thm_2Erat_2ERAT__RDISTRIB (TMZvHUqeBePLHFn5Y1AncPu9mctU8jaYJFY)

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Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$
of type $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 3 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

Definition 4 *We define* c_2Ebool_2EF *to be* $(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 5 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$
of type ι.

Definition 6 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \mathsf{c\_2Emin\_2E\_3D\_3D\_3E}\ V0t)\ \mathsf{c\_2Ebool\_2EF}$

Definition 7 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod \\ A0\ A1)$$

$$\tag{1}$$

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod \\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})})$$

$$\tag{2}$$

Definition 8 *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2E$

Let $ty\_2Efrac\_2Efrac : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Efrac\_2Efrac \tag{3}$$

Let $ty\_2Einteger\_2Eint : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Einteger\_2Eint \tag{4}$$

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Let $c\_2Efrac\_2Eabs\_\_frac : \iota$ be given. Assume the following.

$$c\_2Efrac\_2Eabs\_\_frac \in (ty\_2Efrac\_2Efrac^{(ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)})$$

(5)

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega$$

(6)

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum$$

(7)

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega})$$

(8)

Definition 9 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Let $c\_2Einteger\_2Eint\_\_of\_\_num : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_\_of\_\_num \in (ty\_2Einteger\_2Eint^{ty\_2Enum\_2Enum})$$

(9)

Let $c\_2Einteger\_2Eint\_\_REP\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_\_REP\_\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Einteger\_2}$$

(10)

Definition 10 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^{A}.\textbf{if}\ (\exists x \in A.p\ (ap\ P\ x))\ \textbf{then}\ (the\ (\lambda x.x \in A \wedge$ *of type* $\iota \Rightarrow \iota$.

Definition 11 *We define* c_2Einteger_2Eint__REP *to be* $\lambda V0a \in ty\_2Einteger\_2Eint.(ap\ ($c_2Emin_2E_40$\ (t$

Let $c\_2Einteger\_2Etint\_\_lt : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_}}$$

(11)

Definition 12 *We define* c_2Einteger_2Eint__lt *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_$

Let $c\_2Einteger\_2Etint\_\_add : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum$$
$$ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty}$$

(12)

Let $c\_2Einteger\_2Etint\_\_eq : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum}}$$

(13)

Let $c\_2Einteger\_2Eint\_\_ABS\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_\_ABS\_\_CLASS \in (ty\_2Einteger\_2Eint^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})}}$$

(14)

Definition 13 *We define* c_2Einteger_2Eint__ABS *to be* $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_$

Definition 14 *We define* c_2Einteger_2Eint__add *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteg$

Let $c\_2Efrac\_2Erep\_\_frac : \iota$ be given. Assume the following.

$$c\_2Efrac\_2Erep\_\_frac \in ((ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint \\ ty\_2Einteger\_2Eint)^{ty\_2Efrac\_2Efrac}) \tag{15}$$

Let $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND \\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$$
$$\tag{16}$$

Definition 15 *We define* c_2Efrac_2Efrac__dnm *to be* $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap\ (c\_2Epair\_2ESND\ t$

Let $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST \\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$$
$$\tag{17}$$

Definition 16 *We define* c_2Efrac_2Efrac__nmr *to be* $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap\ (c\_2Epair\_2EFST\ ty$

Let $c\_2Einteger\_2Etint\_\_mul : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum \\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty}$$
$$\tag{18}$$

Definition 17 *We define* c_2Einteger_2Eint__mul *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteg$

Let $ty\_2Erat\_2Erat : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Erat\_2Erat \tag{19}$$

Let $c\_2Erat\_2Erep\_\_rat\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Erat\_2Erep\_\_rat\_\_CLASS \in ((2^{ty\_2Efrac\_2Efrac})^{ty\_2Erat\_2Erat}) \tag{20}$$

Definition 18 *We define* c_2Erat_2Erep__rat *to be* $\lambda V0a \in ty\_2Erat\_2Erat.(ap\ ($c_2Emin_2E_40$\ ty\_2Efrac\_$

Definition 19 *We define* c_2Efrac_2Efrac__add *to be* $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Ej$

Definition 20 *We define* c_2Erat_2Erat__equiv *to be* $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Ef$

Let $c\_2Erat\_2Eabs\_\_rat\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Erat\_2Eabs\_\_rat\_\_CLASS \in (ty\_2Erat\_2Erat^{(2^{ty\_2Efrac\_2Efrac})}) \tag{21}$$

Definition 21 *We define* c_2Erat_2Eabs__rat *to be* $\lambda V0r \in ty\_2Efrac\_2Efrac.(ap\ c\_2Erat\_2Eabs\__rat\__C1$

Definition 22 *We define* c_2Erat_2Erat__add *to be* $\lambda V0r1 \in ty\_2Erat\_2Erat.\lambda V1r2 \in ty\_2Erat\_2Erat.(ap$

Definition 23 *We define* c_2Efrac_2Efrac__mul *to be* $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2Ef$

Definition 24 *We define* c_2Erat_2Erat__mul *to be* $\lambda V0r1 \in ty\_2Erat\_2Erat.\lambda V1r2 \in ty\_2Erat\_2Erat.(ap$

Assume the following.
$$True \tag{22}$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \wedge True) \Leftrightarrow$$
$$(p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge ((((p\ V0t) \wedge False) \Leftrightarrow False) \wedge \tag{23}$$
$$(((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))))$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \Leftrightarrow True) \Leftrightarrow$$
$$(p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg (p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg (\tag{24}$$
$$p\ V0t)))))))$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p\ V0t1) \Rightarrow$$
$$((p\ V1t2) \Rightarrow (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \Rightarrow (p\ V2t3)))))) \tag{25}$$

Assume the following.

$$(\forall V0f \in ty\_2Efrac\_2Efrac.(p\ (ap\ (ap\ \mathsf{c\_2Einteger\_2Eint\__lt}$$
$$(ap\ c\_2Einteger\_2Eint\__of\__num\ \mathsf{c\_2Enum\_2E0}))\ (ap\ \mathsf{c\_2Efrac\_2Efrac\__dnm}$$
$$V0f))))$$
$$\tag{26}$$

Assume the following.

$$(\forall V0a \in ty\_2Einteger\_2Eint.(\forall V1b \in ty\_2Einteger\_2Eint.$$
$$((p\ (ap\ (ap\ \mathsf{c\_2Einteger\_2Eint\__lt}\ (ap\ c\_2Einteger\_2Eint\__of\__num$$
$$\mathsf{c\_2Enum\_2E0}))\ V1b)) \Rightarrow ((ap\ \mathsf{c\_2Efrac\_2Efrac\__nmr}\ (ap\ c\_2Efrac\_2Eabs\__frac$$
$$(ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)$$
$$V0a)\ V1b))) = V0a))))$$
$$\tag{27}$$

Assume the following.

$$(\forall V0a \in ty\_2Einteger\_2Eint.(\forall V1b \in ty\_2Einteger\_2Eint.$$
$$((p\ (ap\ (ap\ \mathsf{c\_2Einteger\_2Eint\__lt}\ (ap\ c\_2Einteger\_2Eint\__of\__num$$
$$\mathsf{c\_2Enum\_2E0}))\ V1b)) \Rightarrow ((ap\ \mathsf{c\_2Efrac\_2Efrac\__dnm}\ (ap\ c\_2Efrac\_2Eabs\__frac$$
$$(ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)$$
$$V0a)\ V1b))) = V1b))))$$
$$\tag{28}$$

Assume the following.

$$(\forall V0a \in ty\_2Einteger\_2Eint.(\forall V1b \in ty\_2Einteger\_2Eint.$$
$$((p \; (ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_lt} \; (ap \; c\_2Einteger\_2Eint\_\_of\_\_num}$$
$$\mathsf{c\_2Enum\_2E0})) \; V0a))\Rightarrow((p \; (ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_lt} \; (ap \; c\_2Einteger\_2Eint\_\_of\_\_num}$$
$$\mathsf{c\_2Enum\_2E0})) \; V1b))\Rightarrow(p \; (ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_lt} \; (ap \; c\_2Einteger\_2Eint\_\_of\_\_num}$$
$$\mathsf{c\_2Enum\_2E0})) \; (ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_mul} \; V0a) \; V1b)))))))))$$

$$(29)$$

Assume the following.

$$(\forall V0y \in ty\_2Einteger\_2Eint.(\forall V1x \in ty\_2Einteger\_2Eint.$$
$$((ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_mul} \; V1x) \; V0y) = (ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_mul}$$
$$V0y) \; V1x))))$$

$$(30)$$

Assume the following.

$$(\forall V0z \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.$$
$$(\forall V2x \in ty\_2Einteger\_2Eint.((ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_mul}$$
$$V2x) \; (ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_mul} \; V1y) \; V0z)) = (ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_mul}$$
$$(ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_mul} \; V2x) \; V1y)) \; V0z)))))$$

$$(31)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.$$
$$(\forall V2z \in ty\_2Einteger\_2Eint.((ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_mul}$$
$$(ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_add} \; V0x) \; V1y)) \; V2z) = (ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_add}$$
$$(ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_mul} \; V0x) \; V2z)) \; (ap \; (ap \; \mathsf{c\_2Einteger\_2Eint\_\_mul}$$
$$V1y) \; V2z)))))))$$

$$(32)$$

Assume the following.

$$(\forall V0f1 \in ty\_2Efrac\_2Efrac.(\forall V1f2 \in ty\_2Efrac\_2Efrac.$$
$$(((ap \; \mathsf{c\_2Erat\_2Eabs\_\_rat} \; V0f1) = (ap \; \mathsf{c\_2Erat\_2Eabs\_\_rat} \; V1f2))\Leftrightarrow \quad (33)$$
$$(p \; (ap \; (ap \; \mathsf{c\_2Erat\_2Erat\_\_equiv} \; V0f1) \; V1f2)))))$$

Assume the following.

$$((\forall V0x \in ty\_2Efrac\_2Efrac.(\forall V1y \in ty\_2Efrac\_2Efrac.$$
$$((ap \; \mathsf{c\_2Erat\_2Eabs\_\_rat} \; (ap \; (ap \; \mathsf{c\_2Efrac\_2Efrac\_\_add} \; (ap \; \mathsf{c\_2Erat\_2Erep\_\_rat}$$
$$(ap \; \mathsf{c\_2Erat\_2Eabs\_\_rat} \; V0x))) \; V1y)) = (ap \; \mathsf{c\_2Erat\_2Eabs\_\_rat} \; ($$
$$ap \; (ap \; \mathsf{c\_2Efrac\_2Efrac\_\_add} \; V0x) \; V1y)))))\wedge(\forall V2x \in ty\_2Efrac\_2Efrac.$$
$$(\forall V3y \in ty\_2Efrac\_2Efrac.((ap \; \mathsf{c\_2Erat\_2Eabs\_\_rat} \; (ap \; ($$
$$ap \; \mathsf{c\_2Efrac\_2Efrac\_\_add} \; V2x) \; (ap \; \mathsf{c\_2Erat\_2Erep\_\_rat} \; (ap \; \mathsf{c\_2Erat\_2Eabs\_\_rat}$$
$$V3y)))) = (ap \; \mathsf{c\_2Erat\_2Eabs\_\_rat} \; (ap \; (ap \; \mathsf{c\_2Efrac\_2Efrac\_\_add}$$
$$V2x) \; V3y))))))$$

$$(34)$$

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Assume the following.

$$((\forall V0x \in ty\_2Efrac\_2Efrac.(\forall V1y \in ty\_2Efrac\_2Efrac.$$
$$((ap\ \mathsf{c\_2Erat\_2Eabs\_\_rat}\ (ap\ (ap\ \mathsf{c\_2Efrac\_2Efrac\_\_mul}\ (ap\ \mathsf{c\_2Erat\_2Erep\_\_rat}$$
$$(ap\ \mathsf{c\_2Erat\_2Eabs\_\_rat}\ V0x)))\ V1y)) = (ap\ \mathsf{c\_2Erat\_2Eabs\_\_rat}\ ($$
$$ap\ (ap\ \mathsf{c\_2Efrac\_2Efrac\_\_mul}\ V0x)\ V1y))))) \wedge (\forall V2x \in ty\_2Efrac\_2Efrac.$$
$$(\forall V3y \in ty\_2Efrac\_2Efrac.((ap\ \mathsf{c\_2Erat\_2Eabs\_\_rat}\ (ap\ ($$
$$ap\ \mathsf{c\_2Efrac\_2Efrac\_\_mul}\ V2x)\ (ap\ \mathsf{c\_2Erat\_2Erep\_\_rat}\ (ap\ \mathsf{c\_2Erat\_2Eabs\_\_rat}$$
$$V3y)))) = (ap\ \mathsf{c\_2Erat\_2Eabs\_\_rat}\ (ap\ (ap\ \mathsf{c\_2Efrac\_2Efrac\_\_mul}$$
$$V2x)\ V3y))))))) \tag{35}$$

Theorem 1

$$(\forall V0a \in ty\_2Erat\_2Erat.(\forall V1b \in ty\_2Erat\_2Erat.($$
$$\forall V2c \in ty\_2Erat\_2Erat.((ap\ (ap\ \mathsf{c\_2Erat\_2Erat\_\_mul}\ (ap\ ($$
$$ap\ \mathsf{c\_2Erat\_2Erat\_\_add}\ V0a)\ V1b))\ V2c) = (ap\ (ap\ \mathsf{c\_2Erat\_2Erat\_\_add}$$
$$(ap\ (ap\ \mathsf{c\_2Erat\_2Erat\_\_mul}\ V0a)\ V2c))\ (ap\ (ap\ \mathsf{c\_2Erat\_2Erat\_\_mul}$$
$$V1b)\ V2c)))))))$$