

# thm\_2Etopology\_2ECLOSED\_\_IN\_\_UNION (TMR9LUNgB3A4YBV7g7uVxnL2itFo8iubYF9)

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**Definition 1** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. \text{inj\_o } (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Ebool_2E_2T` to be  $(\text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^2)) (\lambda V0x \in 2. V0x)) (\lambda V1x \in 2. V1x))$

**Definition 3** We define `c_2Ebool_2E_21` to be  $\lambda A. \lambda 27a : \iota. (\lambda V0P \in (2^{A-27a}). (\text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^{A-27a})) (\lambda V1f \in 2. V1f)) (\lambda V0x \in 2. V0x)))$

**Definition 4** We define `c_2Ebool_2E_2F` to be  $(\text{ap } (\text{c\_2Ebool\_2E\_21 } 2) (\lambda V0t \in 2. V0t))$ .

**Definition 5** We define `c_2Emin_2E_3D_3D_3E` to be  $\lambda P \in 2. \lambda Q \in 2. \text{inj\_o } (p \Rightarrow q)$  of type  $\iota$ .

**Definition 6** We define `c_2Ebool_2E_5C_2F` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (\text{ap } (\text{c\_2Ebool\_2E\_21 } 2) (\lambda V2t \in 2. V2t))))$

**Definition 7** We define `c_2Ebool_2E_2IN` to be  $\lambda A. \lambda 27a : \iota. (\lambda V0x \in A. 27a. (\lambda V1f \in (2^{A-27a}). (\text{ap } V1f V0x)))$

**Definition 8** We define `c_2Ebool_2E_2F_5C` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (\text{ap } (\text{c\_2Ebool\_2E\_21 } 2) (\lambda V2t \in 2. V2t))))$

Let `ty_2Epair_2Eprod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. \text{nonempty } A0 \Rightarrow \forall A1. \text{nonempty } A1 \Rightarrow \text{nonempty } (\text{ty\_2Epair\_2Eprod } A0 A1) \tag{1}$$

Let `c_2Epair_2EABS_prod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. \lambda 27a. \text{nonempty } A. 27a \Rightarrow \forall A. \lambda 27b. \text{nonempty } A. 27b \Rightarrow \text{c\_2Epair\_2EABS\_prod } A. 27a A. 27b \in ((\text{ty\_2Epair\_2Eprod } A. 27a A. 27b))^{((2^{A-27b})^{A-27a})} \tag{2}$$

**Definition 9** We define `c_2Epair_2E_2C` to be  $\lambda A. \lambda 27a : \iota. \lambda A. \lambda 27b : \iota. \lambda V0x \in A. 27a. \lambda V1y \in A. 27b. (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^{A-27a})) (\lambda V2z \in 2. V2z)) (\lambda V3z \in 2. V3z))$

Let `c_2Epred_set_2EGSPEC` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. \lambda 27a. \text{nonempty } A. 27a \Rightarrow \forall A. \lambda 27b. \text{nonempty } A. 27b \Rightarrow \text{c\_2Epred\_set\_2EGSPEC } A. 27a A. 27b \in ((2^{A-27a})^{((\text{ty\_2Epair\_2Eprod } A. 27a 2)^{A-27b})}) \tag{3}$$

**Definition 10** We define  $c\_2Epred\_set\_2EUNION$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A-27a}).\lambda V1t \in (2^{A-27a}).(ap (c\_2E$

**Definition 11** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E$

**Definition 12** We define  $c\_2Epred\_set\_2EINTER$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A-27a}).\lambda V1t \in (2^{A-27a}).(ap (c\_2E$

**Definition 13** We define  $c\_2Epred\_set\_2EDIFF$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A-27a}).\lambda V1t \in (2^{A-27a}).(ap (c\_2E$

Let  $ty\_2Etopology\_2Etopology : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow nonempty (ty\_2Etopology\_2Etopology A0) \quad (4)$$

Let  $c\_2Etopology\_2Eopen\_in : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Etopology\_2Eopen\_in A\_27a \in ((2^{(2^{A-27a})}) (ty\_2Etopology\_2Etopology A\_27a)) \quad (5)$$

**Definition 14** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x))$  then (the  $(\lambda x.x \in A \wedge$  of type  $\iota \Rightarrow \iota$ ).

**Definition 15** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap V0P (ap (c\_2Emin\_2E\_40$

**Definition 16** We define  $c\_2Epred\_set\_2EBIGUNION$  to be  $\lambda A\_27a : \iota.\lambda V0P \in (2^{(2^{A-27a})}).(ap (c\_2Epred\_set\_2E$

**Definition 17** We define  $c\_2Etopology\_2Etopspace$  to be  $\lambda A\_27a : \iota.\lambda V0top \in (ty\_2Etopology\_2Etopology$

**Definition 18** We define  $c\_2Epred\_set\_2ESUBSET$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A-27a}).\lambda V1t \in (2^{A-27a}).(ap (c\_2E$

**Definition 19** We define  $c\_2Etopology\_2Eclosed\_in$  to be  $\lambda A\_27a : \iota.\lambda V0top \in (ty\_2Etopology\_2Etopology$

Assume the following.

$$True \quad (6)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (7)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p V0t))) \quad (8)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p V0t) \Leftrightarrow (p V0t)))) \quad (9)$$

Assume the following.

$$(\forall V0t \in 2.((\neg (p V0t)) \Rightarrow ((p V0t) \Rightarrow False))) \quad (10)$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \quad (11)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (12)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (13)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (14)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p V0A) \wedge (p V1B))) \Leftrightarrow ((\neg(p V0A) \vee \neg(p V1B)))) \wedge (((\neg((p V0A) \vee (p V1B))) \Leftrightarrow ((\neg(p V0A) \wedge \neg(p V1B)))))) \quad (15)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (16)$$

Assume the following.

$$(\forall V0x \in 2. (\forall V1x\_27 \in 2. (\forall V2y \in 2. (\forall V3y\_27 \in 2. (((p V0x) \Leftrightarrow (p V1x\_27)) \wedge ((p V1x\_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y\_27)))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x\_27) \Rightarrow (p V3y\_27)))))) \quad (17)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in (2^{A\_27a}). ((V0s = V1t) \Leftrightarrow (\forall V2x \in A\_27a. ((p (ap (ap (c\_2Ebool\_2EIN A\_27a) V2x) V0s)) \Leftrightarrow (p (ap (ap (c\_2Ebool\_2EIN A\_27a) V2x) V1t)))))) \quad (18)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in (2^{A\_27a}). (\forall V2x \in A\_27a. ((p (ap (ap (c\_2Ebool\_2EIN A\_27a) V2x) (ap (ap (c\_2Ebool\_2EIN A\_27a) V0s) V1t))) \Leftrightarrow ((p (ap (ap (c\_2Ebool\_2EIN A\_27a) V2x) V0s)) \vee (p (ap (ap (c\_2Ebool\_2EIN A\_27a) V2x) V1t)))))) \quad (19)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0s \in (2^{A-27a}). (\forall V1t \in \\ (2^{A-27a}). (\forall V2u \in (2^{A-27a}). ((p\ (ap\ (ap\ (c.2Epred\_set.2ESUBSET \\ A.27a)\ (ap\ (ap\ (c.2Epred\_set.2EUNION\ A.27a)\ V0s)\ V1t))\ V2u)) \Leftrightarrow \\ ((p\ (ap\ (ap\ (c.2Epred\_set.2ESUBSET\ A.27a)\ V0s)\ V2u)) \wedge (p\ (ap\ (ap \\ (c.2Epred\_set.2ESUBSET\ A.27a)\ V1t)\ V2u)))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0s \in (2^{A-27a}). (\forall V1t \in \\ (2^{A-27a}). (\forall V2x \in A.27a. ((p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27a) \\ V2x)\ (ap\ (ap\ (c.2Epred\_set.2EINTER\ A.27a)\ V0s)\ V1t))) \Leftrightarrow ((p\ (ap \\ (ap\ (c.2Ebool.2EIN\ A.27a)\ V2x)\ V0s)) \wedge (p\ (ap\ (ap\ (c.2Ebool.2EIN \\ A.27a)\ V2x)\ V1t)))))) \end{aligned} \quad (21)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0s \in (2^{A-27a}). (\forall V1t \in \\ (2^{A-27a}). (\forall V2x \in A.27a. ((p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27a) \\ V2x)\ (ap\ (ap\ (c.2Epred\_set.2EDIFF\ A.27a)\ V0s)\ V1t))) \Leftrightarrow ((p\ (ap \\ (ap\ (c.2Ebool.2EIN\ A.27a)\ V2x)\ V0s)) \wedge (\neg (p\ (ap\ (ap\ (c.2Ebool.2EIN \\ A.27a)\ V2x)\ V1t)))))) \end{aligned} \quad (22)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0top \in (ty.2Etopology.2Etopology \\ A.27a). (\forall V1s \in (2^{A-27a}). (\forall V2t \in (2^{A-27a}). ((p \\ (ap\ (ap\ (c.2Etopology.2Eopen\_in\ A.27a)\ V0top)\ V1s)) \wedge (p\ (ap\ (ap \\ (c.2Etopology.2Eopen\_in\ A.27a)\ V0top)\ V2t))) \Rightarrow (p\ (ap\ (ap\ (c.2Etopology.2Eopen\_in \\ A.27a)\ V0top)\ (ap\ (ap\ (c.2Epred\_set.2EINTER\ A.27a)\ V1s)\ V2t)))))) \end{aligned} \quad (23)$$

### Theorem 1

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0top \in (ty.2Etopology.2Etopology \\ A.27a). (\forall V1s \in (2^{A-27a}). (\forall V2t \in (2^{A-27a}). ((p \\ (ap\ (ap\ (c.2Etopology.2Eclosed\_in\ A.27a)\ V0top)\ V1s)) \wedge (p\ (ap \\ (ap\ (c.2Etopology.2Eclosed\_in\ A.27a)\ V0top)\ V2t))) \Rightarrow (p\ (ap\ (ap \\ (c.2Etopology.2Eclosed\_in\ A.27a)\ V0top)\ (ap\ (ap\ (c.2Epred\_set.2EUNION \\ A.27a)\ V1s)\ V2t)))))) \end{aligned}$$