

thm_2Erealax_2ETREAL_LT_TRANS
(TMXT6nU2cDNh3nkzH5GUxv7LG1CqKJoaE2)

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Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_2T$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

Definition 4 We define $c_2Ebool_2E_2F$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 5 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 6 We define $c_2Ebool_2E_27E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_2F))$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Epair_2Eprod A0 A1) \tag{1}$$

Let $c_2Epair_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epair_2ESND A_27a A_27b \in (A_27b^{(ty_2Epair_2Eprod A_27a A_27b)}) \tag{2}$$

Let $c_2Epair_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epair_2EFST A_27a A_27b \in (A_27a^{(ty_2Epair_2Eprod A_27a A_27b)}) \tag{3}$$

Definition 7 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t)))$

Let $ty_2Ehrat_2Ehrat : \iota$ be given. Assume the following.

$$nonempty ty_2Ehrat_2Ehrat \tag{4}$$

Let $ty_2Ehreal_2Ehreal : \iota$ be given. Assume the following.

$$nonempty\ ty_2Ehreal_2Ehreal \quad (5)$$

Let $c_2Ehreal_2Ecut : \iota$ be given. Assume the following.

$$c_2Ehreal_2Ecut \in ((2^{ty_2Ehreal_2Ehreal})^{ty_2Ehreal_2Ehreal}) \quad (6)$$

Let $ty_2Enum_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Enum_2Enum \quad (7)$$

Let $c_2Ehreal_2Ecut_2REP_2CLASS : \iota$ be given. Assume the following.

$$c_2Ehreal_2Ecut_2REP_2CLASS \in ((2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})^{ty_2Ehreal_2Ecut}) \quad (8)$$

Definition 8 We define c_2Emin_2E40 to be $\lambda A.\lambda P \in 2^A$. **if** $(\exists x \in A.p (ap\ P\ x))$ **then** (the $(\lambda x.x \in A \wedge p\ x)$ of type $\iota \Rightarrow \iota$).

Definition 9 We define $c_2Ehreal_2Ecut_2REP$ to be $\lambda V0a \in ty_2Ehreal_2Ecut$. $(ap\ (c_2Emin_2E40\ (ty_2Enum_2Enum\ a)))$

Let $c_2Ehreal_2Ecut_2add : \iota$ be given. Assume the following.

$$c_2Ehreal_2Ecut_2add \in ((2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum)}) \quad (9)$$

Let $c_2Ehreal_2Ecut_2eq : \iota$ be given. Assume the following.

$$c_2Ehreal_2Ecut_2eq \in ((2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum)}) \quad (10)$$

Let $c_2Ehreal_2Ecut_2ABS_2CLASS : \iota$ be given. Assume the following.

$$c_2Ehreal_2Ecut_2ABS_2CLASS \in (ty_2Ehreal_2Ecut)^{(2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})} \quad (11)$$

Definition 10 We define $c_2Ehreal_2Ecut_2ABS$ to be $\lambda V0r \in (ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)$

Definition 11 We define $c_2Ehreal_2Ecut_2add$ to be $\lambda V0T1 \in ty_2Ehreal_2Ecut$. $\lambda V1T2 \in ty_2Ehreal_2Ecut$

Definition 12 We define c_2Ebool_2E3F to be $\lambda A.\lambda 27a : \iota$. $(\lambda V0P \in (2^{A-27a}))$. $(ap\ V0P\ (ap\ (c_2Emin_2E40\ (ty_2Enum_2Enum\ a))))$

Let $c_2Ehreal_2Ehreal : \iota$ be given. Assume the following.

$$c_2Ehreal_2Ehreal \in (ty_2Ehreal_2Ehreal)^{(2^{ty_2Ehreal_2Ehreal})} \quad (12)$$

Definition 13 We define $c_2Ehreal_2Ehreal_2add$ to be $\lambda V0X \in ty_2Ehreal_2Ehreal$. $\lambda V1Y \in ty_2Ehreal_2Ehreal$

Definition 14 We define $c_2Ehreal_2Ehreal_2lt$ to be $\lambda V0X \in ty_2Ehreal_2Ehreal$. $\lambda V1Y \in ty_2Ehreal_2Ehreal$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod \\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b})^{A_27a}}) \end{aligned} \quad (13)$$

Definition 15 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0x \in A_27a. \lambda V1y \in A_27b. (ap\ (c_2E$

Let $c_2Erealax_2Etrealt_lt : \iota$ be given. Assume the following.

$$c_2Erealax_2Etrealt_lt \in ((2^{(ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal\ ty_2Ehreal_2Ehreal)})(ty_2Epair_2Eprod\ ty_2Ehreal_2Ehreal)) \quad (14)$$

Assume the following.

$$True \quad (15)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2. (((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow \\ (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow \neg(p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow \neg(\\ p\ V0t)))))) \end{aligned} \quad (16)$$

Assume the following.

$$\begin{aligned} (\forall V0X \in ty_2Ehreal_2Ehreal. (\forall V1Y \in ty_2Ehreal_2Ehreal. \\ ((ap\ (ap\ c_2Ehreal_2Ehreal_add\ V0X)\ V1Y) = (ap\ (ap\ c_2Ehreal_2Ehreal_add \\ V1Y)\ V0X)))) \end{aligned} \quad (17)$$

Assume the following.

$$\begin{aligned} (\forall V0X \in ty_2Ehreal_2Ehreal. (\forall V1Y \in ty_2Ehreal_2Ehreal. \\ (\forall V2Z \in ty_2Ehreal_2Ehreal. ((ap\ (ap\ c_2Ehreal_2Ehreal_add \\ V0X)\ (ap\ (ap\ c_2Ehreal_2Ehreal_add\ V1Y)\ V2Z)) = (ap\ (ap\ c_2Ehreal_2Ehreal_add \\ (ap\ (ap\ c_2Ehreal_2Ehreal_add\ V0X)\ V1Y))\ V2Z)))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ \forall V0x \in (ty_2Epair_2Eprod\ A_27a\ A_27b). ((ap\ (ap\ (c_2Epair_2E_2C \\ A_27a\ A_27b)\ (ap\ (c_2Epair_2EFST\ A_27a\ A_27b)\ V0x))\ (ap\ (c_2Epair_2ESND \\ A_27a\ A_27b)\ V0x)) = V0x)) \end{aligned} \quad (19)$$

Assume the following.

$$\begin{aligned} (\forall V0x1 \in ty_2Ehreal_2Ehreal. (\forall V1x2 \in ty_2Ehreal_2Ehreal. \\ (\forall V2y1 \in ty_2Ehreal_2Ehreal. (\forall V3y2 \in ty_2Ehreal_2Ehreal. \\ (((p\ (ap\ (ap\ c_2Ehreal_2Ehreal_lt\ V0x1)\ V2y1)) \wedge (p\ (ap\ (ap\ c_2Ehreal_2Ehreal_lt \\ V1x2)\ V3y2))) \Rightarrow (p\ (ap\ (ap\ c_2Ehreal_2Ehreal_lt\ (ap\ (ap\ c_2Ehreal_2Ehreal_add \\ V0x1)\ V1x2))\ (ap\ (ap\ c_2Ehreal_2Ehreal_add\ V2y1)\ V3y2)))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty_2Ehreal_2Ehreal. (\forall V1y \in ty_2Ehreal_2Ehreal. \\
& (\forall V2z \in ty_2Ehreal_2Ehreal. ((p (ap (ap c_2Ehreal_2Ehreal_lt \\
& (ap (ap c_2Ehreal_2Ehreal_add V0x) V1y)) (ap (ap c_2Ehreal_2Ehreal_add \\
& V0x) V2z))) \Leftrightarrow (p (ap (ap c_2Ehreal_2Ehreal_lt V1y) V2z))))))
\end{aligned} \tag{21}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x1 \in ty_2Ehreal_2Ehreal. (\forall V1y1 \in ty_2Ehreal_2Ehreal. \\
& (\forall V2x2 \in ty_2Ehreal_2Ehreal. (\forall V3y2 \in ty_2Ehreal_2Ehreal. \\
& ((p (ap (ap c_2Erealax_2Etreal_lt (ap (ap (c_2Epair_2E_2C ty_2Ehreal_2Ehreal \\
& ty_2Ehreal_2Ehreal) V0x1) V1y1)) (ap (ap (c_2Epair_2E_2C ty_2Ehreal_2Ehreal \\
& ty_2Ehreal_2Ehreal) V2x2) V3y2))) \Leftrightarrow (p (ap (ap c_2Ehreal_2Ehreal_lt \\
& (ap (ap c_2Ehreal_2Ehreal_add V0x1) V3y2)) (ap (ap c_2Ehreal_2Ehreal_add \\
& V2x2) V1y1))))))
\end{aligned} \tag{22}$$

Theorem 1

$$\begin{aligned}
& (\forall V0x \in (ty_2Epair_2Eprod ty_2Ehreal_2Ehreal ty_2Ehreal_2Ehreal). \\
& (\forall V1y \in (ty_2Epair_2Eprod ty_2Ehreal_2Ehreal ty_2Ehreal_2Ehreal). \\
& (\forall V2z \in (ty_2Epair_2Eprod ty_2Ehreal_2Ehreal ty_2Ehreal_2Ehreal). \\
& (((p (ap (ap c_2Erealax_2Etreal_lt V0x) V1y)) \wedge (p (ap (ap c_2Erealax_2Etreal_lt \\
& V1y) V2z))) \Rightarrow (p (ap (ap c_2Erealax_2Etreal_lt V0x) V2z))))))
\end{aligned}$$