Automating Formalization by Statistical and Semantic Parsing of Mathematics

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Two Obstacles to Strong AI/Reasoning for Math

1. Low reasoning power of automated reasoning methods, particularly over large complex theories
2. Lack of computer understanding of current human-level (math and exact science) knowledge

- The two are related: human-level math may require nontrivial reasoning to become fully explained. Fully explained math gives us a lot of data for training AI/TP systems.
- And we want to train AI/TP on human-level proofs too. Thus getting interesting formalization/ATP/learning feedback loops.
- In 2014 we have decided that the AI/TP systems are getting strong enough to try this. And we started to combine them with statistical translation of informal-to-formal math.

ProofWiki vs Mizar – our CICM'14 Example

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1 of 1 89,84%

EXAMPLE: PROOFWIKI VS MIZAR VS MIZAR-STYLE AUTOMATED PROOF

== Theorem ==
Let $(S, \circ)$ be an [[Definition:Algebraic Structure|algebraic structure]] that has a [[Definition:Zero Element|zero element]] $z \in S$. Then z is unique.
== Proof ==
Suppose z_1 and z_2 are both zeroes of $(S, \circ)$.
Then by the definition of [[Definition:Zero Element|zero element]]:
$z_2 \circ z_1 = z_1$ by dint of z_1 being a zero;
$z_2 \circ z_1 = z_2$ by dint of z_2 being a zero.
So $z_1 = z_2 \circ z_1 = z_2$.
So $z_1 = z_2$ and there is only one zero after all.
{{qed}}
// *NB: Informal proofs are buggy!*

```
Th9:  e1 is_a_left_unity_wrt o &
e2 is_a_right_unity_wrt o implies e1 = e2
proof
assume that A1:  e1 is_a_left_unity_wrt o and
A2:  e2 is_a_right_unity_wrt o;
thus e1 = o.(e1,e2) by A2,Def6 .= e2 by A1,Def5;
end;

z1 is_a_unity_wrt o & z2 is_a_unity_wrt o
implies z1 = z2 proof
assume that A1:  z1 is_a_unity_wrt o and
A2:  z2 is_a_unity_wrt o;
A3:  o.(z2,z1) = z1 by Th3,A2; ::[ATP]
A4:  o.(z2,z1) = z2 by Def 6,Def 7,A1,A3; ::[ATP]
hence z1 = z2 by Th9,A1,Def 7,A2; ::[ATP]
end;
```

Formal, Informal and Semiformal Corpora

- HOL Light and Flyspeck: some 25,000 toplevel theorems
- The Mizar Mathematical Library: some 60,000 toplevel theorems (most of them rather small lemmas), 10,000 definitions
- Coq: several large projects (Feit-Thompson theorem, ...)
- Isabelle, seL4 and the Archive of Formal Proofs
- Arxiv.org: 1M articles collected over some 20 years (not just math)
- Wikipedia: 25,000 articles in 2010 - collected over 10 years only
- Proofwiki - LaTeX but very semantic, re-invented the Mizar proof style

Our Initial Approach/Plan

- There is not yet much aligned informal/formal data
- So try first with “ambiguated” (informalized) formal corpora
- Try first with non black-box architectures such as probabilistic grammars
- Which can be easily enhanced internally by semantic pruning (e.g. type constraints)
- Develop feedback loops between training statistical parsing and theorem proving
- Start employing more sophisticated ML methods
- Progress to more complicated informal corpora/phenomena
- Both directly: ML/ATP with only cruder alignments (theorems, chapters, etc)
- And indirectly: train statistical/precise alignments across informal and formal corpora, use them to enhance our coverage
- Example: word2vec/Glove/neural learning of synonyms over Arxiv

Work Done So Far: Informalized Flyspeck

- 22000 Flyspeck theorem statements informalized
  - 72 overloaded instances like "+" for `vector_add`
  - 108 infix operators
  - forget "prefixes" `real_, int_, vector_, matrix_, complex_`, etc.
  - REAL_NEGNEG: $\forall x. - -x = x$

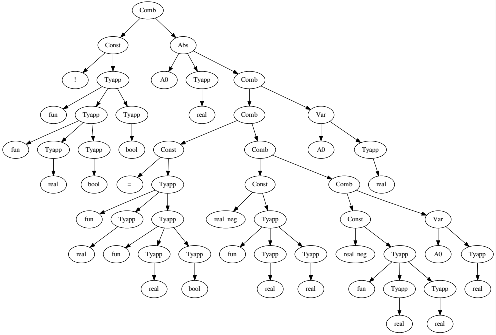
```
(Comb (Const "!" (Tyapp "fun" (Tyapp "fun" (Tyapp "real") (Tyapp "bool"))
(Tyapp "bool"))) (Abs "A0" (Tyapp "real") (Comb (Comb (Const "=" (Tyapp "fu
(Tyapp "real") (Tyapp "fun" (Tyapp "real") (Tyapp "bool")))) (Comb (Const
"real_neg" (Tyapp "fun" (Tyapp "real") (Tyapp "real"))) (Comb (Const
"real_neg" (Tyapp "fun" (Tyapp "real") (Tyapp "real"))) (Var "A0" (Tyapp
"real"))))) (Var "A0" (Tyapp "real")))))
```

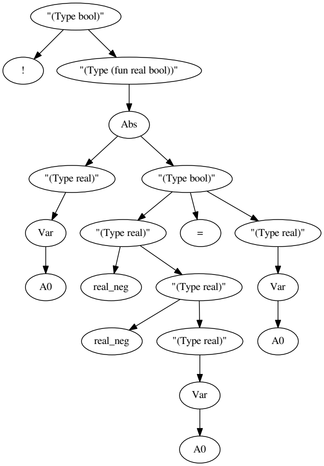
  - becomes

```
("(Type bool)" ! ("(Type (fun real bool))" (Abs ("(Type real)"
(Var A0)) ("(Type bool)" ("(Type real)" real_neg ("(Type real)"
real_neg ("(Type real)" (Var A0)))) = ("(Type real)" (Var A0))))))
```

- Training a probabilistic grammar (context-free, later with deeper context)
- CYK chart parser with semantic pruning (compatible types of variables)
- Using HOL Light and HolyHammer to typecheck and prove the results

Example grammars



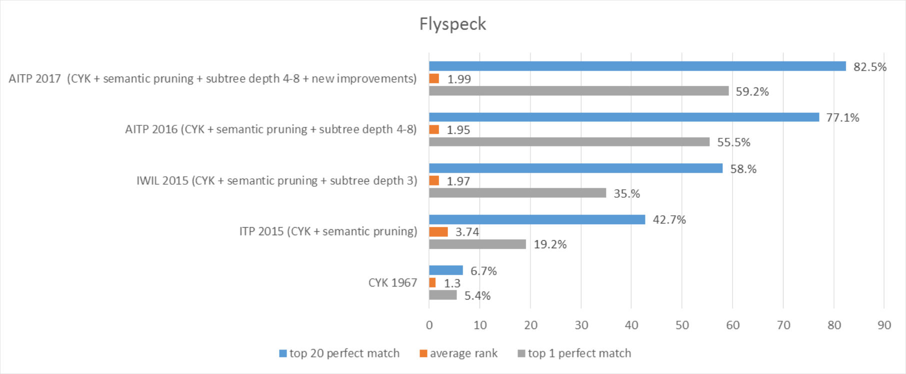


Online parsing system

- “`sin ( 0 * x ) = cos pi / 2`”
- produces 16 parses
- of which 11 get type-checked by HOL Light as follows
- with all but three being proved by HOL(y)Hammer

```
sin (&0 * A0) = cos (pi / &2) where A0:real
sin (&0 * A0) = cos pi / &2 where A0:real
sin (&0 * &A0) = cos (pi / &2) where A0:num
sin (&0 * &A0) = cos pi / &2 where A0:num
sin (&(0 * A0)) = cos (pi / &2) where A0:num
sin (&(0 * A0)) = cos pi / &2 where A0:num
csin (Cx (&0 * A0)) = ccos (Cx (pi / &2)) where A0:real
csin (Cx (&0) * A0) = ccos (Cx (pi / &2)) where A0:real^2
Cx (sin (&0 * A0)) = ccos (Cx (pi / &2)) where A0:real
csin (Cx (&0 * A0)) = Cx (cos (pi / &2)) where A0:real
csin (Cx (&0) * A0) = Cx (cos (pi / &2)) where A0:real^2
```

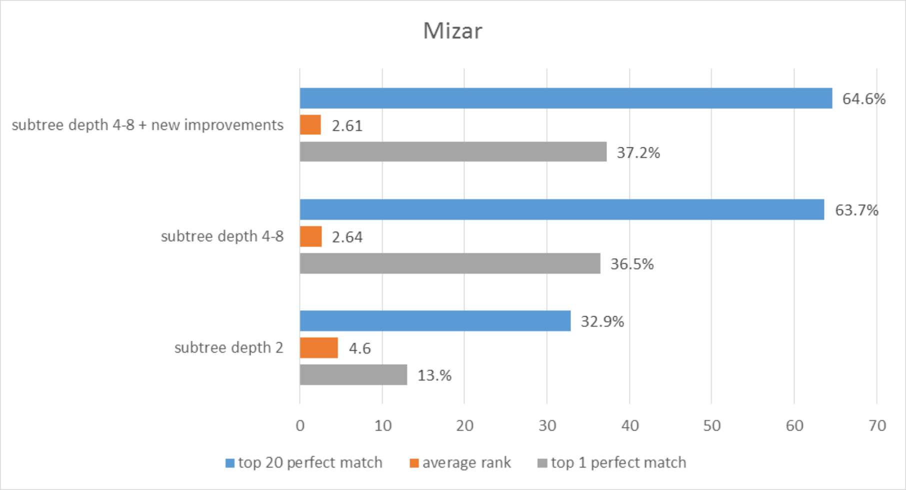
Flyspeck Progress



Tried Also for Mizar

- More natural-language features than HOL (designed by a linguist)
- Pervasive overloading
- Declarative natural-deduction proof style (re-invented in ProofWiki)
- Adjectives, dependent types, hidden arguments, synonyms
- Addressed by using two layers
  - user (pattern) layer - resolves overloading, but no hidden arguments completed, etc.
  - semantic (constructor) layer - hidden arguments computed, types resolved, ATP-ready
  - connected by ATP or a custom elaborator

First Mizar Results (100-fold Cross-validation)



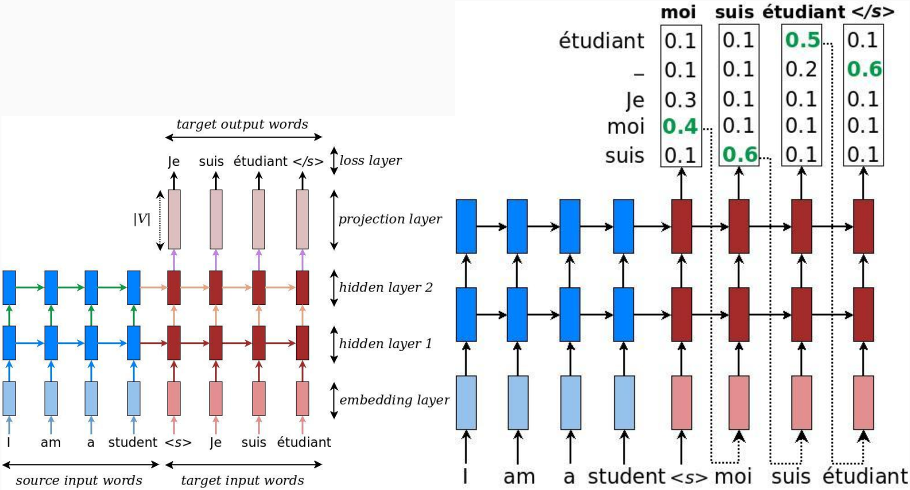
Neural Autoformalization (Wang et al., 2018)

- generate about 1M Latex - Mizar pairs
- Based on Bancerek's work: journal *Formalized Mathematics* `http://fm.mizar.org/`
- train neural seq-to-seq translation models (Luong – NMT)
- evaluate on about 100k examples
- many architectures tested, some work much better than others
- very important latest invention: *attention* in the seq-to-seq models
- more data very important for neural training – our biggest bottleneck (you can help!)

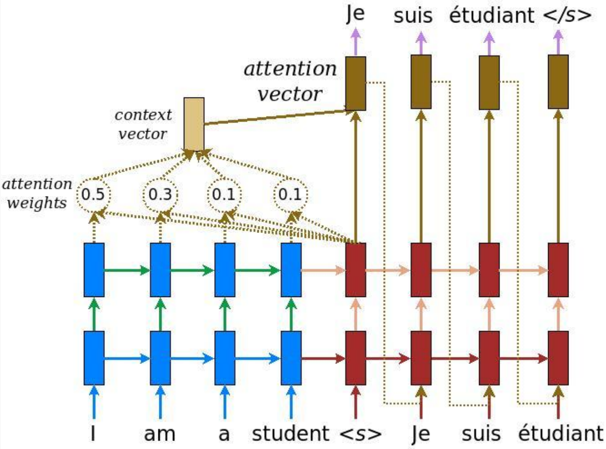
Neural Autoformalization data

Rendered LaTeX	If $X \subseteq Y \subseteq Z$, then $X \subseteq Z$.
Mizar	`X c= Y & Y c= Z implies X c= Z;`
Tokenized Mizar	`X c= Y & Y c= Z implies X c= Z ;`
LaTeX	`If $X \subseteq Y \subseteq Z$, then $X \subseteq Z$.`
Tokenized LaTeX	`If $ X \subseteq Y \subseteq Z $ , then $ X \subseteq Z $ .`

Sequence-to-sequence models - decoder/encoder RNN



Seq2seq with Attention



Initial results - Small Dataset (50k/5k train/test)

Attention	Correct	Percentage
No attention	120	2.5%
Bahdanau	165	3.4%
Normed Bahdanau	1267	26.12%
Luong	1375	28.34%
Scaled Luong	1270	26.18%
Any	1782	36.73%

Sample Statement (50k/5k train/test)

Attention	Statement
Correct	for T being Noetherian sup-Semilattice for I being Ideal of T holds ex_sup_of I , T & sup I in I
No attention	for T being lower-bounded sup-Semilattice for I being Ideal of T holds I is upper-bounded & I is upper-bounded
Bahdanau	for T being T , T being Ideal of T , I being Element of T holds height T in I
Normed Bahdanau	for T being Noetherian adj-structured sup-Semilattice for I being Ideal of T holds ex_sup_of I , T & sup I in I
Luong	for T being Noetherian adj-structured sup-Semilattice for I being Ideal of T holds ex_sup_of I , T & sup I in I
Scaled Luong	for T being Noetherian sup-Semilattice , I being Ideal of T ex I , sup I st ex_sup_of I , T & sup I in I

Full Neural Autoformalization results (1M/100k train/test)

Parameter	Final Test Perplexity	Final Test BLEU	Identical Statements (%)	Identical No-overlap (%)
128 Units	3.06	41.1	40121 (38.12%)	6458 (13.43%)
256 Units	1.59	64.2	63433 (60.27%)	19685 (40.92%)
512 Units	1.6	**67.9**	66361 (63.05%)	21506 (44.71%)
1024 Units	**1.51**	61.6	**69179 (65.73%)**	**22978 (47.77%)**
2048 Units	2.02	60	59637 (56.66%)	16284 (33.85%)

Coverage and Edit Instance

	Identical Statements	0	≤ 1	≤ 2
Best Model - 1024 Units	69179 (total) 22978 (no-overlap)	65.73% 47.77%	74.58% 59.91%	86.07% 70.26%
Top-5 Greedy Cover - 1024 Units - 4-Layer Bi. Res. - 512 Units - 6-Layer Adam Bi. Res. - 2048 Units	78411 (total) 28708 (no-overlap)	74.50% 59.68%	82.07% 70.85%	87.27% 78.84%
Top-10 Greedy Cover - 1024 Units - 4-Layer Bi. Res. - 512 Units - 6-Layer Adam Bi. Res. - 2048 Units - 2-Layer Adam Bi. Res. - 256 Units - 5-Layer Adam Res. - 6-Layer Adam Res. - 2-Layer Bi. Res.	80922 (total) 30426 (no-overlap)	76.89% 63.25%	83.91% 73.74%	88.60% 81.07%
Union of All 39 Models	83321 (total) 32083 (no-overlap)	79.17% 66.70%	85.57% 76.39%	89.73% 82.88%

Caveat

- Our evaluation is strictly syntactic
- Many synonyms in Mizar:
- `for x st P(x) holds Q(x)`
- `for x holds P(x) implies Q(x)`
- ... and much more semantic ones
- We have not done an ATP evaluation yet

Neural Autoformalization - Mizar to LaTeX

Parameter	Final Test Perplexity	Final Test BLEU	Identical Statements	Percentage
512 Units Bidirectional Scaled Luong	2.91	57	54320	51.61%

Neural Fun – Performance after Some Training

Rendered LaTeX	Suppose s_8 is convergent and s_7 is convergent . Then $\lim(s_8+s_7) = \lim s_8 + \lim s_7$
Input LaTeX	`Suppose $ { s _ { 8 } } $ is convergent and $ { s _ { 7 } } $ is convergent . Then $ \mathop { \rm lim } ( { s _ { 8 } } { + } { s _ { 7 } } } ) \mathrel { = } \mathop { \rm lim } { s _ { 8 } } { + } \mathop { \rm lim } { s _ { 7 } } } $ .`
Correct	`seq1 is convergent & seq2 is convergent implies lim ( seq1 + seq2 ) = ( lim seq1 ) + ( lim seq2 ) ;`
Snapshot-1000	`x in dom f implies ( x * y ) * ( f | ( x | ( y | ( y | y ) ) ) ) = ( x | ( y | ( y | ( y | y ) ) ) ) ) ;`
Snapshot-2000	`seq is summable implies seq is summable ;`
Snapshot-3000	`seq is convergent & lim seq = 0c implies seq = seq ;`
Snapshot-4000	`seq is convergent & lim seq = lim seq implies seq1 + seq2 is convergent ;`
Snapshot-5000	`seq1 is convergent & lim seq2 = lim seq2 implies lim_inf seq1 = lim_inf seq2 ;`
Snapshot-6000	`seq is convergent & lim seq = lim seq implies seq1 + seq2 is convergent ;`
Snapshot-7000	`seq is convergent & seq9 is convergent implies lim ( seq + seq9 ) = ( lim seq ) + ( lim seq9 ) ;`

Thanks, references and advertisement

- *Thanks for your attention!*
- *References:*
- C. Kaliszyk, J. Urban, J. Vyskocil: Automating Formalization by Statistical and Semantic Parsing of Mathematics. ITP 2017: 12-27
- Q. Wang, C. Kaliszyk, J. Urban: First Experiments with Neural Translation of Informal to Formal Mathematics. CoRR abs/1805.06502 (2018)
- *Advertisement:*
- To push AI methods in math and theorem proving, we organize:
- AITP – Artificial Intelligence and Theorem Proving
- April 8–12, 2019, Obergurgl, Austria, `aitp-conference.org`
- ATP/ITP/ vs AI/Machine-Learning people, Computational linguists
- Discussion-oriented and experimental