

# thm\_2Ereal\_\_topology\_2EREAL\_\_LT\_\_HAUSDIST\_\_POINT\_\_EXIST

(TMX65irnYJq1PSaUHAdkBAhDHBaSkBZ9vre)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 4** We define  $c\_2Ebool\_2E\_2F$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

Let  $ty\_2Erealx\_2Ereal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erealx\_2Ereal \tag{1}$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{2}$$

Let  $c\_2Ereal\_topology\_2EDist : \iota$  be given. Assume the following.

$$c\_2Ereal\_topology\_2EDist \in (ty\_2Erealx\_2Ereal^{(ty\_2Epair\_2Eprod\ ty\_2Erealx\_2Ereal\ ty\_2Erealx\_2Ereal)}) \tag{3}$$

**Definition 5** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap\ P\ x))$  then (the  $(\lambda x.x \in A \wedge p$  of type  $\iota \Rightarrow \iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P (ap (c\_2Emin\_2E\_40$

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \tag{4}$$

Let  $c\_2Erealx\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealx\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})ty\_2Erealx\_2Ereal) \tag{5}$$

**Definition 7** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap (c\_2Emin\_2E\_40 (ty$   
 Let  $c\_2Erealax\_2Etrealt\_lt : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealt\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal))$$

(6)

**Definition 8** We define  $c\_2Erealax\_2Ereal\_lt$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$

**Definition 9** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow q Q)$   
 of type  $\iota$ .

**Definition 10** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_7E$

Let  $c\_2Ereal\_topology\_2Ehausdist : \iota$  be given. Assume the following.

$$c\_2Ereal\_topology\_2Ehausdist \in (ty\_2Erealax\_2Ereal^{(ty\_2Epair\_2Eprod (2^{ty\_2Erealax\_2Ereal}) (2^{ty\_2Erealax\_2Ereal})})})$$

(7)

**Definition 11** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2E\_7E)$ .

**Definition 12** We define  $c\_2Ebool\_2E\_IN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap V1f V0x)))$

**Definition 13** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2$

**Definition 14** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}})$$

(8)

**Definition 15** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2E$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b \in ((2^{A\_27a})^{((ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b})})$$

(9)

**Definition 16** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap (c\_2E$

Let  $c\_2Ereal\_topology\_2Esetdist : \iota$  be given. Assume the following.

$$c\_2Ereal\_topology\_2Esetdist \in (ty\_2Erealax\_2Ereal^{(ty\_2Epair\_2Eprod (2^{ty\_2Erealax\_2Ereal}) (2^{ty\_2Erealax\_2Ereal})})})$$

(10)

**Definition 17** We define  $c\_2Ereal\_2Ereal\_lte$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal$

Let  $c\_2Erealax\_2Etrealm\_neg : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealm\_neg \in ((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)) \quad (11)$$

Let  $c\_2Erealax\_2Etrealm\_eq : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealm\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})\ (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)) \quad (12)$$

Let  $c\_2Erealax\_2Ereal\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_ABS\_CLASS \in (ty\_2Erealax\_2Ereal)^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})} \quad (13)$$

**Definition 18** We define  $c\_2Erealax\_2Ereal\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)$

**Definition 19** We define  $c\_2Erealax\_2Ereal\_neg$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.(ap\ c\_2Erealax\_2Ereal$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega\omega\omega \quad (14)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (15)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum)^{\omega\omega\omega} \quad (16)$$

**Definition 20** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal)^{ty\_2Enum\_2Enum} \quad (17)$$

**Definition 21** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.$

**Definition 22** We define  $c\_2Ereal\_2Eabs$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.(ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND$

**Definition 23** We define  $c\_2Ereal\_topology\_2Ebounded\_def$  to be  $\lambda V0s \in (2^{ty\_2Erealax\_2Ereal}).(ap\ (c\_2Ebc$

Assume the following.

$$True \quad (18)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (19)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p V0t))) \quad (20)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\ & (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (21)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge \\ & ((\neg False) \Leftrightarrow True))) \end{aligned} \quad (22)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.(\forall V1y \in \\ & A.27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \end{aligned} \quad (23)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\ & p V0t)))))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in ( \\ & 2^{A.27a}).((\exists V2x \in A.27a.((p V0P) \wedge (p (ap V1Q V2x)))) \Leftrightarrow ((p \\ & V0P) \wedge (\exists V3x \in A.27a.(p (ap V1Q V3x)))))) \end{aligned} \quad (25)$$

Assume the following.

$$\begin{aligned} & (\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow \\ & ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in 2.(\forall V1x_{.27} \in 2.(\forall V2y \in 2.(\forall V3y_{.27} \in \\ & 2.(((p V0x) \Leftrightarrow (p V1x_{.27})) \wedge ((p V1x_{.27}) \Rightarrow ((p V2y) \Leftrightarrow (p V3y_{.27})))) \Rightarrow \\ & (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x_{.27}) \Rightarrow (p V3y_{.27})))))) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).(\forall V1a \in \\ & A.27a.((\exists V2x \in A.27a.((V2x = V1a) \wedge (p (ap V0P V2x)))) \Leftrightarrow (p ( \\ & ap V0P V1a)))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.(\forall V1s \in \\ & (2^{A.27a}).(\neg((ap (ap (c.2Epred\_set\_2EINSERT \ A.27a) V0x) V1s) = \\ & (c.2Epred\_set\_2EEMPTY \ A.27a)))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0x \in A.27a. (\forall V1y \in \\ & A.27a. ((p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27a)\ V0x)\ (ap\ (ap\ (c.2Epred\_set.2EINSERT \\ & A.27a)\ V1y)\ (c.2Epred\_set.2EEMPTY\ A.27a)))) \Leftrightarrow (V0x = V1y))) \end{aligned} \quad (30)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. \\ & (\forall V2z \in ty\_2Erealax\_2Ereal. (((p\ (ap\ (ap\ c.2Ereal\_2Ereal\_lte \\ & V0x)\ V1y)) \wedge (p\ (ap\ (ap\ c.2Erealax\_2Ereal\_lt\ V1y)\ V2z))) \Rightarrow (p\ (ap \\ & (ap\ c.2Erealax\_2Ereal\_lt\ V0x)\ V2z)))))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} & (\forall V0s \in (2^{ty\_2Erealax\_2Ereal}). (\forall V1t \in (2^{ty\_2Erealax\_2Ereal}). \\ & (\forall V2b \in ty\_2Erealax\_2Ereal. (((\neg(V0s = (c.2Epred\_set.2EEMPTY \\ & ty\_2Erealax\_2Ereal)))) \wedge ((\neg(V1t = (c.2Epred\_set.2EEMPTY\ ty\_2Erealax\_2Ereal)))) \wedge \\ & (p\ (ap\ (ap\ c.2Erealax\_2Ereal\_lt\ (ap\ c.2Ereal\_topology\_2Esetdist \\ & (ap\ (ap\ (c.2Epair\_2E\_2C\ (2^{ty\_2Erealax\_2Ereal})\ (2^{ty\_2Erealax\_2Ereal})) \\ & V0s)\ V1t)))\ V2b)))) \Rightarrow (\exists V3x \in ty\_2Erealax\_2Ereal. (\exists V4y \in \\ & ty\_2Erealax\_2Ereal. ((p\ (ap\ (ap\ (c.2Ebool.2EIN\ ty\_2Erealax\_2Ereal) \\ & V3x)\ V0s)) \wedge ((p\ (ap\ (ap\ (c.2Ebool.2EIN\ ty\_2Erealax\_2Ereal) \\ & V4y)\ V1t)) \wedge (p\ (ap\ (ap\ c.2Erealax\_2Ereal\_lt\ (ap\ c.2Ereal\_topology\_2EDist \\ & (ap\ (ap\ (c.2Epair\_2E\_2C\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal) \\ & V3x)\ V4y)))\ V2b)))))))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & (\forall V0s \in (2^{ty\_2Erealax\_2Ereal}). (\forall V1t \in (2^{ty\_2Erealax\_2Ereal}). \\ & (\forall V2x \in ty\_2Erealax\_2Ereal. (((p\ (ap\ c.2Ereal\_topology\_2Ebounded\_def \\ & V0s)) \wedge ((p\ (ap\ c.2Ereal\_topology\_2Ebounded\_def\ V1t)) \wedge (p\ (ap \\ & (ap\ (c.2Ebool.2EIN\ ty\_2Erealax\_2Ereal)\ V2x)\ V0s)))) \Rightarrow (p\ (ap\ (ap \\ & c.2Ereal\_2Ereal\_lte\ (ap\ c.2Ereal\_topology\_2Esetdist\ (ap\ ( \\ & ap\ (c.2Epair\_2E\_2C\ (2^{ty\_2Erealax\_2Ereal})\ (2^{ty\_2Erealax\_2Ereal})) \\ & (ap\ (ap\ (c.2Epred\_set.2EINSERT\ ty\_2Erealax\_2Ereal)\ V2x)\ (c.2Epred\_set.2EEMPTY \\ & ty\_2Erealax\_2Ereal)))\ V1t)))\ (ap\ c.2Ereal\_topology\_2Ehausdist \\ & (ap\ (ap\ (c.2Epair\_2E\_2C\ (2^{ty\_2Erealax\_2Ereal})\ (2^{ty\_2Erealax\_2Ereal})) \\ & V0s)\ V1t)))))) \end{aligned} \quad (33)$$

**Theorem 1**

$$\begin{aligned} & (\forall V0s \in (2^{ty\_2Erealax\_2Ereal}).(\forall V1t \in (2^{ty\_2Erealax\_2Ereal}). \\ & (\forall V2x \in ty\_2Erealax\_2Ereal.(\forall V3d \in ty\_2Erealax\_2Ereal. \\ & (((p (ap c\_2Ereal\_topology\_2Ebounded\_def V0s)) \wedge ((p (ap c\_2Ereal\_topology\_2Ebounded\_def \\ & V1t)) \wedge ((\neg(V1t = (c\_2Epred\_set\_2EMPTY ty\_2Erealax\_2Ereal)))) \wedge \\ & ((p (ap (ap c\_2Erealax\_2Ereal\_lt (ap c\_2Ereal\_topology\_2Ehausdist \\ & (ap (ap (c\_2Epair\_2E\_2C (2^{ty\_2Erealax\_2Ereal}) (2^{ty\_2Erealax\_2Ereal})) \\ & V0s) V1t))) V3d)) \wedge (p (ap (ap (c\_2Ebool\_2EIN ty\_2Erealax\_2Ereal) \\ & V2x) V0s)))))) \Rightarrow (\exists V4y \in ty\_2Erealax\_2Ereal.((p (ap (ap ( \\ & c\_2Ebool\_2EIN ty\_2Erealax\_2Ereal) V4y) V1t)) \wedge (p (ap (ap c\_2Erealax\_2Ereal\_lt \\ & (ap c\_2Ereal\_topology\_2EDist (ap (ap (c\_2Epair\_2E\_2C ty\_2Erealax\_2Ereal \\ & ty\_2Erealax\_2Ereal) V2x) V4y))) V3d))))))))) \end{aligned}$$