Solving Hard Mizar Problems with Instantiation and Strategy Invention

Jan Jakubův, Mikoláš Janota, Josef Urban

Czech Technical University in Prague

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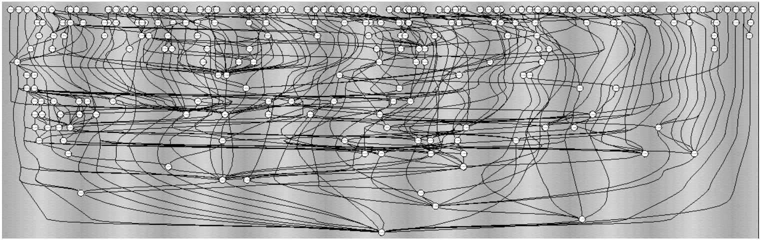
Background - MML and MPTP

- Mizar Mathematical Library (MML): Large library of formal mathematics developed since 1989
- 1465 math articles and 3.7M lines of human-readable proofs in 2024
- In 2003: **MPTP**: Mizar Problems for Theorem Proving
- export MML for automated theorem provers (ATPs)
- Used since for AITP research (MPTP20 talk: `https://t.ly/SFdPA`)
- 2006: the $100 MPTP Challenges (`https://t.ly/clXXe`)
- bushy (easier, smaller) vs chainy (large, *hammer*) MPTP problems



Some MML Theorems

Alexander's Lemma
All Liouville numbers are transcendental
All Primes (1 mod 4) Equal the Sum of Two Squares
Assignment composition
Associativity law
Axiom of Choice
Axiom of Euclid
Axiom schema of continuity
Baire Category Theorem (Banach spaces)
Baire Category Theorem (Hausdorff spaces)
Baire Category Theorem for Continuous Lattices
Banach fixed-point theorem
Banach-Steinhaus theorem (uniform boundedness)
Basel problem
Bayes' theorem
Bertrand's Ballot Theorem
Bertrand's postulate
Bezout's identity
Bezout's lemma
Bing metrization theorem
Binomial Theorem
Birkhoff Variety Theorem
Bolzano theorem (intermediate value)
Bolzano-Weierstrass Theorem (1 dimension)
Borsuk Theorem on Decomposition of Strong Deformation Retracts
Borsuk-Ulam Theorem
Boundary Points of Locally Euclidean Spaces
Branching composition
Brouwer Fixed Point Theorem
Brouwer Fixed Point Theorem for Disks on the Plane
Brouwer Fixed Point Theorem for Intervals
Brown Theorem
CONS_1 rule
CONS_2 rule
Cantor Theorem
Cantor-Bernstein Theorem
Caratheodory's Theorem
Carmichael's Theorem on Prime Divisors
Cauchy Theorem
Cauchy sequence
Cauchy-Schwarz inequality
Cayley Theorem
Centered polygonal number
Ceva's Theorem
Chinese Remainder Theorem
Compactness of Lim-inf Topology
Completeness theorem for Propositional Linear Temporal Logic
Composition rule for sequences of programs
Contraction Lemma
Convergents of continued fraction
Converse 2_dimensional
Converse Desarguesian
Converse Fanoian
Converse Pappian
Converse Vebliean
Converse at_least_3rank
Converse reflexive
Converse transitive
Correctness of Euclid's Algorithm
Correctness of the algorithm of exponentiation by squaring
Cousin's lemma
Cramer's Rule
Cycle composition
DP rule
Darboux Theorem
Darboux's Theorem
De Moivre's Theorem
Deduction Theorem
Deduction theorem

Some MML Theorems

Desargues' Theorem
Dickson Lemma
Dijkstra's shortest path algorithm
Dilworth's Decomposition Theorem
Dimension of the Boundary of Locally Euclidean Spaces
Dimension of the Cartesian Product of Locally Euclidean Spaces
Dimension of the Interior of Locally Euclidean Spaces
Dirichlet Principle
Dirichlet's approximation theorem
Distributivity law
Divergence of the Harmonic Series
Divisibility by 3 Rule
Divisibility rule#Divisibility by 11
Divisibility rule#Divisibility by 13
Divisibility rule#Divisibility by 7
Dynkin Lemma
Egorov's theorem
Emptiness checking predicate
Empty constant function
Empty function
Empty predicate
Erdos-Szekeres Theorem
Euler's Generalization of Fermat's Little Theorem
Euler's Polyhedron Formula
Euler's criterion
Euler's partition theorem
Existence of Cantor Normal Form for ordinal numbers
Extended law of sines
Extreme value theorem
Extreme value theorem#Generalization to arbitrary topological spaces
False constant predicate
Fashoda Meet Theorem
Fatou's Lemma
Feynman's (one-seventh area) Triangle
First Sylow Theorem
First isomorphism theorem for groups
First isomorphism theorem for universal algebras
Fixed-point lemma for normal functions
Ford/Fulkerson maximum flow algorithm
Formula for the Number of Combinations
Frattini subgroup
Friendship theorem
Fubini's theorem
Fubini's theorem
Fundamental Theorem of Algebra
Fundamental Theorem of Arithmetic
Fundamental Theorem of Arithmetic (uniqueness)
Fundamental Theorem of Integral Calculus
Generalized Axiom of Infinity
Generalized Ceva's Theorem
Goedel Completeness Theorem
Grassmann-Pl{?}ucker-Relation in rank 3
Greatest Common Divisor Algorithm
Hahn-Banach Theorem (complex spaces)
Hahn-Banach Theorem (real spaces)
Hahn-Banach's extension theorem (real normed spaces)
Hall Marriage Theorem
Heine-Borel Theorem for intervals
Henrici Theorem
Heron's Formula
Hessenberg's theorem
Hilbert Basis Theorem
Hurwitz's theorem (number theory)
IF rule
Identity composition
Integral of Measurable Function
Integral root theorem
Intermediate Value Theorem
Intersecting chords theorem

Some MML Theorems

Irrationality of e
Jonsson Theorem for lattices
Jordan Curve Theorem for special polygons
Knaster Theorem
Krippenfigur
Kuratowski-Zorn Lemma
Lagrange theorem for addGroups
Law of Cosines
Lebesgue's Monotone Convergence Theorem
Lexicographic_breadth-first_search
Liouville's constant
Little Bezout Theorem (Factor Theorem)
METAMATH: endofsegidand
Main result: Mutual exclusion property of Peterson's algorithm
Markov's inequality
Meister-Gauss formula (for triangles
Modus Barbara
Modus Darii
Morley's trisector theorem
Nachbin theorem for spectra of distributive lattices
Name checking predicate
Niven's Theorem
Partial correctness of GCD algorithm

Isosceles Triangle Theorem
Jonsson Theorem for modular lattices
Jordan Matrix Decomposition Theorem
Koenig Lemma
Kuratowski convergence
Lagrange Theorem
Lagrange's four-square theorem
Lebesgue's Bounded Convergence Theorem
Legendre symbol
Lindenbaum's lemma
Liouville's theorem on diophantine approximation
Lower dimension axiom
METAMATH: segcon2
Main results
Mean value theorem for integrals (first)
Menelaus' Theorem
Modus Celarent
Moebius function
Multiplication of Polynomials using Discrete Fourier Transformation
Nachbin's theorem for bounded distributive lattices
Newman's lemma
Open Mapping Theorem
Partial correctness of a FACTORIAL algorithm

Join-absorbing law
Jordan Curve Theorem
Jordan-Hölder Theorem
Koenig Theorem
Kuratowski's closure-complement problem
Lagrange Theorem for Groups
Laplace expansion
Lebesgue's Covering Lemma
Leibniz's Series for π
Liouville number!irrationality
Lipschitz continuity
Lucas numbers
Main Theorem The AIM Conjecture follows
Makarios: Lemma 6
Meet-absorbing law
Minkowski inequality
Modus Darapti
Monotone Floyd-Hoare composition
Myhill-Nerode theorem
Nagata-Smirnov metrization theorem
Niemytzki plane
Pappus theorem
Partial correctness of a Fibonacci algorithm

Some MML Theorems

- Partial correctness of a Lucas algorithm
- Partial correctness of a POWER algorithm
- Pascal's theorem
- Pepin's test
- Pepin's theorem
- Pigeon Hole Principle
- Pocklington's theorem
- Pre-Routh's Theorem
- Prediction composition
- Prim's Minimum Spanning Tree Algorithm
- Prime Representing Polynomial
- Prime ideal theorem for distributive lattices
- Principle of Inclusion/Exclusion
- Proth's theorem
- Pseudocomplement
- Ptolemy's Theorem
- Pythagorean Theorem
- Quotient ring
- Ramsey's Theorem
- Ramsey's Theorem (finite case)
- Rank-nullity theorem
- Rational root theorem
- Reciprocals of triangular numbers
- Reflection Theorem
- Representation Theorem for Free Continuous Lattices
- Representation theorem for categories as concrete categories
- Rolle Theorem
- Routh's Theorem
- SF rule
- SFID rule
- SFID_1 rule
- SF_1 rule
- Schreier Refinement Theorem
- Schroeder Bernstein theorem
- Schur's criterion
- Second Sylow Theorem
- Second isomorphism theorem for groups
- Sequential composition
- Seven Bridges of Koenigsberg
- Small Fermat's Theorem
- Sorgenfrey line
- Soundness Theorem for LTLB with initial semantics.
- Square triangular number
- Steinitz Theorem
- Stirling numbers of the second kind
- Stone Representation Theorem for Boolean Algebras
- Stone Representation Theorem for Heyting Lattices
- Sum of a Geometric Series
- Sum of an arithmetic series
- Taylor's Theorem
- Telescoping series
- The Cardinality of the Pell's Solutions
- The Denumerability of the Rational Numbers
- The First Implication
- The Infinitude of Primes
- The Irrationality of the Square Root of 2
- The Mean Value Theorem
- The Non-Denumerability of the Continuum
- The Number of Subsets of a Set
- The Perfect Number Theorem
- The Principle of Mathematical Induction
- The Second Implication
- The Small Inductive Dimension of the Sphere
- The Solution of the General Quartic Equation
- The composition of superposition into a function
- The composition of superposition into a function (one function)
- The composition of superposition into a predicate
- The composition of superposition into a predicate (one function)
- The lattice of natural divisors
- The law of quadratic reciprocity
- The ordinal indexing of epsilon num-
- The short(est) axiomatization of ortho-

ATP timeline on MPTP problems

- 2010: Vampire solved 40% of bushy (easier) problems
- 2014: about 40% of chainy (hammer) problems solved by AI/TP methods (also done for Flyspeck)
- **2021**: about 60% of chainy solved with many AI/TP methods:
- E/ENIGMA and Vampire/Deepire (Mizar60 paper at ITP23)
- In total: 75.5% proved (union of bushy and chainy, higher times)
- See `https://github.com/ai4reason/ATP_Proofs` for about 200 interesting proofs found in those experiments
- Our goal here: Solve more of the remaining 14163 *hard* Mizar problems (and thus progress towards my 2014 AITP Challenges)

AITP Challenges/Bets from 2014

- 3 AITP bets for 10k EUR from my 2014 talk at Institut Henri Poincare (`tinyurl.com/yb55b3jv`)
- In 20 years, 80% of Mizar and Flyspeck toplevel theorems will be provable automatically (same hardware, same libraries as in 2014 - about 40% then)
- In 10 years: 60% (DONE already in 2021 - 3 years ahead of schedule)
- In 25 years, 50% of the toplevel statements in LaTeX-written Msc-level math curriculum textbooks will be parsed automatically and with correct formal semantics

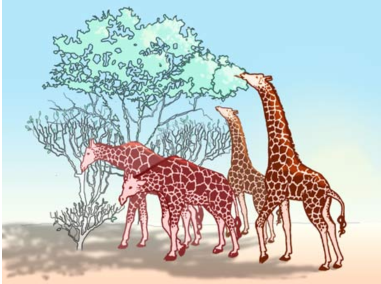
Our Main Results and Methods

- Solved 3,021 (21.3%) of remaining 14,163 hard Mizar problems
- Thus increased percentage of ATP-proved Mizar problems from 75.5% to 80.7%
- We used instantiation-based methods, particularly cvc5 SMT solver
- Note that we did not use any special decision procedures in cvc5
- We invented stronger cvc5 strategies using our Grackle system
- Further improved by different clausification and premise selection
- This has surprisingly high impact on instantiation-based methods

Overview of Instantiation-Based ATP/SMT Methods

- Herbrand (1930): a set of clauses is unsat iff finitely ground-unsat
- Gilmore's procedure (1960) - generate ground instances and check for ground unsat (decidable, inefficient in 1960)
- Efficient SAT/UNSAT: DPLL (1960/61), CDCL (1996, revolutionary)
- 2005: John Harrison: "People now say that problems are NP-easy"
- Since 2000s: renewed development of inst-based methods:
- iProver, Darwin, Equinox, SMTs like Z3, CVC, veriT, etc.
- Satallax (higher-order ATP), AVATAR (Vampire), etc.
- cvc5: SMT solver using instantiation for quantifiers
- Alternates between ground solver and instantiation module
- Generates lemmas by instantiating quantified formulas
- Uses various instantiation heuristics (e-matching, model-based, enumeration, etc.)
- Quite different from saturation-based ATPs; add ML guidance?

Automated Strategy Invention: BliStr and Grackle



- Dawkins: The Blind Watchmaker
- Grow diverse strategies by iterative local search and evolution
- ATP strategies are programs specified in rich DSLs - can be evolved
- The ATP strategies are like giraffes, the problems are their food
- The better the giraffe specializes for eating problems unsolvable by others, the more it gets fed and further evolved
- fast "inductive" training phase, followed (if successful) by a slower "hard thinking" phase, in which the newly trained strategies attempt to solve some more problems, making them into further training data

BliStr: Blind Strategymaker (2012)

- Used for automated invention of saturation-based ATP strategies
- The E strategy with longest specification in Jan 2012

```
G-E--_029_K18_F1_PI_AE_SU_R4_CS_SP_S0Y:

4 * ConjectureGeneralSymbolWeight(
      SimulateSOS,100,100,100,50,50,10,50,1.5,1.5,1),
3 * ConjectureGeneralSymbolWeight(
      PreferNonGoals,200,100,200,50,50,1,100,1.5,1.5,1),
1 * Clauseweight(PreferProcessed,1,1,1),
1 * FIFOWeight(PreferProcessed)
```

The Longest E Strategy After BliStr Evolution

Evolutionarily designed Franken-strategy (29 heuristics combined):

```
6 * ConjectureGeneralSymbolWeight(PreferNonGoals,100,100,100,50,50,1000,100,1.5,1.5
8 * ConjectureGeneralSymbolWeight(PreferNonGoals,200,100,200,50,50,1,100,1.5,1.5,1)
8 * ConjectureGeneralSymbolWeight(SimulateSOS,100,100,100,50,50,50,50,1.5,1.5,1)
4 * ConjectureRelativeSymbolWeight(ConstPrio,0.1, 100, 100, 100, 100, 1.5, 1.5, 1.5
10 * ConjectureRelativeSymbolWeight(PreferNonGoals,0.5, 100, 100, 100, 100, 1.5, 1.
2 * ConjectureRelativeSymbolWeight(SimulateSOS,0.5, 100, 100, 100, 100, 1.5, 1.5, 1
10 * ConjectureSymbolWeight(ConstPrio,10,10,5,5,5,1.5,1.5,1.5)
1 * Clauseweight(ByCreationDate,2,1,0.8)
1 * Clauseweight(ConstPrio,3,1,1)
6 * Clauseweight(ConstPrio,1,1,1)
2 * Clauseweight(PreferProcessed,1,1,1)
6 * FIFOWeight(ByNegLitDist)
1 * FIFOWeight(ConstPrio)
2 * FIFOWeight(SimulateSOS)
8 * OrientLMaxWeight(ConstPrio,2,1,2,1,1)
2 * PNRefinedweight(PreferGoals,1,1,1,2,2,2,0.5)
10 * RelevanceLevelWeight(ConstPrio,2,2,0,2,100,100,100,100,1.5,1.5,1)
8 * RelevanceLevelWeight2(PreferNonGoals,0,2,1,2,100,100,100,400,1.5,1.5,1)
2 * RelevanceLevelWeight2(PreferGoals,1,2,1,2,100,100,100,400,1.5,1.5,1)
6 * RelevanceLevelWeight2(SimulateSOS,0,2,1,2,100,100,100,400,1.5,1.5,1)
8 * RelevanceLevelWeight2(SimulateSOS,1,2,0,2,100,100,100,400,1.5,1.5,1)
5 * rweight21_g
3 * Refinedweight(PreferNonGoals,1,1,2,1.5,1.5)
1 * Refinedweight(PreferNonGoals,2,1,2,2,2)
2 * Refinedweight(PreferNonGoals,2,1,2,3,0.8)
```

Grackle (2022, CICM)

- Successor/generalization of BliStr
- Grackles: birds that evolved different bill sizes for different food
- Uses existing algorithm configuration frameworks
  - ParamILS: Iterative Local Search (Hutter et al.)
  - SMAC3: Bayesian Optimization (Lindauer et al.)

  to improve a strategy on a given set of problems
- Grackle input:
  - initial set of strategies
  - input problems
  - strategy space parametrization: parameters and their values
  - solver wrapper
- Grackle output:
  - portfolio of strategies complementary on input problems

Grackle: Invent Portfolio of Strategies

Repeat the following:

1. Evaluate all strategies on all problems P
2. Select one strategy S to be improved
3. Specialize strategy S for the problems where it performs best
4. Go to 1

Terminate when:

- all strategies has been improved, or ...
- time limit is reached.

cvc5 Strategy Space

- Defined by cvc5's command line options and values
- cvc5 distinguishes regular and expert (experimental) options
- Regular parametrization: 98 parameters, $\sim 10^{35}$ strategies
- Full parametrization: 168 parameters, $\sim 10^{58}$ strategies
- We focused on options relevant to uninterpreted functions with quantifiers

Dataset

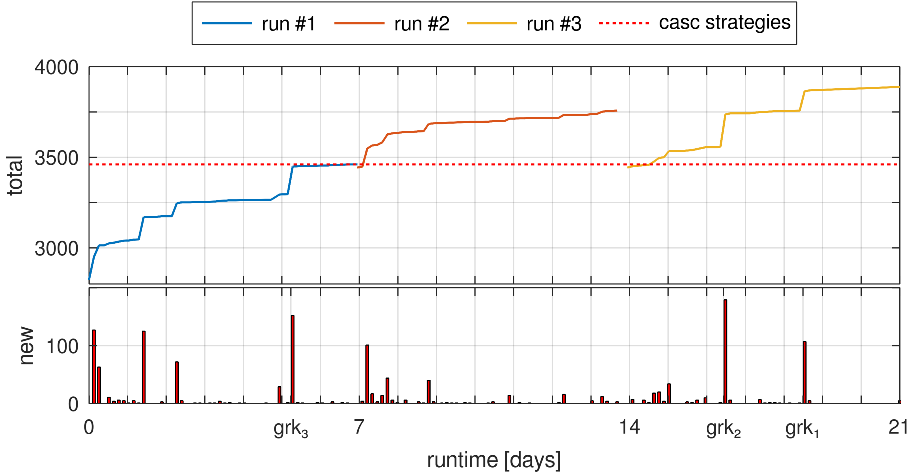
- 14,163 previously ATP-unproved Mizar *bushy* problems
- Extended with 4,283 hard problems proved only in latest ATP experiments
- This is done to give Grackle a bit easier problems to start inventing on
- We also used heuristically premise-minimized versions
- Total of 16,861 hard problems for doing cvc5 strategy development

Grackle Runs

- Three 7-day Grackle runs
- Run #1: regular space, starts with 2 CASC strategies
- Run #2: regular space, starts with 6 best strategies from Run #1
- Run #3: full strategy space, starts with the same as Run #2
- 30 second time limit per problem, 30 minutes per strategy invention
- Run #1: a proof of concept run starting with a weaker portfolio, 345 new probs
- Run #2: more serious, 485 new probs
- Run #3: measure the effect of expert options, 629 new probs

Progress of Three Grackle Runs

Progress in time of problems cumulatively solved by each Grackle run:



Grackle Strategy Invention Results

- 143 new strategies invented
- Best single strategy: 2,796 problems (11.5% improvement)
- Best 16 strategies: 4,039 problems (16.7% improvement)
- Total solved: 4,113 problems

Higher Time Limits

- Evaluated best strategies with 600 second time limit
- Best Grackle strategy: 3,496 problems
- Best CASC strategy: 3,059 problems
- 14.3% improvement for single best strategy
- cvc5 (single strategy grk_1) solves almost 50% more problems when the time limit is increased from 60 to 600 seconds.
- E Prover (auto mode / single strategy) solves only 10% more with the same time limit increase.

Reformulation Experiments

- External clausification using E prover
  - Two variants: default (cnf1) and aggressive definition introduction (cnf2)
  - cnf2: Halved average number of literals, 60% symbols
  - Added 369 newly solved problems
- Tested different premise selection methods:
  - Bushy (original premises)
  - GNN (Graph Neural Networks)
  - LightGBM (Gradient Boosting Decision Trees)
- Highly complementary to other methods
- Added 1,065 newly solved problems

Top 10 Strategies from Greedy Cover

version	*strategy*	*addon*		*total*	*alone*	*new*
min_{fof}	grk_1	+3496	-	3496	3496	1243
min_{cnf1}	grk_2	+738	+21.11%	4234	3231	1192
gnn	grk_1	+535	+12.64%	4769	1215	432
bushy	grk_1	+311	+6.52%	5080	1441	553
min_{fof}	grk_3	+298	+5.87%	5378	3220	1146
lgbm	grk_1	+233	+4.33%	5611	1512	541
min_{cnf1}	grk_3	+161	+2.87%	5772	3223	1092
min_{cnf1}	casc_{10}	+112	+1.94%	5884	3125	999
min_{fof}	grk_2	+90	+1.53%	5974	3146	1131
min_{cnf2}	grk_2	+62	+1.04%	6036	2949	1045

- *addon* = addition to the portfolio; *total* = partial portfolio performance
- *alone* = standalone strategy performance (600 seconds time limit)
- *new* = hard Mizar problems newly solved by each strategy
- Grackle-invented strategies dominate the greedy cover
- The results also transfer to a new (unseen) version of MML

Analysis of Invented Strategies

Best CASC strategies:

casc_7	`full-saturate-quant` `multi-trigger-priority` `multi-trigger-when-single`
casc_{10}	`full-saturate-quant` `enum-inst-interleave` `decision=internal`
casc_{14}	`full-saturate-quant` `cbqi-vo-exp`

Best Grackle strategies:

grk_1	`full-saturate-quant` `cbqi-vo-exp` `relational-triggers` `cond-var-split-quant=agg`
grk_2	`full-saturate-quant` `cbqi-vo-exp` `relevant-triggers` `multi-trigger-priority` `ieval=off` `no-static-learning` `miniscope-quant=off`
grk_3	`full-saturate-quant` `multi-trigger-priority` `multi-trigger-when-single` `term-db-mode=relevant`

- Focus on changing behavior of quantifier instantiation module
- Best strategies combine enumerative instantiations with appropriate trigger selection for e-matching
- grk_1 and grk_2 extend casc_{14}; grk_3 extends casc_7
- repo with the invented strategies and problems solved: `https://github.com/ai4reason/cvc5_grackle_mizar`

Interesting Solved Problems

- KURATO_1:6: Kuratowski's closure-complement problem
  - 131 lines in Mizar
  - Combination of equational reasoning and a large case split (14 cases) That likely makes it hard for the superposition-based systems
  - SMT-style congruence closure likely useful when a more complex term equal to a less complex term
- ASYMPT_1:18: Big O relation for modulo functions
  - functions $f(n) = n \bmod 2$ and $g(n) = n + 1 \bmod 2$ are not in the Big O relation (in any direction).
  - 122 lines in Mizar
  - Only provable with a single Grackle-invented strategy grk_2 and external clausification, taking 62 s.
  - case splits related to the mod 2 values; triggers seems to play a big role
- ROBBINS4:3: Equivalent condition for ortholattices
  - 145 lines in Mizar
  - a lot of equational reasoning (should be good for E/Vampire!)
  - possibly large multi-literal clauses make this hard for saturation systems

Interesting Solved Problems

```
definition let T be non empty TopSpace; let A be Subset of T;
func Kurat14Set A -> Subset-Family of T equals
{ A, A-, A-`, A-`-, A-`-`, A-`-`-, A-`-`-` } \/
{ A`, A`-, A`-`, A`-`-, A`-`-`, A`-`-`-, A`-`-`-` };
end;
theorem :: KURATO_1:6:
for T being non empty TopSpace
for A, Q being Subset of T st Q in Kurat14Set A holds
Q` in Kurat14Set A & Q- in Kurat14Set A;

theorem :: ASYMPT_1:18
 for f,g being Real_Sequence st
   (for n holds f.n = n mod 2) & (for n holds g.n = n+1 mod 2)
  holds ex s,s1 being eventually-nonnegative Real_Sequence
  st s = f & s1 = g & not s in Big_Oh(s1) & not s1 in Big_Oh(s)

theorem :: ROBBINS4:3
for L being non empty OrthoLattStr holds L is Ortholattice iff
  (for a, b, c being Element of L holds
     (a "\/" b) "\/" c = (c` "/\" b`)` "\/" a)
& (for a, b being Element of L holds a = a "/\" (a "\/" b))
& for a, b being Element of L holds a = a "\/" (b "/\" b`)
```

Conclusions

- Significant progress on hard Mizar problems
- Instantiation-based methods today surprisingly good
- Strategy invention (Grackle) very useful for cvc5
- High impact of problem reformulation: different clausifications, premise selection
- Interesting competition (also within our Prague group) between saturation-based (Vampire/Deepire, E/ENIGMA) and instantiation-based (cvc5, iProver, Satallax) ATPs

Future Work

- Apply strategy invention to other problem sets (e.g. TPTP, Isabelle)
- Further explore problem reformulation techniques (rewarding here)
- More learning for guiding instantiation:
  - neural (GNN - LPAR'24)
  - fast non-neural (ECAI'24)
  - choosing formulas, variables, instances ...
  - end-to-end ML-style guessing of instances?

Thanks and Advertisement

- Thanks for your attention!
- To push AI/ML methods in math and theorem proving, we organize:
- AITP – Artificial Intelligence and Theorem Proving
- September 1-6, 2024, Aussois, France, `aitp-conference.org`
- ATP/ITP/Math vs AI/ML/AGI people, Computational linguists
- Discussion-oriented and experimental
- About 50 people in 2024