

# thm\_2Ereal\_\_topology\_2ECONTINUOUS\_\_AT\_\_SEQUENTIALLY (TMYnmi4Sgzo9U7o2SVirVjf1AvSFBRJeY6u)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_2E21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 4** We define  $c\_2Ebool\_2E\_2EF$  to be  $(ap (c\_2Ebool\_2E\_2E21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_2E7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_2EF$

**Definition 7** We define  $c\_2Epred\_set\_2EUNIV$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2E\_2ET)$ .

Let  $ty\_2Ereal\_topology\_2E\_2Enet : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow nonempty (ty\_2Ereal\_topology\_2E\_2Enet A0) \quad (1)$$

**Definition 8** We define  $c\_2Ecombin\_2E\_2Eo$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.\lambda V0f \in (A\_27b^{A\_27c}).\lambda V1g$

Let  $ty\_2Eenum\_2E\_2Enum : \iota$  be given. Assume the following.

$$nonempty ty\_2Eenum\_2E\_2Enum \quad (2)$$

**Definition 9** We define  $c\_2Ebool\_2E\_2E2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_2E21 2) (\lambda V2t \in 2$

Let  $c\_2Enum\_2E\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2E\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (3)$$

Let  $c\_2Enum\_2E\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2E\_2ESUC\_REP \in (\omega^{\omega}) \quad (4)$$

Let  $c\_2Enum\_2E\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2E\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (5)$$

**Definition 10** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_num$

**Definition 11** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p\ (ap\ P\ x)) \text{ then } (the\ (\lambda x.x \in A)\lambda P$   
of type  $\iota \Rightarrow \iota$ .

**Definition 12** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_2Emin\_2E\_40$

**Definition 13** We define  $c\_2Eprim\_rec\_2E\_3C$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

**Definition 14** We define  $c\_2Earithmetic\_2E\_3E$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

**Definition 15** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in$

**Definition 16** We define  $c\_2Earithmetic\_2E\_3E\_3D$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

Let  $c\_2Ereal\_topology\_2Emk\_net : \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ereal\_topology\_2Emk\_net \\ & A\_27a \in ((ty\_2Ereal\_topology\_2Enet\ A\_27a)^{(2^{A\_27a})^{A\_27a}}) \end{aligned} \quad (6)$$

**Definition 17** We define  $c\_2Ereal\_topology\_2Esequentially$  to be  $(ap\ (c\_2Ereal\_topology\_2Emk\_net\ ty\_2Ereal$

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erealax\_2Ereal \quad (7)$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod \\ & A0\ A1) \end{aligned} \quad (8)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod \\ & A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \end{aligned} \quad (9)$$

**Definition 18** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2$

Let  $c\_2Ereal\_topology\_2EDist : \iota$  be given. Assume the following.

$$c\_2Ereal\_topology\_2EDist \in (ty\_2Erealax\_2Ereal^{(ty\_2Epair\_2Eprod\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal)}) \quad (10)$$

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \quad (11)$$

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{ty\_2Erealax\_2Ereal}) \quad (12)$$

**Definition 19** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_2Emin\_2E40 (t$   
Let  $c\_2Erealax\_2Etreall\_lt : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etreall\_lt \in ((2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)})(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal)) \quad (13)$$

**Definition 20** We define  $c\_2Erealax\_2Ereal\_lt$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$   
Let  $c\_2Ereal\_topology\_2Eenetord : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Ereal\_topology\_2Eenetord A\_27a \in ((2^{A\_27a})^{A\_27a})(ty\_2Ereal\_topology\_2Eenet A\_27a) \quad (14)$$

**Definition 21** We define  $c\_2Ereal\_topology\_2Etrivial\_limit$  to be  $\lambda A\_27a : \iota.\lambda V0net \in (ty\_2Ereal\_topology\_2Eenet A\_27a)$

**Definition 22** We define  $c\_2Ereal\_topology\_2Eeventually$  to be  $\lambda A\_27a : \iota.\lambda V0p \in (2^{A\_27a}).\lambda V1net \in (ty\_2Ereal\_topology\_2Eenet A\_27a)$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in omega \quad (15)$$

**Definition 23** We define  $c\_2Enum\_2E0$  to be  $(ap c\_2Enum\_2EABS\_num c\_2Enum\_2EZERO\_REP)$ .

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}) \quad (16)$$

**Definition 24** We define  $c\_2Ereal\_topology\_2E2D\_2D\_3E$  to be  $\lambda A\_27a : \iota.\lambda V0f \in (ty\_2Erealax\_2Ereal^A A\_27a)$

**Definition 25** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap V1f V0x)))$

**Definition 26** We define  $c\_2Ereal\_2Ereal\_lte$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal$

**Definition 27** We define  $c\_2Ereal\_topology\_2Eat$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_2Ereal\_topology\_2Eenet A\_27a))$

**Definition 28** We define  $c\_2Ereal\_topology\_2Ewithin$  to be  $\lambda A\_27a : \iota.\lambda V0net \in (ty\_2Ereal\_topology\_2Eenet A\_27a)$

**Definition 29** We define  $c\_2Ereal\_topology\_2Eenlimit$  to be  $\lambda A\_27a : \iota.\lambda V0net \in (ty\_2Ereal\_topology\_2Eenet A\_27a)$

**Definition 30** We define  $c\_2Ereal\_topology\_2Econtinuous$  to be  $\lambda A\_27a : \iota.\lambda V0f \in (ty\_2Erealax\_2Ereal^A A\_27a)$

Assume the following.

$$True \quad (17)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p V0t)) \Leftrightarrow (p V0t))) \quad (18)$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \wedge (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \wedge True) \Leftrightarrow \\
& (p \ V0t)) \wedge (((False \wedge (p \ V0t)) \Leftrightarrow False) \wedge (((p \ V0t) \wedge False) \Leftrightarrow False) \wedge \\
& (((p \ V0t) \wedge (p \ V0t)) \Leftrightarrow (p \ V0t)))))) \quad (19)
\end{aligned}$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (20)$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (21)$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Leftrightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow True) \Leftrightarrow \\
& (p \ V0t)) \wedge (((False \Leftrightarrow (p \ V0t)) \Leftrightarrow \neg(p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow False) \Leftrightarrow \neg( \\
& p \ V0t)))))) \quad (22)
\end{aligned}$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a.(p \ (ap \ (ap \ (c\_2Ebool\_2EIN \ A\_27a) \ V0x) \ (c\_2Epred\_set\_2EUNIV \ A\_27a)))) \quad (23)$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Erealx\_2Ereal.((ap \ (ap \ (c\_2Ereal\_topology\_2Ewithin \\
& ty\_2Erealx\_2Ereal) \ (ap \ c\_2Ereal\_topology\_2Eat \ V0x)) \ (c\_2Epred\_set\_2EUNIV \\
& ty\_2Erealx\_2Ereal)) = (ap \ c\_2Ereal\_topology\_2Eat \ V0x))) \quad (24)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0f \in (ty\_2Erealx\_2Ereal^{ty\_2Erealx\_2Ereal}).(\forall V1s \in \\
& (2^{ty\_2Erealx\_2Ereal}).(\forall V2a \in ty\_2Erealx\_2Ereal.( \\
& (p \ (ap \ (ap \ (c\_2Ereal\_topology\_2Econtinuous \ ty\_2Erealx\_2Ereal) \\
& V0f) \ (ap \ (ap \ (c\_2Ereal\_topology\_2Ewithin \ ty\_2Erealx\_2Ereal) \\
& (ap \ c\_2Ereal\_topology\_2Eat \ V2a)) \ V1s))) \Leftrightarrow (\forall V3x \in (ty\_2Erealx\_2Ereal^{ty\_2Enum\_2Enum}). \\
& (((\forall V4n \in ty\_2Enum\_2Enum.(p \ (ap \ (ap \ (c\_2Ebool\_2EIN \ ty\_2Erealx\_2Ereal) \\
& (ap \ V3x \ V4n)) \ V1s))) \wedge (p \ (ap \ (ap \ (ap \ (c\_2Ereal\_topology\_2E\_2D\_2D\_3E \\
& ty\_2Enum\_2Enum) \ V3x) \ V2a) \ c\_2Ereal\_topology\_2Esequentially))) \Rightarrow \\
& (p \ (ap \ (ap \ (ap \ (c\_2Ereal\_topology\_2E\_2D\_2D\_3E \ ty\_2Enum\_2Enum) \\
& (ap \ (ap \ (c\_2Ecombin\_2Eo \ ty\_2Enum\_2Enum \ ty\_2Erealx\_2Ereal \ ty\_2Erealx\_2Ereal) \\
& V0f) \ V3x)) \ (ap \ V0f \ V2a)) \ c\_2Ereal\_topology\_2Esequentially)))))) \quad (25)
\end{aligned}$$

**Theorem 1**

$$\begin{aligned} & (\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Erealax\_2Ereal}).(\forall V1a \in \\ & ty\_2Erealax\_2Ereal.((p (ap (ap (c\_2Ereal\_topology\_2Econtinuous \\ & ty\_2Erealax\_2Ereal) V0f) (ap c\_2Ereal\_topology\_2Eat V1a)))) \Leftrightarrow \\ & (\forall V2x \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}).((p (ap (ap \\ & (ap (c\_2Ereal\_topology\_2E\_2D\_2D\_3E ty\_2Enum\_2Enum) V2x) V1a) \\ & c\_2Ereal\_topology\_2Esequentially)) \Rightarrow (p (ap (ap (ap (c\_2Ereal\_topology\_2E\_2D\_2D\_3E \\ & ty\_2Enum\_2Enum) (ap (ap (c\_2Ecombin\_2Eo ty\_2Enum\_2Enum ty\_2Erealax\_2Ereal \\ & ty\_2Erealax\_2Ereal) V0f) V2x)) (ap V0f V1a)) c\_2Ereal\_topology\_2Esequentially)))))) \end{aligned}$$