

thm_2Etoto_2ETO_exists (TMb- vZYEuC67VH3HUb4YggM677kzmwTr1Cdm)

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Let $ty_2EternaryComparisons_2Eordering : \iota$ be given. Assume the following.

$$nonempty\ ty_2EternaryComparisons_2Eordering \quad (1)$$

Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_2T$ to be $(ap (ap (c_2Emin_2E_3D (2^2))) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p \Rightarrow q)$ of type ι .

Definition 4 We define $c_2Ebool_2E_21$ to be $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c_2Emin_2E_3D (2^{A-27a}))) (\lambda V1t \in 2.V1t)) (\lambda V2t \in 2.V2t)))$

Definition 5 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t))))$

Definition 6 We define $c_2Erelation_2Etransitive$ to be $\lambda A.\lambda a : \iota.\lambda V0R \in ((2^{A-27a})^{A-27a}).(ap (c_2Ebool_2E_21 2) (\lambda V1t \in 2.V1t))$

Definition 7 We define $c_2Erelation_2Eantisymmetric$ to be $\lambda A.\lambda a : \iota.\lambda V0R \in ((2^{A-27a})^{A-27a}).(ap (c_2Ebool_2E_21 2) (\lambda V1t \in 2.V1t))$

Definition 8 We define $c_2Erelation_2EOrder$ to be $\lambda A.\lambda a : \iota.\lambda V0Z \in ((2^{A-27a})^{A-27a}).(ap (ap (c_2Ebool_2E_21 2) (\lambda V1t \in 2.V1t)) (\lambda V2t \in 2.V2t)))$

Definition 9 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t))))$

Definition 10 We define $c_2Erelation_2Etrichotomous$ to be $\lambda A.\lambda a : \iota.\lambda V0R \in ((2^{A-27a})^{A-27a}).(ap (c_2Ebool_2E_21 2) (\lambda V1t \in 2.V1t))$

Definition 11 We define $c_2Ebool_2E_2F$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 12 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap (ap (c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_21 2) (\lambda V1t \in 2.V1t)))$

Definition 13 We define $c_2Erelation_2Eirreflexive$ to be $\lambda A.\lambda a : \iota.\lambda V0R \in ((2^{A-27a})^{A-27a}).(ap (c_2Ebool_2E_21 2) (\lambda V1t \in 2.V1t))$

Definition 14 We define $c_2Erelation_2EStrongOrder$ to be $\lambda A.\lambda a : \iota.\lambda V0Z \in ((2^{A-27a})^{A-27a}).(ap (ap (c_2Ebool_2E_21 2) (\lambda V1t \in 2.V1t)) (\lambda V2t \in 2.V2t)))$

Definition 15 We define $c_2Erelation_2EStrongLinearOrder$ to be $\lambda A.\lambda a : \iota.\lambda V0R \in ((2^{A-27a})^{A-27a}).(ap (ap (c_2Ebool_2E_21 2) (\lambda V1t \in 2.V1t)) (\lambda V2t \in 2.V2t)))$

Definition 16 We define `c_2Emin_2E_40` to be $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p \text{ (ap } P \ x)) \text{ then (the } (\lambda x. x \in A) \wedge$
of type $\iota \Rightarrow \iota$.

Definition 17 We define `c_2Ebool_2E_3F` to be $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (\text{ap } V0P \text{ (ap (c_2Emin_2E_40$

Let `c_2EternaryComparisons_2EGREATER` : ι be given. Assume the following.

$$c_2EternaryComparisons_2EGREATER \in ty_2EternaryComparisons_2Eordering \quad (2)$$

Let `c_2EternaryComparisons_2ELESS` : ι be given. Assume the following.

$$c_2EternaryComparisons_2ELESS \in ty_2EternaryComparisons_2Eordering \quad (3)$$

Definition 18 We define `c_2Ebool_2ECOND` to be $\lambda A. 27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A. 27a. (\lambda V2t2 \in A. 27a. ($

Let `c_2EternaryComparisons_2EEQUAL` : ι be given. Assume the following.

$$c_2EternaryComparisons_2EEQUAL \in ty_2EternaryComparisons_2Eordering \quad (4)$$

Definition 19 We define `c_2Etoto_2ETO_of_LinearOrder` to be $\lambda A. 27a : \iota. \lambda V0r \in ((2^{A-27a})^{A-27a}). \lambda V1x \in$

Definition 20 We define `c_2Etoto_2ETotOrd` to be $\lambda A. 27a : \iota. \lambda V0c \in ((ty_2EternaryComparisons_2Eord$

Definition 21 We define `c_2ERelation_2ELinearOrder` to be $\lambda A. 27a : \iota. \lambda V0R \in ((2^{A-27a})^{A-27a}). (\text{ap (ap c_2E$

Assume the following.

$$True \quad (5)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \Rightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p \ V0t)) \Leftrightarrow True) \wedge (((p \ V0t) \Rightarrow (p \ V0t)) \Leftrightarrow True) \wedge ((\\ & (p \ V0t) \Rightarrow False) \Leftrightarrow (\neg (p \ V0t)))))) \end{aligned} \quad (6)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2. ((\neg (\neg (p \ V0t))) \Leftrightarrow (p \ V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ & ((\neg False) \Leftrightarrow True))) \end{aligned} \quad (7)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \Leftrightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p \ V0t)) \wedge (((False \Leftrightarrow (p \ V0t)) \Leftrightarrow (\neg (p \ V0t))) \wedge (((p \ V0t) \Leftrightarrow False) \Leftrightarrow (\neg (\\ & p \ V0t)))))) \end{aligned} \quad (8)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (\forall V2C \in 2. (((p \ V0A) \vee (\\ & (p \ V1B) \wedge (p \ V2C)) \Leftrightarrow (((p \ V0A) \vee (p \ V1B)) \wedge ((p \ V0A) \vee (p \ V2C)))))) \end{aligned} \quad (9)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0R \in ((2^{A.27a})^{A.27a}). \\ & ((p\ (ap\ (c_2Erelation_2EStrongOrder\ A.27a)\ V0R)) \Rightarrow (p\ (ap\ (c_2Erelation_2EOrder \\ & \quad A.27a)\ V0R)))) \end{aligned} \quad (10)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (11)$$

Assume the following.

$$(\forall V0A \in 2.((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (12)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & ((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))) \end{aligned} \quad (13)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((\neg(\neg(p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))) \end{aligned} \quad (14)$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \quad (15)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\ & (p\ V1q) \Leftrightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (p\ V2r))) \wedge (((p\ V0p) \vee (\neg(\\ & p\ V2r))) \vee (\neg(p\ V1q)))) \wedge (((p\ V1q) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V0p)))) \wedge ((p\ V2r) \vee \\ & ((\neg(p\ V1q)) \vee (\neg(p\ V0p)))))))))) \end{aligned} \quad (16)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\ & (p\ V1q) \wedge (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((\neg(p\ V1q)) \vee (\neg(p\ V2r)))) \wedge (((p\ V1q) \vee \\ & (\neg(p\ V0p))) \wedge ((p\ V2r) \vee (\neg(p\ V0p)))))))) \end{aligned} \quad (17)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\ & (p\ V1q) \vee (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee (\neg(p\ V1q))) \wedge (((p\ V0p) \vee (\neg(p\ V2r))) \wedge \\ & ((p\ V1q) \vee ((p\ V2r) \vee (\neg(p\ V0p)))))))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\ & (p\ V1q) \Rightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee (p\ V1q)) \wedge (((p\ V0p) \vee (\neg(p\ V2r))) \wedge ((\\ & \neg(p\ V1q)) \vee ((p\ V2r) \vee (\neg(p\ V0p)))))))))) \end{aligned} \quad (19)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p))))))) \quad (20)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (p V0p)))) \quad (21)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (\neg(p V1q)))))) \quad (22)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\exists V0R \in ((2^{A_27a})^{A_27a}). (p (ap (c_2Erelation_2EStrongLinearOrder A_27a) V0R))) \quad (23)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0r \in ((2^{A_27a})^{A_27a}). ((p (ap (c_2Erelation_2ELinearOrder A_27a) V0r)) \Rightarrow (p (ap (c_2Etoto_2ETotOrd A_27a) (ap (c_2Etoto_2ETO_of_LinearOrder A_27a) V0r)))))) \quad (24)$$

Theorem 1

$$\forall A_27a.nonempty A_27a \Rightarrow (\exists V0x \in ((ty_2EternaryComparisons_2Eordering^{A_27a})^{A_27a}). (p (ap (c_2Etoto_2ETotOrd A_27a) V0x)))$$