

thm\_2Erealax\_2ETREAL\_\_LT\_\_WELLDEFR  
(TMVPexjYJRHbq45QK3ohzcJHArMZPqBC8J6)

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**Definition 1** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 2** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$ .

**Definition 4** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 5** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 6** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2EF$

**Definition 7** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2ESND A\_27a A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod A\_27a A\_27b)}) \tag{2}$$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EFST A\_27a A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod A\_27a A\_27b)}) \tag{3}$$

Let  $ty\_2Ehrat\_2Ehrat : \iota$  be given. Assume the following.

$$nonempty ty\_2Ehrat\_2Ehrat \tag{4}$$

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \quad (5)$$

Let  $c\_2Ehreal\_2Ecut : \iota$  be given. Assume the following.

$$c\_2Ehreal\_2Ecut \in ((2^{ty\_2Ehreal\_2Ehreal})^{ty\_2Ehreal\_2Ehreal}) \quad (6)$$

**Definition 8** We define  $c\_2Ehreal\_2Ehreal\_lt$  to be  $\lambda V0X \in ty\_2Ehreal\_2Ehreal.\lambda V1Y \in ty\_2Ehreal\_2Ehreal$

Let  $c\_2Erealax\_2Etreax\_lt : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etreax\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal)}) \quad (7)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (8)$$

Let  $c\_2Ehrat\_2Ehrat\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Ehrat\_2Ehrat\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Ehrat\_2Ehrat}) \quad (9)$$

**Definition 9** We define  $c\_2Emin\_2E40$  to be  $\lambda A.\lambda P \in 2^A$ .if  $(\exists x \in A.p (ap\ P\ x))$  then (the  $(\lambda x.x \in A \wedge p\ x)$  of type  $\iota \Rightarrow \iota$ ).

**Definition 10** We define  $c\_2Ehrat\_2Ehrat\_REP$  to be  $\lambda V0a \in ty\_2Ehrat\_2Ehrat.(ap\ (c\_2Emin\_2E40\ (ty\_2Ehrat\_2Ehrat)))$

Let  $c\_2Ehrat\_2Etrat\_add : \iota$  be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum})^{ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum}) \quad (10)$$

Let  $c\_2Ehrat\_2Etrat\_eq : \iota$  be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum)}) \quad (11)$$

Let  $c\_2Ehrat\_2Ehrat\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Ehrat\_2Ehrat\_ABS\_CLASS \in (ty\_2Ehrat\_2Ehrat^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})}) \quad (12)$$

**Definition 11** We define  $c\_2Ehrat\_2Ehrat\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)$

**Definition 12** We define  $c\_2Ehrat\_2Ehrat\_add$  to be  $\lambda V0T1 \in ty\_2Ehrat\_2Ehrat.\lambda V1T2 \in ty\_2Ehrat\_2Ehrat$

**Definition 13** We define  $c\_2Ebool\_2E3F$  to be  $\lambda A.\lambda 27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap\ V0P\ (ap\ (c\_2Emin\_2E40\ (ty\_2Ehrat\_2Ehrat))))$

Let  $c\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$c\_2Ehreal\_2Ehreal \in (ty\_2Ehreal\_2Ehreal^{(2^{ty\_2Ehreal\_2Ehreal})}) \quad (13)$$

**Definition 14** We define  $c\_2Ehreal\_2Ehreal\_add$  to be  $\lambda V0X \in ty\_2Ehreal\_2Ehreal.\lambda V1Y \in ty\_2Ehreal\_2Ehreal.$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod \\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \end{aligned} \quad (14)$$

**Definition 15** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b)\ V0x\ V1y)$

Let  $c\_2Erealax\_2Etreax\_eq : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etreax\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal)) \quad (15)$$

Assume the following.

$$True \quad (16)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (17)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow \\ (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow \neg(p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow \neg( \\ p\ V0t)))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} (\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1Y \in ty\_2Ehreal\_2Ehreal. \\ ((ap\ (ap\ c\_2Ehreal\_2Ehreal\_add\ V0X)\ V1Y) = (ap\ (ap\ c\_2Ehreal\_2Ehreal\_add \\ V1Y)\ V0X)))) \end{aligned} \quad (19)$$

Assume the following.

$$\begin{aligned} (\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1Y \in ty\_2Ehreal\_2Ehreal. \\ (\forall V2Z \in ty\_2Ehreal\_2Ehreal.((ap\ (ap\ c\_2Ehreal\_2Ehreal\_add \\ V0X)\ (ap\ (ap\ c\_2Ehreal\_2Ehreal\_add\ V1Y)\ V2Z)) = (ap\ (ap\ c\_2Ehreal\_2Ehreal\_add \\ (ap\ (ap\ c\_2Ehreal\_2Ehreal\_add\ V0X)\ V1Y))\ V2Z)))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ \forall V0x \in (ty\_2Epair\_2Eprod\ A\_27a\ A\_27b).((ap\ (ap\ (c\_2Epair\_2E\_2C \\ A\_27a\ A\_27b)\ (ap\ (c\_2Epair\_2EFST\ A\_27a\ A\_27b)\ V0x))\ (ap\ (c\_2Epair\_2ESND \\ A\_27a\ A\_27b)\ V0x)) = V0x)) \end{aligned} \quad (21)$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Ehreal\_2Ehreal. (\forall V1y \in ty\_2Ehreal\_2Ehreal. \\
& (\forall V2z \in ty\_2Ehreal\_2Ehreal. ((p (ap (ap c\_2Ehreal\_2Ehreal\_lt \\
& (ap (ap c\_2Ehreal\_2Ehreal\_add V0x) V1y)) (ap (ap c\_2Ehreal\_2Ehreal\_add \\
& V0x) V2z))) \Leftrightarrow (p (ap (ap c\_2Ehreal\_2Ehreal\_lt V1y) V2z))))))
\end{aligned} \tag{22}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x1 \in ty\_2Ehreal\_2Ehreal. (\forall V1y1 \in ty\_2Ehreal\_2Ehreal. \\
& (\forall V2x2 \in ty\_2Ehreal\_2Ehreal. (\forall V3y2 \in ty\_2Ehreal\_2Ehreal. \\
& ((p (ap (ap c\_2Erealax\_2Etreal\_lt (ap (ap (c\_2Epair\_2E\_2C ty\_2Ehreal\_2Ehreal \\
& ty\_2Ehreal\_2Ehreal) V0x1) V1y1)) (ap (ap (c\_2Epair\_2E\_2C ty\_2Ehreal\_2Ehreal \\
& ty\_2Ehreal\_2Ehreal) V2x2) V3y2))) \Leftrightarrow (p (ap (ap c\_2Ehreal\_2Ehreal\_lt \\
& (ap (ap c\_2Ehreal\_2Ehreal\_add V0x1) V3y2)) (ap (ap c\_2Ehreal\_2Ehreal\_add \\
& V2x2) V1y1))))))
\end{aligned} \tag{23}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x1 \in ty\_2Ehreal\_2Ehreal. (\forall V1y1 \in ty\_2Ehreal\_2Ehreal. \\
& (\forall V2x2 \in ty\_2Ehreal\_2Ehreal. (\forall V3y2 \in ty\_2Ehreal\_2Ehreal. \\
& ((p (ap (ap c\_2Erealax\_2Etreal\_eq (ap (ap (c\_2Epair\_2E\_2C ty\_2Ehreal\_2Ehreal \\
& ty\_2Ehreal\_2Ehreal) V0x1) V1y1)) (ap (ap (c\_2Epair\_2E\_2C ty\_2Ehreal\_2Ehreal \\
& ty\_2Ehreal\_2Ehreal) V2x2) V3y2))) \Leftrightarrow ((ap (ap c\_2Ehreal\_2Ehreal\_add \\
& V0x1) V3y2) = (ap (ap c\_2Ehreal\_2Ehreal\_add V2x2) V1y1))))))
\end{aligned} \tag{24}$$

**Theorem 1**

$$\begin{aligned}
& (\forall V0x1 \in (ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal). \\
& (\forall V1x2 \in (ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal). \\
& (\forall V2y \in (ty\_2Epair\_2Eprod ty\_2Ehreal\_2Ehreal ty\_2Ehreal\_2Ehreal). \\
& ((p (ap (ap c\_2Erealax\_2Etreal\_eq V0x1) V1x2)) \Rightarrow ((p (ap (ap c\_2Erealax\_2Etreal\_lt \\
& V0x1) V2y)) \Leftrightarrow (p (ap (ap c\_2Erealax\_2Etreal\_lt V1x2) V2y))))))
\end{aligned}$$