thm_2Etoto_2ETO__lexTO (TMQkq7ebhWS1zdLNtaLg6h2Xw79QgktmQR3)

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Let $ty\_2EternaryComparisons\_2Eordering : \iota$ be given. Assume the following.

$$nonempty\ ty\_2EternaryComparisons\_2Eordering \quad (1)$$

Let $c\_2EternaryComparisons\_2ELESS : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2ELESS \in ty\_2EternaryComparisons\_2Eordering \quad (2)$$

Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 3 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (c\_2Emin\_2E\_3D\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 4 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (c\_2Emin\_2E\_3D\ (2^{A\_27}$

Definition 5 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2$

Let $c\_2EternaryComparisons\_2EGREATER : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2EGREATER \in ty\_2EternaryComparisons\_2Eordering \quad (3)$$

Let $c\_2EternaryComparisons\_2EEQUAL : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2EEQUAL \in ty\_2EternaryComparisons\_2Eordering \quad (4)$$

Definition 6 *We define* c_2Etoto_2ETotOrd *to be* $\lambda A\_27a : \iota.\lambda V0c \in ((ty\_2EternaryComparisons\_2Eorde$

Definition 7 *We define* c_2Ebool_2E_5C_2F *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2$

Definition 8 *We define* c_2Erelation_2Etrichotomous *to be* $\lambda A\_27a : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).(ap\ (c\_2Ebo$

Definition 9 *We define* c_2Erelation_2Etransitive *to be* $\lambda A\_27a : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).(ap$ (c_2Ebool_2E

Definition 10 *We define* c_2Ebool_2EF *to be* $(ap$ (c_2Ebool_2E_21 2) $(\lambda V0t \in 2.V0t))$.

Definition 11 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap$ c_2Emin_2E_3D_3D_3E $V0t)$ c_2Ebool_2E

Definition 12 *We define* c_2Erelation_2Eirreflexive *to be* $\lambda A\_27a : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).(ap$ (c_2Ebool_2

Definition 13 *We define* c_2Erelation_2EStrongOrder *to be* $\lambda A\_27g : \iota.\lambda V0Z \in ((2^{A\_27g})^{A\_27g}).(ap\ (ap$ c_2E

Definition 14 *We define* c_2Erelation_2EStrongLinearOrder *to be* $\lambda A\_27a : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).(ap\ (a$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{5}$$

Let $c\_2EternaryComparisons\_2Eordering2num : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2Eordering2num \in (ty\_2Enum\_2Enum^{ty\_2EternaryComparisons\_2Eordering}) \tag{6}$$

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \tag{7}$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \tag{8}$$

Definition 15 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Definition 16 *We define* c_2Earithmetic_2EZERO *to be* c_2Enum_2E0*.*

Let $c\_2Enum\_2EREP\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EREP\_\_num \in (omega^{ty\_2Enum\_2Enum}) \tag{9}$$

Let $c\_2Enum\_2ESUC\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2ESUC\_\_REP \in (omega^{omega}) \tag{10}$$

Definition 17 *We define* c_2Enum_2ESUC *to be* $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_\_num$

Let $c\_2Earithmetic\_2E\_2B : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \tag{11}$$

Definition 18 *We define* c_2Earithmetic_2EBIT1 *to be* $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

Definition 19 *We define* c_2Earithmetic_2ENUMERAL *to be* $\lambda V0x \in ty\_2Enum\_2Enum.V0x$.

Definition 20 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A\wedge p$
of type $\iota{\Rightarrow}\iota$.

Definition 21 *We define* c_2Ebool_2ECOND *to be* $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

Definition 22 *We define* c_2Ebool_2E_3F *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap$ (c_2Emin_2E_40

Definition 23 *We define* c_2Eprim__rec_2E_3C *to be* $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enu$

Definition 24 *We define* c_2EternaryComparisons_2Eordering__CASE *to be* $\lambda A\_27a : \iota.\lambda V0x \in ty\_2Eternary$

Definition 25 *We define* c_2Etoto_2EStrongLinearOrder__of__TO *to be* $\lambda A\_27a : \iota.\lambda V0c \in ((ty\_2EternaryC$

Let $ty\_2Epair\_2Eprod : \iota{\Rightarrow}\iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0{\Rightarrow}\forall A1.nonempty\ A1{\Rightarrow}nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{12}$$

Let $c\_2Epair\_2ESND : \iota{\Rightarrow}\iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}\forall A\_27b.nonempty\ A\_27b{\Rightarrow}c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{13}$$

Let $c\_2Epair\_2EFST : \iota{\Rightarrow}\iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}\forall A\_27b.nonempty\ A\_27b{\Rightarrow}c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{14}$$

Definition 26 *We define* c_2Epair_2EUNCURRY *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.\lambda V0f \in ((A\_27c^{A\_27}$

Definition 27 *We define* c_2Epair_2ELEX *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R1 \in ((2^{A\_27a})^{A\_27a}).\lambda V1R2 \in (($

Definition 28 *We define* c_2Etoto_2ETO__of__LinearOrder *to be* $\lambda A\_27a : \iota.\lambda V0r \in ((2^{A\_27a})^{A\_27a}).\lambda V1x \in$

Definition 29 *We define* c_2Etoto_2ElexTO *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((ty\_2EternaryCompariso$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}(\forall V0c \in ((ty\_2EternaryComparisons\_2Eordering^{A\_27a})^{A\_27a}).((p\ (ap\ (\mathsf{c\_2Etoto\_2ETotOrd}\ A\_27a)\ V0c)){\Rightarrow}(p\ (ap\ (\mathsf{c\_2Erelation\_2EStrongLinearOrder}\ A\_27a)\ (ap\ (\mathsf{c\_2Etoto\_2EStrongLinearOrder\_\_of\_\_TO}\ A\_27a)\ V0c))))) \tag{15}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}(\forall V0r \in ((2^{A\_27a})^{A\_27a}).((p\ (ap\ (\mathsf{c\_2Erelation\_2EStrongLinearOrder}\ A\_27a)\ V0r)){\Rightarrow}(p\ (ap\ (\mathsf{c\_2Etoto\_2ETotOrd}\ A\_27a)\ (ap\ (\mathsf{c\_2Etoto\_2ETO\_\_of\_\_LinearOrder}\ A\_27a)\ V0r))))) \tag{16}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0R \in ((2^{A\_27a})^{A\_27a}).(\forall V1V \in ((2^{A\_27b})^{A\_27b}). (((p\ (ap\ (\mathsf{c\_2Erelation\_2EStrongLinearOrder}\ A\_27a)\ V0R)) \wedge (p\ (ap\ (\mathsf{c\_2Erelation\_2EStrongLinearOrder}\ A\_27b)\ V1V))) \Rightarrow (p\ (ap\ (\mathsf{c\_2Erelation\_2EStrongLinearOrder}\ (ty\_2Epair\_2Eprod\ A\_27a\ A\_27b))\ (ap\ (ap\ (\mathsf{c\_2Epair\_2ELEX}\ A\_27a\ A\_27b)\ V0R)\ V1V)))))) \tag{17}$$

Theorem 1

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0R \in ((ty\_2EternaryComparisons\_2Eordering^{A\_27a})^{A\_27a}). (\forall V1V \in ((ty\_2EternaryComparisons\_2Eordering^{A\_27b})^{A\_27b}). (((p\ (ap\ (\mathsf{c\_2Etoto\_2ETotOrd}\ A\_27a)\ V0R)) \wedge (p\ (ap\ (\mathsf{c\_2Etoto\_2ETotOrd}\ A\_27b)\ V1V))) \Rightarrow (p\ (ap\ (\mathsf{c\_2Etoto\_2ETotOrd}\ (ty\_2Epair\_2Eprod\ A\_27a\ A\_27b))\ (ap\ (ap\ (\mathsf{c\_2Etoto\_2ElexTO}\ A\_27a\ A\_27b)\ V0R)\ V1V))))))$$