

# thm\_2Ereal\_2ESUP\_\_EPSILON (TMKQ5y9PRtYoCPAGWtxwc8EEeSjguHh7UzA)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \tag{2}$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \tag{3}$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \tag{4}$$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A-27a}))$

**Definition 4** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap c\_2Enum\_2EABS\_num ($

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \tag{5}$$

**Definition 5** We define  $c\_2Enum\_2E0$  to be  $(ap c\_2Enum\_2EABS\_num c\_2Enum\_2EZERO\_REP)$ .

**Definition 6** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \tag{6}$$

**Definition 7** We define  $c\_Earithmic\_EBIT1$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap (ap c\_Earithmic\_EBIT1 V0n))$ .

**Definition 8** We define  $c\_Earithmic\_ENUMERAL$  to be  $\lambda V0x \in ty\_2Enum\_2Enum.V0x$ .

**Definition 9** We define  $c\_Ebool\_EF$  to be  $(ap (c\_Ebool\_E21 2)) (\lambda V0t \in 2.V0t)$ .

**Definition 10** We define  $c\_Emin\_E3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 11** We define  $c\_Ebool\_E7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_Emin\_E3D\_3D\_3E V0t) c\_Ebool\_E21))$ .

**Definition 12** We define  $c\_Ebool\_E2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_Ebool\_E21 2)) (\lambda V2t \in 2.V2t)))$ .

**Definition 13** We define  $c\_Emin\_E40$  to be  $\lambda A.\lambda P \in 2^A.\mathbf{if} (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge P x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 14** We define  $c\_Ebool\_E3F$  to be  $\lambda A^{27a} : \iota.(\lambda V0P \in (2^{A-27a}).(ap V0P (ap (c\_Emin\_E40 V0P))))$ .

**Definition 15** We define  $c\_Eprim\_rec\_E3C$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum.V0m$ .

**Definition 16** We define  $c\_Ebool\_E5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_Ebool\_E21 2)) (\lambda V2t \in 2.V2t)))$ .

**Definition 17** We define  $c\_Earithmic\_E3C\_3D$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum.V0m$ .

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \tag{7}$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{8}$$

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erealax\_2Ereal \tag{9}$$

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{ty\_2Erealax\_2Ereal}) \tag{10}$$

**Definition 18** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_Emin\_E40 (ty\_2Erealax\_2Ereal\_REP\_CLASS V0a)))$ .

Let  $c\_2Erealax\_2Etrealm\_inv : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealm\_inv \in ((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)^{ty\_2Erealax\_2Ereal}) \tag{11}$$

Let  $c\_2Erealax\_2Etrealm\_eq : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealm\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal}) \tag{12}$$

Let  $c\_2Erealax\_2Ereal\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_ABS\_CLASS \in (ty\_2Erealax\_2Ereal)^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})} \tag{13}$$

**Definition 19** We define  $c\_2Erealax\_2Ereal\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)$

**Definition 20** We define  $c\_2Erealax\_2Einv$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.(ap\ c\_2Erealax\_2Ereal\_ABS)$

Let  $c\_2Erealax\_2Etrealmul : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealmul \in (((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)))(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal) \quad (14)$$

**Definition 21** We define  $c\_2Erealax\_2Ereal\_mul$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal.$

**Definition 22** We define  $c\_2Ereal\_2E\_2F$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal.$

Let  $c\_2Erealax\_2Etrealmul : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealmul \in ((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)) \quad (15)$$

**Definition 23** We define  $c\_2Erealax\_2Ereal\_neg$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.(ap\ c\_2Erealax\_2Ereal\_mul)$

Let  $c\_2Erealax\_2Etrealmul : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealmul \in (((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)))(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal) \quad (16)$$

**Definition 24** We define  $c\_2Erealax\_2Ereal\_add$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal.$

**Definition 25** We define  $c\_2Ereal\_2Ereal\_sub$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal.$

Let  $c\_2Erealax\_2Etrealmul : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealmul \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)) \quad (17)$$

**Definition 26** We define  $c\_2Erealax\_2Ereal\_lt$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal.$

**Definition 27** We define  $c\_2Ereal\_2Esup$  to be  $\lambda V0P \in (2^{ty\_2Erealax\_2Ereal}).(ap\ (c\_2Emin\_2E\_40\ ty\_2Erealax\_2Ereal\_mul))$

**Definition 28** We define  $c\_2Ereal\_2Ereal\_lte$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal.$

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal^{ty\_2Eenum\_2Eenum}) \quad (18)$$

**Definition 29** We define  $c\_2Ecombin\_2Eo$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.\lambda V0f \in (A\_27b^{A\_27c}).\lambda V1g \in (A\_27a^{A\_27c}).$

Let  $c\_2Ewhile\_2EWHILE : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ewhile\_2EWHILE\ A\_27a \in (((A\_27a^{A\_27a})(A\_27a^{A\_27a}))^{(2^{A\_27a})}) \quad (19)$$

**Definition 30** We define  $c\_2Ewhile\_2ELEAST$  to be  $\lambda V0P \in (2^{ty\_2Enum\_2Enum}).(ap (ap (ap (c\_2Ewhile\_2$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.((V0m = c\_2Enum\_2E0) \vee (\exists V1n \in ty\_2Enum\_2Enum.(V0m = (ap c\_2Enum\_2ESUC V1n)))))) \quad (20)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.(p (ap (ap c\_2Earithmetic\_2E\_3C\_3D V0m) (ap c\_2Enum\_2ESUC V0m)))) \quad (21)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.(((ap c\_2Enum\_2ESUC V0m) = (ap (ap c\_2Earithmetic\_2E\_2B V0m) (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO)))))) \quad (22)$$

Assume the following.

$$True \quad (23)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (24)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p V0t))) \quad (25)$$

Assume the following.

$$(\forall V0t \in 2.(((p V0t) \vee \neg(p V0t)))) \quad (26)$$

Assume the following.

$$(\forall V0t \in 2.(((p V0t) \Rightarrow False) \Rightarrow \neg(p V0t))) \quad (27)$$

Assume the following.

$$(\forall V0t \in 2.(\neg(p V0t) \Rightarrow ((p V0t) \Rightarrow False))) \quad (28)$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \quad (29)$$

Assume the following.

$$(\forall V0t \in 2.(((True \vee (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \vee True) \Leftrightarrow True) \wedge (((False \vee (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee False) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee (p V0t)) \Leftrightarrow (p V0t)))))) \quad (30)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (31)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge ((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True)) \quad (32)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (33)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (34)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (35)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).((\neg(\exists V1x \in A\_27a.(p (ap V0P V1x)))) \Leftrightarrow (\forall V2x \in A\_27a.(\neg(p (ap V0P V2x)))))) \quad (36)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p V0A) \vee (p V1B) \vee (p V2C)) \Leftrightarrow (((p V0A) \vee (p V1B)) \vee (p V2C)))))) \quad (37)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((p V0A) \vee (p V1B)) \Leftrightarrow ((p V1B) \vee (p V0A)))) \quad (38)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \wedge (p V1B))) \Leftrightarrow ((\neg(p V0A)) \vee (\neg(p V1B)))) \wedge ((\neg((p V0A) \vee (p V1B))) \Leftrightarrow ((\neg(p V0A)) \wedge (\neg(p V1B)))))) \quad (39)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((p V0A) \Rightarrow (p V1B)) \Leftrightarrow ((\neg(p V0A)) \vee (p V1B)))) \quad (40)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow (p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (41)$$

Assume the following.

$$(\forall V0x \in 2.(\forall V1x\_27 \in 2.(\forall V2y \in 2.(\forall V3y\_27 \in 2.(((p V0x) \Leftrightarrow (p V1x\_27)) \wedge ((p V1x\_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y\_27)))))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x\_27) \Rightarrow (p V3y\_27)))))) \quad (42)$$

Assume the following.

$$(\forall V0n \in ty\_2Enum\_2Enum.(p (ap (ap c\_2Eprim\_rec\_2E\_3C V0n) (ap c\_2Enum\_2ESUC V0n)))) \quad (43)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.(\forall V1y \in ty\_2Erealax\_2Ereal. ((ap (ap c\_2Erealax\_2Ereal\_add V0x) V1y) = (ap (ap c\_2Erealax\_2Ereal\_add V1y) V0x)))) \quad (44)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.(\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal.((ap (ap c\_2Erealax\_2Ereal\_add V0x) (ap (ap c\_2Erealax\_2Ereal\_add V1y) V2z)) = (ap (ap c\_2Erealax\_2Ereal\_add (ap (ap c\_2Erealax\_2Ereal\_add V0x) V1y)) V2z)))))) \quad (45)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.((ap (ap c\_2Erealax\_2Ereal\_mul (ap c\_2Ereal\_2Ereal\_of\_num (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO)))) V0x) = V0x)) \quad (46)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.(\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal.(((ap (ap c\_2Erealax\_2Ereal\_add V0x) V1y) = (ap (ap c\_2Erealax\_2Ereal\_add V0x) V2z)) \Leftrightarrow (V1y = V2z)))))) \quad (47)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.(\forall V1y \in ty\_2Erealax\_2Ereal. ((ap c\_2Erealax\_2Ereal\_neg (ap (ap c\_2Erealax\_2Ereal\_add V0x) V1y)) = (ap (ap c\_2Erealax\_2Ereal\_add (ap c\_2Erealax\_2Ereal\_neg V0x)) (ap c\_2Erealax\_2Ereal\_neg V1y)))))) \quad (48)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.((ap (ap c\_2Erealax\_2Ereal\_mul (ap c\_2Ereal\_2Ereal\_of\_num c\_2Enum\_2E0)) V0x) = (ap c\_2Ereal\_2Ereal\_of\_num c\_2Enum\_2E0)))) \quad (49)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.((ap c\_2Erealax\_2Ereal\_neg (ap c\_2Erealax\_2Ereal\_neg V0x)) = V0x)) \quad (50)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.(\forall V1y \in ty\_2Erealax\_2Ereal. ((\neg(p (ap (ap c\_2Erealax\_2Ereal\_lt V0x) V1y))) \Leftrightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V1y) V0x)))))) \quad (51)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.(\forall V1y \in ty\_2Erealax\_2Ereal. ((p (ap (ap c\_2Erealax\_2Ereal\_lt V0x) V1y)) \Leftrightarrow ((p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V1y)) \wedge (\neg(V0x = V1y)))))) \quad (52)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.(\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal.(((p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V1y)) \wedge (p (ap (ap c\_2Ereal\_2Ereal\_lte V1y) V2z))) \Rightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V2z)))))) \quad (53)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.(\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal.((p (ap (ap c\_2Ereal\_2Ereal\_lte (ap (ap c\_2Erealax\_2Ereal\_add V0x) V2z)) (ap (ap c\_2Erealax\_2Ereal\_add V1y) V2z))) \Leftrightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V1y)))))) \quad (54)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal.(\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal.(((ap (ap c\_2Erealax\_2Ereal\_mul V0x) V2z) = (ap (ap c\_2Erealax\_2Ereal\_mul V1y) V2z)) \Leftrightarrow ((V2z = (ap c\_2Ereal\_2Ereal\_of\_num c\_2Enum\_2E0)) \vee (V0x = V1y)))))) \quad (55)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum. (p (ap (ap c\_2Ereal\_2Ereal\_lte (ap c\_2Ereal\_2Ereal\_of\_num V0m)) (ap c\_2Ereal\_2Ereal\_of\_num V1n))) \Leftrightarrow (p (ap (ap c\_2Earithmic\_2E\_3C\_3D V0m) V1n)))) \quad (56)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ((ap\ c\_2Ereal\_2Ereal\_of\_num\ V0m) = (ap\ c\_2Ereal\_2Ereal\_of\_num\ V1n)) \Leftrightarrow (V0m = V1n)))) \quad (57)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. (ap\ (ap\ c\_2Erealax\_2Ereal\_add\ (ap\ c\_2Ereal\_2Ereal\_of\_num\ V0m))\ (ap\ c\_2Ereal\_2Ereal\_of\_num\ V1n)) = (ap\ c\_2Ereal\_2Ereal\_of\_num\ (ap\ (ap\ c\_2Earithmetic\_2E\_2B\ V0m)\ V1n)))))) \quad (58)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. ((\neg (V1y = (ap\ c\_2Ereal\_2Ereal\_of\_num\ c\_2Enum\_2E0))) \Rightarrow ((ap\ (ap\ c\_2Erealax\_2Ereal\_mul\ (ap\ (ap\ c\_2Ereal\_2E\_2F\ V0x)\ V1y))\ V1y) = V0x)))))) \quad (59)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal. ((p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte\ (ap\ (ap\ c\_2Ereal\_2Ereal\_sub\ V0x)\ V1y))\ V2z)) \Leftrightarrow (p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte\ V0x)\ (ap\ (ap\ c\_2Erealax\_2Ereal\_add\ V2z)\ V1y))))))) \quad (60)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. ((ap\ (ap\ c\_2Ereal\_2Ereal\_sub\ V0x)\ (ap\ c\_2Ereal\_2Ereal\_of\_num\ c\_2Enum\_2E0)) = V0x)) \quad (61)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal. ((V0x = (ap\ (ap\ c\_2Ereal\_2Ereal\_sub\ V1y)\ V2z)) \Leftrightarrow ((ap\ (ap\ c\_2Erealax\_2Ereal\_add\ V0x)\ V2z) = V1y)))))) \quad (62)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal. ((p\ (ap\ (ap\ c\_2Erealax\_2Ereal\_lt\ (ap\ c\_2Ereal\_2Ereal\_of\_num\ c\_2Enum\_2E0))\ V2z)) \Rightarrow ((p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte\ (ap\ (ap\ c\_2Erealax\_2Ereal\_mul\ V0x)\ V2z))\ (ap\ (ap\ c\_2Erealax\_2Ereal\_mul\ V1y)\ V2z))) \Leftrightarrow (p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte\ V0x)\ V1y))))))) \quad (63)$$



Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. \\
& (((ap\ c\_2Erealax\_2Ereal\_neg\ V0x) = (ap\ c\_2Erealax\_2Ereal\_neg \\
& V1y)) \Leftrightarrow (V0x = V1y)))) \tag{64}
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0P \in (2^{ty\_2Erealax\_2Ereal}). (((\exists V1x \in ty\_2Erealax\_2Ereal. \\
& (p\ (ap\ V0P\ V1x))) \wedge (\exists V2z \in ty\_2Erealax\_2Ereal. (\forall V3x \in \\
& ty\_2Erealax\_2Ereal. ((p\ (ap\ V0P\ V3x)) \Rightarrow (p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte \\
& V3x)\ V2z)))))) \Rightarrow (\forall V4y \in ty\_2Erealax\_2Ereal. ((\exists V5x \in \\
& ty\_2Erealax\_2Ereal. ((p\ (ap\ V0P\ V5x)) \wedge (p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lt \\
& V4y)\ V5x)))))) \Leftrightarrow (p\ (ap\ (ap\ c\_2Erealax\_2Ereal\_lt\ V4y)\ (ap\ c\_2Ereal\_2Esup \\
& V0P)))))) \tag{65}
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. \\
& ((ap\ (ap\ c\_2Erealax\_2Ereal\_mul\ (ap\ c\_2Erealax\_2Ereal\_neg\ V0x)) \\
& V1y) = (ap\ c\_2Erealax\_2Ereal\_neg\ (ap\ (ap\ c\_2Erealax\_2Ereal\_mul \\
& V0x)\ V1y)))) \tag{66}
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0y \in ty\_2Erealax\_2Ereal. (\forall V1x \in ty\_2Erealax\_2Ereal. \\
& ((p\ (ap\ (ap\ c\_2Erealax\_2Ereal\_lt\ V1x)\ V0y)) \Leftrightarrow (\neg (p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte \\
& V0y)\ V1x)))))) \tag{67}
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. \\
& (\forall V2z \in ty\_2Erealax\_2Ereal. ((ap\ (ap\ c\_2Erealax\_2Ereal\_mul \\
& (ap\ (ap\ c\_2Erealax\_2Ereal\_add\ V0x)\ V1y))\ V2z) = (ap\ (ap\ c\_2Erealax\_2Ereal\_add \\
& (ap\ (ap\ c\_2Erealax\_2Ereal\_mul\ V0x)\ V2z))\ (ap\ (ap\ c\_2Erealax\_2Ereal\_mul \\
& V1y)\ V2z)))))) \tag{68}
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0r \in ty\_2Erealax\_2Ereal. (\exists V1n \in ty\_2Enum\_2Enum. \\
& (p\ (ap\ (ap\ c\_2Erealax\_2Ereal\_lt\ V0r)\ (ap\ c\_2Ereal\_2Ereal\_of\_num \\
& V1n)))) \tag{69}
\end{aligned}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \tag{70}$$

Assume the following.

$$(\forall V0A \in 2. ((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \tag{71}$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow ((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \quad (72)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg(\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \quad (73)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (74)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \Leftrightarrow (p V2r)) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee (\neg(p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee (\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee ((\neg(p V1q)) \vee (\neg(p V0p)))))))))) \quad (75)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \wedge (p V2r)) \Leftrightarrow (((p V0p) \vee (\neg(p V1q)) \vee (\neg(p V2r))) \wedge (((p V1q) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee (\neg(p V0p)))))))) \quad (76)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \vee (p V2r)) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((p V1q) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \quad (77)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \Rightarrow (p V2r)) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((\neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \quad (78)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p)))))) \quad (79)$$

Assume the following.

$$(\forall V0p \in (2^{ty\_2Enum\_2Enum}). ((\exists V1n \in ty\_2Enum\_2Enum. (p (ap V0p V1n))) \Rightarrow ((p (ap V0p (ap c\_2Ewhile\_2ELEAST V0p))) \wedge (\forall V2n \in ty\_2Enum\_2Enum. ((p (ap (ap c\_2Eprim\_rec\_2E\_3C V2n) (ap c\_2Ewhile\_2ELEAST V0p))) \Rightarrow (\neg(p (ap V0p V2n)))))))) \quad (80)$$

**Theorem 1**

$$\begin{aligned} & (\forall V0p \in (2^{ty\_2Erealax\_2Ereal}).(\forall V1e \in ty\_2Erealax\_2Ereal. \\ & (((p (ap (ap c\_2Erealax\_2Ereal\_lt (ap c\_2Ereal\_2Ereal\_of\_num \\ & c\_2Enum\_2E0)) V1e)) \wedge ((\exists V2x \in ty\_2Erealax\_2Ereal.(p (ap \\ & V0p V2x))) \wedge (\exists V3z \in ty\_2Erealax\_2Ereal.(\forall V4x \in ty\_2Erealax\_2Ereal. \\ & ((p (ap V0p V4x)) \Rightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V4x) V3z))))))) \Rightarrow \\ & (\exists V5x \in ty\_2Erealax\_2Ereal.((p (ap V0p V5x)) \wedge (p (ap (ap c\_2Ereal\_2Ereal\_lte \\ & (ap c\_2Ereal\_2Esup V0p)) (ap (ap c\_2Erealax\_2Ereal\_add V5x) V1e))))))) \end{aligned}$$