

# thm\_2Ewellorder\_2ETARSKI\_SET (TMGCTC- dRdnv4veXyraoE23DPTGaQ4QVMz2U)

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**Definition 1** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. \text{inj\_o } (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Ebool_2E_2T` to be  $(\text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^2)) (\lambda V 0x \in 2. V 0x)) (\lambda V 1x \in 2. V 1x))$

**Definition 3** We define `c_2Emin_2E_3D_3D_3E` to be  $\lambda P \in 2. \lambda Q \in 2. \text{inj\_o } (p \Rightarrow q)$  of type  $\iota$ .

**Definition 4** We define `c_2Ebool_2E_21` to be  $\lambda A. 27a : \iota. (\lambda V 0P \in (2^{A-27a}). (\text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^{A-27a}))))$

**Definition 5** We define `c_2Ebool_2E_2F_5C` to be  $(\lambda V 0t1 \in 2. (\lambda V 1t2 \in 2. (\text{ap } (\text{c\_2Ebool\_2E\_21 } 2)) (\lambda V 2t \in 2. V 2t)))$

Let `ty_2Epair_2Eprod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A 0. \text{nonempty } A 0 \Rightarrow \forall A 1. \text{nonempty } A 1 \Rightarrow \text{nonempty } (\text{ty\_2Epair\_2Eprod } A 0 A 1) \tag{1}$$

Let `c_2Epair_2EABS_prod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. 27a. \text{nonempty } A. 27a \Rightarrow \forall A. 27b. \text{nonempty } A. 27b \Rightarrow \text{c\_2Epair\_2EABS\_prod } A. 27a A. 27b \in ((\text{ty\_2Epair\_2Eprod } A. 27a A. 27b))^{((2^{A-27b})^{A-27a})} \tag{2}$$

**Definition 6** We define `c_2Epair_2E_2C` to be  $\lambda A. 27a : \iota. \lambda A. 27b : \iota. \lambda V 0x \in A. 27a. \lambda V 1y \in A. 27b. (\text{ap } (\text{c\_2Ebool\_2E\_21 } 2))$

**Definition 7** We define `c_2Emin_2E_40` to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p (\text{ap } P x)) \text{ then } (\text{the } (\lambda x. x \in A \wedge p (\text{ap } P x)))$  of type  $\iota \Rightarrow \iota$ .

**Definition 8** We define `c_2Ebool_2E_3F` to be  $\lambda A. 27a : \iota. (\lambda V 0P \in (2^{A-27a}). (\text{ap } V 0P (\text{ap } (\text{c\_2Emin\_2E\_40 } A. 27a))))$

Let `c_2Epred_set_2EGSPEC` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. 27a. \text{nonempty } A. 27a \Rightarrow \forall A. 27b. \text{nonempty } A. 27b \Rightarrow \text{c\_2Epred\_set\_2EGSPEC } A. 27a A. 27b \in ((2^{A-27a})^{((\text{ty\_2Epair\_2Eprod } A. 27a 2)^{A-27b})}) \tag{3}$$

**Definition 9** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota. (\lambda V0x \in A\_27a. (\lambda V1f \in (2^{A\_27a}). (ap\ V1f\ V0x)))$

**Definition 10** We define  $c\_2Epred\_set\_2ESUBSET$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). \lambda V1t \in (2^{A\_27a}). (ap\ ($

**Definition 11** We define  $c\_2Epred\_set\_2EBIGINTER$  to be  $\lambda A\_27a : \iota. \lambda V0P \in (2^{(2^{A\_27a})}). (ap\ (c\_2Epred\_s$

**Definition 12** We define  $c\_2Ebool\_2E21\_2$  to be  $(ap\ (c\_2Ebool\_2E21\_2)\ (\lambda V0t \in 2. V0t))$ .

**Definition 13** We define  $c\_2Ebool\_2E5C\_2F$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c\_2Ebool\_2E21\_2)\ (\lambda V2t \in$

**Definition 14** We define  $c\_2Ebool\_2E7E$  to be  $(\lambda V0t \in 2. (ap\ (ap\ c\_2Emin\_2E3D\_3D\_3E\ V0t)\ c\_2Ebool\_2E$

Assume the following.

$$True \tag{4}$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \tag{5}$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p\ V0t))) \tag{6}$$

Assume the following.

$$(\forall V0t \in 2. ((p\ V0t) \vee \neg(p\ V0t))) \tag{7}$$

Assume the following.

$$\forall A\_27a. nonempty\ A\_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in A\_27a. (p\ V0t)) \Leftrightarrow (p\ V0t))) \tag{8}$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p\ V0t1) \wedge ((p\ V1t2) \wedge (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \wedge (p\ V2t3)))))) \tag{9}$$

Assume the following.

$$(\forall V0t \in 2. (((p\ V0t) \Rightarrow False) \Rightarrow \neg(p\ V0t))) \tag{10}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(p\ V0t)) \Rightarrow ((p\ V0t) \Rightarrow False))) \tag{11}$$

Assume the following.

$$(\forall V0t \in 2. (((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \tag{12}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \vee (p \vee V0t)) \Leftrightarrow True) \wedge (((p \vee V0t) \vee True) \Leftrightarrow True) \wedge \\
& (((False \vee (p \vee V0t)) \Leftrightarrow (p \vee V0t)) \wedge (((p \vee V0t) \vee False) \Leftrightarrow (p \vee V0t)) \wedge (((p \vee V0t) \vee \\
& (p \vee V0t)) \Leftrightarrow (p \vee V0t))))))
\end{aligned} \tag{13}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Rightarrow (p \vee V0t)) \Leftrightarrow (p \vee V0t)) \wedge (((p \vee V0t) \Rightarrow True) \Leftrightarrow \\
& True) \wedge (((False \Rightarrow (p \vee V0t)) \Leftrightarrow True) \wedge (((p \vee V0t) \Rightarrow (p \vee V0t)) \Leftrightarrow True) \wedge (( \\
& (p \vee V0t) \Rightarrow False) \Leftrightarrow (\neg(p \vee V0t))))))
\end{aligned} \tag{14}$$

Assume the following.

$$\begin{aligned}
& ((\forall V0t \in 2.((\neg(\neg(p \vee V0t))) \Leftrightarrow (p \vee V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge \\
& ((\neg False) \Leftrightarrow True)))
\end{aligned} \tag{15}$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.((V0x = V0x) \Leftrightarrow True)) \tag{16}$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.(\forall V1y \in A.27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \tag{17}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Leftrightarrow (p \vee V0t)) \Leftrightarrow (p \vee V0t)) \wedge (((p \vee V0t) \Leftrightarrow True) \Leftrightarrow \\
& (p \vee V0t)) \wedge (((False \Leftrightarrow (p \vee V0t)) \Leftrightarrow (\neg(p \vee V0t))) \wedge (((p \vee V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\
& p \vee V0t))))))
\end{aligned} \tag{18}$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).((\neg(\forall V1x \in A.27a.(p \ (ap \ V0P \ V1x)))) \Leftrightarrow (\exists V2x \in A.27a.(\neg(p \ (ap \ V0P \ V2x)))))) \tag{19}$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).((\neg(\exists V1x \in A.27a.(p \ (ap \ V0P \ V1x)))) \Leftrightarrow (\forall V2x \in A.27a.(\neg(p \ (ap \ V0P \ V2x)))))) \tag{20}$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in (2^{A.27a}).(((p \ V0P) \wedge (\forall V2x \in A.27a.(p \ (ap \ V1Q \ V2x)))) \Leftrightarrow (\forall V3x \in A.27a.((p \ V0P) \wedge (p \ (ap \ V1Q \ V3x)))))) \tag{21}$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in (2^{A.27a}).(((p \ V0P) \vee (\exists V2x \in A.27a.(p \ (ap \ V1Q \ V2x)))) \Leftrightarrow (\exists V3x \in A.27a.((p \ V0P) \vee (p \ (ap \ V1Q \ V3x)))))) \tag{22}$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A-27a}). (\forall V1Q \in \\ 2. ((\exists V2x \in A\_27a. ((p\ (ap\ V0P\ V2x)) \wedge (p\ V1Q))) \Leftrightarrow ((\exists V3x \in \\ A\_27a. (p\ (ap\ V0P\ V3x)) \wedge (p\ V1Q)))))) \end{aligned} \quad (23)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in 2. (\forall V1Q \in ( \\ 2^{A-27a}). ((\forall V2x \in A\_27a. ((p\ V0P) \vee (p\ (ap\ V1Q\ V2x)))) \Leftrightarrow ((p \\ V0P) \vee (\forall V3x \in A\_27a. (p\ (ap\ V1Q\ V3x)))))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} (\forall V0A \in 2. (\forall V1B \in 2. (\forall V2C \in 2. (((p\ V0A) \vee ( \\ (p\ V1B) \vee (p\ V2C))) \Leftrightarrow (((p\ V0A) \vee (p\ V1B)) \vee (p\ V2C)))))) \end{aligned} \quad (25)$$

Assume the following.

$$\begin{aligned} (\forall V0A \in 2. (\forall V1B \in 2. (((p\ V0A) \vee (p\ V1B)) \Leftrightarrow ((p\ V1B) \vee \\ (p\ V0A)))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned} (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p\ V0A) \wedge (p\ V1B))) \Leftrightarrow ((\neg( \\ p\ V0A) \vee (\neg(p\ V1B)))) \wedge ((\neg((p\ V0A) \vee (p\ V1B))) \Leftrightarrow ((\neg(p\ V0A) \wedge (\neg(p\ V1B)))))))) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A-27a}). (\forall V1a \in \\ A\_27a. ((\exists V2x \in A\_27a. ((V2x = V1a) \wedge (p\ (ap\ V0P\ V2x)))) \Leftrightarrow (p\ ( \\ ap\ V0P\ V1a)))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ \forall V0P \in ((2^{A-27b})^{A-27a}). ((\forall V1x \in A\_27a. (\exists V2y \in \\ A\_27b. (p\ (ap\ (ap\ V0P\ V1x)\ V2y)))) \Leftrightarrow (\exists V3f \in (A\_27b^{A-27a}). ( \\ \forall V4x \in A\_27a. (p\ (ap\ (ap\ V0P\ V4x)\ (ap\ V3f\ V4x)))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ \forall V0x \in A\_27a. (\forall V1y \in A\_27b. (\forall V2a \in A\_27a. (\forall V3b \in \\ A\_27b. (((ap\ (ap\ (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b)\ V0x)\ V1y) = (ap\ (ap \\ (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b)\ V2a)\ V3b)) \Leftrightarrow ((V0x = V2a) \wedge (V1y = V3b)))))) \end{aligned} \quad (30)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow ( \\ & \quad \forall V0f \in ((ty\_2Epair\_2Eprod\ A.27a\ 2)^{A.27b}).(\forall V1v \in \\ & A.27a.((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A.27a)\ V1v)\ (ap\ (c\_2Epred\_set\_2EGSPEC \\ & \quad A.27a\ A.27b)\ V0f))) \Leftrightarrow (\exists V2x \in A.27b.((ap\ (ap\ (c\_2Epair\_2E\_2C \\ & \quad A.27a\ 2)\ V1v)\ c\_2Ebool\_2ET) = (ap\ V0f\ V2x)))))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0s \in (2^{A.27a}).(\forall V1t \in \\ & (2^{A.27a}).(\forall V2u \in (2^{A.27a}).(((p\ (ap\ (ap\ (c\_2Epred\_set\_2ESUBSET \\ & A.27a)\ V0s)\ V1t)) \wedge (p\ (ap\ (ap\ (c\_2Epred\_set\_2ESUBSET\ A.27a)\ V1t) \\ & V2u)))) \Rightarrow (p\ (ap\ (ap\ (c\_2Epred\_set\_2ESUBSET\ A.27a)\ V0s)\ V2u)))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0s \in (2^{A.27a}).(\forall V1t \in \\ & (2^{A.27a}).(((p\ (ap\ (ap\ (c\_2Epred\_set\_2ESUBSET\ A.27a)\ V0s)\ V1t)) \wedge \\ & (p\ (ap\ (ap\ (c\_2Epred\_set\_2ESUBSET\ A.27a)\ V1t)\ V0s))) \Rightarrow (V0s = V1t)))) \end{aligned} \quad (33)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0x \in A.27a.(\forall V1B \in \\ & (2^{(2^{A.27a})}).(((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A.27a)\ V0x)\ (ap\ (c\_2Epred\_set\_2EBIGINTER \\ & A.27a)\ V1B))) \Leftrightarrow (\forall V2P \in (2^{A.27a}).(((p\ (ap\ (ap\ (c\_2Ebool\_2EIN \\ & (2^{A.27a})\ V2P)\ V1B))) \Rightarrow (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A.27a)\ V0x)\ V2P)))))) \end{aligned} \quad (34)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (35)$$

Assume the following.

$$(\forall V0A \in 2.((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (36)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & ((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))) \end{aligned} \quad (37)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((\neg(\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) \end{aligned} \quad (38)$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \quad (39)$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow ( \\
& (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg( \\
& p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\
& ((\neg(p V1q)) \vee (\neg(p V0p))))))))))
\end{aligned} \tag{40}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow ( \\
& (p V1q) \wedge (p V2r))) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \vee (\neg(p V2r)))) \wedge (((p V1q) \vee \\
& (\neg(p V0p))) \wedge ((p V2r) \vee (\neg(p V0p))))))
\end{aligned} \tag{41}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow ( \\
& (p V1q) \vee (p V2r))) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge \\
& ((p V1q) \vee ((p V2r) \vee (\neg(p V0p))))))
\end{aligned} \tag{42}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow ( \\
& (p V1q) \Rightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge (( \\
& \neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p))))))
\end{aligned} \tag{43}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee \\
& (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p))))))
\end{aligned} \tag{44}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (p V0p))) \tag{45}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (\neg(p V1q)))) \tag{46}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V0p)))) \tag{47}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V1q)))) \tag{48}$$

Assume the following.

$$(\forall V0p \in 2. ((\neg(\neg(p V0p))) \Rightarrow (p V0p))) \tag{49}$$

**Theorem 1**

$$\begin{aligned} & \forall A_{.27a}. \text{nonempty } A_{.27a} \Rightarrow (\forall V0f \in ((2^{A_{.27a}})^{(2^{A_{.27a}})}), \\ & ((\forall V1s \in (2^{A_{.27a}}). (\forall V2t \in (2^{A_{.27a}}). ((p (ap (ap ( \\ \text{c\_2Epred\_set\_2ESUBSET } A_{.27a} ) V1s) V2t))) \Rightarrow (p (ap (ap ( \\ \text{c\_2Epred\_set\_2ESUBSET } A_{.27a} ) (ap V0f V1s)) (ap V0f V2t)))))) \Rightarrow (\exists V3s \in (2^{A_{.27a}}). \\ & ((ap V0f V3s) = V3s)))) \end{aligned}$$