First-Order Instantiation-Based Tableau

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Outline

- Brief History: How did we get here?
- First-Order Tableau Rules for Satallax/Lash: Where are we?
  - Instantiation Based
  - No Free Variables, No Unification
  - What are the Rules for Equality?
- Tableau Is Sometimes Better Than Resolution:
  Is it worth being here?
- Completeness: Are we really here?
- Alternative Rules: Where could we go from here?

Outline

A Little History

Beth, Hintikka, Smullyan, Fitting

Rule A: $\dfrac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}}$ *Rule B*: $\dfrac{\beta}{\beta_1 \mid \beta_2}$

Rule C: $\dfrac{\gamma}{\gamma(a)}$, where a is *any* parameter.

Rule D: $\dfrac{\delta}{\delta(a)}$, where a is a *new* parameter.

Smullyan 1995 *First-Order Logic*

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- Only parameters as terms
- No equality
- Is every new rule the "ε-rule"?

Quick Higher-Order ATP History

Andrews, Huet

- Huet: Resolution, 1972

- Andrews: Several theoretical papers in the 1970s

- Leading to a system TPS started in the 1980s:
  TPS (Andrews' Higher-Order "Theorem Proving System")

- Automation based on:
  - expansion proofs (like a compact sequent calculus) and
  - mating method/connection method (close to tableau)

Higher-Order Tableau History

Kohlhase (TABLEAUX 1995) *Higher-Order Tableaux*

Like Smullyan's but with "free variables" for universals:

$$\frac{(\mathbf{A} \lor \mathbf{B})^{\mathrm{T}}}{\mathbf{A}^{\mathrm{T}} \mid \mathbf{B}^{\mathrm{T}}}\,\mathcal{HT}(\land) \qquad \frac{(\mathbf{A} \lor \mathbf{B})^{\mathrm{F}}}{\begin{array}{c}\mathbf{A}^{\mathrm{F}}\\ \mathbf{B}^{\mathrm{F}}\end{array}}\,\mathcal{HT}(\lor) \qquad \frac{(\neg\mathbf{A})^{\mathrm{T}}}{\mathbf{A}^{\mathrm{F}}}\,\mathcal{HT}(\neg^{\mathrm{F}}) \qquad \frac{(\neg\mathbf{A})^{\mathrm{F}}}{\mathbf{A}^{\mathrm{T}}}\,\mathcal{HT}(\neg^{\mathrm{T}})$$

$$\frac{(\Pi^{\alpha}\mathbf{A})^{\mathrm{T}}}{(\mathbf{A}X_{\alpha}^{+})^{\mathrm{T}}}\,\mathcal{HT}(all) \qquad \frac{(\Pi^{\alpha}\mathbf{A})^{\mathrm{F}}}{(\mathbf{A}X_{\alpha}^{-})^{\mathrm{F}}}\,\mathcal{HT}(ex)$$

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New "decomposition" rule:

$$\frac{h\overline{\mathbf{U}^n} \neq^? h\overline{\mathbf{V}^n} \quad h \in \Sigma \cup \mathbf{Dom}(\Gamma^0) \cup \mathbf{Dom}(\Gamma^-)}{\mathbf{U}^1 \neq^? \mathbf{V}^1 \mid \ldots \mid \mathbf{U}^n \neq^? \mathbf{V}^n}\mathcal{HT}(dec)$$

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And other rules...

Higher-Order Tableau History

Konrad (TPHOLs 1998) *HOT: A Concurrent Automated Theorem Prover Based on Higher-Order Tableaux*

Again...usual rules with "free variables" for universals:

$$\frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}}\,alpha \qquad \frac{\beta}{\beta_1 \mid \beta_2}\,beta \qquad \frac{\neg\neg\mathbf{F}}{\mathbf{F}}\,not$$

$$\frac{\delta}{\delta((sk^n X_1,\ldots,X_n))}\,delta \qquad \frac{\gamma}{\gamma(V)}\,gamma$$

$$\frac{h\overline{\mathbf{U}^n} \neq^? h\overline{\mathbf{V}^n}}{\mathbf{U}^1 \neq^? \mathbf{V}^1 \mid \ldots \mid \mathbf{U}^n \neq^? \mathbf{V}^n}\,dec$$

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$$\frac{\delta}{\delta((sk^n X_1, \ldots, X_n))}\ \mathit{delta} \qquad \frac{\gamma}{\gamma(V)}\ \mathit{gamma}$$

$$\frac{h\overline{\mathbf{U}^n} \neq^? h\overline{\mathbf{V}^n}}{\mathbf{U}^1 \neq^? \mathbf{V}^1 \mid \ldots \mid \mathbf{U}^n \neq^? \mathbf{V}^n}\ \mathit{dec}$$

Plus “link” rules (like “general mating” rule):

$$\frac{\begin{array}{c}\mathbf{A}_o\\ \mathbf{B}_o\end{array}}{\neg\mathbf{A} \neq^? \mathbf{B}}\ \mathit{link}_1 \qquad \frac{\begin{array}{c}\mathbf{A}_o\\ \mathbf{B}_o\end{array}}{\mathbf{A} \neq^? \neg\mathbf{B}}\ \mathit{link}_2$$

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Decomposition rule:

$$\mathcal{T}_{\text{DEC}} \quad \frac{xs_1 \ldots s_n \neq_\alpha xt_1 \ldots t_n}{s_1 \neq t_1 \mid \cdots \mid s_n \neq t_n} \quad n \geq 0$$

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Mating rule (instead of Konrad's Link rules):

$$\mathcal{T}_{\text{MAT}} \quad \frac{xs_1 \ldots s_n \,,\, \neg xt_1 \ldots t_n}{s_1 \neq t_1 \mid \cdots \mid s_n \neq t_n} \quad n \geq 0$$

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Confrontation rule for positive equations:

$$\mathcal{T}_{\text{CON}} \quad \frac{s =_\alpha t \,,\, u \neq_\alpha v}{s \neq u \,,\, t \neq u \mid s \neq v \,,\, t \neq v}$$

Flashback: Takahashi's Forgotten Extensionality Rule

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Almost a (sequent calculus) "mating" rule:

$$\frac{S_{1i_1} \cdots S_{1i_m} \quad S_{2i_1} \cdots S_{2i_m}}{(d_1^{\tau_1}, \cdots, d_n^{\tau_n} \in e^{(\tau_1,\cdots,\tau_n)}), \Gamma \to \Delta, (e_1^{\tau_1}, \cdots, e_n^{\tau_n} \in e^{(\tau_1,\cdots,\tau_n)})},$$

where

1) $e^{(\tau_1,\cdots,\tau_n)}$ is a free variable or a constant;
2) at least one of $\tau_1, \cdots, \tau_n$ is $\neq 0$;
3) $\tau_i = 0$ implies $d_i^{\tau_i} = e_i^{\tau_i}$;
4) $i_1, \cdots, i_m$ are all the indices i with $\tau_i \neq 0$;
5) if $\tau_i = 1$, the S_{1i} and S_{2i} denote the sequents

$$d_i^{\tau_i}, \Gamma \to \Delta, e_i^{\tau_i}$$
$$e_i^{\tau_i}, \Gamma \to \Delta, d_i^{\tau_i}$$

respectively;

6) if $\tau_i = (\sigma_{i1}, \cdots, \sigma_{ir})$, then S_{1i} and S_{2i} denote the sequents

$$(a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}} \in d_i^{\tau_i}), \Gamma \to \Delta, (a_{iq}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}} \in e_i^{\tau_i}),$$
$$(a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}} \in e_i^{\tau_i}), \Gamma \to \Delta, (a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}} \in d_i^{\tau_i})$$

respectively ($a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}}$ should not occur in the lower sequent of this inference).

Outline

Satallax/Lash

- Satallax: Higher-order ATP based on a Henkin complete tableau calculus.
- Won TH0 division of CASC most years of the 2010s.
- Instantiation based – used *no* unification in the basic calculus.
- Able to reason with equations without rewriting deeply inside terms.

- Lash is a new implementation of Satallax's calculus.
- Cezary Kaliszyk reimplemented terms/$\beta\eta$-normalization in C
- ...with perfect sharing.
- Lash is much faster than Satallax, but loses CASC.

Satallax/Lash First-Order Tableau

- A branch is a set of closed formulas.
- A branch is *closed* if either
  - φ and $\neg\varphi$ are on the branch for some φ, or
  - $s \neq s$ is on the branch for some s.
- A term t is *discriminating* for a branch if either $s \neq t$ or $t \neq s$ is on the branch (for some term s).

Satallax/Lash First-Order Tableau Rules (1)

Usual rules

$$\frac{\varphi \vee \psi}{\varphi | \psi} \qquad \frac{\neg(\varphi \vee \psi)}{\neg\varphi, \neg\psi} \qquad \frac{\neg\neg\varphi}{\varphi} \qquad \frac{\forall x.\varphi(x)}{\varphi(t)}\ t \text{ discriminating}$$

$$\frac{\neg\forall x.\varphi(x)}{\neg\varphi(c)}\ c \text{ fresh}$$

Note: only use "discriminating" instantiations in $\forall$ rule.

There are no free variables to be instantiated later.

Satallax/Lash First-Order Tableau Rules (2)

Other rules (involving disequations)

$$\text{Mating}\ \frac{p(s_1,\ldots,s_n),\ \neg p(t_1,\ldots,t_n)}{s_1 \neq t_1 | \cdots | s_n \neq t_n}$$

$$\text{Dec}\ \frac{f(s_1,\ldots,s_n) \neq f(t_1,\ldots,t_n)}{s_1 \neq t_1 | \cdots | s_n \neq t_n}$$

Satallax/Lash First-Order Tableau Rules (3)

Equality rule.

What it's *not*:

$$\text{Rewrite}\ \frac{s = t, \varphi[s]}{\varphi[t]}$$

Satallax/Lash First-Order Tableau Rules (3)

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What it's *not*:

$$\text{Rewrite}\ \frac{s = t, \varphi[s]}{\varphi[t]}$$

There is *no* rewrite rule in the calculus.

Satallax/Lash First-Order Tableau Rules (3)

Equality rule.

What it *is*:

$$\text{Con}\ \frac{s = t, u \neq v}{s \neq u, t \neq u \mid s \neq v, t \neq v}$$

Combining with a SAT Solver

- Both Satallax and Lash search by applying tableau rules, generating propositional clauses and incrementally sending the clauses to MiniSat.
- When the clauses are unsatisfiable, there is a tableau refutation.
- Details are here:

Brown (JAR 2013) Reducing Higher-Order Theorem Proving to a Sequence of SAT Problems

Outline

Simple Clausal Problem Set

- 100 (first-order clausal) problems with similar format.
- All have 5 clauses.
- 2 Clauses (model must be infinite):
  - $\forall x. f(x) \neq c$
  - $\forall xy. f(x) \neq f(y) \vee x = y$
- 3 Clauses (model must be finite):
  - Third clause: $\forall xy. x = y \vee r(x, y) \vee b(x, y)$
  - Fourth clause: $\forall x_0 \ldots x_n. \cdots \neg r(x_i, x_j) \cdots$
  - Fifth clause: $\forall x_0 \ldots x_n. \cdots \neg b(x_i, x_j) \cdots$
- The last two clauses always have between 3 and 10 literals.

Simplest of the 100

1. $\forall x. f(x) \neq c$
2. $\forall xy. f(x) \neq f(y) \vee x = y$
3. Third clause: $\forall xy. x = y \vee r(x, y) \vee b(x, y)$
4. Fourth clause: $\forall x_0 x_1 x_2. \neg r(x_0, x_1) \vee \neg r(x_0, x_2) \vee \neg r(x_1, x_2)$
5. Fifth clause: $\forall x_0 x_1 x_2. \neg b(x_0, x_1) \vee \neg b(x_0, x_2) \vee \neg b(x_1, x_2)$

Sketch of Lash Search

- Start with the branch with the 5 formulas.
- Technically there are no discriminating terms, so seed instantiations with the constant c.
- Instantiate $\forall x. f(x) \neq c$ with c.
- Now $f(c) \neq c$ is on the branch and both $f(c)$ and c are discriminating.
- Instantiate all $\forall$'s with c and $f(c)$.
- This gives more discriminating terms and leads to the $\vee$ formulas.
- Split the $\vee$'s and let MiniSat sort out unsatisfiability.
- Easiest of the 100 takes Lash $<$ 50 ms.

Comparison on Problem Set

- 5 minute timeout
- Lash: 82

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- Geo-III: 11 (Hans de Nivelle)
- iProver: 8 (Konstantin Korovin)
- Vampire: 3

Outline

Completeness

- Suppose a branch A is not tableau refutable.
- A can be extended to a Hintikka set H.
- *Model Existence*: H has a model.
- The model construction follows ideas from Schütte 1960, Tait 1966, Takahashi 1967, Prawitz 1968, Takahashi 1968 and Andrews 1971.

Hintikka Set (1)

- $s \neq s \notin H$.
- $s \notin H$ or $\neg s \notin H$.
- If $\neg\neg\varphi \in H$, then $\varphi \in H$.
- If $\varphi \vee \psi \in H$, then $\varphi \in H$ or $\psi \in H$.
- If $\neg(\varphi \vee \psi) \in H$, then $\neg\varphi \in H$ and $\neg\psi \in H$.
- If $\forall x.\varphi(x) \in H$ and t discriminating on H, then $\varphi(t) \in H$.
- If $\neg\forall x.\varphi(x) \in H$, then $\varphi(t) \in H$ for some term t.

Hintikka Set (2)

- Assume signature just has c, f, r and b.
- (Dec) If $f(s) \neq f(t) \in H$, then $(s \neq t) \in H$.
- (Mat) If $r(s_1, s_2) \in H$ and $\neg r(t_1, t_2) \in H$, then $(s_1 \neq t_1) \in H$ or $(s_2 \neq t_2) \in H$.
- (Mat) If $b(s_1, s_2) \in H$ and $\neg b(t_1, t_2) \in H$, then $(s_1 \neq t_1) \in H$ or $(s_2 \neq t_2) \in H$.
- (Con) If $(s = t) \in H$ and $(u \neq v) \in H$, then $(s \neq u) \in H$ and $(t \neq u) \in H$ or $(s \neq v) \in H$ and $(t \neq v) \in H$.

Compatibility and Possible Values

- Terms s and t are *compatible* if $(s \neq t) \notin H$ and $(t \neq s) \notin H$.
- A *discriminant* Δ is a maximal set of compatible discriminating terms.
- Let D (domain of interpretation) be the set of all discriminants.
- Possible values: $s \triangleright \Delta$ means "s has Δ as possible value."
- Def: $s \triangleright \Delta$ if $s \in \Delta$ or s is not discriminating.
- Theorem: Every set of compatible terms has a common possible value.
- Theorem: If $s = t \in H$, then there is one Δ with $s \triangleright \Delta$ and also $t \triangleright \Delta$.

Possible Values For Terms

- Recall: D is the set of all discriminants.
- Let $g : D \to D$.
- Lift $\triangleright$ to unary functions:
- Def: $f \triangleright g$ if $f(s) \triangleright g(\Delta)$ whenever $s \triangleright \Delta$.
- Thm: There is a g such that $f \triangleright g$.
- Interpretation of terms: interpret
  - c as Δ_c where $c \triangleright \Delta_c$ and
  - f by g where $f \triangleright g$.
- Induction: $\theta(s) \triangleright [\![s]\!]_\alpha$ if $\theta(x) \triangleright \alpha(x)$ for all x.
- If $(s \neq t) \in H$, then $[\![s]\!] \neq [\![t]\!]$.
- If $(s = t) \in H$, then $[\![s]\!] = [\![t]\!]$.

Possible Values For Formulas

- Recall: D is the set of all discriminants.
- Let 0 and 1 be possible values for formulas.
- Def:
  - $\varphi \triangleright 0$ if $\varphi \notin H$.
  - $\varphi \triangleright 1$ if $\neg\varphi \notin H$.
- Two formulas φ and ψ are compatible unless $\varphi \in H$ and $\neg\psi \in H$ or $\neg\varphi \in H$ and $\psi \in H$.
- Let $Q : D \times D \to \{0, 1\}$ and q be the relation symbol r or b.
- Def: $q \triangleright Q$ if $q(s, t) \triangleright Q(\Delta, \Delta')$ whenever $s \triangleright \Delta$ and $t \triangleright \Delta'$.
- Interpret r using R such that $r \triangleright R$ and b using B such that $b \triangleright B$.
- Theorem: This is a model of H.

Two Possible Values For Relations

- When choosing Q such that $q \triangleright Q$ there are two obvious choices:
- Minimum: $Q(\Delta, \Delta')$ holds if
  $$q(s, t) \in H \text{ for some } s \in \Delta \text{ and } t \in \Delta'.$$
- Maximum: $Q(\Delta, \Delta')$ holds unless
  $$\neg q(s, t) \in H \text{ for some } s \in \Delta \text{ and } t \in \Delta'.$$
- In the model existence proof above any choice works, but...

Outline

Special Rule for Irreflexivity

- It is possible to avoid some $\forall$ quantifiers by replacing a universally quantified formula with a rule.
- Example: $\forall x.\neg r(x, x)$.
- Incomplete approach: A is closed if $\neg r(s, s) \in H$.
- Instead consider this (complete) rule:

$$\text{Irref}\ \frac{r(s, t)}{s \neq t}$$

- Corresponding Hintikka condition:
- If $r(s, t) \in H$, then $(s \neq t) \in H$.
- If we choose minimum R with $r \triangleright R$, then $\forall x.\neg r(x, x)$ will hold.

Special Rule for Third Clause

- Rule replacing third clause: $\forall xy.x = y \vee r(x, y) \vee b(x, y)$

$$\text{Cover} \; \frac{\neg r(s,t)}{s = t | b(s,t)}$$

- Hintikka condition:
- If $\neg r(s, t) \in H$, then $s = t \in H$ or $b(s, t) \in H$.
- Choosing max R ensures third clause holds.

Special Rule for Last Clauses

- Rule replacing fourth or fifth clauses, e.g., $\forall x_0 x_1 x_2 . \neg b(x_0, x_1) \vee \neg b(x_0, x_2) \vee \neg b(x_1, x_2)$

$$\text{NotHom}\ \frac{b(s_1, t), b(s_2, u)}{s_1 \neq s_2 | \neg b(t, u)}$$

- Hintikka condition:
- If $b(s_1, t), b(s_2, u) \in H$, then $s_1 \neq s_2 \in H$ or $\neg b(t, u) \in H$.
- Choosing min B ensures clause holds.

Results with Special Rules

- Implemented optional use of special rules in Lash.
- Caveat: Restriction to only create rules if they're easy to apply.
- Result: Lash essentially proves the same 82.
- Some are faster and some are slower.
- E.G.: One that took just over 2 minutes before takes only 16 seconds with one kind of special rule, but not the other.

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- Set of problems where it outperforms resolution
- Completeness proof from related higher-order contexts (Takahashi 1967-1968, Prawitz 1968, Andrews 1971)
- Modifications to Completeness Proof justify lifting some formulas to rules

Final Slide

Thank you!

Questions?