thm_2Ereal_2EREAL__SUP__EXISTS (TMTM68U9V4oGH1WfmPYPgd53dCYtbuk9SWA)

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Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \tag{1}$$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{2}$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \tag{3}$$

Definition 2 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Let $ty\_2Erealax\_2Ereal : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Erealax\_2Ereal \tag{4}$$

Let $c\_2Ereal\_2Ereal\_\_of\_\_num : \iota$ be given. Assume the following.

$$c\_2Ereal\_2Ereal\_\_of\_\_num \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}) \tag{5}$$

Definition 3 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 4 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 5 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

Definition 6 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Let $ty\_2Ehreal\_2Ehreal : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \quad (6)$$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \quad (7)$$

Let $c\_2Erealax\_2Ereal\_\_REP\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Erealax\_2Ereal\_\_REP\_\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{ty\_2Erealax} \quad (8)$$

Definition 7 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 8 *We define* c_2Erealax_2Ereal__REP *to be* $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap$ (c_2Emin_2E_40 $(ty$

Let $c\_2Erealax\_2Etreal\_\_add : \iota$ be given. Assume the following.

$$c\_2Erealax\_2Etreal\_\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehr} \quad (9)$$

Let $c\_2Erealax\_2Etreal\_\_eq : \iota$ be given. Assume the following.

$$c\_2Erealax\_2Etreal\_\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Eh} \quad (10)$$

Let $c\_2Erealax\_2Ereal\_\_ABS\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Erealax\_2Ereal\_\_ABS\_\_CLASS \in (ty\_2Erealax\_2Ereal^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehr} \quad (11)$$

Definition 9 *We define* c_2Erealax_2Ereal__ABS *to be* $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_$

Definition 10 *We define* c_2Erealax_2Ereal__add *to be* $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax$

Let $c\_2Erealax\_2Etreal\_\_lt : \iota$ be given. Assume the following.

$$c\_2Erealax\_2Etreal\_\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehr} \quad (12)$$

Definition 11 *We define* c_2Erealax_2Ereal__lt *to be* $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_$

Definition 12 *We define* c_2Ebool_2E_3F *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap$ (c_2Emin_2E_40

Assume the following.

$$(\forall V0P \in (2^{ty\_2Erealax\_2Ereal}).(((\exists V1x \in ty\_2Erealax\_2Ereal.((p\ (ap\ V0P\ V1x))\wedge(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ (ap\ c\_2Ereal\_2Ereal\_\_of\_\_num\ \mathsf{c\_2Enum\_2E0}))\ V1x))))\wedge(\exists V2z \in ty\_2Erealax\_2Ereal.(\forall V3x \in ty\_2Erealax\_2Ereal.((p\ (ap\ V0P\ V3x))\Rightarrow(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V3x)\ V2z))))))\Rightarrow(\exists V4s \in ty\_2Erealax\_2Ereal.(\forall V5y \in ty\_2Erealax\_2Ereal.((\exists V6x \in ty\_2Erealax\_2Ereal.((p\ (ap\ V0P\ V6x))\wedge(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V5y)\ V6x))))\Leftrightarrow(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V5y)\ V4s)))))))\tag{13}$$

Assume the following.

$$(\forall V0P \in (2^{ty\_2Erealax\_2Ereal}).(\forall V1s \in ty\_2Erealax\_2Ereal.(\forall V2d \in ty\_2Erealax\_2Ereal.((\forall V3y \in ty\_2Erealax\_2Ereal.((\exists V4x \in ty\_2Erealax\_2Ereal.((p\ (ap\ (\lambda V5x \in ty\_2Erealax\_2Ereal.(ap\ V0P\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_add}\ V5x)\ V2d)))\ V4x))\wedge(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V3y)\ V4x))))\Leftrightarrow(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V3y)\ V1s))))\Rightarrow(\forall V6y \in ty\_2Erealax\_2Ereal.((\exists V7x \in ty\_2Erealax\_2Ereal.((p\ (ap\ V0P\ V7x))\wedge(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V6y)\ V7x))))\Leftrightarrow(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V6y)\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_add}\ V1s)\ V2d)))))))))\tag{14}$$

Assume the following.

$$(\forall V0P \in (2^{ty\_2Erealax\_2Ereal}).((\exists V1x \in ty\_2Erealax\_2Ereal.(p\ (ap\ V0P\ V1x)))\Rightarrow(\exists V2d \in ty\_2Erealax\_2Ereal.(\exists V3x \in ty\_2Erealax\_2Ereal.((p\ (ap\ (\lambda V4x \in ty\_2Erealax\_2Ereal.(ap\ V0P\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_add}\ V4x)\ V2d)))\ V3x))\wedge(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ (ap\ c\_2Ereal\_2Ereal\_\_of\_\_num\ \mathsf{c\_2Enum\_2E0}))\ V3x)))))))\tag{15}$$

Assume the following.

$$(\forall V0P \in (2^{ty\_2Erealax\_2Ereal}).(\forall V1d \in ty\_2Erealax\_2Ereal.((\exists V2z \in ty\_2Erealax\_2Ereal.(\forall V3x \in ty\_2Erealax\_2Ereal.((p\ (ap\ V0P\ V3x))\Rightarrow(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V3x)\ V2z)))))\Rightarrow(\exists V4z \in ty\_2Erealax\_2Ereal.(\forall V5x \in ty\_2Erealax\_2Ereal.((p\ (ap\ (\lambda V6x \in ty\_2Erealax\_2Ereal.(ap\ V0P\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_add}\ V6x)\ V1d)))\ V5x))\Rightarrow(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V5x)\ V4z))))))))\tag{16}$$

Theorem 1

$$(\forall V0P \in (2^{ty\_2Erealax\_2Ereal}).(((\exists V1x \in ty\_2Erealax\_2Ereal.(p\ (ap\ V0P\ V1x)))\land(\exists V2z \in ty\_2Erealax\_2Ereal.(\forall V3x \in ty\_2Erealax\_2Ereal.((p\ (ap\ V0P\ V3x))\Rightarrow(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V3x)\ V2z))))))\Rightarrow(\exists V4s \in ty\_2Erealax\_2Ereal.(\forall V5y \in ty\_2Erealax\_2Ereal.((\exists V6x \in ty\_2Erealax\_2Ereal.((p\ (ap\ V0P\ V6x))\land(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V5y)\ V6x))))\Leftrightarrow(p\ (ap\ (ap\ \mathsf{c\_2Erealax\_2Ereal\_\_lt}\ V5y)\ V4s)))))))$$