

# thm\_2Ereal\_\_topology\_2ESERIES\_\_LINEAR (TMJsjT8P8G6pVMtPXEp2wtCD8scDTrYwVD)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 4** We define  $c\_2Ebool\_2E\_2F$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_27E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_2F$

**Definition 7** We define  $c\_2Ebool\_2E\_25C\_2E\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t$

**Definition 8** We define  $c\_2Ecombin\_2E\_2Eo$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.\lambda V0f \in (A\_27b^{A\_27c}).\lambda V1g \in (A\_27b^{A\_27c})$

**Definition 9** We define  $c\_2Ebool\_2E\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap V1f V0x)))$

**Definition 10** We define  $c\_2Ebool\_2E\_2F\_2E\_25C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EABS\_prod A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a A\_27b)^{(2^{A\_27b})^{A\_27a}}) \tag{2}$$

**Definition 11** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2Ebool\_2E\_21 2) (c\_2Epair\_2EABS\_prod V0x V1y))$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}})$$
(3)

**Definition 12** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2EGSPEC\ A\_27a\ s)\ x)$

**Definition 13** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2EF)$ .

**Definition 14** We define  $c\_2Epred\_set\_2EFINITE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).(ap\ (c\_2Ebool\_2E21\ 2)\ s)$

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal$$
(4)

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erealax\_2Ereal$$
(5)

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{ty\_2Erealax\_2Ereal})$$
(6)

**Definition 15** We define  $c\_2Emin\_2E40$  to be  $\lambda A.\lambda P \in 2^A.\mathbf{if}\ (\exists x \in A.p\ (ap\ P\ x))\ \mathbf{then}\ (the\ (\lambda x.x \in A \wedge p\ x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 16** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap\ (c\_2Emin\_2E40\ a))$

Let  $c\_2Erealax\_2Etrealmul : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealmul \in (((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})$$
(7)

Let  $c\_2Erealax\_2Etrealeq : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealeq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})$$
(8)

Let  $c\_2Erealax\_2Ereal\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_ABS\_CLASS \in (ty\_2Erealax\_2Ereal)^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})}$$
(9)

**Definition 17** We define  $c\_2Erealax\_2Ereal\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)$

**Definition 18** We define  $c\_2Erealax\_2Ereal\_mul$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$

Let  $c\_2Erealax\_2Ereal\_add : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})$$
(10)

**Definition 19** We define  $c\_Erealax\_Ereal\_add$  to be  $\lambda V0T1 \in ty\_Erealax\_Ereal.\lambda V1T2 \in ty\_Erealax\_Ereal$

**Definition 20** We define  $c\_Ereal\_topology\_Elinear$  to be  $\lambda V0f \in (ty\_Erealax\_Ereal^{ty\_Erealax\_Ereal})$

Let  $ty\_Ereal\_topology\_Eenet : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_Ereal\_topology\_Eenet\ A0) \quad (11)$$

Let  $ty\_Eenum\_Eenum : \iota$  be given. Assume the following.

$$nonempty\ ty\_Eenum\_Eenum \quad (12)$$

Let  $c\_Eenum\_EERP\_num : \iota$  be given. Assume the following.

$$c\_Eenum\_EERP\_num \in (\omega^{ty\_Eenum\_Eenum}) \quad (13)$$

Let  $c\_Eenum\_EESUC\_REP : \iota$  be given. Assume the following.

$$c\_Eenum\_EESUC\_REP \in (\omega^{\omega}) \quad (14)$$

Let  $c\_Eenum\_EEABS\_num : \iota$  be given. Assume the following.

$$c\_Eenum\_EEABS\_num \in (ty\_Eenum\_Eenum^{\omega}) \quad (15)$$

**Definition 21** We define  $c\_Eenum\_EESUC$  to be  $\lambda V0m \in ty\_Eenum\_Eenum.(ap\ c\_Eenum\_EEABS\_num\ m)$

**Definition 22** We define  $c\_Ebool\_E3F$  to be  $\lambda A.27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap\ V0P\ (ap\ (c\_Eemin\_E40\ m)\ P)))$

**Definition 23** We define  $c\_Eprim\_rec\_E3C$  to be  $\lambda V0m \in ty\_Eenum\_Eenum.\lambda V1n \in ty\_Eenum\_Eenum$

**Definition 24** We define  $c\_Earithmic\_E3E$  to be  $\lambda V0m \in ty\_Eenum\_Eenum.\lambda V1n \in ty\_Eenum\_Eenum$

**Definition 25** We define  $c\_Earithmic\_E3E\_3D$  to be  $\lambda V0m \in ty\_Eenum\_Eenum.\lambda V1n \in ty\_Eenum\_Eenum$

Let  $c\_Ereal\_topology\_Emk\_net : \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} &\forall A.27a.nonempty\ A.27a \Rightarrow c\_Ereal\_topology\_Emk\_net \\ &A.27a \in ((ty\_Ereal\_topology\_Eenet\ A.27a)^{(2^{A-27a})^{A-27a}}) \end{aligned} \quad (16)$$

**Definition 26** We define  $c\_Ereal\_topology\_Esequentially$  to be  $(ap\ (c\_Ereal\_topology\_Emk\_net\ ty\_Ereal\_topology\_Eenet))$

Let  $c\_Eenum\_EEZERO\_REP : \iota$  be given. Assume the following.

$$c\_Eenum\_EEZERO\_REP \in \omega \quad (17)$$

**Definition 27** We define  $c\_Eenum\_EE0$  to be  $(ap\ c\_Eenum\_EEABS\_num\ c\_Eenum\_EEZERO\_REP)$ .

**Definition 28** We define  $c\_Earithmic\_E3C\_3D$  to be  $\lambda V0m \in ty\_Eenum\_Eenum.\lambda V1n \in ty\_Eenum\_Eenum$

**Definition 29** We define  $c\_Eiterate\_E2E\_2E$  to be  $\lambda V0m \in ty\_Eenum\_Eenum.\lambda V1n \in ty\_Eenum\_Eenum$

**Definition 30** We define  $c\_2Epred\_set\_2EINTER$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap (c\_2E$

**Definition 31** We define  $c\_2Eiterate\_2Eneutral$  to be  $\lambda A\_27a : \iota.\lambda V0op \in ((A\_27a^{A\_27a})^{A\_27a}).(ap (c\_2Emin$

**Definition 32** We define  $c\_2Eiterate\_2Esupport$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0op \in ((A\_27b^{A\_27b})^{A\_27b}).\lambda V$

**Definition 33** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

**Definition 34** We define  $c\_2Eiterate\_2EITSET$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in ((A\_27a^{A\_27a})^{A\_27b}).\lambda V$

**Definition 35** We define  $c\_2Eiterate\_2Eiterate$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0op \in ((A\_27b^{A\_27b})^{A\_27b}).\lambda V$

**Definition 36** We define  $c\_2Eiterate\_2ESum$  to be  $\lambda A\_27a : \iota.(ap (c\_2Eiterate\_2Eiterate A\_27a ty\_2Erealax$

Let  $c\_2Ereal\_topology\_2EDist : \iota$  be given. Assume the following.

$$c\_2Ereal\_topology\_2EDist \in (ty\_2Erealax\_2Ereal^{(ty\_2Epair\_2Eprod ty\_2Erealax\_2Ereal ty\_2Erealax\_2Ereal)}) \quad (18)$$

Let  $c\_2Erealax\_2Etreallt : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etreallt \in ((2^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal)}) \quad (19)$$

**Definition 37** We define  $c\_2Erealax\_2Ereal\_lt$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax$

Let  $c\_2Ereal\_topology\_2Enetord : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Ereal\_topology\_2Enetord A\_27a \in (((2^{A\_27a})^{A\_27a})^{(ty\_2Ereal\_topology\_2Enet A\_27a)}) \quad (20)$$

**Definition 38** We define  $c\_2Ereal\_topology\_2Etrivial\_limit$  to be  $\lambda A\_27a : \iota.\lambda V0net \in (ty\_2Ereal\_topology$

**Definition 39** We define  $c\_2Ereal\_topology\_2Eeventually$  to be  $\lambda A\_27a : \iota.\lambda V0p \in (2^{A\_27a}).\lambda V1net \in (ty\_2$

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal^{ty\_2Eenum\_2Eenum}) \quad (21)$$

**Definition 40** We define  $c\_2Ereal\_topology\_2E\_2D\_2D\_3E$  to be  $\lambda A\_27a : \iota.\lambda V0f \in (ty\_2Erealax\_2Ereal^A$

**Definition 41** We define  $c\_2Ereal\_topology\_2Esums$  to be  $\lambda V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Eenum\_2Eenum}).\lambda$

Assume the following.

$$True \quad (22)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p V0t)) \Leftrightarrow (p V0t))) \quad (23)$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \wedge (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \wedge True) \Leftrightarrow \\
& (p \ V0t)) \wedge (((False \wedge (p \ V0t)) \Leftrightarrow False) \wedge (((p \ V0t) \wedge False) \Leftrightarrow False) \wedge \\
& (((p \ V0t) \wedge (p \ V0t)) \Leftrightarrow (p \ V0t)))))) \quad (24)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Rightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Rightarrow True) \Leftrightarrow \\
& True) \wedge (((False \Rightarrow (p \ V0t)) \Leftrightarrow True) \wedge (((p \ V0t) \Rightarrow (p \ V0t)) \Leftrightarrow True) \wedge (( \\
& (p \ V0t) \Rightarrow False) \Leftrightarrow (\neg(p \ V0t)))))) \quad (25)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.(\forall V1y \in \\
& A.27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (26)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Leftrightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow True) \Leftrightarrow \\
& (p \ V0t)) \wedge (((False \Leftrightarrow (p \ V0t)) \Leftrightarrow (\neg(p \ V0t))) \wedge (((p \ V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\
& p \ V0t)))))) \quad (27)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p \ V0t1) \Rightarrow \\
& ((p \ V1t2) \Rightarrow (p \ V2t3))) \Leftrightarrow (((p \ V0t1) \wedge (p \ V1t2)) \Rightarrow (p \ V2t3)))))) \quad (28)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in 2.(\forall V1x_{.27} \in 2.(\forall V2y \in 2.(\forall V3y_{.27} \in \\
& 2.(((p \ V0x) \Leftrightarrow (p \ V1x_{.27})) \wedge ((p \ V1x_{.27}) \Rightarrow ((p \ V2y) \Leftrightarrow (p \ V3y_{.27})))) \Rightarrow \\
& (((p \ V0x) \Rightarrow (p \ V2y)) \Leftrightarrow ((p \ V1x_{.27}) \Rightarrow (p \ V3y_{.27})))))) \quad (29)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.( \\
& p \ (ap \ (c\_2Epred\_set\_2EFINITE \ ty\_2Enum\_2Enum) \ (ap \ (ap \ c\_2Eiterate\_2E\_2E \\
& V0m) \ V1n)))) \quad (30)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0s1 \in (2^{A-27a}).(\forall V1s2 \in \\
& (2^{A-27a}).(((p \ (ap \ (c\_2Epred\_set\_2EFINITE \ A.27a) \ V0s1)) \vee (p \\
& (ap \ (c\_2Epred\_set\_2EFINITE \ A.27a) \ V1s2))) \Rightarrow (p \ (ap \ (c\_2Epred\_set\_2EFINITE \\
& A.27a) \ (ap \ (ap \ (c\_2Epred\_set\_2EINTER \ A.27a) \ V0s1) \ V1s2)))))) \quad (31)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Erealax\_2Ereal}). \\
& \quad (\forall V1g \in (ty\_2Erealax\_2Ereal^{A\_27a}). (\forall V2s \in (2^{A\_27a}). \\
& \quad ((p\ (ap\ c\_2Ereal\_topology\_2Elinear\ V0f)) \wedge (p\ (ap\ (c\_2Epred\_set\_2EFINITE \\
& \quad A\_27a)\ V2s))) \Rightarrow ((ap\ V0f\ (ap\ (ap\ (c\_2Eiterate\_2ESum\ A\_27a)\ V2s)\ V1g)) = \\
& \quad (ap\ (ap\ (c\_2Eiterate\_2ESum\ A\_27a)\ V2s)\ (ap\ (ap\ (c\_2Ecombin\_2Eo \\
& \quad A\_27a\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal)\ V0f)\ V1g))))))
\end{aligned} \tag{32}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0net \in (ty\_2Ereal\_topology\_2Enet \\
& \quad A\_27a). (\forall V1h \in (ty\_2Erealax\_2Ereal^{ty\_2Erealax\_2Ereal}). \\
& \quad (\forall V2f \in (ty\_2Erealax\_2Ereal^{A\_27a}). (\forall V3l \in ty\_2Erealax\_2Ereal. \\
& \quad ((p\ (ap\ (ap\ (ap\ (c\_2Ereal\_topology\_2E\_2D\_2D\_3E\ A\_27a)\ V2f)\ V3l) \\
& \quad V0net)) \wedge (p\ (ap\ c\_2Ereal\_topology\_2Elinear\ V1h))) \Rightarrow (p\ (ap\ (ap \\
& \quad (ap\ (c\_2Ereal\_topology\_2E\_2D\_2D\_3E\ A\_27a)\ (\lambda V4x \in A\_27a. \\
& \quad (ap\ V1h\ (ap\ V2f\ V4x))))\ (ap\ V1h\ V3l)\ V0net))))))
\end{aligned} \tag{33}$$

**Theorem 1**

$$\begin{aligned}
& (\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}). (\forall V1h \in \\
& \quad (ty\_2Erealax\_2Ereal^{ty\_2Erealax\_2Ereal}). (\forall V2l \in ty\_2Erealax\_2Ereal. \\
& \quad (\forall V3s \in (2^{ty\_2Enum\_2Enum}). (((p\ (ap\ (ap\ (ap\ c\_2Ereal\_topology\_2ESums \\
& \quad V0f)\ V2l)\ V3s)) \wedge (p\ (ap\ c\_2Ereal\_topology\_2Elinear\ V1h))) \Rightarrow (p \\
& \quad (ap\ (ap\ (ap\ c\_2Ereal\_topology\_2ESums\ (\lambda V4n \in ty\_2Enum\_2Enum. \\
& \quad (ap\ V1h\ (ap\ V0f\ V4n))))\ (ap\ V1h\ V2l)\ V3s))))))
\end{aligned}$$