# SHEAVES OF MODULES

## Contents



## 1. Introduction

In this chapter we work out basic notions of sheaves of modules. This in particular includes the case of abelian sheaves, since these may be viewed as sheaves of $\underline{\mathbf{Z}}$-modules. Basic references are [Ser55], [DG67] and [AGV71].

We work out what happens for sheaves of modules on ringed topoi in another chapter (see Modules on Sites, Section 1), although there we will mostly just duplicate the discussion from this chapter.


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## 2. Pathology

A ringed space is a pair consisting of a topological space $X$ and a sheaf of rings $\mathcal{O}$. We allow $\mathcal{O} = 0$ in the definition. In this case the category of modules has a single object (namely 0). It is still an abelian category etc, but it is a little degenerate. Similarly the sheaf $\mathcal{O}$ may be zero over open subsets of $X$, etc.

This doesn't happen when considering locally ringed spaces (as we will do later).

## 3. The abelian category of sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Definition 25.1. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules, see Sheaves, Definition 10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the map which on each open $U \subset X$ is the sum of the maps induced by $\varphi$, $\psi$. This is clearly again a map of sheaves of $\mathcal{O}_X$-modules. It is also clear that composition of maps of $\mathcal{O}_X$-modules is bilinear with respect to this addition. Thus $\mathit{Mod}(\mathcal{O}_X)$ is a pre-additive category, see Homology, Definition 3.1.

We will denote 0 the sheaf of $\mathcal{O}_X$-modules which has constant value $\{0\}$ for all open $U \subset X$. Clearly this is both a final and an initial object of $\mathit{Mod}(\mathcal{O}_X)$. Given a morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) $\varphi$ is zero, (b) $\varphi$ factors through 0, (c) $\varphi$ is zero on sections over each open $U$, and (d) $\varphi_x = 0$ for all $x \in X$. See Sheaves, Lemma 16.1.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps $(i, j, p, q)$ as in Homology, Definition 3.5. Thus $\mathit{Mod}(\mathcal{O}_X)$ is an additive category, see Homology, Definition 3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. We may define $\mathrm{Ker}(\varphi)$ to be the subsheaf of $\mathcal{F}$ with sections

$$\mathrm{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all open $U \subset X$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}_X$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\mathrm{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$. Moreover, on the level of stalks we have $\mathrm{Ker}(\varphi)_x = \mathrm{Ker}(\varphi_x)$.

On the other hand, we define $\mathrm{Coker}(\varphi)$ as the sheaf of $\mathcal{O}_X$-modules associated to the presheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \mathrm{Coker}(\mathcal{G}(U) \to \mathcal{F}(U)) = \mathcal{F}(U)/\varphi(\mathcal{G}(U)).$$

Since taking stalks commutes with taking sheafification, see Sheaves, Lemma 17.2 we see that $\mathrm{Coker}(\varphi)_x = \mathrm{Coker}(\varphi_x)$. Thus the map $\mathcal{G} \to \mathrm{Coker}(\varphi)$ is surjective (as a map of sheaves of sets), see Sheaves, Section 16. To show that this is a cokernel, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}_X$-modules such that $\beta \circ \varphi$ is zero, then you get for every open $U \subset X$ a map induced by $\beta$ from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 20.1) we obtain a canonical map $\mathrm{Coker}(\varphi) \to \mathcal{H}$ such that the original $\beta$ is equal to the composition $\mathcal{G} \to \mathrm{Coker}(\varphi) \to \mathcal{H}$. The morphism $\mathrm{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

**Lemma 3.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is an abelian category. Moreover a complex*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

*is exact at $\mathcal{G}$ if and only if for all $x \in X$ the complex*

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

*is exact at $\mathcal{G}_x$.*

**Proof.** By Homology, Definition 5.1 we have to show that image and coimage agree. By Sheaves, Lemma 16.1 it is enough to show that image and coimage have the same stalk at every $x \in X$. By the constructions of kernels and cokernels above these stalks are the coimage and image in the categories of $\mathcal{O}_{X,x}$-modules. Thus we get the result from the fact that the category of modules over a ring is abelian. $\square$

Actually the category $Mod(\mathcal{O}_X)$ has many more properties. Here are two constructions we can do.

(1) Given any set $I$ and for each $i \in I$ a $\mathcal{O}_X$-module we can form the product
$$\prod\nolimits_{i \in I} \mathcal{F}_i$$
which is the sheaf that associates to each open $U$ the product of the modules $\mathcal{F}_i(U)$. This is also the categorical product, as in Categories, Definition 14.5.

(2) Given any set $I$ and for each $i \in I$ a $\mathcal{O}_X$-module we can form the direct sum
$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i$$
which is the *sheafification* of the presheaf that associates to each open $U$ the direct sum of the modules $\mathcal{F}_i(U)$. This is also the categorical coproduct, as in Categories, Definition 14.6. To see this you use the universal property of sheafification.

Using these we conclude that all limits and colimits exist in $Mod(\mathcal{O}_X)$.

**Lemma 3.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *All limits exist in $Mod(\mathcal{O}_X)$. Limits are the same as the corresponding limits of presheaves of $\mathcal{O}_X$-modules (i.e., commute with taking sections over opens).*
(2) *All colimits exist in $Mod(\mathcal{O}_X)$. Colimits are the sheafification of the corresponding colimit in the category of presheaves. Taking colimits commutes with taking stalks.*
(3) *Filtered colimits are exact.*
(4) *Finite direct sums are the same as the corresponding finite direct sums of presheaves of $\mathcal{O}_X$-modules.*

**Proof.** As $Mod(\mathcal{O}_X)$ is abelian (Lemma 3.1) it has all finite limits and colimits (Homology, Lemma 5.5). Thus the existence of limits and colimits and their description follows from the existence of products and coproducts and their description (see discussion above) and Categories, Lemmas 14.10 and 14.11. Since sheafification commutes with taking stalks we see that colimits commute with taking stalks. Part (3) signifies that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of $\mathcal{O}_X$-modules over a directed partially ordered set $I$ the sequence $0 \to \operatorname{colim} \mathcal{F}_i \to \operatorname{colim} \mathcal{G}_i \to \operatorname{colim} \mathcal{H}_i \to 0$ is exact as well. Since we can check exactness on stalks (Lemma 3.1) this follows from the case of modules which is Algebra, Lemma 8.9. We omit the proof of (4). $\square$

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 23. See Homology, Lemma 7.1 for a description of exactness properties in terms of short exact sequences.

**Lemma 3.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.*

1. *The functor $f_* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ is left exact. In fact it commutes with all limits.*
2. *The functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is right exact. In fact it commutes with all colimits.*
3. *Pullback $f^{-1} : Ab(Y) \to Ab(X)$ on abelian sheaves is exact.*

**Proof.** Parts (1) and (2) hold because $(f^*, f_*)$ is an adjoint pair of functors, see Sheaves, Lemma 26.2 and Categories, Section 24. Part (3) holds because exactness can be checked on stalks (Lemma 3.1) and the description of stalks of the pullback, see Sheaves, Lemma 22.1. $\square$

**Lemma 3.4.** *Let $j : U \to X$ be an open immersion of topological spaces. The functor $j_! : Ab(U) \to Ab(X)$ is exact.*

**Proof.** Follows from the description of stalks given in Sheaves, Lemma 31.6. $\square$

**Lemma 3.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $I$ be a set. For $i \in I$, let $\mathcal{F}_i$ be a sheaf of $\mathcal{O}_X$-modules. For $U \subset X$ quasi-compact open the map*

$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i(U) \longrightarrow \left(\bigoplus\nolimits_{i \in I} \mathcal{F}_i\right)(U)$$

*is bijective.*

**Proof.** If $s$ is an element of the right hand side, then there exists an open covering $U = \bigcup_{j \in J} U_j$ such that $s|_{U_j}$ is a finite sum $\sum_{i \in I_j} s_{ji}$ with $s_{ji} \in \mathcal{F}_i(U_j)$. Because $U$ is quasi-compact we may assume that the covering is finite, i.e., that $J$ is finite. Then $I' = \bigcup_{j \in J} I_j$ is a finite subset of $I$. Clearly, $s$ is a section of the subsheaf $\bigoplus_{i \in I'} \mathcal{F}_i$. The result follows from the fact that for a finite direct sum sheafification is not needed, see Lemma 3.2 above. $\square$

## 4. Sections of sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $s \in \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ be a global section. There is a unique *map of $\mathcal{O}_X$-modules*

$$\mathcal{O}_X \longrightarrow \mathcal{F}, \ f \longmapsto fs$$

*associated to $s$*. The notation above signifies that a local section $f$ of $\mathcal{O}_X$, i.e., a section $f$ over some open $U$, is mapped to the multiplication of $f$ with the restriction of $s$ to $U$. Conversely, any map $\varphi : \mathcal{O}_X \to \mathcal{F}$ gives rise to a section $s = \varphi(1)$ such that $\varphi$ is the morphism associated to $s$.

**Definition 4.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *generated by global sections* if there exist a set $I$, and global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that the map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}$$

which is the map associated to $s_i$ on the summand corresponding to $i$, is surjective. In this case we say that the sections $s_i$ *generate* $\mathcal{F}$.

We often use the abuse of notation introduced in Sheaves, Section 11 where, given a local section $s$ of $\mathcal{F}$ defined in an open neighbourhood of a point $x \in X$, we denote $s_x$, or even $s$ the image of $s$ in the stalk $\mathcal{F}_x$.

**Lemma 4.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $I$ be a set. Let $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ be global sections. The sections $s_i$ generate $\mathcal{F}$ if and only if for all $x \in X$ the elements $s_{i,x} \in \mathcal{F}_x$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.*

**Proof.** Omitted. ☐

**Lemma 4.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{G}$ are generated by global sections then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*

**Proof.** Omitted. ☐

**Lemma 4.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $I$ be a set. Let $s_i$, $i \in I$ be a collection of local sections of $\mathcal{F}$, i.e., $s_i \in \mathcal{F}(U_i)$ for some opens $U_i \subset X$. There exists a unique smallest subsheaf of $\mathcal{O}_X$-modules $\mathcal{G}$ such that each $s_i$ corresponds to a local section of $\mathcal{G}$.*

**Proof.** Consider the subpresheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \{\text{sums } \sum\nolimits_{i \in J} f_i(s_i|_U) \text{ where } J \text{ is finite, } U \subset U_i \text{ for } i \in J, \text{ and } f_i \in \mathcal{O}_X(U)\}$$

Let $\mathcal{G}$ be the sheafification of this subpresheaf. This is a subsheaf of $\mathcal{F}$ by Sheaves, Lemma 16.3. Since all the finite sums clearly have to be in $\mathcal{G}$ this is the smallest subsheaf as desired. ☐

**Definition 4.5.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set $I$, and local sections $s_i$, $i \in I$ of $\mathcal{F}$ we say that the subsheaf $\mathcal{G}$ of Lemma 4.4 above is the *subsheaf generated by the $s_i$*.

**Lemma 4.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set $I$, and local sections $s_i$, $i \in I$ of $\mathcal{F}$. Let $\mathcal{G}$ be the subsheaf generated by the $s_i$ and let $x \in X$. Then $\mathcal{G}_x$ is the $\mathcal{O}_{X,x}$-submodule of $\mathcal{F}_x$ generated by the elements $s_{i,x}$ for those $i$ such that $s_i$ is defined at $x$.*

**Proof.** This is clear from the construction of $\mathcal{G}$ in the proof of Lemma 4.4. ☐

## 5. Supports of modules and sections

**Definition 5.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

1. The *support of $\mathcal{F}$* is the set of points $x \in X$ such that $\mathcal{F}_x \neq 0$.
2. We denote $\text{Supp}(\mathcal{F})$ the support of $\mathcal{F}$.
3. Let $s \in \Gamma(X, \mathcal{F})$ be a global section. The *support of $s$* is the set of points $x \in X$ such that the image $s_x \in \mathcal{F}_x$ of $s$ is not zero.

Of course the support of a local section is then defined also since a local section is a global section of the restriction of $\mathcal{F}$.

**Lemma 5.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ open.*

1. *The support of $s \in \mathcal{F}(U)$ is closed in $U$.*

(2) *The support of $fs$ is contained in the intersections of the supports of $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$*
(3) *The support of $s + s'$ is contained in the union of the supports of $s, s' \in \mathcal{F}(U)$.*
(4) *The support of $\mathcal{F}$ is the union of the supports of all local sections of $\mathcal{F}$.*
(5) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then the support of $\varphi(s)$ is contained in the support of $s \in \mathcal{F}(U)$.*

**Proof.** This is true because if $s_x = 0$, then $s$ is zero in an open neighbourhood of $x$ by definition of stalks. Similarly for $f$. Details omitted. $\square$

In general the support of a sheaf of modules is not closed. Namely, the sheaf could be an abelian sheaf on $\mathbf{R}$ (with the usual archimedean topology) which is the direct sum of infinitely many nonzero skyscraper sheaves each supported at a single point $p_i$ of $\mathbf{R}$. Then the support would be the set of points $p_i$ which may not be closed.

Another example is to consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 31 the support of this sheaf is exactly $U$.

**Lemma 5.3.** *Let $X$ be a topological space. The support of a sheaf of rings is closed.*

**Proof.** This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. $\square$

## 6. Closed immersions and abelian sheaves

Recall that we think of an abelian sheaf on a topological space $X$ as a sheaf of $\underline{\mathbf{Z}}_X$-modules. Thus we may apply any results, definitions for sheaves of modules to abelian sheaves.

**Lemma 6.1.** *Let $X$ be a topological space. Let $Z \subset X$ be a closed subset. Denote $i : Z \to X$ the inclusion map. The functor*

$$i_* : Ab(Z) \longrightarrow Ab(X)$$

*is exact, fully faithful, with essential image exactly those abelian sheaves whose support is contained in $Z$. The functor $i^{-1}$ is a left inverse to $i_*$.*

**Proof.** Exactness follows from the description of stalks in Sheaves, Lemma 32.1 and Lemma 3.1. The rest was shown in Sheaves, Lemma 32.3. $\square$

Let $\mathcal{F}$ be a sheaf on $X$. There is a canonical subsheaf of $\mathcal{F}$ which consists of exactly those sections whose support is contained in $Z$. Here is the exact statement.

**Lemma 6.2.** *Let $X$ be a topological space. Let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be a sheaf on $X$. For $U \subset X$ open set*

$$\Gamma(U, \mathcal{H}_Z(\mathcal{F})) = \{s \in \mathcal{F}(U) \mid \text{ the support of } s \text{ is contained in } Z \cap U\}$$

*Then $\mathcal{H}_Z(\mathcal{F})$ is an abelian subsheaf of $\mathcal{F}$. It is the largest abelian subsheaf of $\mathcal{F}$ whose support is contained in $Z$. The construction $\mathcal{F} \mapsto \mathcal{H}_Z(\mathcal{F})$ is functorial in the abelian sheaf $\mathcal{F}$.*

**Proof.** This follows from Lemma 5.2. $\square$

This seems like a good opportunity to show that the functor $i_*$ has a right adjoint on abelian sheaves.

**Lemma 6.3.** *Let $i : Z \to X$ be the inclusion of a closed subset into the topological space $X$. Denote*[1] *$i^! : Ab(X) \to Ab(Z)$ the functor $\mathcal{F} \mapsto i^{-1}\mathcal{H}_Z(\mathcal{F})$. Then $i^!$ is a right adjoint to $i_*$, in a formula*

$$\mathrm{Mor}_{Ab(X)}(i_*\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Ab(Z)}(\mathcal{G}, i^!\mathcal{F}).$$

*In particular $i_*$ commutes with arbitrary colimits.*

**Proof.** Note that $i_*i^!\mathcal{F} = \mathcal{H}_Z(\mathcal{F})$. Since $i_*$ is fully faithful we are reduced to showing that

$$\mathrm{Mor}_{Ab(X)}(i_*\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Ab(X)}(i_*\mathcal{G}, \mathcal{H}_Z(\mathcal{F})).$$

This follows since the support of the image via any homomorphism of a section of $i_*\mathcal{G}$ is supported on $Z$, see Lemma 5.2. ☐

**Remark 6.4.** In Sheaves, Remark 32.5 we showed that $i_*$ as a functor on the categories of sheaves of sets does not have a right adjoint simply because it is not exact. However, it is very close to being true, in fact, the functor $i_*$ is exact on sheaves of pointed sets, sections with support in $Z$ can be defined for sheaves of pointed sets, and $i^!$ makes sense and is a right adjoint to $i_*$.

## 7. A canonical exact sequence

We give this exact sequence its own section.

**Lemma 7.1.** *Let $X$ be a topological space. Let $U \subset X$ be an open subset with complement $Z \subset X$. Denote $j : U \to X$ the open immersion and $i : Z \to X$ the closed immersion. For any sheaf of abelian groups $\mathcal{F}$ on $X$ the adjunction mappings $j_!j^*\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to i_*i^*\mathcal{F}$ give a short exact sequence*

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

*of sheaves of abelian groups. For any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of abelian sheaves on $X$ we obtain a morphism of short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & j_!j^*\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & i_*i^*\mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & j_!j^*\mathcal{G} \arrow[r] & \mathcal{G} \arrow[r] & i_*i^*\mathcal{G} \arrow[r] & 0 \end{tikzcd}$$

**Proof.** We may check exactness on stalks (Lemma 3.1). For a description of the stalks in question see Sheaves, Lemmas 31.6 and 32.1. We omit the proof of the functorial behaviour of the exact sequence. ☐

[1] This is likely nonstandard notation.

## 8. Modules locally generated by sections

Let $(X, \mathcal{O}_X)$ be a ringed space. In this and the following section we will often restrict sheaves to open subspaces $U \subset X$, see Sheaves, Section 31. In particular, we will often denote the open subspace by $(U, \mathcal{O}_U)$ instead of the more correct notation $(U, \mathcal{O}_X|_U)$, see Sheaves, Definition 31.2.

Consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 31 the stalk of $j_!\underline{\mathbf{Z}}_U$ at $x = 0$ is 0. In fact the sections of this sheaf over any open interval containing 0 are 0. Thus there is no open neighbourhood of the point 0 over which the sheaf can be generated by sections.

**Definition 8.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *locally generated by sections* if for every $x \in X$ there exists an open neighbourhood $U$ such that $\mathcal{F}|_U$ is globally generated as a sheaf of $\mathcal{O}_U$-modules.

In other words there exists a set $I$ and for each $i$ a section $s_i \in \mathcal{F}(U)$ such that the associated map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U$$

is surjective.

**Lemma 8.2.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ is locally generated by sections if $\mathcal{G}$ is locally generated by sections.*

**Proof.** Given an open subspace $V$ of $Y$ we may consider the commutative diagram of ringed spaces

$$\begin{tikzcd}
(f^{-1}V, \mathcal{O}_{f^{-1}V}) \arrow[r, "j'"'] \arrow[d, "f'"'] & (X, \mathcal{O}_X) \arrow[d, "f"] \\
(V, \mathcal{O}_V) \arrow[r, "j"] & (Y, \mathcal{O}_Y)
\end{tikzcd}$$

We know that $f^*\mathcal{G}|_{f^{-1}V} \cong (f')^*(\mathcal{G}|_V)$, see Sheaves, Lemma 26.3. Thus we may assume that $\mathcal{G}$ is globally generated.

We have seen that $f^*$ commutes with all colimits, and is right exact, see Lemma 3.3. Thus if we have a surjection

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying $f^*$ we obtain the surjection

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. $\square$

## 9. Modules of finite type

**Definition 9.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of *finite type* if for every $x \in X$ there exists an open neighbourhood $U$ such that $\mathcal{F}|_U$ is generated by finitely many sections.

**Lemma 9.2.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a finite type $\mathcal{O}_Y$-module is a finite type $\mathcal{O}_X$-module.*

**Proof.** Arguing as in the proof of Lemma 8.2 we may assume $\mathcal{G}$ is globally generated by finitely many sections. We have seen that $f^*$ commutes with all colimits, and is right exact, see Lemma 3.3. Thus if we have a surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying $f^*$ we obtain the surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. □

**Lemma 9.3.** *Let $X$ be a ringed space. The image of a morphism of $\mathcal{O}_X$-modules of finite type is of finite type. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}_1$ and $\mathcal{F}_3$ are of finite type, so is $\mathcal{F}_2$.*

**Proof.** The statement on images is trivial. The statement on short exact sequences comes from the fact that sections of $\mathcal{F}_3$ locally lift to sections of $\mathcal{F}_2$ and the corresponding result in the category of modules over a ring (applied to the stalks for example). □

**Lemma 9.4.** *Let $X$ be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{F}$ of finite type and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ surjective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is surjective.*

**Proof.** Choose an open neighbourhood $U \subset X$ such that $\mathcal{F}$ is generated by $s_1, \ldots, s_n \in \mathcal{F}(U)$ over $U$. By assumption of surjectivity of $\varphi_x$, after shrinking $U$ we may assume that $s_i = \varphi(t_i)$ for some $t_i \in \mathcal{G}(U)$. Then $U$ works. □

**Lemma 9.5.** *Let $X$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $x \in X$. Assume $\mathcal{F}$ of finite type and $\mathcal{F}_x = 0$. Then there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is zero.*

**Proof.** This is a special case of Lemma 9.4 applied to the morphism $0 \to \mathcal{F}$. □

**Lemma 9.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type then support of $\mathcal{F}$ is closed.*

**Proof.** This is a reformulation of Lemma 9.5. □

**Lemma 9.7.** *Let $X$ be a ringed space. Let $I$ be a partially ordered set and let $(\mathcal{F}_i, f_{ii'})$ be a system over $I$ consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 21). Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ be the colimit. Assume (a) $I$ is directed, (b) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and (c) $X$ is quasi-compact. Then there exists an $i$ such that $\mathcal{F}_i \to \mathcal{F}$ is surjective. If the transition maps $f_{ii'}$ are injective then we conclude that $\mathcal{F} = \mathcal{F}_i$ for some $i \in I$.*

**Proof.** Let $x \in X$. There exists an open neighbourhood $U \subset X$ of $x$ and finitely many sections $s_j \in \mathcal{F}(U)$, $j = 1, \ldots, m$ such that $s_1, \ldots, s_m$ generate $\mathcal{F}$ as $\mathcal{O}_U$-module. After possibly shrinking $U$ to a smaller open neighbourhood of $x$ we may assume that each $s_j$ comes from a section of $\mathcal{F}_i$ for some $i \in I$. Hence, since $X$ is quasi-compact we can find a finite open covering $X = \bigcup_{j=1,\ldots,m} U_j$, and for each $j$ an index $i_j$ and finitely many sections $s_{jl} \in \mathcal{F}_{i_j}(U_j)$ whose images generate the restriction of $\mathcal{F}$ to $U_j$. Clearly, the lemma holds for any index $i \in I$ which is $\geq$ all $i_j$. □

**Lemma 9.8.** *Let $X$ be a ringed space. There exists a set of $\mathcal{O}_X$-modules $\{\mathcal{F}_i\}_{i \in I}$ of finite type such that each finite type $\mathcal{O}_X$-module on $X$ is isomorphic to exactly one of the $\mathcal{F}_i$.*

**Proof.** For each open covering $\mathcal{U} : X = \bigcup U_j$ consider the sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ such that each restriction $\mathcal{F}|_{U_j}$ is a quotient of $\mathcal{O}_{U_j}^{\oplus r}$ for some $r_j \geq 0$. These are parametrized by subsheaves $\mathcal{K}_i \subset \mathcal{O}_{U_j}^{\oplus r_j}$ and glueing data

$$\varphi_{jj'} : \mathcal{O}_{U_j \cap U_{j'}}^{\oplus r_j}/(\mathcal{K}_j|_{U_j \cap U_{j'}}) \longrightarrow \mathcal{O}_{U_j \cap U_{j'}}^{\oplus r_{j'}}/(\mathcal{K}_{j'}|_{U_j \cap U_{j'}})$$

see Sheaves, Section 33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. Hence the collection of all sheaves of $\mathcal{O}_X$-modules gotten from glueing quotients as above forms a set $\mathcal{I}$. By definition every finite type $\mathcal{O}_X$-module is isomorphic to an element of $\mathcal{I}$. Choosing an element out of each isomorphism class inside $\mathcal{I}$ gives the desired set of sheaves (uses axiom of choice). $\square$

## 10. Quasi-coherent modules

In this section we introduce an abstract notion of quasi-coherent $\mathcal{O}_X$-module. This notion is very useful in algebraic geometry, since quasi-coherent modules on a scheme have a good description on any affine open. However, we warn the reader that in the general setting of (locally) ringed spaces this notion is not well behaved at all. The category of quasi-coherent sheaves is not abelian in general, infinite direct sums of quasi-coherent sheaves aren't quasi-coherent, etc, etc.

**Definition 10.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a *quasi-coherent sheaf of $\mathcal{O}_X$-modules* if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_U$$

The category of quasi-coherent $\mathcal{O}_X$-modules is denoted $QCoh(\mathcal{O}_X)$.

The definition means that $X$ is covered by open sets $U$ such that $\mathcal{F}|_U$ has a *presentation* of the form

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_U \to \mathcal{F}|_U \to 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

1. for every point $x$ of $X$ there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by global sections, and
2. for a suitable choice of these sections the kernel of the associated surjection is also generated by global sections.

**Lemma 10.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. The direct sum of two quasi-coherent $\mathcal{O}_X$-modules is a quasi-coherent $\mathcal{O}_X$-module.*

**Proof.** Omitted. $\square$

**Remark 10.3.** Warning: It is not true in general that an infinite direct sum of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent. For more esoteric behaviour of quasi-coherent modules see Example 10.9.

**Lemma 10.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a quasi-coherent $\mathcal{O}_Y$-module is quasi-coherent.*

**Proof.** Arguing as in the proof of Lemma 8.2 we may assume $\mathcal{G}$ has a global presentation by direct sums of copies of $\mathcal{O}_Y$. We have seen that $f^*$ commutes with all colimits, and is right exact, see Lemma 3.3. Thus if we have an exact sequence

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_Y \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying $f^*$ we obtain the exact sequence

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. □

This gives plenty of examples of quasi-coherent sheaves.

**Lemma 10.5.** *Let $(X, \mathcal{O}_X)$ be ringed space. Let $\alpha : R \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism from a ring $R$ into the ring of global sections on $X$. Let $M$ be an $R$-module. The following three constructions give canonically isomorphic sheaves of $\mathcal{O}_X$-modules:*

1. *Let $\pi : (X, \mathcal{O}_X) \longrightarrow (\{*\}, R)$ be the morphism of ringed spaces with $\pi : X \to \{*\}$ the unique map and with $\pi$-map $\pi^\sharp$ the given map $\alpha : R \to \Gamma(X, \mathcal{O}_X)$. Set $\mathcal{F}_1 = \pi^*M$.*
2. *Choose a presentation $\bigoplus_{j\in J} R \to \bigoplus_{i\in I} R \to M \to 0$. Set*
$$\mathcal{F}_2 = Coker\left(\bigoplus\nolimits_{j\in J} \mathcal{O}_X \to \bigoplus\nolimits_{i\in I} \mathcal{O}_X\right).$$
*Here the map on the component $\mathcal{O}_X$ corresponding to $j \in J$ given by the section $\sum_i \alpha(r_{ij})$ where the $r_{ij}$ are the matrix coefficients of the map in the presentation of $M$.*
3. *Set $\mathcal{F}_3$ equal to the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U) \otimes_R M$, where the map $R \to \mathcal{O}_X(U)$ is the composition of $\alpha$ and the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$.*

*This construction has the following properties:*

1. *The resulting sheaf of $\mathcal{O}_X$-modules $\mathcal{F}_M = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$ is quasi-coherent.*
2. *The construction gives a functor from the category of $R$-modules to the category of quasi-coherent sheaves on $X$ which commutes with arbitrary colimits.*
3. *For any $x \in X$ we have $\mathcal{F}_{M,x} = \mathcal{O}_{X,x} \otimes_R M$ functorial in $M$.*
4. *Given any $\mathcal{O}_X$-module $\mathcal{G}$ we have*
$$\mathrm{Mor}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{G}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{G}))$$
*where the $R$-module structure on $\Gamma(X, \mathcal{G})$ comes from the $\Gamma(X, \mathcal{O}_X)$-module structure via $\alpha$.*

**Proof.** The isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_3$ comes from the fact that $\pi^*$ is defined as the sheafification of the presheaf in (3), see Sheaves, Section 26. The isomorphism between the constructions in (2) and (1) comes from the fact that the functor $\pi^*$ is right exact, so $\pi^*(\bigoplus_{j\in J} R) \to \pi^*(\bigoplus_{i\in I} R) \to \pi^*M \to 0$ is exact, $\pi^*$ commutes with arbitrary direct sums, see Lemma 3.3, and finally the fact that $\pi^*(R) = \mathcal{O}_X$.

Assertion (1) is clear from construction (2). Assertion (2) is clear since $\pi^*$ has these properties. Assertion (3) follows from the description of stalks of pullback sheaves, see Sheaves, Lemma 26.4. Assertion (4) follows from adjointness of $\pi_*$ and $\pi^*$. $\square$

**Definition 10.6.** In the situation of Lemma 10.5 we say $\mathcal{F}_M$ is the *sheaf associated to the module $M$ and the ring map $\alpha$*. If $R = \Gamma(X, \mathcal{O}_X)$ and $\alpha = \mathrm{id}_R$ we simply say $\mathcal{F}_M$ is the *sheaf associated to the module $M$*.

**Lemma 10.7.** *Let $(X, \mathcal{O}_X)$ be ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let $M$ be an $R$-module. Let $\mathcal{F}_M$ be the quasi-coherent sheaf of $\mathcal{O}_X$-modules associated to $M$. If $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces, then $g^*\mathcal{F}_M$ is the sheaf associated to the $\Gamma(Y, \mathcal{O}_Y)$-module $\Gamma(Y, \mathcal{O}_Y) \otimes_R M$.*

**Proof.** The assertion follows from the first description of $\mathcal{F}_M$ in Lemma 10.5 as $\pi^*M$, and the following commutative diagram of ringed spaces

$$\begin{tikzcd} (Y, \mathcal{O}_Y) \arrow[r, "\pi"'] \arrow[d, "g"'] & (\{*\}, \Gamma(Y, \mathcal{O}_Y)) \arrow[d, "\text{induced by } g^\sharp"] \\ (X, \mathcal{O}_X) \arrow[r, "\pi"] & (\{*\}, \Gamma(X, \mathcal{O}_X)) \end{tikzcd}$$

(Also use Sheaves, Lemma 26.3.) $\square$

**Lemma 10.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$ be a point. Assume that $x$ has a fundamental system of quasi-compact neighbourhoods. Consider any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Then there exists an open neighbourhood $U$ of $x$ such that $\mathcal{F}|_U$ is isomorphic to the sheaf of modules $\mathcal{F}_M$ on $(U, \mathcal{O}_U)$ associated to some $\Gamma(U, \mathcal{O}_U)$-module $M$.*

**Proof.** First we may replace $X$ by an open neighbourhood of $x$ and assume that $\mathcal{F}$ is isomorphic to the cokernel of a map

$$\Psi : \bigoplus\nolimits_{j \in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_X.$$

The problem is that this map may not be given by a "matrix", because the module of global sections of a direct sum is in general different from the direct sum of the modules of global sections.

Let $x \in E \subset X$ be a quasi-compact neighbourhood of $x$ (note: $E$ may not be open). Let $x \in U \subset E$ be an open neighbourhood of $x$ contained in $E$. Next, we proceed as in the proof of Lemma 3.5. For each $j \in J$ denote $s_j \in \Gamma(X, \bigoplus_{i \in I} \mathcal{O}_X)$ the image of the section 1 in the summand $\mathcal{O}_X$ corresponding to $j$. There exists a finite collection of opens $U_{jk}$, $k \in K_j$ such that $E \subset \bigcup_{k \in K_j} U_{jk}$ and such that each restriction $s_j|_{U_{jk}}$ is a finite sum $\sum_{i \in I_{jk}} f_{jki}$ with $I_{jk} \subset I$, and $f_{jki}$ in the summand $\mathcal{O}_X$ corresponding to $i \in I$. Set $I_j = \bigcup_{k \in k_J} I_{jk}$. This is a finite set. Since $U \subset E \subset \bigcup_{k \in K_j} U_{jk}$ the section $s_j|_U$ is a section of the finite direct sum $\bigoplus_{i \in I_j} \mathcal{O}_X$. By Lemma 3.2 we see that actually $s_j|_U$ is a sum $\sum_{i \in I_j} f_{ij}$ and $f_{ij} \in \mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_U)$.

At this point we can define a module $M$ as the cokernel of the map

$$\bigoplus\nolimits_{j \in J} \Gamma(U, \mathcal{O}_U) \longrightarrow \bigoplus\nolimits_{i \in I} \Gamma(U, \mathcal{O}_U)$$

with matrix given by the $(f_{ij})$. By construction (2) of Lemma 10.5 we see that $\mathcal{F}_M$ has the same presentation as $\mathcal{F}|_U$ and therefore $\mathcal{F}_M \cong \mathcal{F}|_U$. $\square$

**Example 10.9.** Let $X$ be countably many copies $L_1, L_2, L_3, \ldots$ of the real line all glued together at 0; a fundamental system of neighbourhoods of 0 being the collection $\{U_n\}_{n \in \mathbf{N}}$, with $U_n \cap L_i = (-1/n, 1/n)$. Let $\mathcal{O}_X$ be the sheaf of continuous real valued functions. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function which is identically zero on $(-1, 1)$ and identically 1 on $(-\infty, -2) \cup (2, \infty)$. Denote $f_n$ the continuous function on $X$ which is equal to $x \mapsto f(nx)$ on each $L_j = \mathbf{R}$. Let $1_{L_j}$ be the characteristic function of $L_j$. We consider the map

$$\bigoplus\nolimits_{j \in \mathbf{N}} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{j,i \in \mathbf{N}} \mathcal{O}_X, \quad e_j \longmapsto \sum\nolimits_{i \in \mathbf{N}} f_j 1_{L_i} e_{ij}$$

with obvious notation. This makes sense because this sum is locally finite as $f_j$ is zero in a neighbourhood of 0. Over $U_n$ the image of $e_j$, for $j > 2n$ is not a finite linear combination $\sum g_{ij} e_{ij}$ with $g_{ij}$ continuous. Thus there is no neighbourhood of $0 \in X$ such that the displayed map is given by a "matrix" as in the proof of Lemma 10.8 above.

Note that $\bigoplus_{j \in \mathbf{N}} \mathcal{O}_X$ is the sheaf associated to the free module with basis $e_j$ and similarly for the other direct sum. Thus we see that a morphism of sheaves associated to modules in general even locally on $X$ does not come from a morphism of modules. Similarly there should be an example of a ringed space $X$ and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}$ is not locally of the form $\mathcal{F}_M$. (Please email if you find one.) Moreover, there should be examples of locally compact spaces $X$ and maps $\mathcal{F}_M \to \mathcal{F}_N$ which also do not locally come from maps of modules (the proof of Lemma 10.8 shows this cannot happen if $N$ is free).

## 11. Modules of finite presentation

**Definition 11.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of *finite presentation* if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$, and $n, m \in \mathbf{N}$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U$$

This means that $X$ is covered by open sets $U$ such that $\mathcal{F}|_U$ has a *presentation* of the form

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U \to 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point $x$ of $X$ there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by finitely many global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by finitely many global sections.

**Lemma 11.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Any $\mathcal{O}_X$-module of finite presentation is quasi-coherent.*

**Proof.** Immediate from definitions. □

**Lemma 11.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module of finite presentation.*

(1) *If $\psi : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ is a surjection, then $Ker(\psi)$ is of finite type.*
(2) *If $\theta : \mathcal{G} \to \mathcal{F}$ is surjective with $\mathcal{G}$ of finite type, then $Ker(\theta)$ is of finite type.*

**Proof.** Proof of (1). Let $x \in X$. Choose an open neighbourhood $U \subset X$ of $x$ such that there exists a presentation
$$\mathcal{O}_U^{\oplus m} \xrightarrow{\chi} \mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} \mathcal{F}|_U \to 0.$$
Let $e_k$ be the section generating the $k$th factor of $\mathcal{O}_X^{\oplus r}$. For every $k = 1, \ldots, r$ we can, after shrinking $U$ to a small neighbourhood of $x$, lift $\psi(e_k)$ to a section $\tilde{e}_k$ of $\mathcal{O}_U^{\oplus n}$ over $U$. This gives a morphism of sheaves $\alpha : \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n}$ such that $\varphi \circ \alpha = \psi$. Similarly, after shrinking $U$, we can find a morphism $\beta : \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus r}$ such that $\psi \circ \beta = \varphi$. Then the map
$$\mathcal{O}_U^{\oplus m} \oplus \mathcal{O}_U^{\oplus r} \xrightarrow{\beta\circ\chi, 1-\beta\circ\alpha} \mathcal{O}_U^{\oplus r}$$
is a surjection onto the kernel of $\psi$.

To prove (2) we may locally choose a surjection $\eta : \mathcal{O}_X^{\oplus r} \to \mathcal{G}$. By part (1) we see $\mathrm{Ker}(\theta \circ \eta)$ is of finite type. Since $\mathrm{Ker}(\theta) = \eta(\mathrm{Ker}(\theta \circ \eta))$ we win. □

**Lemma 11.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a module of finite presentation is of finite presentation.*

**Proof.** Exactly the same as the proof of Lemma 10.4 but with finite index sets. □

**Lemma 11.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let $M$ be an $R$-module. The $\mathcal{O}_X$-module $\mathcal{F}_M$ associated to $M$ is a directed colimit of finitely presented $\mathcal{O}_X$-modules.*

**Proof.** This follows immediately from Lemma 10.5 and the fact that any module is a directed colimit of finitely presented modules, see Algebra, Lemma 8.13. □

**Lemma 11.6.** *Let $X$ be a ringed space. Let $I$ be a partially ordered set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over $I$ consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 21). Assume*

1. *$I$ is directed,*
2. *$\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation, and*
3. *$X$ has a cofinal system of open coverings $\mathcal{U} : X = \bigcup_{j\in J} U_j$ with $J$ finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$.*

*Then we have*
$$\mathrm{colim}_i \, \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \mathrm{Hom}_X(\mathcal{G}, \mathrm{colim}_i \, \mathcal{F}_i).$$

**Proof.** Let $\alpha$ be an element of the right hand side. For every point $x \in X$ we may choose an open neighbourhood $U \subset X$ and finitely many sections $s_j \in \mathcal{G}(U)$ which generate $\mathcal{G}$ over $U$ and finitely many relations $\sum f_{kj}s_j = 0$, $k = 1, \ldots, n$ with $f_{kj} \in \mathcal{O}_X(U)$ which generate the kernel of $\bigoplus_{j=1,\ldots,m} \mathcal{O}_U \to \mathcal{G}$. After possibly shrinking $U$ to a smaller open neighbourhood of $x$ we may assume there exists an index $i \in I$ such that the sections $\alpha(s_j)$ all come from sections $s'_j \in \mathcal{F}_i(U)$. After possibly shrinking $U$ to a smaller open neighbourhood of $x$ and increasing $i$ we may assume the relations $\sum f_{kj}s'_j = 0$ hold in $\mathcal{F}_i(U)$. Hence we see that $\alpha|_U$ lifts to a morphism $\mathcal{G}|_U \to \mathcal{F}_i|_U$ for some index $i \in I$.

By condition (3) and the preceding arguments, we may choose a finite open covering $X = \bigcup_{j=1,\ldots,m} U_j$ such that (a) $\mathcal{G}|_{U_j}$ is generated by finitely many sections $s_{jk} \in \mathcal{G}(U_j)$, (b) the restriction $\alpha|_{U_j}$ comes from a morphism $\alpha_j : \mathcal{G} \to \mathcal{F}_{i_j}$ for some $i_j \in I$, and (c) the intersections $U_j \cap U_{j'}$ are all quasi-compact. For every pair

$(j, j') \in \{1, \ldots, m\}^2$ and any $k$ we can find we can find an index $i \geq \max(i_j, i_{j'})$ such that

$$\varphi_{i_j i}(\alpha_j(s_{jk}|_{U_j \cap U_{j'}})) = \varphi_{i_{j'} i}(\alpha_{j'}(s_{jk}|_{U_j \cap U_{j'}}))$$

see Sheaves, Lemma 29.1 (2). Since there are finitely many of these pairs $(j, j')$ and finitely many $s_{jk}$ we see that we can find a single $i$ which works for all of them. For this index $i$ all of the maps $\varphi_{i_j i} \circ \alpha_j$ agree on the overlaps $U_j \cap U_{j'}$ as the sections $s_{jk}$ generate $\mathcal{G}$ over this overlap. Hence we get a morphism $\mathcal{G} \to \mathcal{F}_i$ as desired. □

**Remark 11.7.** In the lemma above some condition beyond the condition that $X$ is quasi-compact is necessary. See Sheaves, Example 29.2.

## 12. Coherent modules

The category of coherent sheaves on a ringed space $X$ is a more reasonable object than the category of quasi-coherent sheaves, in the sense that it is at least an abelian subcategory of $\textit{Mod}(\mathcal{O}_X)$ no matter what $X$ is. On the other hand, the pullback of a coherent module is "almost never" coherent in the general setting of ringed spaces.

**Definition 12.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a *coherent $\mathcal{O}_X$-module* if the following two conditions hold:

(1) $\mathcal{F}$ is of finite type, and
(2) for every open $U \subset X$ and every finite collection $s_i \in \mathcal{F}(U)$, $i = 1, \ldots, n$ the kernel of the associated map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type.

The category of coherent $\mathcal{O}_X$-modules is denoted $\textit{Coh}(\mathcal{O}_X)$.

**Lemma 12.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Any coherent $\mathcal{O}_X$-module is of finite presentation and hence quasi-coherent.*

**Proof.** Let $\mathcal{F}$ be a coherent sheaf on $X$. Pick a point $x \in X$. By (1) of the definition of coherent, we may find an open neighbourhood $U$ and sections $s_i$, $i = 1, \ldots, n$ of $\mathcal{F}$ over $U$ such that $\Psi : \bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}$ is surjective. By (2) of the definition of coherent, we may find an open neighbourhood $V$, $x \in V \subset U$ and sections $t_1, \ldots, t_m$ of $\bigoplus_{i=1,\ldots,n} \mathcal{O}_V$ which generate the kernel of $\Psi|_V$. Then over $V$ we get the presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}|_V \to 0$$

as desired. □

**Example 12.3.** Suppose that $X$ is a point. In this case the definition above gives a notion for modules over rings. What does the definition of coherent mean? It is closely related to the notion of Noetherian, but it is not the same: Namely, the ring $R = \mathbf{C}[x_1, x_2, x_3, \ldots]$ is coherent as a module over itself but not Noetherian as a module over itself. See Algebra, Section 87 for more discussion.

**Lemma 12.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *Any finite type subsheaf of a coherent sheaf is coherent.*
(2) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism from a finite type sheaf $\mathcal{F}$ to a coherent sheaf $\mathcal{G}$. Then $\textit{Ker}(\varphi)$ is finite type.*
(3) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent $\mathcal{O}_X$-modules. Then $\textit{Ker}(\varphi)$ and $\textit{Coker}(\varphi)$ are coherent.*

(4) *Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are coherent so is the third.*
(5) *The category $Coh(\mathcal{O}_X)$ is a weak Serre subcategory of $Mod(\mathcal{O}_X)$. In particular, the category of coherent modules is abelian and the inclusion functor $Coh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ is exact.*

**Proof.** Condition (2) of Definition 12.1 holds for any subsheaf of a coherent sheaf. Thus we get (1).

Assume the hypotheses of (2). Let us show that $\mathrm{Ker}(\varphi)$ is of finite type. Pick $x \in X$. Choose an open neighbourhood $U$ of $x$ in $X$ such that $\mathcal{F}|_U$ is generated by $s_1, \ldots, s_n$. By Definition 12.1 the kernel $\mathcal{K}$ of the induced map $\bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$, $e_i \mapsto \varphi(s_i)$ is of finite type. Hence $\mathrm{Ker}(\varphi)$ which is the image of the composition $\mathcal{K} \to \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{F}$ is of finite type.

Assume the hypotheses of (3). By (2) the kernel of $\varphi$ is of finite type and hence by (1) it is coherent.

With the same hypotheses let us show that $\mathrm{Coker}(\varphi)$ is coherent. Since $\mathcal{G}$ is of finite type so is $\mathrm{Coker}(\varphi)$. Let $U \subset X$ be open and let $\overline{s}_i \in \mathrm{Coker}(\varphi)(U)$, $i = 1, \ldots, n$ be sections. We have to show that the kernel of the associated morphism $\overline{\Psi} : \bigoplus_{i=1}^n \mathcal{O}_U \to \mathrm{Coker}(\varphi)$ has finite type. There exists an open covering of $U$ such that on each open all the sections $\overline{s}_i$ lift to sections $s_i$ of $\mathcal{G}$. Hence we may assume this is the case over $U$. Thus $\overline{\Psi}$ lifts to $\Psi : \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$ Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\Psi) \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d, equal] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\overline{\Psi}) \arrow[r] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] & \mathrm{Coker}(\varphi) \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma we get a short exact sequence $0 \to \mathrm{Ker}(\Psi) \to \mathrm{Ker}(\overline{\Psi}) \to \mathrm{Im}(\varphi) \to 0$. Hence by Lemma 9.3 we see that $\mathrm{Ker}(\overline{\Psi})$ has finite type.

Proof of part (4). Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. By part (3) it suffices to prove that if $\mathcal{F}_1$ and $\mathcal{F}_3$ are coherent so is $\mathcal{F}_2$. By Lemma 9.3 we see that $\mathcal{F}_2$ has finite type. Let $s_1, \ldots, s_n$ be finitely many local sections of $\mathcal{F}_2$ defined over a common open $U$ of $X$. We have to show that the module of relations $\mathcal{K}$ between them is of finite type. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0
\end{tikzcd}$$

with obvious notation. By the snake lemma we get a short exact sequence $0 \to \mathcal{K} \to \mathcal{K}_3 \to \mathcal{F}_1$ where $\mathcal{K}_3$ is the module of relations among the images of the sections $s_i$ in $\mathcal{F}_3$. Since $\mathcal{F}_3$ is coherent we see that $\mathcal{K}_3$ is finite type. Since $\mathcal{F}_1$ is coherent we see that the image $\mathcal{I}$ of $\mathcal{K}_3 \to \mathcal{F}_1$ is coherent. Hence $\mathcal{K}$ is the kernel of the map $\mathcal{K}_3 \to \mathcal{I}$ between a finite type sheaf and a coherent sheaves and hence finite type by (2).

Proof of (5). This follows because (3) and (4) show that Homology, Lemma 9.3 applies. □

**Lemma 12.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is coherent if and only if it is of finite presentation.*

**Proof.** Omitted. □

**Lemma 12.6.** *Let $X$ be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{G}$ of finite type, $\mathcal{F}$ coherent and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ injective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is injective.*

**Proof.** Denote $\mathcal{K} \subset \mathcal{G}$ the kernel of $\varphi$. By Lemma 12.4 we see that $\mathcal{K}$ is a finite type $\mathcal{O}_X$-module. Our assumption is that $\mathcal{K}_x = 0$. By Lemma 9.5 there exists an open neighbourhood $U$ of $x$ such that $\mathcal{K}|_U = 0$. Then $U$ works. □

## 13. Closed immersions of ringed spaces

When do we declare a morphism of ringed spaces $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ to be a closed immersion?

Motivated by the example of a closed immersion of normal topological spaces (ringed with the sheaf of continuous functors), or differential manifolds (ringed with the sheaf of differentiable functions), it seems natural to assume at least:

(1) The map $i$ is a closed immersion of topological spaces.
(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.

Already these conditions imply a number of pleasing results: For example we prove that the category of $\mathcal{O}_Z$-modules is equivalent to the category of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ generalizing the result on abelian sheaves of Section 6

However, in the Stacks project we choose the definition that guarantees that if $i$ is a closed immersion and $(X, \mathcal{O}_X)$ is a scheme, then also $(Z, \mathcal{O}_Z)$ is a scheme. Moreover, in this situation we want $i_*$ and $i^*$ to provide an equivalence between the category of quasi-coherent $\mathcal{O}_Z$-modules and the category of quasi-coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$. A minimal condition is that $i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. A good way to guarantee that $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module is to assume that $\mathcal{I}$ is locally generated by sections. We can interpret this condition as saying "$(Z, \mathcal{O}_Z)$ is locally on $(X, \mathcal{O}_X)$ defined by setting some regular functions $f_i$, i.e., local sections of $\mathcal{O}_X$, equal to zero". This leads to the following definition.

**Definition 13.1.** A *closed immersion of ringed spaces*[2] is a morphism $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ with the following properties:

(1) The map $i$ is a closed immersion of topological spaces.
(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

[2] This is nonstandard notation; see discussion above.

Actually, this definition still does not guarantee that $i_*$ of a quasi-coherent $\mathcal{O}_Z$-module is a quasi-coherent $\mathcal{O}_X$-module. The problem is that it is not clear how to convert a local presentation of a quasi-coherent $\mathcal{O}_Z$-module into a local presentation for the pushforward. However, the following is trivial.

**Lemma 13.2.** *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a closed immersion of locally ringed spaces. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is locally on $X$ the cokernel of a map of quasi-coherent $\mathcal{O}_X$-modules.*

**Proof.** This is true because $i_*\mathcal{O}_Z$ is quasi-coherent by definition. And locally on $Z$ the sheaf $\mathcal{F}$ is a cokernel of a map between direct sums of copies of $\mathcal{O}_Z$. Moreover, any direct sum of copies of the *the same* quasi-coherent sheaf is quasi-coherent. And finally, $i_*$ commutes with arbitrary colimits, see Lemma 6.3. Some details omitted. □

**Lemma 13.3.** *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume $i$ is a homeomorphism onto a closed subset of $X$ and that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Let $\mathcal{F}$ be an $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is of finite type if and only if $\mathcal{F}$ is of finite type.*

**Proof.** Suppose that $\mathcal{F}$ is of finite type. Pick $x \in X$. If $x \notin Z$, then $i_*\mathcal{F}$ is zero in a neighbourhood of $x$ and hence finitely generated in a neighbourhood of $x$. If $x = i(z)$, then choose an open neighbourhood $z \in V \subset Z$ and sections $s_1, \ldots, s_n \in \mathcal{F}(V)$ which generate $\mathcal{F}$ over $V$. Write $V = Z \cap U$ for some open $U \subset X$. Note that $U$ is a neighbourhood of $x$. Clearly the sections $s_i$ give sections $s_i$ of $i_*\mathcal{F}$ over $U$. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \longrightarrow i_*\mathcal{F}|_U$$

is surjective by inspection of what it does on stalks (here we use that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective). Hence $i_*\mathcal{F}$ is of finite type.

Conversely, suppose that $i_*\mathcal{F}$ is of finite type. Choose $z \in Z$. Set $x = i(z)$. By assumption there exists an open neighbourhood $U \subset X$ of $x$, and sections $s_1, \ldots, s_n \in (i_*\mathcal{F})(U)$ which generate $i_*\mathcal{F}$ over $U$. Set $V = Z \cap U$. By definition of $i_*$ the sections $s_i$ correspond to sections $s_i$ of $\mathcal{F}$ over $V$. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \longrightarrow \mathcal{F}|_V$$

is surjective by inspection of what it does on stalks. Hence $\mathcal{F}$ is of finite type. □

**Lemma 13.4.** *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume $i$ is a homeomorphism onto a closed subset of $X$ and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote $\mathcal{I} \subset \mathcal{O}_X$ the kernel of $i^\sharp$. The functor*

$$i_* : Mod(\mathcal{O}_Z) \longrightarrow Mod(\mathcal{O}_X)$$

*is exact, fully faithful, with essential image those $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.*

**Proof.** We claim that for a $\mathcal{O}_Z$-module $\mathcal{F}$ the canonical map

$$i^*i_*\mathcal{F} \longrightarrow \mathcal{F}$$

is an isomorphism. We check this on stalks. Say $z \in Z$ and $x = i(z)$. We have

$$(i^*i_*\mathcal{F})_z = (i_*\mathcal{F})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} = \mathcal{F}_z \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} = \mathcal{F}_z$$

by Sheaves, Lemma 26.4, the fact that $\mathcal{O}_{Z,z}$ is a quotient of $\mathcal{O}_{X,x}$, and Sheaves, Lemma 32.1. It follows that $i_*$ is fully faithful.

Let $\mathcal{G}$ be a $\mathcal{O}_X$-module with $\mathcal{I}\mathcal{G} = 0$. If $x \in X$, $x \notin i(Z)$, then $\mathcal{G}_x = 0$ because $\mathcal{I}_x = \mathcal{O}_{X,x}$ in this case. Thus we see that $\mathcal{G}$ us supported on $Z$. By Lemma 6.1 we can write $\mathcal{G} = i_*\mathcal{F}$ for a unique abelian sheaf $\mathcal{F}$ on $Z$. Let $W \subset Z$ be open, $f \in \mathcal{O}_Z(W)$ and $s \in \mathcal{F}(W)$. We define $fs \in \mathcal{F}(W)$. Since $i^\sharp$ is surjective we can find opens $U_j \subset X$ such that $W = \bigcup i^{-1}(U_j)$ and $f|i^{-1}(U_j)$ is the image of $f_j \in \mathcal{O}_X(U_j)$. Note that $s|_{i^{-1}(U_j)}$ is an element of $\mathcal{F}(i^{-1}(U_j)) = \mathcal{G}(U_i)$. Thus we can form $s_j = f_j s \in \mathcal{F}(i^{-1}(U_j)) = \mathcal{G}(U_i)$. By our assumption that $\mathcal{I}\mathcal{G} = 0$ the sections $s_j$ are independent of the choice of $f_j$ lifting $f|i^{-1}(U_j)$ and glue to a section $fs$ of $\mathcal{F}$ over $W$. In this way $\mathcal{F}$ becomes an $\mathcal{O}_Z$-module such that $\mathcal{G} \cong i_*\mathcal{F}$. □

## 14. Locally free sheaves

Let $(X, \mathcal{O}_X)$ be a ringed space. Our conventions allow (some of) the stalks $\mathcal{O}_{X,x}$ to be the zero ring. This means we have to be a little careful when defining the rank of a locally free sheaf.

**Definition 14.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We say $\mathcal{F}$ is *locally free* if for every point $x \in X$ there exists a set $I$ and an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X|_U$ as an $\mathcal{O}_X|_U$-module.
(2) We say $\mathcal{F}$ is *finite locally free* if we may choose the index sets $I$ to be finite.
(3) We say $\mathcal{F}$ is *finite locally free of rank* $r$ if we may choose the index sets $I$ to have cardinality $r$.

A finite direct sum of (finite) locally free sheaves is (finite) locally free. However, it may not be the case that an infinite direct sum of locally free sheaves is locally free.

**Lemma 14.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is locally free then it is quasi-coherent.*

**Proof.** Omitted. □

**Lemma 14.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a locally free $\mathcal{O}_X$-module.*

**Proof.** Omitted. □

**Lemma 14.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that the support of $\mathcal{O}_X$ is $X$, i.e., all stalk of $\mathcal{O}_X$ are nonzero rings. Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules. There exists a locally constant function*

$$rank_{\mathcal{F}} : X \longrightarrow \{0, 1, 2, \ldots\} \cup \{\infty\}$$

*such that for any point $x \in X$ the cardinality of any set $I$ such that $\mathcal{F}$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X$ in a neighbourhood of $x$ is $rank_{\mathcal{F}}(x)$.*

**Proof.** Under the assumption of the lemma the cardinality of $I$ can be read off from the rank of the free module $\mathcal{F}_x$ over the nonzero ring $\mathcal{O}_{X,x}$, and it is constant in a neighbourhood of $x$. □

**Lemma 14.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $r \geq 0$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of finite locally free $\mathcal{O}_X$-modules of rank $r$. Then $\varphi$ is an isomorphism if and only if $\varphi$ is surjective.*

**Proof.** Assume $\varphi$ is surjective. Pick $x \in X$. There exists an open neighbourhood $U$ of $x$ such that both $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are isomorphic to $\mathcal{O}_U^{\oplus r}$. Pick lifts of the free generators of $\mathcal{G}|_U$ to obtain a map $\psi : \mathcal{G}|_U \to \mathcal{F}|_U$ such that $\varphi|_U \circ \psi = \text{id}$. Hence we conclude that the map $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ induced by $\varphi$ is surjective. Since both $\Gamma(U, \mathcal{F})$ and $\Gamma(U, \mathcal{G})$ are isomorphic to $\Gamma(U, \mathcal{O}_U)^{\oplus r}$ as an $\Gamma(U, \mathcal{O}_U)$-module we may apply Algebra, Lemma 15.4 to see that $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ is injective. This finishes the proof. $\square$

## 15. Tensor product

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. We have briefly discussed the tensor product in the setting of change of rings in Sheaves, Sections 6 and 20. In exactly the same way we define first the *tensor product presheaf*

$$\mathcal{F} \otimes_{p, \mathcal{O}_X} \mathcal{G}$$

as the rule which assigns to $U \subset X$ open the $\mathcal{O}_X(U)$-module $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Having defined this we define the *tensor product sheaf* as the sheafification of the above:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F} \otimes_{p, \mathcal{O}_X} \mathcal{G})^{\#}$$

This can be characterized as the sheaf of $\mathcal{O}_X$-modules such that for any third sheaf of $\mathcal{O}_X$-modules $\mathcal{H}$ we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \mathrm{Bilin}_{\mathcal{O}_X}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

Here the right hand side indicates the set of bilinear maps of sheaves of $\mathcal{O}_X$-modules (definition omitted).

The tensor product of modules $M, N$ over a ring $R$ satisfies symmetry, namely $M \otimes_R N = N \otimes_R M$, hence the same holds for tensor products of sheaves of modules, i.e., we have

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

functorial in $\mathcal{F}$, $\mathcal{G}$. And since tensor product of modules satisfies associativity we also get canonical functorial isomorphisms

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$$

functorial in $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$.

**Lemma 15.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $x \in X$. There is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules*

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$$

*functorial in $\mathcal{F}$ and $\mathcal{G}$.*

**Proof.** Omitted. $\square$

**Lemma 15.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}'$, $\mathcal{G}'$ be presheaves of $\mathcal{O}_X$-modules with sheafifications $\mathcal{F}$, $\mathcal{G}$. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F}' \otimes_{p, \mathcal{O}_X} \mathcal{G}')^{\#}$.*

**Proof.** Omitted. $\square$

**Lemma 15.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}_X$-modules then the induced sequence*

$$\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

*is exact.*

**Proof.** This follows from the fact that exactness may be checked at stalks (Lemma 3.1), the description of stalks (Lemma 15.1) and the corresponding result for tensor products of modules (Algebra, Lemma 11.10). □

**Lemma 15.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_Y$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.*

**Proof.** Omitted. □

**Lemma 15.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

1. *If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
2. *If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
3. *If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
4. *If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
5. *If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is coherent.*
6. *If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
7. *If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*

**Proof.** We first prove that the tensor product of locally free $\mathcal{O}_X$-modules is locally free. This follows if we show that $(\bigoplus_{i\in I} \mathcal{O}_X) \otimes_{\mathcal{O}_X} (\bigoplus_{j\in J} \mathcal{O}_X) \cong \bigoplus_{(i,j)\in I\times J} \mathcal{O}_X$. The sheaf $\bigoplus_{i\in I} \mathcal{O}_X$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{i\in I} \mathcal{O}_X(U)$. Hence the tensor product is the sheaf associated to the presheaf

$$U \longmapsto (\bigoplus\nolimits_{i\in I} \mathcal{O}_X(U)) \otimes_{\mathcal{O}_X(U)} (\bigoplus\nolimits_{j\in J} \mathcal{O}_X(U)).$$

We deduce what we want since for any ring $R$ we have $(\bigoplus_{i\in I} R) \otimes_R (\bigoplus_{j\in J} R) = \bigoplus_{(i,j)\in I\times J} R$.

If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is exact, then by Lemma 15.3 the complex $\mathcal{F}_2 \otimes \mathcal{G} \to \mathcal{F}_1 \otimes \mathcal{G} \to \mathcal{F} \otimes \mathcal{G} \to 0$ is exact. Using this we can prove (5). Namely, in this case there exists locally such an exact sequence with $\mathcal{F}_i$, $i = 1, 2$ finite free. Hence the two terms $\mathcal{F}_2 \otimes \mathcal{G}$ are isomorphic to finite direct sums of $\mathcal{G}$. Since finite direct sums are coherent sheaves, these are coherent and so is the cokernel of the map, see Lemma 12.4.

And if also $\mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G} \to 0$ is exact, then we see that

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \oplus \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_2 \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact. Using this we can for example prove (3). Namely, the assumption means that we can locally find presentations as above with $\mathcal{F}_i$ and $\mathcal{G}_i$ free $\mathcal{O}_X$-modules. Hence the displayed presentation is a presentation of the tensor product by free sheaves as well.

The proof of the other statements is omitted. □

**Lemma 15.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. For any $\mathcal{O}_X$-module $\mathcal{F}$ the functor*

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$$

*commutes with arbitrary colimits.*

**Proof.** Let $I$ be a partially ordered set and let $\{\mathcal{G}_i\}$ be a system over $I$. Set $\mathcal{G} = \operatorname{colim}_i \mathcal{G}_i$. Recall that $\mathcal{G}$ is the sheaf associated to the presheaf $\mathcal{G}' : U \mapsto$

$\mathrm{colim}_i\, \mathcal{G}_i(U)$, see Sheaves, Section 29. By Lemma 15.2 the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathrm{colim}_i\, \mathcal{G}_i(U) = \mathrm{colim}_i\, \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}_i(U)$$

where the equality sign is Algebra, Lemma 11.9. Hence the lemma follows from the description of colimits in $\mathit{Mod}(\mathcal{O}_X)$. $\square$

## 16. Flat modules

We can define flat modules exactly as in the case of modules over rings.

**Definition 16.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if the functor

$$\mathit{Mod}(\mathcal{O}_X) \longrightarrow \mathit{Mod}(\mathcal{O}_X), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$$

is exact.

We can characterize flatness by looking at the stalks.

**Lemma 16.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is flat if and only if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$.*

**Proof.** Assume $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$. In this case, if $\mathcal{G} \to \mathcal{H} \to \mathcal{K}$ is exact, then also $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{F}$ is exact because we can check exactness at stalks and because tensor product commutes with taking stalks, see Lemma 15.1. Conversely, suppose that $\mathcal{F}$ is flat, and let $x \in X$. Consider the skyscraper sheaves $i_{x,*}M$ where $M$ is a $\mathcal{O}_{X,x}$-module. Note that

$$M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x = (i_{x,*}M \otimes_{\mathcal{O}_X} \mathcal{F})_x$$

again by Lemma 15.1. Since $i_{x,*}$ is exact, we see that the fact that $\mathcal{F}$ is flat implies that $M \mapsto M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$ is exact. Hence $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module. $\square$

Thus the following definition makes sense.

**Definition 16.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at $x$* if $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module.

Hence we see that $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if it is flat at every point.

**Lemma 16.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. A filtered colimit of flat $\mathcal{O}_X$-modules is flat. A direct sum of flat $\mathcal{O}_X$-modules is flat.*

**Proof.** This follows from Lemma 15.6, Lemma 15.1, Algebra, Lemma 8.9, and the fact that we can check exactness at stalks. $\square$

**Lemma 16.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be open. The sheaf $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}_X$-modules.*

**Proof.** The stalks of $j_{U!}\mathcal{O}_U$ are either zero or equal to $\mathcal{O}_{X,x}$. Apply Lemma 16.2. $\square$

**Lemma 16.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *Any sheaf of $\mathcal{O}_X$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
(2) *Any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module.*

**Proof.** Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For every open $U \subset X$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective, and the source is flat by combining Lemmas 16.4 and 16.5. □

**Lemma 16.7.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

*be a short exact sequence of $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is flat. Then for any $\mathcal{O}_X$-module $\mathcal{G}$ the sequence*

$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$

*is exact.*

**Proof.** Using that $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for every $x \in X$ and that exactness can be checked on stalks, this follows from Algebra, Lemma 38.11. □

**Lemma 16.8.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

*be a short exact sequence of $\mathcal{O}_X$-modules.*

1. *If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.*
2. *If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.*

**Proof.** Since exactness and flatness may be checked at the level of stalks this follows from Algebra, Lemma 38.12. □

**Lemma 16.9.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$

*be an exact complex of $\mathcal{O}_X$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}_X$-modules, then for any $\mathcal{O}_X$-module $\mathcal{G}$ the complex*

$$\ldots \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

*is exact also.*

**Proof.** Follows from Lemma 16.7 by splitting the complex into short exact sequences and using Lemma 16.8 to prove inductively that $\mathrm{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. □

The following lemma gives one direction of the equational criterion of flatness (Algebra, Lemma 38.10).

**Lemma 16.10.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat $\mathcal{O}_X$-module. Let $U \subset X$ be open and let*

$$\mathcal{O}_U \xrightarrow{(f_1, \ldots, f_n)} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} \mathcal{F}|_U$$

*be a complex of $\mathcal{O}_U$-modules. For every $x \in U$ there exists an open neighbourhood $V \subset U$ of $x$ and a factorization*

$$\mathcal{O}_V^{\oplus n} \xrightarrow{A} \mathcal{O}_V^{\oplus m} \xrightarrow{(t_1, \ldots, t_m)} \mathcal{F}|_V$$

*of $(s_1, \ldots, s_n)|_V$ such that $A \circ (f_1, \ldots, f_n)|_V = 0$.*

**Proof.** Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals generated by $f_1, \ldots, f_n$. Then $\sum f_i \otimes s_i$ is a section of $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ which maps to zero in $\mathcal{F}|_U$. As $\mathcal{F}|_U$ is flat the map $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{F}|_U$ is injective. Since $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is the sheaf associated to the presheaf tensor product, we see there exists an open neighbourhood $V \subset U$ of $x$ such that $\sum f_i|_V \otimes s_i|_V$ is zero in $\mathcal{I}(V) \otimes_{\mathcal{O}(V)} \mathcal{F}(V)$. Unwinding the definitions using Algebra, Lemma 103.10 we find $t_1, \ldots, t_m \in \mathcal{F}(V)$ and $a_{ij} \in \mathcal{O}(V)$ such that $\sum a_{ij} f_i|_V = 0$ and $s_i|_V = \sum a_{ij} t_j$. $\square$

**Lemma 16.11.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be locally of finite presentation and flat. Then $\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module.*

**Proof.** After replacing $X$ by the members of an open covering, we may assume there exists a presentation

$$\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{F} \to 0$$

Let $x \in X$. By Lemma 16.10 we can, after shrinking $X$ to an open neighbourhood of $x$, assume there exists a factorization

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_1} \to \mathcal{F}$$

such that the composition $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_1}$ annihilates the first summand of $\mathcal{O}_X^{\oplus r}$. Repeating this argument $r-1$ more times we obtain a factorization

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_r} \to \mathcal{F}$$

such that the composition $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_r}$ is zero. This means that the surjection $\mathcal{O}_X^{\oplus n_r} \to \mathcal{F}$ has a section and we win. $\square$

## 17. Flat morphisms of ringed spaces

The pointwise definition is motivated by Lemma 16.2 and Definition 16.3 above.

**Definition 17.1.** Let $f : X \to Y$ be a morphism of ringed spaces. Let $x \in X$. We say $f$ is said to be *flat at $x$* if the map of rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. We say $f$ is *flat* if $f$ is flat at every $x \in X$.

Consider the map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We see that the stalk at $x$ is the ring map $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Hence $f$ is flat at $x$ if and only if $\mathcal{O}_X$ is flat at $x$ as an $f^{-1}\mathcal{O}_Y$-module. And $f$ is flat if and only if $\mathcal{O}_X$ is flat as an $f^{-1}\mathcal{O}_Y$-module. A very special case of a flat morphism is an open immersion.

**Lemma 17.2.** *Let $f : X \to Y$ be a flat morphism of ringed spaces. Then the pullback functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is exact.*

**Proof.** The functor $f^*$ is the composition of the exact functor $f^{-1} : Mod(\mathcal{O}_Y) \to Mod(f^{-1}\mathcal{O}_Y)$ and the change of rings functor

$$Mod(f^{-1}\mathcal{O}_Y) \to Mod(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Thus the result follows from the discussion following Definition 17.1. $\square$

**Definition 17.3.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

1. We say that $\mathcal{F}$ is *flat over $Y$ at a point $x \in X$* if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module.
2. We say that $\mathcal{F}$ is *flat over $Y$* if $\mathcal{F}$ is flat over $Y$ at every point $x$ of $X$.

With this definition we see that $\mathcal{F}$ is flat over $Y$ at $x$ if and only if $\mathcal{F}$ is flat at $x$ as an $f^{-1}\mathcal{O}_Y$-module because $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)}$ by Sheaves, Lemma 21.5.

## 18. Symmetric and exterior powers

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-algebra. We define the *tensor algebra* of $\mathcal{F}$ to be the sheaf of noncommutative $\mathcal{O}_X$-algebras

$$\mathrm{T}(\mathcal{F}) = \mathrm{T}_{\mathcal{O}_X}(\mathcal{F}) = \bigoplus\nolimits_{n \geq 0} \mathrm{T}^n(\mathcal{F}).$$

Here $\mathrm{T}^0(\mathcal{F}) = \mathcal{O}_X$, $\mathrm{T}^1(\mathcal{F}) = \mathcal{F}$ and for $n \geq 2$ we have

$$\mathrm{T}^n(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{F} \quad (n \text{ factors})$$

We define $\wedge(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections $s \otimes s$ of $\mathrm{T}^2(\mathcal{F})$ where $s$ is a local section of $\mathcal{F}$. This is called the *exterior algebra* of $\mathcal{F}$. Similarly, we define $\mathrm{Sym}(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections of the form $s \otimes t - t \otimes s$ of $\mathrm{T}^2(\mathcal{F})$.

Both $\wedge(\mathcal{F})$ and $\mathrm{Sym}(\mathcal{F})$ are graded $\mathcal{O}_X$-algebras, with grading inherited from $\mathrm{T}(\mathcal{F})$. Moreover $\mathrm{Sym}(\mathcal{F})$ is commutative, and $\wedge(\mathcal{F})$ is graded commutative.

**Lemma 18.1.** *In the situation described above. The sheaf $\wedge^n\mathcal{F}$ is the sheafification of the presheaf*

$$U \longmapsto \wedge^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

*See Algebra, Section 12. Similarly, the sheaf $Sym^n\mathcal{F}$ is the sheafification of the presheaf*

$$U \longmapsto Sym^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

**Proof.** Omitted. It may be more efficient to define $\mathrm{Sym}(\mathcal{F})$ and $\wedge(\mathcal{F})$ in this way instead of the method given above. $\square$

**Lemma 18.2.** *In the situation described above. Let $x \in X$. There are canonical isomorphisms of $\mathcal{O}_{X,x}$-modules $T(\mathcal{F})_x = T(\mathcal{F}_x)$, $Sym(\mathcal{F})_x = Sym(\mathcal{F}_x)$, and $\wedge(\mathcal{F})_x = \wedge(\mathcal{F}_x)$.*

**Proof.** Clear from Lemma 18.1 above, and Algebra, Lemma 12.4. $\square$

**Lemma 18.3.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^*T(\mathcal{F}) = T(f^*\mathcal{F})$, and similarly for the exterior and symmetric algebras associated to $\mathcal{F}$.*

**Proof.** Omitted. $\square$

**Lemma 18.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ be an exact sequence of sheaves of $\mathcal{O}_X$-modules. For each $n \geq 1$ there is an exact sequence*

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} Sym^{n-1}(\mathcal{F}_1) \to Sym^n(\mathcal{F}_1) \to Sym^n(\mathcal{F}) \to 0$$

*and similarly an exact sequence*

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \wedge^{n-1}(\mathcal{F}_1) \to \wedge^n(\mathcal{F}_1) \to \wedge^n(\mathcal{F}) \to 0$$

**Proof.** See Algebra, Lemma 12.2. $\square$

**Lemma 18.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{F}$ is locally generated by sections, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*

(2) *If $\mathcal{F}$ is of finite type, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
(3) *If $\mathcal{F}$ is of finite presentation, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
(4) *If $\mathcal{F}$ is coherent, then for $n > 0$ each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$ is coherent.*
(5) *If $\mathcal{F}$ is quasi-coherent, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
(6) *If $\mathcal{F}$ is locally free, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*

**Proof.** These statements for $\mathrm{T}^n(\mathcal{F})$ follow from Lemma 15.5.

Statements (1) and (2) follow from the fact that $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$ are quotients of $\mathrm{T}^n(\mathcal{F})$.

Statement (6) follows from Algebra, Lemma 12.1.

For (3) and (5) we will use Lemma 18.4 above. By locally choosing a presentation $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ with $\mathcal{F}_i$ free, or finite free and applying the lemma we see that $\mathrm{Sym}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ has a similar presentation; here we use (6) and Lemma 15.5.

To prove (4) we will use Algebra, Lemma 12.3. We may localize on $X$ and assume that $\mathcal{F}$ is generated by a finite set $(s_i)_{i\in I}$ of global sections. The lemma mentioned above combined with Lemma 18.1 above implies that for $n \geq 2$ there exists an exact sequence

$$\bigoplus_{j\in J} \mathrm{T}^{n-2}(\mathcal{F}) \to \mathrm{T}^n(\mathcal{F}) \to \mathrm{Sym}^n(\mathcal{F}) \to 0$$

where the index set $J$ is finite. Now we know that $\mathrm{T}^{n-2}(\mathcal{F})$ is finitely generated and hence the image of the first arrow is a coherent subsheaf of $\mathrm{T}^n(\mathcal{F})$, see Lemma 12.4. By that same lemma we conclude that $\mathrm{Sym}^n(\mathcal{F})$ is coherent. $\square$

**Lemma 18.6.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*
(1) *If $\mathcal{F}$ is quasi-coherent, then so is each $T(\mathcal{F})$, $\wedge(\mathcal{F})$, and $Sym(\mathcal{F})$.*
(2) *If $\mathcal{F}$ is locally free, then so is each $T(\mathcal{F})$, $\wedge(\mathcal{F})$, and $Sym(\mathcal{F})$.*

**Proof.** It is not true that an infinite direct sum $\bigoplus \mathcal{G}_i$ of locally free modules is locally free, or that an infinite direct sum of quasi-coherent modules is quasi-coherent. The problem is that given a point $x \in X$ the open neighbourhoods $U_i$ of $x$ on which $\mathcal{G}_i$ becomes free (resp. has a suitable presentation) may have an intersection which is not an open neighbourhood of $x$. However, in the proof of Lemma 18.5 we saw that once a suitable open neighbourhood for $\mathcal{F}$ has been chosen, then this open neighbourhood works for each of the sheaves $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$. The lemma follows. $\square$

## 19. Internal Hom

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

It follows from the discussion in Sheaves, Section 33 that this is a sheaf of abelian groups. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}_X(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by $f$ on $\mathcal{F}|_U$ or postcomposing with multiplication by $f$ on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a sheaf of $\mathcal{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

which is rarely an isomorphism.

**Lemma 19.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be $\mathcal{O}_X$-modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

*which is functorial in all three entries (sheaf Hom in all three spots). In particular, to give a morphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{H}$ is the same as giving a morphism $\mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})$.*

**Proof.** This is the analogue of Algebra, Lemma 11.8. The proof is the same, and is omitted. □

**Lemma 19.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

1. *If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, then*
$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G})$$
*is exact.*
2. *If $0 \to \mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_2$ is an exact sequence of $\mathcal{O}_X$-modules, then*
$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_1) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_2)$$
*is exact.*

**Proof.** Omitted. □

**Lemma 19.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is finitely presented then the canonical map*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

*is an isomorphism.*

**Proof.** By localizing on $X$ we may assume that $\mathcal{F}$ has a presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to \mathcal{F} \to 0.$$

By Lemma 19.2 this gives an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \bigoplus_{i=1,\ldots,n} \mathcal{G} \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}$. Taking stalks we get an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ and the result follows since $\mathcal{F}_x$ sits in an exact sequence $\bigoplus_{j=1,\ldots,m} \mathcal{O}_{X,x} \longrightarrow \bigoplus_{i=1,\ldots,n} \mathcal{O}_{X,x} \to \mathcal{F}_x \to 0$ which induces the exact sequence $0 \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ which is the same as the one above. □

**Lemma 19.4.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is finitely presented then the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is locally a kernel of a map between finite direct sums of copies of $\mathcal{G}$. In particular, if $\mathcal{G}$ is coherent then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent too.*

**Proof.** The first assertion we saw in the proof of Lemma 19.3. And the result for coherent sheaves then follows from Lemma 12.4. □

**Lemma 19.5.** *Let $X$ be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Then we have*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{F}_{\mathcal{O}_1}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G}))$$

*bifunctorially in $\mathcal{F} \in Mod(\mathcal{O}_2)$ and $\mathcal{G} \in Mod(\mathcal{O}_1)$.*

**Proof.** Omitted. This is the analogue of Algebra, Lemma 13.4 and is proved in exactly the same way. □

## 20. Koszul complexes

We suggest first reading the section on Koszul complexes in More on Algebra, Section 20. We define the Koszul complex in the category of $\mathcal{O}_X$-modules as follows.

**Definition 20.1.** Let $X$ be a ringed space. Let $\varphi : \mathcal{E} \to \mathcal{O}_X$ be an $\mathcal{O}_X$-module map. The *Koszul complex* $K_\bullet(\varphi)$ associated to $\varphi$ is the sheaf of commutative differential graded algebras defined as follows:

(1) the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(\mathcal{E})$,
(2) the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all local sections $e$ of $\mathcal{E} = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is a wedge product of local sections of $\mathcal{E}$, then

$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \wedge e$ and hence factors through the exterior algebra.

**Definition 20.2.** Let $X$ be a ringed space and let $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$. The *Koszul complex on $f_1, \ldots, f_r$* is the Koszul complex associated to the map $(f_1, \ldots, f_n) : \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X$. Notation $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$, or $K_\bullet(\mathcal{O}_X, f_\bullet)$.

Of course, given an $\mathcal{O}_X$-module map $\varphi : \mathcal{E} \to \mathcal{O}_X$, if $\mathcal{E}$ is finite locally free, then $K_\bullet(\varphi)$ is locally on $X$ isomorphic to a Koszul complex $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$.

## 21. Invertible sheaves

**Definition 21.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings[3]. An *invertible $\mathcal{O}_X$-module* is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that for each point $x \in X$ there exists an open neighbourhood $U \subset X$ and an isomorphism $\mathcal{L}|_U \cong \mathcal{O}_X|_U$. We say that $\mathcal{L}$ is *trivial* if it is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{O}_X$.

**Lemma 21.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings.*

(1) *If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}_X$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$.*
(2) *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then so is $\mathcal{L}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.*
(3) *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then the evaluation map $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$ is an isomorphism.*

**Proof.** Omitted. □

[3]We should at least assume that they are nonzero. However, in this generality the stalks $\mathcal{O}_{X,x}$ can have nontrivial Picard groups, and then there are two possible definitions. One were we require $\mathcal{L}$ to be locally free of rank 1, and the other where we require $\mathcal{L}$ to be a flat, finite presentation $\mathcal{O}_X$-module such that there exists a second such sheaf $\mathcal{L}^{\otimes -1}$ with $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \cong \mathcal{O}_X$.

**Definition 21.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Given an invertible sheaf $\mathcal{L}$ on $X$ we define the $n$th *tensor power of* $\mathcal{L}$ by the rule

$$\mathcal{L}^{\otimes n} = \begin{cases} \mathcal{O}_X & \text{if} \quad n = 0 \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) & \text{if} \quad n = -1 \\ \mathcal{L} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L} & \text{if} \quad n > 0 \\ \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} & \text{if} \quad n < -1 \end{cases}$$

With this definition we have canonical isomorphisms $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes n+m}$, and these isomorphisms satisfy a commutativity and an associativity constraint (formulation omitted). Thus we can define a $\mathbf{Z}$-graded ring structure on $\bigoplus \Gamma(X, \mathcal{L}^{\otimes n})$ by mapping $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ to the section corresponding to $s \otimes t$ in $\Gamma(X, \mathcal{L}^{\otimes n+m})$. We omit the verification that this defines a commutative and associative ring with 1. However, by our conventions in Algebra, Section 54 a graded ring has no nonzero elements in negative degrees. This leads to the following definition.

**Definition 21.4.** Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Given an invertible sheaf $\mathcal{L}$ on $X$ we define the *associated graded ring* to be

$$\Gamma_*(X, \mathcal{L}) = \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which we think of as a graded $\Gamma_*(X, \mathcal{L})$-module.

We often write simply $\Gamma_*(\mathcal{L})$ and $\Gamma_*(\mathcal{F})$ (although this is ambiguous if $\mathcal{F}$ is invertible). The multiplication of $\Gamma_*(\mathcal{L})$ on $\Gamma_*(\mathcal{F})$ is defined using the isomorphisms above. If $\gamma : \mathcal{F} \to \mathcal{G}$ is a $\mathcal{O}_X$-module map, then we get an $\Gamma_*(\mathcal{L})$-module homomorphism $\gamma : \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$. If $\alpha : \mathcal{L} \to \mathcal{N}$ is an $\mathcal{O}_X$-module map between invertible $\mathcal{O}_X$-modules, then we obtain a graded ring homomorphism $\Gamma_*(\mathcal{L}) \to \Gamma_*(\mathcal{N})$. If $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of locally ringed spaces (see Schemes, Definition 2.1), and if $\mathcal{L}$ is invertible on $X$, then we get an invertible sheaf $f^*\mathcal{L}$ on $Y$ and an induced homomorphism of graded rings

$$f^* : \Gamma_*(X, \mathcal{L}) \longrightarrow \Gamma_*(Y, f^*\mathcal{L})$$

Furthermore, there are some compatibilities between the constructions above whose statements we omit.

**Lemma 21.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on $X$ is isomorphic to exactly one of the $\mathcal{L}_i$.*

**Proof.** For each open covering $\mathcal{U} : X = \bigcup U_j$ consider the sheaves of $\mathcal{O}_X$-modules gotten from glueing the sheaves $\mathcal{O}_X|_{U_j}$, see Sheaves, Section 33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. Hence the collection of all sheaves of of $\mathcal{O}_X$-modules gotten from glueing trivial invertible $\mathcal{O}_X$-modules forms a set $\mathcal{I}$. By definition every invertible $\mathcal{O}_X$-module is isomorphic to an element of $\mathcal{I}$. Choosing an element out of each isomorphism class inside $\mathcal{I}$ gives the desired set of invertible sheaves (uses axiom of choice). $\square$

This lemma says roughly speaking that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 21.2 says that tensor product defines the structure of an abelian group on this set.

**Definition 21.6.** Let $(X, \mathcal{O}_X)$ be a ringed space. Assume all stalks $\mathcal{O}_{X,x}$ are local rings. The *Picard group* $\text{Pic}(X)$ of $X$ is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}_X$-modules, with addition corresponding to tensor product.

**Lemma 21.7.** *Let $X$ be a ringed space. Assume that each stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For any section $s \in \Gamma(X, \mathcal{L})$ the set*

$$X_s = \{x \in X \mid \textit{image } s \notin \mathfrak{m}_x\mathcal{L}_x\}$$

*is open in $X$. The map $s : \mathcal{O}_{X_s} \to \mathcal{L}|_{X_s}$ is an isomorphism, and there exists a section $s'$ of $\mathcal{L}^{\otimes -1}$ over $X_s$ such that $s'(s|_{X_s}) = 1$.*

**Proof.** Suppose $x \in X_s$. We have an isomorphism

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{L}^{\otimes -1})_x \longrightarrow \mathcal{O}_{X,x}$$

by Lemma 21.2. Both $\mathcal{L}_x$ and $(\mathcal{L}^{\otimes -1})_x$ are free $\mathcal{O}_{X,x}$-modules of rank 1. We conclude from Algebra, Nakayama's Lemma 19.1 that $s_x$ is a basis for $\mathcal{L}_x$. Hence there exists a basis element $t_x \in (\mathcal{L}^{\otimes -1})_x$ such that $s_x \otimes t_x$ maps to 1. Choose an open neighbourhood $U$ of $x$ such that $t_x$ comes from a section $t$ of $(\mathcal{L}^{\otimes -1})_x$ over $U$ and such that $s \otimes t$ maps to $1 \in \mathcal{O}_X(U)$. Clearly, for every $x' \in U$ we see that $s$ generates the module $\mathcal{L}_{x'}$. Hence $U \subset X_s$. This proves that $X_s$ is open. Moreover, the section $t$ constructed over $U$ above is unique, and hence these glue to give te section $s'$ of the lemma. $\square$

It is also true that, given a morphism of locally ringed spaces $f : Y \to X$ (see Schemes, Definition 2.1) that the inverse image $f^{-1}(X_s)$ is equal to $Y_{f^*s}$, where $f^*s \in \Gamma(Y, f^*\mathcal{L})$ is the pullback of $s$.

## 22. Localizing sheaves of rings

Let $X$ be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset, see Algebra, Definition 9.1. In this case we can consider the presheaf of rings

$$\mathcal{S}^{-1}\mathcal{O}_X : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U).$$

The restriction mapping sends the section $f/s$, $f \in \mathcal{O}_X(U)$, $s \in \mathcal{S}(U)$ to $(f|_V)/(s|_V)$ if $V \subset U$ are opens of $X$.

**Lemma 22.1.** *Let $X$ be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset.*

1. *There is a map of presheaves of rings $\mathcal{O}_X \to \mathcal{S}^{-1}\mathcal{O}_X$ such that every local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{O}_X$.*
2. *For any homomorphism of presheaves of rings $\mathcal{O}_X \to \mathcal{A}$ such that each local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{A}$ there exists a unique factorization $\mathcal{S}^{-1}\mathcal{O}_X \to \mathcal{A}$.*

(3) *For any $x \in X$ we have*
$$(\mathcal{S}^{-1}\mathcal{O}_X)_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$
(4) *The sheafification $(\mathcal{S}^{-1}\mathcal{O}_X)^\#$ is a sheaf of rings with a map of sheaves of rings $(\mathcal{O}_X)^\# \to (\mathcal{S}^{-1}\mathcal{O}_X)^\#$ which is universal for maps of $(\mathcal{O}_X)^\#$ into sheaves of rings such that each local section of $\mathcal{S}$ maps to an invertible section.*
(5) *For any $x \in X$ we have*
$$(\mathcal{S}^{-1}\mathcal{O}_X)^\#_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$

**Proof.** Omitted. □

Let $X$ be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules In this case we can consider the presheaf of $\mathcal{S}^{-1}\mathcal{O}_X$-modules
$$\mathcal{S}^{-1}\mathcal{F} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{F}(U).$$
The restriction mapping sends the section $t/s$, $t \in \mathcal{F}(U)$, $s \in \mathcal{S}(U)$ to $(t|_V)/(s|_V)$ if $V \subset U$ are opens of $X$.

**Lemma 22.2.** *Let $X$ be a topological space. Let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. For any presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have*
$$\mathcal{S}^{-1}\mathcal{F} = \mathcal{S}^{-1}\mathcal{O}_X \otimes_{p,\mathcal{O}_X} \mathcal{F}$$
*(see Sheaves, Section 6 for notation) and if $\mathcal{F}$ and $\mathcal{O}_X$ are sheaves then*
$$(\mathcal{S}^{-1}\mathcal{F})^\# = (\mathcal{S}^{-1}\mathcal{O}_X)^\# \otimes_{\mathcal{O}_X} \mathcal{F}$$
*(see Sheaves, Section 20 for notation).*

**Proof.** Omitted. □

## 23. Modules of differentials

In this section we briefly explain how to define the module of relative differentials for a morphism of ringed spaces. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 127).

**Definition 23.1.** Let $X$ be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{O}_2$-module. A *$\mathcal{O}_1$-derivation* or more precisely a *$\varphi$-derivation* into $\mathcal{F}$ is a map $D : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the *Leibniz rule*
$$D(ab) = aD(b) + D(a)b$$
for all $a, b$ local sections of $\mathcal{O}_2$ (wherever they are both defined). We denote $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ the set of $\varphi$-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 23.1. Given a derivation $D : \mathcal{O}_2 \to \mathcal{F}$ as in the definition the map on global sections
$$D : \Gamma(X, \mathcal{O}_2) \longrightarrow \Gamma(X, \mathcal{F})$$

is a $\Gamma(X, \mathcal{O}_1)$-derivation as in the algebra definition. Note that if $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_2$-modules, then there is an induced map

$$\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule $D \mapsto \alpha \circ D$. In other words we obtain a functor.

**Lemma 23.2.** *Let $X$ be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The functor*

$$Mod(\mathcal{O}_2) \longrightarrow Ab, \quad \mathcal{F} \longmapsto Der_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

*is representable.*

**Proof.** This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets $\mathcal{F}$ on $X$, let us denote $\mathcal{O}_2[\mathcal{F}]$ the sheafification of the presheaf $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$ where this denotes the free $\mathcal{O}_1(U)$-module on the set $\mathcal{F}(U)$. For $s \in \mathcal{F}(U)$ we denote $[s]$ the corresponding section of $\mathcal{O}_2[\mathcal{F}]$ over $U$. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which on the presheaf level is given by the rule $\sum f_s[s] \mapsto \sum f_s s$. We will employ the short hand $[s] \mapsto s$ to describe this map and similarly for other maps below. Consider the map of $\mathcal{O}_2$-modules

$$\begin{array}{ccc} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a, b)] \oplus [(f, g)] \oplus [h] & \longmapsto & [a + b] - [a] - [b]+ \\ & & [fg] - g[f] - f[g]+ \\ & & [\varphi(h)] \end{array} \tag{23.2.1}$$

with short hand notation as above. Set $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$\mathrm{d} : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section $f$ to the image of $[f]$ in $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$. By construction d is a $\mathcal{O}_1$-derivation. Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules and let $D : \mathcal{O}_2 \to \mathcal{F}$ be a $\mathcal{O}_1$-derivation. Then we can consider the $\mathcal{O}_2$-linear map $\mathcal{O}_2[\mathcal{O}_2] \to \mathcal{F}$ which sends $[g]$ to $D(g)$. It follows from the definition of a derivation that this map annihilates sections in the image of the map (23.2.1) and hence defines a map

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

Since it is clear that $D = \alpha_D \circ \mathrm{d}$ the lemma is proved. $\square$

**Definition 23.3.** Let $X$ be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on $X$. The *module of differentials* of $\varphi$ is the object representing the functor $\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ which exists by Lemma 23.2. It is denoted $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, and the *universal $\varphi$-derivation* is denoted $\mathrm{d} : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$.

Note that $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (23.2.1) of $\mathcal{O}_2$-modules. Moreover the map d is described by the rule that $\mathrm{d}f$ is the image of the local section $[f]$.

**Lemma 23.4.** *Let $X$ be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on $X$. Then $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the sheaf associated to the presheaf $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$.*

**Proof.** Consider the map (23.2.1). There is a similar map of presheaves whose value on the open $U$ is

$$\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_1(U)] \longrightarrow \mathcal{O}_2(U)[\mathcal{O}_2(U)]$$

The cokernel of this map has value $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ over $U$ by the construction of the module of differentials in Algebra, Definition 127.2. On the other hand, the sheaves in (23.2.1) are the sheafifications of the presheaves above. Thus the result follows as sheafification is exact. $\square$

**Lemma 23.5.** *Let $X$ be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. For $U \subset X$ open there is a canonical isomorphism*

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

*compatible with universal derivations.*

**Proof.** Holds because $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (23.2.1). $\square$

**Lemma 23.6.** *Let $f : Y \to X$ be a continuous map of topological spaces. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on $X$. Then there is a canonical identification $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ compatible with universal derivations.*

**Proof.** This holds because the sheaf $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (23.2.1) and a similar statement holds for $\Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$, because the functor $f^{-1}$ is exact, and because $f^{-1}(\mathcal{O}_2[\mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2]$, $f^{-1}(\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2 \times f^{-1}\mathcal{O}_2]$, and $f^{-1}(\mathcal{O}_2[\mathcal{O}_1]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_1]$. $\square$

**Lemma 23.7.** *Let $X$ be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on $X$. Let $x \in X$. Then we have $\Omega_{\mathcal{O}_2/\mathcal{O}_1,x} = \Omega_{\mathcal{O}_{2,x}/\mathcal{O}_{1,x}}$.*

**Proof.** This is a special case of Lemma 23.6 for the inclusion map $\{x\} \to X$. An alternative proof is the use Lemma 23.4, Sheaves, Lemma 17.2, and Algebra, Lemma 127.4 $\square$

**Lemma 23.8.** *Let $X$ be a topological space. Let*

$$\begin{tikzcd} \mathcal{O}_2 \arrow[r, "\varphi"'] & \mathcal{O}_2' \\ \mathcal{O}_1 \arrow[u] \arrow[r] & \mathcal{O}_1' \arrow[u] \end{tikzcd}$$

*be a commutative diagram of sheaves of rings on $X$. The map $\mathcal{O}_2 \to \mathcal{O}_2'$ composed with the map $d : \mathcal{O}_2' \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$ is a $\mathcal{O}_1$-derivation. Hence we obtain a canonical map of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$. It is uniquely characterized by the property that $d(f)$ mapsto $d(\varphi(f))$ for any local section $f$ of $\mathcal{O}_2$. In this way $\Omega_{-/-}$ becomes a functor on the category of arrows of sheaves of rings.*

**Proof.** This lemma proves itself. $\square$

**Lemma 23.9.** *In Lemma 23.8 suppose that $\mathcal{O}_2 \to \mathcal{O}_2'$ is surjective with kernel $\mathcal{I} \subset \mathcal{O}_2$ and assume that $\mathcal{O}_1 = \mathcal{O}_1'$. Then there is a canonical exact sequence of $\mathcal{O}_2'$-modules*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}_2' \longrightarrow \Omega_{\mathcal{O}_2'/\mathcal{O}_1} \longrightarrow 0$$

*The leftmost map is characterized by the rule that a local section $f$ of $\mathcal{I}$ maps to $df \otimes 1$.*

**Proof.** For a local section $f$ of $\mathcal{I}$ denote $\overline{f}$ the image of $f$ in $\mathcal{I}/\mathcal{I}^2$. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1 f_2 \otimes 1 = 0$ if $f_1, f_2$ are local sections of $\mathcal{I}$. And this is clear from the Leibniz rule $\mathrm{d}f_1 f_2 \otimes 1 = (f_1 \mathrm{d}f_2 + f_2 \mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_2 \otimes f_1 = 0$. A similar computation show this map is $\mathcal{O}'_2 = \mathcal{O}_2/\mathcal{I}$-linear. The map on the right is the one from Lemma 23.8. To see that the sequence is exact, we can check on stalks (Lemma 3.1). By Lemma 23.7 this follows from Algebra, Lemma 127.9. $\square$

**Definition 23.10.** Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. An *$S$-derivation* into $\mathcal{F}$ is a $f^{-1}\mathcal{O}_S$-derivation, or more precisely a $f^\sharp$-derivation in the sense of Definition 23.1. We denote $\mathrm{Der}_S(\mathcal{O}_X, \mathcal{F})$ the set of $S$-derivations into $\mathcal{F}$.
(2) The *sheaf of differentials $\Omega_{X/S}$ of $X$ over $S$* is the module of differentials $\Omega_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}$ endowed with its universal $S$-derivation $\mathrm{d}_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$.

Here is a particular situation where derivations come up naturally.

**Lemma 23.11.** *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

*Here $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I} = Ker(\pi)$ is its kernel. Assume $\mathcal{I}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{I}$ has a natural structure of an $\mathcal{O}_X$-module. A section $s : \mathcal{O}_X \to \mathcal{A}$ of $\pi$ is a $f^{-1}\mathcal{O}_S$-algebra map such that $\pi \circ s = id$. Given any section $s : \mathcal{O}_X \to \mathcal{A}$ of $\pi$ and any $S$-derivation $D : \mathcal{O}_X \to \mathcal{I}$ the map*

$$s + D : \mathcal{O}_X \to \mathcal{A}$$

*is a section of $\pi$ and every section $s'$ is of the form $s + D$ for a unique $S$-derivation $D$.*

**Proof.** Recall that the $\mathcal{O}_X$-module structure on $\mathcal{I}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where $h$ is a local section of $\mathcal{O}_X$, and $\tilde{h}$ is a local lift of $h$ to a local section of $\mathcal{A}$, and $\tau$ is a local section of $\mathcal{I}$. In particular, given $s$, we may use $\tilde{h} = s(h)$. To verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned} (s+D)(ab) &= s(ab) + D(ab) \\ &= s(a)s(b) + aD(b) + D(a)b \\ &= s(a)s(b) + s(a)D(b) + D(a)s(b) \\ &= (s(a) + D(a))(s(b) + D(b)) \end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $f^{-1}\mathcal{O}_S$-algebra map because $D$ is an $S$-derivation. Conversely, given $s'$ we set $D = s' - s$. Details omitted. $\square$

**Lemma 23.12.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "h'"'] & X \arrow[d, "h"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

*be a commutative diagram of ringed spaces.*

(1) *The canonical map* $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ *composed with* $f_*d_{X'/S'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/S'}$ *is a* $S$*-derivation and we obtain a canonical map of* $\mathcal{O}_X$*-modules* $\Omega_{X/S} \to f_*\Omega_{X'/S'}$.
(2) *The commutative diagram*

$$\begin{tikzcd} f^{-1}\mathcal{O}_X \arrow[r] & \mathcal{O}_{X'} \\ f^{-1}h^{-1}\mathcal{O}_S \arrow[r] \arrow[u] & (h')^{-1}\mathcal{O}_{S'} \arrow[u] \end{tikzcd}$$

*induces by Lemmas 23.6 and 23.8 a canonical map* $f^{-1}\Omega_{X/S} \to \Omega_{X'/S'}$.

*These two maps correspond (via adjointness of* $f_*$ *and* $f^*$ *and via* $f^*\Omega_{X/S} = f^{-1}\Omega_{X/S} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'}$ *and Sheaves, Lemma 20.2) to the same* $\mathcal{O}_{X'}$*-module homomorphism*

$$c_f : f^*\Omega_{X/S} \longrightarrow \Omega_{X'/S'}$$

*which is uniquely characterized by the property that* $f^*d_{X/S}(a)$ *mapsto* $d_{X'/S'}(f^*a)$ *for any local section* $a$ *of* $\mathcal{O}_X$.

**Proof.** Omitted. □

**Lemma 23.13.** *Let*

$$\begin{tikzcd} X'' \arrow[r, "g"'] \arrow[d] & X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

*be a commutative diagram of ringed spaces. With notation as in Lemma 23.12 we have*

$$c_{f\circ g} = c_g \circ g^*c_f$$

*as maps* $(f \circ g)^*\Omega_{X/S} \to \Omega_{X''/S''}$.

**Proof.** Omitted. □

## 24. The naive cotangent complex

This section is the analogue of Algebra, Section 129 for morphisms of ringed spaces. We urge the reader to read that section first.

Let $X$ be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. In this section, for any sheaf of sets $\mathcal{E}$ on $X$ we denote $\mathcal{A}[\mathcal{E}]$ the sheafification of the presheaf $U \mapsto \mathcal{A}(U)[\mathcal{E}(U)]$. Here $\mathcal{A}(U)[\mathcal{E}(U)]$ denotes the polynomial algebra over $\mathcal{A}(U)$ whose variables correspond to the elements of $\mathcal{E}(U)$. We denote $[e] \in \mathcal{A}(U)[\mathcal{E}(U)]$ the variable corresponding to $e \in \mathcal{E}(U)$. There is a canonical surjection of $\mathcal{A}$-algebras

$$\mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{B}, \quad [b] \longmapsto b \tag{24.0.1}$$

whose kernel we denote $\mathcal{I} \subset \mathcal{A}[\mathcal{B}]$. It is a simple observation that $\mathcal{I}$ is generated by the local sections $[b][b'] - [bb']$ and $[a] - a$. According to Lemma 23.9 there is a canonical map

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B} \tag{24.0.2}$$

whose cokernel is canonically isomorphic to $\Omega_{\mathcal{B}/\mathcal{A}}$.

**Definition 24.1.** Let $X$ be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. The *naive cotangent complex* $NL_{\mathcal{B}/\mathcal{A}}$ is the chain complex (24.0.2)

$$NL_{\mathcal{B}/\mathcal{A}} = \left(\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}\right)$$

with $\mathcal{I}/\mathcal{I}^2$ placed in (homological) degree 1 and $\Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$ placed in degree 0.

This construction satisfies a functoriality similar to that discussed in Lemma 23.8 for modules of differentials. Namely, given a commutative diagram

(24.1.1)
$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[r] \arrow[u] & \mathcal{A}' \arrow[u] \end{tikzcd}$$

of sheaves of rings on $X$ there is a canonical $\mathcal{B}$-linear map of complexes

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'}$$

Namely, the maps in the commutative diagram give rise to a canonical map $\mathcal{A}[\mathcal{B}] \to \mathcal{A}'[\mathcal{B}']$ which maps $\mathcal{I}$ into $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}'[\mathcal{B}'] \to \mathcal{B}')$. Thus a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{I}'/(\mathcal{I}')^2$ and a map between modules of differentials, which together give the desired map between the naive cotangent complexes.

We can choose a different presentation of $\mathcal{B}$ as a quotient of a polynomial algebra over $\mathcal{A}$ and still obtain the same object of $D(\mathcal{B})$. To explain this, suppose that $\mathcal{E}$ is a sheaves of sets on $X$ and $\alpha : \mathcal{E} \to \mathcal{B}$ a map of sheaves of sets. Then we obtain an $\mathcal{A}$-algebra homomorphism $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. Assume this map is surjective, and let $\mathcal{J} \subset \mathcal{A}[\mathcal{E}]$ be the kernel. Set

$$NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}\right)$$

Here is the result.

**Lemma 24.2.** *In the situation above there is a canonical isomorphism $NL(\alpha) = NL_{\mathcal{B}/\mathcal{A}}$ in $D(\mathcal{B})$.*

**Proof.** Observe that $NL_{\mathcal{B}/\mathcal{A}} = NL(\mathrm{id}_{\mathcal{B}})$. Thus it suffices to show that given two maps $\alpha_i : \mathcal{E}_i \to \mathcal{B}$ as above, there is a canonical quasi-isomorphism $NL(\alpha_1) = NL(\alpha_2)$ in $D(\mathcal{B})$. To see this set $\mathcal{E} = \mathcal{E}_1 \amalg \mathcal{E}_2$ and $\alpha = \alpha_1 \amalg \alpha_2 : \mathcal{E} \to \mathcal{B}$. Set $\mathcal{J}_i = \mathrm{Ker}(\mathcal{A}[\mathcal{E}_i] \to \mathcal{B})$ and $\mathcal{J} = \mathrm{Ker}(\mathcal{A}[\mathcal{E}] \to \mathcal{B})$. We obtain maps $\mathcal{A}[\mathcal{E}_i] \to \mathcal{A}[\mathcal{E}]$ which send $\mathcal{J}_i$ into $\mathcal{J}$. Thus we obtain canonical maps of complexes

$$NL(\alpha_i) \longrightarrow NL(\alpha)$$

and it suffices to show these maps are quasi-isomorphism. To see this it suffices to check on stalks (Lemma 3.1). Here by Lemma 23.7 we see the result holds by Algebra, Lemma 129.2. $\square$

**Lemma 24.3.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $Y$. Then $f^{-1} NL_{\mathcal{B}/\mathcal{A}} = NL_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.*

**Proof.** Omitted. Hint: Use Lemma 23.6. $\square$

The cotangent complex of a morphism of ringed spaces is defined in terms of the cotangent complex we defined above.

**Definition 24.4.** The *naive cotangent complex* $NL_f = NL_{X/Y}$ of a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is $NL_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$.

## 25. Other chapters

## References

[AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, *Theorie de topos et cohomologie etale des schemas I, II, III*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer, 1971.

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[Ser55] Jean-Pierre Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278.