A Finitely Axiomatizable Set Theory in Typed First-Order Logic*

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Abstract

1 Introduction

2 Syntax

We work over a first-order logic with five types: M (sets), $\hat{\mathsf{M}}$ (generalized sets), $\hat{\mathsf{P}}$ (generalized propositions), S (substitutions) and A (assignments). For each type T, let $\mathcal{V}_T$ be an infinite set of variables of type T. We assume that if $T \neq T'$, then $\mathcal{V}_T$ and $\mathcal{V}_{T'}$ are disjoint. We will use x, y to range over variables and more specifically x^T to range over variables of type T.

We define the set of terms $\mathcal{T}_T$ of type T by mutual recursion as follows:

$$\begin{array}{rcll}
(X,Y) & \mathcal{T}_{\mathsf{M}} & := & x^{\mathsf{M}}|\emptyset|\wp(X)|\bigcup(X)|v(U,\alpha) \\
(\alpha,\beta) & \mathcal{T}_{\mathsf{A}} & := & x^{\mathsf{A}}|\mathsf{k}_{\emptyset}|(X \triangleright \alpha)|(\sigma\alpha) \\
(\sigma,\tau) & \mathcal{T}_{\mathsf{S}} & := & x^{\mathsf{S}}|\mathsf{id}|\uparrow|(U \cdot \sigma)|(\sigma \circ \tau) \\
(U,V) & \mathcal{T}_{\hat{\mathsf{M}}} & := & x^{\hat{\mathsf{M}}}|\mathsf{k}(X)|0|U[\sigma]|\hat{\emptyset}|\hat{\wp}(U)|\hat{\bigcup}(U)|\{U|. \in V;\Phi\}|\varepsilon(\Phi) \\
(\Phi,\Psi) & \mathcal{T}_{\hat{\mathsf{P}}} & := & x^{\hat{\mathsf{P}}}|\Phi[\sigma]|(U \hat{\in} V)|(U \hat{=} V)|\hat{\bot}|(\Phi \hat{\to} \Psi)|(\hat{\forall}\Phi)
\end{array}$$

We use s, t to range over terms over any type. Once we have the sets of typed terms, we define the set of formulas as follows:

$$(\varphi,\psi) \quad \text{formulas} \quad p(\Phi,\alpha)|s =_T t|X \in Y|\bot|(\varphi \to \psi)|\forall x.\phi \quad \text{where } s,t \in \mathcal{T}_T$$

We write $\neg\varphi$ for $\varphi \to \bot$, $\varphi \vee \psi$ for $\neg\varphi \to \psi$, $\varphi \wedge \psi$ for $\neg(\varphi \to \neg\psi)$, $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$, and $\exists x.\varphi$ for $\neg\forall x.\neg\varphi$.

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$$(\forall x^{\mathsf{M}}.x^{\mathsf{M}} \in X \to x^{\mathsf{M}} \in Y) \to (\forall x^{\mathsf{M}}.x^{\mathsf{M}} \in Y \to x^{\mathsf{M}} \in X) \to X = Y \qquad \neg(X \in \emptyset)$$

$$(X \in \wp(Y)) \leftrightarrow (\forall x^{\mathsf{M}}.x^{\mathsf{M}} \in X \to x^{\mathsf{M}} \in Y)$$

$$(X \in \bigcup(Y)) \leftrightarrow (\exists x^{\mathsf{M}}.x^{\mathsf{M}} \in Y \wedge X \in x^{\mathsf{M}})$$

$$(X \in v(\{U|. \in V; \Phi\}, \alpha) \leftrightarrow (\exists x^{\mathsf{M}}.x^{\mathsf{M}} \in v(V, \alpha) \wedge X = v(U, x^{\mathsf{M}} \rhd \alpha) \wedge p(\Phi, x^{\mathsf{M}} \rhd \alpha))$$

$$p(\Phi, X \rhd \alpha) \to p(\Phi, \varepsilon(\Phi) \rhd \alpha) \qquad infinity?$$

Figure 1: Set Theoretic Axioms

$$\hat{\bot}[\sigma] = \hat{\bot} \qquad (\Phi \hat{\to} \Psi)[\sigma] = (\Phi[\sigma] \hat{\to} \Psi[\sigma]) \qquad (\hat{\forall}\Phi)[\sigma] = (\hat{\forall}\Phi[0 \cdot (\sigma \circ \uparrow)])$$

$$(U \hat{=} V)[\sigma] = (U[\sigma] \hat{=} V[\sigma]) \qquad (U \hat{\in} V)[\sigma] = (U[\sigma] \hat{\in} V[\sigma])$$

$$\varepsilon(U)[\sigma] = \varepsilon(U[0 \cdot (\sigma \circ \uparrow)]) \qquad \hat{\emptyset}[\sigma] = \hat{\emptyset} \qquad \hat{\wp}(U)[\sigma] = \hat{\wp}(U[\sigma])$$

$$\hat{\bigcup}(U)[\sigma] = \hat{\bigcup}(U[\sigma]) \qquad \{U|. \in V; \Phi\}[\sigma] = \{U[0 \cdot (\sigma \circ \uparrow)]|. \in V[\sigma]; \Phi[0 \cdot (\sigma \circ \uparrow)]\}$$

$$U[\mathsf{id}] = U \qquad \Phi[\mathsf{id}] = \Phi \qquad 0[(U.\sigma)] = U \qquad U[\sigma][\tau] = U[\sigma \circ \tau]$$

$$\Phi[\sigma][\tau] = \Phi[\sigma \circ \tau] \qquad \mathsf{id} \circ \sigma = \sigma \qquad \sigma \circ \mathsf{id} = \sigma \qquad \uparrow \circ (U \cdot \sigma) = \sigma$$

$$(U \cdot \sigma) \circ \tau = U[\tau] \circ (\sigma \circ \tau) \qquad (\sigma_1 \circ \sigma_2) \circ \sigma_3 = \sigma_1 \circ (\sigma_2 \circ \sigma_3) \qquad v(\mathsf{k}(X), \alpha) = X$$

$$v(U[\sigma], \alpha) = v(U, (\sigma\alpha)) \qquad \mathsf{id}\alpha = \alpha \qquad v(0, (X \rhd \alpha)) = X$$

$$(U \cdot \sigma)\alpha = v(U, \alpha) \rhd (\sigma\alpha) \qquad (\sigma \circ \tau)\alpha = \sigma(\tau\alpha) \qquad v(\hat{\emptyset}, \alpha) = \emptyset$$

$$v(\hat{\wp}(U), \alpha) = \wp(v(U, \alpha)) \qquad v(\hat{\bigcup}(U), \alpha) = \bigcup(v(U, \alpha))$$

$$p(\Phi[\sigma], \alpha) \leftrightarrow p(\Phi, (\sigma\alpha)) \qquad p(U \hat{\in} V, \alpha) \leftrightarrow v(U, \alpha) \in v(V, \alpha)$$

$$p(U \hat{=} V, \alpha) \leftrightarrow v(U, \alpha) = v(V, \alpha) \qquad \neg p(\hat{\bot}, \alpha)$$

$$p(\Phi \hat{\to} \Psi, \alpha) \leftrightarrow (p(\Phi, \alpha) \to p(\Psi, \alpha)) \qquad p(\hat{\forall}\Phi, \alpha) \leftrightarrow \forall x^{\mathsf{M}}.p(\Phi, (x^{\mathsf{M}} \rhd \alpha))$$

Figure 2: Axioms for substitutions and assignments

3 Interpretations

When $g : A \to B$ and $f : B \to C$ we will use $f \circ g$ to denote the usual composition of functions, as given by $(f \circ g)(a) = f(g(a))$. Let A be a set. We use Id_A to denote the identity function from A to A, omitting the subscript when it is clear in context. Likewise, we let $\Uparrow_A$ denote the function from A^ω to A^ω given by $\Uparrow_A (f)(n) = f(n+1)$, omitting the subscript A when it is clear in context. Given an element $a \in A$ and function $f \in A^\omega$, we write $(a \blacktriangleright f)$ for the function from ω to A given by $(a \blacktriangleright f)(0) = a$ and $(a \blacktriangleright f)(n+1) = f(n)$. This will serve as the intended interpretation of the $\triangleright$ operation for assignments. Given functions $g : A^\omega \to A$ and $h : A^\omega \to A^\omega$, we write $(g \odot h)$ for the function from A^ω to A^ω given by $(g \odot h)(f)(0) = g(f)$ and $(g \odot h)(f)(n+1) = h(f)(n)$. This will serve as the intended interpretation of the $\cdot$ operation for substitions.

An interpretation of the set theory is given by taking five nonempty sets $\mathcal{D}_T$ for $T \in \{\mathsf{M}, \mathsf{A}, \mathsf{S}, \hat{\mathsf{M}}, \hat{\mathsf{P}}\}$ and interpretations of the basic operators. We will further require the following:

- $\mathcal{D}_\mathsf{M}$ is a model of ZFC, where we also use $\in$, $\emptyset$, $\wp$ and $\bigcup$ to refer to the corresponding semantic notions in this model.
- There is a global choice operator $\overline{\varepsilon} : \wp(\mathcal{D}_\mathsf{M}) \to \mathcal{D}_\mathsf{M}$ where $\overline{\varepsilon}(\emptyset) = \emptyset$ and $\overline{\varepsilon}(u) \in u$ for each nonempty $u \subseteq \mathcal{D}_\mathsf{M}$
- $\mathcal{D}_\mathsf{A} \subseteq (\mathcal{D}_\mathsf{M})^\omega$.
- $\mathcal{D}_\mathsf{S} \subseteq (\mathcal{D}_\mathsf{A})^{\mathcal{D}_\mathsf{A}}$.
- $\mathcal{D}_{\hat{\mathsf{M}}} \subseteq (\mathcal{D}_\mathsf{M})^{\mathcal{D}_\mathsf{A}}$.
- $\mathcal{D}_{\hat{\mathsf{P}}} \subseteq 2^{\mathcal{D}_\mathsf{A}}$.
- $\mathsf{Id} \in \mathcal{D}_\mathsf{S}$, $\Uparrow \in \mathcal{D}_\mathsf{S}$ and $\mathcal{D}_\mathsf{S}$ is closed under composition.
- For $X \in \mathcal{D}_\mathsf{M}$ and $\alpha \in \mathcal{D}_\mathsf{A}$, $(X \blacktriangleright \alpha) \in \mathcal{D}_\mathsf{A}$.
- For $U \in \mathcal{D}_{\hat{\mathsf{M}}}$ and $\sigma \in \mathcal{D}_\mathsf{S}$, $(U \odot \sigma) \in \mathcal{D}_\mathsf{S}$.

Given these restrictions, it is easy to define an evaluation function $[\![-]\!]_-$ for terms by recursion. The evaluation is relative to an environment $\nu : \bigcup_T \mathcal{V}_T \to \bigcup_T \mathcal{D}_T$ where $\phi(x) \in \mathcal{D}_T$ for each $x \in \mathcal{V}_T$ and each type T. The definition is given in Figure 3.

We can now define validity of a formula in such an interpretation in the obvious manner.

- $\models_\nu p(\Phi, \alpha)$ if $[\![\Phi]\!]_\nu([\![\alpha]\!]_\nu) = 1$.
- $\models_\nu s =_T t$ if $[\![s]\!]_\nu = [\![t]\!]_\nu$.
- $\not\models_\nu \bot$.
- $\models_\nu (\phi \to \psi)$ if either $\not\models_\nu \phi$ or $\models_\nu \psi$.
- For $x \in \mathcal{V}_T$ $\models_\nu (\forall x.\phi)$ if $\models_{\nu, x:=v} \phi$ for every $v \in \mathcal{D}_T$ where $\nu, x := v$ is the environment agreeing with ν except sending x to v.

$$[\![x]\!]_\nu = \nu(x) \qquad [\![\emptyset]\!]_\nu = \emptyset \qquad [\![\wp(X)]\!]_\nu = \wp([\![X]\!]_\nu) \qquad [\![\bigcup(X)]\!]_\nu = \bigcup([\![X]\!]_\nu)$$

$$[\![v(U,\alpha)]\!]_\nu = [\![U]\!]_\nu([\![\alpha]\!]_\nu) \qquad [\![\mathsf{k}_\emptyset]\!]_\nu(\gamma) = \emptyset \qquad [\![(X \triangleright \alpha)]\!]_\nu = [\![X]\!]_\nu \blacktriangleright [\![\alpha]\!]_\nu$$

$$[\![(\sigma\alpha)]\!]_\nu = [\![\sigma]\!]_\nu([\![\alpha]\!]_\nu) \qquad [\![\mathsf{id}]\!]_\nu = \mathsf{Id} \qquad [\![\uparrow]\!]_\nu = \Uparrow \qquad [\![(U\cdot\sigma)]\!]_\nu = [\![U]\!]_\nu \odot [\![\sigma]\!]_\nu$$

$$[\![(\sigma \circ \tau)]\!]_\nu = [\![\sigma]\!]_\nu \circ [\![\tau]\!]_\nu \qquad [\![\mathsf{k}(X)]\!]_\nu(\gamma) = [\![X]\!]_\nu \qquad [\![0]\!]_\nu(\gamma) = \gamma(0)$$

$$[\![U[\sigma]]\!]_\nu(\gamma) = [\![U]\!]_\nu([\![\sigma]\!]_\nu(\gamma)) \qquad [\![\hat{\emptyset}]\!]_\nu(\gamma) = \emptyset \qquad [\![\hat{\wp}(U)]\!]_\nu(\gamma) = \wp([\![U]\!]_\nu(\gamma))$$

$$[\![\hat{\bigcup}(U)]\!]_\nu(\gamma) = \bigcup([\![U]\!]_\nu(\gamma))$$

$$[\![\{U|.\in V;\Phi\}]\!]_\nu(\gamma) = \{[\![U]\!]_\nu(v\blacktriangleright\gamma)|v \in [\![V]\!]_\nu(\gamma) \text{ where } [\![\Phi]\!]_\nu(v\blacktriangleright\gamma)\}$$

$$[\![\varepsilon(\Phi)]\!]_\nu(\gamma) = \overline{\varepsilon}\{v \in \mathcal{D}_{\mathsf{M}}|[\![\Phi]\!]_\nu(v\blacktriangleright\gamma)\} \qquad [\![\Phi[\sigma]]\!]_\nu(\gamma) = [\![\Phi]\!]_\nu([\![\sigma]\!]_\nu(\gamma))$$

$$[\![U \hat{\in} V]\!]_\nu(\gamma) = \begin{cases} 1 & \text{if } [\![U]\!]_\nu(\gamma) \in [\![V]\!]_\nu(\gamma) \\ 0 & \text{otherwise} \end{cases}$$

$$[\![U \hat{=} V]\!]_\nu(\gamma) = \begin{cases} 1 & \text{if } [\![U]\!]_\nu(\gamma) = [\![V]\!]_\nu(\gamma) \\ 0 & \text{otherwise} \end{cases} \qquad [\![\hat{\bot}]\!]_\nu(\gamma) = 0$$

$$[\![(\Phi \hat{\to} \Psi)]\!]_\nu(\gamma) = \begin{cases} 1 & \text{if } [\![\Phi]\!]_\nu(\gamma) = 0 \text{ or } [\![\Psi]\!]_\nu(\gamma) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$[\![(\hat{\forall}\Phi)]\!]_\nu(\gamma) = \begin{cases} 1 & \text{if } [\![\Phi]\!]_\nu(v\blacktriangleright\gamma) = 1 \text{ for every } v \in \mathcal{D}_{\mathsf{M}} \\ 0 & \text{otherwise} \end{cases}$$

4 Examples

Consider the informal proposition $\forall XYz.z \in \{x \in X|x \in Y\} \leftrightarrow z \in X \wedge z \in Y$. To represent this as a formula, we must find a way to represent the term $\{x \in X|x \in Y\}$. If we reconsider $\{x \in X|x \in Y\}$ as $\{x|x \in X; x \in Y\}$, then we can use the generalized set construct $\{U|. \in V; \Phi\}$ where the de Bruijn variable 0 will play the role of the bound variable. The sets X and Y will need to be lifted via k to be generalized sets in order use them in this context, giving the term $\{0|. \in \mathsf{k}(X); 0 \in \mathsf{k}(Y)\}$. In order to use this term as a set, we must evaluate it using some assignment α. In this case we simply use the assignment $\mathsf{k}_{\emptyset}$. A formula corresponding to the original proposition is given by

$$\forall XYz.z \in v(\{0|. \in \mathsf{k}(X); 0 \in \mathsf{k}(Y)\}, \mathsf{k}_{\emptyset}) \leftrightarrow z \in X \wedge z \in Y$$

where $X, Y, z \in \mathcal{V}_{\mathsf{M}}$.