

thm\_2Erelation\_2Eestablish\_CR  
(TMdDi6H4suakn5oK71Fw1R5BQHn92wHBK5i)

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**Definition 1** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. \text{inj\_o } (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Ebool_2E_2T` to be  $(\text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^2)) (\lambda V0x \in 2. V0x)) (\lambda V1x \in 2. V1x))$

**Definition 3** We define `c_2Emin_2E_40` to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p (\text{ap } P x)) \text{ then } (\text{the } (\lambda x. x \in A \wedge p x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 4** We define `c_2Ebool_2E_3F` to be  $\lambda A_{27a} : \iota. (\lambda V0P \in (2^{A_{27a}}). (\text{ap } V0P (\text{ap } (\text{c\_2Emin\_2E\_40 } A_{27a}))))$

**Definition 5** We define `c_2Emin_2E_3D_3D_3E` to be  $\lambda P \in 2. \lambda Q \in 2. \text{inj\_o } (p \Rightarrow q)$  of type  $\iota$ .

**Definition 6** We define `c_2Ebool_2E_21` to be  $\lambda A_{27a} : \iota. (\lambda V0P \in (2^{A_{27a}}). (\text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^{A_{27a}})) (\lambda V1t \in 2. V1t))))$

**Definition 7** We define `c_2Ebool_2E_2F_5C` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (\text{ap } (\text{c\_2Ebool\_2E\_21 } 2) (\lambda V2t \in 2. V2t))))$

**Definition 8** We define `c_2Erelation_2ERTC` to be  $\lambda A_{27a} : \iota. \lambda V0R \in ((2^{A_{27a}})^{A_{27a}}). \lambda V1a \in A_{27a}. \lambda V2b \in A_{27a}. \text{ap } (\text{ap } (\text{c\_2Emin\_2E\_40 } A_{27a}) (\lambda V3c \in A_{27a}. \text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^{A_{27a}})) (\lambda V4d \in 2. V4d)) (\lambda V5e \in 2. V5e)))) (\lambda V6f \in 2. V6f))$

**Definition 9** We define `c_2Erelation_2Ediamond` to be  $\lambda A_{27a} : \iota. \lambda V0R \in ((2^{A_{27a}})^{A_{27a}}). (\text{ap } (\text{c\_2Ebool\_2E\_21 } A_{27a}))$

**Definition 10** We define `c_2Erelation_2ECR` to be  $\lambda A_{27a} : \iota. \lambda V0R \in ((2^{A_{27a}})^{A_{27a}}). (\text{ap } (\text{c\_2Erelation\_2Ediamond } A_{27a}))$

**Definition 11** We define `c_2Ebool_2E_5C_2F` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (\text{ap } (\text{c\_2Ebool\_2E\_21 } 2) (\lambda V2t \in 2. V2t))))$

**Definition 12** We define `c_2Erelation_2ERC` to be  $\lambda A_{27a} : \iota. \lambda V0R \in ((2^{A_{27a}})^{A_{27a}}). \lambda V1x \in A_{27a}. \lambda V2y \in A_{27a}. \text{ap } (\text{ap } (\text{c\_2Emin\_2E\_40 } A_{27a}) (\lambda V3z \in A_{27a}. \text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^{A_{27a}})) (\lambda V4w \in 2. V4w)) (\lambda V5v \in 2. V5v)))) (\lambda V6u \in 2. V6u))$

**Definition 13** We define `c_2Erelation_2Ercdiamond` to be  $\lambda A_{27a} : \iota. \lambda V0R \in ((2^{A_{27a}})^{A_{27a}}). (\text{ap } (\text{c\_2Ebool\_2E\_21 } A_{27a}))$

**Definition 14** We define `c_2Erelation_2ETC` to be  $\lambda A_{27a} : \iota. \lambda V0R \in ((2^{A_{27a}})^{A_{27a}}). \lambda V1a \in A_{27a}. \lambda V2b \in A_{27a}. \text{ap } (\text{ap } (\text{c\_2Emin\_2E\_40 } A_{27a}) (\lambda V3c \in A_{27a}. \text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^{A_{27a}})) (\lambda V4d \in 2. V4d)) (\lambda V5e \in 2. V5e)))) (\lambda V6f \in 2. V6f))$

**Definition 15** We define `c_2Ebool_2EF` to be  $(\text{ap } (\text{c\_2Ebool\_2E\_21 } 2) (\lambda V0t \in 2. V0t))$ .

**Definition 16** We define `c_2Ebool_2E_7E` to be  $(\lambda V0t \in 2. (\text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D\_3D\_3E } V0t) (\text{c\_2Ebool\_2E\_21 } 2))))$

Assume the following.

$$True \quad (1)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (2)$$

Assume the following.

$$(\forall V0t \in 2.(((p V0t) \Rightarrow False) \Rightarrow (\neg(p V0t)))) \quad (3)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(p V0t)) \Rightarrow ((p V0t) \Rightarrow False))) \quad (4)$$

Assume the following.

$$(\forall V0t \in 2.(((True) \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))) \quad (5)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge ((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \quad (6)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (7)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (8)$$

Assume the following.

$$(\forall V0t \in 2.(((True) \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False) \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t)))) \quad (9)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).(\neg(\forall V1x \in A\_27a.(p(ap V0P V1x)))) \Leftrightarrow (\exists V2x \in A\_27a.(\neg(p(ap V0P V2x))))) \quad (10)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).(\forall V1Q \in (2^{A\_27a}).((\forall V2x \in A\_27a.((p(ap V0P V2x)) \wedge (p(ap V1Q V2x)))) \Leftrightarrow ((\forall V3x \in A\_27a.(p(ap V0P V3x))) \wedge (\forall V4x \in A\_27a.(p(ap V1Q V4x))))))) \quad (11)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0P \in (2^{A.27a}). (\forall V1Q \in \\ & (2^{A.27a}). ((\exists V2x \in A.27a. ((p\ (ap\ V0P\ V2x)) \vee (p\ (ap\ V1Q\ V2x)))))) \Leftrightarrow \\ & ((\exists V3x \in A.27a. (p\ (ap\ V0P\ V3x))) \vee (\exists V4x \in A.27a. (p\ ( \\ & \quad ap\ V1Q\ V4x)))))) \end{aligned} \quad (12)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((p\ V0A) \vee (p\ V1B)) \Leftrightarrow ((p\ V1B) \vee (p\ V0A)))))) \quad (13)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0R \in ((2^{A.27a})^{A.27a}). \\ & (((ap\ (c.2Erelation.2ERC\ A.27a)\ (ap\ (c.2Erelation.2ETC\ A.27a)\ \\ & V0R)) = (ap\ (c.2Erelation.2ERTC\ A.27a)\ V0R)) \wedge ((ap\ (c.2Erelation.2ETC \\ & A.27a)\ (ap\ (c.2Erelation.2ERC\ A.27a)\ V0R)) = (ap\ (c.2Erelation.2ERTC \\ & \quad A.27a)\ V0R)))) \end{aligned} \quad (14)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0R \in ((2^{A.27a})^{A.27a}). \\ & ((p\ (ap\ (c.2Erelation.2Ercdiamond\ A.27a)\ V0R)) \Leftrightarrow (p\ (ap\ (c.2Erelation.2Ediamond \\ & \quad A.27a)\ (ap\ (c.2Erelation.2ERC\ A.27a)\ V0R)))) \end{aligned} \quad (15)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0R \in ((2^{A.27a})^{A.27a}). \\ & ((p\ (ap\ (c.2Erelation.2Ediamond\ A.27a)\ V0R)) \Rightarrow (p\ (ap\ (c.2Erelation.2Ediamond \\ & \quad A.27a)\ (ap\ (c.2Erelation.2ERC\ A.27a)\ V0R)))) \end{aligned} \quad (16)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0R \in ((2^{A.27a})^{A.27a}). \\ & ((p\ (ap\ (c.2Erelation.2Ediamond\ A.27a)\ V0R)) \Rightarrow (p\ (ap\ (c.2Erelation.2Ediamond \\ & \quad A.27a)\ (ap\ (c.2Erelation.2ETC\ A.27a)\ V0R)))) \end{aligned} \quad (17)$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (18)$$

Assume the following.

$$(\forall V0A \in 2. ((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (19)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & (((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))) \end{aligned} \quad (20)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg(\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow (p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \quad (21)$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (22)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\ & (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg( \\ & p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\ & ((\neg(p V1q)) \vee (\neg(p V0p)))))))))) \end{aligned} \quad (23)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\ & (p V1q) \wedge (p V2r))) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \vee (\neg(p V2r)))) \wedge (((p V1q) \vee \\ & (\neg(p V0p))) \wedge ((p V2r) \vee (\neg(p V0p)))))))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\ & (p V1q) \vee (p V2r))) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge \\ & ((p V1q) \vee ((p V2r) \vee (\neg(p V0p)))))))) \end{aligned} \quad (25)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\ & (p V1q) \Rightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ( \\ & \neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p)))))))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee \\ & (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p)))))) \end{aligned} \quad (27)$$

### Theorem 1

$$\begin{aligned} & \forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0R \in ((2^{A.27a})^{A.27a}). \\ & (((p (ap (c.2Erelation.2Ercdiamond A.27a) V0R)) \Rightarrow (p (ap (c.2Erelation.2ECR \\ & A.27a) V0R))) \wedge ((p (ap (c.2Erelation.2Ediamond A.27a) V0R)) \Rightarrow ( \\ & p (ap (c.2Erelation.2ECR A.27a) V0R)))) \end{aligned}$$