

thm\_2Eset\_\_relation\_2Eextend\_\_linear\_\_order  
(TMaL1SAzvjbWgdJyys9MCsLpDpD5Py4wFcY)

October 26, 2020

**Definition 1** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Ebool_2E_2T` to be  $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define `c_2Ebool_2E_2IN` to be  $\lambda A. 27a : \iota. (\lambda V0x \in A. 27a. (\lambda V1f \in (2^{A-27a}). (ap V1f V0x)))$

**Definition 4** We define `c_2Emin_2E_3D_3D_3E` to be  $\lambda P \in 2. \lambda Q \in 2. inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 5** We define `c_2Ebool_2E_21` to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (ap (ap (c_2Emin_2E_3D (2^{A-27a})) (\lambda V1t \in 2.V1t)) (\lambda V2t \in 2.V2t)))$

**Definition 6** We define `c_2Ebool_2E_5C_2F` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t))))$

**Definition 7** We define `c_2Ebool_2E_2F_5C` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t))))$

Let `ty_2Epair_2Eprod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. nonempty A0 \Rightarrow \forall A1. nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \quad (1)$$

Let `c_2Epair_2EABS_prod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. 27a. nonempty A. 27a \Rightarrow \forall A. 27b. nonempty A. 27b \Rightarrow c\_2Epair\_2EABS\_prod A. 27a A. 27b \in ((ty\_2Epair\_2Eprod A. 27a A. 27b)^{(2^{A-27b})^{A-27a}}) \quad (2)$$

**Definition 8** We define `c_2Epair_2E_2C` to be  $\lambda A. 27a : \iota. \lambda A. 27b : \iota. \lambda V0x \in A. 27a. \lambda V1y \in A. 27b. (ap (c_2Epair_2E_2C V0x V1y))$

Let `c_2Epred_set_2EGSPEC` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. 27a. nonempty A. 27a \Rightarrow \forall A. 27b. nonempty A. 27b \Rightarrow c\_2Epred\_set\_2EGSPEC A. 27a A. 27b \in ((2^{A-27a})^{((ty\_2Epair\_2Eprod A. 27a 2)^{A-27b})}) \quad (3)$$

**Definition 9** We define `c_2Epred_set_2EUNION` to be  $\lambda A. 27a : \iota. \lambda V0s \in (2^{A-27a}). \lambda V1t \in (2^{A-27a}). (ap (c_2Epred\_set\_2EGSPEC V0s V1t))$

**Definition 10** We define `c_2Ebool_2EF` to be  $(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V0t \in 2.V0t))$ .

**Definition 11** We define `c_2Epred_set_2EEMPTY` to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2EF)$ .

**Definition 12** We define `c_2Epred_set_2EINSERT` to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2EEMPTY\ A\_27a\ x)\ V1s)$ .

**Definition 13** We define `c_2Ebool_2E_7E` to be  $(\lambda V0t \in 2.(ap\ (ap\ c\_2Emin\_2E\_3D\_3D\_3E\ V0t)\ c\_2Ebool\_2EF))$ .

**Definition 14** We define `c_2Emin_2E_40` to be  $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p\ (ap\ P\ x)) \text{ then } (the\ (\lambda x.x \in A \wedge P\ x)\ \text{of type } \iota \Rightarrow \iota)$ .

**Definition 15** We define `c_2Ebool_2E_3F` to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_2Emin\_2E\_40\ A\_27a\ P))))$ .

**Definition 16** We define `c_2Eset_relation_2Eantisym` to be  $\lambda A\_27a : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$ .

**Definition 17** We define `c_2Eset_relation_2Etransitive` to be  $\lambda A\_27a : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$ .

**Definition 18** We define `c_2Eset_relation_2ERange` to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$ .

**Definition 19** We define `c_2Eset_relation_2EDomain` to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$ .

**Definition 20** We define `c_2Epred_set_2ESUBSET` to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2EEMPTY\ A\_27a\ s)\ V1t)$ .

**Definition 21** We define `c_2Eset_relation_2Elinear_order` to be  $\lambda A\_27a : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$ .

Assume the following.

$$True \tag{4}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \tag{5}$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p\ V0t))) \tag{6}$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \tag{7}$$

Assume the following.

$$(\forall V0t \in 2.(((True \vee (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \vee True) \Leftrightarrow True) \wedge (((False \vee (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee False) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \tag{8}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (9)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \quad (10)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.((V0x = V0x) \Leftrightarrow True)) \quad (11)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.(\forall V1y \in A.27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (12)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (13)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in (2^{A.27a}).((\forall V2x \in A.27a.((p V0P) \vee (p (ap V1Q V2x)))) \Leftrightarrow ((p V0P) \vee (\forall V3x \in A.27a.(p (ap V1Q V3x))))))) \quad (14)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p V1B) \wedge (p V2C)) \vee (p V0A)) \Leftrightarrow (((p V1B) \vee (p V0A)) \wedge ((p V2C) \vee (p V0A)))))) \quad (15)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0y \in A.27a.(\forall V1P \in (2^{A.27a}).((p (ap (ap (c.2Ebool.2EIN A.27a) V0y) (ap (c.2Epred\_set.2EGSPEC A.27a A.27a) (\lambda V2x \in A.27a.(ap (ap (c.2Epair.2E.2C A.27a 2) V2x) (ap V1P V2x)))))) \Leftrightarrow (p (ap V1P V0y)))))) \quad (16)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow \forall A.27b.nonempty \ A.27b \Rightarrow (\forall V0a \in A.27a.(\forall V1b \in A.27b.(\forall V2x \in A.27b.(\forall V3P \in (2^{A.27a}).((p (ap (ap (c.2Ebool.2EIN (ty.2Epair.2Eprod A.27a A.27b)) (ap (ap (c.2Epair.2E.2C A.27a A.27b) V0a) V1b)) (ap (c.2Epred\_set.2EGSPEC (ty.2Epair.2Eprod A.27a A.27b) A.27a) (\lambda V4y \in A.27a.(ap (ap (c.2Epair.2E.2C (ty.2Epair.2Eprod A.27a A.27b) 2) (ap (ap (c.2Epair.2E.2C A.27a A.27b) V4y) V2x)) (ap V3P V4y)))))) \Leftrightarrow ((p (ap V3P V0a)) \wedge (V1b = V2x)))))) \quad (17)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in \\ & (2^{A\_27a}). (\forall V2x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a) \\ & V2x)\ (ap\ (ap\ (c\_2Epred\_set\_2EUNION\ A\_27a)\ V0s)\ V1t))) \Leftrightarrow ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ V0s)) \vee (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ V1t)))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in \\ & A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V0x)\ (ap\ (ap\ (c\_2Epred\_set\_2EINSERT\ A\_27a)\ V1y)\ (c\_2Epred\_set\_2EEMPTY\ A\_27a)))) \Leftrightarrow (V0x = V1y))) \end{aligned} \quad (19)$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (20)$$

Assume the following.

$$(\forall V0A \in 2. ((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (21)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & (((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))) \end{aligned} \quad (22)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))) \end{aligned} \quad (23)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p\ V0p) \Leftrightarrow ( \\ & (p\ V1q) \Leftrightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (p\ V2r))) \wedge (((p\ V0p) \vee ((\neg \\ & (p\ V2r)) \vee (\neg(p\ V1q)))) \wedge (((p\ V1q) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V0p)))) \wedge ((p\ V2r) \vee \\ & ((\neg(p\ V1q)) \vee (\neg(p\ V0p)))))))))) \end{aligned} \quad (25)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p\ V0p) \Leftrightarrow ( \\ & (p\ V1q) \wedge (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((\neg(p\ V1q)) \vee (\neg(p\ V2r)))) \wedge (((p\ V1q) \vee \\ & (\neg(p\ V0p))) \wedge ((p\ V2r) \vee (\neg(p\ V0p)))))))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow ( \\
& (p V1q) \vee (p V2r))) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge \\
& ((p V1q) \vee ((p V2r) \vee (\neg(p V0p))))))))))
\end{aligned} \tag{27}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow ( \\
& (p V1q) \Rightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ( \\
& \neg(p V1q) \vee ((p V2r) \vee (\neg(p V0p))))))))))
\end{aligned} \tag{28}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee \\
& (p V1q)) \wedge ((\neg(p V1q) \vee (\neg(p V0p))))))
\end{aligned} \tag{29}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (p V0p))) \tag{30}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (\neg(p V1q)))) \tag{31}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V0p)))) \tag{32}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V1q)))) \tag{33}$$

Assume the following.

$$(\forall V0p \in 2. ((\neg(\neg(p V0p))) \Rightarrow (p V0p))) \tag{34}$$

**Theorem 1**

$$\begin{aligned}
& \forall A.27a.nonempty A.27a \Rightarrow (\forall V0r \in (2^{(ty.2Epair.2Eprod A.27a A.27a)}), \\
& (\forall V1s \in (2^{A.27a}). (\forall V2x \in A.27a. (((\neg(p (ap (ap (c.2Ebool.2EIN \\
& A.27a) V2x) V1s))) \wedge (p (ap (ap (c.2Eset\_relation.2Elinear\_order \\
& A.27a) V0r) V1s))) \Rightarrow (p (ap (ap (c.2Eset\_relation.2Elinear\_order \\
& A.27a) (ap (ap (c.2Epred\_set.2EUNION (ty.2Epair.2Eprod A.27a \\
& A.27a)) V0r) (ap (c.2Epred\_set.2EGSPEC (ty.2Epair.2Eprod A.27a \\
& A.27a) A.27a) (\lambda V3y \in A.27a. (ap (ap (c.2Epair.2E.2C (ty.2Epair.2Eprod \\
& A.27a A.27a) 2) (ap (ap (c.2Epair.2E.2C A.27a A.27a) V3y) V2x)) \\
& (ap (ap (c.2Ebool.2EIN A.27a) V3y) (ap (ap (c.2Epred\_set.2EUNION \\
& A.27a) V1s) (ap (ap (c.2Epred\_set.2EINSERT A.27a) V2x) (c.2Epred\_set.2EEMPTY \\
& A.27a)))))))))) (ap (ap (c.2Epred\_set.2EUNION A.27a) V1s) (ap \\
& (ap (c.2Epred\_set.2EINSERT A.27a) V2x) (c.2Epred\_set.2EEMPTY \\
& A.27a))))))))))
\end{aligned}$$