thm_2Esptree_2Elookup__inter__assoc (TMacParFSd3VGwEMHPxk4sTRnaUARGYxZLm)

October 26, 2020

Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 3 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 4 *We define* c_2Ebool_2E_3F *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (\mathsf{c\_2Emin\_2E\_40}\ A$

Definition 5 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 6 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

Definition 7 *We define* c_2Ebool_2E_5C_2F *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Definition 8 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Let $ty\_2Eone\_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Eone\_2Eone \tag{1}$$

Let $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Esum\_2Esum\ A0\ A1) \tag{2}$$

Let $c\_2Esum\_2EABS\_\_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EABS\_\_sum\ A\_27a\ A\_27b \in ((ty\_2Esum\_2Esum\ A\_27a\ A\_27b)^{(((2^{A\_27b})^{A\_27a})^2)}) \tag{3}$$

Definition 9 *We define* c_2Esum_2EINL *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27a.(ap\ (c\_2Esum\_2EABS\_$

Let $ty\_2Eoption\_2Eoption : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Eoption\_2Eoption\ A0) \tag{4}$$

Let $c\_2Eoption\_2Eoption\_\_ABS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eoption\_2Eoption\_\_ABS\ A\_27a \in ((ty\_2Eoption\_2Eoption\ A\_27a)^{(ty\_2Esum\_2Esum\ A\_27a\ ty\_2Eone\_2Eone)}) \tag{5}$$

Definition 10 *We define* c_2Eoption_2ESOME *to be* $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.(ap\ (c\_2Eoption\_2Eoption\_\_$

Definition 11 *We define* c_2Eone_2Eone *to be* $(ap\ (\mathsf{c\_2Emin\_2E\_40}\ ty\_2Eone\_2Eone)\ (\lambda V0x \in ty\_2Eone\_2$

Definition 12 *We define* c_2Ebool_2EF *to be* $(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 13 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \mathsf{c\_2Emin\_2E\_3D\_3D\_3E}\ V0t)\ \mathsf{c\_2Ebool\_2E}$

Definition 14 *We define* c_2Esum_2EINR *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27b.(ap\ (c\_2Esum\_2EABS$

Definition 15 *We define* c_2Eoption_2ENONE *to be* $\lambda A\_27a : \iota.(ap\ (c\_2Eoption\_2Eoption\_\_ABS\ A\_27a)\ ($

Let $c\_2Eoption\_2Eoption\_\_CASE : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Eoption\_2Eoption\_\_CASE\ A\_27a\ A\_27b \in (((A\_27b^{(A\_27b^{A\_27a})})^{A\_27b})^{(ty\_2Eoption\_2Eoption\ A\_27a)}) \tag{6}$$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{7}$$

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})}) \tag{8}$$

Definition 16 *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2$

Let $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{9}$$

Let $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{10}$$

Definition 17 *We define* $\mathsf{c\_2Epair\_2Epair\_\_CASE}$ *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.\lambda V0p \in (ty\_2Epair\_$

Let $ty\_2Esptree\_2Espt : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Esptree\_2Espt\ A0) \tag{11}$$

Let $c\_2Esptree\_2Einter : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esptree\_2Einter\ A\_27a\ A\_27b \in (((ty\_2Esptree\_2Espt\ A\_27a)^{(ty\_2Esptree\_2Espt\ A\_27b)})^{(ty\_2Esptree\_2Espt\ A\_27a)}) \tag{12}$$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{13}$$

Let $c\_2Esptree\_2Elookup : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Esptree\_2Elookup\ A\_27a \in (((ty\_2Eoption\_2Eoption\ A\_27a)^{(ty\_2Esptree\_2Espt\ A\_27a)})^{ty\_2Enum\_2Enum}) \tag{14}$$

Assume the following.

$$True \tag{15}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \tag{16}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0opt \in (ty\_2Eoption\_2Eoption\ A\_27a).((V0opt = (\mathsf{c\_2Eoption\_2ENONE}\ A\_27a)) \lor (\exists V1x \in A\_27a.(V0opt = (ap\ (\mathsf{c\_2Eoption\_2ESOME}\ A\_27a)\ V1x))))) \tag{17}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ((\forall V0v \in A\_27b.(\forall V1f \in (A\_27b^{A\_27a}).((ap\ (ap\ (ap\ (c\_2Eoption\_2Eoption\_\_CASE\ A\_27a\ A\_27b)\ (\mathsf{c\_2Eoption\_2ENONE}\ A\_27a))\ V0v)\ V1f) = V0v))) \land (\forall V2x \in A\_27a.(\forall V3v \in A\_27b.(\forall V4f \in (A\_27b^{A\_27a}).((ap\ (ap\ (ap\ (c\_2Eoption\_2Eoption\_\_CASE\ A\_27a\ A\_27b)\ (ap\ (\mathsf{c\_2Eoption\_2ESOME}\ A\_27a)\ V2x))\ V3v)\ V4f) = (ap\ V4f\ V2x)))))) \tag{18}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.(((ap\ (\mathsf{c\_2Eoption\_2ESOME}\ A\_27a)\ V0x) = (ap\ (\mathsf{c\_2Eoption\_2ESOME}\ A\_27a)\ V1y)) \Leftrightarrow (V0x = V1y)))) \tag{19}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow \forall A\_27c. nonempty\ A\_27c \Rightarrow (\forall V0x \in A\_27b.(\forall V1y \in A\_27c.(\forall V2f \in ((A\_27a^{A\_27c})^{A\_27b}).((ap\ (ap\ (\mathsf{c\_2Epair\_2Epair\_\_CASE}\ A\_27a\ A\_27b\ A\_27c)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ A\_27b\ A\_27c)\ V0x)\ V1y))\ V2f) = (ap\ (ap\ V2f\ V0x)\ V1y))))) \tag{20}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0m1 \in (ty\_2Esptree\_2Espt\ A\_27a).(\forall V1m2 \in (ty\_2Esptree\_2Espt\ A\_27b).(\forall V2k \in ty\_2Enum\_2Enum.((ap\ (ap\ (c\_2Esptree\_2Elookup\ A\_27a)\ V2k)\ (ap\ (ap\ (c\_2Esptree\_2Einter\ A\_27a\ A\_27b)\ V0m1)\ V1m2)) = (ap\ (ap\ (\mathsf{c\_2Epair\_2Epair\_\_CASE}\ (ty\_2Eoption\_2Eoption\ A\_27a)\ (ty\_2Eoption\_2Eoption\ A\_27a)\ (ty\_2Eoption\_2Eoption\ A\_27b))\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Eoption\_2Eoption\ A\_27a)\ (ty\_2Eoption\_2Eoption\ A\_27b))\ (ap\ (ap\ (c\_2Esptree\_2Elookup\ A\_27a)\ V2k)\ V0m1))\ (ap\ (ap\ (c\_2Esptree\_2Elookup\ A\_27b)\ V2k)\ V1m2)))\ (\lambda V3v3 \in (ty\_2Eoption\_2Eoption\ A\_27a).(\lambda V4v4 \in (ty\_2Eoption\_2Eoption\ A\_27b).(ap\ (ap\ (ap\ (c\_2Eoption\_2Eoption\_\_CASE\ A\_27a\ (ty\_2Eoption\_2Eoption\ A\_27a))\ V3v3)\ (\mathsf{c\_2Eoption\_2ENONE}\ A\_27a))\ (\lambda V5v \in A\_27a.(ap\ (ap\ (ap\ (c\_2Eoption\_2Eoption\_\_CASE\ A\_27b\ (ty\_2Eoption\_2Eoption\ A\_27a))\ V4v4)\ (\mathsf{c\_2Eoption\_2ENONE}\ A\_27a))\ (\lambda V6w \in A\_27b.(ap\ (\mathsf{c\_2Eoption\_2ESOME}\ A\_27a)\ V5v))))))))))) \tag{21}$$

Theorem 1

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow \forall A\_27c. nonempty\ A\_27c \Rightarrow (\forall V0x \in ty\_2Enum\_2Enum.(\forall V1t1 \in (ty\_2Esptree\_2Espt\ A\_27a).(\forall V2t2 \in (ty\_2Esptree\_2Espt\ A\_27b).(\forall V3t3 \in (ty\_2Esptree\_2Espt\ A\_27c).((ap\ (ap\ (c\_2Esptree\_2Elookup\ A\_27a)\ V0x)\ (ap\ (ap\ (c\_2Esptree\_2Einter\ A\_27a\ A\_27b)\ V1t1)\ (ap\ (ap\ (c\_2Esptree\_2Einter\ A\_27b\ A\_27c)\ V2t2)\ V3t3))) = (ap\ (ap\ (c\_2Esptree\_2Elookup\ A\_27a)\ V0x)\ (ap\ (ap\ (c\_2Esptree\_2Einter\ A\_27a\ A\_27c)\ (ap\ (ap\ (c\_2Esptree\_2Einter\ A\_27a\ A\_27b)\ V1t1)\ V2t2))\ V3t3)))))))$$