

thm_2Ereal__topology_2ECLOSED__IN__SUBTOPOLOGY__EMPTY (TMSK56aLCpv91PnEX45VN1YS7HYn6wCRdvk)

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Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define c_2Ebool_2ET to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define c_2Ebool_2EIN to be $\lambda A_27a : \iota.(\lambda V0x \in A_27a.(\lambda V1f \in (2^{A_27a}).(ap V1f V0x)))$

Definition 4 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 5 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

Definition 6 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t)))$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Epair_2Eprod A0 A1) \quad (1)$$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epair_2EABS_prod A_27a A_27b \in ((ty_2Epair_2Eprod A_27a A_27b)^{(2^{A_27b})^{A_27a}}) \quad (2)$$

Definition 7 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap (c_2Ebool_2E_21 2) (c_2Epair_2EABS_prod A_27a A_27b))$

Let $c_2Epred_set_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epred_set_2EGSPEC A_27a A_27b \in ((2^{A_27a})^{(ty_2Epair_2Eprod A_27a 2)^{A_27b}}) \quad (3)$$

Definition 8 We define $c_2Epred_set_2EINTER$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1t \in (2^{A_27a}).(ap (c_2Ebool_2E_21 2) (c_2Epair_2EABS_prod A_27a A_27b))$

Definition 9 We define `c_2Emin_2E_40` to be $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p \text{ (ap } P \ x)) \text{ then (the } (\lambda x. x \in A \wedge p \text{ of type } \iota \Rightarrow \iota).$

Definition 10 We define `c_2Ebool_2E_3F` to be $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (\text{ap } V0P \text{ (ap (c_2Emin_2E_40$
Let $ty_2Etopology_2Etopology : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0. \text{nonempty } A0 \Rightarrow \text{nonempty } (ty_2Etopology_2Etopology \ A0) \quad (4)$$

Let $c_2Etopology_2Eopen_in : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A. 27a. \text{nonempty } A. 27a \Rightarrow c_2Etopology_2Eopen_in \ A. 27a \in ((2^{(2^{A-27a})}) (ty_2Etopology_2Etopology \ A. 27a)) \quad (5)$$

Let $c_2Etopology_2Etopology : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A. 27a. \text{nonempty } A. 27a \Rightarrow c_2Etopology_2Etopology \ A. 27a \in ((ty_2Etopology_2Etopology \ A. 27a)^{(2^{(2^{A-27a})})}) \quad (6)$$

Definition 11 We define `c_2Ereal_toplogy_2Esubtopology` to be $\lambda A. 27a : \iota. \lambda V0top \in (ty_2Etopology_2Etopology \ A. 27a).$

Definition 12 We define `c_2Ebool_2E_21_2` to be $(\text{ap (c_2Ebool_2E_21_2) } (\lambda V0t \in 2. V0t)).$

Definition 13 We define `c_2Ebool_2E_5C_2F` to be $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (\text{ap (c_2Ebool_2E_21_2) } (\lambda V2t \in 2. V2t))))$

Definition 14 We define `c_2Ebool_2E_7E` to be $(\lambda V0t \in 2. (\text{ap (ap c_2Emin_2E_3D_3D_3E } V0t) \ c_2Ebool_2E_21_2))$

Definition 15 We define `c_2Epred_set_2EEMPTY` to be $\lambda A. 27a : \iota. (\lambda V0x \in A. 27a. c_2Ebool_2E_21_2).$

Definition 16 We define `c_2Epred_set_2EBIGUNION` to be $\lambda A. 27a : \iota. \lambda V0P \in (2^{(2^{A-27a})}). (\text{ap (c_2Epred_set_2EEMPTY$

Definition 17 We define `c_2Etopology_2Etopspace` to be $\lambda A. 27a : \iota. \lambda V0top \in (ty_2Etopology_2Etopology \ A. 27a).$

Definition 18 We define `c_2Epred_set_2EDIFF` to be $\lambda A. 27a : \iota. \lambda V0s \in (2^{A-27a}). \lambda V1t \in (2^{A-27a}). (\text{ap (c_2Ebool_2E_21_2$

Definition 19 We define `c_2Epred_set_2ESUBSET` to be $\lambda A. 27a : \iota. \lambda V0s \in (2^{A-27a}). \lambda V1t \in (2^{A-27a}). (\text{ap (c_2Ebool_2E_21_2$

Definition 20 We define `c_2Etopology_2Eclosed_in` to be $\lambda A. 27a : \iota. \lambda V0top \in (ty_2Etopology_2Etopology \ A. 27a).$

Assume the following.

$$\text{True} \quad (7)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p \ V0t1) \Rightarrow (p \ V1t2)) \Rightarrow (((p \ V1t2) \Rightarrow (p \ V0t1)) \Rightarrow ((p \ V0t1) \Leftrightarrow (p \ V1t2)))))) \quad (8)$$

Assume the following.

$$(\forall V0t \in 2. (((p \ V0t) \Rightarrow \text{False}) \Rightarrow (\neg (p \ V0t)))) \quad (9)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(p V0t)) \Rightarrow ((p V0t) \Rightarrow False))) \quad (10)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((\\ & (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \end{aligned} \quad (11)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ & ((\neg False) \Leftrightarrow True))) \end{aligned} \quad (12)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0x \in A.27a.(\forall V1y \in A.27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (13)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(\\ & p V0t)))))) \end{aligned} \quad (14)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).((\neg(\forall V1x \in A.27a.(p (ap V0P V1x)))) \Leftrightarrow (\exists V2x \in A.27a.(\neg(p (ap V0P V2x)))))) \quad (15)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).((\neg(\exists V1x \in A.27a.(p (ap V0P V1x)))) \Leftrightarrow (\forall V2x \in A.27a.(\neg(p (ap V0P V2x)))))) \quad (16)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).(\forall V1Q \in \\ & 2.(((\exists V2x \in A.27a.(p (ap V0P V2x))) \vee (p V1Q)) \Leftrightarrow (\exists V3x \in \\ & A.27a.((p (ap V0P V3x)) \vee (p V1Q)))))) \end{aligned} \quad (17)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).(\forall V1Q \in \\ & 2.((\exists V2x \in A.27a.((p (ap V0P V2x)) \wedge (p V1Q))) \Leftrightarrow ((\exists V3x \in \\ & A.27a.(p (ap V0P V3x)) \wedge (p V1Q)))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow ((\forall V0s \in (2^{A.27a}).((ap (\\ & ap (c.2Epred_set_2EINTER A.27a) (c.2Epred_set_2EEMPTY A.27a)) \\ & V0s) = (c.2Epred_set_2EEMPTY A.27a))) \wedge (\forall V1s \in (2^{A.27a}). \\ & ((ap (ap (c.2Epred_set_2EINTER A.27a) V1s) (c.2Epred_set_2EEMPTY \\ & A.27a)) = (c.2Epred_set_2EEMPTY A.27a)))) \end{aligned} \quad (19)$$

Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0top \in (ty_2Etopology_2Etopology \\ A_27a).(\forall V1u \in (2^{A_27a}).(\forall V2s \in (2^{A_27a}).((p (\\ ap (ap (c_2Etopology_2Eclosed_in\ A_27a) (ap (ap (c_2Ereal_topology_2Esubtopology \\ A_27a) V0top) V1u)) V2s)) \Leftrightarrow (\exists V3t \in (2^{A_27a}).((p (ap (ap (\\ c_2Etopology_2Eclosed_in\ A_27a) V0top) V3t)) \wedge (V2s = (ap (ap (\\ c_2Epred_set_2EINTER\ A_27a) V3t) V1u)))))))))) \end{aligned} \quad (20)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (21)$$

Assume the following.

$$(\forall V0A \in 2.((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (22)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ ((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) \quad (23)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) \quad (24)$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \quad (25)$$

Assume the following.

$$\begin{aligned} (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\ (p\ V1q) \Leftrightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (p\ V2r))) \wedge (((p\ V0p) \vee ((\neg \\ p\ V2r)) \vee (\neg(p\ V1q)))) \wedge (((p\ V1q) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V0p)))) \wedge ((p\ V2r) \vee \\ ((\neg(p\ V1q)) \vee (\neg(p\ V0p)))))))))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned} (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\ (p\ V1q) \wedge (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((\neg(p\ V1q)) \vee (\neg(p\ V2r)))) \wedge (((p\ V1q) \vee \\ (\neg(p\ V0p))) \wedge ((p\ V2r) \vee (\neg(p\ V0p)))))))))) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned} (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\ (p\ V1q) \vee (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee (\neg(p\ V1q))) \wedge (((p\ V0p) \vee (\neg(p\ V2r))) \wedge \\ ((p\ V1q) \vee ((p\ V2r) \vee (\neg(p\ V0p)))))))))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow (\\
& (p \ V1q) \Rightarrow (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee (p \ V1q)) \wedge (((p \ V0p) \vee (\neg(p \ V2r))) \wedge (\\
& \neg(p \ V1q)) \vee ((p \ V2r) \vee (\neg(p \ V0p))))))))))
\end{aligned} \tag{29}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (((p \ V0p) \Leftrightarrow (\neg(p \ V1q))) \Leftrightarrow (((p \ V0p) \vee \\
& (p \ V1q)) \wedge ((\neg(p \ V1q)) \vee (\neg(p \ V0p))))))
\end{aligned} \tag{30}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0top \in (ty_2Etopology_2Etopology \\
& A_27a).(p \ (ap \ (ap \ (c_2Etopology_2Eclosed_in \ A_27a) \ V0top) \ (c_2Epred_set_2EEMPTY \\
& A_27a))))
\end{aligned} \tag{31}$$

Theorem 1

$$\begin{aligned}
& \forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0top \in (ty_2Etopology_2Etopology \\
& A_27a).(\forall V1s \in (2^{A_27a}).((p \ (ap \ (ap \ (c_2Etopology_2Eclosed_in \\
& A_27a) \ (ap \ (ap \ (c_2Ereal_topology_2Esubtopology \ A_27a) \ V0top) \\
& (c_2Epred_set_2EEMPTY \ A_27a))) \ V1s)) \Leftrightarrow (V1s = (c_2Epred_set_2EEMPTY \\
& A_27a))))))
\end{aligned}$$