

# thm\_2Ereal\_\_topology\_2EINDEPENDENT\_\_BOUND (TMGceYNHtLgrybpryUEch8rs6iF3rYN9Y5K)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_2E21$  to be  $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A-27a}))$

**Definition 4** We define  $c\_2Ebool\_2E\_2EF$  to be  $(ap (c\_2Ebool\_2E\_2E21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_2E7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_2EF$

**Definition 7** We define  $c\_2Emin\_2E\_2E40$  to be  $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x)) \text{ then } (the (\lambda x.x \in A \wedge p$  of type  $\iota \Rightarrow \iota$ .

**Definition 8** We define  $c\_2Ebool\_2E\_2E3F$  to be  $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap V0P (ap (c\_2Emin\_2E\_2E40 A$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

**Definition 9** We define  $c\_2Epred\_set\_2EUNIV$  to be  $\lambda A.\lambda a : \iota.(\lambda V0x \in A.27a.c\_2Ebool\_2E\_2ET)$ .

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty ty\_2Ehreal\_2Ehreal \tag{2}$$

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty ty\_2Erealax\_2Ereal \tag{3}$$

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)}) ty\_2Erealax \tag{4}$$

**Definition 10** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_2Emin\_2E.40 (t$   
Let  $c\_2Erealax\_2Etrealmul : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealmul \in (((ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)))(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal) \quad (5)$$

Let  $c\_2Erealax\_2Etrealeq : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealeq \in ((2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)})(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)) \quad (6)$$

Let  $c\_2Erealax\_2Ereal\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_ABS\_CLASS \in (ty\_2Erealax\_2Ereal)^{(2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)}} \quad (7)$$

**Definition 11** We define  $c\_2Erealax\_2Ereal\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty$

**Definition 12** We define  $c\_2Erealax\_2Ereal\_mul$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax$

Let  $c\_2Erealax\_2Etrealmul : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealmul \in (((ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)))(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal) \quad (8)$$

**Definition 13** We define  $c\_2Erealax\_2Ereal\_add$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (9)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty ty\_2Enum\_2Enum \quad (10)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum)^{\omega} \quad (11)$$

**Definition 14** We define  $c\_2Enum\_2E0$  to be  $(ap c\_2Enum\_2EABS\_num c\_2Enum\_2EZERO\_REP)$ .

**Definition 15** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap V1f V0x)))$

**Definition 16** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2)))$

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal)^{ty\_2Enum\_2Enum} \quad (12)$$

**Definition 17** We define  $c\_2Ereal\_2Ereal\_topology\_2Esubspace$  to be  $\lambda V0s \in (2^{ty\_2Erealax\_2Ereal}).(ap (ap c\_2Ebool$

**Definition 18** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota. (\lambda V0x \in A\_27a. c\_2Ebool\_2EF)$ .

**Definition 19** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (13)$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \quad (14)$$

**Definition 20** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. (ap\ c\_2Enum\_2EABS\_num)$

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (15)$$

**Definition 21** We define  $c\_2Earithmetic\_2EBIT1$  to be  $\lambda V0n \in ty\_2Enum\_2Enum. (ap\ (ap\ c\_2Earithmetic\_2E\_2B))$

**Definition 22** We define  $c\_2Earithmetic\_2ENUMERAL$  to be  $\lambda V0x \in ty\_2Enum\_2Enum. V0x$ .

**Definition 23** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c\_2Ebool\_2E\_21\ 2))\ (\lambda V2t \in 2. V2t)))$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a. nonempty\ A\_27a \Rightarrow \forall A\_27b. nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod \\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \end{aligned} \quad (16)$$

**Definition 24** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0x \in A\_27a. \lambda V1y \in A\_27b. (ap\ (c\_2Ebool\_2E\_5C\_2F))$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a. nonempty\ A\_27a \Rightarrow \forall A\_27b. nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC \\ A\_27a\ A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}}) \end{aligned} \quad (17)$$

**Definition 25** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota. \lambda V0x \in A\_27a. \lambda V1s \in (2^{A\_27a}). (ap\ (c\_2Ebool\_2E\_5C\_2F))$

Let  $c\_2Epred\_set\_2ECARD : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a. nonempty\ A\_27a \Rightarrow c\_2Epred\_set\_2ECARD\ A\_27a \in (ty\_2Enum\_2Enum^{(2^{A\_27a})}) \quad (18)$$

**Definition 26** We define  $c\_2Eprim\_rec\_2E\_3C$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. \lambda V1n \in ty\_2Enum\_2Enum. (ap\ (c\_2Ebool\_2E\_5C\_2F))$

**Definition 27** We define  $c\_2Earithmetic\_2E\_3C\_3D$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. \lambda V1n \in ty\_2Enum\_2Enum. (ap\ (c\_2Ebool\_2E\_5C\_2F))$

**Definition 28** We define  $c\_2Epred\_set\_2ESUBSET$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). \lambda V1t \in (2^{A\_27a}). (ap\ (c\_2Ebool\_2E\_5C\_2F))$

**Definition 29** We define  $c\_2Epred\_set\_2EBIGINTER$  to be  $\lambda A\_27a : \iota.\lambda V0P \in (2^{(2^A-27^a)}).(ap (c\_2Epred\_s$

**Definition 30** We define  $c\_2Etopology\_2Ehull$  to be  $\lambda A\_27a : \iota.\lambda V0P \in (2^{(2^A-27^a)}).\lambda V1s \in (2^{A-27^a}).(ap (c\_2$

**Definition 31** We define  $c\_2Ereal\_topology\_2Espan$  to be  $\lambda V0s \in (2^{ty-2Erealax-2Ereal}).(ap (ap (c\_2Etopolog$

**Definition 32** We define  $c\_2Epred\_set\_2EDIFF$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A-27^a}).\lambda V1t \in (2^{A-27^a}).(ap (c\_2$

**Definition 33** We define  $c\_2Epred\_set\_2EDELETE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A-27^a}).\lambda V1x \in A\_27a.(ap (ap$

**Definition 34** We define  $c\_2Ereal\_topology\_2Edependent$  to be  $\lambda V0s \in (2^{ty-2Erealax-2Ereal}).(ap (c\_2Ebool$

**Definition 35** We define  $c\_2Ereal\_topology\_2Eindependent$  to be  $\lambda V0s \in (2^{ty-2Erealax-2Ereal}).(ap c\_2Ebool$

**Definition 36** We define  $c\_2Epred\_set\_2EFINITE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A-27^a}).(ap (c\_2Ebool\_2E\_21 (2$

Assume the following.

$$True \tag{19}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \tag{20}$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \tag{21}$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \tag{22}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow \neg(p V0t)) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow \neg(p V0t)))))) \tag{23}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \tag{24}$$

Assume the following.

$$(\forall V0x \in 2.(\forall V1x\_27 \in 2.(\forall V2y \in 2.(\forall V3y\_27 \in 2.(((p V0x) \Leftrightarrow (p V1x\_27)) \wedge ((p V1x\_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y\_27)))))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x\_27) \Rightarrow (p V3y\_27)))))) \tag{25}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}). (\forall V1a \in A\_27a. ((\exists V2x \in A\_27a. ((V2x = V1a) \wedge (p (ap\ V0P\ V2x)))) \Leftrightarrow (p (ap\ V0P\ V1a)))))) \quad (26)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1x \in A\_27a. ((p (ap (ap (c\_2Epred\_set\_2ESUBSET\ A\_27a) (ap (ap (c\_2Epred\_set\_2EINSERT\ A\_27a) V1x) (c\_2Epred\_set\_2EEMPTY\ A\_27a))) V0s)) \Leftrightarrow (p (ap (ap (c\_2Ebool\_2EIN\ A\_27a) V1x) V0s)))))) \quad (27)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27b. (\forall V2a \in A\_27a. (\forall V3b \in A\_27b. (((ap (ap (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b) V0x) V1y) = (ap (ap (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b) V2a) V3b)) \Leftrightarrow ((V0x = V2a) \wedge (V1y = V3b)))))))) \quad (28)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in (2^{A\_27a}). ((V0s = V1t) \Leftrightarrow (\forall V2x \in A\_27a. ((p (ap (ap (c\_2Ebool\_2EIN\ A\_27a) V2x) V0s)) \Leftrightarrow (p (ap (ap (c\_2Ebool\_2EIN\ A\_27a) V2x) V1t))))))) \quad (29)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0f \in ((ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}). (\forall V1v \in A\_27a. ((p (ap (ap (c\_2Ebool\_2EIN\ A\_27a) V1v) (ap (c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b) V0f))) \Leftrightarrow (\exists V2x \in A\_27b. ((ap (ap (c\_2Epair\_2E\_2C\ A\_27a\ 2) V1v) c\_2Ebool\_2ET) = (ap\ V0f\ V2x))))))) \quad (30)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (p (ap (ap (c\_2Ebool\_2EIN\ A\_27a) V0x) (c\_2Epred\_set\_2EUNIV\ A\_27a)))) \quad (31)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (p (ap (ap (c\_2Epred\_set\_2ESUBSET\ A\_27a) V0s) (c\_2Epred\_set\_2EUNIV\ A\_27a)))) \quad (32)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (p (ap (c\_2Epred\_set\_2EFINITE\ A\_27a) (ap (ap (c\_2Epred\_set\_2EINSERT\ A\_27a) V0x) (c\_2Epred\_set\_2EEMPTY\ A\_27a)))))) \quad (33)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0x \in A.27a. (\forall V1B \in \\ & (2^{(2^{A-27a})}). ((p (ap (ap (c.2Ebool.2EIN\ A.27a)\ V0x) (ap (c.2Epred\_set.2EBIGINTER \\ & A.27a)\ V1B))) \Leftrightarrow (\forall V2P \in (2^{A-27a}). ((p (ap (ap (c.2Ebool.2EIN \\ & (2^{A-27a})\ V2P)\ V1B)) \Rightarrow (p (ap (ap (c.2Ebool.2EIN\ A.27a)\ V0x)\ V2P))))))))) \end{aligned} \quad (34)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in ty.2Erealax.2Ereal. ((ap (ap\ c.2Erealax.2Ereal\_mul \\ & V0x) (ap\ c.2Ereal.2Ereal\_of\_num (ap\ c.2Earithmetic.2ENUMERAL \\ & (ap\ c.2Earithmetic.2EBIT1\ c.2Earithmetic.2EZERO)))) = V0x)) \end{aligned} \quad (35)$$

Assume the following.

$$\begin{aligned} & ((ap (c.2Epred\_set.2ECARD\ ty.2Erealax.2Ereal) (ap (ap (c.2Epred\_set.2EINSERT \\ & ty.2Erealax.2Ereal) (ap\ c.2Ereal.2Ereal\_of\_num (ap\ c.2Earithmetic.2ENUMERAL \\ & (ap\ c.2Earithmetic.2EBIT1\ c.2Earithmetic.2EZERO)))) (c.2Epred\_set.2EEMPTY \\ & ty.2Erealax.2Ereal))) = (ap\ c.2Earithmetic.2ENUMERAL (ap\ c.2Earithmetic.2EBIT1 \\ & c.2Earithmetic.2EZERO))) \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned} & (\forall V0s \in (2^{ty.2Erealax.2Ereal}). (\forall V1t \in (2^{ty.2Erealax.2Ereal}). \\ & (((p (ap (c.2Epred\_set.2EFINITE\ ty.2Erealax.2Ereal)\ V1t)) \wedge \\ & ((p (ap\ c.2Ereal\_topology.2Eindependent\ V0s) \wedge (p (ap (ap (c.2Epred\_set.2ESUBSET \\ & ty.2Erealax.2Ereal)\ V0s) (ap\ c.2Ereal\_topology.2Espan\ V1t)))))) \Rightarrow \\ & ((p (ap (c.2Epred\_set.2EFINITE\ ty.2Erealax.2Ereal)\ V0s) \wedge ( \\ & p (ap (ap\ c.2Earithmetic.2E.3C.3D (ap (c.2Epred\_set.2ECARD\ ty.2Erealax.2Ereal) \\ & V0s) (ap (c.2Epred\_set.2ECARD\ ty.2Erealax.2Ereal)\ V1t))))))))) \end{aligned} \quad (37)$$

### Theorem 1

$$\begin{aligned} & (\forall V0s \in (2^{ty.2Erealax.2Ereal}). ((p (ap\ c.2Ereal\_topology.2Eindependent \\ & V0s) \Rightarrow ((p (ap (c.2Epred\_set.2EFINITE\ ty.2Erealax.2Ereal)\ V0s) \wedge \\ & (p (ap (ap\ c.2Earithmetic.2E.3C.3D (ap (c.2Epred\_set.2ECARD \\ & ty.2Erealax.2Ereal)\ V0s) (ap\ c.2Earithmetic.2ENUMERAL (ap\ c.2Earithmetic.2EBIT1 \\ & c.2Earithmetic.2EZERO))))))))) \end{aligned}$$