

# thm\_2Erelation\_2ETFL\_\_INDUCTIVE\_\_INVARIANT\_\_ON\_\_WFRE (TMPn3Hxf6YhcWoXVhkgdLbAiCFDuMNdHJf2)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$   
of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x))$  then (the  $(\lambda x.x \in A \wedge p (ap P x))$ )  
of type  $\iota \Rightarrow \iota$ .

**Definition 4** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap V0P (ap (c\_2Emin\_2E\_40 A\_27a P))))$

**Definition 5** We define  $c\_2Ecombin\_2E\_2K$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0x \in A\_27a.(\lambda V1y \in A\_27b.V0x))$

**Definition 6** We define  $c\_2Ecombin\_2E\_2S$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.(\lambda V0f \in ((A\_27c^{A\_27b})^{A\_27a}))$

**Definition 7** We define  $c\_2Ecombin\_2E\_2I$  to be  $\lambda A\_27a : \iota.(ap (ap (c\_2Ecombin\_2E\_2S A\_27a A\_27a) A\_27a))$

Let  $c\_2Ebool\_2E\_2ARB : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Ebool\_2E\_2ARB A\_27a \in A\_27a \quad (1)$$

**Definition 8** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}) P))))$

**Definition 9** We define  $c\_2Ebool\_2E\_2F$  to be  $(ap (c\_2Ebool\_2E\_21 2)) (\lambda V0t \in 2.V0t)$ .

**Definition 10** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow q Q)$   
of type  $\iota$ .

**Definition 11** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2)) (\lambda V2t \in 2.V2t)))$

**Definition 12** We define  $c\_2Ebool\_2E\_2COND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.(\lambda V3t3 \in A\_27a.V2t2))))$

**Definition 13** We define  $c\_2Erelation\_2E\_2RESTRICT$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0f \in (A\_27b^{A\_27a}).\lambda V1g \in (A\_27b^{A\_27a}).(\lambda V2x \in A\_27a.(\lambda V3y \in A\_27b.V2x)))$

**Definition 14** We define `c_2Erelation_2ETC` to be  $\lambda A\_27a : \iota. \lambda V0R \in ((2^{A\_27a})^{A\_27a}). \lambda V1a \in A\_27a. \lambda V2b$

**Definition 15** We define `c_2Erelation_2Eapprox` to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0R \in ((2^{A\_27a})^{A\_27a}). \lambda V1M$

**Definition 16** We define `c_2Erelation_2Ethe_fun` to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0R \in ((2^{A\_27a})^{A\_27a}). \lambda V1M$

**Definition 17** We define `c_2Erelation_2EWFREC` to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0R \in ((2^{A\_27a})^{A\_27a}). \lambda V1M$

**Definition 18** We define `c_2Erelation_2EINDUCTIVE_INVARIANT_ON` to be

$\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0R \in ((2^{A\_27a})^{A\_27a}). \lambda V1D \in (2^{A\_27a}). \lambda V2P \in ((2^{A\_27b})^{A\_27a}). \lambda V3M \in ((A\_27b)^{A\_27a})$

**Definition 19** We define `c_2Ebool_2E_7E` to be  $(\lambda V0t \in 2. (ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E_7E$

**Definition 20** We define `c_2Erelation_2EWF` to be  $\lambda A\_27a : \iota. \lambda V0R \in ((2^{A\_27a})^{A\_27a}). (ap (c\_2Ebool\_2E_21$

**Definition 21** We define `c_2Ebool_2E_5C_2F` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c\_2Ebool\_2E_21 2) (\lambda V2t \in$

Assume the following.

$$True \tag{2}$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2))))) \tag{3}$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p V0t))) \tag{4}$$

Assume the following.

$$(\forall V0t \in 2. ((p V0t) \vee (\neg (p V0t)))) \tag{5}$$

Assume the following.

$$\forall A\_27a. nonempty A\_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in A\_27a. (p V0t) \Leftrightarrow (p V0t))) \tag{6}$$

Assume the following.

$$(\forall V0t \in 2. (((p V0t) \Rightarrow False) \Rightarrow (\neg (p V0t)))) \tag{7}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg (p V0t)) \Rightarrow ((p V0t) \Rightarrow False))) \tag{8}$$

Assume the following.

$$(\forall V0t \in 2. (((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))) \tag{9}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2. (((True \vee (p \ V0t)) \Leftrightarrow True) \wedge (((p \ V0t) \vee True) \Leftrightarrow True) \wedge \\
& (((False \vee (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \vee False) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \vee \\
& (p \ V0t)) \Leftrightarrow (p \ V0t))))))
\end{aligned} \tag{10}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2. (((True \Rightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Rightarrow True) \Leftrightarrow \\
& True) \wedge (((False \Rightarrow (p \ V0t)) \Leftrightarrow True) \wedge (((p \ V0t) \Rightarrow (p \ V0t)) \Leftrightarrow True) \wedge (( \\
& (p \ V0t) \Rightarrow False) \Leftrightarrow \neg(p \ V0t))))))
\end{aligned} \tag{11}$$

Assume the following.

$$\begin{aligned}
& ((\forall V0t \in 2. ((\neg(\neg(p \ V0t))) \Leftrightarrow (p \ V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge \\
& ((\neg False) \Leftrightarrow True)))
\end{aligned} \tag{12}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a. ((V0x = V0x) \Leftrightarrow \\
& True))
\end{aligned} \tag{13}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in \\
& A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x))))
\end{aligned} \tag{14}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2. (((True \Leftrightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow True) \Leftrightarrow \\
& (p \ V0t)) \wedge (((False \Leftrightarrow (p \ V0t)) \Leftrightarrow \neg(p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow False) \Leftrightarrow \neg( \\
& p \ V0t))))))
\end{aligned} \tag{15}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}). ((\neg(\forall V1x \in \\
& A\_27a. (p \ (ap \ V0P \ V1x)))) \Leftrightarrow (\exists V2x \in A\_27a. (\neg(p \ (ap \ V0P \ V2x))))))
\end{aligned} \tag{16}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2. (\forall V1B \in 2. (\forall V2C \in 2. (((p \ V0A) \vee ( \\
& (p \ V1B) \vee (p \ V2C)) \Leftrightarrow (((p \ V0A) \vee (p \ V1B)) \vee (p \ V2C))))))
\end{aligned} \tag{17}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2. (\forall V1B \in 2. (((p \ V0A) \vee (p \ V1B)) \Leftrightarrow ((p \ V1B) \vee \\
& (p \ V0A))))))
\end{aligned} \tag{18}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p \ V0A) \wedge (p \ V1B))) \Leftrightarrow ((\neg( \\
& p \ V0A)) \vee (\neg(p \ V1B)))) \wedge ((\neg((p \ V0A) \vee (p \ V1B))) \Leftrightarrow ((\neg(p \ V0A)) \wedge (\neg(p \ V1B))))))
\end{aligned} \tag{19}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. ((ap\ (c.2Ecombin\_2EI\ A\_27a)\ V0x) = V0x)) \quad (20)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0R \in ((2^{A\_27a})^{A\_27a}). (\forall V1P \in ((2^{A\_27b})^{A\_27a}). \\ & \quad (\forall V2M \in ((A\_27b^{A\_27a})^{(A\_27b^{A\_27a})}). (\forall V3D \in (2^{A\_27a}). \\ & \quad (\forall V4x \in A\_27a. (((p\ (ap\ (c.2Erelation\_2EWF\ A\_27a)\ V0R)) \wedge \\ & \quad ((p\ (ap\ (ap\ (ap\ (ap\ (ap\ (c.2Erelation\_2EINDUCTIVE\_INVARIANT\_ON \\ & \quad A\_27a\ A\_27b)\ V0R)\ V3D)\ V1P)\ V2M)) \wedge (p\ (ap\ V3D\ V4x)))))) \Rightarrow (p\ (ap\ (ap\ V1P \\ & \quad V4x)\ (ap\ (ap\ (ap\ (c.2Erelation\_2EWFREC\ A\_27a\ A\_27b)\ V0R)\ V2M)\ V4x))))))))) \quad (21) \end{aligned}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (22)$$

Assume the following.

$$(\forall V0A \in 2. ((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (23)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow ((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) \quad (24)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) \quad (25)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \quad (26)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p\ V0p) \Leftrightarrow ( \\ & \quad (p\ V1q) \Leftrightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (p\ V2r))) \wedge (((p\ V0p) \vee ((\neg \\ & \quad p\ V2r)) \vee (\neg(p\ V1q)))) \wedge (((p\ V1q) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V0p)))) \wedge ((p\ V2r) \vee \\ & \quad ((\neg(p\ V1q)) \vee (\neg(p\ V0p)))))))))) \quad (27) \end{aligned}$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p\ V0p) \Leftrightarrow ( \\ & \quad (p\ V1q) \wedge (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((\neg(p\ V1q)) \vee (\neg(p\ V2r)))) \wedge (((p\ V1q) \vee \\ & \quad (\neg(p\ V0p))) \wedge ((p\ V2r) \vee (\neg(p\ V0p)))))))))) \quad (28) \end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \vee (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee \neg(p \ V1q)) \wedge (((p \ V0p) \vee \neg(p \ V2r))) \wedge \\
& ((p \ V1q) \vee ((p \ V2r) \vee \neg(p \ V0p))))))))))
\end{aligned} \tag{29}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \Rightarrow (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee (p \ V1q)) \wedge (((p \ V0p) \vee \neg(p \ V2r))) \wedge (( \\
& \neg(p \ V1q) \vee ((p \ V2r) \vee \neg(p \ V0p))))))))))
\end{aligned} \tag{30}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (((p \ V0p) \Leftrightarrow \neg(p \ V1q))) \Leftrightarrow (((p \ V0p) \vee \\
& (p \ V1q)) \wedge (\neg(p \ V1q) \vee \neg(p \ V0p))))))
\end{aligned} \tag{31}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\neg((p \ V0p) \Rightarrow (p \ V1q))) \Rightarrow (p \ V0p))) \tag{32}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\neg((p \ V0p) \Rightarrow (p \ V1q))) \Rightarrow \neg(p \ V1q))) \tag{33}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\neg((p \ V0p) \vee (p \ V1q))) \Rightarrow \neg(p \ V0p))) \tag{34}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\neg((p \ V0p) \vee (p \ V1q))) \Rightarrow \neg(p \ V1q))) \tag{35}$$

Assume the following.

$$(\forall V0p \in 2. (\neg(\neg(p \ V0p))) \Rightarrow (p \ V0p)) \tag{36}$$

**Theorem 1**

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow \forall A\_27b.nonempty \ A\_27b \Rightarrow ( \\
& \forall V0f \in (A\_27b^{A\_27a}). (\forall V1R \in ((2^{A\_27a})^{A\_27a}). (\forall V2D \in \\
& (2^{A\_27a}). (\forall V3P \in ((2^{A\_27b})^{A\_27a}). (\forall V4M \in ((A\_27b^{A\_27a})^{(A\_27b^{A\_27a})}). \\
& (\forall V5x \in A\_27a. (((V0f = (ap (ap (c\_2Erelation\_2EWFREC \ A\_27a \\
& A\_27b) \ V1R) \ V4M)) \wedge ((p (ap (c\_2Erelation\_2EWF \ A\_27a) \ V1R)) \wedge ((p \\
& (ap (ap (ap (ap (c\_2Erelation\_2EINDUCTIVE\_INVARIANT\_ON \ A\_27a \\
& A\_27b) \ V1R) \ V2D) \ V3P) \ V4M)) \wedge (p (ap \ V2D \ V5x)))))) \Rightarrow (p (ap (ap \ V3P \ V5x) \\
& (ap \ V0f \ V5x)))))))))
\end{aligned}$$