

# thm\_2Ereal\_\_topology\_2ECONTINUOUS\_\_AT\_\_ID (TMPPe3Yhetju3YjjNn3mbqDGSHCTxAyuxAt)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 4** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a})))$

**Definition 5** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t)))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \quad (1)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EABS\_prod A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (2)$$

**Definition 6** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2Epair\_2EABS\_prod$

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty ty\_2Erealax\_2Ereal \quad (3)$$

Let  $c\_2Ereal\_topology\_2EDist : \iota$  be given. Assume the following.

$$c\_2Ereal\_topology\_2EDist \in (ty\_2Erealax\_2Ereal^{(ty\_2Epair\_2Eprod ty\_2Erealax\_2Ereal ty\_2Erealax\_2Ereal)}) \quad (4)$$

**Definition 7** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.$ **if**  $(\exists x \in A.p (ap P x))$  **then**  $(the (\lambda x.x \in A \wedge p$   
of type  $\iota \Rightarrow \iota$ .

**Definition 8** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap V0P (ap (c\_2Emin\_2E\_40 A$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \tag{5}$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{6}$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \tag{7}$$

**Definition 9** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}) \tag{8}$$

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \tag{9}$$

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{ty\_2Erealax}) \tag{10}$$

**Definition 10** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_2Emin\_2E\_40 (t$

Let  $c\_2Erealax\_2Ereal\_lt : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal)}) \tag{11}$$

**Definition 11** We define  $c\_2Erealax\_2Ereal\_lt$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$

**Definition 12** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21\ 2) (\lambda V0t \in 2.V0t))$ .

**Definition 13** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap\ c\_2Emin\_2E\_3D\_3D\_3E\ V0t)\ c\_2Ebool\_2E\_7E))$

**Definition 14** We define  $c\_2Ereal\_2Ereal\_lte$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal$

Let  $ty\_2Ereal\_topology\_2Enet : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Ereal\_topology\_2Enet\ A0) \tag{12}$$

Let  $c\_2Ereal\_topology\_2Emk\_net : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ereal\_topology\_2Emk\_net\ A\_27a \in ((ty\_2Ereal\_topology\_2Enet\ A\_27a)^{(2^{A\_27a})^{A\_27a}}) \tag{13}$$

**Definition 15** We define  $c\_2Ereal\_topology\_2Eat$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_2Ereal\_topology\_2Eat) a)$ .  
Let  $c\_2Ereal\_topology\_2Eenetord : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Ereal\_topology\_2Eenetord A\_27a \in ((2^{A\_27a})^{A\_27a})^{(ty\_2Ereal\_topology\_2Eenet A\_27a)} \quad (14)$$

**Definition 16** We define  $c\_2Ereal\_topology\_2Eenetlimit$  to be  $\lambda A\_27a : \iota.\lambda V0net \in (ty\_2Ereal\_topology\_2Eat A\_27a) \Rightarrow net$

**Definition 17** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21) 2) (\lambda V2t \in 2.(ap (c\_2Ebool\_2E\_21) 2) V2t) V1t2) \Rightarrow (p V0t1)))$

**Definition 18** We define  $c\_2Ereal\_topology\_2Etrivial\_limit$  to be  $\lambda A\_27a : \iota.\lambda V0net \in (ty\_2Ereal\_topology\_2Eat A\_27a) \Rightarrow net$

**Definition 19** We define  $c\_2Ereal\_topology\_2Eeventually$  to be  $\lambda A\_27a : \iota.\lambda V0p \in (2^{A\_27a}).\lambda V1net \in (ty\_2Ereal\_topology\_2Eat A\_27a) \Rightarrow net$

**Definition 20** We define  $c\_2Ereal\_topology\_2E\_2D\_2D\_3E$  to be  $\lambda A\_27a : \iota.\lambda V0f \in (ty\_2Erealax\_2Ereal^{A\_27a}) \Rightarrow f$

**Definition 21** We define  $c\_2Ereal\_topology\_2Econtinuous$  to be  $\lambda A\_27a : \iota.\lambda V0f \in (ty\_2Erealax\_2Ereal^{A\_27a}) \Rightarrow f$

Assume the following.

$$True \quad (15)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (16)$$

Assume the following.

$$(\forall V0t \in 2.(((p V0t) \Rightarrow False) \Rightarrow (\neg(p V0t)))) \quad (17)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(p V0t)) \Rightarrow ((p V0t) \Rightarrow False))) \quad (18)$$

Assume the following.

$$(\forall V0t \in 2.(((True) \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))) \quad (19)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \quad (20)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (21)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow True) \Leftrightarrow (p \ V0t)) \wedge (((False \Leftrightarrow (p \ V0t)) \Leftrightarrow \neg(p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow False) \Leftrightarrow \neg(p \ V0t)))))) \quad (22)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in (2^{A-27a}).(\neg(\forall V1x \in A.27a.(p \ (ap \ V0P \ V1x)))) \Leftrightarrow (\exists V2x \in A.27a.(\neg(p \ (ap \ V0P \ V2x)))))) \quad (23)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in (2^{A-27a}).(\neg(\exists V1x \in A.27a.(p \ (ap \ V0P \ V1x)))) \Leftrightarrow (\forall V2x \in A.27a.(\neg(p \ (ap \ V0P \ V2x)))))) \quad (24)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in (2^{A-27a}).((p \ V0P) \vee (\exists V2x \in A.27a.(p \ (ap \ V1Q \ V2x)))) \Leftrightarrow (\exists V3x \in A.27a.((p \ V0P) \vee (p \ (ap \ V1Q \ V3x)))))) \quad (25)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in (2^{A-27a}).((\exists V2x \in A.27a.((p \ V0P) \wedge (p \ (ap \ V1Q \ V2x)))) \Leftrightarrow ((p \ V0P) \wedge (\exists V3x \in A.27a.(p \ (ap \ V1Q \ V3x)))))) \quad (26)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((p \ V0A) \vee (p \ V1B)) \Leftrightarrow ((p \ V1B) \vee (p \ V0A)))) \quad (27)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p \ V0A) \wedge (p \ V1B))) \Leftrightarrow ((\neg(p \ V0A)) \vee (\neg(p \ V1B)))) \wedge (((\neg(p \ V0A)) \vee (p \ V1B)) \Leftrightarrow ((\neg(p \ V0A)) \wedge (\neg(p \ V1B)))))) \quad (28)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow \forall A.27b.nonempty \ A.27b \Rightarrow (\forall V0P \in ((2^{A-27b})^{A-27a}).((\forall V1x \in A.27a.(\exists V2y \in A.27b.(p \ (ap \ (ap \ V0P \ V1x) \ V2y)))) \Leftrightarrow (\exists V3f \in (A.27b^{A-27a}).(\forall V4x \in A.27a.(p \ (ap \ (ap \ V0P \ V4x) \ (ap \ V3f \ V4x)))))) \quad (29)$$

Assume the following.

$$\begin{aligned}
& (\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Erealax\_2Ereal}).(\forall V1x \in \\
& ty\_2Erealax\_2Ereal.((p (ap (ap (c\_2Ereal\_topology\_2Econtinuous \\
& ty\_2Erealax\_2Ereal) V0f) (ap c\_2Ereal\_topology\_2Eat V1x))) \Leftrightarrow \\
& (\forall V2e \in ty\_2Erealax\_2Ereal.((p (ap (ap c\_2Erealax\_2Ereal\_lt \\
& (ap c\_2Ereal\_2Ereal\_of\_num c\_2Enum\_2E0)) V2e)) \Rightarrow (\exists V3d \in \\
& ty\_2Erealax\_2Ereal.((p (ap (ap c\_2Erealax\_2Ereal\_lt (ap c\_2Ereal\_2Ereal\_of\_num \\
& c\_2Enum\_2E0)) V3d)) \wedge (\forall V4x\_27 \in ty\_2Erealax\_2Ereal.(( \\
& p (ap (ap c\_2Erealax\_2Ereal\_lt (ap c\_2Ereal\_topology\_2EDist \\
& (ap (ap (c\_2Epair\_2E\_2C ty\_2Erealax\_2Ereal ty\_2Erealax\_2Ereal) \\
V4x\_27) V1x))) V3d)) \Rightarrow (p (ap (ap c\_2Erealax\_2Ereal\_lt (ap c\_2Ereal\_topology\_2EDist \\
& (ap (ap (c\_2Epair\_2E\_2C ty\_2Erealax\_2Ereal ty\_2Erealax\_2Ereal) \\
& (ap V0f V4x\_27)) (ap V0f V1x)))) V2e))))))))) \\
& \hspace{15em} (30)
\end{aligned}$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \quad (31)$$

Assume the following.

$$(\forall V0A \in 2.((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \quad (32)$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\
& (((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \\
& \hspace{15em} (33)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\
& ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \\
& \hspace{15em} (34)
\end{aligned}$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (35)$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\
& (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg \\
& p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\
& ((\neg(p V1q)) \vee (\neg(p V0p)))))))))) \\
& \hspace{15em} (36)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\
& (p V1q) \wedge (p V2r))) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \vee (\neg(p V2r)))) \wedge (((p V1q) \vee \\
& (\neg(p V0p))) \wedge ((p V2r) \vee (\neg(p V0p)))))))) \\
& \hspace{15em} (37)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \vee (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee \neg(p \ V1q)) \wedge ((p \ V0p) \vee \neg(p \ V2r))) \wedge \\
& ((p \ V1q) \vee ((p \ V2r) \vee \neg(p \ V0p))))))))))
\end{aligned} \tag{38}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \Rightarrow (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee (p \ V1q)) \wedge ((p \ V0p) \vee \neg(p \ V2r))) \wedge ( \\
& \neg(p \ V1q) \vee ((p \ V2r) \vee \neg(p \ V0p))))))))))
\end{aligned} \tag{39}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (((p \ V0p) \Leftrightarrow \neg(p \ V1q))) \Leftrightarrow (((p \ V0p) \vee \\
& (p \ V1q)) \wedge (\neg(p \ V1q) \vee \neg(p \ V0p))))))
\end{aligned} \tag{40}$$

**Theorem 1**

$$\begin{aligned}
& (\forall V0a \in ty\_2Erealax\_2Ereal. (p \ (ap \ (ap \ (c\_2Ereal\_topology\_2Econtinuous \\
& ty\_2Erealax\_2Ereal) \ (\lambda V1x \in ty\_2Erealax\_2Ereal. V1x)) \ (ap \\
& c\_2Ereal\_topology\_2Eat \ V0a))))
\end{aligned}$$