thm_2Etoto_2Eaplistoto (TMdAvc9hDHWRuJ1r9iyVdzqxZPx6yVg7g8i)

October 26, 2020

Definition 1 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Let $ty\_2EternaryComparisons\_2Eordering : \iota$ be given. Assume the following.

$$nonempty\ ty\_2EternaryComparisons\_2Eordering \quad (1)$$

Let $c\_2EternaryComparisons\_2ELESS : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2ELESS \in ty\_2EternaryComparisons\_2Eordering \quad (2)$$

Definition 2 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 3 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 4 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

Definition 5 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Let $c\_2EternaryComparisons\_2EGREATER : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2EGREATER \in ty\_2EternaryComparisons\_2Eordering \quad (3)$$

Let $c\_2EternaryComparisons\_2EEQUAL : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2EEQUAL \in ty\_2EternaryComparisons\_2Eordering \quad (4)$$

Definition 6 *We define* c_2Etoto_2ETotOrd *to be* $\lambda A\_27a : \iota.\lambda V0c \in ((ty\_2EternaryComparisons\_2Eorde$

Let $ty\_2Etoto\_2Etoto : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Etoto\_2Etoto\ A0) \tag{5}$$

Let $c\_2Etoto\_2Eapto : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Etoto\_2Eapto\ A\_27a \in (((ty\_2EternaryComparisons\_2Eordering^{A\_27a})^{A\_2} \tag{6}$$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{7}$$

Let $c\_2EternaryComparisons\_2Eordering2num : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2Eordering2num \in (ty\_2Enum\_2Enum^{ty\_2EternaryComparisons\_2Eordering}) \tag{8}$$

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \tag{9}$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \tag{10}$$

Definition 7 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Definition 8 *We define* c_2Earithmetic_2EZERO *to be* c_2Enum_2E0.

Let $c\_2Enum\_2EREP\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EREP\_\_num \in (omega^{ty\_2Enum\_2Enum}) \tag{11}$$

Let $c\_2Enum\_2ESUC\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2ESUC\_\_REP \in (omega^{omega}) \tag{12}$$

Definition 9 *We define* c_2Enum_2ESUC *to be* $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_\_num\ ($

Let $c\_2Earithmetic\_2E\_2B : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \tag{13}$$

Definition 10 *We define* c_2Earithmetic_2EBIT1 *to be* $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

Definition 11 *We define* c_2Earithmetic_2ENUMERAL *to be* $\lambda V0x \in ty\_2Enum\_2Enum.V0x$.

Definition 12 *We define* c_2Ebool_2EF *to be* $(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 13 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$
of type $\iota \Rightarrow \iota$.

Definition 14 *We define* c_2Ebool_2ECOND *to be* $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

Definition 15 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap$ c_2Emin_2E_3D_3D_3E $V0t)$ c_2Ebool_2E

Definition 16 *We define* c_2Ebool_2E_3F *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap$ (c_2Emin_2E_40

Definition 17 *We define* c_2Eprim__rec_2E_3C *to be* $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enu$

Definition 18 *We define* c_2EternaryComparisons_2Eordering__CASE *to be* $\lambda A\_27a : \iota.\lambda V0x \in ty\_2Eternary$

Let $ty\_2Elist\_2Elist : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Elist\_2Elist\ A0) \tag{14}$$

Let $c\_2Elist\_2ECONS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Elist\_2ECONS\ A\_27a \in (((ty\_2Elist\_2Elist\ A\_27a)^{(ty\_2Elist\_2Elist\ A\_27a)})^{A\_27a}) \tag{15}$$

Let $c\_2Elist\_2ENIL : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Elist\_2ENIL\ A\_27a \in (ty\_2Elist\_2Elist\ A\_27a) \tag{16}$$

Definition 19 *We define* c_2Etoto_2EStrongLinearOrder__of__TO *to be* $\lambda A\_27a : \iota.\lambda V0c \in ((ty\_2EternaryC$

Let $c\_2Etoto\_2Elistorder : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Etoto\_2Elistorder\ A\_27a \in (((2^{(ty\_2Elist\_2Elist\ A\_27a)})^{(ty\_2Elist\_2Elist\ A\_27a)})^{((2^{A\_27a})^{A\_27a})}) \tag{17}$$

Definition 20 *We define* c_2Etoto_2ETO__of__LinearOrder *to be* $\lambda A\_27a : \iota.\lambda V0r \in ((2^{A\_27a})^{A\_27a}).\lambda V1x \in$

Definition 21 *We define* c_2Etoto_2EListOrd *to be* $\lambda A\_27a : \iota.\lambda V0c \in (ty\_2Etoto\_2Etoto\ A\_27a).(ap$ (c_2E

Let $c\_2Etoto\_2ETO : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Etoto\_2ETO\ A\_27a \in ((ty\_2Etoto\_2Etoto\ A\_27a)^{((ty\_2EternaryComparisons\_2Eordering^{A\_27a})^{A\_27a})}) \tag{18}$$

Definition 22 *We define* c_2Etoto_2Elistoto *to be* $\lambda A\_27a : \iota.\lambda V0c \in (ty\_2Etoto\_2Etoto\ A\_27a).(ap\ (c\_2Et$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0r \in ((ty\_2EternaryComparisons\_2Eordering^{A\_27a})^{A\_27a}).((p\ (ap\ (\mathsf{c\_2Etoto\_2ETotOrd}\ A\_27a)\ V0r)) \Rightarrow ((ap\ (c\_2Etoto\_2Eapto\ A\_27a)\ (ap\ (c\_2Etoto\_2ETO\ A\_27a)\ V0r)) = V0r))) \tag{19}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0c \in (ty\_2Etoto\_2Etoto\ A\_27a).(p\ (ap\ (\mathsf{c\_2Etoto\_2ETotOrd}\ (ty\_2Elist\_2Elist\ A\_27a))\ (ap\ (\mathsf{c\_2Etoto\_2EListOrd}\ A\_27a)\ V0c)))) \tag{20}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0c \in (ty\_2Etoto\_2Etoto\ A\_27a).(((ap\ (ap\ (ap\ (\mathsf{c\_2Etoto\_2EListOrd}\ A\_27a)\ V0c)\ (c\_2Elist\_2ENIL\ A\_27a))\ (c\_2Elist\_2ENIL\ A\_27a)) = c\_2EternaryComparisons\_2EEQUAL) \wedge ((\forall V1b \in A\_27a.(\forall V2y \in (ty\_2Elist\_2Elist\ A\_27a).((ap\ (ap\ (ap\ (\mathsf{c\_2Etoto\_2EListOrd}\ A\_27a)\ V0c)\ (c\_2Elist\_2ENIL\ A\_27a))\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V1b)\ V2y)) = c\_2EternaryComparisons\_2ELESS))) \wedge ((\forall V3a \in A\_27a.(\forall V4x \in (ty\_2Elist\_2Elist\ A\_27a).((ap\ (ap\ (ap\ (\mathsf{c\_2Etoto\_2EListOrd}\ A\_27a)\ V0c)\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V3a)\ V4x))\ (c\_2Elist\_2ENIL\ A\_27a)) = c\_2EternaryComparisons\_2EGREATER))) \wedge (\forall V5a \in A\_27a.(\forall V6x \in (ty\_2Elist\_2Elist\ A\_27a).(\forall V7b \in A\_27a.(\forall V8y \in (ty\_2Elist\_2Elist\ A\_27a).((ap\ (ap\ (ap\ (\mathsf{c\_2Etoto\_2EListOrd}\ A\_27a)\ V0c)\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V5a)\ V6x))\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V7b)\ V8y)) = (ap\ (ap\ (ap\ (ap\ (\mathsf{c\_2EternaryComparisons\_2Eordering\_\_CASE}\ ty\_2EternaryComparisons\_2Eordering)\ (ap\ (ap\ (ap\ (c\_2Etoto\_2Eapto\ A\_27a)\ V0c)\ V5a)\ V7b))\ c\_2EternaryComparisons\_2ELESS)\ (ap\ (ap\ (ap\ (\mathsf{c\_2Etoto\_2EListOrd}\ A\_27a)\ V0c)\ V6x)\ V8y))\ c\_2EternaryComparisons\_2EGREATER)))))) \tag{21}$$

Theorem 1

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0c \in (ty\_2Etoto\_2Etoto\ A\_27a).(((ap\ (ap\ (ap\ (c\_2Etoto\_2Eapto\ (ty\_2Elist\_2Elist\ A\_27a))\ (ap\ (\mathsf{c\_2Etoto\_2Elistoto}\ A\_27a)\ V0c))\ (c\_2Elist\_2ENIL\ A\_27a))\ (c\_2Elist\_2ENIL\ A\_27a)) = c\_2EternaryComparisons\_2EEQUAL) \wedge ((\forall V1b \in A\_27a.(\forall V2y \in (ty\_2Elist\_2Elist\ A\_27a).((ap\ (ap\ (ap\ (c\_2Etoto\_2Eapto\ (ty\_2Elist\_2Elist\ A\_27a))\ (ap\ (\mathsf{c\_2Etoto\_2Elistoto}\ A\_27a)\ V0c))\ (c\_2Elist\_2ENIL\ A\_27a))\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V1b)\ V2y)) = c\_2EternaryComparisons\_2ELESS))) \wedge ((\forall V3a \in A\_27a.(\forall V4x \in (ty\_2Elist\_2Elist\ A\_27a).((ap\ (ap\ (ap\ (c\_2Etoto\_2Eapto\ (ty\_2Elist\_2Elist\ A\_27a))\ (ap\ (\mathsf{c\_2Etoto\_2Elistoto}\ A\_27a)\ V0c))\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V3a)\ V4x))\ (c\_2Elist\_2ENIL\ A\_27a)) = c\_2EternaryComparisons\_2EGREATER))) \wedge (\forall V5a \in A\_27a.(\forall V6x \in (ty\_2Elist\_2Elist\ A\_27a).(\forall V7b \in A\_27a.(\forall V8y \in (ty\_2Elist\_2Elist\ A\_27a).((ap\ (ap\ (ap\ (c\_2Etoto\_2Eapto\ (ty\_2Elist\_2Elist\ A\_27a))\ (ap\ (\mathsf{c\_2Etoto\_2Elistoto}\ A\_27a)\ V0c))\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V5a)\ V6x))\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V7b)\ V8y)) = (ap\ (ap\ (ap\ (ap\ (\mathsf{c\_2EternaryComparisons\_2Eordering\_\_CASE}\ ty\_2EternaryComparisons\_2Eordering)\ (ap\ (ap\ (ap\ (c\_2Etoto\_2Eapto\ A\_27a)\ V0c)\ V5a)\ V7b))\ c\_2EternaryComparisons\_2ELESS)\ (ap\ (ap\ (ap\ (c\_2Etoto\_2Eapto\ (ty\_2Elist\_2Elist\ A\_27a))\ (ap\ (\mathsf{c\_2Etoto\_2Elistoto}\ A\_27a)\ V0c))\ V6x)\ V8y))\ c\_2EternaryComparisons\_2EGREATER))))))))$$