SOME RECENT COMBINATIONS OF AI AND THEOREM PROVING METHODS

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Leibniz's/Hilbert's/Russell's Dream: Let Us Calculate!

Solve all (math, physics, law, economics, society, ...) problems by reduction to logic/computation

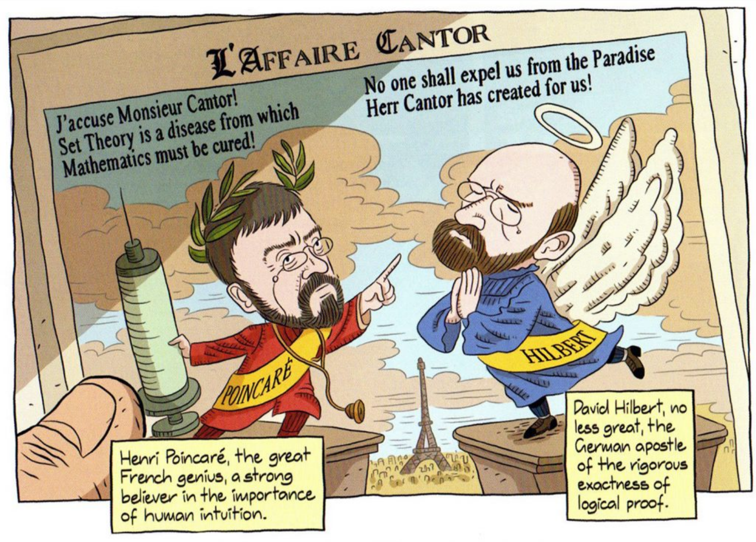
[Adapted from: *Logicomix: An Epic Search for Truth* by A. Doxiadis]

How Do We Automate Math, Science, Programming?

- What is mathematical and scientific thinking?
- Pattern-matching, analogy, induction from examples
- Deductive reasoning
- Complicated feedback loops between induction and deduction
- Using a lot of previous knowledge - both for induction and deduction

- We need to develop such methods on computers
- Are there any large corpora suitable for nontrivial deduction?
- Yes! Large libraries of formal proofs and theories
- So let’s develop strong AI on them!

Intuition vs Formal Reasoning – Poincaré vs Hilbert



[Adapted from: *Logicomix: An Epic Search for Truth* by A. Doxiadis]

What is Formal Mathematics and Theorem Proving?

- 1900s: Mathematics put on formal logic foundations – symbolic logic
- Culmination of a program by Leibniz/Frege/Russell/Hilbert/Church/...
- ... led also to the rise of computers (Turing/Church, 1930s)
- ... and rise of AI - Turing's 1950 paper: Learning Machines, Chess, etc.
- 1950s: First AI program: Logic Theorist by Newell & Simon
- Formalization of math (60s): combine formal foundations and computers
- Proof assistants/Interactive theorem provers and their large libraries:
- Automath (1967), LCF, Mizar, NQTHM, HOL, Coq, Isabelle, ACL2, Lean
- Automated theorem provers - search for proofs automatically:
- Otter, Vampire, E, SPASS, Prover9, CVC4, Z3, Satallax, ...
- more limited logics: SAT, QBF, SMT, UEQ, ... (DPLL, CDCL, ...)
- TP-motivated PLs: ML, Prolog, (*logic programming* - Hayes, Kowalski)
- My MSc (1998): Try ILP to learn explainable rules/heuristics from Mizar
- Since: Do AI/TP over (in)formal math corpora: Mizar, Isabelle, HOL, ...

Why Do This Today?

1. Practically Useful for Verification of Complex HW/SW and Math
    - Formal Proof of the Kepler Conjecture (2014 – Hales – 20k lemmas)
    - Formal Proof of the Feit-Thompson Theorem (2 books, 2012 – Gonthier)
    - Verification of several math textbooks and CS algorithms
    - Verification of compilers (CompCert)
    - Verification of OS microkernels (seL4), HW chips (Intel), transport, finance,
    - Verification of cryptographic protocols (Amazon), etc.
2. Blue Sky AI Visions:
    - Get strong AI by learning/reasoning over large KBs of human thought?
    - Big formal theories: good semantic approximation of such thinking KBs?
    - Deep non-contradictory semantics – better than scanning books?
    - Gradually try learning math/science
    - automate/verify them, include law, etc. (Leibniz, McCarthy, ..)
        - What are the components (inductive/deductive thinking)?
        - How to combine them together?

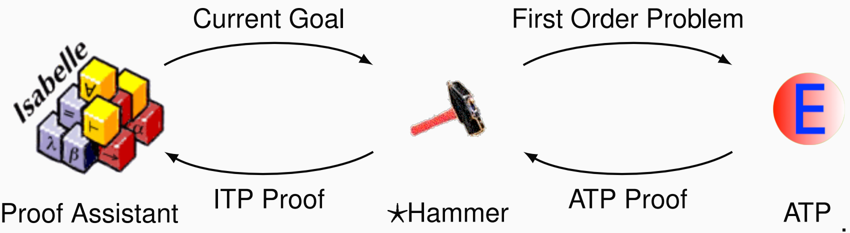
Using Learning to Guide Theorem Proving

- **high-level**: pre-select lemmas from a large library, give them to ATPs
- **high-level**: pre-select a good ATP strategy/portfolio for a problem
- **high-level**: pre-select good *hints* for a problem, use them to guide ATPs
- **low-level**: guide every inference step of ATPs (tableau, superposition)
- **low-level**: guide every kernel step of LCF-style ITPs
- **mid-level**: guide application of tactics in ITPs
- **mid-level**: invent suitable ATP strategies for classes of problems
- **mid-level**: invent suitable conjectures for a problem
- **mid-level**: invent suitable concepts/models for problems/theories
- **proof sketches**: explore stronger/related theories to get proof ideas
- **theory exploration**: develop interesting theories by conjecturing/proving
- **feedback loops**: (dis)prove, learn from it, (dis)prove more, learn more, ...
- **autoformalization**: (semi-)automate translation from LaTeX to formal
- ...

AI/TP Examples and Demos

- ENIGMA/hammer proofs of Pythagoras : https://bit.ly/2MVPAn7 (more at http://grid01.ciirc.cvut.cz/~mptp/enigma-ex.pdf) and simplified Carmichael https://bit.ly/3oGBdRz,
- 3-phase ENIGMA: https://bit.ly/3C0Lwa8, https://bit.ly/3BWqR6K
- Long trig proof from 1k axioms: https://bit.ly/2YZ0OgX
- Extreme Deepire/AVATAR proof of $\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ https://bit.ly/3Ne4WNX
- Hammering demo: http://grid01.ciirc.cvut.cz/~mptp/out4.ogv
- TacticToe on HOL4: http://grid01.ciirc.cvut.cz/~mptp/tactictoe_demo.ogv
- Tactician for Coq: https://blaauwbroek.eu/papers/cicm2020/demo.mp4, https://coq-tactician.github.io/demo.html
- Inf2formal over HOL Light: http://grid01.ciirc.cvut.cz/~mptp/demo.ogv
- QSynt: AI rediscovers the Fermat primality test: https://www.youtube.com/watch?v=24oejR9wsXs

Today's AI-ATP systems (⋆-Hammers)



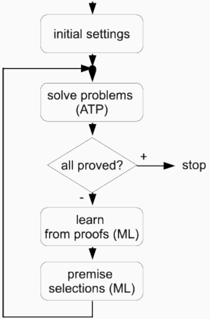
How much can it do?

- Mizar / MML – MizAR
- Isabelle (Auth, Jinja) – Sledgehammer
- Flyspeck (including core HOL Light and Multivariate) – HOL(y)Hammer
- HOL4 (Gauthier and Kaliszyk)
- CoqHammer (Czajka and Kaliszyk) - about 40% on Coq standard library

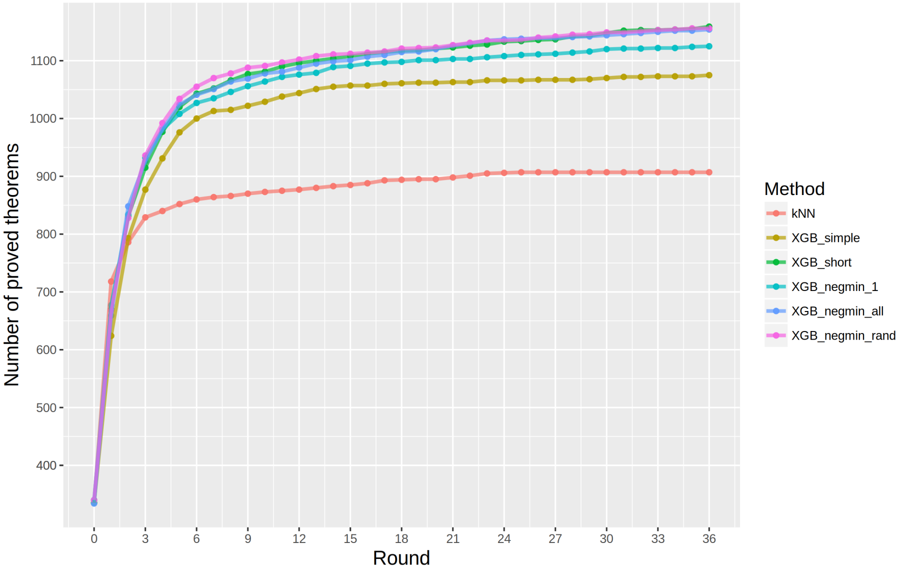
$\approx$ 40-45% success by 2016, 60% on Mizar as of 2021

High-level feedback loops – MALARea, ATPBoost

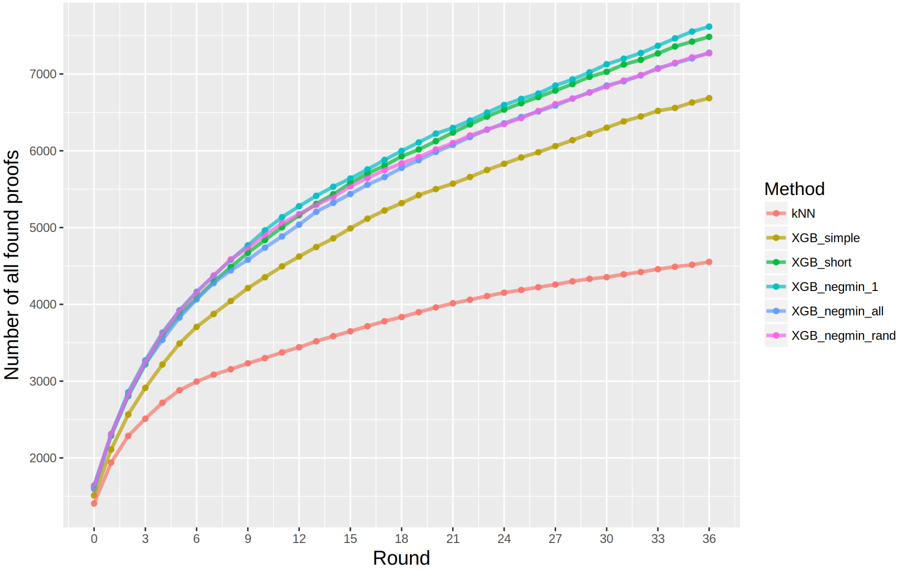
- Machine Learner for Autom. Reasoning (2006) – infinite hammering
- feedback loop interleaving ATP with learning premise selection
- both syntactic and semantic features for characterizing formulas:
- evolving set of finite (counter)models in which formulas evaluated
- winning AI/ATP benchmarks (MPTPChallenge, CASC 2008/12/13/18)
- ATPBoost (Piotrowski) - recent incarnation focusing on multiple proofs

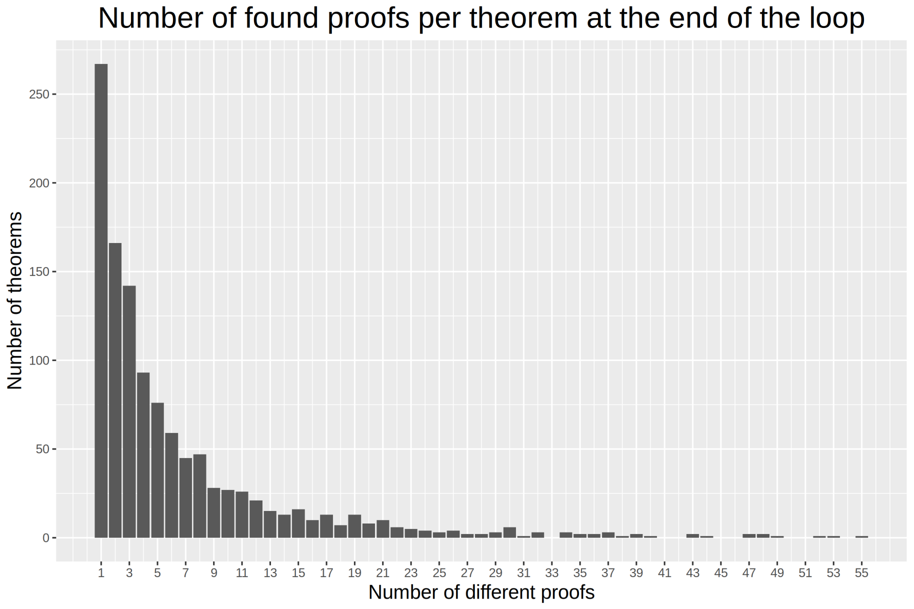


Prove-and-learn loop on MPTP2078 data set



Prove-and-learn loop on MPTP2078 data set




Number of found proofs per theorem at the end of the loop
Number of theorems
0
50
100
150
200
250
Number of different proofs
1
3
5
7
9
11
13
15
17
19
21
23
25
27
29
31
33
35
37
39
41
43
45
47
49
51
53
55

Low-level: Statistical Guidance of Connection Tableau

- learn guidance of every clausal inference in connection tableau (leanCoP)
- set of first-order clauses, *extension* and *reduction* steps
- proof finished when all branches are closed
- a lot of nondeterminism, requires backtracking
- *Iterative deepening* used in leanCoP to ensure completeness
- good for learning – the tableau compactly represents the proof state

Clauses:

$c_1 : P(x)$

$c_2 : R(x, y) \vee \neg P(x) \vee Q(y)$

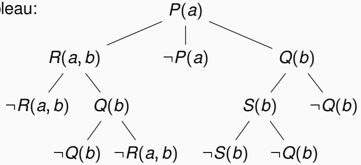
$c_3 : S(x) \vee \neg Q(b)$

$c_4 : \neg S(x) \vee \neg Q(x)$

$c_5 : \neg Q(x) \vee \neg R(a, x)$

$c_6 : \neg R(a, x) \vee Q(x)$

Closed Connection Tableau:



Statistical Guidance of Connection Tableau – rlCoP

- 2018: stronger learners via C interface to OCAML (boosted trees)
- remove iterative deepening, the prover can go arbitrarily deep
- added Monte-Carlo Tree Search (MCTS)
- MCTS search nodes are sequences of clause application
- a good heuristic to explore new vs exploit good nodes:

$$\frac{w_i}{n_i} + c \cdot p_i \cdot \sqrt{\frac{\ln N}{n_i}} \qquad \text{(UCT - Kocsis, Szepesvari 2006)}$$

- learning both *policy* (clause selection) and *value* (state evaluation)
- clauses represented not by names but also by features (generalize!)
- binary learning setting used: | proof state | clause features |
- mostly term walks of length 3 (trigrams), hashed into small integers
- many iterations of proving and learning

Statistical Guidance of Connection Tableau – rlCoP

- On 32k Mizar40 problems using 200k inference limit
- nonlearning CoPs:

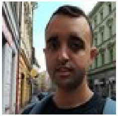
System	leanCoP	bare prover	rlCoP no policy/value (UCT only)
Training problems proved	10438	4184	7348
Testing problems proved	**1143**	431	804
Total problems proved	11581	4615	8152

- rlCoP with policy/value after 5 proving/learning iters on the training data
- $1624/1143 = 42.1$% improvement over leanCoP on the testing problems

Iteration	1	2	3	4	5	6	7	8
Training proved	12325	13749	14155	14363	14403	14431	14342	**14498**
Testing proved	1354	1519	1566	1595	**1624**	1586	1582	1591

ENIGMA (2017): Guiding the Best ATPs like E Prover

- ENIGMA (Jan Jakubuv, Zar Goertzel, Karel Chvalovsky, others)

- The proof state are two large heaps of clauses *processed/unprocessed*
- learn on E's proof search traces, put classifier in E
- positive examples: clauses (lemmas) used in the proof
- negative examples: clauses (lemmas) not used in the proof
- 2021 multi-phase architecture (combination of different methods):
  - fast gradient-boosted decision trees (GBDTs) used in 2 ways
  - fast logic-aware graph neural network (GNN - Olsak) run on a GPU server
  - logic-based subsumption using fast indexing (discrimination trees - Schulz)
- The GNN scores many clauses (context/query) together in a large graph
- Sparse - vastly more efficient than transformers ($\sim$100k symbols)
- 2021: leapfrogging and Split&Merge:
- aiming at learning reasoning/algo components

Feedback prove/learn loop for ENIGMA on Mizar data

- Done on 57880 Mizar problems recently
- Serious ML-guidance breakthrough applied to the best ATPs
- Ultimately a 70% improvement over the original strategy in 2019
- From 14933 proofs to 25397 proofs (all 10s CPU - no cheating)
- Went up to 40k in more iterations and 60s time in 2020
- 75% of the Mizar corpus reached in July 2021 - higher times and many runs: `https://github.com/ai4reason/ATP_Proofs`

	$\mathcal{S}$	$\mathcal{S}\odot\mathcal{M}_9^0$	$\mathcal{S}\oplus\mathcal{M}_9^0$	$\mathcal{S}\odot\mathcal{M}_9^1$	$\mathcal{S}\oplus\mathcal{M}_9^1$	$\mathcal{S}\odot\mathcal{M}_9^2$	$\mathcal{S}\oplus\mathcal{M}_9^2$	$\mathcal{S}\odot\mathcal{M}_9^3$	$\mathcal{S}\oplus\mathcal{M}_9^3$
solved	**14933**	16574	20366	21564	22839	22413	23467	22910	23753
$\mathcal{S}\%$	+0%	+10.5%	+35.8%	+43.8%	+52.3%	+49.4%	+56.5%	+52.8%	+58.4
$\mathcal{S}+$	+0	+4364	+6215	+7774	+8414	+8407	+8964	+8822	+9274
$\mathcal{S}-$	-0	-2723	-782	-1143	-508	-927	-430	-845	-454

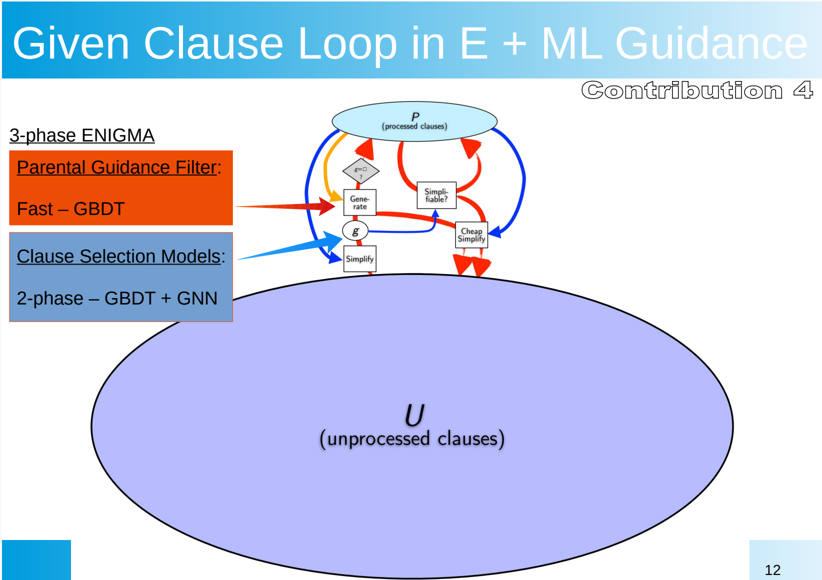
	$\mathcal{S}\odot\mathcal{M}_{12}^3$	$\mathcal{S}\oplus\mathcal{M}_{12}^3$	$\mathcal{S}\odot\mathcal{M}_{16}^3$	$\mathcal{S}\oplus\mathcal{M}_{16}^3$
solved	24159	24701	25100	**25397**
$\mathcal{S}\%$	+61.1%	+64.8%	+68.0%	**+70.0%**
$\mathcal{S}+$	+9761	+10063	+10476	+10647
$\mathcal{S}-$	-535	-295	-309	-183

ENIGMA Anonymous: Learning from patterns only

- The GNN and GBDTs only learn from formula structure, not symbols
- Not from symbols like $+$ and $*$ as Transformer & Co.
- E.g., learning on additive groups thus transfers to multiplicative groups
- Evaluation of old-Mizar ENIGMA on 242 new Mizar articles
- 13370 new theorems, $> 50\%$ of them with new terminology:
- The 3-phase ENIGMA is **58%** better on them than unguided E
- While **53.5%** on the old Mizar (where this ENIGMA was trained)
- Generalizing, analogizing and transfer abilities unusual in the large transformer models
- Recently also trained on 300k Isabelle/AFP problems (Sledgehammer)

3-phase Anonymous ENIGMA

The 3-phase ENIGMA (single strategy) solves in 30s 56.4% of Mizar (bushy)



Neural Clause Selection in Vampire (M. Suda)

Deepire: Similar to ENIGMA:

- build a *classifier* for recognizing *good* clauses
- *good* are those that appeared in past proofs

Deepire's contributions:

- Learn from clause *derivation trees only*
  Not looking at what it says, just who its ancestors were.
- Integrate using *layered clause queues*
  A smooth improvement of the base clause selection strategy.
- Tree Neural Networks: constant work per derived clause
- A signature agnostic approach
- Delayed evaluation trick (not all derived need to be evaluated)

Preliminary Evaluation on Mizar "57880"

- Learn from 63595 proofs of 23071 problems (three 30s runs)
- Deepire solves 26217 (i.e. +4054) problems in a *single 10s run*

More on Conjecturing in Mathematics

- Targeted: generate intermediate lemmas (cuts) for a harder conjecture
- Unrestricted (theory exploration):
- Creation of interesting conjectures based on the previous theory
- One of the most interesting activities mathematicians do (how?)
- Higher-level AI/reasoning task - can we learn it?
- If so, we have solved math:
- ... just (recursively) divide Fermat into many subtasks ...
- ... and conquer (I mean: hammer) them away

Conjecturing and Proof Synthesis by Neural Language models

- Karpathy’15 - RNN experiments with generating fake Math over Stacks
- I have tried to use that for formal math in 2016 but it looked weak
- GPT (-2,3) looks stronger
- Renewed experiments in 2020 on:
- All Mizar articles, stripped of comments and concatenated together (78M)
- Articles with added context/disambiguation (156M) (types, names, thesis)
- TPTP proofs of 28271 Mizar/MPTP theorems by E/ENIGMA (658M)
- Just the conjecture and premises needed for the 28271 proofs printed in prefix notation
- Quite interesting results, server for Mizar authors
- Quickly taken up by others on HOL, Isabelle, MetaMath ...

Can you find the flaw(s) in this fake GPT-2 proof?

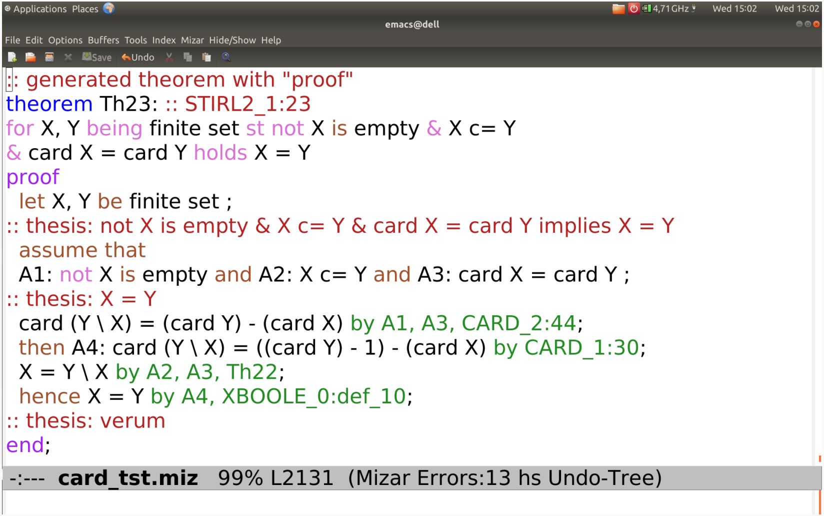


Figure: Fake full declarative GPT-2 “proof” - typechecks!

A correct conjecture that was too hard to prove

- Kinyon and Stanovsky (algebraists) confirmed that this cut is valid:

```
theorem Th10: :: GROUPP_1:10
for G being finite Group for N being normal Subgroup of G st
N is Subgroup of center G & G ./. N is cyclic holds G is commutative

The generalization that avoids finiteness:

for G being Group for N being normal Subgroup of G st
N is Subgroup of center G & G ./. N is cyclic holds G is commutative
```

More cuts

- In total 33100 in this experiment
- Ca 9k proved by trained ENIGMA
- Some are clearly false, yet quite natural to ask:

```
theorem :: SINCOS10:17

sec is increasing on [0, pi/2)
```

leads to conjecturing the following:

```
Every differentiable function is increasing.
```

QSynt: Semantics-Aware Synthesis of Math Objects

- Gauthier'19-22
- Synthesize math expressions based on semantic characterizations
- i.e., not just on the syntactic descriptions (e.g. proof situations)
- Tree Neural Nets and RL (MCTS, policy/value), used for:
- Guiding synthesis of a diophantine equation characterizing a given set
- Guiding synthesis of combinators for a given lambda expression
- 2022: invention of programs for OEIS sequences from scratch
- 50k sequences discovered so far:
  `https://www.youtube.com/watch?v=24oejR9wsXs`,
  `http://grid01.ciirc.cvut.cz/~thibault/qsynt.html`
- Many conjectures invented: 4 different characterizations of primes
- Non-neural (Turing complete) computing and semantics collaborates with the statistical learning

QSynt: synthesizing the programs/expressions

- Inductively defined set P of our *programs and subprograms*,
- and an auxiliary set F of binary functions (higher-order arguments)
- are the smallest sets such that $0, 1, 2, x, y \in P$, and if $a, b, c \in P$ and $f, g \in F$ then:

$$a + b, a - b, a \times b, a \textit{ div } b, a \textit{ mod } b, \textit{cond}(a, b, c) \in P$$

$$\lambda(x, y).a \in F, \ \textit{loop}(f, a, b), \textit{loop2}(f, g, a, b, c), \textit{compr}(f, a) \in P$$

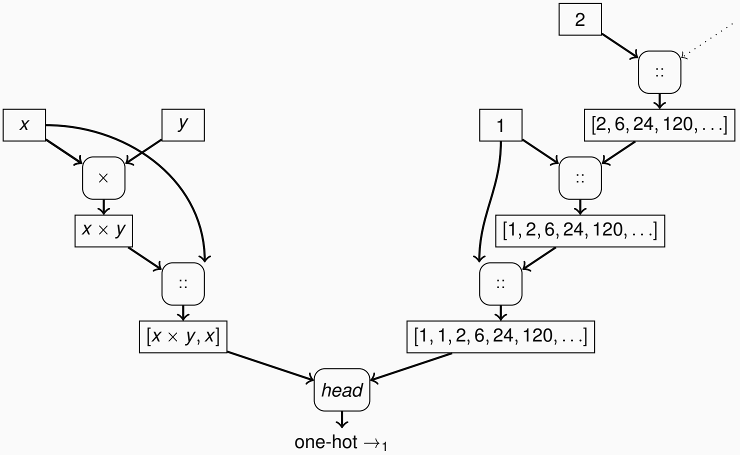
- Programs are built in reverse polish notation
- Start from an empty stack
- Use ML to repeatedly choose the next operator to push on top of a stack
- Example: Factorial is $\textit{loop}(\lambda(x, y).\ x \times y, x, 1)$, built by:

$$[\,] \rightarrow_x [x] \rightarrow_y [x, y] \rightarrow_\times [x \times y] \rightarrow_x [x \times y, x]$$

$$\rightarrow_1 [x \times y, x, 1] \rightarrow_{loop} [\textit{loop}(\lambda(x, y).\ x \times y, x, 1)]$$

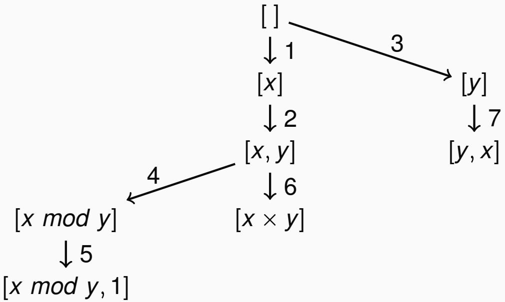
QSynt: Training of the Neural Net Guiding the Search

- The triple $((head([x \times y, x], [1, 1, 2, 6, 24, 120 \ldots]), \rightarrow_1)$ is a training example extracted from the program for factorial $loop(\lambda(x, y).\ x \times y, x, 1)$
- $\rightarrow_1$ is the action (adding 1 to the stack) required on $[x \times y, x]$ to progress towards the construction of $loop(\lambda(x, y).\ x \times y, x, 1)$.

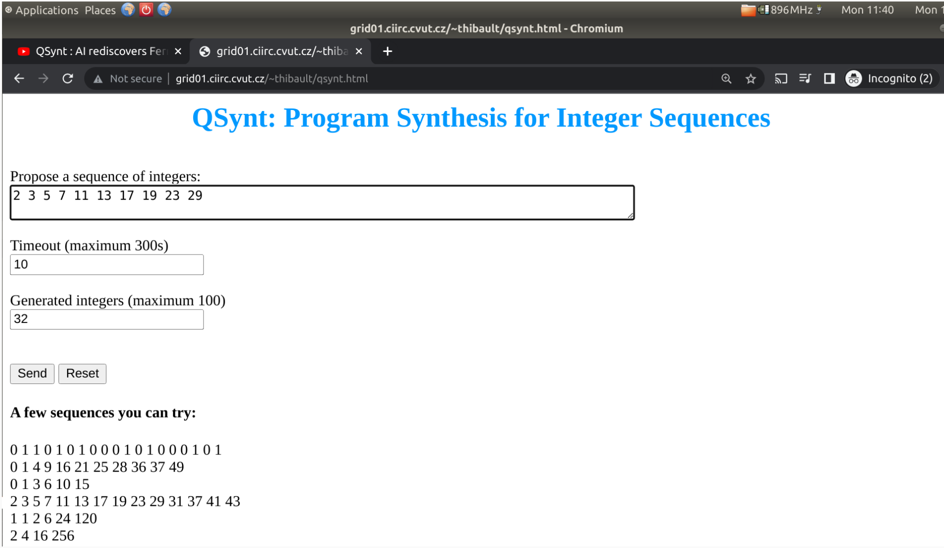


QSynt program search - Monte Carlo search tree

7 iterations of the search loop gradually extending the search tree. The set of the synthesized programs after the 7th iteration is $\{1, x, y, x \times y, x \ mod \ y\}$.



QSynt web interface for program invention



QSynt inventing Fermat pseudoprimes

Positive integers k such that $2^k \equiv 2 \mod k$. ($341 = 11 * 31$ is the first non-prime)

```
First 16 generated numbers (f(0),f(1),f(2),...):
2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53
Generated sequence matches best with: A15919(1-75), A100726(0-59), A40(0-58)

Program found in 5.81 seconds
f(x) := 2 + compr(\x.loop(\(x,i).2*x + 2, x, 2) mod (x + 2), x)
Run the equivalent Python program here or in the window below:
```

Lucas/Fibonacci characterization of (pseudo)primes

```
input sequence: 2,3,5,7,11,13,17,19,23,29

invented output program:
f(x) := compr(\(x,y).(loop2(\(x,y).x + y, \(x,y).x, x, 1, 2) - 1)
              mod (1 + x), x + 1) + 1

human conjecture: x is prime iff? x divides (Lucas(x) - 1)

PARI program:
? lucas(n) = fibonacci(n+1)+fibonacci(n-1)
? b(n) = (lucas(n) - 1) % n

Counterexamples (Bruckman-Lucas pseudoprimes):
? for(n=1,4000,if(b(n)==0,if(isprime(n),0,print(n))))
1
705
2465
2737
3745
```

QSynt inventing primes using Wilson's theorem

n is prime iff $(n-1)!+1$ is divisible by n (i.e.: $(n-1)! \equiv -1 \mod n$)

```
First 32 generated numbers (f(0),f(1),f(2),...):
0 1 1 0 1 0 1 0 0 0 1 0 1 0 0 0 1 0 1 0 0 0 1 0 0 0 0 0 1 0 1 0
Generated sequence matches best with: A10051(0-100), A252233(0-29), A283991(0-24)

Program found in 5.17 seconds
f(x) := (loop(\(x,i).x * i, x, x) mod (x + 1)) mod 2
Run the equivalent Python program here or in the window below:
```

Are two QSynt programs equivalent?

- As with primes, we often find many programs for one OEIS sequence
- It may be quite hard to see that the programs are equivalent
- A simple example for $0, 2, 4, 6, 8, \ldots$ with two programs f and g:
  - $f(0) = 0, f(n) = 2 + f(n-1)$ if $n > 0$
  - $g(n) = 2 * n$
  - conjecture: $\forall n \in \mathbb{N}. g(n) = f(n)$
- We can ask mathematicians, but we have thousands of such problems
- Or we can try to ask our ATPs (and thus create a large ATP benchmark)!
- Here is one SMT encoding by Mikolas Janota:

```
(set-logic UFLIA)
(define-fun-rec f ((x Int)) Int (ite (<= x 0) 0 (+ 2 (f (- x 1)))))
(assert (exists ((c Int)) (and (> c 0) (not (= (f c) (* 2 c))))))
(check-sat)
```

Inductive proof by Vampire of the $f = g$ equivalence

```
% SZS output start Proof for rec2
1. f(X0) = $ite($lesseq(X0,0), 0,$sum(2,f($difference(X0,1)))) [input]
2. ? [X0 : $int] : ($greater(X0,0) & ~f(X0) = $product(2,X0)) [input]
[...]
43. ~$less(0,X0) | iG0(X0) = $sum(2,iG0($sum(X0,-1))) [evaluation 40]
44. (! [X0 : $int] : (($product(2,X0) = iG0(X0) & ~$less(X0,0)) => $product(2,$sum(X0,1)) = iG0($sum(X0,1)))
    &  $product(2,0) = iG0(0))    => ! [X1 : $int] : ($less(0,X1) => $product(2,X1) = iG0(X1)) [induction hypo]
[...]
49. $product(2,0) != iG0(0) | $product(2,$sum(sK3,1)) != iG0($sum(sK3,1)) | ~$less(0,sK1) [resolution 48,41]
50. $product(2,0) != iG0(0) | $product(2,sK3) = iG0(sK3) | ~$less(0,sK1) [resolution 47,41]
51. $product(2,0) != iG0(0) | ~$less(sK3,0) | ~$less(0,sK1) [resolution 46,41]
52. 0 != iG0(0) | $product(2,$sum(sK3,1)) != iG0($sum(sK3,1)) | ~$less(0,sK1) [evaluation 49]
53. 0 != iG0(0) | $product(2,sK3) = iG0(sK3) | ~$less(0,sK1) [evaluation 50]
54. 0 != iG0(0) | ~$less(sK3,0) | ~$less(0,sK1) [evaluation 51]
55. 0 != iG0(0) | ~$less(sK3,0) [subsumption resolution 54,39]
57. 1 <=> $less(sK3,0) [avatar definition]
59. ~$less(sK3,0) <- (~1) [avatar component clause 57]
61. 2 <=> 0 = iG0(0) [avatar definition]
64. ~1 | ~2 [avatar split clause 55,61,57]
65. 0 != iG0(0) | $product(2,sK3) = iG0(sK3) [subsumption resolution 53,39]
67. 3 <=> $product(2,sK3) = iG0(sK3) [avatar definition]
69. $product(2,sK3) = iG0(sK3) <- (3) [avatar component clause 67]
70. 3 | ~2 [avatar split clause 65,61,67]
71. 0 != iG0(0) | $product(2,$sum(sK3,1)) != iG0($sum(sK3,1)) [subsumption resolution 52,39]
72. $product(2,$sum(1,sK3)) != iG0($sum(1,sK3)) | 0 != iG0(0) [forward demodulation 71,5]
74. 4 <=> $product(2,$sum(1,sK3)) = iG0($sum(1,sK3)) [avatar definition]
76. $product(2,$sum(1,sK3)) != iG0($sum(1,sK3)) <- (~4) [avatar component clause 74]
77. ~2 | ~4 [avatar split clause 72,74,61]
82. 0 = iG0(0) [resolution 36,10]
85. 2 [avatar split clause 82,61]
246. iG0($sum(X1,1)) = $sum(2,iG0($sum($sum(X1,1),-1))) | $less(X1,0) [resolution 43,14]
251. $less(X1,0) | iG0($sum(X1,1)) = $sum(2,iG0(X1)) [evaluation 246]
[...]
1176. $false <- (~1, 3, ~4) [subsumption resolution 1175,1052]
1177. 1 | ~3 | 4 [avatar contradiction clause 1176]
1178. $false [avatar sat refutation 64,70,77,85,1177]
% SZS output end Proof for rec2
% Time elapsed: 0.016 s
```

Future: AITP Challenges/Bets from 2014

- 3 AITP bets from my 2014 talk at Institut Henri Poincare
  - In 20 years, 80% of Mizar and Flyspeck toplevel theorems will be provable automatically (same hardware, same libraries as in 2014 - about 40% then)
  - In 10 years: 60% (DONE already in 2021 - 3 years ahead of schedule)
  - In 25 years, 50% of the toplevel statements in LaTeX-written Msc-level math curriculum textbooks will be parsed automatically and with correct formal semantics (this may be faster than I expected)
- My (conservative?) estimate when we will do Fermat:
  - Human-assisted formalization: by 2050
  - Fully automated proof (hard to define precisely): by 2070
  - See the Foundation of Math thread: `https://bit.ly/300k9Pm`
- Big challenge: Learn complicated symbolic algorithms (not black box - motivates also our OEIS research)

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- ... and many more ...
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Thanks and Advertisement

- Thanks for your attention!
- AITP – Artificial Intelligence and Theorem Proving
- September 4–9, 2022, Aussois, France, `aitp-conference.org`
- ATP/ITP/Math vs AI/Machine-Learning people, Computational linguists
- Discussion-oriented and experimental
- Grown to 80 people in 2019
- Will be hybrid in 2022 as in 2021 and 2020
- Invited talks by J. Araujo, K. Buzzard, J. Brandstetter, W. Dean and A. Naibo, M. Rawson, T. Ringer, S. Wolfram