Conjecturing over large corpora

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July 14, 2017

Goal

Automatically discover conjectures in formalized libraries.

Which formalized libraries ?

	theorems	constants	types	theories
Mizar	51086	6462	2710	1230
Coq	23320	3981	860	390
• HOL4	16476	2188	59	126
HOL Light	16191	790	30	68
Isabelle/HOL	14814	1046	30	77
Matita	1712	339	290	101

Why formalized libraries ?

- Easier to learn from.
- Sufficiently large number of theorems.

What for ?

- Improve proof automation, by discovering important intermediate lemmas.

Challenges

How do we conjecture interesting lemmas ?

- Generation: large numbers of possible conjectures.
- Learning: large amount of data.
- Pruning: how to remove false conjectures fast, and select interesting ones.

How to integrate these mechanism in a goal-oriented automatic proof?

Our approach

How do we conjecture interesting lemmas ?

- Generation: **analogies**, probabilistic grammar.
- Learning: **pattern-matching**, genetic algorithm.
- Pruning: **proof**, model-based guidance, neural networks.

How to integrate these mechanism in a goal-oriented automatic proof?

- Copy human reasoning.
- Make high-level inference steps: premise selection + ATPs.

Finding analogies inside libraries

Theorems (first-order, higher-order or type theory):

$\forall x : num.\ x + 0 = x \qquad \forall x : real.\ x = \&(Numeral(BIT1\ 0)) \times x$

Normalization + Conceptualization + Abstraction $\rightarrow$
Properties:

$\lambda num, +, 0.\ \forall x : num\ x = x + 0 \qquad \lambda real, \times, 1.\ \forall x : real.\ x = x \times 1$

Derived constant pairs:

$$num \leftrightarrow real,\ + \leftrightarrow \times,\ 0 \leftrightarrow 1$$

Some similar theorems across libraries

rev_append in Coq
`∀ l, rev l = rev_append l [].`
`∀ l l', rev_append l l' = rev l ++ l'.`

REV in HOL4
`∀ L. REVERSE L = REV L []`
`∀ L1 L2.  REV L1 L2 = REVERSE L1 ++ L2`

Scoring analogies

- Number of common properties.
- TF-IDF to advantage rarer properties.
- Dynamical process (similarity of 0 1 $\rightarrow$ similarity of $+$ *).
- Not greedy. Concepts can have multiple analogues.

Some analogies across libraries with good scores

Prover 1	Prover 2	Constant 1	Constant 2
HOL4	HOL Light	$(prod\ real)\ real$	$complex$
HOL4	Isabelle/HOL	$\frac{\pi}{2}$	$\frac{\pi}{2}$
HOL Light	Isabelle/HOL	$real\_pow$	$power\ real$
Coq	Matita	decidable	decidable
Coq	HOL4	length	LENGTH
Isabelle/HOL	Mizar	$arccos$	$arcos$
Coq	Mizar	$Rlist$	$FinSequence\ REAL$

Other analogies across libraries with good scores

Prover 1	Prover 2	Constant 1	Constant 2
HOL4	HOL Light	*extreal*	*complex*
HOL4	Isabelle/HOL	*modu*	*real_norm complex*
HOL Light	Isabelle/HOL	*FCONS*	*case_nat*
Coq	Matita	*transitive*	*symmetric*
Coq	HOL4	*rev_append*	*REV*
Isabelle/HOL	Mizar	*sqrt*	$\_^2$
Coq	Mizar	*RIneq_Rsqr*	*min*

Best analogies inside one library

Mizar			HOL4		
54494 analogies		Score	5842 analogies		Score
v2_normsp_1	*v8_clvect_1*	0.99	*BIT2*	*BIT1*	0.97
v5_rlvect_1	*v3_normsp_0*	0.99	*real*	*int*	0.96
v6_rlvect_1	*v4_normsp_0*	0.99	*int_of_num*	*real_of_num*	0.95
l1_normsp_1	*l2_clvect_1*	0.99	*real*	*extreal*	0.94
v3_clvect_1	*v6_rlvect_1*	0.99	*semi_ring*	*ring*	0.94
v5_rlvect_1	*v2_clvect_1*	0.99	$\leq$	$<$	0.93

Creating conjectures from analogies

Normalized theorems	Properties	Analogies
$x * (y - z) = x * y - x * z$	$Dist(*, -, i)$	$\{- \leftrightarrow +\}$
$x * (y + z) = x * y + x * z$	$Dist(*, +, i)$	$\{* \leftrightarrow \cup, + \leftrightarrow \cap, i \leftrightarrow s\}$
$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$	$Dist(\cup, \cap, s)$	$\{* \leftrightarrow \cup, - \leftrightarrow \cap, i \leftrightarrow s\}$
$x + 0 = x$	$Neut(+, 0, i)$	$\{- \leftrightarrow +\}$
$x - 0 = x$	$Neut(-, 0, i)$	
$exp(a + b) = exp(a) * exp(b)$	$P(exp, +, *, i, r)$	

Creating conjectures from analogies

Original goal:

- $exp(a+b) = exp(a) * exp(b)$

Substitutions from analogies:

- $+ \rightarrow -$
- $+ \rightarrow \cap, \ * \rightarrow \cup$

Failed conjectures:

- $exp(a-b) = exp(a) * exp(b)$
- $exp(a \cap b) = exp(a) \cup exp(b)$

Expected conjectures (if we had learnt better substitutions):

- $exp(a-b) = exp(a)/exp(b)$
- $complement(a \cap b) = complement(a) \cup complement(b)$

Untargeted conjecture generation

Procedure:

- Generation of "best" 73535 conjectures from the Mizar library.
- Premise selection + Vampire prove 10% in 10 s.
- 4464 are not tautologies or consequences of single lemmas.

Examples:

- convex - circled

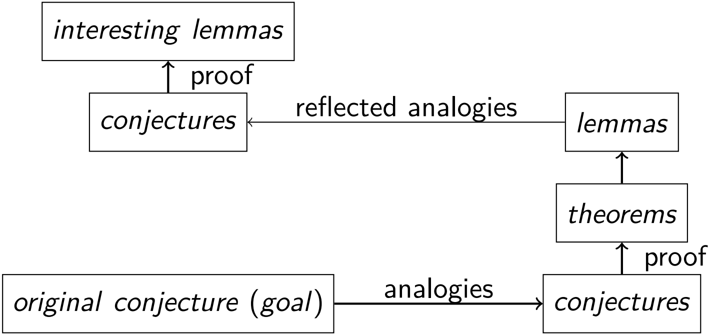
Problem:

- Unlikely to find something useful for a specific goal.
- How to adapt this method in a goal-oriented setting?

Targeted conjecture generation: evaluation settings

	First experiment	Second experiments
Library	Mizar	HOL4
Evaluated theorems	hardest (22069)	all
Accessible library	past theorems	past theorems
Concepts	ground subterms	only constants
Pair creation	pre-computed	fair
Type checking	no	yes
Analogies per theorem	20	20
Premise selection	k-NN 128	-kNN 128
ATP	Vampire 8s	E-prover 8s
Basic strategy	no conjectures	no conjectures
Premise selection	k-NN 128	k-NN 128
ATP	Vampire 3600s	E-prover 16s

First experiment: proof strategy



First experiment: results

	Number	Non-trivial and proven
Hard goals	22069	
Analogous conjectures	441242	3414
Back-translated conjectures	26770	2170
Affected hard goals	500	7
New proven hard goals		1

- Non-trivial theorem: consequences of at least two theorems.
- Affected goal: From the goal, the procedure proves at least one back-translated conjecture.
- Time: 14 hours on a 64-CPU server (proofs)

First experiment: example

```
theorem :: MATHMORP:25
for T being non empty right_complementable Abelian
            add-associative right_zeroed RLSStruct
for X, Y, Z being Subset of T
  holds X (+) (Y (-) Z) c= (X (+) Y) (-) Z
```

Proven using:

- Analogy between + and - in additive structures.
- A conjectured lemma which happens to be MATHMORP:26.

First experiment: limits

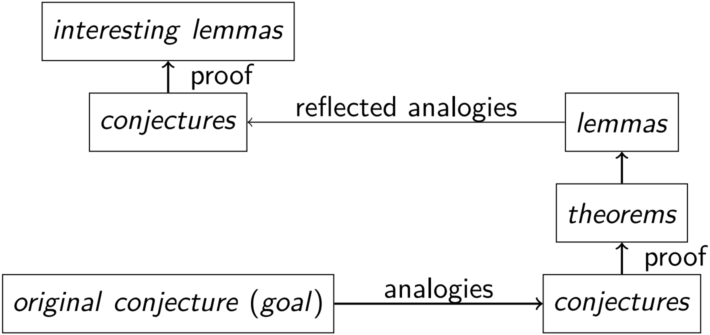
Issues:

- Huge number of proofs.
- Few affected theorems (500).
- Few conjectured lemmas (in average 4 per affected theorems).
- Do not help in proving the goal.

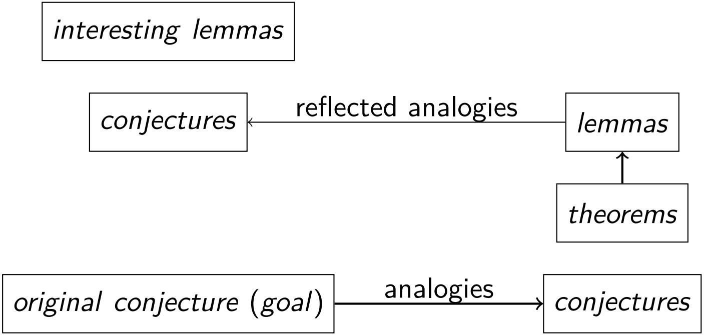
Reasons:

- Design of the strategy.
- Problem set is hard.
- Proof selection is too restrictive.
- Analogies may be too strict.
- No type checking (set theory).
- No understanding of the type hierarchy.

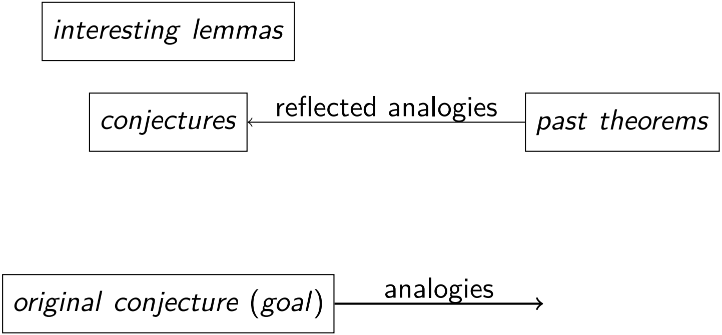
Second experiment: proof strategy



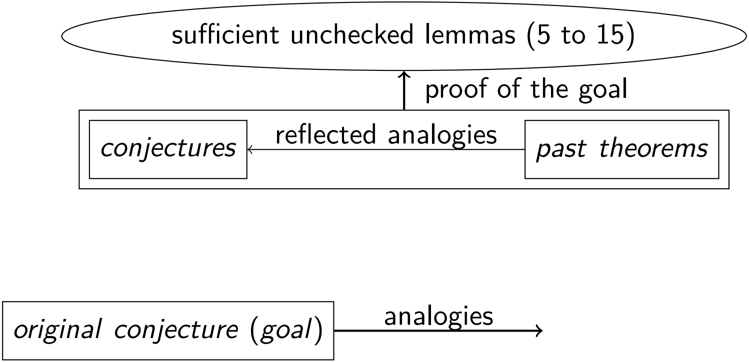
Second experiment: proof strategy



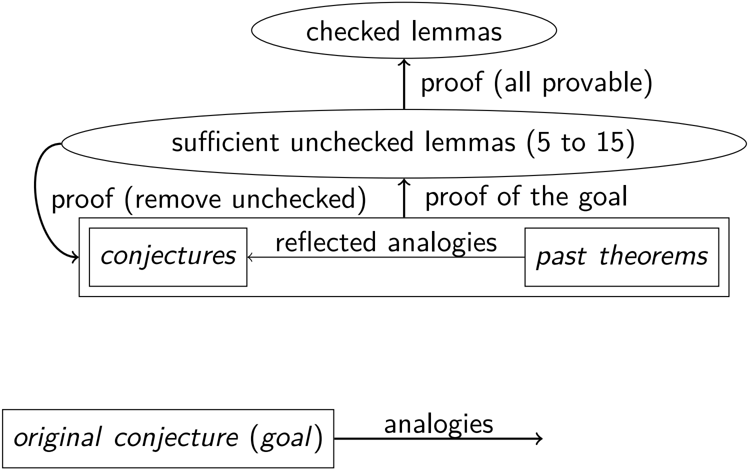
Second experiment: proof strategy



Second experiment: proof strategy



Second experiment: proof strategy



Second experiment: results

Goals	10163
Proven conjectures	8246
Proven goals	2700
Proven goals using one conjecture	724
New proven goals	7

Time: 10 hours on a 40-CPU server
Processes: analogies + premise selection + translation + proof

Second experiment: examples

Theorem	From analogues of
extreal.sub_rdistrib	extreal.sub_ldistrib
pred_set.inter_countable	pred_set.FINITE_DIFF
real.pow_rat_2	real.POW_2_LT
numpair.tri_le	arithmetic.LESS_EQ_SUC_REFL
ratRing.tLRLRRRRRR	integerRing.tLRLRRRRRR
words.word_L2_MULT_e3	words.WORD_NEG_L
real.REAL_EQ_LMUL	intExtension.INT_NO_ZERODIV
	integer.INT_EQ_LMUL2

Conclusion

We designed two conjecture-based proving methods.

- Support many ITP libraries.
- Generate conjectures using analogies.
- Learn analogies by pattern-matching and dynamical scoring.
- Integrated in a proof strategy:
  Combine analogies and standard hammering techniques
  (premise selections and translations to ATPs).

We evaluated them.

- 10% of conjectures from best analogies are provable.
- $+1$ hard Mizar problem.
- $+7$ hard HOL4 problem.

Coming sooner or later

- Conjecture generation:
  - more complex concepts.
  - probabilistic grammar.
  - generalization/specification, weakening/strengthening.
- Learning:
  - faster pattern-matching.
  - genetic algorithm + model evaluation.
  - from proofs.
- Pruning or/and guidance:
  - better scoring mechanism for substitutions,
  - model-based guidance.
  - Truth intuition using machine learning (?).
- Improving proof strategies:
  - Recursion
  - Tree search (Monte-Carlo)