

# thm\_2Ereal\_\_topology\_2EFrontier\_\_Halfspace\_\_GT (TMFpegSzseqb5LfXQgvoPhxM4kRsyZDRDdA)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a})))$

**Definition 4** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \tag{1}$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{2}$$

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erealax\_2Ereal \tag{3}$$

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})\ ty\_2Erealax\_2Ereal) \tag{4}$$

**Definition 5** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge p))$  of type  $\iota \Rightarrow \iota$ .

**Definition 6** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_2Emin\_2E\_40 (ty\_2Erealax\_2Ereal\_REP\_CLASS a)))$

Let  $c\_2Erealax\_2Etrealt\_lt : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealt\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})\ (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)) \tag{5}$$

**Definition 7** We define  $c\_2Erealax\_2Ereal\_lt$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$

**Definition 8** We define  $c\_2Ereal\_2Ereal\_gt$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal$

Let  $c\_2Erealax\_2Etrealm\_neg : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealm\_neg \in ((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)) \quad (6)$$

Let  $c\_2Erealax\_2Etrealm\_eq : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealm\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})\ (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal)) \quad (7)$$

Let  $c\_2Erealax\_2Ereal\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_ABS\_CLASS \in (ty\_2Erealax\_2Ereal)^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})} \quad (8)$$

**Definition 9** We define  $c\_2Erealax\_2Ereal\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)$

**Definition 10** We define  $c\_2Erealax\_2Ereal\_neg$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.(ap\ c\_2Erealax\_2Ereal)$

Let  $c\_2Erealax\_2Etrealm\_mul : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealm\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal))\ (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)) \quad (9)$$

**Definition 11** We define  $c\_2Erealax\_2Ereal\_mul$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$

**Definition 12** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o\ (p\ P \Rightarrow p\ Q)$  of type  $\iota$ .

**Definition 13** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2)))$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (10)$$

**Definition 14** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b))$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}}) \quad (11)$$

**Definition 15** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap\ V1f\ V0x)))$

**Definition 16** We define  $c\_2Epred\_set\_2ESUBSET$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A-27a}).\lambda V1t \in (2^{A-27a}).(ap$

Let  $c\_2Ereal\_topology\_2EDist : \iota$  be given. Assume the following.

$$c\_2Ereal\_topology\_2EDist \in (ty\_2Erealax\_2Ereal^{(ty\_2Epair\_2Eprod\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal)}) \quad (12)$$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (13)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (14)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (15)$$

**Definition 17** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}) \quad (16)$$

**Definition 18** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap\ V0P\ (ap\ (c\_2Emin\_2E\_40$

**Definition 19** We define  $c\_2Ereal\_topology\_2EOpen$  to be  $\lambda V0s \in (2^{ty\_2Erealax\_2Ereal}).(ap\ (c\_2Ebool\_2E\_2$

**Definition 20** We define  $c\_2Ereal\_topology\_2Einterior$  to be  $\lambda V0s \in (2^{ty\_2Erealax\_2Ereal}).(ap\ (c\_2Epred\_set$

**Definition 21** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap\ (ap\ c\_2Emin\_2E\_3D\_3D\_3E\ V0t)\ c\_2Ebool\_2E$

**Definition 22** We define  $c\_2Ereal\_topology\_2Elimit\_point\_of$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1s \in$

**Definition 23** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in$

**Definition 24** We define  $c\_2Epred\_set\_2EUNION$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A-27a}).\lambda V1t \in (2^{A-27a}).(ap\ (c$

**Definition 25** We define  $c\_2Ereal\_topology\_2Eclosure$  to be  $\lambda V0s \in (2^{ty\_2Erealax\_2Ereal}).(ap\ (ap\ (c\_2Epred$

**Definition 26** We define  $c\_2Epred\_set\_2EDIFF$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A-27a}).\lambda V1t \in (2^{A-27a}).(ap\ (c\_2$

**Definition 27** We define  $c\_2Ereal\_topology\_2Efrontier$  to be  $\lambda V0s \in (2^{ty\_2Erealax\_2Ereal}).(ap\ (ap\ (c\_2Epred$

Assume the following.

$$True \quad (17)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (18)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p V0t))) \quad (19)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (( \\ & (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ & ((\neg False) \Leftrightarrow True))) \end{aligned} \quad (21)$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a. ((V0x = V0x) \Leftrightarrow True)) \quad (22)$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (23)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p V0t1) \Rightarrow \\ & (p V1t2) \Rightarrow (p V2t3)) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in 2. (\forall V1x\_27 \in 2. (\forall V2y \in 2. (\forall V3y\_27 \in \\ & 2. (((p V0x) \Leftrightarrow (p V1x\_27)) \wedge ((p V1x\_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y\_27)))) \Rightarrow \\ & (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x\_27) \Rightarrow (p V3y\_27)))))) \end{aligned} \quad (25)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in ty\_2Erealax\_2Ereal. (((ap \ c\_2Erealax\_2Ereal\_neg \\ & V0x) = (ap \ c\_2Ereal\_2Ereal\_of\_num \ c\_2Enum\_2E0)) \Leftrightarrow (V0x = (ap \ c\_2Ereal\_2Ereal\_of\_num \\ & \ c\_2Enum\_2E0)))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. \\ & ((p \ (ap \ (ap \ c\_2Erealax\_2Ereal\_lt \ (ap \ c\_2Erealax\_2Ereal\_neg \\ & V0x)) \ (ap \ c\_2Erealax\_2Ereal\_neg \ V1y))) \Leftrightarrow (p \ (ap \ (ap \ c\_2Erealax\_2Ereal\_lt \\ & \ V1y) \ V0x)))) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. \\
& (((ap\ c\_2Erealax\_2Ereal\_neg\ V0x) = (ap\ c\_2Erealax\_2Ereal\_neg\ V1y)) \Leftrightarrow (V0x = V1y))))
\end{aligned} \tag{28}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. \\
& ((ap\ (ap\ c\_2Erealax\_2Ereal\_mul\ (ap\ c\_2Erealax\_2Ereal\_neg\ V0x)) \\
& V1y) = (ap\ c\_2Erealax\_2Ereal\_neg\ (ap\ (ap\ c\_2Erealax\_2Ereal\_mul \\
& V0x\ V1y))))))
\end{aligned} \tag{29}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty\_2Erealax\_2Ereal. (\forall V1b \in ty\_2Erealax\_2Ereal. \\
& ((\neg((V0a = (ap\ c\_2Ereal\_2Ereal\_of\_num\ c\_2Enum\_2E0)) \wedge (V1b = \\
& (ap\ c\_2Ereal\_2Ereal\_of\_num\ c\_2Enum\_2E0)))) \Rightarrow ((ap\ c\_2Ereal\_topology\_2Efrontier \\
& (ap\ (c\_2Epred\_set\_2EGSPEC\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal) \\
& (\lambda V2x \in ty\_2Erealax\_2Ereal. (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Erealax\_2Ereal \\
& 2)\ V2x) (ap\ (ap\ c\_2Erealax\_2Ereal\_lt\ (ap\ (ap\ c\_2Erealax\_2Ereal\_mul \\
& V0a)\ V2x))\ V1b)))))) = (ap\ (c\_2Epred\_set\_2EGSPEC\ ty\_2Erealax\_2Ereal \\
& ty\_2Erealax\_2Ereal) (\lambda V3x \in ty\_2Erealax\_2Ereal. (ap\ (ap\ (c\_2Epair\_2E\_2C \\
& ty\_2Erealax\_2Ereal\ 2)\ V3x) (ap\ (ap\ (c\_2Emin\_2E\_3D\ ty\_2Erealax\_2Ereal) \\
& (ap\ (ap\ c\_2Erealax\_2Ereal\_mul\ V0a)\ V3x))\ V1b)))))))))
\end{aligned} \tag{30}$$

**Theorem 1**

$$\begin{aligned}
& (\forall V0a \in ty\_2Erealax\_2Ereal. (\forall V1b \in ty\_2Erealax\_2Ereal. \\
& ((\neg((V0a = (ap\ c\_2Ereal\_2Ereal\_of\_num\ c\_2Enum\_2E0)) \wedge (V1b = \\
& (ap\ c\_2Ereal\_2Ereal\_of\_num\ c\_2Enum\_2E0)))) \Rightarrow ((ap\ c\_2Ereal\_topology\_2Efrontier \\
& (ap\ (c\_2Epred\_set\_2EGSPEC\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal) \\
& (\lambda V2x \in ty\_2Erealax\_2Ereal. (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Erealax\_2Ereal \\
& 2)\ V2x) (ap\ (ap\ c\_2Ereal\_2Ereal\_gt\ (ap\ (ap\ c\_2Erealax\_2Ereal\_mul \\
& V0a)\ V2x))\ V1b)))))) = (ap\ (c\_2Epred\_set\_2EGSPEC\ ty\_2Erealax\_2Ereal \\
& ty\_2Erealax\_2Ereal) (\lambda V3x \in ty\_2Erealax\_2Ereal. (ap\ (ap\ (c\_2Epair\_2E\_2C \\
& ty\_2Erealax\_2Ereal\ 2)\ V3x) (ap\ (ap\ (c\_2Emin\_2E\_3D\ ty\_2Erealax\_2Ereal) \\
& (ap\ (ap\ c\_2Erealax\_2Ereal\_mul\ V0a)\ V3x))\ V1b)))))))))
\end{aligned}$$