

# thm\_2Ereal\_\_topology\_2ECBALL\_\_MAX\_\_UNION (TMb2m5LuYNhBnr77HZiVk8zRFfCgM1nC8d2)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 4** We define  $c\_2Emarker\_2E\_2Abbrev$  to be  $\lambda V0x \in 2.V0x$ .

**Definition 5** We define  $c\_2Ebool\_2E\_2IN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap V1f V0x)))$

**Definition 6** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 7** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

**Definition 8** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \quad (1)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EABS\_prod A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (2)$$

**Definition 9** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2Epair\_2EABS\_prod$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC A\_27a A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod A\_27a 2)^{A\_27b}}) \quad (3)$$

**Definition 10** We define  $c\_2Epred\_set\_2EUNION$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). \lambda V1t \in (2^{A\_27a}). (ap (c\_2Epred\_set\_2EUNION) s) t$ .  
Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \quad (4)$$

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erealax\_2Ereal \quad (5)$$

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{ty\_2Erealax\_2Ereal}) \quad (6)$$

**Definition 11** We define  $c\_2Emin\_2E40$  to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p (ap\ P\ x)) \text{ then } (the\ (\lambda x. x \in A \wedge p\ x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 12** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal. (ap (c\_2Emin\_2E40) a)$

Let  $c\_2Erealax\_2Etreallt : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etreallt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal)}) \quad (7)$$

**Definition 13** We define  $c\_2Erealax\_2Ereal\_lt$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal. \lambda V1T2 \in ty\_2Erealax\_2Ereal. (c\_2Erealax\_2Ereal\_lt\ T1\ T2)$

**Definition 14** We define  $c\_2Ebool\_2E21$  to be  $(ap (c\_2Ebool\_2E21) 2) (\lambda V0t \in 2. V0t)$ .

**Definition 15** We define  $c\_2Ebool\_2E7E$  to be  $(\lambda V0t \in 2. (ap (ap\ c\_2Emin\_2E3D\_3D\_3E\ V0t) c\_2Ebool\_2E21) t)$

**Definition 16** We define  $c\_2Ereal\_2Ereal\_lte$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal. \lambda V1y \in ty\_2Erealax\_2Ereal. (c\_2Ereal\_2Ereal\_lte\ x\ y)$

**Definition 17** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A\_27a. (\lambda V2t2 \in A\_27a. (c\_2Ebool\_2ECOND\ t1\ t2)) t))$

**Definition 18** We define  $c\_2Ereal\_2Ereal\_max$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal. \lambda V1y \in ty\_2Erealax\_2Ereal. (c\_2Ereal\_2Ereal\_max\ x\ y)$

Let  $c\_2Erealax\_2Etreallneg : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etreallneg \in ((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)}) \quad (8)$$

Let  $c\_2Erealax\_2Etrealleq : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealleq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal)}) \quad (9)$$

Let  $c\_2Erealax\_2Ereal\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_ABS\_CLASS \in (ty\_2Erealax\_2Ereal)^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})} \quad (10)$$

**Definition 19** We define  $c\_2Erealax\_2Ereal\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)$

**Definition 20** We define  $c\_2Erealax\_2Ereal\_neg$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.(ap\ c\_2Erealax\_2Ereal)$

Let  $c\_2Erealax\_2Etreall\_add : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etreall\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)))(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal) \quad (11)$$

**Definition 21** We define  $c\_2Erealax\_2Ereal\_add$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$

**Definition 22** We define  $c\_2Ereal\_2Ereal\_sub$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (12)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (13)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (14)$$

**Definition 23** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}) \quad (15)$$

**Definition 24** We define  $c\_2Ereal\_2Eabs$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.(ap\ (ap\ (ap\ (c\_2Ebool\_2ECONJ))$

Let  $c\_2Ereal\_topology\_2EDist : \iota$  be given. Assume the following.

$$c\_2Ereal\_topology\_2EDist \in (ty\_2Erealax\_2Ereal^{(ty\_2Epair\_2Eprod\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal)}) \quad (16)$$

Let  $c\_2Ereal\_topology\_2Ecball : \iota$  be given. Assume the following.

$$c\_2Ereal\_topology\_2Ecball \in ((2^{ty\_2Erealax\_2Ereal})^{(ty\_2Epair\_2Eprod\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal)}) \quad (17)$$

Assume the following.

$$True \quad (18)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (19)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in \\ & (2^{A\_27a}). ((V0s = V1t) \Leftrightarrow (\forall V2x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN \\ & A\_27a)\ V2x)\ V0s)) \Leftrightarrow (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ V1t))))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in \\ & (2^{A\_27a}). (\forall V2x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ \\ & V2x)\ (ap\ (ap\ (c\_2Epred\_set\_2EUNION\ A\_27a)\ V0s)\ V1t))) \Leftrightarrow ((p\ (ap\ (21) \\ & (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ V0s)) \vee (p\ (ap\ (ap\ (c\_2Ebool\_2EIN \\ & A\_27a)\ V2x)\ V1t))))))) \end{aligned}$$

Assume the following.

$$\begin{aligned} & (\forall V0z \in ty\_2Erealax\_2Ereal. (\forall V1x \in ty\_2Erealax\_2Ereal. \\ & (\forall V2y \in ty\_2Erealax\_2Ereal. ((p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte \\ & V0z)\ (ap\ (ap\ c\_2Ereal\_2Ereal\_emax\ V1x)\ V2y))) \Leftrightarrow ((p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte \\ & V0z)\ V1x)) \vee (p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte\ V0z)\ V2y)))))) \end{aligned} \quad (22)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. \\ & ((ap\ c\_2Ereal\_topology\_2EDist\ (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Erealax\_2Ereal \\ & ty\_2Erealax\_2Ereal)\ V0x)\ V1y))) = (ap\ c\_2Ereal\_2Eabs\ (ap\ (ap\ c\_2Ereal\_2Ereal\_sub \\ & V0x)\ V1y)))) \end{aligned} \quad (23)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. \\ & (\forall V2e \in ty\_2Erealax\_2Ereal. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ ty\_2Erealax\_2Ereal) \\ & V1y)\ (ap\ c\_2Ereal\_topology\_2Ecball\ (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Erealax\_2Ereal \\ & ty\_2Erealax\_2Ereal)\ V0x)\ V2e)))) \Leftrightarrow (p\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte \\ & (ap\ c\_2Ereal\_topology\_2EDist\ (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Erealax\_2Ereal \\ & ty\_2Erealax\_2Ereal)\ V0x)\ V1y)))\ V2e)))))) \end{aligned} \quad (24)$$

### Theorem 1

$$\begin{aligned} & (\forall V0a \in ty\_2Erealax\_2Ereal. (\forall V1r \in ty\_2Erealax\_2Ereal. \\ & (\forall V2s \in ty\_2Erealax\_2Ereal. ((ap\ c\_2Ereal\_topology\_2Ecball \\ & (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal) \\ & V0a)\ (ap\ (ap\ c\_2Ereal\_2Ereal\_emax\ V1r)\ V2s))) = (ap\ (ap\ (c\_2Epred\_set\_2EUNION \\ & ty\_2Erealax\_2Ereal)\ (ap\ c\_2Ereal\_topology\_2Ecball\ (ap\ (ap\ ( \\ & c\_2Epair\_2E\_2C\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal)\ V0a)\ \\ & V1r)))\ (ap\ c\_2Ereal\_topology\_2Ecball\ (ap\ (ap\ (c\_2Epair\_2E\_2C \\ & ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal)\ V0a)\ V2s)))))) \end{aligned}$$