Developments in AI and Theorem Proving

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How Do We Automate Math and Science?

- What is mathematical and scientific thinking?
- Pattern-matching, analogy, induction from examples
- Deductive reasoning
- Complicated feedback loops between induction and deduction
- Using a lot of previous knowledge - both for induction and deduction

- We need to develop such methods on computers
- Are there any large corpora suitable for nontrivial deduction?
- Yes! Large libraries of formal proofs and theories
- So let's develop strong AI on them!

Why Combine Learning and Reasoning Today?

1. It practically helps!
   - Automated theorem proving for large formal verification is useful:
     - Formal Proof of the Kepler Conjecture (2014 – Hales – 20k lemmas)
     - Formal Proof of the Feit-Thompson Theorem (2012 – Gonthier)
     - Verification of compilers (CompCert) and microkernels (seL4)
     - Verification hardware architectures, transport systems, trading rules
     - ...
   - But good learning/AI methods needed to cope with large theories!
2. Blue Sky AI Visions:
   - Get strong AI by learning/reasoning over large KBs of human thought?
   - Big formal theories: good semantic approximation of such thinking KBs?
   - Deep non-contradictory semantics – better than scanning books?
   - Gradually try learning math/science:
     - What are the components (inductive/deductive thinking)?
     - How to combine them together?
     - Automate/verify math, science, law, ...
     - Leibniz: Calculemus - resolve disputes
     - J. McCarthy: Mathematical Objectivity and the Power of Initiative

What is Formal Mathematics?

- Developed thanks to the Leibniz/Russell/Frege/Hilbert/... program
- Mathematics put on formal logic foundations (*symbolic computation*)
- ... which btw. led also to the rise of computers (Turing/Church, 1930s)
- Formal math (1950/60s): combine formal foundations and the newly available computers
- De Bruijn, Milner, Trybulec, Boyer and Moore, Gordon, Huet, Paulson, ...
- Automath, LCF, Mizar, NQTHM and ACL2, HOL, Coq, Isabelle, ...
- Conceptually very simple:
- Write all your axioms and theorems so that computer understands them
- Write all your inference rules so that computer understands them
- Use the computer to check that your proofs follow the rules
- But in practice, it turns out not to be so simple
- Many approaches, still not mainstream, but big breakthroughs recently

Freek Wiedijk's Example: Irrationality of $\sqrt{2}$ (informal text)

tiny proof from Hardy & Wright:

Theorem 43 (Pythagoras' theorem). $\sqrt{2}$ is irrational.
The traditional proof ascribed to Pythagoras runs as follows. If $\sqrt{2}$ is rational, then the equation

$$a^2 = 2b^2 \tag{4.3.1}$$

is soluble in integers a, b with $(a, b) = 1$. Hence a^2 is even, and therefore a is even. If $a = 2c$, then $4c^2 = 2b^2$, $2c^2 = b^2$, and b is also even, contrary to the hypothesis that $(a, b) = 1$. $\square$

Irrationality of $\sqrt{2}$ (Formal Proof Sketch)

exactly the same text in Mizar syntax:

```
theorem Th43: :: Pythagoras' theorem
  sqrt 2 is irrational
proof
  assume sqrt 2 is rational;
  consider a,b such that
4_3_1: a^2 = 2*b^2 and
    a,b are relative prime;
  a^2 is even;
  a is even;
  consider c such that a = 2*c;
  4*c^2 = 2*b^2;
  2*c^2 = b^2;
  b is even;
  thus contradiction;
end;
```

Irrationality of $\sqrt{2}$ in HOL Light

```
let SQRT_2_IRRATIONAL = prove
 (`~rational(sqrt(&2))`,
  SIMP_TAC[rational; real_abs; SQRT_POS_LE; REAL_POS] THEN
  REWRITE_TAC[NOT_EXISTS_THM] THEN REPEAT GEN_TAC THEN
  DISCH_THEN(CONJUNCTS_THEN2 ASSUME_TAC MP_TAC) THEN
  SUBGOAL_THEN `~((&p / &q) pow 2 = sqrt(&2) pow 2)`
    (fun th -> MESON_TAC[th]) THEN
  SIMP_TAC[SQRT_POW_2; REAL_POS; REAL_POW_DIV] THEN
  ASM_SIMP_TAC[REAL_EQ_LDIV_EQ; REAL_OF_NUM_LT; REAL_POW_LT;
               ARITH_RULE `0 < q <=> ~(q = 0)`] THEN
  ASM_MESON_TAC[NSQRT_2; REAL_OF_NUM_POW; REAL_OF_NUM_MUL; REAL_OF_NUM_EQ]);;
```

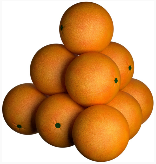
Irrationality of $\sqrt{2}$ in Isabelle/HOL

```
theorem sqrt2_not_rational:
  "sqrt (real 2) ∉ ℚ"
proof
  assume "sqrt (real 2) ∈ ℚ"
  then obtain m n :: nat where
    n_nonzero: "n ≠ 0" and sqrt_rat: "¦sqrt (real 2)¦ = real m / real n"
    and lowest_terms: "gcd m n = 1" ..
  from n_nonzero and sqrt_rat have "real m = ¦sqrt (real 2)¦ * real n" by simp
  then have "real (m²) = (sqrt (real 2))² * real (n²)"
    by (auto simp add: power2_eq_square)
  also have "(sqrt (real 2))² = real 2" by simp
  also have "... * real (m²) = real (2 * n²)" by simp
  finally have eq: "m² = 2 * n²" ..
  hence "2 dvd m²" ..
  with two_is_prime have dvd_m: "2 dvd m" by (rule prime_dvd_power_two)
  then obtain k where "m = 2 * k" ..
  with eq have "2 * n² = 2² * k²" by (auto simp add: power2_eq_square mult_ac)
  hence "n² = 2 * k²" by simp
  hence "2 dvd n²" ..
  with two_is_prime have "2 dvd n" by (rule prime_dvd_power_two)
  with dvd_m have "2 dvd gcd m n" by (rule gcd_greatest)
  with lowest_terms have "2 dvd 1" by simp
  thus False by arith
qed
```

Big Example: The Flyspeck project

- Kepler conjecture (1611): The most compact way of stacking balls of the same size in space is a pyramid.

$$V = \frac{\pi}{\sqrt{18}} \approx 74\%$$



- Formal proof finished in 2014
- 20000 lemmas in geometry, analysis, graph theory
- All of it at `https://code.google.com/p/flyspeck/`
- All of it computer-understandable and verified in HOL Light:
- `polyhedron s /\ c face_of s ==> polyhedron c`
- However, this took 20 – 30 person-years!

What Are Automated Theorem Provers?

- Computer programs that (try to) determine if
  - A conjecture C is a logical consequence of a set of axioms Ax
  - The derivation of conclusions that follow from facts by inference rules
- Systems: Vampire, E, SPASS, Prover9, Z3, CVC4, Satallax, ...
- Brute-force search calculi (resolution, superposition, tableaux, SMT, ...)
- Human-designed heuristics for pruning of the search space
- Fast combinatorial explosion on large knowledge bases like Flyspeck and Mizar
- Need to be equipped with good domain-specific inference guidance ...
- ... and that is what I try to do ...
- ... typically by learning in various ways from the knowledge bases ...

History and Motivation for AI/TP

- Intuition vs Formal Reasoning – Poincaré vs Hilbert, Science & Method
- Turing's 1950 paper: Learning Machines, learn Chess?, undecidability??
- Lenat, Langley, etc: manually-written heuristics, learn Kepler laws,...
- Denzinger, Schulz, Goller, Fuchs – late 90's, ATP-focused:
- *Learning from Previous Proof Experience*
- My MSc (1998): Try ILP to learn rules and heuristics from IMPS/Mizar
- Since: Use large formal math (Big Proof) corpora: Mizar, Isabelle, HOL
- ... to combine/develop symbolic/statistical deductive/inductive ML/TP/AI
- ... hammer-style methods, feedback loops, internal guidance, ...
- More details – AGI'18 keynote: `https://bit.ly/3qifhg4`
- AI vs DL: Ben Goertzel's Prague talk: `https://youtu.be/Zt2HSTuGBn8`
- Big AI visions: automate/verify math, science, law, (Leibniz, McCarthy, ..)
- Practical impact: boost today's large ITP verification projects

Using Learning to Guide Theorem Proving

- **high-level**: pre-select lemmas from a large library, give them to ATPs
- **high-level**: pre-select a good ATP strategy/portfolio for a problem
- **high-level**: pre-select good *hints* for a problem, use them to guide ATPs
- **low-level**: guide every inference step of ATPs (tableau, superposition)
- **low-level**: guide every kernel step of LCF-style ITPs
- **mid-level**: guide application of tactics in ITPs
- **mid-level**: invent suitable ATP strategies for classes of problems
- **mid-level**: invent suitable conjectures for a problem
- **mid-level**: invent suitable concepts/models for problems/theories
- **proof sketches**: explore stronger/related theories to get proof ideas
- **theory exploration**: develop interesting theories by conjecturing/proving
- **feedback loops**: (dis)prove, learn from it, (dis)prove more, learn more, ...
- **autoformalization**: (semi-)automate translation from LaTeX to formal
- ...

Large AI/TP Datasets

- Mizar / MML / MPTP – since 2003
- MPTP Challenge (2006), MPTP2078 (2011), Mizar40 (2013)
- Isabelle (and AFP) – since 2005
- Flyspeck (including core HOL Light and Multivariate) – since 2012
- HOL4 – since 2014, CakeML – 2017, GRUNGE – 2019
- Coq – since 2013/2016
- AIM – Veroff & Kinyon, Loops with Abelian Inner Mappings – long proofs
- Lean?, Stacks?, Arxiv?, ProofWiki?, ...

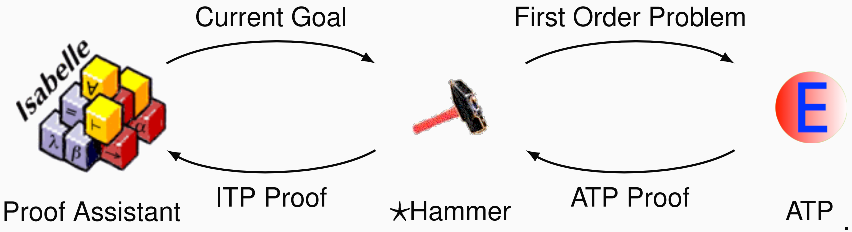
Demos

- ENIGMA/hammer proofs of Pythagoras : `https://bit.ly/2MVPAn7` (more at `http://grid01.ciirc.cvut.cz/~mptp/enigma-ex.pdf`) and simplified Carmichael `https://bit.ly/3oGBdRz`
- Hammering demo: `http://grid01.ciirc.cvut.cz/~mptp/out4.ogv`
- TacticToe on HOL4: `http://grid01.ciirc.cvut.cz/~mptp/tactictoe_demo.ogv`
- Tactician for Coq: `https://blaauwbroek.eu/papers/cicm2020/demo.mp4`, `https://coq-tactician.github.io/demo.html`
- Inf2formal over HOL Light: `http://grid01.ciirc.cvut.cz/~mptp/demo.ogv`

High-level ATP guidance: Premise Selection

- Early 2003: Can existing ATPs be used over the freshly translated Mizar library?
- About 80000 nontrivial math facts at that time – impossible to use them all
- Is good premise selection for proving a new conjecture possible at all?
- Or is it a mysterious power of mathematicians? (Penrose)
- Today: Premise selection is not a mysterious property of mathematicians!
- Reasonably good algorithms started to appear (more below).
- Extensive human (math) knowledge obsolete?? (cf. Watson, Debater, ..)
- Since 2004 (my PhD): many examples of nontrivial alternative proofs proposed by the AIs - in Mizar, Flyspeck, Isabelle, ..
- The premise selection algorithms see *wider* than humans

Today's AI-ATP systems (⋆-Hammers)



How much can it do?

- Mizar / MML – MizAR
- Isabelle (Auth, Jinja) – Sledgehammer
- Flyspeck (including core HOL Light and Multivariate) – HOL(y)Hammer
- HOL4 (Gauthier and Kaliszyk), Coq (Czajka and Kaliszyk)
- Rigorous resource controlled train/test evaluations on toplevel lemmas:

$\approx$ 40-45% success rate by 2016
$\approx$ 60% on Mizar as of 2021

Premise Selection and Hammer Methods

- Many syntactic features (symbols, walks in the parse trees)
- More semantic features encoding
- Term matching/unification, validity in models, latent semantics (LSI)
- Distance-weighted k-nearest neighbor, SVMs, Naive Bayes
- Gradient boosted decision trees (GBDTs - XGBoost, LightGBM)
- Neural models: CNNs, RNNs/Attention/Transformers/GPT, GNNs
- As of 2020, tough competition between GBDTs, GNNs and RNNs/Transformers (and relatives)
- K-NN still very good, Olsak's logic-aware GNN probably best
- RNNs/Transformers good at stateful premise selection (Piotrowski 2019,2020)
- Ensemble methods combining the different predictors help a lot

Premise Selection and Hammer Methods

- Learning in a binary setting from many alternative proofs
- Interleaving many learning and proving runs (*MaLARea loop - 2006*) to get positives/negatives (ATPBoost - Piotrowski 2018)
- Matching and transferring concepts and theorems between libraries (Gauthier & Kaliszyk) – allows “superhammers”, conjecturing, and more
- Lemmatization – extracting and considering millions of low-level lemmas and learning from their proofs (Kaliszyk & JU 2013)
- Hammers combined with guided tactical search: TacticToe (Gauthier - HOL4) and its later relatives

FACE_OF_POLYHEDRON_POLYHEDRON

```
let FACE_OF_POLYHEDRON_POLYHEDRON = prove
 (`!s:real^N->bool c. polyhedron s /\ c face_of s ==> polyhedron c`,
  REPEAT STRIP_TAC THEN FIRST_ASSUM
   (MP_TAC o GEN_REWRITE_RULE I [POLYHEDRON_INTER_AFFINE_MINIMAL]) THEN
  REWRITE_TAC[RIGHT_IMP_EXISTS_THM; SKOLEM_THM] THEN
  SIMP_TAC[LEFT_IMP_EXISTS_THM; RIGHT_AND_EXISTS_THM; LEFT_AND_EXISTS_THM] THEN
  MAP_EVERY X_GEN_TAC
   [`f:(real^N->bool)->bool`; `a:(real^N->bool)->real^N`;
    `b:(real^N->bool)->real`] THEN
  STRIP_TAC THEN
  MP_TAC(ISPECL [`s:real^N->bool`; `f:(real^N->bool)->bool`;
                `a:(real^N->bool)->real^N`; `b:(real^N->bool)->real`]
        FACE_OF_POLYHEDRON_EXPLICIT) THEN
  ANTS_TAC THENL [ASM_REWRITE_TAC[] THEN ASM_MESON_TAC[]; ALL_TAC] THEN
  DISCH_THEN(MP_TAC o SPEC `c:real^N->bool`) THEN ASM_REWRITE_TAC[] THEN
  ASM_CASES_TAC `c:real^N->bool = {}` THEN
  ASM_REWRITE_TAC[POLYHEDRON_EMPTY] THEN
  ASM_CASES_TAC `c:real^N->bool = s` THEN ASM_REWRITE_TAC[] THEN
  DISCH_THEN SUBST1_TAC THEN MATCH_MP_TAC POLYHEDRON_INTERS THEN
  REWRITE_TAC[FORALL_IN_GSPEC] THEN
  ONCE_REWRITE_TAC[SIMPLE_IMAGE_GEN] THEN
  ASM_SIMP_TAC[FINITE_IMAGE; FINITE_RESTRICT] THEN
  REPEAT STRIP_TAC THEN REWRITE_TAC[IMAGE_ID] THEN
  MATCH_MP_TAC POLYHEDRON_INTER THEN
  ASM_REWRITE_TAC[POLYHEDRON_HYPERPLANE]);;
```

FACE_OF_POLYHEDRON_POLYHEDRON

```
polyhedron s /\ c face_of s ==> polyhedron c
```

HOL Light proof: could not be re-played by ATPs.

Alternative proof found by a hammer based on `FACE_OF_STILLCONVEX`:
Face *t* of a convex set *s* is equal to the intersection of *s* with the affine hull of *t*.

```
FACE_OF_STILLCONVEX:
 !s t:real^N->bool. convex s ==>
 (t face_of s <=>
  t SUBSET s /\ convex(s DIFF t) /\ t = (affine hull t) INTER s)
POLYHEDRON_IMP_CONVEX:
 !s:real^N->bool. polyhedron s ==> convex s
POLYHEDRON_INTER:
 !s t:real^N->bool. polyhedron s /\ polyhedron t
   ==> polyhedron (s INTER t)
POLYHEDRON_AFFINE_HULL:
 !s. polyhedron(affine hull s)
```

Statistical Guidance of a Simple Connection Prover

- learn guidance of every clausal inference in connection tableau (leanCoP)
- set of first-order clauses, *extension* and *reduction* steps
- proof finished when all branches are closed
- a lot of nondeterminism, requires backtracking
- the search space quickly explodes
- good for learning – the tableau compactly represents the proof state

Clauses:

$c_1 : P(x)$

$c_2 : R(x, y) \vee \neg P(x) \vee Q(y)$

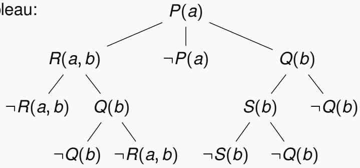
$c_3 : S(x) \vee \neg Q(b)$

$c_4 : \neg S(x) \vee \neg Q(x)$

$c_5 : \neg Q(x) \vee \neg R(a, x)$

$c_6 : \neg R(a, x) \vee Q(x)$

Closed Connection Tableau:



Using Reinforcement Learning to Guide leanCoP

- Monte-Carlo Tree Search (MCTS) – used in AlphaGo
- MCTS search nodes are sequences of clause application
- a good heuristic to explore new vs exploit good nodes:

$$\frac{w_i}{n_i} + c \cdot p_i \cdot \sqrt{\frac{\ln N}{n_i}} \qquad \text{(UCT - Kocsis, Szepesvari 2006)}$$

- we learn the *policy* – clause selection
- ... and the *value* – proof state evaluation
- big issue: representing clauses and proofs for learning
- many approaches - none too good yet, esp. for value
- deep learning not impressive yet and slower than GBDTs
- feedback loop between proving and learning - many iterations

Statistical Guidance of Connection Tableau – rlCoP

- On 32k Mizar40 problems using 200k inference limit
- nonlearning CoPs:

System	leanCoP	bare prover	rlCoP no policy/value (UCT only)
Training problems proved	10438	4184	7348
Testing problems proved	**1143**	431	804
Total problems proved	11581	4615	8152

- rlCoP with policy/value after 5 proving/learning iters on the training data
- $1624/1143 = 42.1\%$ improvement over leanCoP on the testing problems

Iteration	1	2	3	4	5	6	7	8
Training proved	12325	13749	14155	14363	14403	14431	14342	**14498**
Testing proved	1354	1519	1566	1595	**1624**	1586	1582	1591

Recent CoP Mutants: FLoP, GNN, RNN, lazyCoP

- FLoP – Finding Longer Proofs (Zombori et al, 2019)
- Curriculum Learning used for connection tableau over Robinson Arithmetic
- addition and multiplication learned perfectly from $1 * 1 = 1$
- headed towards learning algorithms/decision procedures from math data
- currently black-box, combinations with symbolic methods (ILP) our next target
- Using RNNs for better tableau encoding, prediction of actions ...
- ... even guessing (decoding) next tableau literals (Piotrowski 2020)
- plCoP (Zombori 20), GNN-CoP (Olsak 20), lazyCoP (Rawson)
- Zombori: learning new explainable Prolog actions (tactics) from proofs

ENIGMA: Guiding the Best ATPs like E Prover

- Similar to rlCoP - interleave proving and learning of ENIGMA guidance
- resolution/superposition harder for learning than tableau
- the proof state are two large heaps of clauses *processed/unprocessed*
- Done on 57880 Mizar problems recently - 6 prove/learn iterations
- Feedback loop: 70% improvement over the original strategy in 2019
- From 14933 proofs to 25397 proofs (all 10s CPU - no cheating)
- Went up to 40k proofs in more iterations and 60s time in 2020
- Many proof examples at `https://github.com/ai4reason/ATP_Proofs`

	$\mathcal{S}$	$\mathcal{S} \odot \mathcal{M}_9^0$	$\mathcal{S} \oplus \mathcal{M}_9^0$	$\mathcal{S} \odot \mathcal{M}_9^1$	$\mathcal{S} \oplus \mathcal{M}_9^1$	$\mathcal{S} \odot \mathcal{M}_9^2$	$\mathcal{S} \oplus \mathcal{M}_9^2$	$\mathcal{S} \odot \mathcal{M}_9^3$	$\mathcal{S} \oplus \mathcal{M}_9^3$
solved	14933	16574	20366	21564	22839	22413	23467	22910	23753
$\mathcal{S}\%$	+0%	+10.5%	+35.8%	+43.8%	+52.3%	+49.4%	+56.5%	+52.8%	+58.4
$\mathcal{S}+$	+0	+4364	+6215	+7774	+8414	+8407	+8964	+8822	+9274
$\mathcal{S}-$	-0	-2723	-782	-1143	-508	-927	-430	-845	-454

	$\mathcal{S} \odot \mathcal{M}_{12}^3$	$\mathcal{S} \oplus \mathcal{M}_{12}^3$	$\mathcal{S} \odot \mathcal{M}_{16}^3$	$\mathcal{S} \oplus \mathcal{M}_{16}^3$
solved	24159	24701	25100	25397
$\mathcal{S}\%$	+61.1%	+64.8%	+68.0%	+70.0%
$\mathcal{S}+$	+9761	+10063	+10476	+10647
$\mathcal{S}-$	-535	-295	-309	-183

ENIGMA Proof Example – Knaster fixed-point theorem

```
theorem Th21:
  ex a st a is_a_fixpoint_of f
proof
  set H = {h where h is Element of L: h [= f.h};
  set fH = {f.h where h is Element of L: h [= f.h};
  set uH = "\/"(H, L);
  set fuH = "\/"(fH, L);
  take uH;
  now
    let fh be Element of L;
    assume fh in fH;
    then consider h being Element of L such that
A1: fh = f.h and
A2: h [= f.h;
    h in H by A2;
    then h [= uH by LATTICE3:38;
    hence fh [= f.uH by A1,QUANTAL1:def 12;
  end;
  then fH is_less_than f.uH by LATTICE3:def 17;
  then
A3: fuH [= f.uH by LATTICE3:def 21;
  now
    let a be Element of L;
    assume a in H;
    then consider h being Element of L such that
A4: a = h & h [= f.h;
    reconsider fh = f.h as Element of L;
    take fh;
    thus a [= fh & fh in fH by A4;
  end;
  then uH [= fuH by LATTICE3:47;
  then
A5: uH [= f.uH by A3,LATTICES:7;
  then f.uH [= f.(f.uH) by QUANTAL1:def 12;
  then f.uH in H;
  then f.uH [= uH by LATTICE3:38;
  hence uH = f.uH by A5,LATTICES:8;
end;
```

Low-level Symbolic ATP guidance: Prover9 hints

- The Prover9 community: non-associative algebra, 20-100k long proofs
- Hints (Bob Veroff): extract lemmas from easier proofs to guide new proofs
- The hints behave like checkpoints – you are on the right track
- Gluing the different ideas from many other proofs together by search
- Exploration to get good hints (not really automated yet)
- Very recent huge breakthrough in the AIM project by Kinyon and Veroff

TacticToe: mid-level ITP Guidance (Gauthier'17,18)

- TTT learns from human and its own tactical HOL4 proofs
- No translation or reconstruction needed - native tactical proofs
- Fully integrated with HOL4 and easy to use
- Similar to rlCoP: policy/value learning for applying tactics in a state
- However much more technically challenging - a real breakthrough:
  - tactic and goal state recording
  - tactic argument abstraction
  - absolutization of tactic names
  - nontrivial evaluation issues
  - these issues have often more impact than adding better learners
- policy: which tactic/parameters to choose for a current goal?
- value: how likely is this proof state succeed?
- 66% of HOL4 toplevel proofs in 60s (better than a hammer!)
- similar work for Isabelle (Nagashima 2018), HOL Light (Google), Coq

RL for Normalization and Synthesis Tasks

- Gauthier'19, 20: synthesizing simple programs and conjectures in logic
- Tree Neural Nets and RL (MCTS, policy/value) for:
- Guiding normalization in Robinson arithmetic
- Guiding synthesis of combinators for a given lambda expression
- Guiding synthesis of a diophantine equation characterizing a given set
- Quite encouraging results with a good curriculum (LPAR'20, CICM'20)
- Motivated by TacticToe: argument synthesis/conjecturing is important
- The results are series of applications of correct/explainable rules
- Gauthier's deep RL framework verifies the whole series (proof) in HOL4

More on Synthesis and Conjecturing in Mathematics

- Targeted: generate intermediate lemmas (cuts) for a harder conjecture
- Unrestricted (theory exploration):
- Creation of interesting conjectures based on the previous theory
- One of the most interesting activities mathematicians do (how?)
- Higher-level AI/reasoning task - can we learn it?
- If so, we have solved math:
- ... just (recursively) divide Fermat into many subtasks ...
- ... and conquer (I mean: hammer) them away
- Goes back to Langley (Bacon), Lenat (AM), Fajtlowicz (Graffiti)
- Combined with TP by Colton et al. in early 2000s (HR)
- Statistical methods, RNNs and Transformers by our groups since 2014

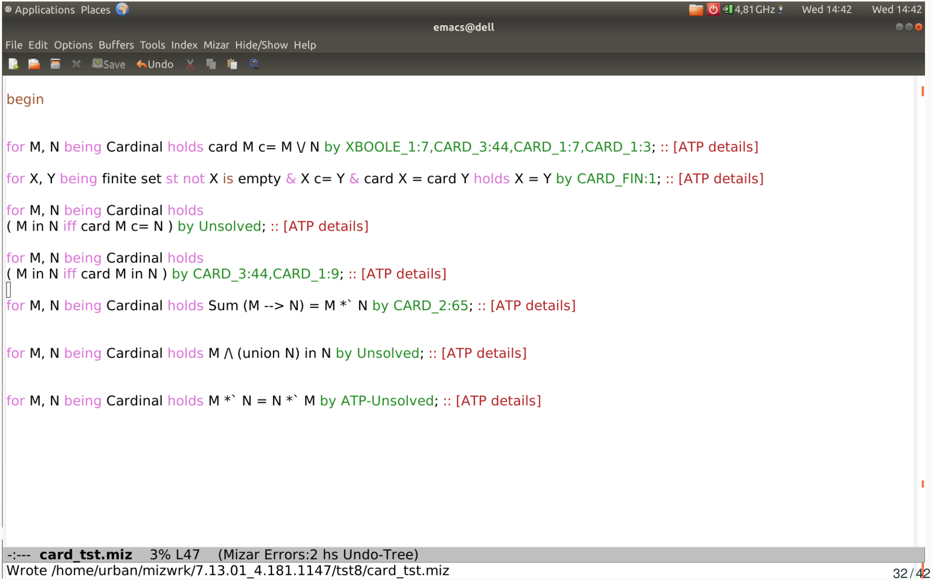
Can you find the flaw(s) in this fake GPT-2 proof?

```
:: generated theorem with "proof"
theorem Th23: :: STIRL2_1:23
for X, Y being finite set st not X is empty & X c= Y
& card X = card Y holds X = Y
proof
  let X, Y be finite set ;
:: thesis: not X is empty & X c= Y & card X = card Y implies X = Y
  assume that
  A1: not X is empty and A2: X c= Y and A3: card X = card Y ;
:: thesis: X = Y
  card (Y \ X) = (card Y) - (card X) by A1, A3, CARD_2:44;
  then A4: card (Y \ X) = ((card Y) - 1) - (card X) by CARD_1:30;
  X = Y \ X by A2, A3, Th22;
  hence X = Y by A4, XBOOLE_0:def_10;
:: thesis: verum
end;
```

-:--- **card_tst.miz** 99% L2131 (Mizar Errors:13 hs Undo-Tree)

Figure: Fake full declarative GPT-2 “proof” - typechecks!

Proving the conditioned completions - MizAR hammer



Some GPT-2 conjectures

- Kinyon and Stanovsky (algebraists) confirmed that this cut is valid:

```
theorem Th10: :: GROUPP_1:10
for G being finite Group for N being normal Subgroup of G st
N is Subgroup of center G & G ./. N is cyclic holds G is commutative

The generalization that avoids finiteness:

for G being Group for N being normal Subgroup of G st
N is Subgroup of center G & G ./. N is cyclic holds G is commutative
```

- Some are clearly false, yet quite natural to ask:

```
theorem :: SINCOS10:17
sec is increasing on [0, pi/2)

leads to conjecturing the following:

Every differentiable function is increasing.
```

Gibberish Generator Provoking Algebraists

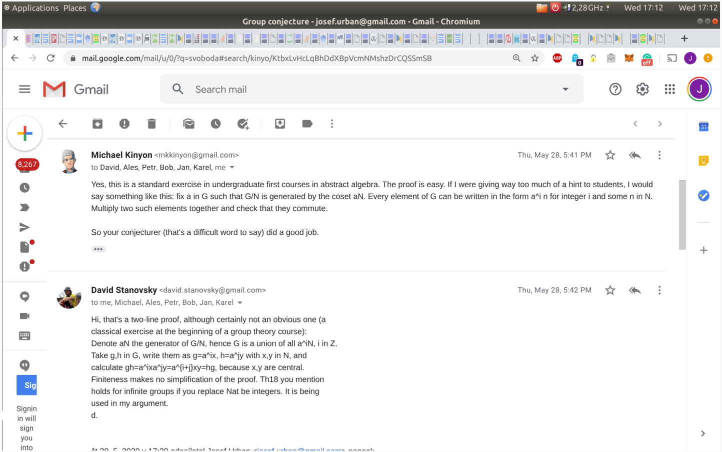


Figure: First successes in making mathematicians comment on AI.

Neural Autoformalization (Wang et al., 2018)

- generate ca 1M Latex/Mizar pairs based on Bancerek's work
- train neural seq-to-seq translation models (Luong – NMT)
- evaluate on about 100k examples, achieves 48% on unseen examples
- many architectures tested, some work much better than others
- very important latest invention: *attention* in the seq-to-seq models
- more data very important for neural training – our biggest bottleneck
- Recent addition: unsupervised methods (Lample et all 2018) – no need for aligned data!

Neural Fun – Performance after Some Training

Rendered LaTeX	Suppose s_8 is convergent and s_7 is convergent . Then $\mathop{\rm lim}(s_8+s_7) = \mathop{\rm lim} s_8 + \mathop{\rm lim} s_7$
Input LaTeX	`Suppose $ { s _ { 8 } } $ is convergent and $ { s _ { 7 } } $ is convergent . Then $ \mathop { \rm lim } ( { s _ { 8 } } { + } { s _ { 7 } } ) \mathrel { = } \mathop { \rm lim } { s _ { 8 } } { + } \mathop { \rm lim } { s _ { 7 } } $ .`
Correct	`seq1 is convergent & seq2 is convergent implies lim ( seq1 + seq2 ) = ( lim seq1 ) + ( lim seq2 ) ;`
Snapshot-1000	`x in dom f implies ( x * y ) * ( f | ( x | ( y | ( y | y ) ) ) ) = ( x | ( y | ( y | ( y | y ) ) ) ) ) ;`
Snapshot-2000	`seq is summable implies seq is summable ;`
Snapshot-3000	`seq is convergent & lim seq = 0c implies seq = seq ;`
Snapshot-4000	`seq is convergent & lim seq = lim seq implies seq1 + seq2 is convergent ;`
Snapshot-5000	`seq1 is convergent & lim seq2 = lim seq2 implies lim_inf seq1 = lim_inf seq2 ;`
Snapshot-6000	`seq is convergent & lim seq = lim seq implies seq1 + seq2 is convergent ;`
Snapshot-7000	`seq is convergent & seq9 is convergent implies lim ( seq + seq9 ) = ( lim seq ) + ( lim seq9 ) ;`

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- ... and many more ...
- Funding: Marie-Curie, NWO, ERC

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Thanks and Advertisement

- Thanks for your attention!
- AITP – Artificial Intelligence and Theorem Proving
- September 5–10, 2021, Aussois, France, `aitp-conference.org`
- ATP/ITP/Math vs AI/Machine-Learning people, Computational linguists
- Discussion-oriented and experimental - submit a talk abstract!
- Grown to 80 people in 2019
- Will be hybrid in 2021 as in 2020