thm_2Erealax_2ETREAL__ADD__ASSOC (TMUGVz-zGc3adwqYCANsMJBdzeX8WEksmTfa)

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Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{1}$$

Let $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{2}$$

Let $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{3}$$

Let $ty\_2Ehrat\_2Ehrat : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Ehrat\_2Ehrat \tag{4}$$

Let $ty\_2Ehreal\_2Ehreal : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \tag{5}$$

Let $c\_2Ehreal\_2Ecut : \iota$ be given. Assume the following.

$$c\_2Ehreal\_2Ecut \in ((2^{ty\_2Ehrat\_2Ehrat})^{ty\_2Ehreal\_2Ehreal}) \tag{6}$$

Definition 3 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 4 *We define* c_2Ebool_2E_21 *to be* $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap\ (ap\ ($c_2Emin_2E_3D $(2^{A\_27$

Definition 5 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ ($c_2Ebool_2E_21 2$)\ (\lambda V2t \in 2$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{7}$$

Let $c\_2Ehrat\_2Ehrat\_\_REP\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Ehrat\_\_REP\_\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Ehrat\_2Ehra} \tag{8}$$

Definition 6 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 7 *We define* c_2Ehrat_2Ehrat__REP *to be* $\lambda V0a \in ty_2Ehrat_2Ehrat.(ap\ ($c_2Emin_2E_40 $(ty\_2E$

Let $c\_2Ehrat\_2Etrat\_\_add : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty} \tag{9}$$

Let $c\_2Ehrat\_2Etrat\_\_eq : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2E} \tag{10}$$

Let $c\_2Ehrat\_2Ehrat\_\_ABS\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Ehrat\_\_ABS\_\_CLASS \in (ty\_2Ehrat\_2Ehrat^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})}) \tag{11}$$

Definition 8 *We define* c_2Ehrat_2Ehrat__ABS *to be* $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2E$

Definition 9 *We define* c_2Ehrat_2Ehrat__add *to be* $\lambda V0T1 \in ty\_2Ehrat\_2Ehrat.\lambda V1T2 \in ty\_2Ehrat\_2Eh$

Definition 10 *We define* c_2Ebool_2E_3F *to be* $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap\ V0P\ (ap\ ($c_2Emin_2E_40

Let $c\_2Ehreal\_2Ehreal : \iota$ be given. Assume the following.

$$c\_2Ehreal\_2Ehreal \in (ty\_2Ehreal\_2Ehreal^{(2^{ty\_2Ehrat\_2Ehrat})}) \tag{12}$$

Definition 11 *We define* c_2Ehreal_2Ehreal__add *to be* $\lambda V0X \in ty\_2Ehreal\_2Ehreal.\lambda V1Y \in ty\_2Ehreal\_2$

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})}) \tag{13}$$

Definition 12 *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2$

Let $c\_2Erealax\_2Etreal\_\_add : \iota$ be given. Assume the following.

$$c\_2Erealax\_2Etreal\_\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehr} \tag{14}$$

Assume the following.

$$True \tag{15}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \tag{16}$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1Y \in ty\_2Ehreal\_2Ehreal.(\forall V2Z \in ty\_2Ehreal\_2Ehreal.((ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V0X)\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V1Y)\ V2Z)) = (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V0X)\ V1Y))\ V2Z))))) \tag{17}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0x \in (ty\_2Epair\_2Eprod\ A\_27a\ A\_27b).((ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ A\_27a\ A\_27b)\ (ap\ (c\_2Epair\_2EFST\ A\_27a\ A\_27b)\ V0x))\ (ap\ (c\_2Epair\_2ESND\ A\_27a\ A\_27b)\ V0x)) = V0x)) \tag{18}$$

Assume the following.

$$(\forall V0x1 \in ty\_2Ehreal\_2Ehreal.(\forall V1y1 \in ty\_2Ehreal\_2Ehreal.(\forall V2x2 \in ty\_2Ehreal\_2Ehreal.(\forall V3y2 \in ty\_2Ehreal\_2Ehreal.((ap\ (ap\ c\_2Erealax\_2Etreal\_\_add\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ V0x1)\ V1y1))\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ V2x2)\ V3y2)) = (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V0x1)\ V2x2))\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V1y1)\ V3y2))))))) \tag{19}$$

Theorem 1

$$(\forall V0x \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal).(\forall V1y \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal).(\forall V2z \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal).((ap\ (ap\ c\_2Erealax\_2Etreal\_\_add\ V0x)\ (ap\ (ap\ c\_2Erealax\_2Etreal\_\_add\ V1y)\ V2z)) = (ap\ (ap\ c\_2Erealax\_2Etreal\_\_add\ (ap\ (ap\ c\_2Erealax\_2Etreal\_\_add\ V0x)\ V1y))\ V2z)))))$$