

thm\_2Ereal\_\_topology\_2EIMAGE\_\_TWIZZLE\_\_INTERVAL  
(TMKdC-  
nVM8BYLppXQx8QaVzHsFKw1WVeDKJV)

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**Definition 1** We define `c_2Emin_2E_3D_3D_3E` to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 2** We define `c_2Emin_2E_3D` to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define `c_2Ebool_2E_2T` to be  $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 4** We define `c_2Ebool_2E_21` to be  $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

**Definition 5** We define `c_2Ebool_2E_2F` to be  $(ap (c_2Ebool_2E_21 2)) (\lambda V0t \in 2.V0t)$ .

**Definition 6** We define `c_2Ebool_2E_27E` to be  $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_2F$

**Definition 7** We define `c_2Emin_2E_40` to be  $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x))$  **then** (the  $(\lambda x.x \in A \wedge p$  of type  $\iota \Rightarrow \iota$ ).

**Definition 8** We define `c_2Ebool_2E_3F` to be  $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap V0P (ap (c_2Emin_2E_40 A_27a$

**Definition 9** We define `c_2Ebool_2E_2IN` to be  $\lambda A_27a : \iota.(\lambda V0x \in A_27a.(\lambda V1f \in (2^{A_27a}).(ap V1f V0x)))$

**Definition 10** We define `c_2Ebool_2E_2F_5C` to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2)) (\lambda V2t \in 2.V2t))$

Let `ty_2Epair_2Eprod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Epair_2Eprod A0 A1) \tag{1}$$

Let `c_2Epair_2EABS__prod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epair_2EABS__prod A_27a A_27b \in ((ty_2Epair_2Eprod A_27a A_27b)^{(2^{A_27b})^{A_27a}}) \tag{2}$$

**Definition 11** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota)$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC A\_27a A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod A\_27a 2)^{A\_27b}})$$
(3)

**Definition 12** We define  $c\_2Epred\_set\_2EIMAGE$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (A\_27b^{A\_27a}).\lambda V1s \in$  Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty ty\_2Ehreal\_2Ehreal$$
(4)

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty ty\_2Erealax\_2Ereal$$
(5)

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)})^{ty\_2Erealax\_2Ereal})$$
(6)

**Definition 13** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_2Emin\_2E\_40 (t$  Let  $c\_2Erealax\_2Ereal\_lt : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_lt \in ((2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal)})$$
(7)

**Definition 14** We define  $c\_2Erealax\_2Ereal\_lt$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$

**Definition 15** We define  $c\_2Ereal\_2Ereal\_lte$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal$

Let  $ty\_2Elist\_2Elist : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow nonempty (ty\_2Elist\_2Elist A0)$$
(8)

Let  $c\_2Elist\_2ENIL : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Elist\_2ENIL A\_27a \in (ty\_2Elist\_2Elist A\_27a)$$
(9)

Let  $c\_2Elist\_2ECONS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Elist\_2ECONS A\_27a \in (((ty\_2Elist\_2Elist A\_27a)^{(ty\_2Elist\_2Elist A\_27a)})^{A\_27a})$$
(10)

Let  $c\_2Elist\_2EHD : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Elist\_2EHD A\_27a \in (A\_27a^{(ty\_2Elist\_2Elist A\_27a)})$$
(11)

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND \\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \end{aligned} \quad (12)$$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST \\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \end{aligned} \quad (13)$$

**Definition 16** We define  $c\_2Ereal\_topology\_2ECLOSED\_interval$  to be  $\lambda V0l \in (ty\_2Elist\_2Elist\ (ty\_2Epa$

Let  $c\_2Ereal\_topology\_2EOPEN\_interval : \iota$  be given. Assume the following.

$$c\_2Ereal\_topology\_2EOPEN\_interval \in ((2^{ty\_2Erealax\_2Ereal})^{(ty\_2Epair\_2Eprod\ ty\_2Erealax\_2Ereal\ ty\_2Epa)} \quad (14)$$

Assume the following.

$$True \quad (15)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (16)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in A\_27a. (p\ V0t) \Leftrightarrow (p\ V0t))) \quad (17)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. ((V0x = V0x) \Leftrightarrow True)) \quad (18)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (19)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2. (((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow \\ (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow \neg(p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow \neg( \\ p\ V0t)))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}). (\forall V1a \in \\ A\_27a. ((\exists V2x \in A\_27a. ((V2x = V1a) \wedge (p\ (ap\ V0P\ V2x)))) \Leftrightarrow (p\ ( \\ ap\ V0P\ V1a)))))) \end{aligned} \quad (21)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0x \in A\_27a. (\forall V1y \in A\_27b. (\forall V2a \in A\_27a. (\forall V3b \in \\ & \quad A\_27b. (((ap\ (ap\ (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b)\ V0x)\ V1y) = (ap\ (ap \\ & \quad (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b)\ V2a)\ V3b)) \Leftrightarrow ((V0x = V2a) \wedge (V1y = V3b)))))) \\ & \end{aligned} \tag{22}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in \\ & \quad (2^{A\_27a}). ((V0s = V1t) \Leftrightarrow (\forall V2x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN \\ & \quad A\_27a)\ V2x)\ V0s)) \Leftrightarrow (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ V1t)))))) \\ & \end{aligned} \tag{23}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0f \in ((ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}). (\forall V1v \in \\ & \quad A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V1v)\ (ap\ (c\_2Epred\_set\_2EGSPEC \\ & \quad A\_27a\ A\_27b)\ V0f))) \Leftrightarrow (\exists V2x \in A\_27b. ((ap\ (ap\ (c\_2Epair\_2E\_2C \\ & \quad A\_27a\ 2)\ V1v)\ c\_2Ebool\_2ET) = (ap\ V0f\ V2x)))))) \\ & \end{aligned} \tag{24}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0y \in A\_27b. (\forall V1s \in (2^{A\_27a}). (\forall V2f \in (A\_27b)^{A\_27a}). \\ & \quad ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27b)\ V0y)\ (ap\ (ap\ (c\_2Epred\_set\_2EIMAGE \\ & \quad A\_27a\ A\_27b)\ V2f)\ V1s))) \Leftrightarrow (\exists V3x \in A\_27a. ((V0y = (ap\ V2f\ V3x)) \wedge \\ & \quad (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V3x)\ V1s)))))) \\ & \end{aligned} \tag{25}$$

Assume the following.

$$\begin{aligned} & (\forall V0a \in ty\_2Erealx\_2Ereal. (\forall V1b \in ty\_2Erealx\_2Ereal. \\ & \quad (((ap\ c\_2Ereal\_topology\_2EOPEN\_interval\ (ap\ (ap\ (c\_2Epair\_2E\_2C \\ & \quad ty\_2Erealx\_2Ereal\ ty\_2Erealx\_2Ereal)\ V0a)\ V1b)) = (ap\ (c\_2Epred\_set\_2EGSPEC \\ & \quad ty\_2Erealx\_2Ereal\ ty\_2Erealx\_2Ereal)\ (\lambda V2x \in ty\_2Erealx\_2Ereal. \\ & \quad (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Erealx\_2Ereal\ 2)\ V2x)\ (ap\ (ap\ c\_2Ebool\_2E\_2F\_5C \\ & \quad (ap\ (ap\ c\_2Erealx\_2Ereal\_lt\ V0a)\ V2x))\ (ap\ (ap\ c\_2Erealx\_2Ereal\_lt \\ & \quad V2x)\ V1b)))))) \wedge ((ap\ c\_2Ereal\_topology\_2ECLOSED\_interval \\ & \quad (ap\ (ap\ (c\_2Elist\_2ECONS\ (ty\_2Epair\_2Eprod\ ty\_2Erealx\_2Ereal \\ & \quad ty\_2Erealx\_2Ereal)\ V0a)\ V1b))\ (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Erealx\_2Ereal \\ & \quad ty\_2Erealx\_2Ereal)\ V0a)\ V1b))\ (c\_2Elist\_2ENIL\ (ty\_2Epair\_2Eprod \\ & \quad ty\_2Erealx\_2Ereal\ ty\_2Erealx\_2Ereal)))) = (ap\ (c\_2Epred\_set\_2EGSPEC \\ & \quad ty\_2Erealx\_2Ereal\ ty\_2Erealx\_2Ereal)\ (\lambda V3x \in ty\_2Erealx\_2Ereal. \\ & \quad (ap\ (ap\ (c\_2Epair\_2E\_2C\ ty\_2Erealx\_2Ereal\ 2)\ V3x)\ (ap\ (ap\ c\_2Ebool\_2E\_2F\_5C \\ & \quad (ap\ (ap\ c\_2Ereal\_2Ereal\_lte\ V0a)\ V3x))\ (ap\ (ap\ c\_2Ereal\_2Ereal\_lte \\ & \quad V3x)\ V1b)))))) \\ & \end{aligned} \tag{26}$$

**Theorem 1**

$$\begin{aligned} & \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow (\forall V_0 p \in A_{27a}. (\forall V_1 a \in \\ & \quad \text{ty\_2Erealax\_2Ereal}. (\forall V_2 b \in \text{ty\_2Erealax\_2Ereal}. ((\text{ap } ( \\ & \quad \text{ap } (\text{c\_2Epred\_set\_2EIMAGE } \text{ty\_2Erealax\_2Ereal } \text{ty\_2Erealax\_2Ereal}) \\ & \quad (\lambda V_3 x \in \text{ty\_2Erealax\_2Ereal}. V_3 x)) (\text{ap } \text{c\_2Ereal\_topology\_2ECLOSED\_interval} \\ & \quad (\text{ap } (\text{ap } (\text{c\_2Elist\_2ECONS } (\text{ty\_2Epair\_2Eprod } \text{ty\_2Erealax\_2Ereal} \\ & \quad \text{ty\_2Erealax\_2Ereal})) (\text{ap } (\text{ap } (\text{c\_2Epair\_2E\_2C } \text{ty\_2Erealax\_2Ereal} \\ & \quad \text{ty\_2Erealax\_2Ereal}) V_1 a) V_2 b)) (\text{c\_2Elist\_2ENIL } (\text{ty\_2Epair\_2Eprod} \\ & \quad \text{ty\_2Erealax\_2Ereal } \text{ty\_2Erealax\_2Ereal})))))) = (\text{ap } \text{c\_2Ereal\_topology\_2ECLOSED\_interval} \\ & \quad (\text{ap } (\text{ap } (\text{c\_2Elist\_2ECONS } (\text{ty\_2Epair\_2Eprod } \text{ty\_2Erealax\_2Ereal} \\ & \quad \text{ty\_2Erealax\_2Ereal})) (\text{ap } (\text{ap } (\text{c\_2Epair\_2E\_2C } \text{ty\_2Erealax\_2Ereal} \\ & \quad \text{ty\_2Erealax\_2Ereal}) V_1 a) V_2 b)) (\text{c\_2Elist\_2ENIL } (\text{ty\_2Epair\_2Eprod} \\ & \quad \text{ty\_2Erealax\_2Ereal } \text{ty\_2Erealax\_2Ereal})))))) \end{aligned}$$