

thm\_2Esptree\_2Edomain\_\_inter  
(TMJiABr3YiiX7EGAJnFdV8p8cB7sJEFtEnh)

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**Definition 1** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p \Rightarrow Q)$  of type  $\iota$ .

**Definition 2** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 4** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 5** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 6** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

**Definition 7** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2EF$

**Definition 8** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

**Definition 9** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap V1f V0x)))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EABS\_prod A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a A\_27b)^{(2^{A\_27b})^{A\_27a}}) \tag{2}$$

**Definition 10** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2E$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC A\_27a A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod A\_27a 2)^{A\_27b}}) \tag{3}$$

**Definition 11** We define  $c\_2Epred\_set\_2EINTER$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap (c\_2Eone\_2Eone : \iota$   
 Let  $ty\_2Eone\_2Eone : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Eone\_2Eone \tag{4}$$

**Definition 12** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.\mathbf{if} (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge P x))$   
 of type  $\iota \Rightarrow \iota$ .

**Definition 13** We define  $c\_2Eone\_2Eone$  to be  $(ap (c\_2Emin\_2E\_40\ ty\_2Eone\_2Eone) (\lambda V0x \in ty\_2Eone\_2Eone.2Eone\_2Eone))$

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Esum\_2Esum\ A0\ A1) \tag{5}$$

Let  $c\_2Esum\_2EABS\_sum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b \in ((ty\_2Esum\_2Esum\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})^2}) \tag{6}$$

**Definition 14** We define  $c\_2Esum\_2EINR$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27b.(ap (c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b))$

Let  $ty\_2Eoption\_2Eoption : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Eoption\_2Eoption\ A0) \tag{7}$$

Let  $c\_2Eoption\_2Eoption\_ABS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eoption\_2Eoption\_ABS\ A\_27a \in ((ty\_2Eoption\_2Eoption\ A\_27a)^{(ty\_2Esum\_2Esum\ A\_27a\ ty\_2Eone\_2Eone)}) \tag{8}$$

**Definition 15** We define  $c\_2Eoption\_2ENONE$  to be  $\lambda A\_27a : \iota.(ap (c\_2Eoption\_2Eoption\_ABS\ A\_27a))$

Let  $c\_2Eoption\_2Eoption\_CASE : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Eoption\_2Eoption\_CASE\ A\_27a\ A\_27b \in (((A\_27b^{(A\_27b^{A\_27a})})^{A\_27b})^{(ty\_2Eoption\_2Eoption\ A\_27a)}) \tag{9}$$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{10}$$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{11}$$

**Definition 16** We define  $c\_2Epair\_2Epair\_CASE$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda A\_27c : \iota. \lambda V0p \in (ty\_2Epair$ .  
Let  $ty\_2Esptree\_2Espt : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Esptree\_2Espt\ A0) \quad (12)$$

Let  $c\_2Esptree\_2Einter : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esptree\_2Einter \\ & A\_27a\ A\_27b \in (((ty\_2Esptree\_2Espt\ A\_27a)^{(ty\_2Esptree\_2Espt\ A\_27b)})^{(ty\_2Esptree\_2Espt\ A\_27a)}) \end{aligned} \quad (13)$$

**Definition 17** We define  $c\_2Esum\_2EINL$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0e \in A\_27a.(ap\ (c\_2Esum\_2EABS$

**Definition 18** We define  $c\_2Eoption\_2ESOME$  to be  $\lambda A\_27a : \iota. \lambda V0x \in A\_27a.(ap\ (c\_2Eoption\_2Eoption$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (14)$$

Let  $c\_2Esptree\_2Elookup : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Esptree\_2Elookup\ A\_27a \in (((ty\_2Eoption\_2Eoption\ A\_27a)^{(ty\_2Esptree\_2Espt\ A\_27a)})^{ty\_2Enum\_2Enum}) \quad (15)$$

**Definition 19** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota. (\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_2Emin\_2E\_40$

Let  $c\_2Esptree\_2Edomain : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Esptree\_2Edomain\ A\_27a \in ((2^{ty\_2Enum\_2Enum})^{(ty\_2Esptree\_2Espt\ A\_27a)}) \quad (16)$$

Assume the following.

$$True \quad (17)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (18)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p\ V0t))) \quad (19)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2. ((\exists V1x \in A\_27a. (p\ V0t)) \Leftrightarrow (p\ V0t))) \quad (20)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge \\ & (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \end{aligned} \quad (21)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \quad (22)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (23)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t))))) \quad (24)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0a \in A\_27a. (\exists V1x \in A\_27a. (V1x = V0a))) \quad (25)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0opt \in (ty\_2Eoption\_2Eoption A\_27a). ((V0opt = (c\_2Eoption\_2ENONE A\_27a)) \vee (\exists V1x \in A\_27a. (V0opt = (ap (c\_2Eoption\_2ESOME A\_27a) V1x)))))) \quad (26)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow ((\forall V0v \in A\_27b. (\forall V1f \in (A\_27b^{A\_27a}). ((ap (ap (ap (c\_2Eoption\_2Eoption\_CASE A\_27a A\_27b) (c\_2Eoption\_2ENONE A\_27a)) V0v) V1f) = V0v))) \wedge (\forall V2x \in A\_27a. (\forall V3v \in A\_27b. (\forall V4f \in (A\_27b^{A\_27a}). ((ap (ap (ap (c\_2Eoption\_2Eoption\_CASE A\_27a A\_27b) (ap (c\_2Eoption\_2ESOME A\_27a) V2x)) V3v) V4f) = (ap V4f V2x))))))) \quad (27)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27a. (((ap (c\_2Eoption\_2ESOME A\_27a) V0x) = (ap (c\_2Eoption\_2ESOME A\_27a) V1y)) \Leftrightarrow (V0x = V1y)))) \quad (28)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a. (\neg((c\_2Eoption\_2ENONE A\_27a) = (ap (c\_2Eoption\_2ESOME A\_27a) V0x)))) \quad (29)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow \forall A\_27c.nonempty A\_27c \Rightarrow (\forall V0x \in A\_27b. (\forall V1y \in A\_27c. (\forall V2f \in ((A\_27a^{A\_27c})^{A\_27b}). ((ap (ap (c\_2Epair\_2Epair\_CASE A\_27a A\_27b A\_27c) (ap (ap (c\_2Epair\_2E\_2C A\_27b A\_27c) V0x) V1y)) V2f) = (ap (ap V2f V0x) V1y)))))) \quad (30)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in \\ & (2^{A\_27a}). ((V0s = V1t) \Leftrightarrow (\forall V2x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN \\ & A\_27a)\ V2x)\ V0s)) \Leftrightarrow (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ V1t))))))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in \\ & (2^{A\_27a}). (\forall V2x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ \\ & (ap\ (ap\ (c\_2Epred\_set\_2EINTER\ A\_27a)\ V0s)\ V1t))) \Leftrightarrow ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ V0s)) \wedge (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ V1t)))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \forall V0m1 \in (ty\_2Esptree\_2Espt\ A\_27a). (\forall V1m2 \in (ty\_2Esptree\_2Espt\ A\_27b). (\forall V2k \in ty\_2Enum\_2Enum. ((ap\ (ap\ (c\_2Esptree\_2Elookup\ A\_27a)\ V2k)\ (ap\ (ap\ (c\_2Esptree\_2Einter\ A\_27a\ A\_27b)\ V0m1)\ V1m2))) = \\ & (ap\ (ap\ (c\_2Epair\_2Epair\_CASE\ (ty\_2Eoption\_2Eoption\ A\_27a)\ (ty\_2Eoption\_2Eoption\ A\_27a)\ (ty\_2Eoption\_2Eoption\ A\_27b))) \\ & (ap\ (ap\ (c\_2Epair\_2E\_2C\ (ty\_2Eoption\_2Eoption\ A\_27a)\ (ty\_2Eoption\_2Eoption\ A\_27b))) (ap\ (ap\ (c\_2Esptree\_2Elookup\ A\_27a)\ V2k)\ V0m1)) (ap\ (ap\ (c\_2Esptree\_2Elookup\ A\_27b)\ V2k)\ V1m2))) (\lambda V3v3 \in (ty\_2Eoption\_2Eoption\ A\_27a). (\lambda V4v4 \in (ty\_2Eoption\_2Eoption\ A\_27b). (ap\ (ap\ (ap\ (c\_2Eoption\_2Eoption\_CASE\ A\_27a\ (ty\_2Eoption\_2Eoption\ A\_27a))\ V3v3)\ (c\_2Eoption\_2ENONE\ A\_27a)) (\lambda V5v \in A\_27a. (ap\ (ap\ (ap\ (c\_2Eoption\_2Eoption\_CASE\ A\_27b\ (ty\_2Eoption\_2Eoption\ A\_27a))\ V4v4)\ (c\_2Eoption\_2ENONE\ A\_27a)) (\lambda V6w \in A\_27b. (ap\ (c\_2Eoption\_2ESOME\ A\_27a)\ V5v)))))))))) \end{aligned} \quad (33)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in (ty\_2Esptree\_2Espt\ A\_27a). (\forall V1k \in ty\_2Enum\_2Enum. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ ty\_2Enum\_2Enum)\ V1k)\ (ap\ (c\_2Esptree\_2Edomain\ A\_27a)\ V0t))) \Leftrightarrow \\ & (\exists V2v \in A\_27a. ((ap\ (ap\ (c\_2Esptree\_2Elookup\ A\_27a)\ V1k)\ V0t) = (ap\ (c\_2Eoption\_2ESOME\ A\_27a)\ V2v)))))) \end{aligned} \quad (34)$$

**Theorem 1**

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \forall V0t1 \in (ty\_2Esptree\_2Espt\ A\_27a). (\forall V1t2 \in (ty\_2Esptree\_2Espt\ A\_27b). ((ap\ (c\_2Esptree\_2Edomain\ A\_27a)\ (ap\ (ap\ (c\_2Esptree\_2Einter\ A\_27a\ A\_27b)\ V0t1)\ V1t2))) = (ap\ (ap\ (c\_2Epred\_set\_2EINTER\ ty\_2Enum\_2Enum)\ (ap\ (c\_2Esptree\_2Edomain\ A\_27a)\ V0t1)) (ap\ (c\_2Esptree\_2Edomain\ A\_27b)\ V1t2)))) \end{aligned}$$