# thm_2Ewords_2EFST__ADD__WITH__CARRY (TMPveW-FyBy2qDSNBitc1w4ppDsZw3Lb5XdE)

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**Definition 1** *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

**Definition 2** *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

**Definition 3** *We define* c_2Ecombin_2EK *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0x \in A\_27a.(\lambda V1y \in A\_27b.V0x))$

**Definition 4** *We define* c_2Ecombin_2ES *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.(\lambda V0f \in ((A\_27c^{A\_27b})^{A\_27a}$

**Definition 5** *We define* c_2Ecombin_2EI *to be* $\lambda A\_27a : \iota.(ap\ (ap\ (\mathsf{c\_2Ecombin\_2ES}\ A\_27a\ (A\_27a^{A\_27a})\ A\_$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{1}$$

Let $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{2}$$

**Definition 6** *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

**Definition 7** *We define* c_2Ebool_2EF *to be* $(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

**Definition 8** *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* $\iota$.

**Definition 9** *We define* c_2Ebool_2E_5C_2F *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

**Definition 10** *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in$

**Definition 11** *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^{A}.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$
*of type* $\iota \Rightarrow \iota$.

**Definition 12** *We define* c_2Ebool_2ECOND *to be* $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \tag{3}$$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{4}$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \tag{5}$$

**Definition 13** *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

**Definition 14** *We define* c_2Earithmetic_2EZERO *to be* c_2Enum_2E0.

Let $c\_2Enum\_2EREP\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EREP\_\_num \in (omega^{ty\_2Enum\_2Enum}) \tag{6}$$

Let $c\_2Enum\_2ESUC\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2ESUC\_\_REP \in (omega^{omega}) \tag{7}$$

**Definition 15** *We define* c_2Enum_2ESUC *to be* $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_\_num$

Let $c\_2Earithmetic\_2E\_2B : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \tag{8}$$

**Definition 16** *We define* c_2Earithmetic_2EBIT1 *to be* $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

**Definition 17** *We define* c_2Earithmetic_2ENUMERAL *to be* $\lambda V0x \in ty\_2Enum\_2Enum.V0x$.

Let $ty\_2Ebool\_2Eitself : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Ebool\_2Eitself\ A0) \tag{9}$$

Let $c\_2Ebool\_2Ethe\_\_value : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ebool\_2Ethe\_\_value\ A\_27a \in (ty\_2Ebool\_2Eitself\ A\_27a) \tag{10}$$

Let $c\_2Efcp\_2Edimindex : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Efcp\_2Edimindex\ A\_27a \in (ty\_2Enum\_2Enum^{(ty\_2Ebool\_2Eitself\ A\_27a)}) \tag{11}$$

Let $c\_2Earithmetic\_2E\_2D : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2E\_2D \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (12)$$

Let $ty\_2Efcp\_2Efinite\_\_image : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Efcp\_2Efinite\_\_image\ A0) \quad (13)$$

**Definition 18** *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \mathsf{c\_2Emin\_2E\_3D\_3D\_3E}\ V0t)\ \mathsf{c\_2Ebool\_2E}$

**Definition 19** *We define* c_2Ebool_2E_3F *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (\mathsf{c\_2Emin\_2E\_40}$

**Definition 20** *We define* c_2Eprim__rec_2E_3C *to be* $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enu$

**Definition 21** *We define* c_2Ebool_2E_3F_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ \mathsf{c\_2Ebool\_2E\_2F\_5C}$

**Definition 22** *We define* c_2Efcp_2Efinite__index *to be* $\lambda A\_27a : \iota.(ap\ (\mathsf{c\_2Emin\_2E\_40}\ (A\_27a^{ty\_2Enum\_2Enu}$

Let $ty\_2Efcp\_2Ecart : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Efcp\_2Ecart\ A0\ A1) \quad (14)$$

Let $c\_2Efcp\_2Edest\_\_cart : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Efcp\_2Edest\_\_cart\ A\_27a\ A\_27b \in ((A\_27a^{(ty\_2Efcp\_2Efinite\_\_image\ A\_27b)})^{(ty\_2Efcp\_2Ecart\ A\_27a\ A\_27b)}) \quad (15)$$

**Definition 23** *We define* c_2Efcp_2Efcp__index *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in (ty\_2Efcp\_2Ecart\ A\_27$

**Definition 24** *We define* c_2Ewords_2Eword__msb *to be* $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(a$

**Definition 25** *We define* c_2Earithmetic_2EBIT2 *to be* $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

Let $c\_2Earithmetic\_2EEXP : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2EEXP \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (16)$$

**Definition 26** *We define* c_2Ebit_2ESBIT *to be* $\lambda V0b \in 2.\lambda V1n \in ty\_2Enum\_2Enum.(ap\ (ap\ (ap\ (\mathsf{c\_2Eboo}$

Let $c\_2Esum\_\_num\_2ESUM : \iota$ be given. Assume the following.

$$c\_2Esum\_\_num\_2ESUM \in ((ty\_2Enum\_2Enum^{(ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})})^{ty\_2Enum\_2Enum}) \quad (17)$$

**Definition 27** *We define* c_2Ewords_2Ew2n *to be* $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(ap\ (ap$

**Definition 28** *We define* c_2Ebool_2ELET *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0f \in (A\_27b^{A\_27a}).(\lambda V1x \in A\_27$

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})}) \tag{18}$$

**Definition 29** *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2$

Let $c\_2Ewords\_2Eadd\_\_with\_\_carry : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ewords\_2Eadd\_\_with\_\_carry\ A\_27a \in ((ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ 2\ 2))^{(ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ 2))}) \tag{19}$$

Let $c\_2Earithmetic\_2EDIV : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2EDIV \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \tag{20}$$

**Definition 30** *We define* c_2Ebit_2EDIV__2EXP *to be* $\lambda V0x \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2En$

Let $c\_2Earithmetic\_2EMOD : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2EMOD \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \tag{21}$$

**Definition 31** *We define* c_2Ebit_2EMOD__2EXP *to be* $\lambda V0x \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2E$

**Definition 32** *We define* c_2Ebit_2EBITS *to be* $\lambda V0h \in ty\_2Enum\_2Enum.\lambda V1l \in ty\_2Enum\_2Enum.\lambda V$

**Definition 33** *We define* c_2Ebit_2EBIT *to be* $\lambda V0b \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum.(ap$

**Definition 34** *We define* c_2Efcp_2EFCP *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0g \in (A\_27a^{ty\_2Enum\_2Enum}).(ap$

**Definition 35** *We define* c_2Ewords_2En2w *to be* $\lambda A\_27a : \iota.\lambda V0n \in ty\_2Enum\_2Enum.(ap$ (c_2Efcp_2EFC

**Definition 36** *We define* c_2Ewords_2Eword__add *to be* $\lambda A\_27a : \iota.\lambda V0v \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).\lambda V$

Let $c\_2Ewords\_2Edimword : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ewords\_2Edimword\ A\_27a \in (ty\_2Enum\_2Enum^{(ty\_2Ebool\_2Eitself\ A\_27a)}) \tag{22}$$

**Definition 37** *We define* c_2Ewords_2Eword__2comp *to be* $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).$

**Definition 38** *We define* c_2Ewords_2Eword__sub *to be* $\lambda A\_27a : \iota.\lambda V0v \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).\lambda V$

**Definition 39** *We define* c_2Ewords_2Eword__1comp *to be* $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.((ap\ (ap\ c\_2Earithmetic\_2E\_2B\ V0m)\ \mathsf{c\_2Enum\_2E0}) = V0m)) \tag{23}$$

Assume the following.

$$True \tag{24}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \tag{25}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0f \in (A\_27b^{A\_27a}).(\forall V1x \in A\_27a.((ap\ (ap\ (\mathsf{c\_2Ebool\_2ELET}\ A\_27a\ A\_27b)\ V0f)\ V1x) = (ap\ V0f\ V1x)))) \tag{26}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \land ((((p\ V0t) \Rightarrow True) \Leftrightarrow True) \land (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \land ((((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \land (((p\ V0t) \Rightarrow False) \Leftrightarrow (\neg (p\ V0t)))))))) \tag{27}$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \land (((\neg True) \Leftrightarrow False) \land ((\neg False) \Leftrightarrow True))) \tag{28}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.(V0x = V0x)) \tag{29}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \tag{30}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \tag{31}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \land ((((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \land (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg (p\ V0t))) \land (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg (p\ V0t))))))) \tag{32}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).(\forall V1Q \in (2^{A\_27a}).((\forall V2x \in A\_27a.((p\ (ap\ V0P\ V2x)) \wedge (p\ (ap\ V1Q\ V2x)))) \Leftrightarrow ((\forall V3x \in A\_27a.(p\ (ap\ V0P\ V3x))) \wedge (\forall V4x \in A\_27a.(p\ (ap\ V1Q\ V4x)))))))) \tag{33}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow ((\forall V0t1 \in A\_27a.(\forall V1t2 \in A\_27a.((ap\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ECOND}\ A\_27a)\ \mathsf{c\_2Ebool\_2ET})\ V0t1)\ V1t2) = V0t1))) \wedge (\forall V2t1 \in A\_27a.(\forall V3t2 \in A\_27a.((ap\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ECOND}\ A\_27a)\ \mathsf{c\_2Ebool\_2EF})\ V2t1)\ V3t2) = V3t2)))) \tag{34}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((ap\ (\mathsf{c\_2Ecombin\_2EI}\ A\_27a)\ V0x) = V0x)) \tag{35}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27b.((ap\ (c\_2Epair\_2EFST\ A\_27a\ A\_27b)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ A\_27a\ A\_27b)\ V0x)\ V1y)) = V0x))) \tag{36}$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \tag{37}$$

Assume the following.

$$(\forall V0A \in 2.((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \tag{38}$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow (((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False))))) \tag{39}$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False))))) \tag{40}$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \tag{41}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow ((p\ V1q) \Leftrightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (p\ V2r))) \wedge (((p\ V0p) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V1q)))) \wedge (((p\ V1q) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V0p)))) \wedge ((p\ V2r) \vee ((\neg(p\ V1q)) \vee (\neg(p\ V0p))))))))))) \tag{42}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p)\Leftrightarrow((p\ V1q)\wedge(p\ V2r)))\Leftrightarrow(((p\ V0p)\vee((\neg(p\ V1q))\vee(\neg(p\ V2r))))\wedge(((p\ V1q)\vee(\neg(p\ V0p)))\wedge((p\ V2r)\vee(\neg(p\ V0p))))))))) \tag{43}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p)\Leftrightarrow((p\ V1q)\vee(p\ V2r)))\Leftrightarrow(((p\ V0p)\vee(\neg(p\ V1q)))\wedge(((p\ V0p)\vee(\neg(p\ V2r)))\wedge((p\ V1q)\vee((p\ V2r)\vee(\neg(p\ V0p)))))))))) \tag{44}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p)\Leftrightarrow((p\ V1q)\Rightarrow(p\ V2r)))\Leftrightarrow(((p\ V0p)\vee(p\ V1q))\wedge(((p\ V0p)\vee(\neg(p\ V2r)))\wedge((\neg(p\ V1q))\vee((p\ V2r)\vee(\neg(p\ V0p)))))))))) \tag{45}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(((p\ V0p)\Leftrightarrow(\neg(p\ V1q)))\Leftrightarrow(((p\ V0p)\vee(p\ V1q))\wedge((\neg(p\ V1q))\vee(\neg(p\ V0p))))))) \tag{46}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p\ V0p)\Rightarrow(p\ V1q)))\Rightarrow(p\ V0p)))) \tag{47}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p\ V0p)\Rightarrow(p\ V1q)))\Rightarrow(\neg(p\ V1q))))) \tag{48}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p\ V0p)\vee(p\ V1q)))\Rightarrow(\neg(p\ V0p))))) \tag{49}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p\ V0p)\vee(p\ V1q)))\Rightarrow(\neg(p\ V1q))))) \tag{50}$$

Assume the following.

$$(\forall V0p \in 2.((\neg(\neg(p\ V0p)))\Rightarrow(p\ V0p))) \tag{51}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(\forall V1y \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(\forall V2carry\_\_in \in 2.((ap\ (c\_2Ewords\_2Eadd\_\_with\_\_carry\ A\_27a)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ 2))\ V0x)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ 2)\ V1y)\ V2carry\_\_in))) = (ap\ (ap\ (\mathsf{c\_2Ebool\_2ELET}\ ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ 2\ 2)))\ (\lambda V3unsigned\_\_sum \in ty\_2Enum\_2Enum.(ap\ (ap\ (\mathsf{c\_2Ebool\_2ELET}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ 2\ 2)))\ (\lambda V4result \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(ap\ (ap\ (\mathsf{c\_2Ebool\_2ELET}\ 2\ (ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ 2\ 2)))\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ELET}\ 2\ ((ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ 2\ 2))^{2}))\ (\lambda V5carry\_\_out \in 2.(\lambda V6overflow \in 2.(ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ 2\ 2))\ V4result)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ 2\ 2)\ V5carry\_\_out)\ V6overflow)))))\ (ap\ \mathsf{c\_2Ebool\_2E\_7E}\ (ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ ty\_2Enum\_2Enum)\ (ap\ (\mathsf{c\_2Ewords\_2Ew2n}\ A\_27a)\ V4result))\ V3unsigned\_\_sum))))\ (ap\ (ap\ \mathsf{c\_2Ebool\_2E\_2F\_5C}\ (ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ 2)\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_msb}\ A\_27a)\ V0x))\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_msb}\ A\_27a)\ V1y)))\ (ap\ \mathsf{c\_2Ebool\_2E\_7E}\ (ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ 2)\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_msb}\ A\_27a)\ V0x))\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_msb}\ A\_27a)\ V4result)))))))\ (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27a)\ V3unsigned\_\_sum))))\ (ap\ (ap\ c\_2Earithmetic\_2E\_2B\ (ap\ (ap\ c\_2Earithmetic\_2E\_2B\ (ap\ (\mathsf{c\_2Ewords\_2Ew2n}\ A\_27a)\ V0x))\ (ap\ (\mathsf{c\_2Ewords\_2Ew2n}\ A\_27a)\ V1y)))\ (ap\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ECOND}\ ty\_2Enum\_2Enum)\ V2carry\_\_in)\ (ap\ \mathsf{c\_2Earithmetic\_2ENUMERAL}\ (ap\ \mathsf{c\_2Earithmetic\_2EBIT1}\ \mathsf{c\_2Earithmetic\_2EZERO})))\ \mathsf{c\_2Enum\_2E0})))))) \tag{52}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.((ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27a)\ V0m))\ (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27a)\ V1n)) = (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27a)\ (ap\ (ap\ c\_2Earithmetic\_2E\_2B\ V0m)\ V1n))))) \tag{53}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0a \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).((ap\ (\mathsf{c\_2Ewords\_2Eword\_\_1comp}\ A\_27a)\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_1comp}\ A\_27a)\ V0a)) = V0a)) \tag{54}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow ((\forall V0w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).((ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ V0w)\ (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27a)\ \mathsf{c\_2Enum\_2E0})) = V0w)) \land (\forall V1w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).((ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27a)\ \mathsf{c\_2Enum\_2E0}))\ V1w) = V1w))) \tag{55}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0v \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(\forall V1w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(\forall V2x \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).((ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ V0v)\ (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ V1w)\ V2x)) = (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ V0v)\ V1w))\ V2x))))) \tag{56}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0v \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(\forall V1w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).((ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ V0v)\ V1w) = (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ V1w)\ V0v)))) \tag{57}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).((ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ V0w)\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_2comp}\ A\_27a)\ V0w)) = (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27a)\ \mathsf{c\_2Enum\_2E0}))) \tag{58}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0v \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(\forall V1w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(\forall V2x \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(((ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ V0v)\ V1w) = (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ V0v)\ V2x)) \Leftrightarrow (V1w = V2x))))) \tag{59}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0w \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).((ap\ (\mathsf{c\_2Ewords\_2Eword\_\_1comp}\ A\_27a)\ V0w) = (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_sub}\ A\_27a)\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_2comp}\ A\_27a)\ V0w))\ (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27a)\ (ap\ \mathsf{c\_2Earithmetic\_2ENUMERAL}\ (ap\ \mathsf{c\_2Earithmetic\_2EBIT1}\ \mathsf{c\_2Earithmetic\_2EZERO})))))) \tag{60}$$

**Theorem 1**

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow \forall A\_27c.\ nonempty\ A\_27c \Rightarrow (((\forall V0a \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).(\forall V1b \in (ty\_2Efcp\_2Ecart\ 2\ A\_27a).((ap\ (c\_2Epair\_2EFST\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ 2\ 2))\ (ap\ (c\_2Ewords\_2Eadd\_\_with\_\_carry\ A\_27a)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ (ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ 2))\ V0a)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27a)\ 2)\ V1b)\ \mathsf{c\_2Ebool\_2EF})))) = (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_add}\ A\_27a)\ V0a)\ V1b)))) \land ((\forall V2a \in (ty\_2Efcp\_2Ecart\ 2\ A\_27b).(\forall V3b \in (ty\_2Efcp\_2Ecart\ 2\ A\_27b).((ap\ (c\_2Epair\_2EFST\ (ty\_2Efcp\_2Ecart\ 2\ A\_27b)\ (ty\_2Epair\_2Eprod\ 2\ 2))\ (ap\ (c\_2Ewords\_2Eadd\_\_with\_\_carry\ A\_27b)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27b)\ (ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27b)\ 2))\ V2a)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27b)\ 2)\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_1comp}\ A\_27b)\ V3b))\ \mathsf{c\_2Ebool\_2ET})))) = (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_sub}\ A\_27b)\ V2a)\ V3b)))) \land (\forall V4a \in (ty\_2Efcp\_2Ecart\ 2\ A\_27c).(\forall V5b \in (ty\_2Efcp\_2Ecart\ 2\ A\_27c).((ap\ (c\_2Epair\_2EFST\ (ty\_2Efcp\_2Ecart\ 2\ A\_27c)\ (ty\_2Epair\_2Eprod\ 2\ 2))\ (ap\ (c\_2Ewords\_2Eadd\_\_with\_\_carry\ A\_27c)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27c)\ (ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27c)\ 2))\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_1comp}\ A\_27c)\ V4a))\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27c)\ 2)\ V5b)\ \mathsf{c\_2Ebool\_2ET})))) = (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_sub}\ A\_27c)\ V5b)\ V4a)))))) \land ((\forall V6n \in ty\_2Enum\_2Enum.(\forall V7a \in (ty\_2Efcp\_2Ecart\ 2\ A\_27b).((ap\ (c\_2Epair\_2EFST\ (ty\_2Efcp\_2Ecart\ 2\ A\_27b)\ (ty\_2Epair\_2Eprod\ 2\ 2))\ (ap\ (c\_2Ewords\_2Eadd\_\_with\_\_carry\ A\_27b)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27b)\ (ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27b)\ 2))\ V7a)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27b)\ 2)\ (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27b)\ V6n))\ \mathsf{c\_2Ebool\_2ET})))) = (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_sub}\ A\_27b)\ V7a)\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_1comp}\ A\_27b)\ (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27b)\ V6n)))))) \land (\forall V8n \in ty\_2Enum\_2Enum.(\forall V9b \in (ty\_2Efcp\_2Ecart\ 2\ A\_27c).((ap\ (c\_2Epair\_2EFST\ (ty\_2Efcp\_2Ecart\ 2\ A\_27c)\ (ty\_2Epair\_2Eprod\ 2\ 2))\ (ap\ (c\_2Ewords\_2Eadd\_\_with\_\_carry\ A\_27c)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27c)\ (ty\_2Epair\_2Eprod\ (ty\_2Efcp\_2Ecart\ 2\ A\_27c)\ 2))\ (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27c)\ V8n))\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ (ty\_2Efcp\_2Ecart\ 2\ A\_27c)\ 2)\ V9b)\ \mathsf{c\_2Ebool\_2ET})))) = (ap\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_sub}\ A\_27c)\ V9b)\ (ap\ (\mathsf{c\_2Ewords\_2Eword\_\_1comp}\ A\_27c)\ (ap\ (\mathsf{c\_2Ewords\_2En2w}\ A\_27c)\ V8n))))))))$$