

# thm\_2Eseq\_2ENEST\_\_LEMMA

(TMN2cPXsKpFteKXdCig5YGtKw7BLRuSPgCS)

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Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum)^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum} \tag{2}$$

**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A.27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A-27a})) (P) (V0t)) (V1t))$

**Definition 4** We define  $c\_2Ebool\_2E\_2F$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p \Rightarrow q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_27E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_2F))$

**Definition 7** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (omega^{ty\_2Enum\_2Enum}) \tag{3}$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (omega^{omega}) \tag{4}$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{omega}) \tag{5}$$

**Definition 8** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_num$

**Definition 9** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p (ap\ P\ x)) \text{ then } (the\ (\lambda x.x \in A \wedge p$   
of type  $\iota \Rightarrow \iota$ .

**Definition 10** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap\ V0P (ap\ (c\_2Emin\_2E\_40$

**Definition 11** We define  $c\_2Eprim\_rec\_2E\_3C$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

**Definition 12** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in$

**Definition 13** We define  $c\_2Earithmetic\_2E\_3C\_3D$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

Let  $ty\_2Ehreal\_2Ehreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \quad (6)$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \quad (7)$$

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erealax\_2Ereal \quad (8)$$

Let  $c\_2Erealax\_2Ereal\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})_{ty\_2Erealax}) \quad (9)$$

**Definition 14** We define  $c\_2Erealax\_2Ereal\_REP$  to be  $\lambda V0a \in ty\_2Erealax\_2Ereal.(ap\ (c\_2Emin\_2E\_40\ ($

Let  $c\_2Erealax\_2Etrealm\_add : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealm\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)_{ty\_2Ehreal\_2Ehreal})_{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})_{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal)} \quad (10)$$

Let  $c\_2Erealax\_2Etrealm\_eq : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etrealm\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})_{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})_{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal)} \quad (11)$$

Let  $c\_2Erealax\_2Ereal\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Ereal\_ABS\_CLASS \in (ty\_2Erealax\_2Ereal)^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})_{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)}} \quad (12)$$

**Definition 15** We define  $c\_2Erealax\_2Ereal\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)$

**Definition 16** We define  $c\_2Erealax\_2Ereal\_add$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax$

Let  $c\_2Erealax\_2Etreal\_neg : \iota$  be given. Assume the following.

$$\begin{aligned} c\_2Erealax\_2Etreal\_neg \in ((ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal \\ ty\_2Ehreal\_2Ehreal)^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)}) \end{aligned} \quad (13)$$

**Definition 17** We define  $c\_2Erealax\_2Ereal\_neg$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.(ap\ c\_2Erealax\_2Ereal$

**Definition 18** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

**Definition 19** We define  $c\_2Ereal\_2Ereal\_sub$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (14)$$

**Definition 20** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}) \quad (15)$$

Let  $c\_2Erealax\_2Etreal\_lt : \iota$  be given. Assume the following.

$$c\_2Erealax\_2Etreal\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal\ ty\_2Ehreal\_2Ehreal)})^{(ty\_2Epair\_2Eprod\ ty\_2Ehreal\_2Ehreal)}) \quad (16)$$

**Definition 21** We define  $c\_2Erealax\_2Ereal\_lt$  to be  $\lambda V0T1 \in ty\_2Erealax\_2Ereal.\lambda V1T2 \in ty\_2Erealax\_2Ereal$

**Definition 22** We define  $c\_2Ereal\_2Ereal\_lte$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.\lambda V1y \in ty\_2Erealax\_2Ereal$

**Definition 23** We define  $c\_2Ereal\_2Eabs$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal.(ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND$

**Definition 24** We define  $c\_2Earithmetic\_2E\_3E$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

**Definition 25** We define  $c\_2Earithmetic\_2E\_3E\_3D$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND \\ A\_27a\ A\_27b \in (A\_27b)^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)} \end{aligned} \quad (17)$$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST \\ A\_27a\ A\_27b \in (A\_27a)^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)} \end{aligned} \quad (18)$$

**Definition 26** We define  $c\_2Epair\_2EUNCURRY$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.\lambda V0f \in ((A\_27c)^{A\_27a$

Let  $ty\_2Emetric\_2Emetric : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Emetric\_2Emetric\ A0) \quad (19)$$

Let  $c\_2Emetric\_2Emetric : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Emetric\_2Emetric\ A\_27a \in ((ty\_2Emetric\_2Emetric\ A\_27a)^{(ty\_2Erealax\_2Ereal\ (ty\_2Epair\_2Eprod\ A\_27a\ A\_27a))}) \quad (20)$$

**Definition 27** We define  $c\_2Emetric\_2Emr1$  to be  $(ap\ (c\_2Emetric\_2Emetric\ ty\_2Erealax\_2Ereal)\ (ap\ (c\_2Epair\_2EABS\_prod\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal)))$ .

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (21)$$

**Definition 28** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0x \in A\_27a. \lambda V1y \in A\_27b. (ap\ (c\_2Epair\_2EABS\_prod\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal))$ .

Let  $c\_2Emetric\_2Edist : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Emetric\_2Edist\ A\_27a \in ((ty\_2Erealax\_2Ereal\ (ty\_2Epair\_2Eprod\ A\_27a\ A\_27a))^{(c\_2Emetric\_2Edist\ ty\_2Erealax\_2Ereal)}) \quad (22)$$

Let  $ty\_2Etopology\_2Etopology : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Etopology\_2Etopology\ A0) \quad (23)$$

Let  $c\_2Etopology\_2Etopology : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Etopology\_2Etopology\ A\_27a \in ((ty\_2Etopology\_2Etopology\ A\_27a)^{(2^{(2^{A\_27a})})}) \quad (24)$$

**Definition 29** We define  $c\_2Emetric\_2Emtop$  to be  $\lambda A\_27a : \iota. \lambda V0m \in (ty\_2Emetric\_2Emetric\ A\_27a). (ap\ (c\_2Epair\_2EABS\_prod\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal))$ .

Let  $c\_2Enets\_2Etends : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Enets\_2Etends\ A\_27a\ A\_27b \in (((2^{(ty\_2Epair\_2Eprod\ (ty\_2Etopology\_2Etopology\ A\_27a)\ (2^{A\_27b})^{A\_27b}))})^{A\_27a})^{(A\_27a)^{A\_27b}}) \quad (25)$$

**Definition 30** We define  $c\_2Eseq\_2E\_2D\_2D\_3E$  to be  $\lambda V0x \in (ty\_2Erealax\_2Ereal\ ty\_2Eenum\_2Eenum). \lambda V1x \in (ty\_2Erealax\_2Ereal\ ty\_2Eenum\_2Eenum)$ .

**Definition 31** We define  $c\_2Eseq\_2Elim$  to be  $\lambda V0f \in (ty\_2Erealax\_2Ereal\ ty\_2Eenum\_2Eenum). (ap\ (c\_2Emin\ ty\_2Erealax\_2Ereal\ ty\_2Eenum\_2Eenum))$ .

**Definition 32** We define  $c\_2Ereal\_2Ereal\_ge$  to be  $\lambda V0x \in ty\_2Erealax\_2Ereal. \lambda V1y \in ty\_2Erealax\_2Ereal. (c\_2Ereal\_2Ereal\_ge\ ty\_2Erealax\_2Ereal\ ty\_2Erealax\_2Ereal)$ .

**Definition 33** We define  $c\_2Eseq\_2Econvergent$  to be  $\lambda V0f \in (ty\_2Erealax\_2Ereal\ ty\_2Eenum\_2Eenum). (ap\ (c\_2Eseq\_2Econvergent\ ty\_2Erealax\_2Ereal\ ty\_2Eenum\_2Eenum))$ .

**Definition 34** We define  $c\_2Eseq\_2Emono$  to be  $\lambda V0f \in (ty\_2Erealax\_2Ereal\ ty\_2Eenum\_2Eenum). (ap\ (ap\ (c\_2Eseq\_2Econvergent\ ty\_2Erealax\_2Ereal\ ty\_2Eenum\_2Eenum)\ ty\_2Eenum\_2Eenum))$ .

Let  $c\_2Enets\_2Ebunded : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Enets\_2Ebunded \\ & A\_27a\ A\_27b \in ((2^{(A\_27a^{A\_27b})})(ty\_2Epair\_2Eprod\ (ty\_2Emetric\_2Emetric\ A\_27a)\ ((2^{A\_27b})^{A\_27b}))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & ((ap\ (ap\ c\_2Earithmetic\_2E\_2B\ c\_2Enum\_2E0)\ V0m) = V0m) \wedge (((ap\ ( \\ & ap\ c\_2Earithmetic\_2E\_2B\ V0m)\ c\_2Enum\_2E0) = V0m) \wedge (((ap\ (ap\ c\_2Earithmetic\_2E\_2B \\ & (ap\ c\_2Enum\_2ESUC\ V0m))\ V1n) = (ap\ c\_2Enum\_2ESUC\ (ap\ (ap\ c\_2Earithmetic\_2E\_2B \\ & V0m)\ V1n))) \wedge ((ap\ (ap\ c\_2Earithmetic\_2E\_2B\ V0m)\ (ap\ c\_2Enum\_2ESUC \\ & V1n)) = (ap\ c\_2Enum\_2ESUC\ (ap\ (ap\ c\_2Earithmetic\_2E\_2B\ V0m)\ V1n)))))) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & p\ (ap\ (ap\ c\_2Earithmetic\_2E\_3C\_3D\ V0m)\ (ap\ (ap\ c\_2Earithmetic\_2E\_2B \\ & V0m)\ V1n)))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} & (\forall V0n \in ty\_2Enum\_2Enum. (\forall V1m \in ty\_2Enum\_2Enum. ( \\ & (p\ (ap\ (ap\ c\_2Earithmetic\_2E\_3E\_3D\ V0n)\ V1m)) \Leftrightarrow (p\ (ap\ (ap\ c\_2Earithmetic\_2E\_3C\_3D \\ & V1m)\ V0n)))) \end{aligned} \quad (29)$$

Assume the following.

$$True \quad (30)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in \\ & A\_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2. ((\exists V1x \in \\ & A\_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge \\ & (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))) \end{aligned} \quad (33)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \vee (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \vee True) \Leftrightarrow True) \wedge \\ & (((False \vee (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee False) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee \\ & (p\ V0t)) \Leftrightarrow (p\ V0t)))) \end{aligned} \quad (34)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((p V0t) \Rightarrow False) \Leftrightarrow (\neg (p V0t)))))) \quad (35)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \quad (36)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (37)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg (p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg (p V0t)))))) \quad (38)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t1 \in A\_27a. (\forall V1t2 \in A\_27a. (((ap (ap (ap (c\_2Ebool\_2ECOND A\_27a) c\_2Ebool\_2ET) V0t1) V1t2) = V0t1) \wedge ((ap (ap (ap (c\_2Ebool\_2ECOND A\_27a) c\_2Ebool\_2EF) V0t1) V1t2) = V1t2)))))) \quad (39)$$

Assume the following.

$$(\forall V0P \in (2^{ty\_2Enum\_2Enum}). (((p (ap V0P c\_2Enum\_2E0)) \wedge (\forall V1n \in ty\_2Enum\_2Enum. ((p (ap V0P V1n)) \Rightarrow (p (ap V0P (ap c\_2Enum\_2ESUC V1n)))))) \Rightarrow (\forall V2n \in ty\_2Enum\_2Enum. (p (ap V0P V2n)))))) \quad (40)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. ((\neg (p (ap (ap c\_2Erealax\_2Ereal\_lte V0x) V1y))) \Leftrightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V1y) V0x)))))) \quad (41)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. ((\neg (p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V1y))) \Leftrightarrow (p (ap (ap c\_2Erealax\_2Ereal\_lte V1y) V0x)))))) \quad (42)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V0x))) \quad (43)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. ((p (ap (ap c\_2Erealax\_2Ereal\_lt V0x) V1y)) \Rightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V1y)))))) \quad (44)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal. (((p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V1y)) \wedge (p (ap (ap c\_2Erealax\_2Ereal\_lt V1y) V2z))) \Rightarrow (p (ap (ap c\_2Erealax\_2Ereal\_lt V0x) V2z)))))) \quad (45)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal. (((p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V1y)) \wedge (p (ap (ap c\_2Ereal\_2Ereal\_lte V1y) V2z))) \Rightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V2z)))))) \quad (46)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal. (((p (ap (ap c\_2Ereal\_2Ereal\_lte (ap (ap c\_2Erealax\_2Ereal\_add V0x) V1y)) (ap (ap c\_2Erealax\_2Ereal\_add V0x) V2z))) \Leftrightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V1y) V2z)))))) \quad (47)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. (\forall V2z \in ty\_2Erealax\_2Ereal. (((p (ap (ap c\_2Ereal\_2Ereal\_lte (ap (ap c\_2Erealax\_2Ereal\_add V0x) V2z)) (ap (ap c\_2Erealax\_2Ereal\_add V1y) V2z))) \Leftrightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V0x) V1y)))))) \quad (48)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. ((ap c\_2Erealax\_2Ereal\_neg (ap (ap c\_2Ereal\_2Ereal\_sub V0x) V1y)) = (ap (ap c\_2Ereal\_2Ereal\_sub V1y) V0x)))) \quad (49)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. ((p (ap (ap c\_2Erealax\_2Ereal\_lt (ap c\_2Ereal\_2Ereal\_of\_num c\_2Enum\_2E0)) (ap (ap c\_2Ereal\_2Ereal\_sub V0x) V1y))) \Leftrightarrow (p (ap (ap c\_2Erealax\_2Ereal\_lt V1y) V0x)))))) \quad (50)$$

Assume the following.

$$(\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. ((p (ap (ap c\_2Ereal\_2Ereal\_lte (ap c\_2Ereal\_2Ereal\_of\_num c\_2Enum\_2E0)) (ap (ap c\_2Ereal\_2Ereal\_sub V0x) V1y))) \Leftrightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V1y) V0x)))))) \quad (51)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in ty\_2Erealax\_2Ereal. (\forall V1y \in ty\_2Erealax\_2Ereal. \\ & ((p (ap (ap c\_2Ereal\_2Ereal\_lte (ap c\_2Erealax\_2Ereal\_neg V0x)) \\ & (ap c\_2Erealax\_2Ereal\_neg V1y))) \Leftrightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte \\ & V1y) V0x)))))) \end{aligned} \quad (52)$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \quad (53)$$

Assume the following.

$$(\forall V0A \in 2. ((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \quad (54)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\ & (((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))))) \end{aligned} \quad (55)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg(\neg(\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\ & ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))))) \end{aligned} \quad (56)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (57)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow ( \\ & (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg( \\ & p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\ & ((\neg(p V1q)) \vee (\neg(p V0p)))))))))) \end{aligned} \quad (58)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee \\ & (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p)))))) \end{aligned} \quad (59)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}). (\forall V1x0 \in \\ & ty\_2Erealax\_2Ereal. ((p (ap (ap c\_2Eseq\_2E\_2D\_2D\_3E V0x) V1x0)) \Leftrightarrow \\ & (\forall V2e \in ty\_2Erealax\_2Ereal. ((p (ap (ap c\_2Erealax\_2Ereal\_lt \\ & (ap c\_2Ereal\_2Ereal\_of\_num c\_2Enum\_2E0) V2e)) \Rightarrow (\exists V3N \in \\ & ty\_2Enum\_2Enum. (\forall V4n \in ty\_2Enum\_2Enum. ((p (ap (ap c\_2Earithmic\_2E\_3E\_3D \\ & V4n) V3N)) \Rightarrow (p (ap (ap c\_2Erealax\_2Ereal\_lt (ap c\_2Ereal\_2Eabs \\ & (ap (ap c\_2Ereal\_2Ereal\_sub (ap V0x V4n)) V1x0)) V2e)))))))))) \end{aligned} \quad (60)$$



Assume the following.

$$(\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}).((p (ap c\_2Eseq\_2Econvergent V0f)) \Leftrightarrow (p (ap (ap c\_2Eseq\_2E\_2D\_2D\_3E V0f) (ap c\_2Eseq\_2Elim V0f)))))) \quad (61)$$

Assume the following.

$$(\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}).((p (ap c\_2Eseq\_2Emono V0f)) \Leftrightarrow ((\forall V1n \in ty\_2Enum\_2Enum.(p (ap (ap c\_2Ereal\_2Ereal\_ge (ap V0f (ap c\_2Enum\_2ESUC V1n))) (ap V0f V1n)))) \vee (\forall V2n \in ty\_2Enum\_2Enum.(p (ap (ap c\_2Ereal\_2Ereal\_lte (ap V0f (ap c\_2Enum\_2ESUC V2n))) (ap V0f V2n)))))))))) \quad (62)$$

Assume the following.

$$(\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}).(\forall V1k \in ty\_2Erealax\_2Ereal.(\forall V2k\_27 \in ty\_2Erealax\_2Ereal.((\forall V3n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Ereal\_2Ereal\_lte V1k) (ap V0f V3n))) \wedge (p (ap (ap c\_2Ereal\_2Ereal\_lte (ap V0f V3n) V2k\_27)))))) \Rightarrow (p (ap (ap (c\_2Enets\_2Ebounded ty\_2Erealax\_2Ereal ty\_2Enum\_2Enum) (ap (ap (c\_2Epair\_2E\_2C (ty\_2Emetric\_2Emetric ty\_2Erealax\_2Ereal) ((2^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum})) c\_2Emetric\_2Emr1 c\_2Earithmic\_2E\_3E\_3D)) V0f)))))) \quad (63)$$

Assume the following.

$$(\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}).(((p (ap (ap (c\_2Enets\_2Ebounded ty\_2Erealax\_2Ereal ty\_2Enum\_2Enum) (ap (ap (c\_2Epair\_2E\_2C (ty\_2Emetric\_2Emetric ty\_2Erealax\_2Ereal) ((2^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum})) c\_2Emetric\_2Emr1 c\_2Earithmic\_2E\_3E\_3D)) V0f)) \wedge (p (ap c\_2Eseq\_2Emono V0f))) \Rightarrow (p (ap c\_2Eseq\_2Econvergent V0f)))))) \quad (64)$$

Assume the following.

$$(\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}).(\forall V1g \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}).(\forall V2l \in ty\_2Erealax\_2Ereal.(\forall V3m \in ty\_2Erealax\_2Ereal.(((p (ap (ap c\_2Eseq\_2E\_2D\_2D\_3E V0f) V2l)) \wedge ((p (ap (ap c\_2Eseq\_2E\_2D\_2D\_3E V1g) V3m)) \wedge (\exists V4N \in ty\_2Enum\_2Enum.(\forall V5n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmic\_2E\_3E\_3D V5n) V4N)) \Rightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte (ap V0f V5n)) (ap V1g V5n)))))))))) \Rightarrow (p (ap (ap c\_2Ereal\_2Ereal\_lte V2l) V3m)))))) \quad (65)$$

**Theorem 1**

$$\begin{aligned} & (\forall V0f \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}).(\forall V1g \in \\ & (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}).((\forall V2n \in ty\_2Enum\_2Enum. \\ & (p (ap (ap c\_2Ereal\_2Ereal\_ge (ap V0f (ap c\_2Enum\_2ESUC V2n))) \\ & (ap V0f V2n)))))) \wedge ((\forall V3n \in ty\_2Enum\_2Enum.(p (ap (ap c\_2Ereal\_2Ereal\_lte \\ & (ap V1g (ap c\_2Enum\_2ESUC V3n))) (ap V1g V3n)))))) \wedge (\forall V4n \in ty\_2Enum\_2Enum. \\ & (p (ap (ap c\_2Ereal\_2Ereal\_lte (ap V0f V4n)) (ap V1g V4n)))))) \Rightarrow \\ & (\exists V5l \in ty\_2Erealax\_2Ereal.(\exists V6m \in ty\_2Erealax\_2Ereal. \\ & ((p (ap (ap c\_2Ereal\_2Ereal\_lte V5l) V6m)) \wedge ((\forall V7n \in ty\_2Enum\_2Enum. \\ & (p (ap (ap c\_2Ereal\_2Ereal\_lte (ap V0f V7n)) V5l))) \wedge (p (ap (ap c\_2Eseq\_2E\_2D\_2D\_3E \\ & V0f) V5l))) \wedge ((\forall V8n \in ty\_2Enum\_2Enum.(p (ap (ap c\_2Ereal\_2Ereal\_lte \\ & V6m) (ap V1g V8n)))) \wedge (p (ap (ap c\_2Eseq\_2E\_2D\_2D\_3E V1g) V6m)))))))))) \end{aligned}$$