$$\begin{aligned}
 & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \vee V0p) \Leftrightarrow ( \\
 & (p \vee V1q) \Rightarrow (p \vee V2r))) \Leftrightarrow (((p \vee V0p) \vee (p \vee V1q)) \wedge (((p \vee V0p) \vee \neg(p \vee V2r))) \wedge ( \\
 & \neg(p \vee V1q) \vee ((p \vee V2r) \vee \neg(p \vee V0p)))))))))) \quad (57)
 \end{aligned}$$

Assume the following.

$$\begin{aligned}
 & (\forall V0p \in 2. (\forall V1q \in 2. (((p \vee V0p) \Leftrightarrow \neg(p \vee V1q))) \Leftrightarrow (((p \vee V0p) \vee \\
 & (p \vee V1q)) \wedge (\neg(p \vee V1q) \vee \neg(p \vee V0p)))))) \quad (58)
 \end{aligned}$$

**Theorem 1**

$$\begin{aligned}
 & (\forall V0r1 \in ty\_2Erat\_2Erat. ((ap (ap c\_2Erat\_2Erat\_mul (ap \\
 & c\_2Erat\_2Erat\_of\_num c\_2Enum\_2E0)) V0r1) = (ap c\_2Erat\_2Erat\_of\_num \\
 & c\_2Enum\_2E0)))
 \end{aligned}$$