

thm\_2Ereal\_\_topology\_2ECOMPONENTS\_\_MAXIMAL  
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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 4** We define  $c\_2Ebool\_2E\_EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2E\_EF)$ .

**Definition 6** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 7** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_EF))$

**Definition 8** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge p x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 9** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap V0P (ap (c\_2Emin\_2E\_40 A\_27a) P)))$

**Definition 10** We define  $c\_2Ebool\_2E\_EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap V1f V0x)))$

**Definition 11** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EABS\_prod A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a A\_27b)^{(2^{A\_27b})^{A\_27a}}) \tag{2}$$

**Definition 12** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota)$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC A\_27a A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod A\_27a A\_27b)}) \quad (3)$$

**Definition 13** We define  $c\_2Epred\_set\_2EINTER$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap (c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota)$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2ESND A\_27a A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod A\_27a A\_27b)}) \quad (4)$$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EFST A\_27a A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod A\_27a A\_27b)}) \quad (5)$$

**Definition 14** We define  $c\_2Epair\_2EUNCURRY$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.\lambda V0f \in ((A\_27c)^{A\_27a})$

**Definition 15** We define  $c\_2Epred\_set\_2ESUBSET$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap (c\_2Ereal\_topology\_2EDist : \iota)$  be given. Assume the following.

$$nonempty ty\_2Erealax\_2Ereal \quad (6)$$

**Definition 16** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.(ap (c\_2Ereal\_topology\_2EDist : \iota)$  be given. Assume the following.

**Definition 17** We define  $c\_2Epred\_set\_2EUNION$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap (c\_2Ereal\_topology\_2EDist : \iota)$  be given. Assume the following.

$$c\_2Ereal\_topology\_2EDist \in (ty\_2Erealax\_2Ereal^{(ty\_2Epair\_2Eprod ty\_2Erealax\_2Ereal ty\_2Erealax\_2Ereal)}) \quad (7)$$

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty ty\_2Ehreal\_2Ehreal \quad (8)$$

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)})^{ty\_2Erealax\_2Ereal}) \quad (9)$$

**Definition 18** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_2Emin\_2E\_40 (t$

Let  $c\_2Erealax\_2Etreall\_lt : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etreall\_lt \in ((2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal)}) \quad (10)$$

**Definition 19** We define  $c\_2Erealax\_2Ereal\_lt$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$ . Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \tag{11}$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{12}$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega\omega}) \tag{13}$$

**Definition 20** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}) \tag{14}$$

**Definition 21** We define  $c\_2Ereal\_topology\_2EOpen$  to be  $\lambda V0s \in (2^{ty\_2Erealax\_2Ereal}).(ap\ (c\_2Ebool\_2E2Ereal))$ .

**Definition 22** We define  $c\_2Ereal\_topology\_2Econnected$  to be  $\lambda V0s \in (2^{ty\_2Erealax\_2Ereal}).(ap\ c\_2Ebool\_2E2Ereal)$ .

**Definition 23** We define  $c\_2Ereal\_topology\_2Econnected\_component$  to be  $\lambda V0s \in (2^{ty\_2Erealax\_2Ereal}).\lambda V1t \in ty\_2Erealax\_2Ereal$ .

**Definition 24** We define  $c\_2Ereal\_topology\_2Ecomponents$  to be  $\lambda V0s \in (2^{ty\_2Erealax\_2Ereal}).(ap\ (c\_2Epre\_2Epre\_2Epre))$ .

Assume the following.

$$True \tag{15}$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge \\ & (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \tag{16} \end{aligned}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \tag{17}$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow \neg(p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow \neg \\ & (p\ V0t)))))) \tag{18} \end{aligned}$$

Assume the following.

$$\begin{aligned} & (\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p\ V0t1) \Rightarrow \\ & ((p\ V1t2) \Rightarrow (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \Rightarrow (p\ V2t3)))))) \tag{19} \end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in 2. (\forall V1x_{27} \in 2. (\forall V2y \in 2. (\forall V3y_{27} \in \\
& 2. (((p V0x) \Leftrightarrow (p V1x_{27})) \wedge ((p V1x_{27}) \Rightarrow ((p V2y) \Leftrightarrow (p V3y_{27})))))) \Rightarrow \quad (20) \\
& (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x_{27}) \Rightarrow (p V3y_{27}))))))
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow (\forall V0P \in (2^{A_{27a}}). (\forall V1Q \in \\
& 2. (((\exists V2x \in A_{27a}. (p (ap V0P V2x))) \Rightarrow (p V1Q)) \Leftrightarrow (\forall V3x \in \quad (21) \\
& A_{27a}. ((p (ap V0P V3x)) \Rightarrow (p V1Q))))))
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0r \in 2. (\forall V1p \in 2. (\forall V2q \in 2. (((p V1p) \wedge \\
& (p V2q) \Rightarrow (p V0r)) \Leftrightarrow ((p V1p) \Rightarrow ((p V2q) \Rightarrow (p V0r)))))) \quad (22)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow (\forall V0s \in (2^{A_{27a}}). ((\exists V1x \in \\
& A_{27a}. (p (ap (ap (c\_2Ebool\_2EIN A_{27a}) V1x) V0s))) \Leftrightarrow (\neg (V0s = (c\_2Epred\_set\_2EEMPTY \\
& A_{27a})))))) \quad (23)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow (\forall V0s \in (2^{A_{27a}}). (\forall V1t \in \\
& (2^{A_{27a}}). (\forall V2x \in A_{27a}. ((p (ap (ap (c\_2Ebool\_2EIN A_{27a}) \\
& V2x) (ap (ap (c\_2Epred\_set\_2EINTER A_{27a}) V0s) V1t))) \Leftrightarrow ((p (ap \quad (24) \\
& (ap (c\_2Ebool\_2EIN A_{27a}) V2x) V0s)) \wedge (p (ap (ap (c\_2Ebool\_2EIN \\
& A_{27a}) V2x) V1t))))))
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow \forall A.27c. \\
& nonempty\ A.27c \Rightarrow \forall A.27d.nonempty\ A.27d \Rightarrow \forall A.27e.nonempty \\
& A.27e \Rightarrow \forall A.27f.nonempty\ A.27f \Rightarrow \forall A.27g.nonempty\ A.27g \Rightarrow \\
& (\forall V0Q \in (2^{A.27b}).((\forall V1P \in (2^{A.27a}).(\forall V2f \in \\
& (A.27b^{A.27a}).(\forall V3z \in A.27b.((p\ (ap\ (ap\ (c.2Ebool.2EIN \\
& A.27b)\ V3z)\ (ap\ (c.2Epred\_set.2EGSPEC\ A.27b\ A.27a)\ (\lambda V4x \in \\
& A.27a.(ap\ (ap\ (c.2Epair.2E.2C\ A.27b\ 2)\ (ap\ V2f\ V4x))\ (ap\ V1P\ V4x)))))) \Rightarrow \\
& (p\ (ap\ V0Q\ V3z)))) \Leftrightarrow (\forall V5x \in A.27a.((p\ (ap\ V1P\ V5x)) \Rightarrow (p\ (ap\ V0Q \\
& (ap\ V2f\ V5x)))))) \wedge ((\forall V6P \in ((2^{A.27d})^{A.27c}).(\forall V7f \in \\
& ((A.27b^{A.27d})^{A.27c}).(\forall V8z \in A.27b.((p\ (ap\ (ap\ (c.2Ebool.2EIN \\
& A.27b)\ V8z)\ (ap\ (c.2Epred\_set.2EGSPEC\ A.27b\ (ty.2Epair.2Eprod \\
& A.27c\ A.27d))\ (ap\ (c.2Epair.2EUNCURRY\ A.27c\ A.27d\ (ty.2Epair.2Eprod \\
& A.27b\ 2))\ (\lambda V9x \in A.27c.(\lambda V10y \in A.27d.(ap\ (ap\ (c.2Epair.2E.2C \\
& A.27b\ 2)\ (ap\ (ap\ V7f\ V9x)\ V10y))\ (ap\ (ap\ V6P\ V9x)\ V10y)))))) \Rightarrow (p \\
& (ap\ V0Q\ V8z)))) \Leftrightarrow (\forall V11x \in A.27c.(\forall V12y \in A.27d.((p \\
& (ap\ (ap\ V6P\ V11x)\ V12y)) \Rightarrow (p\ (ap\ V0Q\ (ap\ (ap\ V7f\ V11x)\ V12y)))))) \wedge \\
& (\forall V13P \in (((2^{A.27g})^{A.27f})^{A.27e}).(\forall V14f \in (((A.27b^{A.27g})^{A.27f})^{A.27e}). \\
& (\forall V15z \in A.27b.((p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27b)\ V15z)\ ( \\
& ap\ (c.2Epred\_set.2EGSPEC\ A.27b\ (ty.2Epair.2Eprod\ A.27e\ (ty.2Epair.2Eprod \\
& A.27f\ A.27g)))\ (ap\ (c.2Epair.2EUNCURRY\ A.27e\ (ty.2Epair.2Eprod \\
& A.27f\ A.27g)\ (ty.2Epair.2Eprod\ A.27b\ 2))\ (\lambda V16w \in A.27e.(ap \\
& (c.2Epair.2EUNCURRY\ A.27f\ A.27g\ (ty.2Epair.2Eprod\ A.27b\ 2)) \\
& (\lambda V17x \in A.27f.(\lambda V18y \in A.27g.(ap\ (ap\ (c.2Epair.2E.2C\ A.27b \\
& 2)\ (ap\ (ap\ (ap\ V14f\ V16w)\ V17x)\ V18y))\ (ap\ (ap\ (ap\ V13P\ V16w)\ V17x) \\
& V18y)))))) \Rightarrow (p\ (ap\ V0Q\ V15z)))) \Leftrightarrow (\forall V19w \in A.27e.(\forall V20x \in \\
& A.27f.(\forall V21y \in A.27g.((p\ (ap\ (ap\ (ap\ V13P\ V19w)\ V20x)\ V21y)) \Rightarrow \\
& (p\ (ap\ V0Q\ (ap\ (ap\ (ap\ V14f\ V19w)\ V20x)\ V21y)))))))))
\end{aligned} \tag{25}$$

Assume the following.

$$\begin{aligned}
& (\forall V0s \in (2^{ty.2Erealax.2Ereal}).(\forall V1t \in (2^{ty.2Erealax.2Ereal}). \\
& (\forall V2x \in ty.2Erealax.2Ereal.(((p\ (ap\ (ap\ (c.2Ebool.2EIN \\
& ty.2Erealax.2Ereal)\ V2x)\ V1t)) \wedge ((p\ (ap\ c.2Ereal\_topology.2Econnected \\
& V1t)) \wedge (p\ (ap\ (ap\ (c.2Epred\_set.2ESUBSET\ ty.2Erealax.2Ereal) \\
& V1t)\ V0s)))) \Rightarrow (p\ (ap\ (ap\ (c.2Epred\_set.2ESUBSET\ ty.2Erealax.2Ereal) \\
& V1t)\ (ap\ (ap\ c.2Ereal\_topology.2Econnected\_component\ V0s) \\
& V2x))))))
\end{aligned} \tag{26}$$

Assume the following.

$$\begin{aligned}
& (\forall V0s \in (2^{ty\_2Erealax\_2Ereal}).(\forall V1x \in ty\_2Erealax\_2Ereal. \\
& (\forall V2y \in ty\_2Erealax\_2Ereal.((p (ap (ap (c\_2Ebool\_2EIN ty\_2Erealax\_2Ereal) \\
& V2y) (ap (ap c\_2Ereal\_topology\_2Econnected\_component V0s) \\
& V1x))) \Rightarrow ((ap (ap c\_2Ereal\_topology\_2Econnected\_component \\
& V0s) V2y) = (ap (ap c\_2Ereal\_topology\_2Econnected\_component \\
& V0s) V1x))))))
\end{aligned} \tag{27}$$

**Theorem 1**

$$\begin{aligned}
& (\forall V0s \in (2^{ty\_2Erealax\_2Ereal}).(\forall V1t \in (2^{ty\_2Erealax\_2Ereal}). \\
& (\forall V2c \in (2^{ty\_2Erealax\_2Ereal}).(((p (ap (ap (c\_2Ebool\_2EIN \\
& (2^{ty\_2Erealax\_2Ereal})) V2c) (ap c\_2Ereal\_topology\_2Ecomponents \\
& V0s))) \wedge ((p (ap c\_2Ereal\_topology\_2Econnected V1t)) \wedge ((p (ap \\
& (ap (c\_2Epred\_set\_2ESUBSET ty\_2Erealax\_2Ereal) V1t) V0s)) \wedge \\
& (\neg((ap (ap (c\_2Epred\_set\_2EINTER ty\_2Erealax\_2Ereal) V2c) V1t) = \\
& (c\_2Epred\_set\_2EEMPTY ty\_2Erealax\_2Ereal)))))) \Rightarrow (p (ap (ap \\
& (c\_2Epred\_set\_2ESUBSET ty\_2Erealax\_2Ereal) V1t) V2c))))))
\end{aligned}$$