CHOW HOMOLOGY AND CHERN CLASSES

Contents

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1. Introduction

In this chapter we discuss Chow homology groups and the construction of chern classes of vector bundles as elements of operational Chow cohomology groups (everything with **Z**-coefficients). We follow the first few chapters of [Ful98], except that we have been less precise about the supports of the cycles involved. More classical discussions of Chow groups can be found in [Sam56], [Che58a], and [Che58b]. Of course there are many others.

To make the material a little bit more challenging we decided to treat a somewhat more general case than is usually done. Namely we assume our schemes X are locally of finite type over a fixed locally Noetherian base scheme which is universally catenary and has a given dimension function. This seems to be all that is needed to be able to define the Chow homology groups $A_*(X)$ and the action of capping with chern classes on them. This is an indication that we should be able to define these also for algebraic stacks locally of finite type over such a base.

In another chapter we will define the intersection products on $A_*(X)$ using Serre's Tor-formula in case X is nonsingular (see [Ser00], or [Ser65]) and we have a good moving lemma. See (insert future reference here).

2. Determinants of finite length modules

The material in this section is related to the material in the paper [KM76] and to the material in the thesis [Ros09]. If you have a good reference then please email stacks.project@gmail.com.

Given any field κ and any finite dimensional κ-vector space V we set $\det_\kappa(V) = \wedge^n(V)$ where $n = \dim_\kappa(V)$. We want to generalize this slightly.

Definition 2.1. Let R be a local ring with maximal ideal $\mathfrak m$ and residue field κ. Let M be a finite length R-module. Say $l = \text{length}_R(M)$.

(1) Given elements $x_1, \ldots, x_r \in M$ we denote $\langle x_1, \ldots, x_r\rangle = Rx_1 + \ldots + Rx_r$ the R-submodule of R generated by $x_1, \ldots, x_r$.
(2) We will say an l-tuple of elements $(e_1, \ldots, e_l)$ of M is *admissible* if $\mathfrak m e_i \in \langle e_1, \ldots, e_{i-1}\rangle$ for $i = 1, \ldots, l$.
(3) A *symbol* $[e_1, \ldots, e_l]$ will mean $(e_1, \ldots, e_l)$ is an admissible l-tuple.
(4) An *admissible relation* between symbols is one of the following:
    (a) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$, then $[e_1, \ldots, e_l] = 0$,
    (b) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1}\rangle$, then

$$[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l]$$

where $\overline{\lambda} \in \kappa^*$ is the image of λ in the residue field, and

(c) if $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m}e_a \subset \langle e_1, \ldots, e_{a-2}\rangle$ then
$$[e_1, \ldots, e_l] = -[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l].$$

(5) We define the *determinant of the finite length R-module M* to be
$$\det_\kappa(M) = \left\{\frac{\kappa\text{-vector space generated by symbols}}{\kappa\text{-linear combinations of admissible relations}}\right\}$$

We stress that always $l = \text{length}_R(M)$. We also stress that it does not follow that the symbol $[e_1, \ldots, e_l]$ is additive in the entries (this will typically not be the case). Before we can show that the determinant $\det_\kappa(M)$ actually has dimension 1 we have to show that it has dimension at most 1.

Lemma 2.2. *With notations as above we have* $\dim_\kappa(\det_\kappa(M)) \leq 1$.

Proof. Fix an admissible sequence $(f_1, \ldots, f_l)$ of M such that
$$\text{length}_R(\langle f_1, \ldots, f_i\rangle) = i$$
for $i = 1, \ldots, l$. Such an admissible sequence exists exactly because M has length l. We will show that any element of $\det_\kappa(M)$ is a κ-multiple of the symbol $[f_1, \ldots, f_l]$. This will prove the lemma.

Let $(e_1, \ldots, e_l)$ be an admissible sequence of M. It suffices to show that $[e_1, \ldots, e_l]$ is a multiple of $[f_1, \ldots, f_l]$. First assume that $\langle e_1, \ldots, e_l\rangle \neq M$. Then there exists an $i \in [1, \ldots, l]$ such that $e_i \in \langle e_1, \ldots, e_{i-1}\rangle$. It immediately follows from the first admissible relation that $[e_1, \ldots, e_n] = 0$ in $\det_\kappa(M)$. Hence we may assume that $\langle e_1, \ldots, e_l\rangle = M$. In particular there exists a smallest index $i \in \{1, \ldots, l\}$ such that $f_1 \in \langle e_1, \ldots, e_i\rangle$. This means that $e_i = \lambda f_1 + x$ with $x \in \langle e_1, \ldots, e_{i-1}\rangle$ and $\lambda \in R^*$. By the second admissible relation this means that $[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{i-1}, f_1, e_{i+1}, \ldots, e_l]$. Note that $\mathfrak{m}f_1 = 0$. Hence by applying the third admissible relation $i-1$ times we see that
$$[e_1, \ldots, e_l] = (-1)^{i-1}\overline{\lambda}[f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l].$$
Note that it is also the case that $\langle f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l\rangle = M$. By induction suppose we have proven that our original symbol is equal to a scalar times
$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l]$$
for some admissible sequence $(f_1, \ldots, f_j, e_{j+1}, \ldots, e_l)$ whose elements generate M, i.e., with $\langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_l\rangle = M$. Then we find the smallest i such that $f_{j+1} \in \langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_i\rangle$ and we go through the same process as above to see that
$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l] = (\text{scalar})[f_1, \ldots, f_j, f_{j+1}, e_{j+1}, \ldots, \hat{e_i}, \ldots, e_l]$$
Continuing in this vein we obtain the desired result. □

Before we show that $\det_\kappa(M)$ always has dimension 1, let us show that it agrees with the usual top exterior power in the case the module is a vector space over κ.

Lemma 2.3. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module which is annihilated by $\mathfrak{m}$. Let $l = n = \dim_\kappa(M)$. Then the map*
$$\det_\kappa(M) \longrightarrow \wedge^l_\kappa(M), \quad [e_1, \ldots, e_l] \longmapsto e_1 \wedge \ldots \wedge e_l$$
is an isomorphism.

Proof. It is clear that the rule described in the lemma gives a κ-linear map since all of the admissible relations are satisfied by the usual symbols $e_1 \wedge \ldots \wedge e_l$. It is also clearly a surjective map. Since by Lemma 2.2 the left hand side has dimension at most one we see that the map is an isomorphism. $\square$

Lemma 2.4. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module. The determinant $\det_\kappa(M)$ defined above is a κ-vector space of dimension 1. It is generated by the symbol $[f_1, \ldots, f_l]$ for any admissible sequence such that $\langle f_1, \ldots f_l \rangle = M$.*

Proof. We know $\det_\kappa(M)$ has dimension at most 1, and in fact that it is generated by $[f_1, \ldots, f_l]$, by Lemma 2.2 and its proof. We will show by induction on $l = \text{length}(M)$ that it is nonzero. For $l = 1$ it follows from Lemma 2.3. Choose a nonzero element $f \in M$ with $\mathfrak{m}f = 0$. Set $\overline{M} = M/\langle f \rangle$, and denote the quotient map $x \mapsto \overline{x}$. We will define a surjective map

$$\psi : \det_k(M) \to \det_\kappa(\overline{M})$$

which will prove the lemma since by induction the determinant of $\overline{M}$ is nonzero.

We define ψ on symbols as follows. Let $(e_1, \ldots, e_l)$ be an admissible sequence. If $f \notin \langle e_1, \ldots, e_l \rangle$ then we simply set $\psi([e_1, \ldots, e_l]) = 0$. If $f \in \langle e_1, \ldots, e_l \rangle$ then we choose an i minimal such that $f \in \langle e_1, \ldots, e_i \rangle$ and write $e_i = \lambda f + x$ for some $\lambda \in R$ and $x \in \langle e_1, \ldots, e_{i-1} \rangle$. In this case we set

$$\psi([e_1, \ldots, e_l]) = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l].$$

Note that it is indeed the case that $(\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l)$ is an admissible sequence in $\overline{M}$, so this makes sense. Let us show that extending this rule κ-linearly to linear combinations of symbols does indeed lead to a map on determinants. To do this we have to show that the admissible relations are mapped to zero.

Type (a) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1} \rangle$. Suppose that $f \in \langle e_1, \ldots, e_i \rangle$ with i minimal. Then it is immediate that $i \neq a$. Since it is also the case that $\overline{e}_a \in \langle \overline{e}_1, \ldots, \hat{\overline{e}}_i, \ldots, \overline{e}_{a-1} \rangle$ we see immediately that the same admissible relation for $\det_\kappa(\overline{M})$ forces the symbol $[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$ to be zero as desired.

Type (b) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1} \rangle$. Suppose that $f \in \langle e_1, \ldots, e_i \rangle$ with i minimal. Say $e_i = \mu f + y$ with $y \in \langle e_1, \ldots, e_{i-1} \rangle$. If $i < a$ then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda \overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. If $i > a$ then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda \overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. The interesting case is when $i = a$. In this case we have $e_a = \lambda e'_a + x = \mu f + y$. Hence also $e'_a = \lambda^{-1}(\mu f + y - x)$. Thus we see that

$$\psi([e_1, \ldots, e_l]) = \overline{\mu}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \psi(\overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l])$$

as desired.

Type (c) relations. Suppose that $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m}e_a \subset \langle e_1, \ldots, e_{a-2}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Say $e_i = \lambda f + x$ with $x \in \langle e_1, \ldots, e_{i-1}\rangle$. If $i < a-1$, then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$. Similarly, if $i > a$, then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$. If $i = a$, then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which is tautological. Finally, the interesting case is $i = a-1$. This case itself splits into two cases as to whether $f \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$ or not. If not, then we see that the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which is tautological since after switching e_{a-1} and e_a the smallest index such that f is in the becomes equal to $i' = a$ and it is again e_a which is removed. On the other hand, suppose that $f \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$. In this case we see that we can, besides the equality $e_{a-1} = \lambda f + x$ of above, also write $e_a = \mu f + y$ with $y \in \langle e_1, \ldots, e_{a-2}\rangle$. Clearly this means that both $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$ and $e_{a-1} \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$. Thus we can use relations of type (a) and the compatibility of ψ with these shown above to see that both

$$\psi([e_1, \ldots, e_l]) \quad \text{and} \quad \psi([e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l])$$

are zero, as desired.

At this point we have shown that ψ is well defined, and all that remains is to show that it is surjective. To see this let $(\overline{f}_2, \ldots, \overline{f}_l)$ be an admissible sequence in $\overline{M}$. We can choose lifts $f_2, \ldots, f_l \in M$, and then $(f, f_2, \ldots, f_l)$ is an admissible sequence in M. Since $\psi([f, f_2, \ldots, f_l]) = [f_2, \ldots, f_l]$ we win. □

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Note that if $\varphi : M \to N$ is an isomorphism of finite length R-modules, then we get an isomorphism

$$\det_\kappa(\varphi) : \det_\kappa(M) \to \det_\kappa(N)$$

simply by the rule

$$\det_\kappa(\varphi)([e_1, \ldots, e_l]) = [\varphi(e_1), \ldots, \varphi(e_l)]$$

for any symbol $[e_1, \ldots, e_l]$ for M. Hence we see that $\det_\kappa$ is a functor

$$(2.4.1) \qquad \left\{ \begin{matrix} \text{finite length } R\text{-modules} \\ \text{with isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} 1\text{-dimensional } \kappa\text{-vector spaces} \\ \text{with isomorphisms} \end{matrix} \right\}$$

This is typical for a "determinant functor" (see [Knu02]), as is the following additivity property.

Lemma 2.5. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. For every short exact sequence*

$$0 \to K \to L \to M \to 0$$

of finite length R-modules there exists a canonical isomorphism

$$\gamma_{K\to L\to M} : \det_\kappa(K) \otimes_\kappa \det_\kappa(M) \longrightarrow \det_\kappa(L)$$

defined by the rule on nonzero symbols

$$[e_1, \ldots, e_k] \otimes [\overline{f}_1, \ldots, \overline{f}_m] \longrightarrow [e_1, \ldots, e_k, f_1, \ldots, f_m]$$

with the following properties:

(1) *For every isomorphism of short exact sequences, i.e., for every commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "u"] & L \arrow[r] \arrow[d, "v"] & M \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & K' \arrow[r] & L' \arrow[r] & M' \arrow[r] & 0 \end{tikzcd}$$

with short exact rows and isomorphisms u, v, w we have

$$\gamma_{K'\to L'\to M'} \circ (\det_\kappa(u) \otimes \det_\kappa(w)) = \det_\kappa(v) \circ \gamma_{K\to L\to M},$$

(2) *for every commutative square of finite length R-modules with exact rows and columns*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

the following diagram is commutative

$$\begin{tikzcd} \det_\kappa(A) \otimes \det_\kappa(C) \otimes \det_\kappa(G) \otimes \det_\kappa(I) \arrow[r, "\gamma_{A\to B\to C} \otimes \gamma_{G\to H\to I}"'] \arrow[dd, "\epsilon"'] & \det_\kappa(B) \otimes \det_\kappa(H) \arrow[d, "\gamma_{B\to E\to H}"] \\ & \det_\kappa(E) \\ \det_\kappa(A) \otimes \det_\kappa(G) \otimes \det_\kappa(C) \otimes \det_\kappa(I) \arrow[r, "\gamma_{A\to D\to G} \otimes \gamma_{C\to F\to I}"] & \det_\kappa(D) \otimes \det_\kappa(F) \arrow[u, "\gamma_{D\to E\to F}"'] \end{tikzcd}$$

where ϵ is the switch of the factors in the tensor product times $(-1)^{cg}$ with $c = length_R(C)$ and $g = length_R(G)$, and

(3) *the map $\gamma_{K\to L\to M}$ agrees with the usual isomorphism if $0 \to K \to L \to M \to 0$ is actually a short exact sequence of κ-vector spaces.*

Proof. The significance of taking nonzero symbols in the explicit description of the map $\gamma_{K\to L\to M}$ is simply that if $(e_1, \ldots, e_l)$ is an admissible sequence in K, and $(\overline{f}_1, \ldots, \overline{f}_m)$ is an admissible sequence in M, then it is not guaranteed that

$(e_1, \ldots, e_l, f_1, \ldots, f_m)$ is an admissible sequence in L (where of course $f_i \in L$ signifies a lift of $\overline{f}_i$). However, if the symbol $[e_1, \ldots, e_l]$ is nonzero in $\det_\kappa(K)$, then necessarily $K = \langle e_1, \ldots, e_k \rangle$ (see proof of Lemma 2.2), and in this case it is true that $(e_1, \ldots, e_k, f_1, \ldots, f_m)$ is an admissible sequence. Moreover, by the admissible relations of type (b) for $\det_\kappa(L)$ we see that the value of $[e_1, \ldots, e_k, f_1, \ldots, f_m]$ in $\det_\kappa(L)$ is independent of the choice of the lifts f_i in this case also. Given this remark, it is clear that an admissible relation for $e_1, \ldots, e_k$ in K translates into an admissible relation among $e_1, \ldots, e_k, f_1, \ldots, f_m$ in L, and similarly for an admissible relation among the $\overline{f}_1, \ldots, \overline{f}_m$. Thus γ defines a linear map of vector spaces as claimed in the lemma.

By Lemma 2.4 we know $\det_\kappa(L)$ is generated by any single symbol $[x_1, \ldots, x_{k+m}]$ such that $(x_1, \ldots, x_{k+m})$ is an admissible sequence with $L = \langle x_1, \ldots, x_{k+m} \rangle$. Hence it is clear that the map $\gamma_{K \to L \to M}$ is surjective and hence an isomorphism.

Property (1) holds because

$$\begin{aligned} & \det_\kappa(v)([e_1, \ldots, e_k, f_1, \ldots, f_m]) \\ = \ & [v(e_1), \ldots, v(e_k), v(f_1), \ldots, v(f_m)] \\ = \ & \gamma_{K' \to L' \to M'}([u(e_1), \ldots, u(e_k)] \otimes [w(f_1), \ldots, w(f_m)]). \end{aligned}$$

Property (2) means that given a symbol $[\alpha_1, \ldots, \alpha_a]$ generating $\det_\kappa(A)$, a symbol $[\gamma_1, \ldots, \gamma_c]$ generating $\det_\kappa(C)$, a symbol $[\zeta_1, \ldots, \zeta_g]$ generating $\det_\kappa(G)$, and a symbol $[\iota_1, \ldots, \iota_i]$ generating $\det_\kappa(I)$ we have

$$\begin{aligned} & [\alpha_1, \ldots, \alpha_a, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_c, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_g, \tilde{\iota}_1, \ldots, \tilde{\iota}_i] \\ = \ & (-1)^{cg}[\alpha_1, \ldots, \alpha_a, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_g, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_c, \tilde{\iota}_1, \ldots, \tilde{\iota}_i] \end{aligned}$$

(for suitable lifts $\tilde{x}$ in E) in $\det_\kappa(E)$. This holds because we may use the admissible relations of type (c) cg times in the following order: move the $\tilde{\zeta}_1$ past the elements $\tilde{\gamma}_c, \ldots, \tilde{\gamma}_1$ (allowed since $\mathfrak{m}\tilde{\zeta}_1 \subset A$), then move $\tilde{\zeta}_2$ past the elements $\tilde{\gamma}_c, \ldots, \tilde{\gamma}_1$ (allowed since $\mathfrak{m}\tilde{\zeta}_2 \subset A + R\tilde{\zeta}_1$), and so on.

Part (3) of the lemma is obvious. This finishes the proof. $\square$

We can use the maps γ of the lemma to define more general maps γ as follows. Suppose that $(R, \mathfrak{m}, \kappa)$ is a local ring. Let M be a finite length R-module and suppose we are given a finite filtration (see Homology, Definition 16.1)

$$M = F^n \supset F^{n+1} \supset \ldots \supset F^{m-1} \supset F^m = 0.$$

Then there is a canonical isomorphism

$$\gamma_{(M,F)} : \bigotimes\nolimits_i \det_\kappa(F^i/F^{i+1}) \longrightarrow \det_\kappa(M)$$

well defined up to sign(!). One can make the sign explicit either by giving a well defined order of the terms in the tensor product (starting with higher indices unfortunately), and by thinking of the target category for the functor $\det_\kappa$ as the category of 1-dimensional super vector spaces. See [KM76, Section 1].

Here is another typical result for determinant functors. It is not hard to show. The tricky part is usually to show the existence of a determinant functor.

Lemma 2.6. *Let $(R, \mathfrak{m}, \kappa)$ be any local ring. The functor*

$$\det_\kappa : \left\{ \begin{matrix} \textit{finite length } R\textit{-modules} \\ \textit{with isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} 1\textit{-dimensional } \kappa\textit{-vector spaces} \\ \textit{with isomorphisms} \end{matrix} \right\}$$

endowed with the maps $\gamma_{K \to L \to M}$ is characterized by the following properties

1. *its restriction to the subcategory of modules annihilated by $\mathfrak{m}$ is isomorphic to the usual determinant functor (see Lemma 2.3), and*
2. *(1), (2) and (3) of Lemma 2.5 hold.*

Proof. Omitted. □

Lemma 2.7. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $I \subset \mathfrak{m}$ be an ideal, and set $R' = R/I$. Let $\det_{R,\kappa}$ denote the determinant functor on the category Mod^f_R finite length R-modules and denote $\det_{R',\kappa}$ the determinant on the category $Mod^f_{R'}$ of finite length R'-modules. Then $Mod^f_{R'} \subset Mod^f_R$ is a full subcategory and there exists an isomorphism of functors*

$$\det_{R,\kappa}|_{Mod^f_{R'}} = \det_{R',\kappa}$$

compatible with the isomorphisms $\gamma_{K \to L \to M}$ for either of these functors.

Proof. This can be shown by using the characterization of the pair $(\det_{R',\kappa}, \gamma)$ in Lemma 2.6. But really the isomorphism is obtained by mapping a symbol $[x_1, \ldots, x_l] \in \det_{R,\kappa}(M)$ to the corresponding symbol $[x_1, \ldots, x_l] \in \det_{R',\kappa}(M)$ which "obviously" works. □

Here is a case where we can compute the determinant of a linear map. In fact there is nothing mysterious about this in any case, see Example 2.9 for a random example.

Lemma 2.8. *Let R be a local ring with residue field κ. Let $u \in R^*$ be a unit. Let M be a module of finite length over R. Denote $u_M : M \to M$ the map multiplication by u. Then*

$$\det_\kappa(u_M) : \det_\kappa(M) \longrightarrow \det_\kappa(M)$$

is multiplication by $\overline{u}^l$ where $l = length_R(M)$ and $\overline{u} \in \kappa^*$ is the image of u.

Proof. Denote $f_M \in \kappa^*$ the element such that $\det_\kappa(u_M) = f_M \mathrm{id}_{\det_\kappa(M)}$. Suppose that $0 \to K \to L \to M \to 0$ is a short exact sequence of finite R-modules. Then we see that u_k, u_L, u_M give an isomorphism of short exact sequences. Hence by Lemma 2.5 (1) we conclude that $f_K f_M = f_L$. This means that by induction on length it suffices to prove the lemma in the case of length 1 where it is trivial. □

Example 2.9. Consider the local ring $R = \mathbf{Z}_p$. Set $M = \mathbf{Z}_p/(p^2) \oplus \mathbf{Z}_p/(p^3)$. Let $u : M \to M$ be the map given by the matrix

$$u = \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$$

where $a, b, c, d \in \mathbf{Z}_p$, and $a, d \in \mathbf{Z}_p^*$. In this case $\det_\kappa(u)$ equals multiplication by $a^2 d^3 \bmod p \in \mathbf{F}_p^*$. This can easily be seen by consider the effect of u on the symbol $[p^2 e, pe, pf, e, f]$ where $e = (0, 1) \in M$ and $f = (1, 0) \in M$.

3. Periodic complexes

Of course there is a very general notion of periodic complexes. We can require periodicity of the maps, or periodicity of the objects. We will add these here as needed. For the moment we only need the following cases.

Definition 3.1. Let R be a ring.

(1) A 2-*periodic complex* over R is given by a quadruple (M, N, φ, ψ) consisting of R-modules M, N and R-module maps $\varphi : M \to N$, $\psi : N \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow \ldots$$

is a complex. In this setting we define the *cohomology modules* of the complex to be the R-modules

$$H^0(M, N, \varphi, \psi) = \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi), \quad \text{and} \quad H^1(M, N, \varphi, \psi) = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi).$$

We say the 2-periodic complex is *exact* if the cohomology groups are zero.

(2) A $(2,1)$-*periodic complex* over R is given by a triple (M, φ, ψ) consisting of an R-module M and R-module maps $\varphi : M \to M$, $\psi : M \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots$$

is a complex. Since this is a special case of a 2-periodic complex we have its *cohomology modules* $H^0(M, \varphi, \psi)$, $H^1(M, \varphi, \psi)$ and a notion of exactness.

In the following we will use any result proved for 2-periodic complexes without further mention for $(2,1)$-periodic complexes. It is clear that the collection of 2-periodic complexes (resp. $(2,1)$-periodic complexes) forms a category with morphisms $(f, g) : (M, N, \varphi, \psi) \to (M', N', \varphi', \psi')$ pairs of morphisms $f : M \to M'$ and $g : N \to N'$ such that $\varphi' \circ f = f \circ \varphi$ and $\psi' \circ g = g \circ \psi$. In fact it is an abelian category, with kernels and cokernels as in Homology, Lemma 12.3. Also, note that a special case are the $(2,1)$-periodic complexes of the form $(M, 0, \psi)$. In this special case we have

$$H^0(M, 0, \psi) = \mathrm{Coker}(\psi), \quad \text{and} \quad H^1(M, 0, \psi) = \mathrm{Ker}(\psi).$$

Definition 3.2. Let R be a local ring. Let (M, N, φ, ψ) be a 2-periodic complex over R whose cohomology groups have finite length over R. In this case we define the *multiplicity* of (M, N, φ, ψ) to be the integer

$$e_R(M, N, \varphi, \psi) = \mathrm{length}_R(H^0(M, N, \varphi, \psi)) - \mathrm{length}_R(H^1(M, N, \varphi, \psi))$$

We will sometimes (especially in the case of a $(2,1)$-periodic complex with $\varphi = 0$) call this the *Herbrand quotient*[1].

Lemma 3.3. *Let R be a local ring.*

(1) *If (M, N, φ, ψ) is a 2-periodic complex such that M, N have finite length. Then $e_R(M, N, \varphi, \psi) = \mathrm{length}_R(M) - \mathrm{length}_R(N)$.*
(2) *If (M, φ, ψ) is a $(2,1)$-periodic complex such that M has finite length. Then $e_R(M, \varphi, \psi) = 0$.*

---

[1] If the residue field of R is finite with q elements it is customary to call the Herbrand quotient $h(M, N, \varphi, \psi) = q^{e_R(M,N,\varphi,\psi)}$ which is equal to the number of elements of H^0 divided by the number of elements of H^1.

(3) *Suppose that we have a short exact sequence of* $(2,1)$*-periodic complexes*

$$0 \to (M_1, N_1, \varphi_1, \psi_1) \to (M_2, N_2, \varphi_2, \psi_2) \to (M_3, N_3, \varphi_3, \psi_3) \to 0$$

If two out of three have cohomology modules of finite length so does the third and we have

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

Proof. Proof of (3). Abbreviate $A = (M_1, N_1, \varphi_1, \psi_1)$, $B = (M_2, N_2, \varphi_2, \psi_2)$ and $C = (M_3, N_3, \varphi_3, \psi_3)$. We have a long exact cohomology sequence

$$\ldots \to H^1(C) \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to H^1(C) \to \ldots$$

This gives a finite exact sequence

$$0 \to I \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to K \to 0$$

with $0 \to K \to H^1(C) \to I \to 0$ a filtration. By additivity of the length function (Algebra, Lemma 50.3) we see the result. The proofs of (1) and (2) are omitted. □

Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. We are going to use the determinant construction to define an invariant of this situation. See Section 2. Let us abbreviate $K_\varphi = \operatorname{Ker}(\varphi)$, $I_\varphi = \operatorname{Im}(\varphi)$, $K_\psi = \operatorname{Ker}(\psi)$, and $I_\psi = \operatorname{Im}(\psi)$. The short exact sequences

$$0 \to K_\varphi \to M \to I_\varphi \to 0, \quad 0 \to K_\psi \to M \to I_\psi \to 0$$

give isomorphisms

$$\gamma_\varphi : \det\nolimits_\kappa(K_\varphi) \otimes \det\nolimits_\kappa(I_\varphi) \longrightarrow \det\nolimits_\kappa(M), \quad \gamma_\psi : \det\nolimits_\kappa(K_\psi) \otimes \det\nolimits_\kappa(I_\psi) \longrightarrow \det\nolimits_\kappa(M),$$

see Lemma 2.5. On the other hand the exactness of the complex gives equalities $K_\varphi = I_\psi$, and $K_\psi = I_\varphi$ and hence an isomorphism

$$\sigma : \det\nolimits_\kappa(K_\varphi) \otimes \det\nolimits_\kappa(I_\varphi) \longrightarrow \det\nolimits_\kappa(K_\psi) \otimes \det\nolimits_\kappa(I_\psi)$$

by switching the factors. Using this notation we can define our invariant.

Definition 3.4. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. The *determinant of* (M, φ, ψ) is the element

$$\det\nolimits_\kappa(M, \varphi, \psi) \in \kappa^*$$

such that the composition

$$\det\nolimits_\kappa(M) \xrightarrow{\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}} \det\nolimits_\kappa(M)$$

is multiplication by $(-1)^{\text{length}_R(I_\varphi)\text{length}_R(I_\psi)} \det_\kappa(M, \varphi, \psi)$.

Remark 3.5. Here is a more down to earth description of the determinant introduced above. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Let us abbreviate $I_\varphi = \operatorname{Im}(\varphi)$, $I_\psi = \operatorname{Im}(\psi)$ as above. Assume that $\text{length}_R(I_\varphi) = a$ and $\text{length}_R(I_\psi) = b$, so that $a + b = \text{length}_R(M)$ by exactness. Choose admissible sequences $x_1, \ldots, x_a \in I_\varphi$ and $y_1, \ldots, y_b \in I_\psi$ such that the symbol $[x_1, \ldots, x_a]$ generates $\det_\kappa(I_\varphi)$ and the symbol $[x_1, \ldots, x_b]$ generates $\det_\kappa(I_\psi)$.

Choose $\tilde{x}_i \in M$ such that $\varphi(\tilde{x}_i) = x_i$. Choose $\tilde{y}_j \in M$ such that $\psi(\tilde{y}_j) = y_j$. Then $\det_\kappa(M, \varphi, \psi)$ is characterized by the equality

$$[x_1, \ldots, x_a, \tilde{y}_1, \ldots, \tilde{y}_b] = (-1)^{ab} \det_\kappa(M, \varphi, \psi)[y_1, \ldots, y_b, \tilde{x}_1, \ldots, \tilde{x}_a]$$

in $\det_\kappa(M)$. This also explains the sign.

Lemma 3.6. *Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Then*

$$\det_\kappa(M, \varphi, \psi) \det_\kappa(M, \psi, \varphi) = 1.$$

Proof. Omitted. □

Lemma 3.7. *Let R be a local ring with residue field κ. Let (M, φ, φ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, φ) is exact. Then $length_R(M) = 2 length_R(Im(\varphi))$ and*

$$\det_\kappa(M, \varphi, \psi) = (-1)^{length_R(Im(\varphi))} = (-1)^{\frac{1}{2} length_R(M)}$$

Proof. Follows directly from the sign rule in the definitions. □

Lemma 3.8. *Let R be a local ring with residue field κ. Let M be a finite length R-module.*

(1) *if $\varphi : M \to M$ is an isomorphism then* $\det_\kappa(M, \varphi, 0) = \det_\kappa(\varphi)$.
(2) *if $\psi : M \to M$ is an isomorphism then* $\det_\kappa(M, 0, \psi) = \det_\kappa(\psi)^{-1}$.

Proof. Let us prove (1). Set $\psi = 0$. Then we may, with notation as above Definition 3.4, identify $K_\varphi = I_\psi = 0$, $I_\varphi = K_\psi = M$. With these identifications, the map

$$\gamma_\varphi : \kappa \otimes \det_\kappa(M) = \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M)$$

is identified with $\det_\kappa(\varphi^{-1})$. On the other hand the map γ_ψ is identified with the identity map. Hence $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ is equal to $\det_\kappa(\varphi)$ in this case. Whence the result. We omit the proof of (2). □

Lemma 3.9. *Let R be a local ring with residue field κ. Suppose that we have a short exact sequence of $(2,1)$-periodic complexes*

$$0 \to (M_1, \varphi_1, \psi_1) \to (M_2, \varphi_2, \psi_2) \to (M_3, \varphi_3, \psi_3) \to 0$$

with all M_i of finite length, and each (M_1, φ_1, ψ_1) exact. Then

$$\det_\kappa(M_2, \varphi_2, \psi_2) = \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3).$$

in κ^*.

Proof. Let us abbreviate $I_{\varphi,i} = \mathrm{Im}(\varphi_i)$, $K_{\varphi,i} = \mathrm{Ker}(\varphi_i)$, $I_{\psi,i} = \mathrm{Im}(\psi_i)$, and $K_{\psi,i} = \mathrm{Ker}(\psi_i)$. Observe that we have a commutative square

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_{\varphi,1} \arrow[r] \arrow[d] & K_{\varphi,2} \arrow[r] \arrow[d] & K_{\varphi,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{\varphi,1} \arrow[r] \arrow[d] & I_{\varphi,2} \arrow[r] \arrow[d] & I_{\varphi,3} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

of finite length R-modules with exact rows and columns. The top row is exact since it can be identified with the sequence $I_{\psi,1} \to I_{\psi,2} \to I_{\psi,3} \to 0$ of images, and similarly for the bottom row. There is a similar diagram involving the modules $I_{\psi,i}$ and $K_{\psi,i}$. By definition $\det_\kappa(M_2, \varphi_2, \psi_2)$ corresponds, up to a sign, to the composition of the left vertical maps in the following diagram

$$\begin{tikzcd}
\det_\kappa(M_1) \otimes \det_\kappa(M_3) \arrow[r, "\gamma"] \arrow[d, "\gamma^{-1}\otimes\gamma^{-1}"] & \det_\kappa(M_2) \arrow[d, "\gamma^{-1}"] \\
\det_\kappa(K_{\varphi,1}) \otimes \det_\kappa(I_{\varphi,1}) \otimes \det_\kappa(K_{\varphi,3}) \otimes \det_\kappa(I_{\varphi,3}) \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\sigma\otimes\sigma"] & \det_\kappa(K_{\varphi,2}) \otimes \det_\kappa(I_{\varphi,2}) \arrow[d, "\sigma"] \\
\det_\kappa(K_{\psi,1}) \otimes \det_\kappa(I_{\psi,1}) \otimes \det_\kappa(K_{\psi,3}) \otimes \det_\kappa(I_{\psi,3}) \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\gamma\otimes\gamma"] & \det_\kappa(K_{\psi,2}) \otimes \det_\kappa(I_{\psi,2}) \arrow[d, "\gamma"] \\
\det_\kappa(M_1) \otimes \det_\kappa(M_3) \arrow[r, "\gamma"] & \det_\kappa(M_2)
\end{tikzcd}$$

The top and bottom squares are commutative up to sign by applying Lemma 2.5 (2). The middle square is trivially commutative (we are just switching factors). Hence we see that $\det_\kappa(M_2, \varphi_2, \psi_2) = \epsilon \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3)$ for some sign ϵ. And the sign can be worked out, namely the outer rectangle in the diagram above commutes up to

$$\begin{aligned}
\epsilon &= (-1)^{\mathrm{length}(I_{\varphi,1})\mathrm{length}(K_{\varphi,3}) + \mathrm{length}(I_{\psi,1})\mathrm{length}(K_{\psi,3})} \\
&= (-1)^{\mathrm{length}(I_{\varphi,1})\mathrm{length}(I_{\psi,3}) + \mathrm{length}(I_{\psi,1})\mathrm{length}(I_{\varphi,3})}
\end{aligned}$$

(proof omitted). It follows easily from this that the signs work out as well. □

Example 3.10. Let k be a field. Consider the ring $R = k[T]/(T^2)$ of dual numbers over k. Denote t the class of T in R. Let $M = R$ and $\varphi = ut$, $\psi = vt$ with $u, v \in k^*$. In this case $\det_k(M)$ has generator $e = [t, 1]$. We identify $I_\varphi = K_\varphi = I_\psi = K_\psi = (t)$. Then $\gamma_\varphi(t \otimes t) = u^{-1}[t, 1]$ (since $u^{-1} \in M$ is a lift of $t \in I_\varphi$) and $\gamma_\psi(t \otimes t) = v^{-1}[t, 1]$ (same reason). Hence we see that $\det_k(M, \varphi, \psi) = -u/v \in k^*$.

Example 3.11. Let $R = \mathbf{Z}_p$ and let $M = \mathbf{Z}_p/(p^l)$. Let $\varphi = p^b u$ and $\varphi = p^a v$ with $a, b \geq 0$, $a + b = l$ and $u, v \in \mathbf{Z}_p^*$. Then a computation as in Example 3.10 shows that

$$\begin{aligned} \det_{\mathbf{F}_p}(\mathbf{Z}_p/(p^l), p^b u, p^a v) &= (-1)^{ab} u^a/v^b \bmod p \\ &= (-1)^{\mathrm{ord}_p(\alpha)\mathrm{ord}_p(\beta)} \frac{\alpha^{\mathrm{ord}_p(\beta)}}{\beta^{\mathrm{ord}_p(\alpha)}} \bmod p \end{aligned}$$

with $\alpha = p^b u, \beta = p^a v \in \mathbf{Z}_p$. See Lemma 4.10 for a more general case (and a proof).

Example 3.12. Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus b}$ be $l = a+b$ dimensional. Let φ and ψ be the following diagonal matrices

$$\varphi = \mathrm{diag}(u_1, \ldots, u_a, 0, \ldots, 0), \quad \psi = \mathrm{diag}(0, \ldots, 0, v_1, \ldots, v_b)$$

with $u_i, v_j \in k^*$. In this case we have

$$\det_k(M, \varphi, \psi) = \frac{u_1 \ldots u_a}{v_1 \ldots v_b}.$$

This can be seen by a direct computation or by computing in case $l = 1$ and using the additivity of Lemma 3.9.

Example 3.13. Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus a}$ be $l = 2a$ dimensional. Let φ and ψ be the following block matrices

$$\varphi = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

with $U, V \in \mathrm{Mat}(a \times a, k)$ invertible. In this case we have

$$\det_k(M, \varphi, \psi) = (-1)^a \frac{\det(U)}{\det(V)}.$$

This can be seen by a direct computation. The case $a = 1$ is similar to the computation in Example 3.10.

Example 3.14. Let $R = k$ be a field. Let $M = k^{\oplus 4}$. Let

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_2 & 0 \end{pmatrix} \quad \varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 \end{pmatrix}$$

with $u_1, u_2, v_1, v_2 \in k^*$. Then we have

$$\det_k(M, \varphi, \psi) = -\frac{u_1 u_2}{v_1 v_2}.$$

Next we come to the analogue of the fact that the determinant of a composition of linear endomorphisms is the product of the determinants. To avoid very long formulae we write $I_\varphi = \mathrm{Im}(\varphi)$, and $K_\varphi = \mathrm{Ker}(\varphi)$ for any R-module map $\varphi : M \to M$. We also denote $\varphi\psi = \varphi \circ \psi$ for a pair of morphisms $\varphi, \psi : M \to M$.

Lemma 3.15. *Let R be a local ring with residue field κ. Let M be a finite length R-module. Let α, β, γ be endomorphisms of M. Assume that*

1. *$I_\alpha = K_{\beta\gamma}$, and similarly for any permutation of α, β, γ,*
2. *$K_\alpha = I_{\beta\gamma}$, and similarly for any permutation of α, β, γ.*

Then

1. *The triple $(M, \alpha, \beta\gamma)$ is an exact $(2,1)$-periodic complex.*

(2) *The triple* $(I_\gamma, \alpha, \beta)$ *is an exact* $(2,1)$*-periodic complex.*
(3) *The triple* $(M/K_\beta, \alpha, \gamma)$ *is an exact* $(2,1)$*-periodic complex.*
(4) *We have*

$$\det_\kappa(M, \alpha, \beta\gamma) = \det_\kappa(I_\gamma, \alpha, \beta)\det_\kappa(M/K_\beta, \alpha, \gamma).$$

Proof. It is clear that the assumptions imply part (1) of the lemma.

To see part (1) note that the assumptions imply that $I_{\gamma\alpha} = I_{\alpha\gamma}$, and similarly for kernels and any other pair of morphisms. Moreover, we see that $I_{\gamma\beta} = I_{\beta\gamma} = K_\alpha \subset I_\gamma$ and similarly for any other pair. In particular we get a short exact sequence

$$0 \to I_{\beta\gamma} \to I_\gamma \xrightarrow{\alpha} I_{\alpha\gamma} \to 0$$

and similarly we get a short exact sequence

$$0 \to I_{\alpha\gamma} \to I_\gamma \xrightarrow{\beta} I_{\beta\gamma} \to 0.$$

This proves $(I_\gamma, \alpha, \beta)$ is an exact $(2,1)$-periodic complex. Hence part (2) of the lemma holds.

To see that α, γ give well defined endomorphisms of M/K_β we have to check that $\alpha(K_\beta) \subset K_\beta$ and $\gamma(K_\beta) \subset K_\beta$. This is true because $\alpha(K_\beta) = \alpha(I_{\gamma\alpha}) = I_{\alpha\gamma\alpha} \subset I_{\alpha\gamma} = K_\beta$, and similarly in the other case. The kernel of the map $\alpha : M/K_\beta \to M/K_\beta$ is $K_{\beta\alpha}/K_\beta = I_\gamma/K_\beta$. Similarly, the kernel of $\gamma : M/K_\beta \to M/K_\beta$ is equal to I_α/K_β. Hence we conclude that (3) holds.

We introduce $r = \text{length}_R(K_\alpha)$, $s = \text{length}_R(K_\beta)$ and $t = \text{length}_R(K_\gamma)$. By the exact sequences above and our hypotheses we have $\text{length}_R(I_\alpha) = s + t$, $\text{length}_R(I_\beta) = r + t$, $\text{length}_R(I_\gamma) = r + s$, and $\text{length}(M) = r + s + t$. Choose

(1) an admissible sequence $x_1, \ldots, x_r \in K_\alpha$ generating K_α
(2) an admissible sequence $y_1, \ldots, y_s \in K_\beta$ generating K_β,
(3) an admissible sequence $z_1, \ldots, z_t \in K_\gamma$ generating K_γ,
(4) elements $\tilde{x}_i \in M$ such that $\beta\gamma\tilde{x}_i = x_i$,
(5) elements $\tilde{y}_i \in M$ such that $\alpha\gamma\tilde{y}_i = y_i$,
(6) elements $\tilde{z}_i \in M$ such that $\beta\alpha\tilde{z}_i = z_i$.

With these choices the sequence $y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t$ is an admissible sequence in I_α generating it. Hence, by Remark 3.5 the determinant $D = \det_\kappa(M, \alpha, \beta\gamma)$ is the unique element of κ^* such that

$$\begin{aligned}&[y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_s, \tilde{x}_1, \ldots, \tilde{x}_r]\\ &= (-1)^{r(s+t)} D[x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t]\end{aligned}$$

By the same remark, we see that $D_1 = \det_\kappa(M/K_\beta, \alpha, \gamma)$ is characterized by

$$[y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t, \tilde{x}_1, \ldots, \tilde{x}_r] = (-1)^{rt} D_1[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t]$$

By the same remark, we see that $D_2 = \det_\kappa(I_\gamma, \alpha, \beta)$ is characterized by

$$[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t] = (-1)^{rs} D_2[x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t]$$

Combining the formulas above we see that $D = D_1 D_2$ as desired. ☐

Lemma 3.16. *Let* R *be a local ring with residue field* κ. *Let* $\alpha : (M, \varphi, \psi) \to (M', \varphi', \psi')$ *be a morphism of* $(2,1)$*-periodic complexes over* R. *Assume*

(1) M, M' *have finite length,*
(2) (M, φ, ψ), (M', φ', ψ') *are exact,*

(3) *the maps φ, ψ induce the zero map on $K = Ker(\alpha)$, and*
(4) *the maps φ, ψ induce the zero map on $Q = Coker(\alpha)$.*

Denote $N = \alpha(M) \subset M'$. We obtain two short exact sequences of $(2,1)$-periodic complexes

$$0 \to (N, \varphi', \psi') \to (M', \varphi', \psi') \to (Q, 0, 0) \to 0$$
$$0 \to (K, 0, 0) \to (M, \varphi, \psi) \to (N, \varphi', \psi') \to 0$$

which induce two isomorphisms $\alpha_i : Q \to K$, $i = 0, 1$. Then

$$\det_\kappa(M, \varphi, \psi) = \det_\kappa(\alpha_0^{-1} \circ \alpha_1) \det_\kappa(M', \varphi', \psi')$$

In particular, if $\alpha_0 = \alpha_1$, then $\det_\kappa(M, \varphi, \psi) = \det_\kappa(M', \varphi', \psi')$.

Proof. There are (at least) two ways to prove this lemma. One is to produce an enormous commutative diagram using the properties of the determinants. The other is to use the characterization of the determinants in terms of admissible sequences of elements. It is the second approach that we will use.

First let us explain precisely what the maps α_i are. Namely, α_0 is the composition

$$\alpha_0 : Q = H^0(Q, 0, 0) \to H^1(N, \varphi', \psi') \to H^2(K, 0, 0) = K$$

and α_1 is the composition

$$\alpha_1 : Q = H^1(Q, 0, 0) \to H^2(N, \varphi', \psi') \to H^3(K, 0, 0) = K$$

coming from the boundary maps of the short exact sequences of complexes displayed in the lemma. The fact that the complexes (M, φ, ψ), (M', φ', ψ') are exact implies these maps are isomorphisms.

We will use the notation $I_\varphi = \mathrm{Im}(\varphi)$, $K_\varphi = \mathrm{Ker}(\varphi)$ and similarly for the other maps. Exactness for M and M' means that $K_\varphi = I_\psi$ and three similar equalities. We introduce $k = \mathrm{length}_R(K)$, $a = \mathrm{length}_R(I_\varphi)$, $b = \mathrm{length}_R(I_\psi)$. Then we see that $\mathrm{length}_R(M) = a + b$, and $\mathrm{length}_R(N) = a + b - k$, $\mathrm{length}_R(Q) = k$ and $\mathrm{length}_R(M') = a+b$. The exact sequences below will show that also $\mathrm{length}_R(I_{\varphi'}) = a$ and $\mathrm{length}_R(I_{\psi'}) = b$.

The assumption that $K \subset K_\varphi = I_\psi$ means that φ factors through N to give an exact sequence

$$0 \to \alpha(I_\psi) \to N \xrightarrow{\varphi\alpha^{-1}} I_\psi \to 0.$$

Here $\varphi\alpha^{-1}(x') = y$ means $x' = \alpha(x)$ and $y = \varphi(x)$. Similarly, we have

$$0 \to \alpha(I_\varphi) \to N \xrightarrow{\psi\alpha^{-1}} I_\varphi \to 0.$$

The assumption that ψ' induces the zero map on Q means that $I_{\psi'} = K_{\varphi'} \subset N$. This means the quotient $\varphi'(N) \subset I_{\varphi'}$ is identified with Q. Note that $\varphi'(N) = \alpha(I_\varphi)$. Hence we conclude there is an isomorphism

$$\varphi' : Q \to I_{\varphi'}/\alpha(I_\varphi)$$

simply described by $\varphi'(x' \bmod N) = \varphi'(x') \bmod \alpha(I_\varphi)$. In exactly the same way we get

$$\psi' : Q \to I_{\psi'}/\alpha(I_\psi)$$

Finally, note that α_0 is the composition

$$Q \xrightarrow{\varphi'} I_{\varphi'}/\alpha(I_\varphi) \xrightarrow{\psi\alpha^{-1}|_{I_{\varphi'}/\alpha(I_\varphi)}} K$$

and similarly $\alpha_1 = \varphi\alpha^{-1}|_{I_{\psi'}/\alpha(I_\psi)} \circ \psi'$.

To shorten the formulas below we are going to write αx instead of $\alpha(x)$ in the following. No confusion should result since all maps are indicated by greek letters and elements by roman letters. We are going to choose

(1) an admissible sequence $z_1, \ldots, z_k \in K$ generating K,
(2) elements $z'_i \in M$ such that $\varphi z'_i = z_i$,
(3) elements $z''_i \in M$ such that $\psi z''_i = z_i$,
(4) elements $x_{k+1}, \ldots, x_a \in I_\varphi$ such that $z_1, \ldots, z_k, x_{k+1}, \ldots, x_a$ is an admissible sequence generating I_φ,
(5) elements $\tilde{x}_i \in M$ such that $\varphi\tilde{x}_i = x_i$,
(6) elements $y_{k+1}, \ldots, y_b \in I_\psi$ such that $z_1, \ldots, z_k, y_{k+1}, \ldots, y_b$ is an admissible sequence generating I_ψ,
(7) elements $\tilde{y}_i \in M$ such that $\psi\tilde{y}_i = y_i$, and
(8) elements $w_1, \ldots, w_k \in M'$ such that $w_1 \bmod N, \ldots, w_k \bmod N$ are an admissible sequence in Q generating Q.

By Remark 3.5 the element $D = \det_\kappa(M, \varphi, \psi) \in \kappa^*$ is characterized by

$$\begin{aligned} & [z_1, \ldots, z_k, x_{k+1}, \ldots, x_a, z''_1, \ldots, z''_k, \tilde{y}_{k+1}, \ldots, \tilde{y}_b] \\ = \quad & (-1)^{ab} D[z_1, \ldots, z_k, y_{k+1}, \ldots, y_b, z'_1, \ldots, z'_k, \tilde{x}_{k+1}, \ldots, \tilde{x}_a] \end{aligned}$$

Note that by the discussion above $\alpha x_{k+1}, \ldots, \alpha x_a, \varphi w_1, \ldots, \varphi w_k$ is an admissible sequence generating $I_{\varphi'}$ and $\alpha y_{k+1}, \ldots, \alpha y_b, \psi w_1, \ldots, \psi w_k$ is an admissible sequence generating $I_{\psi'}$. Hence by Remark 3.5 the element $D' = \det_\kappa(M', \varphi', \psi') \in \kappa^*$ is characterized by

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b, w_1, \ldots, w_k] \\ = \quad & (-1)^{ab} D'[\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a, w_1, \ldots, w_k] \end{aligned}$$

Note how in the first, resp. second displayed formula the the first, resp. last k entries of the symbols on both sides are the same. Hence these formulas are really equivalent to the equalities

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z''_1, \ldots, \alpha z''_k, \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b] \\ = \quad & (-1)^{ab} D[\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z'_1, \ldots, \alpha z'_k, \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a] \end{aligned}$$

and

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b] \\ = \quad & (-1)^{ab} D'[\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a] \end{aligned}$$

in $\det_\kappa(N)$. Note that $\varphi' w_1, \ldots, \varphi' w_k$ and $\alpha z''_1, \ldots, z''_k$ are admissible sequences generating the module $I_{\varphi'}/\alpha(I_\varphi)$. Write

$$[\varphi' w_1, \ldots, \varphi' w_k] = \lambda_0[\alpha z''_1, \ldots, \alpha z''_k]$$

in $\det_\kappa(I_{\varphi'}/\alpha(I_\varphi))$ for some $\lambda_0 \in \kappa^*$. Similarly, write

$$[\psi' w_1, \ldots, \psi' w_k] = \lambda_1[\alpha z'_1, \ldots, \alpha z'_k]$$

in $\det_\kappa(I_{\psi'}/\alpha(I_\psi))$ for some $\lambda_1 \in \kappa^*$. On the one hand it is clear that

$$\alpha_i([w_1, \ldots, w_k]) = \lambda_i[z_1, \ldots, z_k]$$

for $i = 0, 1$ by our description of α_i above, which means that

$$\det_\kappa(\alpha_0^{-1} \circ \alpha_1) = \lambda_1/\lambda_0$$

and on the other hand it is clear that

$$\begin{aligned} & \lambda_0[\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z_1'', \ldots, \alpha z_k'', \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \\ = \quad & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \end{aligned}$$

and

$$\begin{aligned} & \lambda_1[\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z_1', \ldots, \alpha z_k', \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \\ = \quad & [\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \end{aligned}$$

which imply $\lambda_0 D = \lambda_1 D'$. The lemma follows. □

4. Symbols

The correct generality for this construction is perhaps the situation of the following lemma.

Lemma 4.1. *Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Assume $\varphi, \psi : M \to M$ are two injective A-module maps, and assume $\varphi(\psi(M)) = \psi(\varphi(M))$, for example if φ and ψ commute. Then $\text{length}_R(M/\varphi\psi M) < \infty$ and $(M/\varphi\psi M, \varphi, \psi)$ is an exact $(2,1)$-periodic complex.*

Proof. Let $\mathfrak{q}$ be a minimal prime of the support of M. Then $M_{\mathfrak{q}}$ is a finite length $A_{\mathfrak{q}}$-module, see Algebra, Lemma 61.3. Hence both φ and ψ induce isomorphisms $M_{\mathfrak{q}} \to M_{\mathfrak{q}}$. Thus the support of $M/\varphi\psi M$ is $\{\mathfrak{m}_A\}$ and hence it has finite length (see lemma cited above). Finally, the kernel of φ on $M/\varphi\psi M$ is clearly $\psi M/\varphi\psi M$, and hence the kernel of φ is the image of ψ on $M/\varphi\psi M$. Similarly the other way since $M/\varphi\psi M = M/\psi\varphi M$ by assumption. □

Lemma 4.2. *Let A be a Noetherian local ring. Let $a, b \in A$.*

1. *If M is a finite A-module of dimension 1 such that a, b are nonzerodivisors on M, then $\text{length}_A(M/abM) < \infty$ and $(M/abM, a, b)$ is a $(2,1)$-periodic exact complex.*
2. *If a, b are nonzerodivisors and $\dim(A) = 1$ then $\text{length}_A(A/(ab)) < \infty$ and $(A/(ab), a, b)$ is a $(2,1)$-periodic exact complex.*

In particular, in these cases $\det_\kappa(M/abM, a, b) \in \kappa^*$, resp. $\det_\kappa(A/(ab), a, b) \in \kappa^*$ are defined.

Proof. Follows from Lemma 4.1. □

Definition 4.3. Let A be a Noetherian local ring with residue field κ. Let $a, b \in A$. Let M be a finite A-module of dimension 1 such that a, b are nonzerodivisors on M. We define the *symbol associated to M, a, b* to be the element

$$d_M(a, b) = \det_\kappa(M/abM, a, b) \in \kappa^*$$

Lemma 4.4. *Let A be a Noetherian local ring. Let $a, b, c \in A$. Let M be a finite A-module with $\dim(M) = 1$. Assume a, b, c are nonzerodivisors on M. Then*

$$d_M(a, bc) = d_M(a, b) d_M(a, c)$$

and $d_M(a, b) d_M(b, a) = 1$.

Proof. The first statement is immediate from Lemma 3.15 above. The second comes from Lemma 3.6. □

Definition 4.5. Let A be a Noetherian local domain of dimension 1 with residue field κ. Let K be the fraction field of A. We define the *tame symbol* of A to be the map

$$K^* \times K^* \longrightarrow \kappa^*, \quad (x, y) \longmapsto d_A(x, y)$$

where $d_A(x, y)$ is extended to $K^* \times K^*$ by the multiplicativity of Lemma 4.4.

It is clear that we may extend more generally $d_M(-, -)$ to certain rings of fractions of A (even if A is not a domain).

Lemma 4.6. *Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Let $b \in A$ be a nonzerodivisor on M, and let $u \in A^*$. Then*

$$d_M(u, b) = u^{length_M(M/bM)} \bmod \mathfrak{m}_A.$$

In particular, if $M = A$, then $d_A(u, b) = u^{ord_A(b)} \bmod \mathfrak{m}_A$.

Proof. Note that in this case $M/ubM = M/bM$ on which multiplication by b is zero. Hence $d_M(u, b) = \det_\kappa(u|_{M/bM})$ by Lemma 3.8. The lemma then follows from Lemma 2.8. $\square$

Lemma 4.7. *Let A be a Noetherian local ring. Let $a, b \in A$. Let*

$$0 \to M \to M' \to M'' \to 0$$

be a short exact sequence of A-modules of dimension 1 such that a, b are nonzerodivisors on all three A-modules. Then

$$d_{M'}(a, b) = d_M(a, b) d_{M''}(a, b)$$

in κ^*.

Proof. It is easy to see that this leads to a short exact sequence of exact $(2, 1)$-periodic complexes

$$0 \to (M/abM, a, b) \to (M'/abM', a, b) \to (M''/abM'', a, b) \to 0$$

Hence the lemma follows from Lemma 3.9. $\square$

Lemma 4.8. *Let A be a Noetherian local ring. Let $\alpha : M \to M'$ be a homomorphism of finite A-modules of dimension 1. Let $a, b \in A$. Assume*

1. *a, b are nonzerodivisors on both M and M', and*
2. $\dim(\mathit{Ker}(\alpha)), \dim(\mathit{Coker}(\alpha)) \leq 0$.

Then $d_M(a, b) = d_{M'}(a, b)$.

Proof. If $a \in A^*$, then the equality follows from the equality $\text{length}(M/bM) = \text{length}(M'/bM')$ and Lemma 4.6. Similarly if b is a unit the lemma holds as well (by the symmetry of Lemma 4.4). Hence we may assume that $a, b \in \mathfrak{m}_A$. This in particular implies that $\mathfrak{m}$ is not an associated prime of M, and hence $\alpha : M \to M'$ is injective. This permits us to think of M as a submodule of M'. By assumption M'/M is a finite A-module with support $\{\mathfrak{m}_A\}$ and hence has finite length. Note that for any third module M'' with $M \subset M'' \subset M'$ the maps $M \to M''$ and $M'' \to M'$ satisfy the assumptions of the lemma as well. This reduces us, by induction on the length of M'/M, to the case where $\text{length}_A(M'/M) = 1$. Finally, in this case consider the map

$$\overline{\alpha} : M/abM \longrightarrow M'/abM'.$$

By construction the cokernel Q of $\overline{\alpha}$ has length 1. Since $a, b \in \mathfrak{m}_A$, they act trivially on Q. It also follows that the kernel K of $\overline{\alpha}$ has length 1 and hence also a, b act trivially on K. Hence we may apply Lemma 3.16. Thus it suffices to see that the two maps $\alpha_i : Q \to K$ are the same. In fact, both maps are equal to the map $q = x' \bmod \mathrm{Im}(\overline{\alpha}) \mapsto abx' \in K$. We omit the verification. $\square$

Lemma 4.9. *Let A be a Noetherian local ring. Let M be a finite A-module with $\dim(M) = 1$. Let $a, b \in A$ nonzerodivisors on M. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then*

$$d_M(a,b) = \prod\nolimits_{i=1,\ldots,t} d_{A/\mathfrak{q}_i}(a,b)^{length_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})}$$

as elements of κ^*.

Proof. Choose a filtration by A-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_j/M_{j-1} is isomorphic to $A/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j$ of A. See Algebra, Lemma 61.1. For each j we have either $\mathfrak{p}_j = \mathfrak{q}_i$ for some i, or $\mathfrak{p}_j = \mathfrak{m}_A$. Moreover, for a fixed i, the number of j such that $\mathfrak{p}_j = \mathfrak{q}_i$ is equal to $\mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$ by Algebra, Lemma 61.5. Hence $d_{M_j}(a,b)$ is defined for each j and

$$d_{M_j}(a,b) = \begin{cases} d_{M_{j-1}}(a,b) d_{A/\mathfrak{q}_i}(a,b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{q}_i \\ d_{M_{j-1}}(a,b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{m}_A \end{cases}$$

by Lemma 4.7 in the first instance and Lemma 4.8 in the second. Hence the lemma. $\square$

Lemma 4.10. *Let A be a discrete valuation ring with fraction field K. For nonzero $x, y \in K$ we have*

$$d_A(x,y) = (-1)^{ord_A(x) ord_A(y)} \frac{x^{ord_A(y)}}{y^{ord_A(x)}} \bmod \mathfrak{m}_A,$$

in other words the symbol is equal to the usual tame symbol.

Proof. By multiplicativity it suffices to prove this when $x, y \in A$. Let $t \in A$ be a uniformizer. Write $x = t^b u$ and $y = t^b v$ for some $a, b \geq 0$ and $u, v \in A^*$. Set $l = a + b$. Then $t^{l-1}, \ldots, t^b$ is an admissible sequence in $(x)/(xy)$ and $t^{l-1}, \ldots, t^a$ is an admissible sequence in $(y)/(xy)$. Hence by Remark 3.5 we see that $d_A(x,y)$ is characterized by the equation

$$[t^{l-1}, \ldots, t^b, v^{-1}t^{b-1}, \ldots, v^{-1}] = (-1)^{ab} d_A(x,y) [t^{l-1}, \ldots, t^a, u^{-1}t^{a-1}, \ldots, u^{-1}].$$

Hence by the admissible relations for the symbols $[x_1, \ldots, x_l]$ we see that

$$d_A(x,y) = (-1)^{ab} u^a / v^b \bmod \mathfrak{m}_A$$

as desired. $\square$

We add the following lemma here. It is very similar to Algebra, Lemma 115.2.

Lemma 4.11. *Let R be a local Noetherian domain of dimension 1 with maximal ideal $\mathfrak{m}$. Let $a, b \in \mathfrak{m}$ be nonzero. There exists a finite ring extension $R \subset R'$ with same field of fractions, and $t, a', b' \in R'$ such that $a = ta'$ and $b = tb'$ and $R' = a'R' + b'R'$.*

Proof. Set $I = (a, b)$. The idea is to blow up R in I as in the proof of Algebra, Lemma 115.2. Instead of doing the algebraic argument we work geometrically. Let $X = \text{Proj}(\bigoplus I^d/I^{d+1})$. By Divisors, Lemma 18.7 this is an integral scheme. The morphism $X \to \text{Spec}(R)$ is projective by Divisors, Lemma 18.11. By Algebra, Lemma 109.2 and the fact that X is quasi-compact we see that the fibre of $X \to \text{Spec}(R)$ over $\mathfrak{m}$ is finite. By Properties, Lemma 27.5 there exists an affine open $U \subset X$ containing this fibre. Hence $X = U$ because $X \to \text{Spec}(R)$ is closed. In other words X is affine, say $X = \text{Spec}(R')$. By Morphisms, Lemma 16.2 we see that $R \to R'$ is of finite type. Since $X \to \text{Spec}(R)$ is proper and affine it is integral (see Morphisms, Lemma 44.7). Hence $R \to R'$ is of finite type and integral, hence finite (Algebra, Lemma 35.5). By Divisors, Lemma 18.4 we see that IR' is a locally principal ideal. Since R' is semi-local we see that IR' is principal, see Algebra, Lemma 75.6, say $IR' = (t)$. Then we have $a = a't$ and $b = b't$ and everything is clear. □

Lemma 4.12. *Let A be a Noetherian local ring. Let $a, b \in A$. Let M be a finite A-module of dimension 1 on which each of a, b, $b - a$ are nonzerodivisors. Then*

$$d_M(a, b - a)d_M(b, b) = d_M(b, b - a)d_M(a, b)$$

in κ^*.

Proof. By Lemma 4.9 it suffices to show the relation when $M = A/\mathfrak{q}$ for some prime $\mathfrak{q} \subset A$ with $\dim(A/\mathfrak{q}) = 1$.

In case $M = A/\mathfrak{q}$ we may replace A by $A/\mathfrak{q}$ and a, b by their images in $A/\mathfrak{q}$. Hence we may assume $A = M$ and A a local Noetherian domain of dimension 1. The reason is that the residue field κ of A and $A/\mathfrak{q}$ are the same and that for any $A/\mathfrak{q}$-module M the determinant taken over A or over $A/\mathfrak{q}$ are canonically identified. See Lemma 2.7.

It suffices to show the relation when both a, b are in the maximal ideal. Namely, the case where one or both are units follows from Lemma 4.6.

Choose an extension $A \subset A'$ and factorizations $a = ta'$, $b = tb'$ as in Lemma 4.11. Note that also $b - a = t(b' - a')$ and that $A' = (a', b') = (a', b' - a') = (b' - a', b')$. Here and in the following we think of A' as an A-module and a, b, a', b', t as A-module endomorphisms of A'. We will use the notation $d^A_{A'}(a', b')$ and so on to indicate

$$d^A_{A'}(a', b') = \det_\kappa(A'/a'b'A', a', b')$$

which is defined by Lemma 4.1. The upper index ^{A} is used to distinguish this from the already defined symbol $d_{A'}(a', b')$ which is different (for example because it has values in the residue field of A' which may be different from κ). By Lemma 4.8 we see that $d_A(a, b) = d^A_{A'}(a, b)$, and similarly for the other combinations. Using this and multiplicativity we see that it suffices to prove

$$d^A_{A'}(a', b' - a')d^A_{A'}(b', b') = d^A_{A'}(b', b' - a')d^A_{A'}(a', b')$$

Now, since $(a', b') = A'$ and so on we have

$$\begin{array}{ccc} A'/(a'(b' - a')) & \cong & A'/(a') \oplus A'/(b' - a') \\ A'/(b'(b' - a')) & \cong & A'/(b') \oplus A'/(b' - a') \\ A'/(a'b') & \cong & A'/(a') \oplus A'/(b') \end{array}$$

Moreover, note that multiplication by $b' - a'$ on $A/(a')$ is equal to multiplication by b', and that multiplication by $b' - a'$ on $A/(b')$ is equal to multiplication by $-a'$. Using Lemmas 3.8 and 3.9 we conclude

$$\begin{array}{ccc} d^A_{A'}(a', b' - a') & = & \det_\kappa(b'|_{A'/(a')})^{-1}\det_\kappa(a'|_{A'/(b'-a')}) \\ d^A_{A'}(b', b' - a') & = & \det_\kappa(-a'|_{A'/(b')})^{-1}\det_\kappa(b'|_{A'/(b'-a')}) \\ d^A_{A'}(a', b') & = & \det_\kappa(b'|_{A'/(a')})^{-1}\det_\kappa(a'|_{A'/(b')}) \end{array}$$

Hence we conclude that

$$(-1)^{\text{length}_A(A'/(b'))}d^A_{A'}(a', b' - a') = d^A_{A'}(b', b' - a')d^A_{A'}(a', b')$$

the sign coming from the $-a'$ in the second equality above. On the other hand, by Lemma 3.7 we have $d^A_{A'}(b', b') = (-1)^{\text{length}_A(A'/(b'))}$, and the lemma is proved. $\square$

The tame symbol is a Steinberg symbol.

Lemma 4.13. *Let A be a Noetherian local domain of dimension 1. Let $K = f.f.(A)$. For $x \in K \setminus \{0, 1\}$ we have*

$$d_A(x, 1 - x) = 1$$

Proof. Write $x = a/b$ with $a, b \in A$. The hypothesis implies, since $1-x = (b-a)/b$, that also $b - a \neq 0$. Hence we compute

$$d_A(x, 1 - x) = d_A(a, b - a)d_A(a, b)^{-1}d_A(b, b - a)^{-1}d_A(b, b)$$

Thus we have to show that $d_A(a, b - a)d_A(b, b) = d_A(b, b - a)d_A(a, b)$. This is Lemma 4.12. $\square$

5. Lengths and determinants

In this section we use the determinant to compare lattices. The key lemma is the following.

Lemma 5.1. *Let R be a noetherian local ring. Let $\mathfrak{q} \subset R$ be a prime with $\dim(R/\mathfrak{q}) = 1$. Let $\varphi : M \to N$ be a homomorphism of finite R-modules. Assume there exist $x_1, \ldots, x_l \in M$ and $y_1, \ldots, y_l \in M$ with the following properties*

1. *$M = \langle x_1, \ldots, x_l\rangle$,*
2. *$\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle \cong R/\mathfrak{q}$ for $i = 1, \ldots, l$,*
3. *$N = \langle y_1, \ldots, y_l\rangle$, and*
4. *$\langle y_1, \ldots, y_i\rangle/\langle y_1, \ldots, y_{i-1}\rangle \cong R/\mathfrak{q}$ for $i = 1, \ldots, l$.*

Then φ is injective if and only if $\varphi_\mathfrak{q}$ is an isomorphism, and in this case we have

$$\mathit{length}_R(\mathit{Coker}(\varphi)) = \mathit{ord}_{R/\mathfrak{q}}(f)$$

where $f \in \kappa(\mathfrak{q})$ is the element such that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l]$$

in $\det_{\kappa(\mathfrak{q})}(N_\mathfrak{q})$.

Proof. First, note that the lemma holds in case $l = 1$. Namely, in this case x_1 is a basis of M over $R/\mathfrak{q}$ and y_1 is a basis of N over $R/\mathfrak{q}$ and we have $\varphi(x_1) = fy_1$ for some $f \in R$. Thus φ is injective if and only if $f \notin \mathfrak{q}$. Moreover, $\text{Coker}(\varphi) = R/(f, \mathfrak{q})$ and hence the lemma holds by definition of $\text{ord}_{R/q}(f)$ (see Algebra, Definition 117.2).

In fact, suppose more generally that $\varphi(x_i) = f_i y_i$ for some $f_i \in R$, $f_i \not\in \mathfrak q$. Then the induced maps

$$\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle \longrightarrow \langle y_1, \ldots, y_i\rangle/\langle y_1, \ldots, y_{i-1}\rangle$$

are all injective and have cokernels isomorphic to $R/(f_i, \mathfrak q)$. Hence we see that

$$\text{length}_R(\text{Coker}(\varphi)) = \sum \text{ord}_{R/\mathfrak q}(f_i).$$

On the other hand it is clear that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f_1 \ldots f_l [y_1, \ldots, y_l]$$

in this case from the admissible relation (b) for symbols. Hence we see the result holds in this case also.

We prove the general case by induction on l. Assume $l > 1$. Let $i \in \{1, \ldots, l\}$ be minimal such that $\varphi(x_1) \in \langle y_1, \ldots, y_i\rangle$. We will argue by induction on i. If $i = 1$, then we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \langle x_1\rangle \arrow[r] \arrow[d] & \langle x_1, \ldots, x_l\rangle \arrow[r] \arrow[d] & \langle x_1, \ldots, x_l\rangle/\langle x_1\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \langle y_1\rangle \arrow[r] & \langle y_1, \ldots, y_l\rangle \arrow[r] & \langle y_1, \ldots, y_l\rangle/\langle y_1\rangle \arrow[r] & 0
\end{tikzcd}$$

and the lemma follows from the snake lemma and induction on l. Assume now that $i > 1$. Write $\varphi(x_1) = a_1 y_1 + \ldots + a_{i-1} y_{i-1} + a y_i$ with $a_j, a \in R$ and $a \not\in \mathfrak q$ (since otherwise i was not minimal). Set

$$x'_j = \begin{cases} x_j & \text{if} \quad j = 1 \\ a x_j & \text{if} \quad j \geq 2 \end{cases} \quad \text{and} \quad y'_j = \begin{cases} y_j & \text{if} \quad j < i \\ a y_j & \text{if} \quad j \geq i \end{cases}$$

Let $M' = \langle x'_1, \ldots, x'_l\rangle$ and $N' = \langle y'_1, \ldots, y'_l\rangle$. Since $\varphi(x'_1) = a_1 y'_1 + \ldots + a_{i-1} y'_{i-1} + y'_i$ by construction and since for $j > 1$ we have $\varphi(x'_j) = a\varphi(x_i) \in \langle y'_1, \ldots, y'_l\rangle$ we get a commutative diagram of R-modules and maps

$$\begin{tikzcd}
M' \arrow[r, "\varphi'"'] \arrow[d] & N' \arrow[d] \\
M \arrow[r, "\varphi"] & N
\end{tikzcd}$$

By the result of the second paragraph of the proof we know that $\text{length}_R(M/M') = (l-1)\text{ord}_{R/\mathfrak q}(a)$ and similarly $\text{length}_R(M/M') = (l-i+1)\text{ord}_{R/\mathfrak q}(a)$. By a diagram chase this implies that

$$\text{length}_R(\text{Coker}(\varphi')) = \text{length}_R(\text{Coker}(\varphi)) + i \ \text{ord}_{R/\mathfrak q}(a).$$

On the other hand, it is clear that writing

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l], \quad [\varphi'(x'_1), \ldots, \varphi(x'_l)] = f'[y'_1, \ldots, y'_l]$$

we have $f' = a^i f$. Hence it suffices to prove the lemma for the case that $\varphi(x_1) = a_1 y_1 + \ldots a_{i-1} y_{i-1} + y_i$, i.e., in the case that $a = 1$. Next, recall that

$$[y_1, \ldots, y_l] = [y_1, \ldots, y_{i-1}, a_1 y_1 + \ldots a_{i-1} y_{i-1} + y_i, y_{i+1}, \ldots, y_l]$$

by the admissible relations for symbols. The sequence $y_1, \ldots, y_{i-1}, a_1 y_1 + \ldots + a_{i-1} y_{i-1} + y_i, y_{i+1}, \ldots, y_l$ satisfies the conditions (3), (4) of the lemma also. Hence, we may actually assume that $\varphi(x_1) = y_i$. In this case, note that we have $\mathfrak{q}x_1 = 0$ which implies also $\mathfrak{q}y_i = 0$. We have

$$[y_1, \ldots, y_l] = -[y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l]$$

by the third of the admissible relations defining $\det_{\kappa(\mathfrak{q})}(N_{\mathfrak{q}})$. Hence we may replace $y_1, \ldots, y_l$ by the sequence $y'_1, \ldots, y'_l = y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l$ (which also satisfies conditions (3) and (4) of the lemma). Clearly this decreases the invariant i by 1 and we win by induction on i. □

To use the previous lemma we show that often sequences of elements with the required properties exist.

Lemma 5.2. *Let R be a local Noetherian ring. Let $\mathfrak{q} \subset R$ be a prime ideal. Let M be a finite R-module such that $\mathfrak{q}$ is one of the minimal primes of the support of M. Then there exist $x_1, \ldots, x_l \in M$ such that*

1. *the support of $M/\langle x_1, \ldots, x_l\rangle$ does not contain $\mathfrak{q}$, and*
2. *$\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle \cong R/\mathfrak{q}$ for $i = 1, \ldots, l$.*

Moreover, in this case $l = length_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$.

Proof. The condition that $\mathfrak{q}$ is a minimal prime in the support of M implies that $l = \text{length}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$ is finite (see Algebra, Lemma 61.3). Hence we can find $y_1, \ldots, y_l \in M_{\mathfrak{q}}$ such that $\langle y_1, \ldots, y_i\rangle/\langle y_1, \ldots, y_{i-1}\rangle \cong \kappa(\mathfrak{q})$ for $i = 1, \ldots, l$. We can find $f_i \in R$, $f_i \notin \mathfrak{q}$ such that $f_i y_i$ is the image of some element $z_i \in M$. Moreover, as R is Noetherian we can write $\mathfrak{q} = (g_1, \ldots, g_t)$ for some $g_j \in R$. By assumption $g_j y_i \in \langle y_1, \ldots, y_{i-1}\rangle$ inside the module $M_{\mathfrak{q}}$. By our choice of z_i we can find some further elements $f_{ji} \in R$, $f_{ij} \notin \mathfrak{q}$ such that $f_{ij} g_j z_i \in \langle z_1, \ldots, z_{i-1}\rangle$ (equality in the module M). The lemma follows by taking

$$x_1 = f_{11} f_{12} \ldots f_{1t} z_1, \quad x_2 = f_{11} f_{12} \ldots f_{1t} f_{21} f_{22} \ldots f_{2t} z_2,$$

and so on. Namely, since all the elements f_i, f_{ij} are invertible in $R_{\mathfrak{q}}$ we still have that $R_{\mathfrak{q}}x_1 + \ldots + R_{\mathfrak{q}}x_i / R_{\mathfrak{q}}x_1 + \ldots + R_{\mathfrak{q}}x_{i-1} \cong \kappa(\mathfrak{q})$ for $i = 1, \ldots, l$. By construction, $\mathfrak{q}x_i \in \langle x_1, \ldots, x_{i-1}\rangle$. Thus $\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle$ is an R-module generated by one element, annihilated $\mathfrak{q}$ such that localizing at $\mathfrak{q}$ gives a q-dimensional vector space over $\kappa(\mathfrak{q})$. Hence it is isomorphic to $R/\mathfrak{q}$. □

Here is the main result of this section. We will see below the various different consequences of this proposition. The reader is encouraged to first prove the easier Lemma 5.4 his/herself.

Proposition 5.3. *Let R be a local Noetherian ring with residue field κ. Suppose that (M, φ, ψ) is a $(2,1)$-periodic complex over R. Assume*

1. *M is a finite R-module,*
2. *the cohomology modules of (M, φ, ψ) are of finite length, and*
3. *$\dim(Supp(M)) = 1$.*

Let $\mathfrak{q}_i$, $i = 1, \ldots, t$ be the minimal primes of the support of M. Then we have[2]

$$-e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} ord_{R/\mathfrak{q}_i}\left(\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi_{\mathfrak{q}_i}, \psi_{\mathfrak{q}_i})\right)$$

[2] Obviously we could get rid of the minus sign by redefining $\det_\kappa(M, \varphi, \psi)$ as the inverse of its current value, see Definition 3.4.

Proof. We first reduce to the case $t = 1$ in the following way. Note that $\text{Supp}(M) = \{\mathfrak m, \mathfrak q_1, \ldots, \mathfrak q_t\}$, where $\mathfrak m \subset R$ is the maximal ideal. Let M_i denote the image of $M \to M_{\mathfrak q_i}$, so $\text{Supp}(M_i) = \{\mathfrak m, \mathfrak q_i\}$. The map φ (resp. ψ) induces an R-module map $\varphi_i : M_i \to M_i$ (resp. $\psi_i : M_i \to M_i$). Thus we get a morphism of $(2,1)$-periodic complexes

$$(M, \varphi, \psi) \longrightarrow \bigoplus\nolimits_{i=1,\ldots,t} (M_i, \varphi_i, \psi_i).$$

The kernel and cokernel of this map have support equal to $\{\mathfrak m\}$ (or are zero). Hence by Lemma 3.3 these $(2,1)$-periodic complexes have multiplicity 0. In other words we have

$$e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} e_R(M_i, \varphi_i, \psi_i)$$

On the other hand we clearly have $M_{\mathfrak q_i} = M_{i,\mathfrak q_i}$, and hence the terms of the right hand side of the formula of the lemma are equal to the expressions

$$\text{ord}_{R/\mathfrak q_i}\left(\det\nolimits_{\kappa(\mathfrak q_i)}(M_{i,\mathfrak q_i}, \varphi_{i,\mathfrak q_i}, \psi_{i,\mathfrak q_i})\right)$$

In other words, if we can prove the lemma for each of the modules M_i, then the lemma holds. This reduces us to the case $t = 1$.

Assume we have a $(2,1)$-periodic complex (M, φ, ψ) over a Noetherian local ring with M a finite R-module, $\text{Supp}(M) = \{\mathfrak m, \mathfrak q\}$, and finite length cohomology modules. The proof in this case follows from Lemma 5.1 and careful bookkeeping. Denote $K_\varphi = \text{Ker}(\varphi)$, $I_\varphi = \text{Im}(\varphi)$, $K_\psi = \text{Ker}(\psi)$, and $I_\psi = \text{Im}(\psi)$. Since R is Noetherian these are all finite R-modules. Set

$$a = \text{length}_{R_\mathfrak q}(I_{\varphi,\mathfrak q}) = \text{length}_{R_\mathfrak q}(K_{\psi,\mathfrak q}), \quad b = \text{length}_{R_\mathfrak q}(I_{\psi,\mathfrak q}) = \text{length}_{R_\mathfrak q}(K_{\varphi,\mathfrak q}).$$

Equalities because the complex becomes exact after localizing at $\mathfrak q$. Note that $l = \text{length}_{R_\mathfrak q}(M_\mathfrak q)$ is equal to $l = a + b$.

We are going to use Lemma 5.2 to choose sequences of elements in finite R-modules N with support contained in $\{\mathfrak m, \mathfrak q\}$. In this case $N_\mathfrak q$ has finite length, say $n \in \mathbf N$. Let us call a sequence $w_1, \ldots, w_n \in N$ with properties (1) and (2) of Lemma 5.2 a "good sequence". Note that the quotient $N/\langle w_1, \ldots, w_n\rangle$ of N by the submodule generated by a good sequence has support (contained in) $\{\mathfrak m\}$ and hence has finite length (Algebra, Lemma 61.3). Moreover, the symbol $[w_1, \ldots, w_n] \in \det_{\kappa(\mathfrak q)}(N_\mathfrak q)$ is a generator, see Lemma 2.4.

Having said this we choose good sequences

$$\begin{array}{llllll} x_1, \ldots, x_b & \text{in} & K_\varphi, & t_1, \ldots, t_a & \text{in} & K_\psi, \\ y_1, \ldots, y_a & \text{in} & I_\varphi \cap \langle t_1, \ldots t_a\rangle, & s_1, \ldots, s_b & \text{in} & I_\psi \cap \langle x_1, \ldots, x_b\rangle. \end{array}$$

We will adjust our choices a little bit as follows. Choose lifts $\tilde y_i \in M$ of $y_i \in I_\varphi$ and $\tilde s_i \in M$ of $s_i \in I_\psi$. It may not be the case that $\mathfrak q \tilde y_1 \subset \langle x_1, \ldots, x_b\rangle$ and it may not be the case that $\mathfrak q \tilde s_1 \subset \langle t_1, \ldots, t_a\rangle$. However, using that $\mathfrak q$ is finitely generated (as in the proof of Lemma 5.2) we can find a $d \in R$, $d \notin \mathfrak q$ such that $\mathfrak q d\tilde y_1 \subset \langle x_1, \ldots, x_b\rangle$ and $\mathfrak q d\tilde s_1 \subset \langle t_1, \ldots, t_a\rangle$. Thus after replacing y_i by dy_i, $\tilde y_i$ by $d\tilde y_i$, s_i by ds_i and $\tilde s_i$ by $d\tilde s_i$ we see that we may assume also that $x_1, \ldots, x_b, \tilde y_1, \ldots, \tilde y_b$ and $t_1, \ldots, t_a, \tilde s_1, \ldots, \tilde s_b$ are good sequences in M.

Finally, we choose a good sequence $z_1, \ldots, z_l$ in the finite R-module

$$\langle x_1, \ldots, x_b, \tilde y_1, \ldots, \tilde y_a\rangle \cap \langle t_1, \ldots, t_a, \tilde s_1, \ldots, \tilde s_b\rangle.$$

Note that this is also a good sequence in M.

Since $I_{\varphi,\mathfrak{q}} = K_{\psi,\mathfrak{q}}$ there is a unique element $h \in \kappa(\mathfrak{q})$ such that $[y_1, \ldots, y_a] = h[t_1, \ldots, t_a]$ inside $\det_{\kappa(\mathfrak{q})}(K_{\psi,\mathfrak{q}})$. Similarly, as $I_{\psi,\mathfrak{q}} = K_{\varphi,\mathfrak{q}}$ there is a unique element $h \in \kappa(\mathfrak{q})$ such that $[s_1, \ldots, s_b] = g[x_1, \ldots, x_b]$ inside $\det_{\kappa(\mathfrak{q})}(K_{\varphi,\mathfrak{q}})$. We can also do this with the three good sequences we have in M. All in all we get the following identities

$$\begin{aligned}
[y_1, \ldots, y_a] &= h[t_1, \ldots, t_a] \\
[s_1, \ldots, s_b] &= g[x_1, \ldots, x_b] \\
[z_1, \ldots, z_l] &= f_\varphi[x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a] \\
[z_1, \ldots, z_l] &= f_\psi[t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b]
\end{aligned}$$

for some $g, h, f_\varphi, f_\psi \in \kappa(\mathfrak{q})$.

Having set up all this notation let us compute $\det_{\kappa(\mathfrak{q})}(M, \varphi, \psi)$. Namely, consider the element $[z_1, \ldots, z_l]$. Under the map $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ of Definition 3.4 we have

$$\begin{array}{rcl}
[z_1, \ldots, z_l] & = & f_\varphi[x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a] \\
& \mapsto & f_\varphi[x_1, \ldots, x_b] \otimes [y_1, \ldots, y_a] \\
& \mapsto & f_\varphi h/g[t_1, \ldots, t_a] \otimes [s_1, \ldots, s_b] \\
& \mapsto & f_\varphi h/g[t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b] \\
& = & f_\varphi h/f_\psi g[z_1, \ldots, z_l]
\end{array}$$

This means that $\det_{\kappa(\mathfrak{q})}(M_\mathfrak{q}, \varphi_\mathfrak{q}, \psi_\mathfrak{q})$ is equal to $f_\varphi h/f_\psi g$ up to a sign.

We abbreviate the following quantities

$$\begin{array}{rcl}
k_\varphi & = & \mathrm{length}_R(K_\varphi/\langle x_1, \ldots, x_b\rangle) \\
k_\psi & = & \mathrm{length}_R(K_\psi/\langle t_1, \ldots, t_a\rangle) \\
i_\varphi & = & \mathrm{length}_R(I_\varphi/\langle y_1, \ldots, y_a\rangle) \\
i_\psi & = & \mathrm{length}_R(I_\psi/\langle s_1, \ldots, s_a\rangle) \\
m_\varphi & = & \mathrm{length}_R(M/\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle) \\
m_\psi & = & \mathrm{length}_R(M/\langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle) \\
\delta_\varphi & = & \mathrm{length}_R(\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle\langle z_1, \ldots, z_l\rangle) \\
\delta_\psi & = & \mathrm{length}_R(\langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle\langle z_1, \ldots, z_l\rangle)
\end{array}$$

Using the exact sequences $0 \to K_\varphi \to M \to I_\varphi \to 0$ we get $m_\varphi = k_\varphi + i_\varphi$. Similarly we have $m_\psi = k_\psi + i_\psi$. We have $\delta_\varphi + m_\varphi = \delta_\psi + m_\psi$ since this is equal to the colength of $\langle z_1, \ldots, z_l\rangle$ in M. Finally, we have

$$\delta_\varphi = \mathrm{ord}_{R/\mathfrak{q}}(f_\varphi), \quad \delta_\psi = \mathrm{ord}_{R/\mathfrak{q}}(f_\psi)$$

by our first application of the key Lemma 5.1.

Next, let us compute the multiplicity of the periodic complex

$$\begin{aligned} e_R(M,\varphi,\psi) &= \text{length}_R(K_\varphi/I_\psi) - \text{length}_R(K_\psi/I_\varphi) \\ &= \text{length}_R(\langle x_1,\ldots,x_b\rangle/\langle s_1,\ldots,s_b\rangle) + k_\varphi - i_\psi \\ &\quad -\text{length}_R(\langle t_1,\ldots,t_a\rangle/\langle y_1,\ldots,y_a\rangle) - k_\psi + i_\varphi \\ &= \text{ord}_{R/\mathfrak q}(g/h) + k_\varphi - i_\psi - k_\psi + i_\varphi \\ &= \text{ord}_{R/\mathfrak q}(g/h) + m_\varphi - m_\psi \\ &= \text{ord}_{R/\mathfrak q}(g/h) + \delta_\psi - \delta_\varphi \\ &= \text{ord}_{R/\mathfrak q}(f_\psi g/f_\varphi h) \end{aligned}$$

where we used the key Lemma 5.1 twice in the third equality. By our computation of $\det_{\kappa(\mathfrak q)}(M_\mathfrak q, \varphi_\mathfrak q, \psi_\mathfrak q)$ this proves the proposition. □

In most applications the following lemma suffices.

Lemma 5.4. *Let R be a Noetherian local ring with maximal ideal $\mathfrak m$. Let M be a finite R-module, and let $\psi : M \to M$ be an R-module map. Assume that*

(1) *Ker(ψ) and Coker(ψ) have finite length, and*
(2) $\dim(\textit{Supp}(M)) \leq 1$.

Write Supp(M) $= \{\mathfrak m, \mathfrak q_1, \ldots, \mathfrak q_t\}$ and denote $f_i \in \kappa(\mathfrak q_i)^*$ the element such that $\det_{\kappa(\mathfrak q_i)}(\psi_{\mathfrak q_i}) : \det_{\kappa(\mathfrak q_i)}(M_{\mathfrak q_i}) \to \det_{\kappa(\mathfrak q_i)}(M_{\mathfrak q_i})$ is multiplication by f_i. Then we have

$$\textit{length}_R(\textit{Coker}(\psi)) - \textit{length}_R(\textit{Ker}(\psi)) = \sum\nolimits_{i=1,\ldots,t} \textit{ord}_{R/\mathfrak q_i}(f_i).$$

Proof. Recall that $H^0(M, 0, \psi) = \text{Coker}(\psi)$ and $H^1(M, 0, \psi) = \text{Ker}(\psi)$, see remarks above Definition 3.2. The lemma follows by combining Proposition 5.3 with Lemma 3.8.

Alternative proof. Reduce to the case $\text{Supp}(M) = \{\mathfrak m, \mathfrak q\}$ as in the proof of Proposition 5.3. Then directly combine Lemmas 5.1 and 5.2 to prove this specific case of Proposition 5.3. There is much less bookkeeping in this case, and the reader is encouraged to work this out. Details omitted. □

Lemma 5.5. *Let R be a Noetherian local ring with maximal ideal $\mathfrak m$. Let M be a finite R-module. Let $x \in R$. Assume that*

(1) $\dim(\textit{Supp}(M)) \leq 1$, *and*
(2) $\dim(M/xM) \leq 0$.

Write Supp(M) $= \{\mathfrak m, \mathfrak q_1, \ldots, \mathfrak q_t\}$. Then

$$\textit{length}_R(M_x) - \textit{length}_R({}_xM) = \sum\nolimits_{i=1,\ldots,t} \textit{ord}_{R/\mathfrak q_i}(x)\textit{length}_{R_{\mathfrak q_i}}(M_{\mathfrak q_i}).$$

where $M_x = M/xM$ and ${}_xM = \textit{Ker}(x : M \to M)$.

Proof. This is a special case of Lemma 5.4. To see that $f_i = x^{\text{length}_{R_{\mathfrak q_i}}(M_{\mathfrak q_i})}$ see Lemma 2.8. □

Lemma 5.6. *Let R be a Noetherian local ring with maximal ideal $\mathfrak m$. Let $I \subset R$ be an ideal and let $x \in R$. Assume x is a nonzerodivisor on R/I and that $\dim(R/I) = 1$. Then*

$$\textit{length}_R(R/(x, I)) = \sum\nolimits_i \textit{length}_R(R/(x, \mathfrak q_i))\textit{length}_{R_{\mathfrak q_i}}((R/I)_{\mathfrak q_i})$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are the minimal primes over I. More generally if M is any finite Cohen-Macaulay module of dimension 1 over R and $\dim(M/xM) = 0$, then

$$length_R(M/xM) = \sum\nolimits_i length_R(R/(x, \mathfrak{q}_i)) length_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the support of M.

Proof. These are special cases of Lemma 5.5. □

Lemma 5.7. *Let A be a Noetherian local ring. Let M be a finite A-module. Let $a, b \in A$. Assume*

1. $\dim(A) = 1$,
2. *both a and b are nonzerodivisors in A,*
3. *A has no embedded primes,*
4. *M has no embedded associated primes,*
5. *$Supp(M) = \mathrm{Spec}(A)$.*

Let $I = \{x \in A \mid x(a/b) \in A\}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes of A. Then $(a/b)IM \subset M$ and

$$length_A(M/(a/b)IM) - length_A(M/IM) = \sum\nolimits_i length_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) ord_{A/\mathfrak{q}_i}(a/b)$$

Proof. Since M has no embedded associated primes, and since the support of M is $\mathrm{Spec}(A)$ we see that $\mathrm{Ass}(M) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Hence a, b are nonzerodivisors on M. Note that

$$\begin{aligned}
&\text{length}_A(M/(a/b)IM) \\
&= \text{length}_A(bM/aIM) \\
&= \text{length}_A(M/aIM) - \text{length}_A(M/bM) \\
&= \text{length}_A(M/aM) + \text{length}_A(aM/aIM) - \text{length}_A(M/bM) \\
&= \text{length}_A(M/aM) + \text{length}_A(M/IM) - \text{length}_A(M/bM)
\end{aligned}$$

as the injective map $b : M \to bM$ maps $(a/b)IM$ to aIM and the injective map $a : M \to aM$ maps IM to aIM. Hence the left hand side of the equation of the lemma is equal to

$$\text{length}_A(M/aM) - \text{length}_A(M/bM).$$

Applying the second formula of Lemma 5.6 with $x = a, b$ respectively and using Algebra, Definition 117.2 of the ord-functions we get the result. □

6. Application to tame symbol

In this section we apply the results above to show the following lemma.

Lemma 6.1. *Let A be a 2-dimensional Noetherian local domain. Let $K = f.f.(A)$. Let $f, g \in K^*$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the height 1 primes $\mathfrak{q}$ of A such that either f or g is not an element of $A^*_{\mathfrak{q}}$. Then we have*

$$\sum\nolimits_{i=1,\ldots,t} ord_{A/\mathfrak{q}_i}(d_{A_{\mathfrak{q}_i}}(f, g)) = 0$$

We can also write this as

$$\sum\nolimits_{height(\mathfrak{q})=1} ord_{A/\mathfrak{q}}(d_{A_{\mathfrak{q}}}(f, g)) = 0$$

since at any height one prime $\mathfrak{q}$ of A where $f, g \in A^*_{\mathfrak{q}}$ we have $d_{A_{\mathfrak{q}}}(f, g) = 1$ by Lemma 4.6.

Proof. Since the tame symbols $d_{A_\mathfrak{q}}(f, g)$ are additive (Lemma 4.4) and the order functions $\text{ord}_{A/\mathfrak{q}}$ are additive (Algebra, Lemma 117.1) it suffices to prove the formula when $f = a \in A$ and $g = b \in A$. In this case we see that we have to show

$$\sum\nolimits_{\text{height}(\mathfrak{q})=1} \text{ord}_{A/\mathfrak{q}}(\det\nolimits_\kappa(A_\mathfrak{q}/(ab), a, b)) = 0$$

By Proposition 5.3 this is equivalent to showing that

$$e_A(A/(ab), a, b) = 0.$$

Since the complex $A/(ab) \xrightarrow{a} A/(ab) \xrightarrow{b} A/(ab) \xrightarrow{a} A/(ab)$ is exact we win. □

7. Setup

We will throughout work over a locally Noetherian universally catenary base S endowed with a dimension function δ. Although it is likely possible to generalize (parts of) the discussion in the chapter, it seems that this is a good first approximation. We usually do not assume our schemes are separated or quasi-compact. Many interesting algebraic stacks are non-separated and/or non-quasi-compact and this is a good case study to see how to develop a reasonable theory for those as well. In order to reference these hypotheses we give it a number.

Situation 7.1. Here S is a locally Noetherian, and universally catenary scheme. Moreover, we assume S is endowed with a dimension function $\delta : S \longrightarrow \mathbf{Z}$.

See Morphisms, Definition 18.1 for the notion of a universally catenary scheme, and see Topology, Definition 19.1 for the notion of a dimension function. Recall that any locally Noetherian catenary scheme locally has a dimension function, see Properties, Lemma 11.3. Moreover, there are lots of schemes which are universally catenary, see Morphisms, Lemma 18.4.

Let (S, δ) be as in Situation 7.1. Any scheme X locally of finite type over S is locally Noetherian and catenary. In fact, X has a canonical dimension function

$$\delta = \delta_{X/S} : X \longrightarrow \mathbf{Z}$$

associated to $(f : X \to S, \delta)$ given by the rule $\delta_{X/S}(x) = \delta(f(x)) + \text{trdeg}_{\kappa(f(x))}\kappa(x)$. See Morphisms, Lemma 31.2. Moreover, if $h : X \to Y$ is a morphism of schemes locally of finite type over S, and $x \in X$, $y = h(x)$, then obviously $\delta_{X/S}(x) = \delta_{Y/S}(y) + \text{trdeg}_{\kappa(y)}\kappa(x)$. We will freely use this function and its properties in the following.

Here are the basic examples of setups as above. In fact, the main interest lies in the case where the base is the spectrum of a field, or the case where the base is the spectrum of a Dedekind ring (e.g. $\mathbf{Z}$, or a discrete valuation ring).

Example 7.2. Here $S = \text{Spec}(k)$ and k is a field. We set $\delta(pt) = 0$ where pt indicates the unique point of S. The pair (S, δ) is an example of a situation as in Situation 7.1 by Morphisms, Lemma 18.4.

Example 7.3. Here $S = \text{Spec}(A)$, where A is a Noetherian domain of dimension 1. For example we could consider $A = \mathbf{Z}$. We set $\delta(\mathfrak{p}) = 0$ if $\mathfrak{p}$ is a maximal ideal and $\delta(\mathfrak{p}) = 1$ if $\mathfrak{p} = (0)$ corresponds to the generic point. This is an example of Situation 7.1 by Morphisms, Lemma 18.4.

In good cases δ corresponds to the dimension function.

Lemma 7.4. *Let (S, δ) be as in Situation 7.1. Assume in addition S is a Jacobson scheme, and $\delta(s) = 0$ for every closed point s of S. Let X be locally of finite type over S. Let $Z \subset X$ be an integral closed subscheme and let $\xi \in Z$ be its generic point. The following integers are the same:*

(1) $\delta_{X/S}(\xi)$,
(2) $\dim(Z)$, *and*
(3) $\dim(\mathcal{O}_{Z,z})$ *where z is a closed point of Z.*

Proof. Let $X \to S$, $\xi \in Z \subset X$ be as in the lemma. Since X is locally of finite type over S we see that X is Jacobson, see Morphisms, Lemma 17.9. Hence closed points of X are dense in every closed subset of Z and map to closed points of S. Hence given any chain of irreducible closed subsets of Z we can end it with a closed point of Z. It follows that $\dim(Z) = \sup_z(\dim(\mathcal{O}_{Z,z})$ (see Properties, Lemma 11.4) where $z \in Z$ runs over the closed points of Z. Note that $\dim(\mathcal{O}_{Z,z}) = \delta(\xi) - \delta(z))$ by the properties of a dimension function. For each closed $z \in Z$ the field extension $\kappa(z) \supset \kappa(f(z))$ is finite, see Morphisms, Lemma 17.8. Hence $\delta_{X/S}(z) = \delta(f(z)) = 0$ for $z \in Z$ closed. It follows that all three integers are equal. $\square$

In the situation of the lemma above the value of δ at the generic point of a closed irreducible subset is the dimension of the irreducible closed subset. However, in general we cannot expect the equality to hold. For example if $S = \text{Spec}(\mathbf{C}[[t]])$ and $X = \text{Spec}(\mathbf{C}((t)))$ then we would get $\delta(x) = 1$ for the unique point of X, but $\dim(X) = 0$. Still we want to think of $\delta_{X/S}$ as giving the dimension of the irreducible closed subschemes. Thus we introduce the following terminology.

Definition 7.5. Let (S, δ) as in Situation 7.1. For any scheme X locally of finite type over S and any irreducible closed subset $Z \subset X$ we define

$$\dim_\delta(Z) = \delta(\xi)$$

where $\xi \in Z$ is the generic point of Z. We will call this the *δ-dimension of Z*. If Z is a closed subscheme of X, then we define $\dim_\delta(Z)$ as the supremum of the δ-dimensions of its irreducible components.

8. Cycles

Since we are not assuming our schemes are quasi-compact we have to be a little careful when defining cycles. We have to allow infinite sums because a rational function may have infinitely many poles for example. In any case, if X is quasi-compact then a cycle is a finite sum as usual.

Definition 8.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) A collection of closed subschemes $\{Z_i\}_{i \in I}$ of X is said to be *locally finite* if for every quasi-compact open $U \subset X$ the set
$$\#\{i \in I \mid Z_i \cap U \neq \emptyset\}$$
is finite.
(2) A *cycle on X* is a formal sum
$$\alpha = \sum n_Z[Z]$$
where the sum is over integral closed subschemes $Z \subset X$, each $n_Z \in \mathbf{Z}$, and the collection $\{Z; n_Z \neq 0\}$ is locally finite.

(3) A *k-cycle*, on X is a cycle

$$\alpha = \sum n_Z[Z]$$

where $n_Z \neq 0 \Rightarrow \dim_\delta(Z) = k$.

(4) The abelian group of all k-cycles on X is denoted $Z_k(X)$.

In other words, a k-cycle on X is a locally finite formal $\mathbf{Z}$-linear combination of integral closed subschemes of δ-dimension k. Addition of k-cycles $\alpha = \sum n_Z[Z]$ and $\beta = \sum m_Z[Z]$ is given by

$$\alpha + \beta = \sum (n_Z + m_Z)[Z],$$

i.e., by adding the coefficients.

9. Cycle associated to a closed subscheme

Lemma 9.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.*

(1) *The collection of irreducible components of Z is locally finite.*
(2) *Let $Z' \subset Z$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then*

$$\textit{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi} < \infty$$

(3) *If $\dim_\delta(Z) \leq k$ and $\xi \in Z$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of Z.*

Proof. Let $U \subset X$ be a quasi-compact open subscheme. Then U is a Noetherian scheme, and hence has a Noetherian underlying topological space (Properties, Lemma 5.5). Hence every subspace is Noetherian and has finitely many irreducible components (see Topology, Lemma 8.2). This proves (1).

Let $Z' \subset Z$, $\xi \in Z'$ be as in (2). Then $\dim(\mathcal{O}_{Z,\xi}) = 0$ (for example by Properties, Lemma 11.4). Hence $\mathcal{O}_{Z,\xi}$ is Noetherian local ring of dimension zero, and hence has finite length over itself (see Algebra, Proposition 59.6). Hence, it also has finite length over $\mathcal{O}_{X,\xi}$, see Algebra, Lemma 50.12.

Assume $\xi \in Z$ and $\delta(\xi) = k$. Consider the closure $Z' = \overline{\{\xi\}}$. It is an irreducible closed subscheme with $\dim_\delta(Z') = k$ by definition. Since $\dim_\delta(Z) = k$ it must be an irreducible component of Z. Hence we see (3) holds. □

Definition 9.2. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.

(1) For any irreducible component $Z' \subset Z$ with generic point ξ the integer $m_{Z',Z} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi}$ (Lemma 9.1) is called the *multiplicity of Z' in Z*.
(2) Assume $\dim_\delta(Z) \leq k$. The *k-cycle associated to Z* is

$$[Z]_k = \sum m_{Z',Z}[Z']$$

where the sum is over the irreducible components of Z of δ-dimension k. (This is a k-cycle by Lemma 9.1.)

It is important to note that we only define $[Z]_k$ if the δ-dimension of Z does not exceed k. In other words, by convention, if we write $[Z]_k$ then this implies that $\dim_\delta(Z) \leq k$.

10. Cycle associated to a coherent sheaf

Lemma 10.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

1. *The collection of irreducible components of the support of $\mathcal{F}$ is locally finite.*
2. *Let $Z' \subset Supp(\mathcal{F})$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then*
$$length_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi < \infty$$
3. *If $\dim_\delta(Supp(\mathcal{F})) \leq k$ and $\xi \in Z$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of $Supp(\mathcal{F})$.*

Proof. By Cohomology of Schemes, Lemma 9.7 the support Z of $\mathcal{F}$ is a closed subset of X. We may think of Z as a reduced closed subscheme of X (Schemes, Lemma 12.4). Hence (1) and (3) follow immediately by applying Lemma 9.1 to $Z \subset X$.

Let $\xi \in Z'$ be as in (2). In this case for any specialization $\xi' \rightsquigarrow \xi$ in X we have $\mathcal{F}_{\xi'} = 0$. Recall that the non-maximal primes of $\mathcal{O}_{X,\xi}$ correspond to the points of X specializing to ξ (Schemes, Lemma 13.2). Hence $\mathcal{F}_\xi$ is a finite $\mathcal{O}_{X,\xi}$-module whose support is $\{\mathfrak{m}_\xi\}$. Hence it has finite length by Algebra, Lemma 61.3. □

Definition 10.2. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

1. For any irreducible component $Z' \subset \text{Supp}(\mathcal{F})$ with generic point ξ the integer $m_{Z',\mathcal{F}} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi$ (Lemma 10.1) is called the *multiplicity of Z' in $\mathcal{F}$*.
2. Assume $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$. The *$k$-cycle associated to $\mathcal{F}$* is
$$[\mathcal{F}]_k = \sum m_{Z',\mathcal{F}}[Z']$$
where the sum is over the irreducible components of $\text{Supp}(\mathcal{F})$ of δ-dimension k. (This is a k-cycle by Lemma 10.1.)

It is important to note that we only define $[\mathcal{F}]_k$ if $\mathcal{F}$ is coherent and the δ-dimension of $\text{Supp}(\mathcal{F})$ does not exceed k. In other words, by convention, if we write $[\mathcal{F}]_k$ then this implies that $\mathcal{F}$ is coherent on X and $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

Lemma 10.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme. If $\dim_\delta(Z) \leq k$, then $[Z]_k = [\mathcal{O}_Z]_k$.*

Proof. This is because in this case the multiplicities $m_{Z',Z}$ and $m_{Z',\mathcal{O}_Z}$ agree by definition. □

Lemma 10.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of coherent sheaves on X. Assume that the δ-dimension of the supports of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ is $\leq k$. Then $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$.*

Proof. Follows immediately from additivity of lengths, see Algebra, Lemma 50.3. □

11. Preparation for proper pushforward

Lemma 11.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume X, Y integral and $\dim_\delta(X) = \dim_\delta(Y)$. Then either $f(X)$ is contained in a proper closed subscheme of Y, or f is dominant and the extension of function fields $R(Y) \subset R(X)$ is finite.*

Proof. The closure $\overline{f(X)} \subset Y$ is irreducible as X is irreducible (Topology, Lemmas 7.2 and 7.3). If $\overline{f(X)} \neq Y$, then we are done. If $\overline{f(X)} = Y$, then f is dominant and by Morphisms, Lemma 8.5 we see that the generic point η_Y of Y is in the image of f. Of course this implies that $f(\eta_X) = \eta_Y$, where $\eta_X \in X$ is the generic point of X. Since $\delta(\eta_X) = \delta(\eta_Y)$ we see that $R(Y) = \kappa(\eta_Y) \subset \kappa(\eta_X) = R(X)$ is an extension of transcendence degree 0. Hence $R(Y) \subset R(X)$ is a finite extension by Morphisms, Lemma 47.4 (which applies by Morphisms, Lemma 16.8). $\square$

Lemma 11.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is quasi-compact, and $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of X. Then $\{\overline{f(Z_i)}\}_{i \in I}$ is a locally finite collection of closed subsets of Y.*

Proof. Let $V \subset Y$ be a quasi-compact open subset. Since f is quasi-compact the open $f^{-1}(V)$ is quasi-compact. Hence the set $\{i \in I \mid Z_i \cap f^{-1}(V) \neq \emptyset\}$ is finite by assumption (Definition 8.1). Since this is the same as the set

$$\{i \in I \mid f(Z_i) \cap V \neq \emptyset\} = \{i \in I \mid \overline{f(Z_i)} \cap V \neq \emptyset\}$$

the lemma is proved. $\square$

12. Proper pushforward

Definition 12.1. Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is proper.

(1) Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = k$. We define

$$f_*[Z] = \begin{cases} 0 & \text{if} \quad \dim_\delta(f(Z)) < k, \\ \deg(Z/f(Z))[f(Z)] & \text{if} \quad \dim_\delta(f(Z)) = k. \end{cases}$$

Here we think of $f(Z) \subset Y$ as an integral closed subscheme. The degree of Z over $f(Z)$ is finite if $\dim_\delta(f(Z)) = \dim_\delta(Z)$ by Lemma 11.1.

(2) Let $\alpha = \sum n_Z[Z]$ be a k-cycle on X. The *pushforward* of α as the sum

$$f_*\alpha = \sum n_Z f_*[Z]$$

where each $f_*[Z]$ is defined as above. The sum is locally finite by Lemma 11.2 above.

By definition the proper pushforward of cycles

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

is a homomorphism of abelian groups. It turns $X \mapsto Z_k(X)$ into a covariant functor on the category of schemes locally of finite type over S with morphisms equal to proper morphisms.

Lemma 12.2. *Let (S, δ) be as in Situation 7.1. Let X, Y, and Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \to Z_k(Z)$.*

Proof. Let $W \subset X$ be an integral closed subscheme of dimension k. Consider $W' = f(Z) \subset Y$ and $W'' = g(f(Z)) \subset Z$. Since f, g are proper we see that W' (resp. W'') is an integral closed subscheme of Y (resp. Z). We have to show that $g_*(f_*[W]) = (f \circ g)_*[W]$. If $\dim_\delta(W'') < k$, then both sides are zero. If $\dim_\delta(W'') = k$, then we see the induced morphisms

$$W \longrightarrow W' \longrightarrow W''$$

both satisfy the hypotheses of Lemma 11.1. Hence

$$g_*(f_*[W]) = \deg(W/W')\deg(W'/W'')[W''], \quad (f \circ g)_*[W] = \deg(W/W'')[W''].$$

Then we can apply Morphisms, Lemma 47.6 to conclude. □

Lemma 12.3. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is proper.*

(1) *Let $Z \subset X$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then*

$$f_*[Z]_k = [f_*\mathcal{O}_Z]_k.$$

(2) *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then*

$$f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.$$

Note that the statement makes sense since $f_*\mathcal{F}$ and $f_*\mathcal{O}_Z$ are coherent $\mathcal{O}_Y$-modules by Cohomology of Schemes, Proposition 17.2.

Proof. Part (1) follows from (2) and Lemma 10.3. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$. By Cohomology of Schemes, Lemma 9.7 there exists a closed subscheme $i : Z \to X$ and a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} \cong \mathcal{F}$ and such that the support of $\mathcal{F}$ is Z. Let $Z' \subset Y$ be the scheme theoretic image of $f|_Z : Z \to Y$.Consider the commutative diagram of schemes

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f|_Z"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "i'"] & Y \end{tikzcd}$$

We have $f_*\mathcal{F} = f_*i_*\mathcal{G} = i'_*(f|_Z)_*\mathcal{G}$ by going around the diagram in two ways. Suppose we know the result holds for closed immersions and for $f|_Z$. Then we see that

$$f_*[\mathcal{F}]_k = f_*i_*[\mathcal{G}]_k = (i')_*(f|_Z)_*[\mathcal{G}]_k = (i')_*[(f|_Z)_*\mathcal{G}]_k = [(i')_*(f|_Z)_*\mathcal{G}]_k = [f_*\mathcal{F}]_k$$

as desired. The case of a closed immersion is straightforward (omitted). Note that $f|_Z : Z \to Z'$ is a dominant morphism (see Morphisms, Lemma 6.3). Thus we have reduced to the case where $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant.

Assume $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant. Since f is dominant, for every irreducible component $Z \subset Y$ with generic point η there exists a point $\xi \in X$ such that $f(\xi) = \eta$. Hence $\delta(\eta) \leq \delta(\xi) \leq k$. Thus we see that in the expressions

$$f_*[\mathcal{F}]_k = \sum n_Z[Z], \quad \text{and} \quad [f_*\mathcal{F}]_k = \sum m_Z[Z].$$

whenever $n_Z \neq 0$, or $m_Z \neq 0$ the integral closed subscheme Z is actually an irreducible component of Y of δ-dimension k. Pick such an integral closed subscheme $Z \subset Y$ and denote η its generic point. Note that for any $\xi \in X$ with $f(\xi) = \eta$ we have $\delta(\xi) \geq k$ and hence ξ is a generic point of an irreducible component of X of

δ-dimension k as well (see Lemma 9.1). Since f is quasi-compact and X is locally Noetherian, there can be only finitely many of these and hence $f^{-1}(\{\eta\})$ is finite. By Morphisms, Lemma 47.1 there exists an open neighbourhood $\eta \in V \subset Y$ such that $f^{-1}(V) \to V$ is finite. Replacing Y by V and X by $f^{-1}(V)$ we reduce to the case where Y is affine, and f is finite.

Write $Y = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ (possible as a finite morphism is affine). Then R and A are Noetherian rings and A is finite over R. Moreover $\mathcal{F} = \widetilde{M}$ for some finite A-module M. Note that $f_*\mathcal{F}$ corresponds to M viewed as an R-module. Let $\mathfrak p \subset R$ be the minimal prime corresponding to $\eta \in Y$. The coefficient of Z in $[f_*\mathcal{F}]_k$ is clearly $\text{length}_{R_\mathfrak p}(M_\mathfrak p)$. Let $\mathfrak q_i$, $i = 1, \ldots, t$ be the primes of A lying over $\mathfrak p$. Then $A_\mathfrak p = \prod A_{\mathfrak q_i}$ since $A_\mathfrak p$ is an Artinian ring being finite over the dimension zero local Noetherian ring $R_\mathfrak p$. Clearly the coefficient of Z in $f_*[\mathcal{F}]_k$ is

$$\sum\nolimits_{i=1,\ldots,t} [\kappa(\mathfrak q_i) : \kappa(\mathfrak p)] \text{length}_{A_{\mathfrak q_i}}(M_{\mathfrak q_i})$$

Hence the desired equality follows from Algebra, Lemma 50.12. □

13. Preparation for flat pullback

Recall that a morphism $f : X \to Y$ which is locally of finite type is said to have relative dimension r if every nonempty fibre is equidimensional of dimension r. See Morphisms, Definition 30.1.

Lemma 13.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r. For any closed subset $Z \subset Y$ we have*

$$\dim_\delta(f^{-1}(Z)) = \dim_\delta(Z) + r.$$

If Z is irreducible and $Z' \subset f^{-1}(Z)$ is an irreducible component, then Z' dominates Z and $\dim_\delta(Z') = \dim_\delta(Z) + r$.

Proof. It suffices to prove the final statement. We may replace Y by the integral closed subscheme Z and X by the scheme theoretic inverse image $f^{-1}(Z) = Z \times_Y X$. Hence we may assume $Z = Y$ is integral and f is a flat morphism of relative dimension r. Since Y is locally Noetherian the morphism f which is locally of finite type, is actually locally of finite presentation. Hence Morphisms, Lemma 26.9 applies and we see that f is open. Let $\xi \in X$ be a generic point of an irreducible component of X. By the openness of f we see that $f(\xi)$ is the generic point η of $Z = Y$. Note that $\dim_\xi(X_\eta) = r$ by assumption that f has relative dimension r. On the other hand, since ξ is a generic point of X we see that $\mathcal{O}_{X,\xi} = \mathcal{O}_{X_\eta,\xi}$ has only one prime ideal and hence has dimension 0. Thus by Morphisms, Lemma 29.1 we conclude that the transcendence degree of $\kappa(\xi)$ over $\kappa(\eta)$ is r. In other words, $\delta(\xi) = \delta(\eta) + r$ as desired. □

Here is the lemma that we will use to prove that the flat pullback of a locally finite collection of closed subschemes is locally finite.

Lemma 13.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of Y. Then $\{f^{-1}(Z_i)\}_{i \in I}$ is a locally finite collection of closed subsets of Y.*

Proof. Let $U \subset X$ be a quasi-compact open subset. Since the image $f(U) \subset Y$ is a quasi-compact subset there exists a quasi-compact open $V \subset Y$ such that $f(U) \subset V$. Note that

$$\{i \in I \mid f^{-1}(Z_i) \cap U \neq \emptyset\} \subset \{i \in I \mid Z_i \cap V \neq \emptyset\}.$$

Since the right hand side is finite by assumption we win. $\square$

14. Flat pullback

In the following we use $f^{-1}(Z)$ to denote the *scheme theoretic inverse image* of a closed subscheme $Z \subset Y$ for a morphism of schemes $f : X \to Y$. We recall that the scheme theoretic inverse image is the fibre product

$$\begin{tikzcd} f^{-1}(Z) \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

and it is also the closed subscheme of X cut out by the quasi-coherent sheaf of ideals $f^{-1}(\mathcal{I})\mathcal{O}_X$, if $\mathcal{I} \subset \mathcal{O}_Y$ is the quasi-coherent sheaf of ideals corresponding to Z in Y. (This is discussed in Schemes, Section 4 and Lemma 17.6 and Definition 17.7.)

Definition 14.1. Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r.

(1) Let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. We define $f^*[Z]$ to be the $(k + r)$-cycle on X to the scheme theoretic inverse image
$$f^*[Z] = [f^{-1}(Z)]_{k+r}.$$
This makes sense since $\dim_\delta(f^{-1}(Z)) = k + r$ by Lemma 13.1.

(2) Let $\alpha = \sum n_i[Z_i]$ be a k-cycle on Y. The *flat pullback of α by f* is the sum
$$f^*\alpha = \sum n_i f^*[Z_i]$$
where each $f^*[Z_i]$ is defined as above. The sum is locally finite by Lemma 13.2.

(3) We denote $f^* : Z_k(Y) \to Z_{k+r}(X)$ the map of abelian groups so obtained.

An open immersion is flat. This is an important though trivial special case of a flat morphism. If $U \subset X$ is open then sometimes the pullback by $j : U \to X$ of a cycle is called the *restriction* of the cycle to U. Note that in this case the maps

$$j^* : Z_k(X) \longrightarrow Z_k(U)$$

are all *surjective*. The reason is that given any integral closed subscheme $Z' \subset U$, we can take the closure of Z of Z' in X and think of it as a reduced closed subscheme of X (see Schemes, Lemma 12.4). And clearly $Z \cap U = Z'$, in other words $j^*[Z] = [Z']$ whence the surjectivity. In fact a little bit more is true.

Lemma 14.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. For every $k \in \mathbf{Z}$ the sequence*

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. By the description above the basis elements $[Z]$ of the free abelian group $Z_k(X)$ map either to (distinct) basis elements $[Z \cap U]$ or to zero if $Z \subset Y$. Hence the lemma is clear. ☐

Lemma 14.3. *Let (S, δ) be as in Situation 7.1. Let X, Y, Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of relative dimensions r and s. Then $g \circ f$ is flat of relative dimension $r + s$ and*

$$f^* \circ g^* = (g \circ f)^*$$

as maps $Z_k(Z) \to Z_{k+r+s}(X)$.

Proof. The composition is flat of relative dimension $r + s$ by Morphisms, Lemma 30.3. Suppose that

(1) $W \subset Z$ is a closed integral subscheme of δ-dimension k,
(2) $W' \subset Y$ is a closed integral subscheme of δ-dimension $k + s$ with $W' \subset g^{-1}(W)$, and
(3) $W'' \subset Y$ is a closed integral subscheme of δ-dimension $k + s + r$ with $W'' \subset f^{-1}(W')$.

We have to show that the coefficient n of $[W'']$ in $(g \circ f)^*[W]$ agrees with the coefficient m of $[W'']$ in $f^*(g^*[W])$. That it suffices to check the lemma in these cases follows from Lemma 13.1. Let $\xi'' \in W''$, $\xi' \in W'$ and $\xi \in W$ be the generic points. Consider the local rings $A = \mathcal{O}_{Z,\xi}$, $B = \mathcal{O}_{Y,\xi'}$ and $C = \mathcal{O}_{X,\xi''}$. Then we have local flat ring maps $A \to B$, $B \to C$ and moreover

$$n = \text{length}_C(C/\mathfrak{m}_A C), \quad \text{and} \quad m = \text{length}_C(C/\mathfrak{m}_B C)\text{length}_B(B/\mathfrak{m}_A B)$$

Hence the equality follows from Algebra, Lemma 50.14. ☐

Lemma 14.4. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r.*

(1) *Let $Z \subset Y$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then we have $\dim_\delta(f^{-1}(Z)) \leq k + r$ and $[f^{-1}(Z)]_{k+r} = f^*[Z]_k$ in $Z_{k+r}(X)$.*
(2) *Let $\mathcal{F}$ be a coherent sheaf on Y with $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then we have $\dim_\delta(Supp(f^*\mathcal{F})) \leq k + r$ and*

$$f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$$

in $Z_{k+r}(X)$.

Proof. Part (1) follows from part (2) by Lemma 10.3 and the fact that $f^*\mathcal{O}_Z = \mathcal{O}_{f^{-1}(Z)}$.

Proof of (2). As X, Y are locally Noetherian we may apply Cohomology of Schemes, Lemma 9.1 to see that $\mathcal{F}$ is of finite type, hence $f^*\mathcal{F}$ is of finite type (Modules, Lemma 9.2), hence $f^*\mathcal{F}$ is coherent (Cohomology of Schemes, Lemma 9.1 again). Thus the lemma makes sense. Let $W \subset Y$ be an integral closed subscheme of δ-dimension k, and let $W' \subset X$ be an integral closed subscheme of dimension $k + r$ mapping into W under f. We have to show that the coefficient n of $[W]$ in $f^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W]$ in $[f^*\mathcal{F}]_{k+r}$. Let $\xi \in W$ and $\xi' \in W'$ be the generic points. Let $A = \mathcal{O}_{Y,\xi}$, $B = \mathcal{O}_{X,\xi'}$ and set $M = \mathcal{F}_\xi$ as an A-module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 10.1.) We have $f^*\mathcal{F}_{\mathcal{E}'} = B \otimes_A M$. Thus we see that

$$n = \text{length}_B(B \otimes_A M) \quad \text{and} \quad m = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B)$$

Thus the equality follows from Algebra, Lemma 50.13. □

15. Push and pull

In this section we verify that proper pushforward and flat pullback are compatible when this makes sense. By the work we did above this is a consequence of cohomology and base change.

Lemma 15.1. *Let (S, δ) be as in Situation 7.1. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a fibre product diagram of schemes locally of finite type over S. Assume $f : X \to Y$ proper and $g : Y' \to Y$ flat of relative dimension r. Then also f' is proper and g' is flat of relative dimension r. For any k-cycle α on X we have

$$g^* f_* \alpha = f'_*(g')^* \alpha$$

in $Z_{k+r}(Y')$.

Proof. The assertion that f' is proper follows from Morphisms, Lemma 42.5. The assertion that g' is flat of relative dimension r follows from Morphisms, Lemmas 30.2 and 26.7. It suffices to prove the equality of cycles when $\alpha = [W]$ for some integral closed subscheme $W \subset X$ of δ-dimension k. Note that in this case we have $\alpha = [\mathcal{O}_W]_k$, see Lemma 10.3. By Lemmas 12.3 and 14.4 it therefore suffices to show that $f'_*(g')^*\mathcal{O}_W$ is isomorphic to $g^*f_*\mathcal{O}_W$. This follows from cohomology and base change, see Cohomology of Schemes, Lemma 5.2. □

Lemma 15.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a finite locally free morphism of degree d (see Morphisms, Definition 46.1). Then f is both proper and flat of relative dimension 0, and*

$$f_* f^* \alpha = d\alpha$$

for every $\alpha \in Z_k(Y)$.

Proof. A finite locally free morphism is flat and finite by Morphisms, Lemma 46.2, and a finite morphism is proper by Morphisms, Lemma 44.10. We omit showing that a finite morphism has relative dimension 0. Thus the formula makes sense. To prove it, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. It suffices to prove the formula for $\alpha = [Z]$. Since the base change of a finite locally free morphism is finite locally free (Morphisms, Lemma 46.4) we see that $f_*f^*\mathcal{O}_Z$ is a finite locally free sheaf of rank d on Z. Hence

$$f_*f^*[Z] = f_*f^*[\mathcal{O}_Z]_k = [f_*f^*\mathcal{O}_Z]_k = d[Z]$$

where we have used Lemmas 14.4 and 12.3. □

16. Preparation for principal divisors

Recall that if Z is an irreducible closed subset of a scheme X, then the codimension of Z in X is equal to the dimension of the local ring $\mathcal{O}_{X,\xi}$, where $\xi \in Z$ is the generic point. See Properties, Lemma 11.4.

Definition 16.1. Let X be a locally Noetherian scheme. Assume X is integral. Let $f \in R(X)^*$. For every integral closed subscheme $Z \subset X$ of codimension 1 we define the *order of vanishing of f along Z* as the integer

$$\text{ord}_Z(f) = \text{ord}_{\mathcal{O}_{X,\xi}}(f)$$

where the right hand side is the notion of Algebra, Definition 117.2 and ξ is the generic point of Z.

Of course it can happen that $\text{ord}_Z(f) < 0$. In this case we say that f has a *pole* along Z and that $-\text{ord}_Z(f) > 0$ is the *order of pole of f along Z*. Note that for $f, g \in R(X)^*$ we have

$$\text{ord}_Z(fg) = \text{ord}_Z(f) + \text{ord}_Z(g).$$

Lemma 16.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral. If $Z \subset X$ is an integral closed subscheme of codimension 1, then* $\dim_\delta(Z) = \dim_\delta(X) - 1$.

Proof. This is more or less the defining property of a dimension function. □

Lemma 16.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral. Let $f \in R(X)^*$. Then the set*

$$\{Z \subset X \mid Z \textit{ is integral, closed of codimension } 1 \textit{ and } \text{ord}_Z(f) \neq 0\}$$

is locally finite in X.

Proof. This is true simply because there exists a nonempty open subscheme $U \subset X$ such that f corresponds to a section of $\Gamma(U, \mathcal{O}_X^*)$, and hence the codimension 1 irreducibles which can occur in the set of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 9.1 gives the desired result. □

Lemma 16.4. *Let $f : X \to Y$ be a morphism of schemes. Let $\xi \in Y$ be a point. Assume that*

1. *X, Y are integral,*
2. *X is locally Noetherian*
3. *f is proper, dominant and $R(X) \subset R(Y)$ is finite, and*
4. $\dim(\mathcal{O}_{Y,\xi}) = 1$.

Then there exists an open neighbourhood $V \subset Y$ of ξ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. This lemma is a special case of Varieties, Lemma 24.2. Here is a direct argument in this case. By Cohomology of Schemes, Lemma 19.2 it suffices to prove that $f^{-1}(\{\xi\})$ is finite. We replace Y by an affine open, say $Y = \text{Spec}(R)$. Note that R is Noetherian, as X is assumed locally Noetherian. Since f is proper it is quasi-compact. Hence we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_n$ with each $U_i = \text{Spec}(A_i)$. Note that $R \to A_i$ is a finite type injective homomorphism of domains with $f.f.(R) \subset f.f.(A_i)$ finite. Thus the lemma follows from Algebra, Lemma 109.2. □

17. Principal divisors

The following definition is the key to everything that follows.

Definition 17.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral with $\dim_\delta(X) = n$. Let $f \in R(X)^*$. The *principal divisor associated to* f is the $(n-1)$-cycle

$$\text{div}(f) = \text{div}_X(f) = \sum \text{ord}_Z(f)[Z]$$

where the sum is over integral closed subschemes of codimension 1 and $\text{ord}_Z(f)$ is as in Definition 16.1. This makes sense by Lemmas 16.2 and 16.3.

Lemma 17.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral with $\dim_\delta(X) = n$. Let $f, g \in R(X)^*$. Then*

$$div(fg) = div(f) + div(g)$$

in $Z_{n-1}(X)$.

Proof. This is clear from the additivity of the ord functions. □

An important role in the discussion of principal divisors is played by the "universal" principal divisor $[0] - [\infty]$ on $\mathbf{P}^1_S$. To make this more precise, let us denote

$$D_0, D_\infty \subset \mathbf{P}^1_S = \underline{\text{Proj}}_S(\mathcal{O}_S[X_0, X_1])$$

the closed subscheme cut out by the section X_1, resp. X_0 of $\mathcal{O}(1)$. These are effective Cartier divisors, see Divisors, Definition 9.1 and Lemma 9.20. The following lemma says that loosely speaking we have "$\text{div}(X_1/X_0) = [D_0] - [D_1]$" and that this is the universal principal divisor.

Lemma 17.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $f \in R(X)^*$. Let $U \subset X$ be a nonempty open such that f corresponds to a section $f \in \Gamma(U, \mathcal{O}_X^*)$. Let $Y \subset X \times_S \mathbf{P}^1_S$ be the closure of the graph of $f : U \to \mathbf{P}^1_S$. Then*

1. *the projection morphism $p : Y \to X$ is proper,*
2. *$p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is an isomorphism,*
3. *the pullbacks $q^{-1}D_0$ and $q^{-1}D_\infty$ via the morphism $q : Y \to \mathbf{P}^1_S$ are effective Cartier divisors on Y,*
4. *we have*
$$div_Y(f) = [q^{-1}D_0]_{n-1} - [q^{-1}D_\infty]_{n-1}$$
5. *we have*
$$div_X(f) = p_* div_Y(f)$$
6. *if we view $Y_0 = q^{-1}D_0$, and $Y_\infty = q^{-1}D_\infty$ as closed subschemes of X via the morphism p then we have*
$$div_X(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$

Proof. Since X is integral, we see that U is integral. Hence Y is integral, and $(1, f)(U) \subset Y$ is an open dense subscheme. Also, note that the closed subscheme $Y \subset X \times_S \mathbf{P}^1_S$ does not depend on the choice of the open U, since after all it is the closure of the one point set $\{\eta'\} = \{(1, f)(\eta)\}$ where $\eta \in X$ is the generic point. Having said this let us prove the assertions of the lemma.

For (1) note that p is the composition of the closed immersion $Y \to X \times_S \mathbf{P}^1_S = \mathbf{P}^1_X$ with the proper morphism $\mathbf{P}^1_X \to X$. As a composition of proper morphisms is proper (Morphisms, Lemma 42.4) we conclude.

It is clear that $Y \cap U \times_S \mathbf{P}^1_S = (1, f)(U)$. Thus (2) follows. It also follows that $\dim_\delta(Y) = n$.

Note that $q(\eta') = f(\eta)$ is not contained in D_0 or D_∞ since $f \in R(X)^*$. Hence $q^{-1}D_0$ and $q^{-1}D_\infty$ are effective Cartier divisors on Y by Divisors, Lemma 9.12. Thus we see (3). It also follows that $\dim_\delta(q^{-1}D_0) = n - 1$ and $\dim_\delta(q^{-1}D_\infty) = n - 1$.

Consider the effective Cartier divisor $q^{-1}D_0$. At every point $\xi \in q^{-1}D_0$ we have $f \in \mathcal{O}_{Y,\xi}$ and the local equation for $q^{-1}D_0$ is given by f. In particular, if $\delta(\xi) = n-1$ so ξ is the generic point of a integral closed subscheme Z of δ-dimension $n-1$, then we see that the coefficient of $[Z]$ in $\text{div}_Y(f)$ is

$$\text{ord}_Z(f) = \text{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y,\xi}/f\mathcal{O}_{Y,\xi}) = \text{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{q^{-1}D_0,\xi})$$

which is the coefficient of $[Z]$ in $[q^{-1}D_0]_{n-1}$. A similar argument using the rational function $1/f$ shows that $-[q^{-1}D_\infty]$ agrees with the terms with negative coefficients in the expression for $\text{div}_Y(f)$. Hence (4) follows.

Note that $D_0 \to S$ is an isomorphism. Hence we see that $X \times_S D_0 \to X$ is an isomorphism as well. Clearly we have $q^{-1}D_0 = Y \cap X \times_S D_0$ (scheme theoretic intersection) inside $X \times_S \mathbf{P}^1_S$. Hence it is really the case that $Y_0 \to X$ is a closed immersion. By the same token we see that

$$p_*\mathcal{O}_{q^{-1}D_0} = \mathcal{O}_{Y_0}$$

and hence by Lemma 12.3 we have $p_*[q^{-1}D_0]_{n-1} = [Y_0]_{n-1}$. Of course the same is true for D_∞ and Y_∞. Hence to finish the proof of the lemma it suffices to prove the last assertion.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. We want to show that the coefficient of $[Z]$ in $\text{div}(f)$ is the same as the coefficient of $[Z]$ in $[Y_0]_{n-1} - [Y_\infty]_{n-1}$. We may apply Lemma 16.4 to the morphism $p : Y \to X$ and the generic point $\xi \in Z$. Hence we may replace X by an affine open neighbourhood of ξ and assume that $p : Y \to X$ is finite. Write $X = \text{Spec}(R)$ and $Y = \text{Spec}(A)$ with p induced by a finite homomorphism $R \to A$ of Noetherian domains which induces an isomorphism $f.f.(R) \cong f.f.(A)$ of fraction fields. Now we have $f \in f.f.(R)$ and a prime $\mathfrak{p} \subset R$ with $\dim(R_\mathfrak{p}) = 1$. The coefficient of $[Z]$ in $\text{div}_X(f)$ is $\text{ord}_{R_\mathfrak{p}}(f)$. The coefficient of $[Z]$ in $p_*\text{div}_Y(f)$ is

$$\sum\nolimits_{\mathfrak{q} \text{ lying over } \mathfrak{p}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})] \text{ord}_{A_\mathfrak{q}}(f)$$

The desired equality therefore follows from Algebra, Lemma 117.8. □

This lemma will be superseded by the more general Lemma 20.1.

Lemma 17.4. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $g \in R(Y)^*$. Then*

$$f^*(div_Y(g)) = div_X(g)$$

in $Z_{n+r-1}(X)$.

Proof. Note that since f is flat it is dominant so that f induces an embedding $R(Y) \subset R(X)$, and hence we may think of g as an element of $R(X)^*$. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n + r - 1$. Let $\xi \in Z$ be its generic point. If $\dim_\delta(f(Z)) > n - 1$, then we see that the coefficient of $[Z]$ in the left and right hand side of the equation is zero. Hence we may assume that $Z' = \overline{f(Z)}$ is an integral closed subscheme of Y of δ-dimension $n - 1$. Let $\xi' = f(\xi)$. It is the generic point of Z'. Set $A = \mathcal{O}_{Y,\xi'}$, $B = \mathcal{O}_{X,\xi}$. The ring map $A \to B$ is a flat local homomorphism of Noetherian local domains of dimension 1. We have $g \in f.f.(A)$. What we have to show is that

$$\text{ord}_A(g)\text{length}_B(B/\mathfrak{m}_A B) = \text{ord}_B(g).$$

This follows from Algebra, Lemma 50.13 (details omitted). $\square$

18. Two fun results on principal divisors

The first lemma implies that the pushforward of a principal divisor along a generically finite morphism is a principal divisor.

Lemma 18.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(X) = \dim_\delta(Y)$. Let $p : X \to Y$ be a dominant proper morphism. Let $f \in R(X)^*$. Set*

$$g = Nm_{R(X)/R(Y)}(f).$$

Then we have $p_* div(f) = div(g)$.

Proof. Let $Z \subset Y$ be an integral closed subscheme of δ-dimension $n - 1$. We want to show that the coefficient of $[Z]$ in $p_*\text{div}(f)$ and $\text{div}(g)$ are equal. We may apply Lemma 16.4 to the morphism $p : X \to X$ and the generic point $\xi \in Z$. Hence we may replace X by an affine open neighbourhood of ξ and assume that $p : Y \to X$ is finite. Write $X = \text{Spec}(R)$ and $Y = \text{Spec}(A)$ with p induced by a finite homomorphism $R \to A$ of Noetherian domains which induces an finite field extension $f.f.(R) \subset f.f.(A)$ of fraction fields. Now we have $f \in f.f.(A)$, $g = \text{Nm}(f) \in f.f.(R)$, and a prime $\mathfrak{p} \subset R$ with $\dim(R_\mathfrak{p}) = 1$. The coefficient of $[Z]$ in $\text{div}_Y(g)$ is $\text{ord}_{R_\mathfrak{p}}(g)$. The coefficient of $[Z]$ in $p_*\text{div}_X(f)$ is

$$\sum\nolimits_{\mathfrak{q} \text{ lying over } \mathfrak{p}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})]\text{ord}_{A_\mathfrak{q}}(f)$$

The desired equality therefore follows from Algebra, Lemma 117.8. $\square$

The following lemma says that the degree of a principal divisor on a proper curve is zero.

Lemma 18.2. *Let K be any field. Let X be a 1-dimensional integral scheme endowed with a proper morphism $c : X \to \text{Spec}(K)$. Let $f \in K(X)^*$ be an invertible rational function. Then*

$$\sum\nolimits_{x \in X \ closed} [\kappa(x) : K] ord_{\mathcal{O}_{X,x}}(f) = 0$$

where ord is as in Algebra, Definition 117.2. In other words, $c_* div(f) = 0$.

Proof. Consider the diagram

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "c"] \\ \mathbf{P}^1_K \arrow[r, "c'"] & \mathrm{Spec}(K) \end{tikzcd}$$

that we constructed in Lemma 17.3 starting with X and the rational function f over $S = \mathrm{Spec}(K)$. We will use all the results of this lemma without further mention. We have to show that $c_*\mathrm{div}_X(f) = c_*p_*\mathrm{div}_Y(f) = 0$. This is the same as proving that $c'_*q_*\mathrm{div}_Y(f) = 0$. If $q(Y)$ is a closed point of $\mathbf{P}^1_K$ then we see that $\mathrm{div}_X(f) = 0$ and the lemma holds. Thus we may assume that q is dominant. Since $\mathrm{div}_Y(f) = [q^{-1}D_0]_0 - [q^{-1}D_\infty]_0$ we see (by definition of flat pullback) that $\mathrm{div}_Y(f) = q^*([D_0]_0 - [D_\infty]_0)$. Suppose we can show that $q : Y \to \mathbf{P}^1_K$ is finite locally free of degree d (see Morphisms, Definition 46.1). Then byy Lemma 15.2 we get $q_*\mathrm{div}_Y(f) = d([D_0]_0 - [D_\infty]_0)$. Since clearly $c'_*[D_0]_0 = c'_*[D_\infty]_0$ we win.

It remains to show that q is finite locally free. (It will automatically have some given degree as $\mathbf{P}^1_K$ is connected.) Since $\dim(\mathbf{P}^1_K) = 1$ we see that q is finite for example by Lemma 16.4. All local rings of $\mathbf{P}^1_K$ at closed points are regular local rings of dimension 1 (in other words discrete valuation rings), since they are localizations of $K[T]$ (see Algebra, Lemma 110.1). Hence for $y \in Y$ closed the local ring $\mathcal{O}_{Y,y}$ will be flat over $\mathcal{O}_{\mathbf{P}^1_K, q(y)}$ as soon as it is torsion free. This is obviously the case as $\mathcal{O}_{Y,y}$ is a domain and q is dominant. Thus q is flat. Hence q is finite locally free by Morphisms, Lemma 46.2. $\square$

19. Rational equivalence

In this section we define *rational equivalence* on k-cycles. We will allow locally finite sums of images of principal divisors (under closed immersions). This leads to some pretty strange phenomena, see Example 19.3. However, if we do not allow these then we do not know how to prove that capping with chern classes of line bundles factors through rational equivalence.

Definition 19.1. Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) Given any locally finite collection $\{W_j \subset X\}$ of integral closed subschemes with $\dim_\delta(W_j) = k+1$, and any $f_j \in R(W_j)^*$ we may consider
$$\sum (i_j)_*\mathrm{div}(f_j) \in Z_k(X)$$
where $i_j : W_j \to X$ is the inclusion morphism. This makes sense as the morphism $\coprod i_j : \coprod W_j \to X$ is proper.
(2) We say that $\alpha \in Z_k(X)$ is *rationally equivalent to zero* if α is a cycle of the form displayed above.
(3) We say $\alpha, \beta \in Z_k(X)$ are *rationally equivalent* and we write $\alpha \sim_{rat} \beta$ if $\alpha - \beta$ is rationally equivalent to zero.
(4) We define
$$A_k(X) = Z_k(X)/\sim_{rat}$$
to be the *Chow group of k-cycles on X*. This is sometimes called the *Chow group of k-cycles module rational equivalence on X*.

There are many other interesting (adequate) equivalence relations. Rational equivalence is the coarsest one of them all. A very simple but important lemma is the following.

Lemma 19.2. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. Let $k \in \mathbf{Z}$. Suppose $\alpha, \beta \in Z_k(X)$. If $\alpha|_U \sim_{rat} \beta|_U$ then there exist a cycle $\gamma \in Z_k(Y)$ such that*

$$\alpha \sim_{rat} \beta + i_*\gamma.$$

In other words, the sequence

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j_*} A_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. Let $\{W_j\}_{j\in J}$ be a locally finite collection of integral closed subschemes of δ-dimension $k+1$, and let $f_j \in R(W_j)^*$ be elements such that $(\alpha - \beta)|_U = \sum (i_j)_*\text{div}(f_j)$ as in the definition. Set $W_j' \subset X$ equal to the closure of W_j. Suppose that $V \subset X$ is a quasi-compact open. Then also $V \cap U$ is quasi-compact open in U as V is Noetherian. Hence the set $\{j \in J \mid W_j \cap V \neq \emptyset\} = \{j \in J \mid W_j' \cap V \neq \emptyset\}$ is finite since $\{W_j\}$ is locally finite. In other words we see that $\{W_j'\}$ is also locally finite. Since $R(W_j) = R(W_j')$ we see that

$$\alpha - \beta - \sum (i_j')_*\text{div}(f_j)$$

is a cycle supported on Y and the lemma follows (see Lemma 14.2). □

Example 19.3. Here is a "strange" example. Suppose that S is the spectrum of a field k with δ as in Example 7.2. Suppose that $X = C_1 \cup C_2 \cup \ldots$ is an infinite union of curves $C_j \cong \mathbf{P}^1_k$ glued together in the following way: The point $\infty \in C_j$ is glued transversally to the point $0 \in C_{j+1}$ for $j = 1, 2, 3, \ldots$. Take the point $0 \in C_1$. This gives a zero cycle $[0] \in Z_0(X)$. The "strangeness" in this situation is that actually $[0] \sim_{rat} 0$! Namely we can choose the rational function $f_j \in R(C_j)$ to be the function which has a simple zero at 0 and a simple pole at ∞ and no other zeros or poles. Then we see that the sum $\sum (i_j)_*\text{div}(f_j)$ is exactly the 0-cycle $[0]$. In fact it turns out that $A_0(X) = 0$ in this example. If you find this too bizarre, then you can just make sure your spaces are always quasi-compact (so X does not even exist for you).

Remark 19.4. Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Suppose we have infinite collections $\alpha_i, \beta_i \in Z_k(X)$, $i \in I$ of k-cycles on X. Suppose that the supports of α_i and β_i form locally finite collections of closed subsets of X so that $\sum \alpha_i$ and $\sum \beta_i$ are defined as cycles. Moreover, assume that $\alpha_i \sim_{rat} \beta_i$ for each i. Then it is not clear that $\sum \alpha_i \sim_{rat} \sum \beta_i$. Namely, the problem is that the rational equivalences may be given by locally finite families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ but the union $\{W_{i,j}\}_{i\in I, j\in J_i}$ may not be locally finite.

In many cases in practice, one has a locally finite family of closed subsets $\{T_i\}_{i\in I}$ such that α_i, β_i are supported on T_i and such that $\alpha_i = \beta_i$ in $A_k(T_i)$, in other words, the families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j\in J_i}$ consist of subschemes $W_{i,j} \subset T_i$. In this case it is true that $\sum \alpha_i \sim_{rat} \sum \beta_i$ on X, simply because the family $\{W_{i,j}\}_{i\in I, j\in J_i}$ is automatically locally finite in this case.

20. Properties of rational equivalence

Lemma 20.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be schemes locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\alpha \sim_{rat} \beta$ be rationally equivalent k-cycles on Y. Then $f^*\alpha \sim_{rat} f^*\beta$ as $(k+r)$-cycles on X.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow Y$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on Y. Then we have to show that

$$f^*(\sum i_{j,*}\text{div}(f_j))$$

is rationally equivalent to zero on X.

Consider the fibre products

$$i'_j : W'_j = W_j \times_Y X \longrightarrow X.$$

For each j, consider the collection $\{W'_{j,l}\}_{l \in L_j}$ of irreducible components $W'_{j,l} \subset W'_j$ having δ-dimension $k+1$. We may write

$$[W'_j]_{k+1} = \sum\nolimits_{l \in L_j} n_{j,l}[W'_{j,l}]_{k+1}$$

for some $n_{j,l} > 0$. By Lemma 13.1 we see that $W'_{j,l} \to W_j$ is dominant and hence we can let $f_{j,l} \in R(W'_{j,l})^*$ denote the image of f_j under the map of fields $R(W_j) \to R(W'_{j,l})$. We claim that

1. the collection $\{W'_{j,l}\}_{j \in J, l \in L_j}$ is locally finite on X, and
2. with obvious notation $f^*(\sum i_{j,*}\text{div}(f_j)) = \sum i'_{j,l,*}\text{div}(f_{j,l}^{n_{j,l}})$.

Clearly this claim implies the lemma.

To show (1), note that $\{W'_j\}$ is a locally finite collection of closed subschemes of X by Lemma 13.2. Hence if $U \subset X$ is quasi-compact, then U meets only finitely many W'_j. By Lemma 9.1 the collection of irreducible components of each W_j is locally finite as well. Hence we see only finitely many $W'_{j,l}$ meet U as desired.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension $k+r$. We have to show that the coefficient n of $[Z]$ in $f^*(\sum i_{j,*}\text{div}(f_j))$ is equal to the coefficient m of $[Z]$ in $\sum i'_{j,l,*}\text{div}(f_{j,l}^{n_{j,l}})$. Let Z' be the closure of $f(Z)$ which is an integral closed subscheme of Y. By Lemma 13.1 we have $\dim_\delta(Z') \geq k$. If $\dim_\delta(Z') > k$, then the coefficients n and m are both zero, since the generic point of Z will not be contained in any W'_j or $W'_{j,l}$. Hence we may assume that $\dim_\delta(Z') = k$.

We are going to translate the equality of n and m into algebra. Namely, let $\xi' \in Z'$ and $\xi \in Z$ be the generic points. Set $A = \mathcal{O}_{Y,\xi'}$ and $B = \mathcal{O}_{X,\xi}$. Note that A, B are Noetherian, $A \to B$ is flat, local, and that $\mathfrak{m}_A B$ is an ideal of definition of the local ring B. There are finitely many j such that W_j passes through ξ', and these correspond to prime ideals

$$\mathfrak{p}_1, \ldots, \mathfrak{p}_T \subset A$$

with the property that $\dim(A/\mathfrak{p}_t) = 1$ for each $t = 1, \ldots, T$. The rational functions f_j correspond to elements $f_t \in \kappa(\mathfrak{p}_t)^*$. Say $\mathfrak{p}_t$ corresponds to W_j. By construction, the closed subschemes $W'_{j,l}$ which meet ξ correspond $1-1$ with minimal primes

$$\mathfrak{p}_t B \subset \mathfrak{q}_{t,1}, \ldots, \mathfrak{q}_{t,S_t} \subset B$$

over $\mathfrak{p}_t B$. The integers $n_{j,l}$ correspond to the integers

$$n_{t,s} = \text{length}_{B_{\mathfrak{q}_{t,s}}}((B/\mathfrak{p}_t B)_{B_{\mathfrak{q}_{t,s}}})$$

The rational functions $f_{j,l}$ correspond to the images $f_{t,s} \in \kappa(\mathfrak{q}_{t,s})^*$ of the elements $f_t \in \kappa(\mathfrak{p}_t)^*$. Putting everything together we see that

$$n = \sum \text{ord}_{A/\mathfrak{p}_t}(f_t)\text{length}_B(B/\mathfrak{m}_A B)$$

and that

$$m = \sum \text{ord}_{B/\mathfrak{q}_{t,s}}(f_{t,s})\text{length}_{B_{\mathfrak{q}_{t,s}}}((B/\mathfrak{p}_t B)_{B_{\mathfrak{q}_{t,s}}})$$

Note that it suffices to prove the equality for each $t \in \{1, \ldots, T\}$ separately. Writing $f_t = x/y$ for some nonzero $\overline{x}, \overline{y} \in A/\mathfrak{p}_t$ coming from $x, y \in A$ we see that it suffices to prove

$$\text{length}_{A/\mathfrak{p}_t}(A/(\mathfrak{p}_t, x))\text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(B/(x, \mathfrak{p}_t)B)$$

(equality uses Algebra, Lemma 50.13) equals

$$\sum\nolimits_{s=1,\ldots,S_t} \text{ord}_{B/\mathfrak{q}_{t,s}}(B/(x, \mathfrak{q}_{t,s}))\text{length}_{B_{\mathfrak{q}_{t,s}}}((B/\mathfrak{p}_t B)_{B_{\mathfrak{q}_{t,s}}})$$

and similarly for y. Note that as $x \notin \mathfrak{p}_t$ we see that x is a nonzerodivisor on $A/\mathfrak{p}_t$. As $A \to B$ is flat it follows that x is a nonzerodivisor on the module $M = B/\mathfrak{p}_t B$. Hence the equality above follows from Lemma 5.6. $\square$

Lemma 20.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be schemes locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Suppose $\alpha, \beta \in Z_k(X)$ are rationally equivalent. Then $p_*\alpha$ is rationally equivalent to $p_*\beta$.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow X$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on X. Then we have to show that

$$p_*\left(\sum i_{j,*}\text{div}(f_j)\right)$$

is rationally equivalent to zero on X.

Note that the sum is equal to

$$\sum p_* i_{j,*}\text{div}(f_j).$$

Let $W'_j \subset Y$ be the integral closed subscheme which is the image of $p \circ i_j$. The collection $\{W'_j\}$ is locally finite in Y by Lemma 11.2. Hence it suffices to show, for a given j, that either $p_* i_{j,*}\text{div}(f_j) = 0$ or that it is equal to $i'_{j,*}\text{div}(g_j)$ for some $g_j \in R(W'_j)^*$.

The arguments above therefore reduce us to the case of a since integral closed subscheme $W \subset X$ of δ-dimension $k+1$. Let $f \in R(W)^*$. Let $W' = p(W)$ as above. We get a commutative diagram of morphisms

$$\begin{tikzcd} W \arrow[r, "i"'] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ W' \arrow[r, "i'"] & Y \end{tikzcd}$$

Note that $p_* i_* \text{div}(f) = i'_*(p')_* \text{div}(f)$ by Lemma 12.2. As explained above we have to show that $(p')_* \text{div}(f)$ is the divisor of a rational function on W' or zero. There are three cases to distinguish.

The case $\dim_\delta(W') < k$. In this case automatically $(p')_* \text{div}(f) = 0$ and there is nothing to prove.

The case $\dim_\delta(W') = k$. Let us show that $(p')_* \text{div}(f) = 0$ in this case. Let $\eta \in W'$ be the generic point. Note that $c : W_\eta \to \text{Spec}(K)$ is a proper integral curve over $K = \kappa(\eta)$ whose function field $K(W_\eta)$ is identified with $R(W)$. Here is a diagram

$$\begin{tikzcd} W_\eta \arrow[r] \arrow[d, "c"'] & W \arrow[d, "p'"] \\ \text{Spec}(K) \arrow[r] & W' \end{tikzcd}$$

Let us denote $f_\eta \in K(W_\eta)^*$ the rational function corresponding to $f \in R(W)^*$. Moreover, the closed points ξ of W_η correspond $1-1$ to the closed integral subschemes $Z = Z_\xi \subset W$ of δ-dimension k with $p'(Z) = W'$. Note that the multiplicity of Z_ξ in $\text{div}(f)$ is equal to $\text{ord}_{\mathcal{O}_{W_\eta,\xi}}(f_\eta)$ simply because the local rings $\mathcal{O}_{W_\eta,\xi}$ and $\mathcal{O}_{W,\xi}$ are identified (as subrings of their fraction fields). Hence we see that the multiplicity of $[W']$ in $(p')_* \text{div}(f)$ is equal to the multiplicity of $[\text{Spec}(K)]$ in $c_* \text{div}(f_\eta)$. By Lemma 18.2 this is zero.

The case $\dim_\delta(W') = k+1$. In this case Lemma 18.1 applies, and we see that indeed $p'_* \text{div}(f) = \text{div}(g)$ for some $g \in R(W')^*$ as desired. $\square$

21. Different characterizations of rational equivalence

Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Given any closed subscheme $Z \subset X \times_S \mathbf{P}^1_S = X \times \mathbf{P}^1$ we let Z_0, resp. Z_∞ be the scheme theoretic closed subscheme $Z_0 = \text{pr}_2^{-1}(D_0)$, resp. $Z_\infty = \text{pr}_2^{-1}(D_\infty)$. Here D_0, D_∞ are as defined just above Lemma 17.3.

Lemma 21.1. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $W \subset X \times_S \mathbf{P}^1_S$ be an integral closed subscheme of δ-dimension $k+1$. Assume $W \neq W_0$, and $W \neq W_\infty$. Then*

(1) *W_0, W_∞ are effective Cartier divisors of W,*
(2) *W_0, W_∞ can be viewed as closed subschemes of X and*
$$[W_0]_k \sim_{rat} [W_\infty]_k,$$
(3) *for any locally finite family of integral closed subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ of δ-dimension $k+1$ with $W_i \neq (W_i)_0$ and $W_i \neq (W_i)_\infty$ we have $\sum([(W_i)_0]_k - [(W_i)_\infty]_k) \sim_{rat} 0$ on X, and*

(4) *for any* $\alpha \in Z_k(X)$ *with* $\alpha \sim_{rat} 0$ *there exists a locally finite family of integral closed subschemes* $W_i \subset X \times_S \mathbf{P}^1_S$ *as above such that* $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$.

Proof. Part (1) follows from Divisors, Lemma 9.12 since the generic point of W is not mapped into D_0 or D_∞ under the projection $X \times_S \mathbf{P}^1_S \to \mathbf{P}^1_S$ by assumption.

Since $X \times_S D_0 \to X$ is an isomorphism we see that W_0 is isomorphic to a closed subscheme of X. Similarly for W_∞. Consider the morphism $p : W \to X$. It is proper and on W we have $[W_0]_k \sim_{rat} [W_\infty]_k$. Hence part (2) follows from Lemma 20.2 as clearly $p_*[W_0]_k = [W_0]_k$ and similarly for W_∞.

The only content of statement (3) is, given parts (1) and (2), that the collection $\{(W_i)_0, (W_i)_\infty\}$ is a locally finite collection of closed subschemes of X. This is clear.

Suppose that $\alpha \sim_{rat} 0$. By definition this means there exist integral closed subschemes $V_i \subset X$ of δ-dimension $k+1$ and rational functions $f_i \in R(V_i)^*$ such that the family $\{V_i\}_{i\in I}$ is locally finite in X and such that $\alpha = \sum (V_i \to X)_*\mathrm{div}(f_i)$. Let

$$W_i \subset V_i \times_S \mathbf{P}^1_S \subset X \times_S \mathbf{P}^1_S$$

be the closure of the graph of the rational map f_i as in Lemma 17.3. Then we have that $(V_i \to X)_*\mathrm{div}(f_i)$ is equal to $[(W_i)_0]_k - [(W_i)_\infty]_k$ by that same lemma. Hence the result is clear. □

Lemma 21.2. *Let* (S, δ) *be as in Situation 7.1. Let* X *be a scheme locally of finite type over* S. *Let* Z *be a closed subscheme of* $X \times \mathbf{P}^1$. *Assume* $\dim_\delta(Z) \leq k+1$, $\dim_\delta(Z_0) \leq k$, $\dim_\delta(Z_\infty) \leq k$ *and assume any embedded point* ξ *(Divisors, Definition 4.1) of* Z *has* $\delta(\xi) < k$. *Then*

$$[Z_0]_k \sim_{rat} [Z_\infty]_k$$

as k*-cycles on* X.

Proof. Let $\{W_i\}_{i\in I}$ be the collection of irreducible components of Z which have δ-dimension $k+1$. Write

$$[Z]_{k+1} = \sum n_i[W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Lemma 9.1. We claim that

$$[Z_0]_k = \sum n_i[(W_i)_0]_k$$

and similarly for $[Z_\infty]_k$. If we prove this then the lemma follows from Lemma 21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[Z_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i[(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X\times_S\mathbf{P}^1_S,\xi}$. Let $I \subset A$ be the ideal cutting out Z, in other words so that $A/I = \mathcal{O}_{Z,\xi}$. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(A/I) = 1$. Since ξ is not an embedded point by definition we see that A/I is Cohen-Macaulay. Since $\dim_\delta(Z_0) = k$ we see that $\dim(A/(t, I)) = 0$ which implies that t is a nonzerodivisor on A/I. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal primes $I \subset \mathfrak{q}_i$ over I. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(A/(t, I))$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i))\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 5.6. □

Lemma 21.3. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on $X \times \mathbf{P}^1$. Let $i_0, i_\infty : X \to X \times \mathbf{P}^1$ be the closed immersion such that $i_t(x) = (x, t)$. Denote $\mathcal{F}_0 = i_0^*\mathcal{F}$ and $\mathcal{F}_\infty = i_\infty^*\mathcal{F}$. Assume*

1. $\dim_\delta(\mathit{Supp}(\mathcal{F})) \leq k + 1$,
2. $\dim_\delta(\mathit{Supp}(\mathcal{F}_0)) \leq k$, $\dim_\delta(\mathit{Supp}(\mathcal{F}_\infty)) \leq k$, *and*
3. *any nonmaximal associated point (insert future reference here)* $\xi \in \mathit{Supp}(\mathcal{F})$ *of* $\mathcal{F}$ *has* $\delta(\xi) < k$.

Then

$$[\mathcal{F}_0]_k \sim_{rat} [\mathcal{F}_\infty]_k$$

as k-cycles on X.

Proof. Let $\{W_i\}_{i \in I}$ be the collection of irreducible components of $\text{Supp}(\mathcal{F})$ which have δ-dimension $k + 1$. Write

$$[\mathcal{F}]_{k+1} = \sum n_i[W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Lemma 10.1. We claim that

$$[\mathcal{F}_0]_k = \sum n_i[(W_i)_0]_k$$

and similarly for $[\mathcal{F}_\infty]_k$. If we prove this then the lemma follows from Lemma 21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[\mathcal{F}_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i[(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X \times_S \mathbf{P}^1_S, \xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(M) = 1$. Since ξ is not an associated point of $\mathcal{F}$ by definition we see that M is Cohen-Macaulay module. Since $\dim_\delta(\text{Supp}(\mathcal{F}_0)) = k$ we see that $\dim(M/tM) = 0$ which implies that t is a nonzerodivisor on M. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\text{Ass}(M)$. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(M/tM)$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i)A)\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 5.6. □

22. Rational equivalence and K-groups

In this section we compare the cycle groups $Z_k(X)$ and the Chow groups $A_k(X)$ with certain K_0-groups of abelian categories of coherent sheaves on X. We avoid having to talk about $K_1(\mathcal{A})$ for an abelian category $\mathcal{A}$ by dint of Homology, Lemma 10.3. In particular, the motivation for the precise form of Lemma 22.4 is that lemma.

Let us introduce the following notation. Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. We denote $\textit{Coh}(X) = \textit{Coh}(\mathcal{O}_X)$ the category of coherent sheaves on X. It is an abelian category, see Cohomology of Schemes, Lemma 9.2. For any $k \in \mathbf{Z}$ we let $\textit{Coh}_{\leq k}(X)$ be the full subcategory of $\textit{Coh}(X)$ consisting of those coherent sheaves $\mathcal{F}$ having $\dim_\delta(\text{Supp}(\mathcal{F})) \leq k$.

Lemma 22.1. *Let us introduce the following notation. Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. The categories $\textit{Coh}_{\leq k}(X)$ are Serre subcategories of the abelian category $\textit{Coh}(X)$.*

Proof. Omitted. The definition of a Serre subcategory is Homology, Definition 9.1. $\square$

Lemma 22.2. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. There are maps*

$$Z_k(X) \longrightarrow K_0(\textit{Coh}_{\leq k}(X)/\textit{Coh}_{\leq k-1}(X)) \longrightarrow Z_k(X)$$

whose composition is the identity. The first is the map

$$\sum n_Z[Z] \mapsto \left[\bigoplus\nolimits_{n_Z > 0} \mathcal{O}_Z^{\oplus n_Z}\right] - \left[\bigoplus\nolimits_{n_Z < 0} \mathcal{O}_Z^{\oplus -n_Z}\right]$$

and the second comes from the map $\mathcal{F} \mapsto [\mathcal{F}]_k$. If X is quasi-compact, then both maps are isomorphisms.

Proof. Note that the direct sum $\bigoplus_{n_Z>0} \mathcal{O}_Z^{\oplus n_Z}$ is indeed a coherent sheaf on X since the family $\{Z \mid n_Z > 0\}$ is locally finite on X. The map $\mathcal{F} \to [\mathcal{F}]_k$ is additive on $\textit{Coh}_{\leq k}(X)$, see Lemma 10.4. And $[\mathcal{F}]_k = 0$ if $\mathcal{F} \in \textit{Coh}_{\leq k-1}(X)$. This implies we have the left map as shown in the lemma. It is clear that their composition is the identity.

In case X is quasi-compact we will show that the right arrow is injective. Suppose that $q \in K_0(\textit{Coh}_{\leq k}(X)/\textit{Coh}_{\leq k+1}(X))$ maps to zero in $Z_k(X)$. By Homology, Lemma 10.3 we can find a $\tilde{q} \in K_0(\textit{Coh}_{\leq k}(X))$ mapping to q. Write $\tilde{q} = [\mathcal{F}] - [\mathcal{G}]$ for some $\mathcal{F}, \mathcal{G} \in K_0(\textit{Coh}_{\leq k}(X))$. Since X is quasi-compact we may apply Cohomology of Schemes, Lemma 12.3. This shows that there exist integral closed subschemes $Z_j, T_i \subset X$ and (nonzero) ideal sheaves $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$, $\mathcal{I}_i \subset \mathcal{O}_{T_i}$ such that $\mathcal{F}$, resp. $\mathcal{G}$ have filtrations whose successive quotients are the sheaves $\mathcal{I}_j$, resp. $\mathcal{I}_i$. In particular we see that $\dim_\delta(Z_j), \dim_\delta(T_i) \leq k$. In other words we have

$$[\mathcal{F}] = \sum\nolimits_j [\mathcal{I}_j], \quad [\mathcal{G}] = \sum\nolimits_i [\mathcal{I}_i],$$

in $K_0(\textit{Coh}_{\leq k}(X))$. Our assumption is that $\sum_j [\mathcal{I}_j]_k - \sum_i [\mathcal{I}_i]_k = 0$. It is clear that we may throw out the indices j, resp. i such that $\dim_\delta(Z_j) < k$, resp. $\dim_\delta(T_i) < k$, since the corresponding sheaves are in $\textit{Coh}_{k-1}(X)$ and also do not contribute to the cycle. Moreover, the exact sequences $0 \to \mathcal{I}_j \to \mathcal{O}_{Z_j} \to \mathcal{O}_{Z_j}/\mathcal{I}_j \to 0$ and

$0 \to \mathcal{I}_i \to \mathcal{O}_{T_i} \to \mathcal{O}_{Z_i}/\mathcal{I}_i \to 0$ show similarly that we may replace $\mathcal{I}_j$, resp. $\mathcal{I}_i$ by $\mathcal{O}_{Z_j}$, resp. $\mathcal{O}_{T_i}$. OK, and finally, at this point it is clear that our assumption

$$\sum\nolimits_j [\mathcal{O}_{Z_j}]_k - \sum\nolimits_i [\mathcal{O}_{T_i}]_k = 0$$

implies that in $K_0(\textit{Coh}_k(X))$ we have also $\sum_j [\mathcal{O}_{Z_j}] - \sum_i [\mathcal{O}_{T_i}] = 0$ as desired. $\square$

Remark 22.3. It seems likely that the arrows of Lemma 22.2 are not isomorphisms if X is not quasi-compact. For example, suppose X is an infinite disjoint union $X = \coprod_{n \in \mathbf{N}} \mathbf{P}^1_k$ over a field k. Let $\mathcal{F}$, resp. $\mathcal{G}$ be the coherent sheaf on X whose restriction to the nth summand is equal to the skyscraper sheaf at 0 associated to $\mathcal{O}_{\mathbf{P}^1_k, 0}/\mathfrak{m}_0^n$, resp. $\kappa(0)^{\oplus n}$. The cycle associated to $\mathcal{F}$ is equal to the cycle associated to $\mathcal{G}$, namely both are equal to $\sum n[0_n]$ where $0_n \in X$ denotes 0 on the nth component of X. But there seems to be no way to show that $[\mathcal{F}] = [\mathcal{G}]$ in $K_0(\textit{Coh}(X))$ since any proof we can envision uses infinitely many relations.

Lemma 22.4. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let*

$$\ldots \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \longrightarrow \ldots$$

be a complex as in Homology, Equation (10.2.1). Assume that

1. $\dim_\delta(\textit{Supp}(\mathcal{F})) \leq k+1$.
2. $\dim_\delta(\textit{Supp}(H^i(\mathcal{F}, \varphi, \psi))) \leq k$ *for* $i = 0, 1$.

Then we have

$$[H^0(\mathcal{F}, \varphi, \psi)]_k \sim_{rat} [H^1(\mathcal{F}, \varphi, \psi)]_k$$

as k-cycles on X.

Proof. Let $\{W_j\}_{j \in J}$ be the collection of irreducible components of $\text{Supp}(\mathcal{F})$ which have δ-dimension $k+1$. Note that $\{W_j\}$ is a locally finite collection of closed subsets of X by Lemma 10.1. For every j, let $\xi_j \in W_j$ be the generic point. Set

$$f_j = \det\nolimits_{\kappa(\xi_j)}(\mathcal{F}_{\xi_j}, \varphi_{\xi_j}, \psi_{\xi_j}) \in R(W_j)^*.$$

See Definition 3.4 for notation. We claim that

$$-[H^0(\mathcal{F}, \varphi, \psi)]_k + [H^1(\mathcal{F}, \varphi, \psi)]_k = \sum (W_j \to X)_* \text{div}(f_j)$$

If we prove this then the lemma follows.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z]$ in $[H^0(\mathcal{F}, \varphi, \psi)]_k - [H^1(\mathcal{F}, \varphi, \psi)]_k$ is the same as the coefficient m of $[Z]$ in $\sum (W_j \to X)_* \text{div}(f_j)$. Let $\xi \in Z$ be the generic point. Consider the local ring $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Denote $\varphi, \psi : M \to M$ the action of φ, ψ on the stalk. By our choice of $\xi \in Z$ we have $\delta(\xi) = k$ and hence $\dim(M) = 1$. Finally, the integral closed subschemes W_j passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\text{Supp}(M)$. In each case the element $f_j \in R(W_j)^*$ corresponds to the element $\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi)$ in $\kappa(\mathfrak{q}_i)^*$. Hence we see that

$$n = -e_A(M, \varphi, \psi)$$

and

$$m = \sum \text{ord}_{A/\mathfrak{q}_i}(\det\nolimits_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi))$$

Thus the result follows from Proposition 5.3. $\square$

Lemma 22.5. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Denote $B_k(X)$ the image of the map*

$$K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X)).$$

There is a commutative diagram

$$\begin{tikzcd} K_0\left(\frac{Coh_{\leq k}(X)}{Coh_{\leq k-1}(X)}\right) \arrow[r] \arrow[d] & B_k(X) \arrow[r, hook] \arrow[d] & K_0\left(\frac{Coh_{\leq k+1}(X)}{Coh_{\leq k-1}(X)}\right) \\ Z_k(X) \arrow[r] & A_k(X) & \end{tikzcd}$$

where the left vertical arrow is the one from Lemma 22.2. If X is quasi-compact then both vertical arrows are isomorphisms.

Proof. Suppose we have an element $[A] - [B]$ of $K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$ which maps to zero in $B_k(X)$, i.e., in $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$. Suppose $[A] = [\mathcal{A}]$ and $[B] = [\mathcal{B}]$ for some coherent sheaves $\mathcal{A}, \mathcal{B}$ on X supported in δ-dimension $\leq k$. The assumption that $[A] - [B]$ maps to zero in the group $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$ means that there exists coherent sheaves $\mathcal{A}', \mathcal{B}'$ on X supported in δ-dimension $\leq k-1$ such that $[\mathcal{A} \oplus \mathcal{A}'] - [\mathcal{B} \oplus \mathcal{B}']$ is zero in $K_0(Coh_{k+1}(X))$ (use part (1) of Homology, Lemma 10.3). By part (2) of Homology, Lemma 10.3 this means there exists a $(2,1)$-periodic complex $(\mathcal{F}, \varphi, \psi)$ in the category $Coh_{\leq k+1}(X)$ such that $\mathcal{A} \oplus \mathcal{A}' = H^0(\mathcal{F}, \varphi, \psi)$ and $\mathcal{B} \oplus \mathcal{B}' = H^1(\mathcal{F}, \varphi, \psi)$. By Lemma 22.4 this implies that

$$[\mathcal{A} \oplus \mathcal{A}']_k \sim_{rat} [\mathcal{B} \oplus \mathcal{B}']_k$$

This proves that $[A] - [B]$ maps to zero via the composition

$$K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow Z_k(X) \longrightarrow A_k(X).$$

In other words this proves the commutative diagram exists.

Next, assume that X is quasi-compact. By Lemma 22.2 the left vertical arrow is bijective. Hence it suffices to show any $\alpha \in Z_k(X)$ which is rationally equivalent to zero maps to zero in $B_k(X)$ via the inverse of the left vertical arrow composed with the horizontal arrow. By Lemma 21.1 we see that $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$ for some closed integral subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ of δ-dimension $k+1$. Moreover the family $\{W_i\}$ is finite because X is quasi-compact. Note that the ideal sheaves $\mathcal{I}_i, \mathcal{J}_i \subset \mathcal{O}_{W_i}$ of the effective Cartier divisors $(W_i)_0, (W_i)_\infty$ are isomorphic (as $\mathcal{O}_{W_i}$-modules). This is true because the ideal sheaves of D_0 and D_∞ on $\mathbf{P}^1$ are isomorphic and $\mathcal{I}_i, \mathcal{J}_i$ are the pullbacks of these. (Some details omitted.) Hence we have short exact sequences

$$0 \to \mathcal{I}_i \to \mathcal{O}_{W_i} \to \mathcal{O}_{(W_i)_0} \to 0, \quad 0 \to \mathcal{J}_i \to \mathcal{O}_{W_i} \to \mathcal{O}_{(W_i)_\infty} \to 0$$

of coherent $\mathcal{O}_{W_i}$-modules. Also, since $[(W_i)_0]_k = [p_*\mathcal{O}_{(W_i)_0}]_k$ in $Z_k(X)$ we see that the inverse of the left vertical arrow maps $[(W_i)_0]_k$ to the element $[p_*\mathcal{O}_{(W_i)_0}]$ in $K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X))$. Thus we have

$$\begin{aligned} \alpha &= \sum([(W_i)_0]_k - [(W_i)_\infty]_k) \\ &\mapsto \sum\left([p_*\mathcal{O}_{(W_i)_0}] - [p_*\mathcal{O}_{(W_i)_\infty}]\right) \\ &= \sum([p_*\mathcal{O}_{W_i}] - [p_*\mathcal{I}_i] - [p_*\mathcal{O}_{W_i}] + [p_*\mathcal{J}_i]) \end{aligned}$$

in $K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X))$. By what was said above this is zero, and we win. $\square$

Remark 22.6. Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Assume X is quasi-compact. The result of Lemma 22.5 in particular gives a map

$$A_k(X) \longrightarrow K_0(Coh(X)/Coh_{\leq k-1}(X)).$$

We have not been able to find a statement or conjecture in the literature as to whether this map is should be injective or not. If X is connected nonsingular, then, using the isomorphism $K_0(X) = K^0(X)$ (see insert future reference here) and chern classes (see below), one can show that the map is an isomorphism up to $(p-1)!$-torsion where $p = \dim_\delta(X) - k$.

23. Preparation for the divisor associated to an invertible sheaf

For the following remarks, see Divisors, Section 15. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\xi \in X$ be a point. If $s_\xi, s'_\xi \in \mathcal{L}_\xi$ generate $\mathcal{L}_\xi$ as $\mathcal{O}_{X,\xi}$-module, then there exists a unit $u \in \mathcal{O}^*_{X,\xi}$ such that $s_\xi = us'_\xi$. The stalk of the sheaf of meromorphic sections $\mathcal{K}_X(\mathcal{L})$ of $\mathcal{L}$ at x is equal to $\mathcal{K}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x$. Thus the image of any meromorphic section s of $\mathcal{L}$ in the stalk at x can be written as $s = fs_\xi$ with $f \in \mathcal{K}_{X,x}$. Below we will abbreviate this by saying $f = s/s_\xi$. Also, if X is integral we have $\mathcal{K}_{X,x} = R(X)$ is equal to the function field of X, so $s/s_\xi \in R(X)$. If s is a *regular* meromorphic section (see Divisors, Definition 15.11), then actually $f \in R(X)^*$. (On an integral scheme a regular meromorphic section is the same thing as a nonzero meromorphic section.) Hence the following definition makes sense.

Definition 23.1. Let X be a locally Noetherian scheme. Assume X is integral. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a regular meromorphic section of $\mathcal{L}$. For every integral closed subscheme $Z \subset X$ of codimension 1 we define the *order of vanishing of s along Z* as the integer

$$\text{ord}_{Z,\mathcal{L}}(s) = \text{ord}_{\mathcal{O}_{X,\xi}}(s/s_\xi)$$

where the right hand side is the notion of Algebra, Definition 117.2, $\xi \in Z$ is the generic point, and $s_\xi \in \mathcal{L}_\xi$ is a generator.

Lemma 23.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \mathcal{K}_X(\mathcal{L})$ be a regular (i.e., nonzero) meromorphic section of $\mathcal{L}$. Then the set*

$$\{Z \subset X \mid Z \text{ is irreducible, closed of codimension } 1 \text{ and } ord_{Z,\mathcal{L}}(s) \neq 0\}$$

is locally finite in X.

Proof. This is true simply because there exists a nonempty open subscheme $U \subset X$ such that s corresponds to a section of $\Gamma(U, \mathcal{L})$ which generates $\mathcal{L}$ over U. Hence the codimension 1 irreducibles which can occur in the set of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 9.1 gives the desired result. $\square$

Lemma 23.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let*

$s, s' \in \mathcal{K}_X(\mathcal{L})$ be nonzero meromorphic sections of $\mathcal{L}$. Then $f = s/s'$ is an element of $R(X)^*$ and we have

$$\sum ord_{Z,\mathcal{L}}(s)[Z] = \sum ord_{Z,\mathcal{L}}(s')[Z] + div(f)$$

(where the sums are over integral closed subschemes $Z \subset X$ of δ-dimension $n-1$) as elements of $Z_{n-1}(X)$.

Proof. This is clear from the definitions. Note that Lemma 23.2 guarantees that the sums are indeed elements of $Z_{n-1}(X)$. □

24. The divisor associated to an invertible sheaf

The material above allows us to define the divisor associated to an invertible sheaf.

Definition 24.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the *Weil divisor associated to s* as

$$\mathrm{div}_\mathcal{L}(s) := \sum \mathrm{ord}_{Z,\mathcal{L}}(s)[Z] \in Z_{n-1}(X)$$

where the sum is over integral closed subschemes $Z \subset X$ of δ-dimension $n-1$.

(2) We define *Weil divisor associated to $\mathcal{L}$*

$$c_1(\mathcal{L}) \cap [X] = \text{class of } \mathrm{div}_\mathcal{L}(s) \in A_{n-1}(X)$$

where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Lemma 23.3.

There are some cases where it is easy to compute the Weil divisor associated to an invertible sheaf.

Lemma 24.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a nonzero global section. Then*

$$div_\mathcal{L}(s) = [Z(s)]_{n-1}$$

in $Z_{n-1}(X)$ and

$$c_1(\mathcal{L}) \cap [X] = [Z(s)]_{n-1}$$

in $A_{n-1}(X)$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. Let $\xi \in Z$ be its generic point. Choose a generator $s_\xi \in \mathcal{L}_\xi$. Write $s = f s_\xi$ for some $f \in \mathcal{O}_{X,\xi}$. By definition of $Z(s)$, see Divisors, Definition 9.18 we see that $Z(s)$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I}_\xi = (f)$. Hence $\mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{Z(s),\xi}) = \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,\xi}/(f)) = \mathrm{ord}_{\mathcal{O}_{X,x}}(f)$ as desired. □

Lemma 24.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Then*

(1) *Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$, and*
$$div_{\mathcal{L}\otimes\mathcal{N}}(st) = div_{\mathcal{L}}(s) + div_{\mathcal{N}}(t)$$
in $Z_{n-1}(X)$.

(2) *We have*
$$c_1(\mathcal{L}) \cap [X] + c_1(\mathcal{N}) \cap [X] = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap [X]$$
in $A_{n-1}(X)$.

Proof. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L}\otimes\mathcal{N}$. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. Let $\xi \in Z$ be its generic point. Choose generators $s_\xi \in \mathcal{L}_\xi$, and $t_\xi \in \mathcal{N}_\xi$. Then $s_\xi t_\xi$ is a generator for $(\mathcal{L} \otimes \mathcal{N})_\xi$. So $st/(s_\xi t_\xi) = (s/s_\xi)(t/t_\xi)$. Hence we see that
$$\text{div}_{\mathcal{L}\otimes\mathcal{N},Z}(st) = \text{div}_{\mathcal{L},Z}(s) + \text{div}_{\mathcal{N},Z}(t)$$
by the additivity of the ord_Z function. $\square$

The following lemma will be superseded by the more general Lemma 25.4.

Lemma 24.4. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Then*
$$f^*(c_1(\mathcal{L}) \cap [Y]) = c_1(f^*\mathcal{L}) \cap [X]$$
in $A_{n+r-1}(X)$.

Proof. Let s be a nonzero meromorphic section of $\mathcal{L}$. We will show that actually $f^*\text{div}_{\mathcal{L}}(s) = \text{div}_{f^*\mathcal{L}}(f^*s)$ and hence the lemma holds. To see this let $\xi \in Y$ be a point and let $s_\xi \in \mathcal{L}_\xi$ be a generator. Write $s = gs_\xi$ with $g \in R(X)^*$. Then there is an open neighbourhood $V \subset Y$ of ξ such that $s_\xi \in \mathcal{L}(V)$ and such that s_ξ generates $\mathcal{L}|_V$. Hence we see that
$$\text{div}_{\mathcal{L}}(s)|_V = \text{div}(g)|_V.$$
In exactly the same way, since f^*s_ξ generates $\mathcal{L}$ over $f^{-1}(V)$ and since $f^*s = gf^*s_\xi$ we also have
$$\text{div}_{\mathcal{L}}(f^*s)|_{f^{-1}(V)} = \text{div}(g)|_{f^{-1}(V)}.$$
Thus the desired equality of cycles over $f^{-1}(V)$ follows from the corresponding result for pullbacks of principal divisors, see Lemma 17.4. $\square$

25. Intersecting with Cartier divisors

Definition 25.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We define, for every integer k, an operation
$$c_1(\mathcal{L}) \cap - : Z_{k+1}(X) \to A_k(X)$$
called *intersection with the first chern class of $\mathcal{L}$*.

(1) Given an integral closed subscheme $i : W \to X$ with $\dim_\delta(W) = k+1$ we define
$$c_1(\mathcal{L}) \cap [W] = i_*(c_1(i^*\mathcal{L}) \cap [W])$$
where the right hand side is defined in Definition 24.1.

(2) For a general $(k+1)$-cycle $\alpha = \sum n_i[W_i]$ we set
$$c_1(\mathcal{L}) \cap \alpha = \sum n_i c_1(\mathcal{L}) \cap [W_i]$$

Write each $c_1(\mathcal{L}) \cap W_i = \sum_j n_{i,j}[Z_{i,j}]$ with $\{Z_{i,j}\}_j$ a locally finite sum of integral closed subschemes of W_i. Since $\{W_i\}$ is a locally finite collection of integral closed subschemes on X, it follows easily that $\{Z_{i,j}\}_{i,j}$ is a locally finite collection of closed subschemes of X. Hence $c_1(\mathcal{L}) \cap \alpha = \sum n_i n_{i,j}[Z_{i,j}]$ is a cycle. Another, more convenient, way to think about this is to observe that the morphism $\coprod W_i \to X$ is proper. Hence $c_1(\mathcal{L}) \cap \alpha$ can be viewed as the pushforward of a class in $A_k(\coprod W_i) = \prod A_k(W_i)$. This also explains why the result is well defined up to rational equivalence on X.

The main goal for the next few sections is to show that intersecting with $c_1(\mathcal{L})$ factors through rational equivalence, and is commutative. This is not a triviality.

Lemma 25.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be an invertible sheaves on X. Then*
$$c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap \alpha$$
in $A_k(X)$ for every $\alpha \in Z_{k-1}(X)$. Moreover, $c_1(\mathcal{O}_X) \cap \alpha = 0$ for all α.

Proof. The additivity follows directly from Lemma 24.3 and the definitions. To see that $c_1(\mathcal{O}_X) \cap \alpha = 0$ consider the section $1 \in \Gamma(X, \mathcal{O}_X)$. This restricts to an everywhere nonzero section on any integral closed subscheme $W \subset X$. Hence $c_1(\mathcal{O}_X) \cap [W] = 0$ as desired. $\square$

The following lemma is a useful result in order to compute the intersection product of the c_1 of an invertible sheaf and the cycle associated to a closed subscheme. Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 9.18.

Lemma 25.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a closed subscheme. Assume $\dim_\delta(Z) \leq k+1$. Let $s \in \Gamma(Z, \mathcal{L}|_Z)$. Assume*

1. *$\dim_\delta(Z(s)) \leq k$, and*
2. *for every generic point ξ of an irreducible component of $Z(s)$ of dimension k the multiplication by s induces an injection $\mathcal{O}_{Z,\xi} \to (\mathcal{L}|_Z)_\xi$.*

This holds for example if s is a regular section of $\mathcal{L}|_Z$. Then
$$[Z(s)]_k = c_1(\mathcal{L}) \cap [Z]_{k+1}$$
in $A_k(X)$.

Proof. Write
$$[Z]_{k+1} = \sum n_i[W_i]$$
where $W_i \subset Z$ are the irreducible components of Z of δ-dimension $k+1$ and $n_i > 0$. By assumption the restriction $s_i = s|_{W_i} \in \Gamma(W_i, \mathcal{L}|_{W_i})$ is not zero, and hence is a regular section. By Lemma 24.2 we see that $[Z(s_i)]_k$ represents $c_1(\mathcal{L}|_{W_i})$. Hence by definition
$$c_1(\mathcal{L}) \cap [Z]_{k+1} = \sum n_i[Z(s_i)]_k$$
In fact, the proof below will show that we have
$$[Z(s)]_k = \sum n_i[Z(s_i)]_k \tag{25.3.1}$$

as k-cycles on X.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. Let $\xi' \in Z'$ be its generic point. We want to compare the coefficient n of $[Z']$ in the expression $\sum n_i[Z(s_i)]_k$ with the coefficient m of $[Z']$ in the expression $[Z(s)]_k$. Choose a generator $s_{\xi'} \in \mathcal{L}_\xi$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Write $A = \mathcal{O}_{X,\xi'}$, $L = \mathcal{L}_{\xi'}$ and $I = \mathcal{I}_{\xi'}$. Then $L = As_{\xi'}$ and $L/IL = (A/I)s_{\xi'} = (\mathcal{L}|_Z)_{\xi'}$. Write $s = fs_{\xi'}$ for some (unique) $f \in A/I$. Hypothesis (2) means that $f : A/I \to A/I$ is injective. Since $\dim_\delta(Z) \leq k + 1$ and $\dim_\delta(Z') = k$ we have $\dim(A/I) = 0$ or 1. We have

$$m = \text{length}_A(A/(f, I))$$

which is finite in either case.

If $\dim(A/I) = 0$, then $f : A/I \to A/I$ being injective implies that $f \in (A/I)^*$. Hence in this case m is zero. Moreover, the condition $\dim(A/I) = 0$ means that ξ' does not lie on any irreducible component of δ-dimension $k + 1$, i.e., $n = 0$ as well.

Now, let $\dim(A/I) = 1$. Since A is a Noetherian local ring there are finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t \supset I$ over I. These correspond 1-1 with W_i passing through ξ'. Moreover $n_i = \text{length}_{A_{\mathfrak{q}_i}}((A/I)_{\mathfrak{q}_i})$. Also, the multiplicity of $[Z']$ in $[Z(s_i)]_k$ is $\text{length}_A(A/(f, \mathfrak{q}_i))$. Hence the equation to prove in this case is

$$\text{length}_A(A/(f, I)) = \sum \text{length}_{A_{\mathfrak{q}_i}}((A/I)_{\mathfrak{q}_i})\text{length}_A(A/(f, \mathfrak{q}_i))$$

which follows from Lemma 5.6. $\square$

Lemma 25.4. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Let α be a k-cycle on Y. Then*

$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^*\mathcal{L}) \cap f^*\alpha$$

in $A_{k+r-1}(X)$.

Proof. Write $\alpha = \sum n_i[W_i]$. We claim it suffices to show that $f^*(c_1(\mathcal{L}) \cap [W_i]) = c_1(f^*\mathcal{L}) \cap f^*[W_i]$ for each i. Proof of this claim is omitted. (Remarks: it is clear in the quasi-compact case. Something similar happened in the proof of Lemma 20.1, and one can copy the method used there here. Another possibility is to check the cycles and rational equivalences used for all W_i combined at each step form a locally finite collection).

Let $W \subset Y$ be an integral closed subscheme of δ-dimension k. We have to show that $f^*(c_1(\mathcal{L}) \cap [W]) = c_1(f^*\mathcal{L}) \cap f^*[W]$. Consider the following fibre product diagram

$$\begin{tikzcd} W' = W \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ W \arrow[r] & Y \end{tikzcd}$$

and let $W'_i \subset W'$ be the irreducible components of δ-dimension $k + r$. Write $[W']_{k+r} = \sum n_i[W'_i]$ with $n_i > 0$ as per definition. So $f^*[W] = \sum n_i[W'_i]$. Choose a nonzero meromorphic section s of $\mathcal{L}|_W$. Since each $W'_i \to W$ is dominant we see that $s_i = s|_{W'_i}$ is a nonzero meromorphic section for each i. We claim that we have the following equality of cycles

$$\sum n_i \text{div}_{\mathcal{L}|_{W_i}}(s_i) = f^*\text{div}_{\mathcal{L}|_W}(s)$$

in $Z_{k+r-1}(X)$.

Having formulated the problem as an equality of cycles we may work locally on Y. Hence we may assume Y and also W affine, and $s = p/q$ for some nonzero sections $p \in \Gamma(W, \mathcal{L})$ and $q \in \Gamma(W, \mathcal{O})$. If we can show both

$$\sum n_i \text{div}_{\mathcal{L}|_{W_i}}(p_i) = f^*\text{div}_{\mathcal{L}|_W}(p), \quad \text{and} \quad \sum n_i \text{div}_{\mathcal{O}|_{W_i}}(q_i) = f^*\text{div}_{\mathcal{O}|_W}(q)$$

(with obvious notations) then we win by the additivity, see Lemma 24.3. Thus we may assume that $s \in \Gamma(W, \mathcal{L}|_W)$. In this case we may apply the equality (25.3.1) obtained in the proof of Lemma 25.3 to see that

$$\sum n_i \text{div}_{\mathcal{L}|_{W_i}}(s_i) = [Z(s')]_{k+r-1}$$

where $s' \in f^*\mathcal{L}|_{W'}$ denotes the pullback of s to W'. On the other hand we have

$$f^*\text{div}_{\mathcal{L}|_W}(s) = f^*[Z(s)]_{k-1} = [f^{-1}(Z(s))]_{k+r-1},$$

by Lemmas 24.2 and 14.4. Since $Z(s') = f^{-1}(Z(s))$ we win. □

Lemma 25.5. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a proper morphism. Let $\mathcal{L}$ be an invertible sheaf on Y. Let s be a nonzero meromorphic section s of $\mathcal{L}$ on Y. Assume X, Y integral, f dominant, and $\dim_\delta(X) = \dim_\delta(Y)$. Then*

$$f_*\left(div_{f^*\mathcal{L}}(f^*s)\right) = [R(X) : R(Y)]\, div_{\mathcal{L}}(s).$$

In particular

$$f_*(c_1(f^*\mathcal{L}) \cap [X]) = c_1(\mathcal{L}) \cap f_*[Y].$$

Proof. The last equation follows from the first since $f_*[X] = [R(X) : R(Y)][Y]$ by definition. It turns out that we can re-use Lemma 18.1 to prove this. Namely, since we are trying to prove an equality of cycles, we may work locally on Y. Hence we may assume that $\mathcal{L} = \mathcal{O}_Y$. In this case s corresponds to a rational function $g \in R(Y)$, and we are simply trying to prove

$$f_*\left(\text{div}_X(g)\right) = [R(X) : R(Y)]\text{div}_Y(g).$$

Comparing with the result of the aforementioned Lemma 18.1 we see this true since $\text{Nm}_{R(X)/R(Y)}(g) = g^{[R(X):R(Y)]}$ as $g \in R(Y)^*$. □

Lemma 25.6. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Let $\alpha \in Z_{k+1}(X)$. Let $\mathcal{L}$ be an invertible sheaf on Y. Then*

$$p_*(c_1(p^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap p_*\alpha$$

in $A_k(Y)$.

Proof. Suppose that p has the property that for every integral closed subscheme $W \subset X$ the map $p|_W : W \to Y$ is a closed immersion. Then, by definition of capping with $c_1(\mathcal{L})$ the lemma holds.

We will use this remark to reduce to a special case. Namely, write $\alpha = \sum n_i[W_i]$ with $n_i \neq 0$ and W_i pairwise distinct. Let $W_i' \subset Y$ be the image of W_i (as an integral closed subscheme). Consider the diagram

$$\begin{tikzcd} X' = \coprod W_i \arrow[r, "q"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' = \coprod W_i' \arrow[r, "q'"] & Y. \end{tikzcd}$$

Since $\{W_i\}$ is locally finite on X, and p is proper we see that $\{W_i'\}$ is locally finite on Y and that q, q', p' are also proper morphisms. We may think of $\sum n_i[W_i]$ also as a k-cycle $\alpha' \in Z_k(X')$. Clearly $q_*\alpha' = \alpha$. We have $q_*(c_1(q^*p^*\mathcal{L}) \cap \alpha') = c_1(p^*\mathcal{L}) \cap q_*\alpha'$ and $(q')_*(c_1((q')^*\mathcal{L}) \cap p'_*\alpha') = c_1(\mathcal{L}) \cap q'_*p'_*\alpha'$ by the initial remark of the proof. Hence it suffices to prove the lemma for the morphism p' and the cycle $\sum n_i[W_i]$. Clearly, this means we may assume X, Y integral, $f : X \to Y$ dominant and $\alpha = [X]$. In this case the result follows from Lemma 25.5. $\square$

26. Cartier divisors and K-groups

In this section we describe how the intersection with the first chern class of an invertible sheaf $\mathcal{L}$ corresponds to tensoring with $\mathcal{L} - \mathcal{O}$ in K-groups.

Lemma 26.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a meromorphic section of $\mathcal{L}$. Assume*

1. $\dim_\delta(X) \leq k + 1$,
2. *X has no embedded points,*
3. *$\mathcal{F}$ has no embedded associated points,*
4. *the support of $\mathcal{F}$ is X, and*
5. *the section s is regular meromorphic.*

In this situation let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s, see Divisors, Definition 15.15. Then we have the following:

1. *there are short exact sequences*
$$\begin{matrix} 0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{1} & \mathcal{F} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{s} & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} & \to & \mathcal{Q}_2 & \to & 0 \end{matrix}$$
2. *the coherent sheaves $\mathcal{Q}_1$, $\mathcal{Q}_2$ are supported in δ-dimension $\leq k$,*
3. *the section s restricts to a regular meromorphic section s_i on every irreducible component X_i of X of δ-dimension $k + 1$, and*
4. *writing $[\mathcal{F}]_{k+1} = \sum m_i[X_i]$ we have*
$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i(X_i \to X)_* div_{\mathcal{L}|_{X_i}}(s_i)$$
in $Z_k(X)$, in particular
$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$$
in $A_k(X)$.

Proof. Recall from Divisors, Lemma 15.16 the existence of injective maps $1 : \mathcal{I}\mathcal{F} \to \mathcal{F}$ and $s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ whose cokernels are supported on a closed nowhere dense subsets T. Denote $\mathcal{Q}_i$ there cokernels as in the lemma. We conclude that $\dim_\delta(\text{Supp}(\mathcal{Q}_i)) \leq k$. By Divisors, Lemmas 15.4 and 15.12 the pullbacks s_i are defined and are regular meromorphic sections for $\mathcal{L}|_{X_i}$. The equality of cycles in (4) implies the equality of cycle classes in (4). Hence the only remaining thing to show is that

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i (X_i \to X)_* \mathrm{div}_{\mathcal{L}|_{X_i}}(s_i)$$

holds in $Z_k(X)$. To see this, let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be the generic point. Let $A = \mathcal{O}_{X,\xi}$ and $M = \mathcal{F}_\xi$. Moreover, choose a generator $s_\xi \in \mathcal{L}_\xi$. Then we can write $s = (a/b)s_\xi$ where $a, b \in A$ are nonzerodivisors. In this case $I = \mathcal{I}_\xi = \{x \in A \mid x(a/b) \in A\}$. In this case the coefficient of $[Z]$ in the left hand side is

$$\mathrm{length}_A(M/(a/b)IM) - \mathrm{length}_A(M/IM)$$

and the coefficient of $[Z]$ in the right hand side is

$$\sum \mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \mathrm{ord}_{A/\mathfrak{q}_i}(a/b)$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the 1-dimensional local ring A. Hence the result follows from Lemma 5.7. $\square$

Lemma 26.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume $\dim_\delta(\mathit{Support}(\mathcal{F})) \leq k+1$. Then the element*

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] \in K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$$

lies in the subgroup $B_k(X)$ of Lemma 22.5 and maps to the element $c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$ via the map $B_k(X) \to A_k(X)$.

Proof. Let

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}' \to 0$$

be the short exact sequence constructed in Divisors, Lemma 4.5. This in particular means that $\mathcal{F}'$ has no embedded associated points. Since the support of $\mathcal{K}$ is nowhere dense in the support of $\mathcal{F}$ we see that $\dim_\delta(\mathrm{Supp}(\mathcal{K})) \leq k$. We may re-apply Divisors, Lemma 4.5 starting with $\mathcal{K}$ to get a short exact sequence

$$0 \to \mathcal{K}'' \to \mathcal{K} \to \mathcal{K}' \to 0$$

where now $\dim_\delta(\mathrm{Supp}(\mathcal{K}'')) < k$ and $\mathcal{K}'$ has no embedded associated points. Suppose we can prove the lemma for the coherent sheaves $\mathcal{F}'$ and $\mathcal{K}'$. Then we see from the equations

$$[\mathcal{F}]_{k+1} = [\mathcal{F}']_{k+1} + [\mathcal{K}']_{k+1} + [\mathcal{K}'']_{k+1}$$

(use Lemma 10.4),

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [\mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}'] + [\mathcal{K}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'] + [\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$$

(use the $\otimes\mathcal{L}$ is exact) and the trivial vanishing of $[\mathcal{K}'']_{k+1}$ and $[\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$ in $K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$ that the result holds for $\mathcal{F}$. What this means is that we may assume that the sheaf $\mathcal{F}$ has no embedded associated points.

Assume X, $\mathcal{F}$ as in the lemma, and assume in addition that $\mathcal{F}$ has no embedded associated points. Consider the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, the corresponding closed subscheme $i : Z \to X$ and the coherent $\mathcal{O}_Z$-module $\mathcal{G}$ constructed in Divisors, Lemma 4.6. Recall that Z is a locally Noetherian scheme without embedded points, $\mathcal{G}$ is a coherent sheaf without embedded associated points, with $\mathrm{Supp}(\mathcal{G}) = Z$ and such that $i_*\mathcal{G} = \mathcal{F}$. Moreover, set $\mathcal{N} = \mathcal{L}|_Z$.

By Divisors, Lemma 15.13 the invertible sheaf $\mathcal{N}$ has a regular meromorphic section s over Z. Let us denote $\mathcal{J} \subset \mathcal{O}_Z$ the sheaf of denominators of s. By Lemma 26.1 there exist short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{1} & \mathcal{G} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{s} & \mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{N} & \to & \mathcal{Q}_2 & \to & 0 \end{array}$$

such that $\dim_\delta(\mathrm{Supp}(\mathcal{Q}_i)) \leq k$ and such that the cycle $[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k$ is a representative of $c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}$. We see (using the fact that $i_*(\mathcal{G} \otimes \mathcal{N}) = \mathcal{F} \otimes \mathcal{L}$ by the projection formula, see Cohomology, Lemma 8.2) that

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [i_*\mathcal{Q}_2] - [i_*\mathcal{Q}_1]$$

in $K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$. This already shows that $[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}]$ is an element of $B_k(X)$. Moreover we have

$$\begin{aligned} [i_*\mathcal{Q}_2]_k - [i_*\mathcal{Q}_1]_k &= i_*\left([\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k\right) \\ &= i_*\left(c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}\right) \\ &= c_1(\mathcal{L}) \cap i_*[\mathcal{G}]_{k+1} \\ &= c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1} \end{aligned}$$

by the above and Lemmas 25.6 and 12.3. And this agree with the image of the element under $B_k(X) \to A_k(X)$ by definition. Hence the lemma is proved. $\square$

27. Blowing up lemmas

In this section we prove some lemmas on representing Cartier divisors by suitable effective Cartier divisors on blow-ups. These lemmas can be found in [Ful98, Section 2.4]. We have adapted the formulation so they also work in the non-finite type setting. It may happen that the morphism b of Lemma 27.7 is a composition of infinitely many blow ups, but over any given quasi-compact open $W \subset X$ one needs only finitely many blow-ups (and this is the result of loc. cit.).

Lemma 27.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a proper morphism. Let $D \subset Y$ be an effective Cartier divisor. Assume X, Y integral, $n = \dim_\delta(X) = \dim_\delta(Y)$ and f dominant. Then*

$$f_*[f^{-1}(D)]_{n-1} = [R(X) : R(Y)][D]_{n-1}.$$

In particular if f is birational then $f_*[f^{-1}(D)]_{n-1} = [D]_{n-1}$.

Proof. Immediate from Lemma 25.5 and the fact that D is the zero scheme of the canonical section 1_D of $\mathcal{O}_X(D)$. $\square$

Lemma 27.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X integral with $\dim_\delta(X) = n$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a nonzero meromorphic section of $\mathcal{L}$. Let $U \subset X$ be the maximal open subscheme such that s corresponds to a section of $\mathcal{L}$ over U. There exists a projective morphism*

$$\pi : X' \longrightarrow X$$

such that

1. *X' is integral,*
2. *$\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism,*
3. *there exist effective Cartier divisors $D, E \subset X'$ such that*

$$\pi^*\mathcal{L} = \mathcal{O}_{X'}(D - E),$$

(4) *the meromorphic section s corresponds, via the isomorphism above, to the meromorphic section $1_D \otimes (1_E)^{-1}$ (see Divisors, Definition 9.14),*
(5) *we have*

$$\pi_*([D]_{n-1} - [E]_{n-1}) = div_{\mathcal{L}}(s)$$

in $Z_{n-1}(X)$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of denominators of s. Namely, we declare a local section f of $\mathcal{O}_X$ to be a local section of $\mathcal{I}$ if and only if fs is a local section of $\mathcal{L}$. On any affine open $U = \text{Spec}(A)$ of X write $\mathcal{L}|_U = \widetilde{L}$ for some invertible A-module L. Then A is a Noetherian domain with fraction field $K = R(X)$ and we may think of $s|_U$ as an element of $L \otimes_A K$ (see Divisors, Lemma 15.7). Let $I = \{x \in A \mid xs \in L\}$. Then we see that $\mathcal{I}|_U = \widetilde{I}$ (details omitted) and hence $\mathcal{I}$ is quasi-coherent.

Consider the closed subscheme $Z \subset X$ defined by $\mathcal{I}$. It is clear that $U = X \setminus Z$. This suggests we should blow up Z. Let

$$\pi : X' = \underline{\text{Proj}}_X \left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n\right) \longrightarrow X$$

be the blowing up of X along Z. The quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n$ is generated in degree 1 over $\mathcal{O}_X$. Moreover, the degree 1 part is a coherent $\mathcal{O}_X$-module, in particular of finite type. Hence we see that π is projective and $\mathcal{O}_{X'}(1)$ is relatively very ample.

By Divisors, Lemma 18.7 we have X' is integral. By Divisors, Lemma 18.4 there exists an effective Cartier divisor $E \subset X'$ such that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{X'} = \mathcal{I}_E$. Also, by the same lemma we see that $\pi^{-1}(U) \cong U$.

Denote s' the pullback of the meromorphic section s to a meromorphic section of $\mathcal{L}' = \pi^*\mathcal{L}$ over X'. It follows from the fact that $\mathcal{I}s \subset \mathcal{L}$ that $\mathcal{I}_E s' \subset \mathcal{L}'$. In other words, s' gives rise to an $\mathcal{O}_{X'}$-linear map $\mathcal{I}_E \to \mathcal{L}'$, or in other words a section $t \in \mathcal{L}' \otimes \mathcal{O}_{X'}(E)$. By Divisors, Lemma 9.20 we obtain a unique effective Cartier divisor $D \subset X'$ such that $\mathcal{L}' \otimes \mathcal{O}_{X'}(E) \cong \mathcal{O}_{X'}(D)$ with t corresponding to 1_D. Reversing this procedure we conclude that $\mathcal{L}' = \mathcal{O}_{X'}(-E) \cong \mathcal{O}_{X'}(D)$ with s' corresponding to $1_D \otimes 1_E^{-1}$ as in (4).

We still have to prove (5). By Lemma 25.5 we have

$$\pi_*(\text{div}_{\mathcal{L}'}(s')) = \text{div}_{\mathcal{L}}(s).$$

Hence it suffices to show that $\text{div}_{\mathcal{L}'}(s') = [D]_{n-1} - [E]_{n-1}$. This follows from the equality $s' = 1_D \otimes 1_E^{-1}$ and additivity, see Lemma 24.3. □

Definition 27.3. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X. Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = n - 1$. The *ϵ-invariant* of this situation is

$$\epsilon_Z(D_1, D_2) = n_Z \cdot m_Z$$

where n_Z, resp. m_Z is the coefficient of Z in the $(n-1)$-cycle $[D_1]_{n-1}$, resp. $[D_2]_{n-1}$.

Lemma 27.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X. Let Z be an open and closed subscheme of the scheme $D_1 \cap D_2$.*

Assume $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n-2$. *Then there exists a morphism* $b : X' \to X$, *and Cartier divisors* D_1', D_2', E *on* X' *with the following properties*

1. X' *is integral,*
2. b *is projective,*
3. b *is the blow up of* X *in the closed subscheme* Z,
4. $E = b^{-1}(Z)$,
5. $b^{-1}(D_1) = D_1' + E$, *and* $b^{-1}D_2 = D_2' + E$,
6. $\dim_\delta(D_1' \cap D_2') \leq n-2$, *and if* $Z = D_1 \cap D_2$ *then* $D_1' \cap D_2' = \emptyset$,
7. *for every integral closed subscheme* W' *with* $\dim_\delta(W') = n-1$ *we have*
   (a) *if* $\epsilon_{W'}(D_1', E) > 0$, *then setting* $W = b(W')$ *we have* $\dim_\delta(W) = n-1$ *and*
   $$\epsilon_{W'}(D_1', E) < \epsilon_W(D_1, D_2),$$
   (b) *if* $\epsilon_{W'}(D_2', E) > 0$, *then setting* $W = b(W')$ *we have* $\dim_\delta(W) = n-1$ *and*
   $$\epsilon_{W'}(D_2', E) < \epsilon_W(D_1, D_2),$$

Proof. Note that the quasi-coherent ideal sheaf $\mathcal{I} = \mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ defines the scheme theoretic intersection $D_1 \cap D_2 \subset X$. Since Z is a union of connected components of $D_1 \cap D_2$ we see that for every $z \in Z$ the kernel of $\mathcal{O}_{X,z} \to \mathcal{O}_{Z,z}$ is equal to $\mathcal{I}_z$. Let $b : X' \to X$ be the blow up of X in Z. (So Zariski locally around Z it is the blow up of X in $\mathcal{I}$.) Denote $E = b^{-1}(Z)$ the corresponding effective Cartier divisor, see Divisors, Lemma 18.4. Since $Z \subset D_1$ we have $E \subset f^{-1}(D_1)$ and hence $D_1 = D_1' + E$ for some effective Cartier divisor $D_1' \subset X'$, see Divisors, Lemma 9.8. Similarly $D_2 = D_2' + E$. This takes care of assertions (1) – (5).

Note that if W' is as in (7) (a) or (7) (b), then the image W of W' is contained in $D_1 \cap D_2$. If W is not contained in Z, then b is an isomorphism at the generic point of W and we see that $\dim_\delta(W) = \dim_\delta(W') = n-1$ which contradicts the assumption that $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n-2$. Hence $W \subset Z$. This means that to prove (6) and (7) we may work locally around Z on X.

Thus we may assume that $X = \mathrm{Spec}(A)$ with A a Noetherian domain, and $D_1 = \mathrm{Spec}(A/a)$, $D_2 = \mathrm{Spec}(A/b)$ and $Z = D_1 \cap D_2$. Set $I = (a, b)$. Since A is a domain and $a, b \neq 0$ we can cover the blow up by two patches, namely $U = \mathrm{Spec}(A[s]/(as - b))$ and $V = \mathrm{Spec}(A[t]/(bt-a))$. These patches are glued using the isomorphism $A[s, s^{-1}]/(as-b) \cong A[t, t^{-1}]/(bt-a)$ which maps s to t^{-1}. The effective Cartier divisor E is described by $\mathrm{Spec}(A[s]/(as-b, a)) \subset U$ and $\mathrm{Spec}(A[t]/(bt-a, b)) \subset V$. The closed subscheme D_1' corresponds to $\mathrm{Spec}(A[t]/(bt-a, t)) \subset U$. The closed subscheme D_2' corresponds to $\mathrm{Spec}(A[s]/(as-b, s)) \subset V$. Since "$ts = 1$" we see that $D_1' \cap D_2' = \emptyset$.

Suppose we have a prime $\mathfrak{q} \subset A[s]/(as-b)$ of height one with $s, a \in \mathfrak{q}$. Let $\mathfrak{p} \subset A$ be the corresponding prime of A. Observe that $a, b \in \mathfrak{p}$. By the dimension formula we see that $\dim(A_\mathfrak{p}) = 1$ as well. The final assertion to be shown is that

$$\mathrm{ord}_{A_\mathfrak{p}}(a)\mathrm{ord}_{A_\mathfrak{p}}(b) > \mathrm{ord}_{B_\mathfrak{q}}(a)\mathrm{ord}_{B_\mathfrak{q}}(s)$$

where $B = A[s]/(as-b)$. By Algebra, Lemma 120.1 we have $\mathrm{ord}_{A_\mathfrak{p}}(x) \geq \mathrm{ord}_{B_\mathfrak{q}}(x)$ for $x = a, b$. Since $\mathrm{ord}_{B_\mathfrak{q}}(s) > 0$ we win by additivity of the ord function and the fact that $as = b$. $\square$

Definition 27.5. Let X be a scheme. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Suppose given a function $I \to \mathbf{Z}_{\geq 0}$, $i \mapsto n_i$. The *sum of the effective Cartier divisors* $D = \sum n_i D_i$, is the unique effective Cartier divisor $D \subset X$ such that on any quasi-compact open $U \subset X$ we have $D|_U = \sum_{D_i \cap U \neq \emptyset} n_i D_i|_U$ is the sum as in Divisors, Definition 9.6.

Lemma 27.6. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Suppose given $n_i \geq 0$ for $i \in I$. Then*

$$[D]_{n-1} = \sum\nolimits_i n_i [D_i]_{n-1}$$

in $Z_{n-1}(X)$.

Proof. Since we are proving an equality of cycles we may work locally on X. Hence this reduces to a finite sum, and by induction to a sum of two effective Cartier divisors $D = D_1 + D_2$. By Lemma 24.2 we see that $D_1 = \text{div}_{\mathcal{O}_X(D_1)}(1_{D_1})$ where 1_{D_1} denotes the canonical section of $\mathcal{O}_X(D_1)$. Of course we have the same statement for D_2 and D. Since $1_D = 1_{D_1} \otimes 1_{D_2}$ via the identification $\mathcal{O}_X(D) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ we win by Lemma 24.3. $\square$

Lemma 27.7. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = d$. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Assume that for all $\{i, j, k\} \subset I$, $\#\{i, j, k\} = 3$ we have $D_i \cap D_j \cap D_k = \emptyset$. Then there exist*

1. *an open subscheme $U \subset X$ with $\dim_\delta(X \setminus U) \leq d - 3$,*
2. *a morphism $b : U' \to U$, and*
3. *effective Cartier divisors $\{D'_j\}_{j \in J}$ on U'*

with the following properties:

1. *b is proper morphism $b : U' \to U$,*
2. *U' is integral,*
3. *b is an isomorphism over the complement of the union of the pairwise intersections of the $D_i|_U$,*
4. *$\{D'_j\}_{j \in J}$ is a locally finite collection of effective Cartier divisors on U',*
5. *$\dim_\delta(D'_j \cap D'_{j'}) \leq d - 2$ if $j \neq j'$, and*
6. *$b^{-1}(D_i|_U) = \sum n_{ij} D'_j$ for certain $n_{ij} \geq 0$.*

Moreover, if X is quasi-compact, then we may assume $U = X$ in the above.

Proof. Let us first prove this in the quasi-compact case, since it is perhaps the most interesting case. In this case we produce inductively a sequence of blowups

$$X = X_0 \xleftarrow{b_0} X_1 \xleftarrow{b_1} X_2 \leftarrow \ldots$$

and finite sets of effective Cartier divisors $\{D_{n,i}\}_{i \in I_n}$. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \coprod P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n. Finally, we will have

$$b_n^{-1}(D_{n,i}) = D_{n+1,i} + \sum\nolimits_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$$

We conclude that for each $n \geq 0$ we have $(b_0 \circ \ldots \circ b_n)^{-1}(D_i)$ is a nonnegative integer combination of the divisors $D_{n+1,j}$, $j \in I_{n+1}$.

To start the induction we set $X_0 = X$ and $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\}_{i \in I_n})$ let X_{n+1} be the blow up of X_n in the closed subscheme $Z_n = \bigcup_{\{i,i'\} \in P(I_n)} D_{n,i} \cap D_{n,i'}$. Note that the closed subschemes $D_{n,i} \cap D_{n,i'}$ are pairwise disjoint by our assumption on triple intersections. In other words we may write $Z_n = \coprod_{\{i,i'\} \in P(I_n)} D_{n,i} \cap D_{n,i'}$. Moreover, in a Zariski neighbourhood of $D_{n,i} \cap D_{n,i'}$ the morphism b_n is equal to the blow up of the scheme X_n in the closed subscheme $D_{n,i} \cap D_{n,i'}$, and the results of Lemma 27.4 apply. Hence setting $D_{n+1,\{i,i'\}} = b_n^{-1}(D_i \cap D_{i'})$ we get an effective Cartier divisor. The Cartier divisors $D_{n+1,\{i,i'\}}$ are pairwise disjoint. Clearly we have $b_n^{-1}(D_{n,i}) \supset D_{n+1,\{i,i'\}}$ for every $i' \in I_n$, $i' \neq i$. Hence, applying Divisors, Lemma 9.8 we see that indeed $b^{-1}(D_{n,i}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$ for some effective Cartier divisor $D_{n+1,i}$ on X_{n+1}. In a neighbourhood of $D_{n+1,\{i,i'\}}$ these divisors $D_{n+1,i}$ play the role of the primed divisors of Lemma 27.4. In particular we conclude that $D_{n+1,i} \cap D_{n+1,i'} = \emptyset$ if $i \neq i'$, $i, i' \in I_n$ by part (6) of Lemma 27.4. This already implies that triple intersections of the divisors $D_{n+1,i}$ are zero.

OK, and at this point we can use the quasi-compactness of X to conclude that the invariant

(27.7.1)
$$\epsilon(X, \{D_i\}_{i \in I}) = \max\{\epsilon_Z(D_i, D_{i'}) \mid Z \subset X, \dim_\delta(Z) = d - 1, \{i, i'\} \in P(I)\}$$

is finite, since after all each D_i has at most finitely many irreducible components. We claim that for some n the invariant $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$ is zero. Namely, if not then by Lemma 27.4 we have a strictly decreasing sequence

$$\epsilon(X, \{D_i\}_{i \in I}) = \epsilon(X_0, \{D_{0,i}\}_{i \in I_0}) > \epsilon(X_1, \{D_{1,i}\}_{i \in I_1}) > \ldots$$

of positive integers which is a contradiction. Take n with invariant $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$ equal to zero. This means that there is no integral closed subscheme $Z \subset X_n$ and no pair of indices $i, i' \in I_n$ such that $\epsilon_Z(D_{n,i}, D_{n,i'}) > 0$. In other words, $\dim_\delta(D_{n,i}, D_{n,i'}) \leq d - 2$ for all pairs $\{i, i'\} \in P(I_n)$ as desired.

Next, we come to the general case where we no longer assume that the scheme X is quasi-compact. The problem with the idea from the first part of the proof is that we may get and infinite sequence of blow ups with centers dominating a fixed point of X. In order to avoid this we cut out suitable closed subsets of codimension ≥ 3 at each stage. Namely, we will construct by induction a sequence of morphisms having the following shape

$$\begin{tikzcd}
X = X_0 & & & \\
U_0 \arrow[u, "j_0"] & X_1 \arrow[l, "b_0"'] & & \\
& U_1 \arrow[u, "j_1"] & X_2 \arrow[l, "b_1"'] & \\
& & U_2 \arrow[u, "j_2"] & X_3 \arrow[l, "b_2"']
\end{tikzcd}$$

Each of the morphisms $j_n : U_n \to X_n$ will be an open immersion. Each of the morphisms $b_n : X_{n+1} \to U_n$ will be a proper birational morphism of integral schemes.

As in the quasi-compact case we will have effective Cartier divisors $\{D_{n,i}\}_{i \in I_n}$ on X_n. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \coprod P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n. Finally, we will arrange it so that

$$b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum\nolimits_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$$

We start the induction by setting $X_0 = X$, $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\})$ we construct the open subscheme U_n as follows. For each pair $\{i, i'\} \in P(I_n)$ consider the closed subscheme $D_{n,i} \cap D_{n,i'}$. This has "good" irreducible components which have δ-dimension $d-2$ and "bad" irreducible components which have δ-dimension $d-1$. Let us set

$$\text{Bad}(i, i') = \bigcup\nolimits_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W) = d-1} W$$

and similarly

$$\text{Good}(i, i') = \bigcup\nolimits_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W) = d-2} W.$$

Then $D_{n,i} \cap D_{n,i'} = \text{Bad}(i,i') \cup \text{Good}(i,i')$ and moreover we have $\dim_\delta(\text{Bad}(i,i') \cap \text{Good}(i,i')) \leq d-3$. Here is our choice of U_n:

$$U_n = X_n \setminus \bigcup\nolimits_{\{i,i'\} \in P(I_n)} \text{Bad}(i,i') \cap \text{Good}(i,i').$$

By our condition on triple intersections of the divisors $D_{n,i}$ we see that the union is actually a disjoint union. Moreover, we see that (as a scheme)

$$D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n} = Z_{n,i,i'} \coprod G_{n,i,i'}$$

where $Z_{n,i,i'}$ is δ-equidimension of dimension $d-1$ and $G_{n,i,i'}$ is δ-equidimensional of dimension $d-2$. (So topologically $Z_{n,i,i'}$ is the union of the bad components but throw out intersections with good components.) Finally we set

$$Z_n = \bigcup\nolimits_{\{i,i'\} \in P(I_n)} Z_{n,i,i'} = \coprod\nolimits_{\{i,i'\} \in P(I_n)} Z_{n,i,i'},$$

and we let $b_n : X_{n+1} \to X_n$ be the blow up in Z_n. Note that Lemma 27.4 applies to the morphism $b_n : X_{n+1} \to X_n$ locally around each of the loci $D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n}$. Hence, exactly as in the first part of the proof we obtain effective Cartier divisors $D_{n+1,\{i,i'\}}$ for $\{i,i'\} \in P(I_n)$ and effective Cartier divisors $D_{n+1,i}$ for $i \in I_n$ such that $b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$. For each n denote $\pi_n : X_n \to X$ the morphism obtained as the composition $j_0 \circ \ldots \circ j_{n-1} \circ b_{n-1}$.

Claim: given any quasi-compact open $V \subset X$ for all sufficiently large n the maps

$$\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V) \leftarrow \ldots$$

are all isomorphisms. Namely, if the map $\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V)$ is not an isomorphism, then $Z_{n,i,i'} \cap \pi_n^{-1}(V) \neq \emptyset$ for some $\{i,i'\} \in P(I_n)$. Hence there exists an irreducible component $W \subset D_{n,i} \cap D_{n,i'}$ with $\dim_\delta(W) = d-1$. In particular we see that $\epsilon_W(D_{n,i}, D_{n,i'}) > 0$. Applying Lemma 27.4 repeatedly we see that

$$\epsilon_W(D_{n,i}, D_{n,i'}) < \epsilon(V, \{D_i|_V\}) - n$$

with $\epsilon(V, \{D_i|_V\})$ as in (27.7.1). Since V is quasi-compact, we have $\epsilon(V, \{D_i|_V\}) < \infty$ and taking $n > \epsilon(V, \{D_i|_V\})$ we see the result.

Note that by construction the difference $X_n \setminus U_n$ has $\dim_\delta(X_n \setminus U_n) \leq d-3$. Let $T_n = \pi_n(X_n \setminus U_n)$ be its image in X. Traversing in the diagram of maps above using each b_n is closed it follows that $T_0 \cup \ldots \cup T_n$ is a closed subset of X for each n. Any $t \in T_n$ satisfies $\delta(t) \leq d-3$ by construction. Hence $\overline{T_n} \subset X$ is a closed subset with $\dim_\delta(T_n) \leq d-3$. By the claim above we see that for any quasi-compact open $V \subset X$ we have $T_n \cap V \neq \emptyset$ for at most finitely many n. Hence $\{\overline{T_n}\}_{n \geq 0}$ is a locally finite collection of closed subsets, and we may set $U = X \setminus \bigcup \overline{T_n}$. This will be U as in the lemma.

Note that $U_n \cap \pi_n^{-1}(U) = \pi_n^{-1}(U)$ by construction of U. Hence all the morphisms

$$b_n : \pi_{n+1}^{-1}(U) \longrightarrow \pi_n^{-1}(U)$$

are proper. Moreover, by the claim they eventually become isomorphisms over each quasi-compact open of X. Hence we can define

$$U' = \lim_n \pi_n^{-1}(U).$$

The induced morphism $b : U' \to U$ is proper since this is local on U, and over each compact open the limit stabilizes. Similarly we set $J = \bigcup_{n \geq 0} I_n$ using the inclusions $I_n \to I_{n+1}$ from the construction. For $j \in J$ choose an n_0 such that j corresponds to $i \in I_{n_0}$ and define $D'_j = \lim_{n \geq n_0} D_{n,i}$. Again this makes sense as locally over X the morphisms stabilize. The other claims of the lemma are verified as in the case of a quasi-compact X. $\square$

28. Intersecting with effective Cartier divisors

To be able to prove the commutativity of intersection products we need a little more precision in terms of supports of the cycles. Here is the relevant notion.

Definition 28.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let D be an effective Cartier divisor on X, and denote $i : D \to X$ the closed immersion. We define, for every integer k, a *Gysin homomorphism*

$$i^* : Z_{k+1}(X) \to A_k(D).$$

(1) Given a integral closed subscheme $W \subset X$ with $\dim_\delta(W) = k+1$ we define
  (a) if $W \not\subset D$, then $i^*[W] = [D \cap W]_k$ as a k-cycle on D, and
  (b) if $W \subset D$, then $i^*[W] = i'_*(c_1(\mathcal{O}_X(D)|_W) \cap [W])$, where $i' : W \to D$ is the induced closed immersion.

(2) For a general $(k+1)$-cycle $\alpha = \sum n_j[W_j]$ we set

$$i^*\alpha = \sum n_j i^*[W_j]$$

(3) We denote $D \cdot \alpha = i_* i^* \alpha$ the pushforward of the class to a class on X.

In fact, as we will see later, this Gysin homomorphism i^* can be viewed as an example of a non-flat pullback. Thus we will sometimes informally call the class $i^*\alpha$ the *pullback* of the class α.

Lemma 28.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let D be an effective Cartier divisor on X. Let α be a $(k+1)$-cycle on X. Then $D \cdot \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$ in $A_k(X)$.*

Proof. Write $\alpha = \sum n_j[W_j]$ where $i_j : W_j \to X$ are integral closed subschemes with $\dim_\delta(W_j) = k$. Since D is the zero scheme of the canonical section 1_D of $\mathcal{O}_X(D)$ we see that $D \cap W_j$ is the zero scheme of the restriction $1_D|_{W_j}$. Hence for each j such that $W_j \not\subset D$ we have $c_1(\mathcal{O}_X(D)) \cap [W_j] = [D \cap W_j]_k$ by Lemma 25.3. So we have

$$c_1(\mathcal{O}_X(D)) \cap \alpha = \sum\nolimits_{W_j \not\subset D} n_j[D \cap W_j]_k + \sum\nolimits_{W_j \subset D} n_j i_{j,*}(c_1(\mathcal{O}_X(D)|_{W_j}) \cap [W_j])$$

in $A_k(X)$ by Definition 25.1. The right hand side matches (termwise) the pushforward of the class $i^*\alpha$ on D from Definition 28.1. Hence we win. $\square$

The following lemma will be superseded later.

Lemma 28.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let D be an effective Cartier divisor on X. Let $W \subset X$ be a closed subscheme such that $D' = W \cap D$ is an effective Cartier divisor on W.*

$$\begin{tikzcd} D' \arrow[r, "i'"] \arrow[d, "i''"'] & W \arrow[d] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

For any $(k+1)$-cycle on W we have $i^*\alpha = (i'')_*(i')^*\alpha$ in $A_k(D)$.

Proof. Suppose $\alpha = [Z]$ for some integral closed subscheme $Z \subset W$. In case $Z \not\subset D$ we have $Z \cap D' = Z \cap D$ scheme theoretically. Hence the equality holds as cycles. In case $Z \subset D$ we also have $Z \subset D'$ and the equality holds since $\mathcal{O}_X(D)|_Z \cong \mathcal{O}_W(D')|_Z$ and the definition of i^* and $(i')^*$ in these cases. $\square$

Lemma 28.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $i : D \to X$ be an effective Cartier divisor on X.*

1. *Let $Z \subset X$ be a closed subscheme such that $\dim_\delta(Z) \leq k+1$ and such that $D \cap Z$ is an effective Cartier divisor on Z. Then $i^*[Z]_{k+1} = [D \cap Z]_k$.*
2. *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(Support(\mathcal{F})) \leq k+1$ and $1_D : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is injective. Then*
$$i^*[\mathcal{F}]_{k+1} = [i^*\mathcal{F}]_k$$
in $A_k(D)$.

Proof. Assume $Z \subset X$ as in (1). Then set $\mathcal{F} = \mathcal{O}_Z$. The assumption that $D \cap Z$ is an effective Cartier divisor is equivalent to the assumption that $1_D : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is injective. Moreover $[Z]_{k+1} = [\mathcal{F}]_{k+1}]$ and $[D \cap Z]_k = [\mathcal{O}_{D \cap Z}]_k = [i^*\mathcal{F}]_k$. See Lemma 10.3. Hence part (1) follows from part (2).

Write $[\mathcal{F}]_{k+1} = \sum m_j[W_j]$ with $m_j > 0$ and pairwise distinct integral closed subschemes $W_j \subset X$ of δ-dimension $k+1$. The assumption that $1_D : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is injective implies that $W_j \not\subset D$ for all j. By definition we see that

$$i^*[\mathcal{F}]_{k+1} = \sum [D \cap W_j]_k.$$

We claim that

$$\sum [D \cap W_j]_k = [i^*\mathcal{F}]_k$$

as cycles. Let $Z \subset D$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be its generic point. Let $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$. Let $f \in A$ be an element generating the ideal of D, i.e., such that $\mathcal{O}_{D,\xi} = A/fA$. By assumption $\dim(M) = 1$, $f : M \to M$ is injective, and $\text{length}_A(M/fM) < \infty$. Moreover, $\text{length}_A(M/fM)$ is the coefficient of $[Z]$ in $[i^*\mathcal{F}]_k$. On the other hand, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then

$$\sum \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})\text{ord}_{A/\mathfrak{q}_i}(f)$$

is the coefficient of $[Z]$ in $\sum [D \cap W_j]_k$. Hence we see the equality by Lemma 5.6. □

Lemma 28.5. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\{i_j : D_j \to X\}_{j \in J}$ be a locally finite collection of effective Cartier divisors on X. Let $n_j > 0$, $j \in J$. Set $D = \sum_{j \in J} n_j D_j$, and denote $i : D \to X$ the inclusion morphism. Let $\alpha \in Z_{k+1}(X)$. Then*

$$p : \coprod\nolimits_{j \in J} D_j \longrightarrow D$$

is proper and

$$i^*\alpha = p_*\left(\sum n_j i_j^*\alpha\right)$$

in $A_k(D)$.

Proof. The proof of this lemma is made a bit longer than expected by a subtlety concerning infinite sums of rational equivalences. In the quasi-compact case the family D_j is finite and the result is altogether easy and a straightforward consequence of Lemmas 24.2 and 24.3 and the definitions.

The morphism p is proper since the family $\{D_j\}_{j \in J}$ is locally finite. Write $\alpha = \sum_{a \in A} m_a [W_a]$ with $W_a \subset X$ an integral closed subscheme of δ-dimension $k+1$. Denote $i_a : W_a \to X$ the closed immersion. We assume that $m_a \neq 0$ for all $a \in A$ such that $\{W_a\}_{a \in A}$ is locally finite on X.

Observe that by Definition 28.1 the class $i^*\alpha$ is the class of a cycle $\sum m_a \beta_a$ for certain $\beta_a \in Z_k(W_a \cap D)$. Namely, if $W_a \not\subset D$ then $\beta_a = [D \cap W_a]_k$ and if $W_a \subset D$, then β_a is a cycle representing $c_1(\mathcal{O}_X(D)) \cap [W_a]$.

For each $a \in A$ write $J = J_{a,1} \coprod J_{a,2} \coprod J_{a,3}$ where

(1) $j \in J_{a,1}$ if and only if $W_a \cap D_j = \emptyset$,
(2) $j \in J_{a,2}$ if and only if $W_a \neq W_a \cap D_1 \neq \emptyset$, and
(3) $j \in J_{a,3}$ if and only if $W_a \subset D_j$.

Since the family $\{D_j\}$ is locally finite we see that $J_{a,3}$ is a finite set. For every $a \in A$ and $j \in J$ we choose a cycle $\beta_{a,j} \in Z_k(W_a \cap D_j)$ as follows

(1) if $j \in J_{a,1}$ we set $\beta_{a,j} = 0$,
(2) if $j \in J_{a,2}$ we set $\beta_{a,j} = [D_j \cap W_a]_k$, and
(3) if $j \in J_{a,3}$ we choose $\beta_{a,j} \in Z_k(W_a)$ representing $c_1(i_a^*\mathcal{O}_X(D_j)) \cap [W_j]$.

We claim that

$$\beta_a \sim_{rat} \sum\nolimits_{j \in J} n_j \beta_{a,j}$$

in $A_k(W_a \cap D)$.

Case I: $W_a \not\subset D$. In this case $J_{a,3} = \emptyset$. Thus it suffices to show that $[D \cap W_a]_k = \sum n_j [D_j \cap W_a]_k$ as cycles. This is Lemma 27.6.

Case II: $W_a \subset D$. In this case β_a is a cycle representing $c_1(i_a^*\mathcal{O}_X(D)) \cap [W_a]$. Write $D = D_{a,1} + D_{a,2} + D_{a,3}$ with $D_{a,s} = \sum_{j \in J_{a,s}} n_j D_j$. By Lemma 24.3 we have

$$\begin{aligned} c_1(i_a^*\mathcal{O}_X(D)) \cap [W_a] &= c_1(i_a^*\mathcal{O}_X(D_{a,1})) \cap [W_a] + c_1(i_a^*\mathcal{O}_X(D_{a,2})) \cap [W_a] \\ &\quad + c_1(i_a^*\mathcal{O}_X(D_{a,3})) \cap [W_a]. \end{aligned}$$

It is clear that the first term of the sum is zero. Since $J_{a,3}$ is finite we see that the last term agrees with $\sum_{j \in J_{a,3}} n_j c_1(i_a^*\mathcal{L}_j) \cap [W_a]$, see Lemma 24.3. This is represented by $\sum_{j \in J_{a,3}} n_j \beta_{a,j}$. Finally, by Case I we see that the middle term is represented by the cycle $\sum_{j \in J_{a,2}} n_j [D_j \cap W_a]_k = \sum_{j \in J_{a,2}} n_j \beta_{a,j}$. Whence the claim in this case.

At this point we are ready to finish the proof of the lemma. Namely, we have $i^*D \sim_{rat} \sum m_a \beta_a$ by our choice of β_a. For each a we have $\beta_a \sim_{rat} \sum_j \beta_{a,j}$ with the rational equivalence taking place on $D \cap W_a$. Since the collection of closed subschemes $D \cap W_a$ is locally finite on D, we see that also $\sum m_a \beta_a \sim_{rat} \sum_{a,j} m_a \beta_{a,j}$ on D! (See Remark 19.4.) Ok, and now it is clear that $\sum_a m_a \beta_{a,j}$ (viewed as a cycle on D_j) represents $i_j^*\alpha$ and hence $\sum_{a,j} m_a \beta_{a,j}$ represents $p_* \sum_j i_j^*\alpha$ and we win. $\square$

Lemma 28.6. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X. Assume $\dim_\delta(D \cap D') = n - 2$. Let $i : D \to X$, resp. $i' : D' \to X$ be the corresponding closed immersions. Then*

(1) *there exists a cycle $\alpha \in Z_{n-2}(D \cap D')$ whose pushforward to D represents $i^*[D']_{n-1} \in A_{n-2}(D)$ and whose pushforward to D' represents $(i')^*[D]_{n-1} \in A_{n-2}(D')$, and*
(2) *we have*
$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$
in $A_{n-2}(X)$.

Proof. Part (2) is a trivial consequence of part (1). Let us write $[D]_{n-1} = \sum n_a [Z_a]$ and $[D']_{n-1} = \sum m_b [Z_b]$ with Z_a the irreducible components of D and $[Z_b]$ the irreducible components of D'. According to Definition 28.1, we have $i^*D' = \sum m_b i^*[Z_b]$ and $(i')^*D = \sum n_a (i')^*[Z_a]$. By assumption, none of the irreducible components Z_b is contained in D, and hence $i^*[Z_b] = [Z_b \cap D]_{n-2}$ by definition. Similarly $(i')^*[Z_a] = [Z_a \cap D']_{n-2}$. Hence we are trying to prove the equality of cycles

$$\sum n_a [Z_a \cap D']_{n-2} = \sum m_b [Z_b \cap D]_{n-2}$$

which are indeed supported on $D \cap D'$. Let $W \subset X$ be an integral closed subscheme with $\dim_\delta(W) = n - 2$. Let $\xi \in W$ be its generic point. Set $R = \mathcal{O}_{X,\xi}$. It is a Noetherian local domain. Note that $\dim(R) = 2$. Let $f \in R$, resp. $f' \in R$ be an element defining the ideal of D, resp. D'. By assumption $\dim(R/(f, f')) = 0$. Let $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_t \subset R$ be the minimal primes over (f'), let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset R$ be the minimal primes over (f). The equality above comes down to the equality

$$\sum_{i=1,\ldots,s} \text{length}_{R_{\mathfrak{q}_i}}(R_{\mathfrak{q}_i}/(f)) \text{ord}_{R/\mathfrak{q}_i}(f') = \sum_{j=1,\ldots,t} \text{length}_{R_{\mathfrak{q}'_j}}(R_{\mathfrak{q}'_j}/(f')) \text{ord}_{R/\mathfrak{q}'_j}(f).$$

By Lemma 5.5 applied with $M = R/(f)$ the left hand side of this equation is equal to

$$\text{length}_R(R/(f, f')) - \text{length}_R(\text{Ker}(f' : R/(f) \to R/(f)))$$

OK, and now we note that $\mathrm{Ker}(f' : R/(f) \to R/(f))$ is canonically isomorphic to $((f) \cap (f'))/(ff')$ via the map $x \bmod (f) \mapsto f'x \bmod (ff')$. Hence the left hand side is

$$\mathrm{length}_R(R/(f, f')) - \mathrm{length}_R((f) \cap (f')/(ff'))$$

Since this is symmetric in f and f' we win. □

Lemma 28.7. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\{D_j\}_{j \in J}$ be a locally finite collection of effective Cartier divisors on X. Let $n_j, m_j \geq 0$ be collections of nonnegative integers. Set $D = \sum n_j D_j$ and $D' = \sum m_j D_j$. Assume that $\dim_\delta(D_j \cap D_{j'}) = n-2$ for every $j \neq j'$. Then $D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$ in $A_{n-2}(X)$.*

Proof. This lemma is a trivial consequence of Lemmas 27.6 and 28.6 in case the sums are finite, e.g., if X is quasi-compact. Hence we suggest the reader skip the proof.

Here is the proof in the general case. Let $i_j : D_j \to X$ be the closed immersions Let $p : \coprod D_j \to X$ denote coproduct of the morphisms i_j. Let $\{Z_a\}_{a \in A}$ be the collection of irreducible components of $\bigcup D_j$. For each j we write

$$[D_j]_{n-1} = \sum d_{j,a}[Z_a].$$

By Lemma 27.6 we have

$$[D]_{n-1} = \sum n_j d_{j,a}[Z_a], \quad [D']_{n-1} = \sum m_j d_{j,a}[Z_a].$$

By Lemma 28.5 we have

$$D \cdot [D']_{n-1} = p_*\left(\sum n_j i_j^*[D']_{n-1}\right), \quad D' \cdot [D]_{n-1} = p_*\left(\sum m_{j'} i_{j'}^*[D]_{n-1}\right).$$

As in the definition of the Gysin homomorphisms (see Definition 28.1) we choose cycles $\beta_{a,j}$ on $D_j \cap Z_a$ representing $i_j^*[Z_a]$. (Note that in fact $\beta_{a,j} = [D_j \cap Z_a]_{n-2}$ if Z_a is not contained in D_j, i.e., there is no choice in that case.) Now since p is a closed immersion when restricted to each of the D_j we can (and we will) view $\beta_{a,j}$ as a cycle on X. Plugging in the formulas for $[D]_{n-1}$ and $[D']_{n-1}$ obtained above we see that

$$D \cdot [D']_{n-1} = \sum\nolimits_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j}, \quad D' \cdot [D]_{n-1} = \sum\nolimits_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'}.$$

Moreover, with the same conventions we also have

$$D_j \cdot [D_{j'}]_{n-1} = \sum d_{j',a} \beta_{a,j}.$$

In these terms Lemma 28.6 (see also its proof) says that for $j \neq j'$ the cycles $\sum d_{j',a}\beta_{a,j}$ and $\sum d_{j,a}\beta_{a,j'}$ are equal as cycles! Hence we see that

$$\begin{aligned}
D \cdot [D']_{n-1} &= \sum\nolimits_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j} \\
&= \sum\nolimits_{j \neq j'} n_j m_{j'} \left(\sum\nolimits_a d_{j',a}\beta_{a,j}\right) + \sum\nolimits_{j,a} n_j m_j d_{j,a} \beta_{a,j} \\
&= \sum\nolimits_{j \neq j'} n_j m_{j'} \left(\sum\nolimits_a d_{j,a}\beta_{a,j'}\right) + \sum\nolimits_{j,a} n_j m_j d_{j,a} \beta_{a,j} \\
&= \sum\nolimits_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'} \\
&= D' \cdot [D]_{n-1}
\end{aligned}$$

and we win. □

Here is the key lemma of this chapter. A stronger version of this lemma asserts that $D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$ holds in $A_{n-2}(D \cap D')$ for suitable representatives of the dot products involved. The first proof of the lemma together with Lemmas 28.5, 28.6, and 28.7 can be modified to show this (see [Ful98]). It is not so clear how to modify the second proof to prove the refined version. An application of the refined version is a proof that the Gysin homomorphism factors through rational equivalence. We will show this by another method later.

Lemma 28.8. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X. Then*
$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$
in $A_{n-2}(X)$.

First proof of Lemma 28.8. First, let us prove this in case X is quasi-compact. In this case, apply Lemma 27.7 to X and the two element set $\{D, D'\}$ of effective Cartier divisors. Thus we get a proper morphism $b : X' \to X$, a finite collection of effective Cartier divisors $D'_j \subset X'$ intersecting pairwise in codimension ≥ 2, with $b^{-1}(D) = \sum n_j D'_j$, and $b^{-1}(D') = \sum m_j D'_j$. Note that $b_*[b^{-1}(D)]_{n-1} = [D]_{n-1}$ in $Z_{n-1}(X)$ and similarly for D', see Lemma 27.1. Hence, by Lemma 25.6 we have
$$D \cdot [D']_{n-1} = b_*\left(b^{-1}(D) \cdot [b^{-1}(D')]_{n-1}\right)$$
in $A_{n-2}(X)$ and similarly for the other term. Hence the lemma follows from the equality $b^{-1}(D) \cdot [b^{-1}(D')]_{n-1} = b^{-1}(D') \cdot [b^{-1}(D)]_{n-1}$ in $A_{n-2}(X')$ of Lemma 28.7.

Note that in the proof above, each referenced lemma works also in the general case (when X is not assumed quasi-compact). The only minor change in the general case is that the morphism $b : U' \to U$ we get from applying Lemma 27.7 has as its target an open $U \subset X$ whose complement has codimension ≥ 3. Hence by Lemma 19.2 we see that $A_{n-2}(U) = A_{n-2}(X)$ and after replacing X by U the rest of the proof goes through unchanged. $\square$

Second proof of Lemma 28.8. Let $\mathcal{I} = \mathcal{O}_X(-D)$ and $\mathcal{I}' = \mathcal{O}_X(-D')$ be the invertible ideal sheaves of D and D'. We denote $\mathcal{I}_{D'} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_{D'}$ and $\mathcal{I}'_D = \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{O}_D$. We can restrict the inclusion map $\mathcal{I} \to \mathcal{O}_X$ to D' to get a map
$$\varphi : \mathcal{I}_{D'} \longrightarrow \mathcal{O}_{D'}$$
and similarly
$$\psi : \mathcal{I}'_D \longrightarrow \mathcal{O}_D$$
It is clear that
$$\mathrm{Coker}(\varphi) \cong \mathcal{O}_{D \cap D'} \cong \mathrm{Coker}(\psi)$$
and
$$\mathrm{Ker}(\varphi) \cong \frac{\mathcal{I} \cap \mathcal{I}'}{\mathcal{I}\mathcal{I}'} \cong \mathrm{Ker}(\psi).$$
Hence we see that
$$\gamma = [\mathcal{I}_{D'}] - [\mathcal{O}_{D'}] = [\mathcal{I}'_D] - [\mathcal{O}_D]$$
in $K_0(\mathit{Coh}_{\leq n-1}(X))$. On the other hand it is clear that
$$[\mathcal{I}'_D]_{n-1} = [D]_{n-1}, \quad [\mathcal{I}_{D'}]_{n-1} = [D']_{n-1}.$$
and that
$$\mathcal{O}_X(D') \otimes \mathcal{I}'_D = \mathcal{O}_D, \quad \mathcal{O}_X(D) \otimes \mathcal{I}_{D'} = \mathcal{O}_{D'}.$$

By Lemma 26.2 (applied two times) this means that the element γ is an element of $B_{n-2}(X)$, and maps to both $c_1(\mathcal{O}_X(D')) \cap [D]_{n-1}$ and to $c_1(\mathcal{O}_X(D)) \cap [D']_{n-1}$ and we win (since the map $B_{n-2}(X) \to A_{n-2}(X)$ is well defined – which is the key to this proof). $\square$

29. Commutativity

At this point we can start using the material above and start proving more interesting results.

Lemma 29.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. Choose a nonzero meromorphic section s of $\mathcal{L}$ and a nonzero meromorphic section t of $\mathcal{N}$. Set $\alpha = div_{\mathcal{L}}(s)$ and $\beta = div_{\mathcal{N}}(t)$. Then*

$$c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L}) \cap \beta$$

in $A_{n-2}(X)$.

Proof. By Lemma 27.2 (applied twice) there exists a proper morphism $\pi : X' \to X$ and effective Cartier divisors D_1, E_1, D_2, E_2 on X' such that

$$b^*\mathcal{L} = \mathcal{O}_{X'}(D_1 - E_1), \quad b^*\mathcal{N} = \mathcal{O}_{X'}(D_2 - E_2),$$

and such that

$$\alpha = \pi_*([D_1]_{n-1} - [E_1]_{n-1}), \quad \beta = \pi_*([D_2]_{n-1} - [E_2]_{n-1}).$$

By the projection formula of Lemma 25.6 and the additivity of Lemma 25.2 it is enough to show the equality

$$c_1(\mathcal{O}_{X'}(D_1)) \cap [D_2]_{n-1} = c_1(\mathcal{O}_{X'}(D_2)) \cap [D_1]_{n-1}$$

and three other similar equalities involving D_i and E_j. By Lemma 28.2 this is the same as showing that $D_1 \cdot [D_2]_{n-1} = D_2 \cdot [D_1]_{n-1}$ and so on. Thus the result follows from Lemma 28.8. $\square$

Lemma 29.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be invertible on X. The operation $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$ factors through rational equivalence to give an operation*

$$c_1(\mathcal{L}) \cap - : A_{k+1}(X) \to A_k(X)$$

Proof. Let $\alpha \in Z_{k+1}(X)$, and $\alpha \sim_{rat} 0$. We have to show that $c_1(\mathcal{L}) \cap \alpha$ as defined in Definition 25.1 is zero. By Definition 19.1 there exists a locally finite family $\{W_j\}$ of integral closed subschemes with $\dim_\delta(W_j) = k + 2$ and rational functions $f_j \in R(W_j)^*$ such that

$$\alpha = \sum (i_j)_* \text{div}_{W_j}(f_j)$$

Note that $p : \coprod W_j \to X$ is a proper morphism, and hence $\alpha = p_*\alpha'$ where $\alpha' \in Z_{k+1}(\coprod W_j)$ is the sum of the principal divisors $\text{div}_{W_j}(f_j)$. By the projection formula (Lemma 25.6) we have $c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to show that each $c_1(\mathcal{L}|_{W_j}) \cap \text{div}_{W_j}(f_j)$ is zero. In other words we may assume that X is integral and $\alpha = \text{div}_X(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \text{div}_X(f)$ for some $f \in R(X)^*$. We can think of f as a regular meromorphic section of the invertible sheaf $\mathcal{N} = \mathcal{O}_X$. Choose a meromorphic section s of $\mathcal{L}$ and denote $\beta = \text{div}_{\mathcal{L}}(s)$. By Lemma 29.1 we conclude that

$$c_1(\mathcal{L}) \cap \alpha = c_1(\mathcal{O}_X) \cap \beta.$$

However, by Lemma 25.2 we see that the right hand side is zero in $A_k(X)$ as desired. □

For any integer $s \geq 0$ we will denote

$$c_1(\mathcal{L})^s \cap - : A_{k+s}(X) \to A_k(X)$$

the s-fold iterate of the operation $c_1(\mathcal{L}) \cap -$. This makes sense by the lemma above.

Lemma 29.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. For any $\alpha \in A_{k+2}(X)$ we have*

$$c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha$$

as elements of $A_k(X)$.

Proof. Write $\alpha = \sum m_j[Z_j]$ for some locally finite collection of integral closed subschemes $Z_j \subset X$ with $\dim_\delta(Z_j) = k + 2$. Consider the proper morphism $p : \coprod Z_j \to X$. Set $\alpha' = \sum m_j[Z_j]$ as a $(k+2)$-cycle on $\coprod Z_j$. By several applications of Lemma 25.6 we see that $c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap c_1(p^*\mathcal{N}) \cap \alpha')$ and $c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{N}) \cap c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to prove the formula in case X is integral and $\alpha = [X]$. In this case the result follows from Lemma 29.1 and the definitions. □

30. Gysin homomorphisms

We want to show the Gysin homomorphisms factor through rational equivalence. One method (see [Ful98]) is to prove a more precise version of the key Lemma 28.8 keeping track of supports. Having obtained this one can find analogues of the lemmas of Section 29 for the Gysin homomorphism and get the result. We will use another method.

Lemma 30.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $a \in \Gamma(X, \mathcal{O}_X)$ be a nonzero function. Let $i : D = Z(a) \to X$ be the closed immersion of the zero scheme of a. Let $f \in R(X)^*$. In this case $i^* \, div_X(f) = 0$ in $A_{n-2}(D)$.*

Proof. Write $\text{div}_X(f) = \sum n_j[Z_j]$ for some integral closed subschemes $Z_j \subset X$ of δ-dimension $n - 1$. We may assume that the family $\{Z_j\}_{j \in J}$ is locally finite and that $f \in \Gamma(U, \mathcal{O}_U^*)$ where $U = X \setminus \bigcup Z_j$ (see Lemma 16.3 and its proof).

Write $J = J_1 \coprod J_2$ where $J_1 = \{j \in J \mid Z_j \subset D\}$. Note that $\mathcal{O}_X(D) \cong \mathcal{O}_X$ because a^{-1} is a trivializing global section. Hence by Definition 28.1 of i^* we see that $i^*\text{div}_X(f)$ is represented by

$$\sum\nolimits_{j \in J_2} n_j[D \cap Z_j]_{n-2}.$$

Namely, the terms involving $c_1(\mathcal{O}_X(D)|_{Z_j}) \cap Z_j$ may be dropped since $c_1(\mathcal{O}) \cap -$ is the zero operation anyway (see Lemma 25.2).

For each j let $\xi_j \in Z_j$ be its generic point. Let $B_j = \mathcal{O}_{X,\xi_j}$, which has residue field $\kappa_j = \kappa(\xi_j) = R(Z_j)$. For $j \in J_1$, let

$$f_j = d_{B_j}(f, a)$$

be the tame symbol, see Definition 4.5. We claim that we have the following equality of cycles

$$\sum\nolimits_{j \in J_2} n_j [D \cap Z_j]_{n-2} = \sum\nolimits_{j \in J_1} (Z_j \to D)_* \text{div}_{Z_j}(f_j)$$

on D. Indeed, note that $[D \cap Z_j]_{n-2} = \text{div}_{Z_j}(a)$. Hence $n_j[D \cap Z_j]_{n-2} = \text{div}_{Z_j}(a^{n_j})$. Since $n_j = \text{ord}_{B_j}(f)$ we see that in fact also $n_j[D \cap Z_j]_{n-2} = \text{div}_{Z_j}(d_{B_j}(a, f))$, as a is a unit in B_j see Lemma 4.6. Note that $d_{B_j}(f, a) = d_{B_j}(a, f)^{-1}$, see Lemma 4.4. Hence altogether we are trying to show that

$$\sum\nolimits_{j \in J} (Z_j \to D)_* \text{div}_{Z_j}(d_{B_j}(a, f)) = 0$$

as an $(n-2)$-cycle. Consider any codimension 2 integral closed subscheme $W \subset X$ with generic point $\zeta \in X$. Set $A = \mathcal{O}_{X,\zeta}$. Applying Lemma 6.1 to (A, a, f) we see that the coefficient of $[W]$ in the expression above is zero as desired. □

Lemma 30.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $i : D \to X$ be an effective Cartier divisor. Let $f \in R(X)^*$. In this case $i^* div_X(f) = 0$ in $A_{n-2}(D)$.*

Proof. This proof is a repeat of the proof of Lemma 30.1. So make sure you've read that one first.

Write $\text{div}_X(f) = \sum n_j[Z_j]$ for some integral closed subschemes $Z_j \subset X$ of δ-dimension $n-1$. We may assume that the family $\{Z_j\}_{j \in J}$ is locally finite and that $f \in \Gamma(U, \mathcal{O}_U^*)$ where $U = X \setminus \bigcup Z_j$ (see Lemma 16.3 and its proof).

Write $J = J_1 \coprod J_2$ where $J_1 = \{j \in J \mid Z_j \subset D\}$. For each j let $\xi_j \in Z_j$ be its generic point. Let us write $\mathcal{L} = \mathcal{O}_X(D)$. Choose $\tilde{s}_j \in \mathcal{L}_{\xi_j}$ a generator. Denote $s_j \in \mathcal{L}_{\xi_j} \otimes \kappa(\xi_j)$ the corresponding nonzero meromorphic section of $\mathcal{L}|_{Z_j}$. Then by Definition 28.1 of i^* we see that $i^*\text{div}_X(f)$ is represented by the cycle

$$\sum\nolimits_{j \in J_2} n_j [D \cap Z_j]_{n-2} + \sum\nolimits_{j \in J_2} n_j \text{div}_{\mathcal{L}|_{Z_j}}(s_j)$$

on D. Our goal is to show that this is rationally equivalent to zero on D.

Let $B_j = \mathcal{O}_{X,\xi_j}$, which has residue field $\kappa_j = \kappa(\xi_j) = R(Z_j)$. Write $s = a_j \tilde{s}_j$ for some $a_j \in B_j$. For $j \in J_1$ let

$$f_j = d_{B_j}(f, a_j) \in \kappa_j^* = R(Z_j)^*$$

be the tame symbol, see Definition 4.5. We claim that we have the following equality of cycles

$$\sum\nolimits_{j \in J_2} n_j [D \cap Z_j]_{n-2} + \sum\nolimits_{j \in J_2} n_j \text{div}_{\mathcal{L}|_{Z_j}}(s_j) = \sum\nolimits_{j \in J_1} (Z_j \to D)_* \text{div}_{Z_j}(f_j)$$

on D. This will clearly prove the lemma.

Note that for $j \in J_2$ we have $[D \cap Z_j]_{n-2} = \text{div}_{\mathcal{L}|_{Z_j}}(s|_{Z_j})$. Since $s|_{Z_j} = a_j|_{Z_j} s_j$ we see that $[D \cap Z_j]_{n-2} = \text{div}_{\mathcal{L}|_{Z_j}}(s_j) + \text{div}_{Z_j}(a_j|_{Z_j})$. Hence, still for $j \in J_2$, we have

$$n_j [D \cap Z_j]_{n-2} = n_j \text{div}_{\mathcal{L}|_{Z_j}}(s_j) + \text{div}_{Z_j}((a_j|_{Z_j})^{n_j})$$

Since $n_j = \text{ord}_{B_j}(f)$ we see that $\text{div}_{Z_j}((a_j|_{Z_j})^{n_j}) = \text{div}_{Z_j}(d_{B_j}(a_j, f))$, as a_j is a unit in B_j (since $j \in J_2$), see Lemma 4.6. Note that $d_{B_j}(f, a_j) = d_{B_j}(a_j, f)^{-1}$, see Lemma 4.4. Hence altogether we are trying to show that

$$\sum\nolimits_{j \in J} n_j \text{div}_{\mathcal{L}|_{Z_j}}(s_j) = \sum\nolimits_{j \in J} (Z_j \to D)_* \text{div}_{Z_j}(d_{B_j}(a_j, f)) \tag{30.2.1}$$

as an $(n-2)$-cycle.

Consider any codimension 2 integral closed subscheme $W \subset X$ with generic point $\zeta \in X$. Set $A = \mathcal{O}_{X,\zeta}$. Choose a generator $s_\zeta \in \mathcal{L}_\zeta$. For those j such that $\zeta \in Z_j$ we may write $\tilde{s}_j = b_j s_\zeta$ with $b_j \in B_j^*$. We may also write $s = a_\zeta s_\zeta$ for some $a_\zeta \in A$. Then we see that $a_j = b_j a_\zeta$. The coefficient of $[W]$ on the right hand side of Equation (30.2.1) is

$$\sum\nolimits_{\zeta \in Z_j} n_j \text{ord}_{A/\mathfrak{q}_j}(\overline{b_j}).$$

where $\mathfrak{q}_j \subset A$ is the height one prime corresponding to Z_j. Note that $B_j = A_{\mathfrak{q}_j}$ in this case. The coefficient of $[W]$ on the left hand side of Equation (30.2.1) is

$$\sum\nolimits_{\zeta \in Z_j} \text{ord}_{A/\mathfrak{q}_j}(d_{A_{\mathfrak{q}_j}}(b_j a_\zeta, f)).$$

Since b_j is a unit, and $n_j = \text{ord}_{A_{\mathfrak{q}_j}}(f)$ we see that $d_{A_{\mathfrak{q}_j}}(b_j a_\zeta, f) = \overline{b_j}^{n_j} d_{A_{\mathfrak{q}_j}}(a_\zeta, f)$ by Lemmas 4.4 and 4.6. By additivity of ord we see that it suffices to prove

$$0 = \sum\nolimits_{\zeta \in Z_j} \text{ord}_{A/\mathfrak{q}_j}(d_{A_{\mathfrak{q}_j}}(a_\zeta, f))$$

which is Lemma 6.1. □

Lemma 30.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $i : D \to X$ be an effective Cartier divisor on X. The Gysin homomorphism factors through rational equivalence to give a map $i^* : A_{k+1}(X) \to A_k(D)$.*

Proof. Let $\alpha \in Z_{k+1}(X)$ and assume that $\alpha \sim_{rat} 0$. This means there exists a locally finite collection of integral closed subschemes $W_j \subset X$ of δ-dimension $k+2$ and $f_j \in R(W_j)^*$ such that $\alpha = \sum i_{j,*}\text{div}_{W_j}(f_j)$. By construction of the map i^* we see that $i^*\alpha = \sum i^* i_{j,*}\text{div}_{W_j}(f_j)$ where each cycle $i^* i_{j,*}\text{div}_{W_j}(f_j)$ is supported on $D \cap W_j$. If we can show that each $i^* i_{j,*}\text{div}_{W_j}(f_j)$ is rationally equivalent on $W_j \cap D$, then we see that $i^*\alpha \sim_{rat} 0$ (this is clear if the sum is finite, in general see Remark 19.4).

Pick an index j. If $W_j \subset D$, then we see that $i^* i_{j,*}\text{div}_{W_j}(f_j)$ is simply equal to

$$i'_{j,*} c_1(\mathcal{O}_X(D)|_{W_j}) \cap \text{div}_{W_j}(f_j)$$

where $i'_j : W_j \to D$ is the inclusion map. This is rationally equivalent to zero by Lemma 29.2. If $W_j \not\subset D$, then we see that $i^* i_{j,*}\text{div}_{W_j}(f_j)$ is simply equal to

$$(i')^*\text{div}_{W_j}(f_j)$$

where $i' : D \cap W_j \to W_j$ is the corresponding closed immersion (see Lemma 28.3). Hence in this case Lemma 30.2 applies, and we win. □

31. Relative effective Cartier divisors

Lemma 31.1. *Let $A \to B$ be a ring map. Let $f \in B$. Assume that*

(1) *$A \to B$ is flat,*
(2) *f is a nonzerodivisor, and*
(3) *$A \to B/fB$ is flat.*

Then for every ideal $I \subset A$ the map $f : B/IB \to B/IB$ is injective.

Proof. Note that $IB = I \otimes_A B$ and $I(B/fB) = I \otimes_A B/fB$ by the flatness of B and B/fB over A. In particular $IB/fIB \cong I \otimes_A B/fB$ maps injectively into B/fB. Hence the result follows from the snake lemma applied to the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_A B \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "f"] & B/IB \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I \otimes_A B \arrow[r] & B \arrow[r] & B/IB \arrow[r] & 0 \end{tikzcd}$$

with exact rows. □

Lemma 31.2. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a flat morphism of relative dimension r. Let $i : D \to X$ be an effective Cartier divisor with the property that $p|_D : D \to Y$ is flat of relative dimension $r - 1$. Let $\mathcal{L} = \mathcal{O}_X(D)$. For any $\alpha \in A_{k+1}(Y)$ we have*

$$i^*p^*\alpha = (p|_D)^*\alpha$$

in $A_{k+r}(D)$ and

$$c_1(\mathcal{L}) \cap p^*\alpha = i_*((p|_D)^*\alpha)$$

in $A_{k+r}(X)$.

Proof. Let $W \subset Y$ be an integral closed subvariety of δ-dimension $k+1$. By Lemma 31.1 we see that $D \cap p^{-1}W$ is an effective Cartier divisor on $p^{-1}W$. By Lemma 28.4 we see that $i^*[p^{-1}W]_{k+r+1} = [D \cap W]_{k+r} = [(p|_D)^{-1}(W)]_{k+r}$. Since by definition $p^*[W] = [p^{-1}W]_{k+r+1}$ and $(p|_D)^*[W] = [(p|_D)^{-1}(W)]_{k+r}$ we see we have equality of cycles. Hence if $\alpha = \sum m_j[W_j]$, then we get $i^*\alpha = \sum m_j i^*[W_j] = \sum m_j (p|_D)^*[W_j]$ as cycles. This proves then first equality. To deduce the second from the first apply Lemma 28.2. □

32. Affine bundles

Lemma 32.1. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Assume that for every $y \in Y$, there exists an open neighbourhood $U \subset Y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is identified with the morphism $U \times \mathbf{A}^r \to U$. Then $f^* : A_k(Y) \to A_{k+r}(X)$ is surjective for all $k \in \mathbf{Z}$.*

Proof. Let $\alpha \in A_{k+r}(X)$. Write $\alpha = \sum m_j[W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subschemes of δ-dimension $k + r$. Then the family $\{W_j\}$ is locally finite in X. For any quasi-compact open $V \subset Y$ we see that $f^{-1}(V) \cap W_j$ is nonempty only for finitely many j. Hence the collection $Z_j = \overline{f(W_j)}$ of closures of images is a locally finite collection of integral closed subschemes of Y.

Consider the fibre product diagrams

$$\begin{tikzcd} f^{-1}(Z_j) \arrow[r] \arrow[d, "f_j"'] & X \arrow[d, "f"] \\ Z_j \arrow[r] & Y \end{tikzcd}$$

Suppose that $[W_j] \in Z_{k+r}(f^{-1}(Z_j))$ is rationally equivalent to $f_j^*\beta_j$ for some k-cycle $\beta_j \in A_k(Z_j)$. Then $\beta = \sum m_j\beta_j$ will be a k-cycle on Y and $f^*\beta = \sum m_j f_j^*\beta_j$

will be rationally equivalent to α (see Remark 19.4). This reduces us to the case Y integral, and $\alpha = [W]$ for some integral closed subscheme of X dominating Y. In particular we may assume that $d = \dim_\delta(Y) < \infty$.

Hence we can use induction on $d = \dim_\delta(Y)$. If $d < k$, then $A_{k+r}(X) = 0$ and the lemma holds. By assumption there exists a dense open $V \subset Y$ such that $f^{-1}(V) \cong V \times \mathbf{A}^r$ as schemes over V. Suppose that we can show that $\alpha|_{f^{-1}(V)} = f^*\beta$ for some $\beta \in Z_k(V)$. By Lemma 14.2 we see that $\beta = \beta'|_V$ for some $\beta' \in Z_k(Y)$. By the exact sequence $A_k(f^{-1}(Y \setminus V)) \to A_k(X) \to A_k(f^{-1}(V))$ of Lemma 19.2 we see that $\alpha - f^*\beta'$ comes from a cycle $\alpha' \in A_{k+r}(f^{-1}(Y \setminus V))$. Since $\dim_\delta(Y \setminus V) < d$ we win by induction on d.

Thus we may assume that $X = Y \times \mathbf{A}^r$. In this case we can factor f as

$$X = Y \times \mathbf{A}^r \to Y \times \mathbf{A}^{r-1} \to \ldots \to Y \times \mathbf{A}^1 \to Y.$$

Hence it suffices to do the case $r = 1$. By the argument in the second paragraph of the proof we are reduced to the case $\alpha = [W]$, Y integral, and $W \to Y$ dominant. Again we can do induction on $d = \dim_\delta(Y)$. If $W = Y \times \mathbf{A}^1$, then $[W] = f^*[Y]$. Lastly, $W \subset Y \times \mathbf{A}^1$ is a proper inclusion, then $W \to Y$ induces a finite field extension $R(Y) \subset R(W)$. Let $P(T) \in R(Y)[T]$ be the monic irreducible polynomial such that the generic fibre of $W \to Y$ is cut out by P in $\mathbf{A}^1_{R(Y)}$. Let $V \subset Y$ be a nonempty open such that $P \in \Gamma(V, \mathcal{O}_Y)[T]$, and such that $W \cap f^{-1}(V)$ is still cut out by P. Then we see that $\alpha|_{f^{-1}(V)} \sim_{rat} 0$ and hence $\alpha \sim_{rat} \alpha'$ for some cycle α' on $(Y \setminus V) \times \mathbf{A}^1$. By induction on the dimension we win. $\square$

Remark 32.2. We will see later (Lemma 33.3) that if X is a vectorbundle over Y then the pullback map $A_k(Y) \to A_{k+r}(X)$ is an isomorphism. Is this true in general?

33. Projective space bundle formula

Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Consider a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Our convention is that the *projective bundle associated to* $\mathcal{E}$ is the morphism

$$\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{\pi} X$$

over X with $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ normalized so that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$. In particular there is a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. We will say informally "let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$" to denote the situation where $P = \mathbf{P}(\mathcal{E})$ and $\mathcal{O}_P(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Lemma 33.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For any $\alpha \in A_k(X)$ the element*

$$\pi_* \left(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha\right) \in A_{k+r-1-s}(X)$$

is 0 if $s < r - 1$ and is equal to α when $s = r - 1$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Note that $\pi^*[Z] = [\pi^{-1}(Z)]$ as $\pi^{-1}(Z)$ is integral of δ-dimension $r - 1$. If $s < r - 1$, then by construction $c_1(\mathcal{O}_P(1))^s \cap \pi^*[Z]$ is represented by a $(k + r - 1 - s)$-cycle supported on $\pi^{-1}(Z)$. Hence the pushforward of this cycle is zero for dimension reasons.

Let $s = r-1$. By the argument given above we see that $\pi_*(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha) = n[Z]$ for some $n \in \mathbf{Z}$. We want to show that $n = 1$. For the same dimension reasons as above it suffices to prove this result after replacing X by $X \setminus T$ where $T \subset Z$ is a proper closed subset. Let ξ be the generic point of Z. We can choose elements $e_1, \ldots, e_{r-1} \in \mathcal{E}_\xi$ which form part of a basis of $\mathcal{E}_\xi$. These give rational sections $s_1, \ldots, s_{r-1}$ of $\mathcal{O}_P(1)|_{\pi^{-1}(Z)}$ whose common zero set is the closure of the image a rational section of $\mathbf{P}(\mathcal{E}|_Z) \to Z$ union a closed subset whose support maps to a proper closed subset T of Z. After removing T from X (and correspondingly $\pi^{-1}(T)$ from P), we see that $s_1, \ldots, s_n$ form a sequence of global sections $s_i \in \Gamma(\pi^{-1}(Z), \mathcal{O}_{\pi^{-1}(Z)}(1))$ whose common zero set is the image of a section $Z \to \pi^{-1}(Z)$. Hence we see successively that

$$\begin{array}{rcl} \pi^*[Z] & = & [\pi^{-1}(Z)] \\ c_1(\mathcal{O}_P(1)) \cap \pi^*[Z] & = & [Z(s_1)] \\ c_1(\mathcal{O}_P(1))^2 \cap \pi^*[Z] & = & [Z(s_1) \cap Z(s_2)] \\ \ldots & = & \ldots \\ c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*[Z] & = & [Z(s_1) \cap \ldots \cap Z(s_{r-1})] \end{array}$$

by repeated applications of Lemma 25.3. Since the pushforward by π of the image of a section of π over Z is clearly $[Z]$ we see the result when $\alpha = [Z]$. We omit the verification that these arguments imply the result for a general cycle $\alpha = \sum n_j[Z_j]$. $\square$

Lemma 33.2 (Projective space bundle formula)**.** *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. The map*

$$\bigoplus\nolimits_{i=0}^{r-1} A_{k+i}(X) \longrightarrow A_{k+r-1}(P),$$

$$(\alpha_0, \ldots, \alpha_{r-1}) \longmapsto \pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

is an isomorphism.

Proof. Fix $k \in \mathbf{Z}$. We first show the map is injective. Suppose that $(\alpha_0, \ldots, \alpha_{r-1})$ is an element of the left hand side that maps to zero. By Lemma 33.1 we see that

$$0 = \pi_*(\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}) = \alpha_{r-1}$$

Next, we see that

$$0 = \pi_*(c_1(\mathcal{O}_P(1)) \cap (\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-2} \cap \pi^*\alpha_{r-2})) = \alpha_{r-2}$$

and so on. Hence the map is injective.

It remains to show the map is surjective. Let X_i, $i \in I$ be the irreducible components of X. Then $P_i = \mathbf{P}(\mathcal{E}|_{X_i})$, $i \in I$ are the irreducible components of P. If the map is surjective for each of the morphisms $P_i \to X_i$, then the map is surjective for $\pi : P \to X$. Details omitted. Hence we may assume X is irreducible. Thus $\dim_\delta(X) < \infty$ and in particular we may use induction on $\dim_\delta(X)$.

The result is clear if $\dim_\delta(X) < k$. Let $\alpha \in A_{k+r-1}(P)$. For any locally closed subscheme $T \subset X$ denote $\gamma_T : \bigoplus A_{k+i}(T) \to A_{k+r-1}(\pi^{-1}(T))$ the map

$$\gamma_T(\alpha_0, \ldots, \alpha_{r-1}) = \pi^*\alpha_0 + \ldots + c_1(\mathcal{O}_{\pi^{-1}(T)}(1))^{r-1} \cap \pi^*\alpha_{r-1}.$$

Suppose for some nonempty open $U \subset X$ we have $\alpha|_{\pi^{-1}(U)} = \gamma_U(\alpha_0, \ldots, \alpha_{r-1})$. Then we may choose lifts $\alpha'_i \in A_{k+i}(X)$ and we see that $\alpha - \gamma_X(\alpha'_0, \ldots, \alpha'_{r-1})$ is by Lemma 19.2 rationally equivalent to a k-cycle on $P_Y = \mathbf{P}(\mathcal{E}|_Y)$ where $Y = X \setminus U$ as a reduced closed subscheme. Note that $\dim_\delta(Y) < \dim_\delta(X)$. By induction the result holds for $P_Y \to Y$ and hence the result holds for α. Hence we may replace X by any nonempty open of X.

In particular we may assume that $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$. In this case $\mathbf{P}(\mathcal{E}) = X \times \mathbf{P}^{r-1}$. Let us use the stratification

$$\mathbf{P}^{r-1} = \mathbf{A}^{r-1} \coprod \mathbf{A}^{r-2} \coprod \ldots \coprod \mathbf{A}^0$$

The closure of each stratum is a $\mathbf{P}^{r-1-i}$ which is a representative of $c_1(\mathcal{O}(1))^i \cap [\mathbf{P}^{r-1}]$. Hence P has a similar stratification

$$P = U^{r-1} \coprod U^{r-2} \coprod \ldots \coprod U^0$$

Let P^i be the closure of U^i. Let $\pi^i : P^i \to X$ be the restriction of π to P^i. Let $\alpha \in A_{k+r-1}(P)$. By Lemma 32.1 we can write $\alpha|_{U^{r-1}} = \pi^*\alpha_0|_{U^{r-1}}$ for some $\alpha_0 \in A_k(X)$. Hence the difference $\alpha - \pi^*\alpha_0$ is the image of some $\alpha' \in A_{k+r-1}(P^{r-2})$. By Lemma 32.1 again we can write $\alpha'|_{U^{r-2}} = (\pi^{r-2})^*\alpha_1|_{U^{r-2}}$ for some $\alpha_1 \in A_{k+1}(X)$. By Lemma 31.2 we see that the image of $(\pi^{r-2})^*\alpha_1$ represents $c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$. We also see that $\alpha - \pi^*\alpha_0 - c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$ is the image of some $\alpha'' \in A_{k+r-1}(P^{r-3})$. And so on. $\square$

Lemma 33.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let*

$$p : E = \underline{\mathrm{Spec}}(Sym^*(\mathcal{E})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : A_k(X) \to A_{k+r}(E)$ is an isomorphism for all k.

Proof. For surjectivity see Lemma 32.1. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to the finite locally free sheaf $\mathcal{E} \oplus \mathcal{O}_X$. Let $s \in \Gamma(P, \mathcal{O}_P(1))$ correspond to the global section $(0, 1) \in \Gamma(X, \mathcal{E} \oplus \mathcal{O}_X)$. Let $D = Z(s) \subset P$. Note that $(\pi|_D : D \to X, \mathcal{O}_P(1)|_D)$ is the projective space bundle associated to $\mathcal{E}$. We denote $\pi_D = \pi|_D$ and $\mathcal{O}_D(1) = \mathcal{O}_P(1)|_D$. Moreover, D is an effective Cartier divisor on P. Hence $\mathcal{O}_P(D) = \mathcal{O}_P(1)$ (see Divisors, Lemma 9.20). Also there is an isomorphism $E \cong P \setminus D$. Denote $j : E \to P$ the corresponding open immersion. For injectivity we use that the kernel of

$$j^* : A_{k+r}(P) \longrightarrow A_{k+r}(E)$$

are the cycles supported in the effective Cartier divisor D, see Lemma 19.2. So if $p^*\alpha = 0$, then $\pi^*\alpha = i_*\beta$ for some $\beta \in A_{k+r}(D)$. By Lemma 33.2 we may write

$$\beta = \pi_D^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^{r-1} \cap \pi_D^*\beta_{r-1}.$$

for some $\beta_i \in A_{k+i}(X)$. By Lemmas 31.2 and 25.6 this implies

$$\pi^*\alpha = i_*\beta = c_1(\mathcal{O}_P(1)) \cap \pi^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^r \cap \pi^*\beta_{r-1}.$$

Since the rank of $\mathcal{E} \oplus \mathcal{O}_X$ is $r + 1$ this contradicts Lemma 25.6 unless all α and all β_i are zero. $\square$

34. The Chern classes of a vector bundle

We can use the projective space bundle formula to define the chern classes of a rank r vector bundle in terms of the expansion of $c_1(\mathcal{O}(1))^r$ in terms of the lower powers, see formula (34.1.1). The reason for the signs will be explained later.

Definition 34.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to $\mathcal{E}$.

(1) By Lemma 33.2 there are elements $c_i \in A_{n-i}(X)$, $i = 0, \ldots, r$ such that $c_0 = [X]$, and

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^* c_{r-i} = 0. \tag{34.1.1}$$

(2) With notation as above we set $c_i(\mathcal{E}) \cap [X] = c_i$ as an element of $A_{n-i}(X)$. We call these the *chern classes of $\mathcal{E}$ on X*.

(3) The *total chern class of $\mathcal{E}$ on X* is the combination

$$c(\mathcal{E}) \cap [X] = c_0(\mathcal{E}) \cap [X] + c_1(\mathcal{E}) \cap [X] + \ldots + c_r(\mathcal{E}) \cap [X]$$

which is an element of $A_*(X) = \bigoplus_{k \in \mathbf{Z}} A_k(X)$.

Let us check that this does not give a new notion in case the vector bundle has rank 1.

Lemma 34.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The first chern class of $\mathcal{L}$ on X of Definition 34.1 is equal to the Weil divisor associated to $\mathcal{L}$ by Definition 24.1.*

Proof. In this proof we use $c_1(\mathcal{L}) \cap [X]$ to denote the construction of Definition 24.1. Since $\mathcal{L}$ has rank 1 we have $\mathbf{P}(\mathcal{L}) = X$ and $\mathcal{O}_{\mathbf{P}(\mathcal{L})}(1) = \mathcal{L}$ by our normalizations. Hence (34.1.1) reads

$$(-1)^1 c_1(\mathcal{L}) \cap c_0 + (-1)^0 c_1 = 0$$

Since $c_0 = [X]$, we conclude $c_1 = c_1(\mathcal{L}) \cap [X]$ as desired. $\square$

Remark 34.3. We could also rewrite equation 34.1.1 as

$$\sum\nolimits_{i=0}^{r} c_1(\mathcal{O}_P(-1))^i \cap \pi^* c_{r-i} = 0. \tag{34.3.1}$$

but we find it easier to work with the tautological quotient sheaf $\mathcal{O}_P(1)$ instead of its dual.

35. Intersecting with chern classes

Definition 35.1. Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. We define, for every integer k and any $0 \leq j \leq r$, an operation

$$c_j(\mathcal{E}) \cap - : Z_k(X) \to A_{k-j}(X)$$

called *intersection with the jth chern class of $\mathcal{E}$*.

(1) Given an integral closed subscheme $i : W \to X$ of δ-dimension k we define

$$c_j(\mathcal{E}) \cap [W] = i_*(c_j(i^*\mathcal{E}) \cap [W]) \in A_{k-j}(X)$$

where $c_j(i^*\mathcal{E}) \cap [W]$ is as defined in Definition 34.1.

(2) For a general k-cycle $\alpha = \sum n_i[W_i]$ we set

$$c_j(\mathcal{E}) \cap \alpha = \sum n_i c_j(\mathcal{E}) \cap [W_i]$$

Again, if $\mathcal{E}$ has rank 1 then this agrees with our previous definition.

Lemma 35.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For $\alpha \in Z_k(X)$ the elements $c_j(\mathcal{E}) \cap \alpha$ are the unique elements α_j of $A_{k-j}(X)$ such that $\alpha_0 = \alpha$ and*

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

holds in the Chow group of P.

Proof. The uniqueness of $\alpha_0, \ldots, \alpha_r$ such that $\alpha_0 = \alpha$ and such that the displayed equation holds follows from the projective space bundle formula Lemma 33.2. The identity holds by definition for $\alpha = [X]$. For a general k-cycle α on X write $\alpha = \sum n_a[W_a]$ with $n_a \neq 0$, and $i_a : W_a \to X$ pairwise distinct integral closed subschemes. Then the family $\{W_a\}$ is locally finite on X. Set $P_a = \pi^{-1}(W_a) = \mathbf{P}(\mathcal{E}|_{W_a})$. Denote $i'_a : P_a \to P$ the corresponding closed immersions. Consider the fibre product diagram

$$\begin{tikzcd} P' \arrow[r, equal] \arrow[d, "\pi'"'] & \coprod P_a \arrow[r, "i'_a"'] \arrow[d, "\pi_a"'] & P \arrow[d, "\pi"] \\ X' \arrow[r, equal] & \coprod W_a \arrow[r, "i_a"] & X \end{tikzcd}$$

The morphism $p : X' \to X$ is proper. Moreover $\pi' : P' \to X'$ together with the invertible sheaf $\mathcal{O}_{P'}(1) = \coprod \mathcal{O}_{P_a}(1)$ which is also the pullback of $\mathcal{O}_P(1)$ is the projective bundle associated to $\mathcal{E}' = p^*\mathcal{E}$. By definition

$$c_j(\mathcal{E}) \cap [\alpha] = \sum i_{a,*}(c_j(\mathcal{E}|_{W_a}) \cap [W_a]).$$

Write $\beta_{a,j} = c_j(\mathcal{E}|_{W_a}) \cap [W_a]$ which is an element of $A_{k-j}(W_a)$. We have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_a}(1))^i \cap \pi_a^*(\beta_{a,r-i}) = 0$$

for each a by definition. Thus clearly we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(\beta_{r-i}) = 0$$

with $\beta_j = \sum n_a \beta_{a,j} \in A_{k-j}(X')$. Denote $p' : P' \to P$ the morphism $\coprod i'_a$. We have $\pi^* p_* \beta_j = p'_*(\pi')^* \beta_j$ by Lemma 15.1. By the projection formula of Lemma 25.6 we conclude that

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_*\beta_j) = 0$$

Since $p_*\beta_j$ is a representative of $c_j(\mathcal{E}) \cap \alpha$ we win. $\square$

This characterization of chern classes allows us to prove many more properties.

Lemma 35.3. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. If $\alpha \sim_{rat} \beta$ are rationally equivalent k-cycles on X then $c_j(\mathcal{E}) \cap \alpha = c_j(\mathcal{E}) \cap \beta$ in $A_{k-j}(X)$.*

Proof. By Lemma 35.2 the elements $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, $j \geq 1$ and $\beta_j = c_j(\mathcal{E}) \cap \beta$, $j \geq 1$ are uniquely determined by the *same* equation in the chow group of the projective bundle associated to $\mathcal{E}$. (This of course relies on the fact that flat pullback is compatible with rational equivalence, see Lemma 20.1.) Hence they are equal. $\square$

In other words capping with chern classes of finite locally free sheaves factors through rational equivalence to give maps

$$c_j(\mathcal{E}) \cap - : A_k(X) \to A_{k-j}(X).$$

Lemma 35.4. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on Y. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let α be a k-cycle on Y. Then*

$$f^*(c_j(\mathcal{E}) \cap \alpha) = c_j(f^*\mathcal{E}) \cap f^*\alpha$$

Proof. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 35.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to Y, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{tikzcd}
P_X = \mathbf{P}(f^*\mathcal{E}) \arrow[r, "f_P"'] \arrow[d, "\pi_X"'] & P \arrow[d, "\pi"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Note that $\mathcal{O}_{P_X}(1) = f_P^*\mathcal{O}_P(1)$. By Lemmas 25.4 and 14.3 we see that

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_X}(1))^i \cap \pi_X^*(f^*\alpha_{r-i}) = 0$$

holds in the chow group of P_X. Since $f^*\alpha_0 = f^*\alpha$ the lemma follows from the uniqueness in Lemma 35.2. $\square$

Lemma 35.5. *Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $p : X \to Y$ be a proper morphism. Let α be a k-cycle on X. Let $\mathcal{E}$ be a finite locally free sheaf on Y. Then*

$$p_*(c_j(p^*\mathcal{E}) \cap \alpha) = c_j(\mathcal{E}) \cap p_*\alpha$$

Proof. Write $\alpha_j = c_j(p^*\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 35.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi_X^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi_X : P_X \to X, \mathcal{O}_{P_X}(1))$ associated to $p^*\mathcal{E}$. Let $(\pi : P \to Y, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{tikzcd}
P_X = \mathbf{P}(p^*\mathcal{E}) \arrow[r, "p_P"'] \arrow[d, "\pi_X"'] & P \arrow[d, "\pi"] \\
X \arrow[r, "p"] & Y
\end{tikzcd}$$

Note that $\mathcal{O}_{P_X}(1) = p_P^*\mathcal{O}_P(1)$. Pushing the displayed equality above to P and using Lemmas 15.1, 25.6 and 14.3 we see that

$$\sum\nolimits_{i=0}^r (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_*\alpha_{r-i}) = 0$$

holds in the chow group of P. Since $p_*\alpha_0 = p_*\alpha$ the lemma follows from the uniqueness in Lemma 35.2. □

Lemma 35.6. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r and s. For any $\alpha \in A_k(X)$ we have*

$$c_i(\mathcal{E}) \cap c_j(\mathcal{F}) \cap \alpha = c_j(\mathcal{F}) \cap c_i(\mathcal{E}) \cap \alpha$$

as elements of $A_{k-i-j}(X)$.

Proof. Consider

$$\pi : \mathbf{P}(\mathcal{E}) \times_X \mathbf{P}(\mathcal{F}) \longrightarrow X$$

with invertible sheaves $\mathcal{L} = \mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ and $\mathcal{N} = \mathrm{pr}_2^*\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$. Write $\alpha_{i,j}$ for the left hand side and $\beta_{i,j}$ for the right hand side. Also write $\alpha_j = c_j(\mathcal{F}) \cap \alpha$ and $\beta_i = c_i(\mathcal{E}) \cap \alpha$. In particular this means that $\alpha_0 = \alpha = \beta_0$, and $\alpha_{0,j} = \alpha_j = \beta_{0,j}$, $\alpha_{i,0} = \beta_i = \beta_{i,0}$. From Lemma 35.2 (pulled back to the space above using Lemma 25.4 for the first two) we see that

$$\begin{aligned}
0 &= \sum\nolimits_{j=0,\ldots,s} (-1)^j c_1(\mathcal{N})^j \cap \pi^*\alpha_{s-j} \\
0 &= \sum\nolimits_{i=0,\ldots,r} (-1)^i c_1(\mathcal{L})^i \cap \pi^*\beta_{r-i} \\
0 &= \sum\nolimits_{i=0,\ldots,r} (-1)^i c_1(\mathcal{L})^i \cap \pi^*\alpha_{r-i,s-j} \\
0 &= \sum\nolimits_{j=0,\ldots,s} (-1)^j c_1(\mathcal{N})^j \cap \pi^*\beta_{r-i,s-j}
\end{aligned}$$

We can combine the first and the third of these to get

$$\begin{aligned}
&(-1)^{r+s} c_1(\mathcal{L})^r \cap c_1(\mathcal{N})^s \cap \pi^*\alpha \\
&= \sum\nolimits_{j=1,\ldots,s} (-1)^{r+j-1} c_1(\mathcal{L})^r \cap c_1(\mathcal{N})^j \cap \pi^*\alpha_{s-j} \\
&= \sum\nolimits_{j=1,\ldots,s} (-1)^{j-1+r} c_1(\mathcal{N})^j \cap c_1(\mathcal{L})^r \cap \pi^*\alpha_{0,s-j} \\
&= \sum\nolimits_{j=1}^s \sum\nolimits_{i=1}^r (-1)^{i+j} c_1(\mathcal{N})^j \cap c_1(\mathcal{L})^i \cap \pi^*\alpha_{r-i,s-j}
\end{aligned}$$

using that capping with $c_1(\mathcal{L})$ commutes with capping with $c_1(\mathcal{N})$. In exactly the same way one shows that

$$(-1)^{r+s} c_1(\mathcal{L})^r \cap c_1(\mathcal{N})^s \cap \pi^*\alpha = \sum\nolimits_{j=1}^s \sum\nolimits_{i=1}^r (-1)^{i+j} c_1(\mathcal{N})^j \cap c_1(\mathcal{L})^i \cap \pi^*\beta_{r-i,s-j}$$

By the projective space bundle formula Lemma 33.2 applied twice these representations are unique. Whence the result. □

36. Polynomial relations among chern classes

Definition 36.1. Let $P(x_{i,j}) \in \mathbf{Z}[x_{i,j}]$ be a polynomial. We write P as a finite sum

$$\sum\nolimits_s \sum\nolimits_{I=((i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s))} a_I x_{i_1,j_1} \ldots x_{i_s,j_s}.$$

Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}_i$ be a finite collection of finite locally free sheaves on X. We say that P is a *polynomial relation among the chern classes* and we write $P(c_j(\mathcal{E}_i)) = 0$ if for any morphism $f : Y \to X$ of an integral scheme locally of finite type over S the cycle

$$\sum\nolimits_s \sum\nolimits_{I=((i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s))} a_I \; c_{j_1}(f^*\mathcal{E}_{i_1}) \cap \ldots \cap c_{j_s}(f^*\mathcal{E}_{i_s}) \cap [Y]$$

is zero in $A_*(Y)$.

This is not an elegant definition but it will do for now. It makes sense because we showed in Lemma 35.6 that capping with chern classes of vector bundles is commutative. By our definitions and results above this is equivalent with requiring all the operations

$$\sum\nolimits_s \sum\nolimits_I a_I \; c_{j_1}(f^*\mathcal{E}_{i_1}) \cap \ldots \cap c_{j_s}(f^*\mathcal{E}_{i_s}) \cap - : A_*(Y) \to A_*(Y)$$

to be zero for all morphisms $f : Y \to X$ which are locally of finite type.

An example of such a relation is the relation

$$c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) = c_1(\mathcal{L}) + c_1(\mathcal{N})$$

proved in Lemma 25.2. More generally, here is what happens when we tensor an arbitrary locally free sheaf by an invertible sheaf.

Lemma 36.2. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $\mathcal{L}$ be an invertible sheaf on X. Then*

$$c_i(\mathcal{E} \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j \tag{36.2.1}$$

is a valid polynomial relation in the sense described above.

Proof. This should hold for any triple $(X, \mathcal{E}, \mathcal{L})$. In particular it should hold when X is integral, and in fact by definition of a polynomial relation it is enough to prove it holds when capping with $[X]$ for such X. Thus assume that X is integral. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ be the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{E} \otimes \mathcal{L}$. Consider the canonical morphism

$$\begin{tikzcd} P \arrow[rr, "g"'] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

see Constructions, Lemma 20.1. It has the property that $g^*\mathcal{O}_{P'}(1) = \mathcal{O}_P(1) \otimes \pi^*\mathcal{L}$. This means that we have

$$\sum\nolimits_{i=0}^{r} (-1)^i (\xi + x)^i \cap \pi^*(c_{r-i}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) = 0$$

in $A_*(P)$, where ξ represents $c_1(\mathcal{O}_P(1))$ and x represents $c_1(\pi^*\mathcal{L})$. By simple algebra this is equivalent to

$$\sum\nolimits_{i=0}^{r} (-1)^i \xi^i \left(\sum\nolimits_{j=i}^{r} (-1)^{j-i} \binom{j}{i} x^{j-i} \cap \pi^*(c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) \right) = 0$$

Comparing with Equation (34.1.1) it follows from this that

$$c_{r-i}(\mathcal{E}) \cap [X] = \sum\nolimits_{j=i}^{r} \binom{j}{i} (-c_1(\mathcal{L}))^{j-i} \cap c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]$$

Reworking this (getting rid of minus signs, and renumbering) we get the desired relation. □

Some example cases of (36.2.1) are

$$c_1(\mathcal{E}\otimes\mathcal{L}) = c_1(\mathcal{E}) + rc_1(\mathcal{L})$$
$$c_2(\mathcal{E}\otimes\mathcal{L}) = c_2(\mathcal{E}) + (r-1)c_1(\mathcal{E})c_1(\mathcal{L}) + \binom{r}{2}c_1(\mathcal{L})^2$$
$$c_3(\mathcal{E}\otimes\mathcal{L}) = c_3(\mathcal{E}) + (r-2)c_2(\mathcal{E})c_1(\mathcal{L}) + \binom{r-1}{2}c_1(\mathcal{E})c_1(\mathcal{L})^2 + \binom{r}{3}c_1(\mathcal{L})^3$$

37. Additivity of chern classes

All of the preliminary lemmas follow trivially from the final result.

Lemma 37.1. *Let (S,δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$$

Then

$$c_r(\mathcal{E}) = 0, \quad c_j(\mathcal{E}) = c_j(\mathcal{F}), \quad j = 0, \ldots, r-1$$

are valid polynomial relations among chern classes.

Proof. By Definition 36.1 it suffices to show that if X is integral then $c_j(\mathcal{E})\cap[X] = c_j(\mathcal{F})\cap[X]$. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ denote the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{F}$. The surjection $\mathcal{E} \to \mathcal{F}$ gives rise to a closed immersion

$$i : P' \longrightarrow P$$

over X. Moreover, the element $1 \in \Gamma(X, \mathcal{O}_X) \subset \Gamma(X, \mathcal{E})$ gives rise to a global section $s \in \Gamma(P, \mathcal{O}_P(1))$ whose zero set is exactly P'. Hence P' is an effective Cartier divisor on P such that $\mathcal{O}_P(P') \cong \mathcal{O}_P(1)$. Hence we see that

$$c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha = i_*((\pi')^*\alpha)$$

for any cycle class α on X by Lemma 31.2. By Lemma 35.2 we see that $\alpha_j = c_j(\mathcal{F})\cap[X]$, $j = 0, \ldots, r-1$ satisfy

$$\sum\nolimits_{j=0}^{r-1}(-1)^j c_1(\mathcal{O}_{P'}(1))^j \cap (\pi')^*\alpha_j = 0$$

Pushing this to P and using the remark above as well as Lemma 25.6 we get

$$\sum\nolimits_{j=0}^{r-1}(-1)^j c_1(\mathcal{O}_P(1))^{j+1} \cap \pi^*\alpha_j = 0$$

By the uniqueness of Lemma 35.2 we conclude that $c_r(\mathcal{E})\cap[X] = 0$ and $c_j(\mathcal{E})\cap[X] = \alpha_j = c_j(\mathcal{F})\cap[X]$ for $j = 0, \ldots, r-1$. Hence the lemma holds. □

Lemma 37.2. *Let (S,δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is an invertible sheaf Then

$$c(\mathcal{E}) = c(\mathcal{L})c(\mathcal{F})$$

is a valid polynomial relation among chern classes.

Proof. This relation really just says that $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$. By Lemma 37.1 we have $c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1}) = c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1})$ for $j = 0, \ldots, r$ (were we set $c_r(\mathcal{F}) = 0$ by convention). Applying Lemma 36.2 we deduce

$$\sum_{j=0}^{i} \binom{r-i+j}{j} (-1)^j c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j = \sum_{j=0}^{i} \binom{r-1-i+j}{j} (-1)^j c_{i-j}(\mathcal{F}) c_1(\mathcal{L})^j$$

Setting $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$ gives a "solution" of this equation. The lemma follows if we show that this is the only possible solution. We omit the verification. □

Lemma 37.3. *Let (S, δ) be as in Situation 7.1. Let X be a scheme locally of finite type over S. Suppose that $\mathcal{E}$ sits in an exact sequence*

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

of finite locally free sheaves $\mathcal{E}_i$ of rank r_i. Then

$$c(\mathcal{E}) = c(\mathcal{E}_1)c(\mathcal{E}_2)$$

is a polynomial relation among chern classes.

Proof. We may assume that X is integral and we have to show the identity when capping against $[X]$. By induction on r_1. The case $r_1 = 1$ is Lemma 37.2. Assume $r_1 > 1$. Let $(\pi : P \to X, \mathcal{O}_P(1))$ denote the projective space bundle associated to $\mathcal{E}_1$. Note that

(1) $\pi^* : A_*(X) \to A_*(P)$ is injective, and
(2) $\pi^*\mathcal{E}_1$ sits in a short exact sequence $0 \to \mathcal{F} \to \pi^*\mathcal{E}_1 \to \mathcal{L} \to 0$ where $\mathcal{L}$ is invertible.

The first assertion follows from the projective space bundle formula and the second follows from the definition of a projective space bundle. (In fact $\mathcal{L} = \mathcal{O}_P(1)$.) Let $Q = \pi^*\mathcal{E}/\mathcal{F}$, which sits in an exact sequence $0 \to \mathcal{L} \to Q \to \pi^*\mathcal{E}_2 \to 0$. By induction we have

$$\begin{aligned} c(\pi^*\mathcal{E}) \cap [P] &= c(\mathcal{F}) \cap c(\pi^*\mathcal{E}/\mathcal{F}) \cap [P] \\ &= c(\mathcal{F}) \cap c(\mathcal{L}) \cap c(\pi^*\mathcal{E}_2) \cap [P] \\ &= c(\pi^*\mathcal{E}_1) \cap c(\pi^*\mathcal{E}_2) \cap [P] \end{aligned}$$

Since $[P] = \pi^*[X]$ we win by Lemma 35.4. □

Lemma 37.4. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{L}_i$, $i = 1, \ldots, r$ be invertible $\mathcal{O}_X$-modules on X. Let $\mathcal{E}$ be a finite locally free rank r $\mathcal{O}_X$-module endowed with a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Set $c_1(\mathcal{L}_i) = x_i$. Then

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^{r} (1 + x_i)$$

is a valid polynomial relation among chern classes in the sense of Definition 36.1.

Proof. Apply Lemma 37.2 and induction. □

38. The splitting principle

In our setting it is not so easy to say what the splitting principle exactly says/is. Here is a possible formulation.

Lemma 38.1. *Let (S, δ) be as in Situation 7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf $\mathcal{E}$ on X of rank r. There exists a projective flat morphism of relative dimension d $\pi : P \to X$ such that*

1. *for any morphism $f : Y \to X$ the map $\pi_Y^* : A_*(Y) \to A_{*+r}(Y \times_X P)$ is injective, and*
2. *$\pi^*\mathcal{E}$ has a filtration with successive quotients $\mathcal{L}_1, \ldots, \mathcal{L}_r$ for some invertible $\mathcal{O}_P$-modules $\mathcal{L}_i$.*

Proof. Omitted. Hint: Use a composition of projective space bundles, i.e., a flag variety over X. □

The splitting principle refers to the practice of symbolically writing

$$c(\mathcal{E}) = \prod (1 + x_i)$$

with $x_i = c_1(\mathcal{L}_i)$. The expressions x_i are then called the *Chern roots* of $\mathcal{E}$. In order to prove polynomial relations among chern classes of vector bundles it is permissible to do calculations using the chern roots.

For example, let us calculate the chern classes of the dual vector bundle $\mathcal{E}^\wedge$. Note that if $\mathcal{E}$ has a filtration with subquotients invertible sheaves $\mathcal{L}_i$ then $\mathcal{E}^\wedge$ has a filtration with subquotients the invertible sheaves $\mathcal{L}_i^{-1}$. Hence if x_i are the chern roots of $\mathcal{E}$, then the $-x_i$ are the chern roots of $\mathcal{E}^\wedge$. It follows that

$$c_j(\mathcal{E}^\wedge) = (-1)^j c_j(\mathcal{E})$$

is a valid polynomial relation among chern classes.

In the same vain, let us compute the chern classes of a tensor product of vector bundles. Namely, suppose that $\mathcal{E}$, $\mathcal{F}$ are finite locally free of ranks r, s. Write

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^{r} (1 + x_i), \quad c(\mathcal{E}) = \prod\nolimits_{j=1}^{s} (1 + y_j)$$

where x_i, y_j are the chern roots of $\mathcal{E}$, $\mathcal{F}$. Then we see that

$$c(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \prod\nolimits_{i,j} (1 + x_i + y_j)$$

Here are some examples of what this means in terms of chern classes

$$c_1(\mathcal{E} \otimes \mathcal{F}) = rc_1(\mathcal{F}) + sc_1(\mathcal{E})$$
$$c_2(\mathcal{E} \otimes \mathcal{F}) = r^2 c_2(\mathcal{F}) + rsc_1(\mathcal{F})c_1(\mathcal{E}) + s^2 c_2(\mathcal{E})$$

39. Chern classes and tensor product

We define the *Chern character* of a finite locally free sheaf of rank r to be the formal expression

$$ch(\mathcal{E}) := \sum\nolimits_{i=1}^{r} e^{x_i}$$

if the x_i are the chern roots of $\mathcal{E}$. Writing this in terms of chern classes $c_i = c_i(\mathcal{E})$ we see that

$$ch(\mathcal{E}) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \ldots$$

What does it mean that the coefficients are rational numbers? Well this simply means that we think of these as operations

$$ch_j(\mathcal{E}) \cap - : A_k(X) \longrightarrow A_{k-j}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$$

and we think of polynomial relations among them as relations between these operations with values in the groups $A_{k-j}(Y) \otimes_{\mathbf{Z}} \mathbf{Q}$ for varying Y. By the above we have in case of an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

that

$$ch(\mathcal{E}) = ch(\mathcal{E}_1) + ch(\mathcal{E}_2)$$

Using the Chern character we can express the compatibility of the chern classes and tensor product as follows:

$$ch(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = ch(\mathcal{E}_1)ch(\mathcal{E}_2)$$

This follows directly from the discussion of the chern roots of the tensor product in the previous section.

40. Todd classes

A final class associated to a vector bundle $\mathcal{E}$ of rank r is its *Todd class* $Todd(\mathcal{E})$. In terms of the chern roots $x_1, \ldots, x_r$ it is defined as

$$Todd(\mathcal{E}) = \prod\nolimits_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}}$$

In terms of the chern classes $c_i = c_i(\mathcal{E})$ we have

$$Todd(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \ldots$$

We have made the appropriate remarks about denominators in the previous section. It is the case that given an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

we have

$$Todd(\mathcal{E}) = Todd(\mathcal{E}_1)Todd(\mathcal{E}_2).$$

41. Grothendieck-Riemann-Roch

Let (S, δ) be as in Situation 7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf $\mathcal{E}$ on X of rank r. Let $f : X \to Y$ be a proper smooth morphism. Assume that $R^if_*\mathcal{E}$ are locally free sheaves on Y of finite rank (for example if Y is a point). The Grothendieck-Riemann-Roch theorem implies that in this case we have

$$f_*(Todd(T_{X/Y})ch(\mathcal{E})) = \sum (-1)^i ch(R^if_*\mathcal{E})$$

Here

$$T_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$$

is the relative tangent bundle of X over Y. The theorem is more general and becomes easier to prove when formulated in correct generality. We will return to this elsewhere (insert future reference here).

42. Other chapters

References

[Che58a] Claude Chevalley, *Les classes d'equivalence rationnelles i*, S'eminair Claude Chevalley (1958), 14.

[Che58b] ———, *Les classes d'equivalence rationnelles II*, S'eminair Claude Chevalley (1958), 18.

[Ful98] William Fulton, *Intersection theory*, 2 ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 2, Springer-Verlag, 1998.

[KM76] Finn Faye Knudsen and David Mumford, *The projectivity of the moduli space of stable curves, I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55.

[Knu02] Finn Faye Knudsen, *Determinant functors on exact categories and their extensions to categories of bounded complexes*, Michigan Math. J. **50** (2002), no. 2, 407–444.

[Ros09] Joseph Ross, *The hilbert-chow morphism and the incidence divisor*, Columbia University PhD thesis, 2009.

[Sam56] Pierre Samuel, *Rational equivalence of arbitrary cycles*, Amer. J. Math. **78** (1956), 383–400.

[Ser65] Jean-Pierre Serre, *Algèbre locale. Multiplicités*, Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel. Seconde édition, 1965. Lecture Notes in Mathematics, vol. 11, Springer-Verlag, Berlin, 1965.

[Ser00] ———, *Local algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000, Translated from the French by CheeWhye Chin and revised by the author.