A Tale of Two Set Theories

Chad E. Brown, Karol Pąk

Czech Technical University in Prague, Czech Republic

University of Białystok, Poland

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Introduction

Tarski vs. Grothendieck

Higher-Order Tarski-Grothendieck

Proving Axiom A in Egal

Constructing Grothendieck Universes in Mizar

Conclusion

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Mizar and Egal

- Mizar 1973-now
  - First-Order Tarski-Grothendieck
  - Scheme for Replacement
  - Universes via Tarski's Axiom A
  - Big Library (MML) $>$ 60K theorems
- Egal 2014-now
  - Higher-Order Tarski-Grothendieck
  - Replacement as a single HO formula
  - Universes via a Grothendieck universe operator
  - Small Library $<$ 1000 theorems
- Goal: Compare the two set theories.

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Tarski's Axiom A

- Tarski 1938. Über Unerreichbare Kardinalzahlen.
- Axiom A: every set N is a member of some Tarski universe M.
- For M to be a Tarski universe it must satisfy:
  1. $X \in M$ and $Y \subseteq X$ imply $Y \in M$
  2. $X \in M$ implies $\wp X \in M$ (sort of)
  3. $X \subseteq M$ implies X and M are equipotent or $X \in M$

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- Equipotence of X and M is defined via existence of a set of Kuratowski pairs that is, essentially, a bijection from X to M.
- $X \subseteq M$ and $X \notin M$ implies X "is as big as" M.

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- Equipotence of X and M is defined via existence of a set of Kuratowski pairs that is, essentially, a bijection from X to M.
- $X \subseteq M$ and $X \notin M$ implies X "is as big as" M.
- Note: Axiom A implies Choice

Grothendieck Universes

- Grothendieck, Verdier 1972. Théorie des topos et cohomologie étale des schémas - (SGA 4) - vol. 1
- Universe Axiom: every set N is in a Grothendieck universe U.
- For U to be a Grothendieck universe it must satisfy:
  1. U is a transitive set ($X \in U$ implies $X \subseteq U$).
  2. U is closed under ZF operations.

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- Not every Tarski universe is transitive.

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- With Choice, every Grothendieck universe is Tarski.

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- With Choice, every Grothendieck universe is Tarski.
- ZFC $\vdash$ Axiom A $\Leftrightarrow$ Universe Axiom

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Higher-Order Set Theory

- ι - base type of sets
- o - type of propositions
- $\sigma \to \tau$ - type of functions from σ to τ

Typed Terms:

- $\mathcal{V}_\sigma$ - variables x of type σ
- $\mathcal{C}_\sigma$ - constants c of type σ
- Λ_σ - terms of type σ generated by

  $$s, t ::= x|c|st|\lambda x.s|s \Rightarrow t|\forall x.s$$

  restricted to well-typed terms.
- $(\lambda x.s)$ has type $\sigma \to \tau$ where $x \in \mathcal{V}_\sigma$ and $s \in \Lambda_\tau$. It means the function sending x to s.
- Formula - term of type o
- Definable: $\wedge$, $\vee$, $\equiv$, $=$, $\exists$, $\exists!$ (Russell-Prawitz)

Higher-Order Set Theory (Constants)

- $\varepsilon_\sigma : (\sigma \to o) \to \sigma$
- In : $\iota \to \iota \to o$
- Empty : ι
- Union : $\iota \to \iota$
- Power : $\iota \to \iota$
- Repl : $\iota \to (\iota \to \iota) \to \iota$

- UnivOf : $\iota \to \iota$

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  - Repl s $(\lambda x.t)$ means $\{t|x \in s\}$
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- Repl : $\iota \to (\iota \to \iota) \to \iota$
  - Repl s $(\lambda x.t)$ means $\{t | x \in s\}$
- UnivOf : $\iota \to \iota$
  - UnivOf s means the least Grothendieck universe with s as a member.

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Higher-Order Set Theory (Axioms)

- Propositional and Functional Extensionality
- Choice
- Set Extensionality
- $\in$-induction (higher-order, but equivalent to regularity)

$$\forall P : \iota \to o.(\forall X.(\forall x \in X.Px) \Rightarrow PX) \Rightarrow \forall X.PX$$

- Empty
- Union
- Power
- Replacement (higher-order formula, not a scheme)

$$\forall X : \iota.\forall F : \iota \to \iota.\forall y : \iota.y \in \{Fx|x \in X\} \Leftrightarrow \exists x \in X.y = Fx$$

  where $\{Fx|x \in X\}$ is Repl X $(\lambda x.Fx)$.
- Universes. Write $\mathcal{U}_N$ for UnivOf N.

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  - $\mathcal{U}_N$ is transitive
  - $\mathcal{U}_N$ is ZF-closed (Union, Power, Repl)
  - $\mathcal{U}_N$ is the least such set.

Egal

- Egal is a proof assistant based on Higher-Order Tarski-Grothendieck
- Natural Deduction proofs – with proof terms
- Meant to satisfy de Bruijn criteria (independently checkable proof terms)
- Format of proofs is similar to Coq

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- Could we translate Mizar's MML into Egal?
- First step: Are Mizar's Axioms provable in Egal?
- Axiom A?

Egal Library

- Small: About 65 definitions and 600 theorems.
- A few definitions
  - ordinal : $\iota \to o$.
  - Sep : $\iota \to (\iota \to o) \to \iota$ giving $\{x \in X | P\ x\}$
  - ReplSep : $\iota \to (\iota \to o) \to (\iota \to \iota) \to \iota$ giving $\{F\ x | x \in X, P\ x\}$
  - Unordered pairs, singletons, etc.
  - $\mathbf{R} : (\iota \to (\iota \to \iota) \to \iota) \to \iota \to \iota$ definition by $\in$-recursion.
- A few theorems:
  - $x \notin x$
  - Regularity: $x \in X \Rightarrow \exists Y \in X. \neg\exists z \in X. z \in Y$
  - ordinal $\alpha \Rightarrow \forall\beta \in \alpha.$ordinal β
  - ordinal $\alpha \Rightarrow$ ordinal $\beta \Rightarrow \alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha$

Definition by Epsilon (Membership) Recursion

Functions from sets to sets can be defined by $\in$-recursion.
Suppose $\Phi : \iota(\iota\iota)\iota$ satisfies

$$\forall XFG.(\forall x.x \in X \to Fx = Gx) \to \Phi XF = \Phi XG.$$

Under this condition, Φ defines a function $\mathbf{R}\Phi$ satisfying

$$\forall X.\mathbf{R}\Phi X = \Phi X(\lambda x.\mathbf{R}\Phi x)$$

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Proving Axiom A

- Main Lemma Let U be ZF-closed, transitive set. If $X \subseteq U$ and $X \notin U$, then there is a bijection $f : \iota \to \iota$ taking $\{\alpha \in U | \text{ordinal } \alpha\}$ onto X.
- Axiom A easily follows applying the Main Lemma twice to obtain:
  - Bijection g from $\{\alpha \in U | \text{ordinal } \alpha\}$ onto X.
  - Bijection h from $\{\alpha \in U | \text{ordinal } \alpha\}$ onto U.

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Proving the Main Lemma

- Start by defining von Neumann hierarchy: $\mathbf{V} : \iota \to \iota$ such that
$$\forall X.\mathbf{V}_X = \bigcup_{x \in X} \wp(\mathbf{V}_x)$$
- If U is transitive and ZF-closed, then it is $\mathbf{V}$-closed.
- Now, let U be transitive and ZF-closed.
- Let $X \subseteq U$ such that $X \notin U$.

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- Define $\boldsymbol{\lambda}$ be $\{\alpha \in U | \text{ordinal } \alpha\}$.

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- Define $\mathbf{P}$ such that $\mathbf{P}\ \alpha\ x\ f$ means $x \in X$ and $f \beta \neq x$ for $\beta \in \alpha$.

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- Define $\mathbf{Q}$ such that $\mathbf{Q}\ \alpha\ f\ x$ means x is a sort of $\mathbf{V}$-minimal set satisfying $\mathbf{P}\ \alpha\ x\ f$.

Proving the Main Lemma

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- Define $\mathbf{f}$ to be $\mathbf{RF}$ and
- $\mathbf{g}$ to be $\lambda y. \varepsilon \alpha \in \boldsymbol{\lambda}. \mathbf{f}\alpha = y$.

Proving the Main Lemma

- Define $\boldsymbol{\lambda}$ be $\{\alpha \in U | \text{ordinal } \alpha\}$.
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- With some work, get $\forall \alpha \in \boldsymbol{\lambda}.\mathbf{Q}\ \alpha\ \mathbf{f}\ (\mathbf{f}\ \alpha)$.

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- Easy to see **f** maps injectively into X.

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- With some work, get $\forall\alpha \in \boldsymbol{\lambda}.\mathbf{Q}\ \alpha\ \mathbf{f}\ (\mathbf{f}\ \alpha)$.
- Easy to see **f** maps injectively into X.
- With more work, **g** is an inverse so **f** is bijective.

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Grothendieck Universes in Mizar

- What about porting from Egal to Mizar?
- Can we construct a Grothendieck universe operator in Mizar?

Grothendieck Universes in Mizar

- What about porting from Egal to Mizar?
- Can we construct a Grothendieck universe operator in Mizar?
- Yes. And we have done it

Tarski-Class in Mizar

- From the MML (Bancerek) we have a $\mathrm{Tarski-Class}$ operator taking a set to a Tarski universe containing it.
- $A \in \mathrm{Tarski-Class}'A$.
- $\mathrm{Tarski-Class}'A$ is a Tarski universe.

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- Problem: $\mathrm{Tarski-Class}'A$ may not be transitive.

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- MML: If A is transitive, then Tarski−Class'A is transitive.

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- Problem: $\mathrm{Tarski-Class}'A$ may not be transitive.
- MML: If A is transitive, then $\mathrm{Tarski-Class}'A$ is transitive.
- Solution: Use $\mathrm{Tarski-Class}(\mathrm{the\_transitive-closure\_of}\ \{A\})$

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- $\mathrm{Tarski}-\mathrm{Class}'A$ is a Tarski universe.
- Problem: $\mathrm{Tarski}-\mathrm{Class}'A$ may not be transitive.
- MML: If A is transitive, then $\mathrm{Tarski}-\mathrm{Class}'A$ is transitive.
- Solution: Use $\mathrm{Tarski}-\mathrm{Class}(\mathrm{the\_transitive}-\mathrm{closure\_of}\ \{A\})$
- The result is transitive and a Grothendieck universe.

Grothendieck universes operator in Mizar

- Define a Mizar type: Grothendieck of' A.
- Type of all Grothendieck universes of A.
- Nonempty by previous slide.

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- Define a Mizar type: Grothendieck of'A.
- Type of all Grothendieck universes of A.
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- The type is closed under arbitrary intersections.
- Use intersection to define an operator giving the least Grothendieck universe of A.

Outline

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Higher-Order Tarski-Grothendieck

Proving Axiom A in Egal

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- Mizar uses Tarski's Axiom A.
- Egal uses Grothendieck universes.
- We proved Axiom A in Egal.
- We constructed Grothendieck universes in Mizar.
- This provides the first steps towards porting the MML to Egal or future Egal formalizations to the MML.

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