thm_2Erealax_2ETREAL__INV__WELLDEF (TMdz3LWeKRTrUqXh19BpMWsRPzZWJDdcXe4)

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Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (c\_2Emin\_2E\_3D\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 3 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (c\_2Emin\_2E\_3D\ (2^{A\_27}$

Definition 4 *We define* c_2Ebool_2EF *to be* $(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 5 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 6 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ c\_2Emin\_2E\_3D\_3D\_3E\ V0t)\ c\_2Ebool\_2EF$

Definition 7 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2$

Definition 8 *We define* c_2Ebool_2E_5C_2F *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2$

Definition 9 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 10 *We define* c_2Ebool_2E_3F *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_2Emin\_2E\_40$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{1}$$

Let $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{2}$$

Let $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{3}$$

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \tag{4}$$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{5}$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \tag{6}$$

Definition 11 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})}) \tag{7}$$

Definition 12 *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2$

Definition 13 *We define* c_2Ehrat_2Etrat__1 *to be* $(ap\ (ap\ ($c_2Epair_2E_2C $ty\_2Enum\_2Enum\ ty\_2Enum\_$

Let $c\_2Ehrat\_2Etrat\_\_eq : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2E} \tag{8}$$

Let $ty\_2Ehrat\_2Ehrat : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Ehrat\_2Ehrat \tag{9}$$

Let $c\_2Ehrat\_2Ehrat\_\_ABS\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Ehrat\_\_ABS\_\_CLASS \in (ty\_2Ehrat\_2Ehrat^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})}) \tag{10}$$

Definition 14 *We define* c_2Ehrat_2Ehrat__ABS *to be* $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2$

Definition 15 *We define* c_2Ehrat_2Ehrat__1 *to be* $(ap$ c_2Ehrat_2Ehrat__ABS c_2Ehrat_2Etrat__1$)$.

Let $c\_2Ehrat\_2Ehrat\_\_REP\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Ehrat\_\_REP\_\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Ehrat\_2Ehra} \tag{11}$$

Definition 16 *We define* c_2Ehrat_2Ehrat__REP *to be* $\lambda V0a \in ty\_2Ehrat\_2Ehrat.(ap$ (c_2Emin_2E_40 $(ty\_2$

Let $c\_2Ehrat\_2Etrat\_\_add : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty} \tag{12}$$

Definition 17 *We define* c_2Ehrat_2Ehrat__add *to be* $\lambda V0T1 \in ty\_2Ehrat\_2Ehrat.\lambda V1T2 \in ty\_2Ehrat\_2E$

Definition 18 *We define* c_2Ehreal_2Ehrat__lt *to be* $\lambda V0x \in ty\_2Ehrat\_2Ehrat.\lambda V1y \in ty\_2Ehrat\_2Ehrat$

Definition 19 *We define* c_2Ehreal_2Ecut__of__hrat *to be* $\lambda V0x \in ty\_2Ehrat\_2Ehrat.(\lambda V1y \in ty\_2Ehrat\_2$

Let $ty\_2Ehreal\_2Ehreal : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \tag{13}$$

Let $c\_2Ehreal\_2Ehreal : \iota$ be given. Assume the following.

$$c\_2Ehreal\_2Ehreal \in (ty\_2Ehreal\_2Ehreal^{(2^{ty\_2Ehrat\_2Ehrat})}) \tag{14}$$

Definition 20 *We define* c_2Ehreal_2Ehreal__1 *to be* $(ap\ c\_2Ehreal\_2Ehreal\ (ap$ c_2Ehreal_2Ecut__of__hrat

Let $c\_2Ehreal\_2Ecut : \iota$ be given. Assume the following.

$$c\_2Ehreal\_2Ecut \in ((2^{ty\_2Ehrat\_2Ehrat})^{ty\_2Ehreal\_2Ehreal}) \tag{15}$$

Definition 21 *We define* c_2Ehreal_2Ehreal__sub *to be* $\lambda V0Y \in ty\_2Ehreal\_2Ehreal.\lambda V1X \in ty\_2Ehreal\_2$

Let $c\_2Ehrat\_2Etrat\_\_mul : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty} \tag{16}$$

Definition 22 *We define* c_2Ehrat_2Ehrat__mul *to be* $\lambda V0T1 \in ty\_2Ehrat\_2Ehrat.\lambda V1T2 \in ty\_2Ehrat\_2E$

Definition 23 *We define* c_2Ehreal_2Ehreal__inv *to be* $\lambda V0X \in ty\_2Ehreal\_2Ehreal.(ap\ c\_2Ehreal\_2Ehrea$

Definition 24 *We define* c_2Ehreal_2Ehreal__lt *to be* $\lambda V0X \in ty\_2Ehreal\_2Ehreal.\lambda V1Y \in ty\_2Ehreal\_2E$

Definition 25 *We define* c_2Erealax_2Etreal__0 *to be* $(ap\ (ap$ (c_2Epair_2E_2C $ty\_2Ehreal\_2Ehreal\ ty\_2Eh$

Definition 26 *We define* c_2Ebool_2ECOND *to be* $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

Let $c\_2Erealax\_2Etreal\_\_inv : \iota$ be given. Assume the following.

$$c\_2Erealax\_2Etreal\_\_inv \in ((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)}) \tag{17}$$

Definition 27 *We define* c_2Ehreal_2Ehreal__add *to be* $\lambda V0X \in ty\_2Ehreal\_2Ehreal.\lambda V1Y \in ty\_2Ehreal\_2$

Let $c\_2Erealax\_2Etreal\_\_eq : \iota$ be given. Assume the following.

$$c\_2Erealax\_2Etreal\_\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Eh} \tag{18}$$

Assume the following.

$$True \tag{19}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \tag{20}$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p\ V0t))) \tag{21}$$

Assume the following.

$$(\forall V0t \in 2.((p\ V0t) \vee (\neg (p\ V0t)))) \tag{22}$$

Assume the following.

$$(\forall V0t \in 2.(((True \vee (p\ V0t)) \Leftrightarrow True) \wedge ((((p\ V0t) \vee True) \Leftrightarrow True) \wedge (((False \vee (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \vee False) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee (p\ V0t)) \Leftrightarrow (p\ V0t))))))) \tag{23}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge ((((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow False) \Leftrightarrow (\neg (p\ V0t)))))))) \tag{24}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \tag{25}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg (p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg (p\ V0t))))))) \tag{26}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t1 \in A\_27a.(\forall V1t2 \in A\_27a.(((ap\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ECOND}\ A\_27a)\ \mathsf{c\_2Ebool\_2ET})\ V0t1)\ V1t2) = V0t1) \wedge ((ap\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ECOND}\ A\_27a)\ \mathsf{c\_2Ebool\_2EF})\ V0t1)\ V1t2) = V1t2)))) \tag{27}$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1Y \in ty\_2Ehreal\_2Ehreal.\\ ((ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V0X)\ V1Y) = (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\\ V1Y)\ V0X))))\tag{28}$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1Y \in ty\_2Ehreal\_2Ehreal.\\ (\forall V2Z \in ty\_2Ehreal\_2Ehreal.((ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\\ V0X)\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V1Y)\ V2Z)) = (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V0X)\ V1Y))\ V2Z)))))\tag{29}$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1Y \in ty\_2Ehreal\_2Ehreal.\\ ((p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_lt}\ V0X)\ V1Y)) \Rightarrow ((ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_sub}\ V1Y)\ V0X))\ V0X) = V1Y))))\tag{30}$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1Y \in ty\_2Ehreal\_2Ehreal.\\ ((V0X = V1Y) \vee ((p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_lt}\ V0X)\ V1Y)) \vee (p\ (ap\\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_lt}\ V1Y)\ V0X))))))\tag{31}$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1Y \in ty\_2Ehreal\_2Ehreal.\\ ((p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_lt}\ V0X)\ V1Y)) \Leftrightarrow (\exists V2D \in ty\_2Ehreal\_2Ehreal.\\ (V1Y = (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V0X)\ V2D))))))\tag{32}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\\ \forall V0x \in (ty\_2Epair\_2Eprod\ A\_27a\ A\_27b).((ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\\ A\_27a\ A\_27b)\ (ap\ (c\_2Epair\_2EFST\ A\_27a\ A\_27b)\ V0x))\ (ap\ (c\_2Epair\_2ESND\\ A\_27a\ A\_27b)\ V0x)) = V0x))\tag{33}$$

Assume the following.

$$(\forall V0x \in ty\_2Ehreal\_2Ehreal.(\forall V1y \in ty\_2Ehreal\_2Ehreal.\\ (\forall V2z \in ty\_2Ehreal\_2Ehreal.(((ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\\ V0x)\ V1y) = (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V0x)\ V2z)) \Leftrightarrow (V1y = V2z)))))\tag{34}$$

Assume the following.

$$(\forall V0x \in ty\_2Ehreal\_2Ehreal.(\forall V1y \in ty\_2Ehreal\_2Ehreal.\\ (p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_lt}\ V0x)\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\\ V0x)\ V1y)))))\tag{35}$$

Assume the following.

$$(\forall V0x \in ty\_2Ehreal\_2Ehreal.(\forall V1y \in ty\_2Ehreal\_2Ehreal. ((ap\ c\_2Erealax\_2Etreal\_\_inv\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ V0x)\ V1y)) = (ap\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ECOND}\ (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal))\ (ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ ty\_2Ehreal\_2Ehreal)\ V0x)\ V1y))\ \mathsf{c\_2Erealax\_2Etreal\_\_0})\ (ap\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ECOND}\ (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal))\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_lt}\ V1y)\ V0x))\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_inv}\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_sub}\ V0x)\ V1y)))\ \mathsf{c\_2Ehreal\_2Ehreal\_\_1}))\ \mathsf{c\_2Ehreal\_2Ehreal\_\_1}))\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ \mathsf{c\_2Ehreal\_2Ehreal\_\_1})\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_inv}\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_sub}\ V1y)\ V0x)))\ \mathsf{c\_2Ehreal\_2Ehreal\_\_1})))))))) \tag{36}$$

Assume the following.

$$(\forall V0x1 \in ty\_2Ehreal\_2Ehreal.(\forall V1y1 \in ty\_2Ehreal\_2Ehreal. (\forall V2x2 \in ty\_2Ehreal\_2Ehreal.(\forall V3y2 \in ty\_2Ehreal\_2Ehreal. ((p\ (ap\ (ap\ c\_2Erealax\_2Etreal\_\_eq\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ V0x1)\ V1y1))\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)\ V2x2)\ V3y2)))\Leftrightarrow((ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V0x1)\ V3y2) = (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_add}\ V2x2)\ V1y1))))))) \tag{37}$$

Assume the following.

$$(\forall V0x \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal). (p\ (ap\ (ap\ c\_2Erealax\_2Etreal\_\_eq\ V0x)\ V0x))) \tag{38}$$

Theorem 1

$$(\forall V0x1 \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal). (\forall V1x2 \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal). ((p\ (ap\ (ap\ c\_2Erealax\_2Etreal\_\_eq\ V0x1)\ V1x2))\Rightarrow(p\ (ap\ (ap\ c\_2Erealax\_2Etreal\_\_eq\ (ap\ c\_2Erealax\_2Etreal\_\_inv\ V0x1))\ (ap\ c\_2Erealax\_2Etreal\_\_inv\ V1x2))))))$$