

thm_2Erealax_2EREAL_LT_TRANS
(TMXYF1bG5H29SL41LTDgVXStSbixjkgNGyP)

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Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$
of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_ET$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a})$

Definition 4 We define $c_2Ebool_2E_EF$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 5 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p \Rightarrow q)$
of type ι .

Definition 6 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_EF$

Definition 7 We define $c_2Ecombin_2EK$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.(\lambda V0x \in A_27a.(\lambda V1y \in A_27b.V0x))$

Definition 8 We define $c_2Ecombin_2ES$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.(\lambda V0f \in ((A_27c^{A_27b})^{A_27a})$

Definition 9 We define $c_2Ecombin_2EI$ to be $\lambda A_27a : \iota.(ap (ap (c_2Ecombin_2ES A_27a (A_27a^{A_27a}) A_27a$

Definition 10 We define $c_2Equotient_2E_2D_2D_3E$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda A_27d : \iota.\lambda V0f$

Definition 11 We define $c_2Equotient_2E_3D_3D_3D_3E$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0R1 \in ((2^{A_27a})^{A_27b})$

Definition 12 We define $c_2Ecombin_2EW$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.(\lambda V0f \in ((A_27b^{A_27a})^{A_27a}).(\lambda V1x$

Definition 13 We define $c_2Equotient_2Erespects$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.(c_2Ecombin_2EW A_27a A_27b$

Definition 14 We define $c_2Ebool_2E_EIN$ to be $\lambda A_27a : \iota.(\lambda V0x \in A_27a.(\lambda V1f \in (2^{A_27a}).(ap V1f V0x))$

Definition 15 We define $c_2Ebool_2E_ERES_FORALL$ to be $\lambda A_27a : \iota.(\lambda V0p \in (2^{A_27a}).(\lambda V1m \in (2^{A_27a}).$

Definition 16 We define $c_2Equotient_2EEQUIV$ to be $\lambda A_27a : \iota.\lambda V0E \in ((2^{A_27a})^{A_27a}).(ap (c_2Ebool_2E_ERES_FORALL$

Definition 17 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t$

Let $ty_2Ehreal_2Ehreal : \iota$ be given. Assume the following.

$$nonempty\ ty_2Ehreal_2Ehreal \quad (1)$$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \quad (2)$$

Let $c_2Erealax_2Etreall_eq : \iota$ be given. Assume the following.

$$c_2Erealax_2Etreall_eq \in ((2^{(ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal)})_{ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal}) \quad (3)$$

Let $ty_2Erealax_2Ereal : \iota$ be given. Assume the following.

$$nonempty\ ty_2Erealax_2Ereal \quad (4)$$

Let $c_2Erealax_2Ereal_ABS_CLASS : \iota$ be given. Assume the following.

$$c_2Erealax_2Ereal_ABS_CLASS \in (ty_2Erealax_2Ereal)^{(2^{(ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal)})} \quad (5)$$

Definition 18 We define $c_2Erealax_2Ereal_ABS$ to be $\lambda V0r \in (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal)$

Definition 19 We define $c_2Equotient_2EQUOTIENT$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0R \in ((2^{A_27a})^{A_27a})$

Let $c_2Erealax_2Ereal_REP_CLASS : \iota$ be given. Assume the following.

$$c_2Erealax_2Ereal_REP_CLASS \in ((2^{(ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal)})_{ty_2Erealax_2Ereal}) \quad (6)$$

Definition 20 We define c_2Emin_2E40 to be $\lambda A.\lambda P \in 2^A$. **if** $(\exists x \in A.p (ap\ P\ x))$ **then** (the $(\lambda x.x \in A \wedge P\ x)$ of type $\iota \Rightarrow \iota$).

Definition 21 We define $c_2Erealax_2Ereal_REP$ to be $\lambda V0a \in ty_2Erealax_2Ereal.(ap\ (c_2Emin_2E40\ (ty_2Erealax_2Ereal\ V0a)))$

Let $c_2Erealax_2Etreall_lt : \iota$ be given. Assume the following.

$$c_2Erealax_2Etreall_lt \in ((2^{(ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal)})_{ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal}) \quad (7)$$

Definition 22 We define $c_2Erealax_2Ereal_lt$ to be $\lambda V0T1 \in ty_2Erealax_2Ereal.\lambda V1T2 \in ty_2Erealax_2Ereal$

Assume the following.

$$True \quad (8)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge ((\\ & (p\ V0t) \Rightarrow False) \Leftrightarrow (\neg(p\ V0t)))))) \end{aligned} \quad (9)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. (\forall V1y \in A_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (10)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p\ V0t1) \Rightarrow ((p\ V1t2) \Rightarrow (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \Rightarrow (p\ V2t3)))))) \quad (11)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. ((ap\ (c_2Ecombin_2EI\ A_27a)\ V0x) = V0x)) \quad (12)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT\ A_27a\ A_27a)\ (c_2Emin_2E_3D\ A_27a))\ (c_2Ecombin_2EI\ A_27a))\ (c_2Ecombin_2EI\ A_27a))) \quad (13)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow \forall A_27c. \\ & \quad nonempty\ A_27c \Rightarrow \forall A_27d.nonempty\ A_27d \Rightarrow (\forall V0R1 \in (\\ & \quad (2^{A_27a})^{A_27a}). (\forall V1abs1 \in (A_27c^{A_27a}). (\forall V2rep1 \in \\ & \quad (A_27a^{A_27c}). ((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT\ A_27a\ A_27c) \\ & \quad V0R1)\ V1abs1)\ V2rep1)) \Rightarrow (\forall V3R2 \in ((2^{A_27b})^{A_27b}). (\forall V4abs2 \in \\ & \quad (A_27d^{A_27b}). (\forall V5rep2 \in (A_27b^{A_27d}). ((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\ & \quad A_27b\ A_27d)\ V3R2)\ V4abs2)\ V5rep2)) \Rightarrow (p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\ & \quad (A_27b^{A_27a})\ (A_27d^{A_27c}))\ (ap\ (ap\ (c_2Equotient_2E_2D_2D_3E\ A_27c \\ & \quad A_27a\ A_27b)\ V0R1)\ V3R2))\ (ap\ (ap\ (c_2Equotient_2E_2D_2D_3E\ A_27c \\ & \quad A_27b\ A_27a\ A_27d)\ V2rep1)\ V4abs2))\ (ap\ (ap\ (c_2Equotient_2E_2D_2D_3E \\ & \quad A_27a\ A_27d\ A_27c\ A_27b)\ V1abs1)\ V5rep2)))))))))) \quad (14) \end{aligned}$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow \forall A_27c. \\ & \quad nonempty\ A_27c \Rightarrow \forall A_27d.nonempty\ A_27d \Rightarrow (\forall V0R1 \in (\\ & \quad (2^{A_27a})^{A_27a}). (\forall V1abs1 \in (A_27c^{A_27a}). (\forall V2rep1 \in \\ & \quad (A_27a^{A_27c}). ((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT\ A_27a\ A_27c) \\ & \quad V0R1)\ V1abs1)\ V2rep1)) \Rightarrow (\forall V3R2 \in ((2^{A_27b})^{A_27b}). (\forall V4abs2 \in \\ & \quad (A_27d^{A_27b}). (\forall V5rep2 \in (A_27b^{A_27d}). ((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\ & \quad A_27b\ A_27d)\ V3R2)\ V4abs2)\ V5rep2)) \Rightarrow (\forall V6f \in (A_27d^{A_27c}). \\ & \quad ((\lambda V7x \in A_27c. (ap\ V6f\ V7x)) = (ap\ (ap\ (ap\ (c_2Equotient_2E_2D_2D_3E \\ & \quad A_27c\ A_27b\ A_27a\ A_27d)\ V2rep1)\ V4abs2)\ (\lambda V8x \in A_27a. (ap\ V5rep2 \\ & \quad (ap\ V6f\ (ap\ V1abs1\ V8x)))))))))))))) \quad (15) \end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& \quad \forall V0REL \in ((2^{A_27a})^{A_27a}).(\forall V1abs \in (A_27b^{A_27a}). \\
& \quad (\forall V2rep \in (A_27a^{A_27b}).((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\
& \quad A_27a\ A_27b)\ V0REL)\ V1abs)\ V2rep))) \Rightarrow (\forall V3x1 \in A_27a.(\forall V4x2 \in \\
& \quad A_27a.((p\ (ap\ (ap\ V0REL\ V3x1)\ V4x2)) \Rightarrow (p\ (ap\ (ap\ V0REL\ V3x1)\ (ap\ V2rep \\
& \quad (ap\ V1abs\ V4x2))))))))))
\end{aligned} \tag{16}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& \quad \forall V0R \in ((2^{A_27a})^{A_27a}).(\forall V1abs \in (A_27b^{A_27a}). \\
& \quad (\forall V2rep \in (A_27a^{A_27b}).((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\
& \quad A_27a\ A_27b)\ V0R)\ V1abs)\ V2rep))) \Rightarrow (\forall V3f \in (2^{A_27b}).((p\ (\\
& \quad ap\ (c_2Ebool_2E_21\ A_27b)\ V3f)) \Leftrightarrow (p\ (ap\ (ap\ (c_2Ebool_2ERES_FORALL \\
& \quad A_27a)\ (ap\ (c_2Equotient_2ERespects\ A_27a\ 2)\ V0R))\ (ap\ (ap\ (ap \\
& \quad (c_2Equotient_2E_2D_2D_3E\ A_27a\ 2\ A_27b\ 2)\ V1abs)\ (c_2Ecombin_2EI \\
& \quad 2))\ V3f))))))
\end{aligned} \tag{17}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& \quad \forall V0R \in ((2^{A_27a})^{A_27a}).(\forall V1abs \in (A_27b^{A_27a}). \\
& \quad (\forall V2rep \in (A_27a^{A_27b}).((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\
& \quad A_27a\ A_27b)\ V0R)\ V1abs)\ V2rep))) \Rightarrow (\forall V3f \in (2^{A_27a}).(\forall V4g \in \\
& \quad (2^{A_27a}).((p\ (ap\ (ap\ (ap\ (ap\ (c_2Equotient_2E_3D_3D_3D_3E\ A_27a \\
& \quad 2)\ V0R)\ (c_2Emin_2E_3D\ 2)\ V3f)\ V4g)) \Rightarrow ((p\ (ap\ (ap\ (c_2Ebool_2ERES_FORALL \\
& \quad A_27a)\ (ap\ (c_2Equotient_2ERespects\ A_27a\ 2)\ V0R))\ V3f)) \Leftrightarrow (p\ (\\
& \quad ap\ (ap\ (c_2Ebool_2ERES_FORALL\ A_27a)\ (ap\ (c_2Equotient_2ERespects \\
& \quad A_27a\ 2)\ V0R))\ V4g))))))
\end{aligned} \tag{18}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow \forall A_27c. \\
& \quad nonempty\ A_27c \Rightarrow \forall A_27d.nonempty\ A_27d \Rightarrow (\forall V0R1 \in (\\
& \quad (2^{A_27a})^{A_27a}).(\forall V1abs1 \in (A_27c^{A_27a}).(\forall V2rep1 \in \\
& \quad (A_27a^{A_27c}).((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT\ A_27a\ A_27c) \\
& \quad V0R1)\ V1abs1)\ V2rep1))) \Rightarrow (\forall V3R2 \in ((2^{A_27b})^{A_27b}).(\forall V4abs2 \in \\
& \quad (A_27d^{A_27b}).(\forall V5rep2 \in (A_27b^{A_27d}).((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\
& \quad A_27b\ A_27d)\ V3R2)\ V4abs2)\ V5rep2))) \Rightarrow (\forall V6f \in (A_27b^{A_27a}). \\
& \quad (\forall V7g \in (A_27b^{A_27a}).(\forall V8x \in A_27a.(\forall V9y \in \\
& \quad A_27a.(((p\ (ap\ (ap\ (ap\ (c_2Equotient_2E_3D_3D_3D_3E\ A_27a \\
& \quad A_27b)\ V0R1)\ V3R2)\ V6f)\ V7g)) \wedge (p\ (ap\ (ap\ V0R1\ V8x)\ V9y))) \Rightarrow (p\ (ap\ (\\
& \quad ap\ V3R2\ (ap\ V6f\ V8x))\ (ap\ V7g\ V9y))))))))))
\end{aligned} \tag{19}$$

Assume the following.

$$\begin{aligned}
& \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow (\forall V0E \in ((2^{A_{.27a}})^{A_{.27a}}). \\
& (\forall V1P \in (2^{A_{.27a}}).((p\ (ap\ (c_2Equotient_2EEQUIV\ A_{.27a}) \\
V0E)) \Rightarrow ((p\ (ap\ (ap\ (c_2Ebool_2ERES_FORALL\ A_{.27a})\ (ap\ (c_2Equotient_2Erespects \\
A_{.27a}\ 2)\ V0E))\ V1P)) \Leftrightarrow (p\ (ap\ (c_2Ebool_2E_21\ A_{.27a})\ V1P))))))
\end{aligned} \tag{20}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal). \\
& (\forall V1q \in (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal). \\
& ((p\ (ap\ (ap\ c_2Erealax_2Etreal_eq\ V0p)\ V1q)) \Leftrightarrow ((ap\ c_2Erealax_2Etreal_eq \\
V0p) = (ap\ c_2Erealax_2Etreal_eq\ V1q))))))
\end{aligned} \tag{21}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal). \\
& (\forall V1y \in (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal). \\
& (\forall V2z \in (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal). \\
& (((p\ (ap\ (ap\ c_2Erealax_2Etreal_lt\ V0x)\ V1y)) \wedge (p\ (ap\ (ap\ c_2Erealax_2Etreal_lt \\
V1y)\ V2z))) \Rightarrow (p\ (ap\ (ap\ c_2Erealax_2Etreal_lt\ V0x)\ V2z))))))
\end{aligned} \tag{22}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x1 \in (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal). \\
& (\forall V1x2 \in (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal). \\
& (\forall V2y1 \in (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal). \\
& (\forall V3y2 \in (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal). \\
& (((p\ (ap\ (ap\ c_2Erealax_2Etreal_eq\ V0x1)\ V1x2)) \wedge (p\ (ap\ (ap\ c_2Erealax_2Etreal_eq \\
V2y1)\ V3y2))) \Rightarrow ((p\ (ap\ (ap\ c_2Erealax_2Etreal_lt\ V0x1)\ V2y1)) \Leftrightarrow \\
(p\ (ap\ (ap\ c_2Erealax_2Etreal_lt\ V1x2)\ V3y2))))))
\end{aligned} \tag{23}$$

Assume the following.

$$\begin{aligned}
& (p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT\ (ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal \\
ty_2Ehreal_2Ehreal)\ ty_2Erealax_2Ereal)\ c_2Erealax_2Etreal_eq) \\
c_2Erealax_2Ereal_ABS)\ c_2Erealax_2Ereal_REP))
\end{aligned} \tag{24}$$

Theorem 1

$$\begin{aligned}
& (\forall V0x \in ty_2Erealax_2Ereal. (\forall V1y \in ty_2Erealax_2Ereal. \\
& (\forall V2z \in ty_2Erealax_2Ereal. (((p\ (ap\ (ap\ c_2Erealax_2Ereal_lt \\
V0x)\ V1y)) \wedge (p\ (ap\ (ap\ c_2Erealax_2Ereal_lt\ V1y)\ V2z))) \Rightarrow (p\ (ap \\
(ap\ c_2Erealax_2Ereal_lt\ V0x)\ V2z))))))
\end{aligned}$$