CATEGORIES

Contents

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1. Introduction

Categories were first introduced in [EM45]. The category of categories (which is a proper class) is a 2-category. Similarly, the category of stacks forms a 2-category. If you already know about categories, but not about 2-categories you should read Section 26 as an introduction to the formal definitions later on.

2. Definitions

We recall the definitions, partly to fix notation.

Definition 2.1. A *category* $\mathcal{C}$ consists of the following data:

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a set of morphisms $\mathrm{Mor}_\mathcal{C}(x, y)$.
(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a composition map $\mathrm{Mor}_\mathcal{C}(y, z) \times \mathrm{Mor}_\mathcal{C}(x, y) \to \mathrm{Mor}_\mathcal{C}(x, z)$, denoted $(\phi, \psi) \mapsto \phi \circ \psi$.

These data are to satisfy the following rules:

(1) For every element $x \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $\mathrm{id}_x \in \mathrm{Mor}_\mathcal{C}(x, x)$ such that $\mathrm{id}_x \circ \phi = \phi$ and $\psi \circ \mathrm{id}_x = \psi$ whenever these compositions make sense.
(2) Composition is associative, i.e., $(\phi \circ \psi) \circ \chi = \phi \circ (\psi \circ \chi)$ whenever these compositions make sense.

It is customary to require all the morphism sets $\mathrm{Mor}_\mathcal{C}(x, y)$ to be disjoint. In this way a morphism $\phi : x \to y$ has a unique *source* x and a unique *target* y. This is not strictly necessary, although care has to be taken in formulating condition (2) above if it is not the case. It is convenient and we will often assume this is the case. In this case we say that ϕ and ψ are *composable* if the source of ϕ is equal to the target of ψ, in which case $\phi \circ \psi$ is defined. An equivalent definition would be to define a category as a quintuple $(\mathrm{Ob}, \mathrm{Arrows}, s, t, \circ)$ consisting of a set of objects, a set of morphisms (arrows), source, target and composition subject to a long list of axioms. We will occasionally use this point of view.

Remark 2.2. Big categories. In some texts a category is allowed to have a proper class of objects. We will allow this as well in these notes but only in the following list of cases (to be updated as we go along). In particular, when we say: "Let $\mathcal{C}$ be a category" then it is understood that $\mathrm{Ob}(\mathcal{C})$ is a set.

(1) The category *Sets* of sets.
(2) The category *Ab* of abelian groups.
(3) The category *Groups* of groups.
(4) Given a group G the category G-*Sets* of sets with a left G-action.
(5) Given a ring R the category Mod_R of R-modules.
(6) Given a field k the category of vector spaces over k.
(7) The category of rings.
(8) The category of schemes.

(9) The category *Top* of topological spaces.
(10) Given a topological space X the category $PSh(X)$ of presheaves of sets over X.
(11) Given a topological space X the category $Sh(X)$ of sheaves of sets over X.
(12) Given a topological space X the category $PAb(X)$ of presheaves of abelian groups over X.
(13) Given a topological space X the category $Ab(X)$ of sheaves of abelian groups over X.
(14) Given a small category $\mathcal{C}$ the category of functors from $\mathcal{C}$ to *Sets*.
(15) Given a category $\mathcal{C}$ the category of presheaves of sets over $\mathcal{C}$.
(16) Given a site $\mathcal{C}$ the category of sheaves of sets over $\mathcal{C}$.

One of the reason to enumerate these here is to try and avoid working with something like the "collection" of "big" categories which would be like working with the collection of all classes which I think definitively is a meta-mathematical object.

Remark 2.3. It follows directly from the definition that any two identity morphisms of an object x of $\mathcal{A}$ are the same. Thus we may and will speak of *the* identity morphism id_x of x.

Definition 2.4. A morphism $\phi : x \to y$ is an *isomorphism* of the category $\mathcal{C}$ if there exists a morphism $\psi : y \to x$ such that $\phi \circ \psi = \mathrm{id}_y$ and $\psi \circ \phi = \mathrm{id}_x$.

An isomorphism ϕ is also sometimes called an *invertible* morphism, and the morphism ψ of the definition is called the *inverse* and denoted ϕ^{-1}. It is unique if it exists. Note that given an object x of a category $\mathcal{A}$ the set of invertible elements $\mathrm{Aut}_{\mathcal{A}}(x)$ of $\mathrm{Mor}_{\mathcal{A}}(x, x)$ forms a group under composition. This group is called the *automorphism* group of x in $\mathcal{A}$.

Definition 2.5. A *groupoid* is a category where every morphism is an isomorphism.

Example 2.6. A group G gives rise to a groupoid with a single object x and morphisms $\mathrm{Mor}(x, x) = G$, with the composition rule given by the group law in G. Every groupoid with a single object is of this form.

Example 2.7. A set C gives rise to a groupoid $\mathcal{C}$ defined as follows: As objects we take $\mathrm{Ob}(\mathcal{C}) := C$ and for morphisms we take $\mathrm{Mor}(x, y)$ empty if $x \neq y$ and equal to $\{\mathrm{id}_x\}$ if $x = y$.

Definition 2.8. A *functor* $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is given by the following data:

(1) A map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$.
(2) For every $x, y \in \mathrm{Ob}(\mathcal{A})$ a map $F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$, denoted $\phi \mapsto F(\phi)$.

These data should be compatible with composition and identity morphisms in the following manner: $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ for a composable pair (ϕ, ψ) of morphisms of $\mathcal{A}$ and $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

Note that every category $\mathcal{A}$ has an *identity* functor $\mathrm{id}_{\mathcal{A}}$. In addition, given a functor $G : \mathcal{B} \to \mathcal{C}$ and a functor $F : \mathcal{A} \to \mathcal{B}$ there is a *composition* functor $G \circ F : \mathcal{A} \to \mathcal{C}$ defined in an obvious manner.

Definition 2.9. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) We say F is *faithful* if for any objects x, y of $\mathrm{Ob}(\mathcal{A})$ the map
$$F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$$
is injective.
(2) If these maps are all bijective then F is called *fully faithful*.
(3) The functor F is called *essentially surjective* if for any object $y \in \mathrm{Ob}(\mathcal{B})$ there exists an object $x \in \mathrm{Ob}(\mathcal{A})$ such that $F(x)$ is isomorphic to y in $\mathcal{B}$.

Definition 2.10. A *subcategory* of a category $\mathcal{B}$ is a category $\mathcal{A}$ whose objects and arrows form subsets of the objects and arrows of $\mathcal{B}$ and such that source, target and composition in $\mathcal{A}$ agree with those of $\mathcal{B}$. We say $\mathcal{A}$ is a *full subcategory* of $\mathcal{B}$ if $\mathrm{Mor}_{\mathcal{A}}(x, y) = \mathrm{Mor}_{\mathcal{B}}(x, y)$ for all $x, y \in \mathrm{Ob}(\mathcal{A})$. We say $\mathcal{A}$ is a *strictly full* subcategory of $\mathcal{B}$ if it is a full subcategory and given $x \in \mathrm{Ob}(\mathcal{A})$ any object of $\mathcal{B}$ which is isomorphic to x is also in $\mathcal{A}$.

If $\mathcal{A} \subset \mathcal{B}$ is a subcategory then the identity map is a functor from $\mathcal{A}$ to $\mathcal{B}$. Furthermore a subcategory $\mathcal{A} \subset \mathcal{B}$ is full if and only if the inclusion functor is fully faithful. Note that given a category $\mathcal{B}$ the set of full subcategories of $\mathcal{B}$ is the same as the set of subsets of $\mathrm{Ob}(\mathcal{B})$.

Remark 2.11. Suppose that $\mathcal{A}$ is a category. A functor F from $\mathcal{A}$ to *Sets* is a mathematical object (i.e., it is a set not a class or a formula of set theory, see Sets, Section 2) even though the category of sets is "big". Namely, the range of F on objects will be a set $F(\mathrm{Ob}(\mathcal{A}))$ and then we may think of F as a functor between $\mathcal{A}$ and the full subcategory of the category of sets whose objects are elements of $F(\mathrm{Ob}(\mathcal{A}))$.

Example 2.12. A homomorphism $p : G \to H$ of groups gives rise to a functor between the associated groupoids in Example 2.6. It is faithful (resp. fully faithful) if and only if p is injective (resp. an isomorphism).

Example 2.13. Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$ we define the *category of objects over* X, denoted $\mathcal{C}/X$ as follows. The objects of $\mathcal{C}/X$ are morphisms $Y \to X$ for some $Y \in \mathrm{Ob}(\mathcal{C})$. Morphisms between objects $Y \to X$ and $Y' \to X$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : \mathcal{C}/X \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : \mathcal{C}/X' \to \mathcal{C}/X$ obtained by composition with f, and $p_X \circ F = p_{X'}$.

Example 2.14. Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$ we define the *category of objects under* X, denoted $X/\mathcal{C}$ as follows. The objects of $X/\mathcal{C}$ are morphisms $X \to Y$ for some $Y \in \mathrm{Ob}(\mathcal{C})$. Morphisms between objects $X \to Y$ and $X \to Y'$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : X/\mathcal{C} \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : X/\mathcal{C} \to X'/\mathcal{C}$ obtained by composition with f, and $p_{X'} \circ F = p_X$.

Definition 2.15. Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A *natural transformation*, or a *morphism of functors* $t : F \to G$, is a collection $\{t_x\}_{x \in \mathrm{Ob}(\mathcal{A})}$ such that

(1) $t_x : F(x) \to G(x)$ is a morphism in the category $\mathcal{B}$, and

(2) for every morphism $\phi : x \to y$ of $\mathcal{A}$ the following diagram is commutative

$$\begin{tikzcd} F(x) \arrow[r, "t_x"] \arrow[d, "F(\phi)"'] & G(x) \arrow[d, "G(\phi)"] \\ F(y) \arrow[r, "t_y"] & G(y) \end{tikzcd}$$

Sometimes we use the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow t"] & \mathcal{B} \end{tikzcd}$$

to indicate that t is a morphism from F to G.

Note that every functor F comes with the *identity* transformation $\mathrm{id}_F : F \to F$. In addition, given a morphism of functors $t : F \to G$ and a morphism of functors $s : E \to F$ then the *composition* $t \circ s$ is defined by the rule

$$(t \circ s)_x = t_x \circ s_x : E(x) \to G(x)$$

for $x \in \mathrm{Ob}(\mathcal{A})$. It is easy to verify that this is indeed a morphism of functors from E to G. In this way, given categories $\mathcal{A}$ and $\mathcal{B}$ we obtain a new category, namely the category of functors between $\mathcal{A}$ and $\mathcal{B}$.

Remark 2.16. This is one instance where the same thing does not hold if $\mathcal{A}$ is a "big" category. For example consider functors *Sets* $\to$ *Sets*. As we have currently defined it such a functor is a class and not a set. In other words, it is given by a formula in set theory (with some variables equal to specified sets)! It is not a good idea to try to consider all possible formulae of set theory as part of the definition of a mathematical object. The same problem presents itself when considering sheaves on the category of schemes for example. We will come back to this point later.

Definition 2.17. An *equivalence of categories* $F : \mathcal{A} \to \mathcal{B}$ is a functor such that there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that the compositions $F \circ G$ and $G \circ F$ are isomorphic to the identity functors $\mathrm{id}_\mathcal{B}$, respectively $\mathrm{id}_\mathcal{A}$. In this case we say that G is a *quasi-inverse* to F.

Lemma 2.18. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor. Suppose for every $X \in \mathrm{Ob}(\mathcal{B})$ given an object $j(X)$ of $\mathcal{A}$ and an isomorphism $i_X : X \to F(j(X))$. Then there is a unique functor $j : \mathcal{B} \to \mathcal{A}$ such that j extends the rule on objects, and the isomorphisms i_X define an isomorphism of functors $\mathrm{id}_\mathcal{B} \to F \circ j$. Moreover, j and F are quasi-inverse equivalences of categories.*

Proof. This lemma proves itself. $\square$

Lemma 2.19. *A functor is an equivalence of categories if and only if it is both fully faithful and essentially surjective.*

Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be essentially surjective and fully faithful. As by convention all categories are small and as F is essentially surjective we can, using the axiom of choice, choose for every $X \in \mathrm{Ob}(\mathcal{B})$ an object $j(X)$ of $\mathcal{A}$ and an isomorphism $i_X : X \to F(j(X))$. Then we apply Lemma 2.18 using that F is fully faithful. $\square$

Definition 2.20. Let $\mathcal{A}$, $\mathcal{B}$ be categories. We define the *product category* $\mathcal{A} \times \mathcal{B}$ to be the category with objects $\mathrm{Ob}(\mathcal{A} \times \mathcal{B}) = \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$ and

$$\mathrm{Mor}_{\mathcal{A}\times\mathcal{B}}((x,y),(x',y')) := \mathrm{Mor}_{\mathcal{A}}(x,x') \times \mathrm{Mor}_{\mathcal{B}}(y,y').$$

Composition is defined componentwise.

3. Opposite Categories and the Yoneda Lemma

Definition 3.1. Given a category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{opp}$ is the category with the same objects as $\mathcal{C}$ but all morphisms reversed.

In other words $\mathrm{Mor}_{\mathcal{C}^{opp}}(x,y) = \mathrm{Mor}_{\mathcal{C}}(y,x)$. Composition in $\mathcal{C}^{opp}$ is the same as in $\mathcal{C}$ except backwards: if $\phi : y \to z$ and $\psi : x \to y$ in $\mathcal{C}^{opp}$ then $\phi \circ^{opp} \psi := \psi \circ \phi$.

Definition 3.2. Let $\mathcal{C}$, $\mathcal{S}$ be categories. A *contravariant* functor F from $\mathcal{C}$ to $\mathcal{S}$ is a functor $\mathcal{C}^{opp} \to \mathcal{S}$.

Concretely, a contravariant functor F is given by a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{S})$ and for every morphism $\psi : x \to y$ in $\mathcal{C}$ a morphism $F(\psi) : F(y) \to F(x)$. These should satisfy the property that, given another morphism $\phi : y \to z$, we have $F(\phi \circ \psi) = F(\psi) \circ F(\phi)$ as morphisms $F(z) \to F(x)$. (Note the reverse of order.)

Definition 3.3. Let $\mathcal{C}$ be a category.

(1) A *presheaf of sets on* $\mathcal{C}$ or simply a *presheaf* is a contravariant functor F from $\mathcal{C}$ to *Sets*.
(2) The category of presheaves is denoted $PSh(\mathcal{C})$.

Of course the category of presheaves is a proper class.

Example 3.4. Functor of points. For any $U \in \mathrm{Ob}(\mathcal{C})$ there is a contravariant functor

$$\begin{array}{rccc} h_U & : & \mathcal{C} & \longrightarrow & \mathit{Sets} \\ & & X & \longmapsto & \mathrm{Mor}_{\mathcal{C}}(X,U) \end{array}$$

which takes an object X to the set $\mathrm{Mor}_{\mathcal{C}}(X,U)$. In other words h_U is a presheaf. Given a morphism $f : X \to Y$ the corresponding map $h_U(f) : \mathrm{Mor}_{\mathcal{C}}(Y,U) \to \mathrm{Mor}_{\mathcal{C}}(X,U)$ takes ϕ to $\phi \circ f$. We will always denote this presheaf $h_U : \mathcal{C}^{opp} \to \mathit{Sets}$. It is called the *representable presheaf* associated to U. If $\mathcal{C}$ is the category of schemes this functor is sometimes referred to as the *functor of points* of U.

Note that given a morphism $\phi : U \to V$ in $\mathcal{C}$ we get a corresponding natural transformation of functors $h(\phi) : h_U \to h_V$ defined simply by composing with the morphism $U \to V$. It is trivial to see that this turns composition of morphisms in $\mathcal{C}$ into composition of transformations of functors. In other words we get a functor

$$h : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{opp}, \mathit{Sets}) = PSh(\mathcal{C})$$

Note that the target is a "big" category, see Remark 2.2. On the other hand, h is an actual mathematical object (i.e. a set), compare Remark 2.11.

Lemma 3.5 (Yoneda lemma)**.** *Let $U, V \in \mathrm{Ob}(\mathcal{C})$. Given any morphism of functors $s : h_U \to h_V$ there is a unique morphism $\phi : U \to V$ such that $h(\phi) = s$. In other words the functor h is fully faithful. More generally, given any contravariant functor F and any object U of $\mathcal{C}$ we have a natural bijection*

$$\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, F) \longrightarrow F(U), \quad s \longmapsto s_U(id_U).$$

Proof. Just take $\phi = s_U(\mathrm{id}_U) \in \mathrm{Mor}_{\mathcal{C}}(U, V)$. $\square$

Definition 3.6. A contravariant functor $F : \mathcal{C} \to \mathit{Sets}$ is said to be *representable* if it is isomorphic to the functor of points h_U for some object U of $\mathcal{C}$.

Choose an object U of $\mathcal{C}$ and an isomorphism $s : h_U \to F$. The Yoneda lemma guarantees that the pair (U, s) is unique up to unique isomorphism. The object U is called an object *representing* F.

4. Products of pairs

Definition 4.1. Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A *product* of x and y is an object $x \times y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x \times y, x)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(x \times y, y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(w, x)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(w, y)$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(w, x \times y)$ making the diagram

$$\begin{tikzcd}
w \arrow[drr, dashed, "\gamma"] \arrow[drrr, "\beta"] \arrow[ddrr, "\alpha"'] & & & \\
& & x \times y \arrow[r, "q"'] \arrow[d, "p"'] & y \\
& & x &
\end{tikzcd}$$

commute.

If a product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times y$ to be an object of $\mathcal{C}$ such that

$$h_{x \times y}(w) = h_x(w) \times h_y(w)$$

functorially in w. In other words the product $x \times y$ is an object representing the functor $w \mapsto h_x(w) \times h_y(w)$.

Definition 4.2. We say the category $\mathcal{C}$ *has products of pairs of objects* if a product $x \times y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

We use this terminology to distinguish this notion from the notion of "having products" or "having finite products" which usually means something else (in particular it always implies there exists a final object).

5. Coproducts of pairs

Definition 5.1. Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A *coproduct*, or *amalgamated sum* of x and y is an object $x \amalg y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $i \in \mathrm{Mor}_{\mathcal{C}}(x, x \amalg y)$ and $j \in \mathrm{Mor}_{\mathcal{C}}(y, x \amalg y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(x, w)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(y, w)$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(x \amalg y, w)$ making the diagram

$$\begin{tikzcd}
& y \arrow[d, "j"] \arrow[ddr, "\beta"] & \\
x \arrow[r, "i"] \arrow[drr, "\alpha"'] & x \amalg y \arrow[dr, dashed, "\gamma"] & \\
& & w
\end{tikzcd}$$

commute.

If a coproduct exists it is unique up to unique isomorphism. This follows from the Yoneda lemma (applied to the opposite category) as the definition requires $x \amalg y$ to be an object of $\mathcal{C}$ such that

$$\mathrm{Mor}_\mathcal{C}(x \amalg y, w) = \mathrm{Mor}_\mathcal{C}(x, w) \times \mathrm{Mor}_\mathcal{C}(y, w)$$

functorially in w.

Definition 5.2. We say the category $\mathcal{C}$ *has coproducts of pairs of objects* if a coproduct $x \amalg y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

We use this terminology to distinguish this notion from the notion of "having coproducts" or "having finite coproducts" which usually means something else (in particular it always implies there exists an initial object in $\mathcal{C}$).

6. Fibre products

Definition 6.1. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_\mathcal{C}(x, y)$ and $g \in \mathrm{Mor}_\mathcal{C}(z, y)$. A *fibre product* of f and g is an object $x \times_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_\mathcal{C}(x \times_y z, x)$ and $q \in \mathrm{Mor}_\mathcal{C}(x \times_y z, z)$ making the diagram

$$\begin{tikzcd}
x \times_y z \arrow[r, "q"'] \arrow[d, "p"'] & z \arrow[d, "g"] \\
x \arrow[r, "f"] & y
\end{tikzcd}$$

commute, and such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_\mathcal{C}(w, x)$ and $\beta \in \mathrm{Mor}_\mathcal{C}(w, z)$ with $f \circ \alpha = g \circ \beta$ there is a unique $\gamma \in \mathrm{Mor}_\mathcal{C}(w, x \times_y z)$ making the diagram

$$\begin{tikzcd}
w \arrow[dr, dashed, "\gamma"] \arrow[drr, "\beta"] \arrow[ddr, "\alpha"'] & & \\
& x \times_y z \arrow[r, "q"'] \arrow[d, "p"] & z \arrow[d, "g"] \\
& x \arrow[r, "f"] & y
\end{tikzcd}$$

commute.

If a fibre product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times_y z$ to be an object of $\mathcal{C}$ such that

$$h_{x \times_y z}(w) = h_x(w) \times_{h_y(w)} h_z(w)$$

functorially in w. In other words the fibre product $x \times_y z$ is an object representing the functor $w \mapsto h_x(w) \times_{h_y(w)} h_z(w)$.

Definition 6.2. We say a commutative diagram

$$\begin{tikzcd}
w \arrow[r] \arrow[d] & z \arrow[d] \\
x \arrow[r] & y
\end{tikzcd}$$

in a category is *cartesian* if w and the morphisms $w \to x$ and $w \to z$ form a fibre product of the morphisms $x \to y$ and $z \to y$.

Definition 6.3. We say the category $\mathcal{C}$ *has fibre products* if the fibre product exists for any $f \in \mathrm{Mor}_{\mathcal{C}}(x, y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(z, y)$.

Definition 6.4. A morphism $f : x \to y$ of a category $\mathcal{C}$ is said to be *representable*, if and only if for every morphism $z \to y$ in $\mathcal{C}$ the fibre product $x \times_y z$ exists.

Lemma 6.5. *Let $\mathcal{C}$ be a category. Let $f : x \to y$, and $g : y \to z$ be representable. Then $g \circ f : x \to z$ is representable.*

Proof. Omitted. □

Lemma 6.6. *Let $\mathcal{C}$ be a category. Let $f : x \to y$ be representable. Let $y' \to y$ be a morphism of $\mathcal{C}$. Then the morphism $x' := x \times_y y' \to y'$ is representable also.*

Proof. Let $z \to y'$ be a morphism. The fibre product $x' \times_{y'} z$ is supposed to represent the functor

$$\begin{aligned} w & \mapsto h_{x'}(w) \times_{h_{y'}(w)} h_z(w) \\ & = (h_x(w) \times_{h_y(w)} h_{y'}(w)) \times_{h_{y'}(w)} h_z(w) \\ & = h_x(w) \times_{h_y(w)} h_z(w) \end{aligned}$$

which is representable by assumption. □

7. Examples of fibre products

In this section we list examples of fibre products and we describe them.

As a really trivial first example we observe that the category of sets has fibred products and hence every morphism is representable. Namely, if $f : X \to Y$ and $g : Z \to Y$ are maps of sets then we define $X \times_Y Z$ as the subset of $X \times Z$ consisting of pairs (x, z) such that $f(x) = g(z)$. The morphisms $p : X \times_Y Z \to X$ and $q : X \times_U Z \to Z$ are the projection maps $(x, z) \mapsto x$, and $(x, z) \mapsto z$. Finally, if $\alpha : W \to X$ and $\beta : W \to Z$ are morphisms such that $f \circ \alpha = g \circ \beta$ then the map $W \to X \times Y$, $w \mapsto (\alpha(w), \beta(w))$ obviously ends up in $X \times_Y Z$ as desired.

In many categories whose objects are sets endowed with certain types of algebraic structures the fibre product of the underlying sets also provides the fibre product in the category. For example, suppose that X, Y and Z above are groups and that f, g are homomorphisms of groups. Then the set-theoretic fibre product $X \times_Y Z$ inherits the structure of a group, simply by defining the product of two pairs by the formula $(x, z) \cdot (x', z') = (xx', zz')$. Here we list those categories for which a similar reasoning works.

(1) The category *Groups* of groups.
(2) The category *G-Sets* of sets endowed with a left G-action for some fixed group G.
(3) The category of rings.
(4) The category of R-modules given a ring R.

8. Fibre products and representability

In this section we work out fibre products in the category of contravariant functors from a category to the category of sets. This will later be superseded during the discussion of sites, presheaves, sheaves. Of some interest is the notion of a "representable morphism" between such functors.

Lemma 8.1. *Let $\mathcal{C}$ be a category. Let $F, G, H : \mathcal{C}^{opp} \to Sets$ be functors. Let $a : F \to G$ and $b : H \to G$ be transformations of functors. Then the fibre product $F \times_{a,G,b} H$ in the category $Fun(\mathcal{C}^{opp}, Sets)$ exists and is given by the formula*

$$(F \times_{a,G,b} H)(X) = F(X) \times_{a_X, G(X), b_X} H(X)$$

for any object X of $\mathcal{C}$.

Proof. Omitted. □

As a special case suppose we have a morphism $a : F \to G$, an object $U \in \mathrm{Ob}(\mathcal{C})$ and an element $\xi \in G(U)$. According to the Yoneda Lemma 3.5 this gives a transformation $\xi : h_U \to G$. The fibre product in this case is described by the rule

$$(h_U \times_{\xi,G,a} F)(X) = \{(f, \xi') \mid f : X \to U,\ \xi' \in F(X),\ G(f)(\xi) = a_X(\xi')\}$$

If F, G are also representable, then this is the functor representing the fibre product, if it exists, see Section 6. The analogy with Definition 6.4 prompts us to define a notion of representable transformations.

Definition 8.2. Let $\mathcal{C}$ be a category. Let $F, G : \mathcal{C}^{opp} \to Sets$ be functors. We say a morphism $a : F \to G$ is *representable*, or that *F is relatively representable over G*, if for every $U \in \mathrm{Ob}(\mathcal{C})$ and any $\xi \in G(U)$ the functor $h_U \times_G F$ is representable.

Lemma 8.3. *Let $\mathcal{C}$ be a category. Let $a : F \to G$ be a morphism of contravariant functors from $\mathcal{C}$ to Sets. If a is representable, and G is a representable functor, then F is representable.*

Proof. Omitted. □

Lemma 8.4. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{C}^{opp} \to Sets$ be a functor. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:*

1. *The diagonal $F \to F \times F$ is representable.*
2. *For every U in $\mathcal{C}$, and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable.*

Proof. Suppose the diagonal is representable, and let U, ξ be given. Consider any $V \in \mathrm{Ob}(\mathcal{C})$ and any $\xi' \in F(V)$. Note that $h_U \times h_V = h_{U \times V}$ is representable. Hence the fibre product of the maps $(\xi, \xi') : h_U \times h_V \to F \times F$ and $F \to F \times F$ is representable by assumption. This means there exists $W \in \mathrm{Ob}(\mathcal{C})$, morphisms $W \to U$, $W \to V$ and $h_W \to F$ such that

$$\begin{tikzcd} h_W \arrow[r] \arrow[d] & F \arrow[d] \\ h_U \times h_V \arrow[r] & F \times F \end{tikzcd}$$

is cartesian. We leave it to the reader to see that this implies that $h_W = h_U \times_F h_V$ as desired.

Assume (2) holds. Consider any $V \in \mathrm{Ob}(\mathcal{C})$ and any $(\xi, \xi') \in (F \times F)(V)$. We have to show that $h_V \times_{F \times F} F$ is representable. What we know is that $h_V \times_{\xi, F, \xi'} h_V$ is representable, say by W in $\mathcal{C}$ with corresponding morphisms $a, a' : W \to V$ (such that $\xi \circ a = \xi' \circ a'$). Consider $W' = W \times_{(a,a'), V \times V} V$. It is formal to show that W' represents $h_V \times_{F \times F} F$ because

$$h_{W'} = h_W \times_{h_V \times h_V} h_V = (h_V \times_{\xi, F, \xi'} h_V) \times_{h_V \times h_V} h_V = F \times_{F \times F} h_V.$$

□

9. Pushouts

The dual notion to fibre products is that of pushouts.

Definition 9.1. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_\mathcal{C}(y, x)$ and $g \in \mathrm{Mor}_\mathcal{C}(y, z)$. A *pushout* of f and g is an object $x \amalg_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_\mathcal{C}(x, x \amalg_y z)$ and $q \in \mathrm{Mor}_\mathcal{C}(z, x \amalg_y z)$ making the diagram

$$\begin{tikzcd} y \arrow[r, "g"'] \arrow[d, "f"'] & z \arrow[d, "q"] \\ x \arrow[r, "p"] & x \amalg_y z \end{tikzcd}$$

commute, and such that the following universal property holds: For any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_\mathcal{C}(x, w)$ and $\beta \in \mathrm{Mor}_\mathcal{C}(z, w)$ with $\alpha \circ f = \beta \circ g$ there is a unique $\gamma \in \mathrm{Mor}_\mathcal{C}(x \amalg_y z, w)$ making the diagram

$$\begin{tikzcd} y \arrow[r, "g"'] \arrow[d, "f"'] & z \arrow[d, "q"] \arrow[rdd, "\beta"] & \\ x \arrow[r, "p"] \arrow[rrd, "\alpha"'] & x \amalg_y z \arrow[rd, dashed, "\gamma"] & \\ & & w \end{tikzcd}$$

commute.

It is possible and straightforward to prove the uniqueness of the triple $(x \amalg_y z, p, q)$ up to unique isomorphism (if it exists) by direct arguments. Another possibility is to think of the coproduct as the product in the opposite category, thereby getting this uniqueness for free from the discussion in Section 6.

Definition 9.2. We say a commutative diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & z \arrow[d] \\ x \arrow[r] & w \end{tikzcd}$$

in a category is *cocartesian* if w and the morphisms $x \to w$ and $z \to w$ form a pushout of the morphisms $y \to x$ and $y \to z$.

10. Equalizers

Definition 10.1. Suppose that X, Y are objects of a category $\mathcal{C}$ and that $a, b : X \to Y$ are morphisms. We say a morphism $e : Z \to X$ is an *equalizer* for the pair (a, b) if $a \circ e = b \circ e$ and if (Z, e) satisfies the following universal property: For every morphism $t : W \to X$ in $\mathcal{C}$ such that $a \circ t = b \circ t$ there exists a unique morphism $s : W \to Z$ such that $t = e \circ s$.

As in the case of the fibre product above, equalizers when they exist are unique up to unique isomorphism. There is a straightforward generalization of this definition to the case where we have more than 2 morphisms.

11. Coequalizers

Definition 11.1. Suppose that X, Y are objects of a category $\mathcal{C}$ and that $a, b : X \to Y$ are morphisms. We say a morphism $c : Y \to Z$ is a *coequalizer* for the pair (a, b) if $c \circ a = c \circ b$ and if (Z, c) satisfies the following universal property: For every morphism $t : Y \to W$ in $\mathcal{C}$ such that $t \circ a = t \circ b$ there exists a unique morphism $s : Z \to W$ such that $t = s \circ c$.

As in the case of the pushouts above, coequalizers when they exist are unique up to unique isomorphism, and this follows from the uniqueness of equalizers upon considering the opposite category. There is a straightforward generalization of this definition to the case where we have more than 2 morphisms.

12. Initial and final objects

Definition 12.1. Let $\mathcal{C}$ be a category.

(1) An object x of the category $\mathcal{C}$ is called an *initial* object if for every object y of $\mathcal{C}$ there is exactly one morphism $x \to y$.
(2) An object x of the category $\mathcal{C}$ is called a *final* object if for every object y of $\mathcal{C}$ there is exactly one morphism $y \to x$.

In the category of sets the empty set $\emptyset$ is an initial object, and in fact the only initial object. Also, any *singleton*, i.e., a set with one element, is a final object (so it is not unique).

13. Monomorphisms and Epimorphisms

Definition 13.1. Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a morphism of $\mathcal{C}$.

(1) We say that f is a *monomorphism* if for every object W and every pair of morphisms $a, b : W \to X$ such that $f \circ a = f \circ b$ we have $a = b$.
(2) We say that f is an *epimorphism* if for every object W and every pair of morphisms $a, b : Y \to W$ such that $a \circ f = b \circ f$ we have $a = b$.

Example 13.2. In the category of sets the monomorphisms correspond to injective maps and the epimorphisms correspond to surjective maps.

Lemma 13.3. *Let $\mathcal{C}$ be a category, and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then*

(1) *f is a monomorphism if and only if X is the fibre product $X \times_Y X$, and*
(2) *f is an epimorphism if and only if Y is the pushout $Y \amalg_X Y$.*

Proof. Omitted. □

14. Limits and colimits

Let $\mathcal{C}$ be a category. A *diagram* in $\mathcal{C}$ is simply a functor $M : \mathcal{I} \to \mathcal{C}$. We say that $\mathcal{I}$ is the *index category* or that M is an $\mathcal{I}$-diagram. We will use the notation M_i to denote the image of the object i of $\mathcal{I}$. Hence for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $M(\phi) : M_i \to M_{i'}$.

Definition 14.1. A *limit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\lim_I M$ in $\mathcal{C}$ together with morphisms $p_i : \lim_I M \to M_i$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $p_{i'} = M(\phi) \circ p_i$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $q_i : W \to M_i$ such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $q_{i'} = M(\phi) \circ q_i$ there exists a unique morphism $q : W \to \lim_I M$ such that $q_i = p_i \circ q$ for every object i of $\mathcal{I}$.

Limits $(\lim_I M, (p_i)_{i \in \mathrm{Ob}(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Products of pairs, fibred products, and equalizers are examples of limits. The limit over the empty diagram is a final object of $\mathcal{C}$. In the category of sets all limits exist. The dual notion is that of colimits.

Definition 14.2. A *colimit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\mathrm{colim}_I M$ in $\mathcal{C}$ together with morphisms $s_i : M_i \to \mathrm{colim}_I M$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $s_i = s_{i'} \circ M(\phi)$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $t_i : M_i \to W$ such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $t_i = t_{i'} \circ M(\phi)$ there exists a unique morphism $t : \mathrm{colim}_I M \to W$ such that $t_i = t \circ s_i$ for every object i of $\mathcal{I}$.

Colimits $(\mathrm{colim}_I M, (s_i)_{i \in \mathrm{Ob}(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Coproducts of pairs, pushouts, and coequalizers are examples of colimits. The colimit over an empty diagram is an initial object of $\mathcal{C}$. In the category of sets all colimits exist.

Remark 14.3. The index category of a (co)limit will never be allowed to have a proper class of objects. In this project it means that it cannot be one of the categories listed in Remark 2.2

Remark 14.4. We often write $\lim_i M_i$, $\mathrm{colim}_i M_i$, $\lim_{i \in \mathcal{I}} M_i$, or $\mathrm{colim}_{i \in \mathcal{I}} M_i$ instead of the versions indexed by $\mathcal{I}$. Using this notation, and using the description of limits and colimits of sets in Section 15 below, we can say the following. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram.

(1) The object $\lim_i M_i$ if it exists satisfies the following property
$$\mathrm{Mor}_\mathcal{C}(W, \lim_i M_i) = \lim_i \mathrm{Mor}_\mathcal{C}(W, M_i)$$
where the limit on the right takes place in the category of sets.
(2) The object $\mathrm{colim}_i M_i$ if it exists satisfies the following property
$$\mathrm{Mor}_\mathcal{C}(\mathrm{colim}_i M_i, W) = \lim_{i \in \mathcal{I}^{opp}} \mathrm{Mor}_\mathcal{C}(M_i, W)$$
where on the right we have the limit over the opposite category with value in the category of sets.

By the Yoneda lemma (and its dual) this formula completely determines the limit, respectively the colimit.

As an application of the notions of limits and colimits we define products and coproducts.

Definition 14.5. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *product* $\prod_{i \in I} M_i$ is by definition $\lim_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the product is a final object of the category. The morphisms $p_i : \prod M_i \to M_i$ are called the *projection morphisms*.

Definition 14.6. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *coproduct* $\coprod_{i \in I} M_i$ is by definition $\operatorname{colim}_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the product is an initial object of the category. Note that the coproduct comes equipped with morphisms $M_i \to \coprod M_i$. These are sometimes called the *coprojections*.

Lemma 14.7. *Suppose that $M : \mathcal{I} \to \mathcal{C}$, and $N : \mathcal{J} \to \mathcal{C}$ are diagrams whose colimits exist. Suppose $H : \mathcal{I} \to \mathcal{J}$ is a functor, and suppose $t : M \to N \circ H$ is a transformation of functors. Then there is a unique morphism*

$$\theta : \operatorname{colim}_{\mathcal{I}} M \longrightarrow \operatorname{colim}_{\mathcal{J}} N$$

such that all the diagrams

$$\begin{tikzcd} M_i \arrow[r] \arrow[d, "t_i"'] & \operatorname{colim}_{\mathcal{I}} M \arrow[d, "\theta"] \\ N_{H(i)} \arrow[r] & \operatorname{colim}_{\mathcal{J}} N \end{tikzcd}$$

commute.

Proof. Omitted. □

Lemma 14.8. *Suppose that $M : \mathcal{I} \to \mathcal{C}$, and $N : \mathcal{J} \to \mathcal{C}$ are diagrams whose limits exist. Suppose $H : \mathcal{I} \to \mathcal{J}$ is a functor, and suppose $t : N \circ H \to M$ is a transformation of functors. Then there is a unique morphism*

$$\theta : \lim_{\mathcal{J}} N \longrightarrow \lim_{\mathcal{I}} M$$

such that all the diagrams

$$\begin{tikzcd} \lim_{\mathcal{J}} N \arrow[r] \arrow[d, "\theta"] & N_{H(i)} \arrow[d, "t_i"'] \\ \lim_{\mathcal{I}} M \arrow[r] & M_i \end{tikzcd}$$

commute.

Proof. Omitted. □

Lemma 14.9. *Let $\mathcal{I}$, $\mathcal{J}$ be index categories. Let $M : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ be a functor. We have*

$$\operatorname{colim}_i \operatorname{colim}_j M_{i,j} = \operatorname{colim}_{i,j} M_{i,j} = \operatorname{colim}_j \operatorname{colim}_i M_{i,j}$$

provided all the indicated colimits exist. Similar for limits.

Proof. Omitted. ☐

Lemma 14.10. *Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Write $I = \mathrm{Ob}(\mathcal{I})$ and $A = Arrow(\mathcal{I})$. Denote $s, t : A \to I$ the source and target maps. Suppose that $\prod_{i \in I} M_i$ and $\prod_{a \in A} M_{t(a)}$ exist. Suppose that the equalizer of*

$$\begin{tikzcd} \prod_{i \in I} M_i \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \prod_{a \in A} M_{t(a)} \end{tikzcd}$$

exists, where the morphisms are determined by their components as follows: $p_a \circ \psi = M(a) \circ p_{s(a)}$ and $p_a \circ \phi = p_{t(a)}$. Then this equalizer is the limit of the diagram.

Proof. Omitted. ☐

Lemma 14.11. *Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Write $I = \mathrm{Ob}(\mathcal{I})$ and $A = Arrow(\mathcal{I})$. Denote $s, t : A \to I$ the source and target maps. Suppose that $\coprod_{i \in I} M_i$ and $\coprod_{a \in A} M_{s(a)}$ exist. Suppose that the coequalizer of*

$$\begin{tikzcd} \coprod_{a \in A} M_{s(a)} \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \coprod_{i \in I} M_i \end{tikzcd}$$

exists, where the morphisms are determined by their components as follows: The component $M_{s(a)}$ maps via ψ to the component $M_{t(a)}$ via the morphism a. The component $M_{s(a)}$ maps via ϕ to the component $M_{s(a)}$ by the identity morphism. Then this coequalizer is the colimit of the diagram.

Proof. Omitted. ☐

15. Limits and colimits in the category of sets

Not only do limits and colimits exist in *Sets* but they are also easy to describe. Namely, let $M : \mathcal{I} \to Sets$, $i \mapsto M_i$ be a diagram of sets. Denote $I = \mathrm{Ob}(\mathcal{I})$. The limit is described as

$$\lim_{\mathcal{I}} M = \{(m_i)_{i \in I} \in \prod\nolimits_{i \in I} M_i \mid \forall \phi : i \to i' \text{ in } \mathcal{I}, M(\phi)(m_i) = m_{i'}\}.$$

So we think of an element of the limit as a compatible system of elements of all the sets M_i.

On the other hand, the colimit is

$$\mathrm{colim}_{\mathcal{I}} M = (\coprod\nolimits_{i \in I} M_i)/\sim$$

where the equivalence relation $\sim$ is the equivalence relation generated by setting $m_i \sim m_{i'}$ if $m_i \in M_i$, $m_{i'} \in M_{i'}$ and $M(\phi)(m_i) = m_{i'}$ for some $\phi : i \to i'$. In other words, $m_i \in M_i$ and $m_{i'} \in M_{i'}$ are equivalent if there is a chain of morphisms in $\mathcal{I}$

$$\begin{tikzcd} & i_1 \arrow[dl] \arrow[dr] & & i_3 \arrow[dl] & & i_{2n-1} \arrow[dr] & \\ i = i_0 & & i_2 & & \ldots & & i_{2n} = i' \end{tikzcd}$$

and elements $m_{i_j} \in M_{i_j}$ mapping to each other under the maps $M_{i_{2k-1}} \to M_{i_{2k-2}}$ and $M_{i_{2k-1}} \to M_{i_{2k}}$ induced from the maps in $\mathcal{I}$ above.

This is not a very pleasant type of object to work with. But if the diagram is filtered then it is much easier to describe. We will explain this in Section 19.

16. Connected limits

A (co)limit is called connected if its index category is connected.

Definition 16.1. We say that a category $\mathcal{I}$ is *connected* if the equivalence relation generated by $x \sim y \Leftrightarrow \mathrm{Mor}_{\mathcal{I}}(x, y) \neq \emptyset$ has exactly one equivalence class.

Here we follow the convention of Topology, Definition 6.1 that connected spaces are nonempty. The following in some vague sense characterizes connected limits.

Lemma 16.2. *Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $M : \mathcal{I} \to \mathcal{C}/X$ be a diagram in the category of objects over X. If the index category $\mathcal{I}$ is connected and the limit of M exists in $\mathcal{C}/X$, then the limit of the composition $\mathcal{I} \to \mathcal{C}/X \to \mathcal{C}$ exists and is the same.*

Proof. Let $M \to X$ be an object representing the limit in $\mathcal{C}/X$. Consider the functor

$$W \longmapsto \lim_i \mathrm{Mor}_{\mathcal{C}}(W, M_i).$$

Let (φ_i) be an element of the set on the right. Since each M_i comes equipped with a morphism $s_i : M_i \to X$ we get morphisms $f_i = s_i \circ \varphi_i : W \to X$. But as $\mathcal{I}$ is connected we see that all f_i are equal. Since $\mathcal{I}$ is nonempty there is at least one f_i. Hence this common value $W \to X$ defines the structure of an object of W in $\mathcal{C}/X$ and (φ_i) defines is an element of $\lim_i \mathrm{Mor}_{\mathcal{C}/X}(W, M_i)$. Thus we obtain a unique morphism $\phi : W \to M$ such that φ_i is the composition of ϕ with $M \to M_i$ as desired. $\square$

Lemma 16.3. *Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $M : \mathcal{I} \to X/\mathcal{C}$ be a diagram in the category of objects under X. If the index category $\mathcal{I}$ is connected and the colimit of M exists in $X/\mathcal{C}$, then the colimit of the composition $\mathcal{I} \to X/\mathcal{C} \to \mathcal{C}$ exists and is the same.*

Proof. Omitted. Hint: This lemma is dual to Lemma 16.2. $\square$

17. Cofinal and initial categories

In the literature sometimes the word "final" is used instead of cofinal in the following definition.

Definition 17.1. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor between categories. We say $\mathcal{I}$ *is cofinal in* $\mathcal{J}$ or that H is *cofinal* if

(1) for all $y \in \mathrm{Ob}(\mathcal{J})$ there exists a $x \in \mathrm{Ob}(\mathcal{I})$ and a morphism $y \to H(x)$, and
(2) given $y \in \mathrm{Ob}(\mathcal{J})$, $x, x' \in \mathrm{Ob}(\mathcal{I})$ and morphisms $y \to H(x)$ and $y \to H(x')$ there exists a sequence of morphisms

$$x = x_0 \leftarrow x_1 \to x_2 \leftarrow x_3 \to \ldots \to x_{2n} = x'$$

in $\mathcal{I}$ and morphisms $y \to H(x_i)$ in $\mathcal{J}$ such that the diagrams

$$\begin{tikzcd} & y \arrow[ld] \arrow[d] \arrow[rd] & \\ H(x_{2k}) & H(x_{2k+1}) \arrow[l] \arrow[r] & H(x_{2k+2}) \end{tikzcd}$$

commute for $k = 0, \ldots, n-1$.

Lemma 17.2. *Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of categories. Assume $\mathcal{I}$ is cofinal in $\mathcal{J}$. Then for every diagram $M : \mathcal{J} \to \mathcal{C}$ we have a canonical isomorphism*

$$\mathrm{colim}_\mathcal{I} M \circ H = \mathrm{colim}_\mathcal{J} M$$

if either side exists.

Proof. Omitted. □

Definition 17.3. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor between categories. We say $\mathcal{I}$ *is initial in* $\mathcal{J}$ or that H is *initial* if

(1) for all $y \in \mathrm{Ob}(\mathcal{J})$ there exists a $x \in \mathrm{Ob}(\mathcal{I})$ and a morphism $H(x) \to y$,
(2) for any $y \in \mathrm{Ob}(\mathcal{J})$, $x, x' \in \mathrm{Ob}(\mathcal{I})$ and morphisms $H(x) \to y$, $H(x') \to y$ in $\mathcal{J}$ there exists $n \geq 0$ and a commutative diagram

$$\begin{tikzcd} H(x) \arrow[drrr] & H(x_1) \arrow[l] \arrow[r] \arrow[drr] & H(x_2) \arrow[dr] & \ldots \arrow[l] \arrow[r] & H(x_{2n-2}) \arrow[dl] & H(x_{2n-1}) \arrow[l] \arrow[r] \arrow[dll] & H(x') \arrow[dlll] \\ & & & y & & & \end{tikzcd}$$

where all the horizontal morphisms come from morphisms in $\mathcal{I}$, and the vertical arrows in $\mathcal{J}$.

This is just the dual notion to "cofinal" functors.

Lemma 17.4. *Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of categories. Assume $\mathcal{I}$ is initial in $\mathcal{J}$. Then for every diagram $M : \mathcal{J} \to \mathcal{C}$ the limit $\lim_\mathcal{J} M$ exists if and only if $\lim_\mathcal{I} M$ exists and if so these limits agree.*

Proof. Omitted. □

Lemma 17.5. *Let $F : \mathcal{I} \to \mathcal{I}'$ be a functor. Assume*

(1) *the fibre categories (see Definition 30.2) of $\mathcal{I}$ over $\mathcal{I}'$ are all connected, and*
(2) *for every morphism $\alpha' : x' \to y'$ in $\mathcal{I}'$ there exist a morphism $\alpha : x \to y$ in $\mathcal{I}$ such that $F(\alpha) = \alpha'$.*

Then for every diagram $M : \mathcal{I}' \to \mathcal{C}$ the colimit $\mathrm{colim}_\mathcal{I} M \circ F$ exists if and only if $\mathrm{colim}_{\mathcal{I}'} M$ exists and if so these colimits agree.

Proof. One can prove this by showing that $\mathcal{I}$ is cofinal in $\mathcal{I}'$ and applying Lemma 17.2. But we can also prove it directly as follows. It suffices to show that for any object T of $\mathcal{C}$ we have

$$\lim_\mathcal{I} \mathrm{Mor}_\mathcal{C}(M_{F(i)}, T) = \lim_{\mathcal{I}'} \mathrm{Mor}_\mathcal{C}(M_{i'}, T)$$

If $(g_{i'})_{i' \in \mathrm{Ob}(\mathcal{I}')}$ is an element of the right hand side, then setting $f_i = g_{F(i)}$ we obtain an element $(f_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ of the left hand side. Conversely, let $(f_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ be an element of the left hand side. Note that on each (connected) fibre category $\mathcal{I}_{i'}$ the functor $M \circ F$ is constant with value $M_{i'}$. Hence the morphisms f_i for $i \in \mathrm{Ob}(\mathcal{I})$ with $F(i) = i'$ are all the same and determine a well defined morphism $g_{i'} : M_{i'} \to T$. By assumption (2) the collection $(g_{i'})_{i' \in \mathrm{Ob}(\mathcal{I}')}$ defines an element of the right hand side. □

Lemma 17.6. *Let $\mathcal{I}$ and $\mathcal{J}$ be a categories and denote $p : \mathcal{I} \times \mathcal{J} \to \mathcal{J}$ the projection. If $\mathcal{I}$ is connected, then for a diagram $M : \mathcal{J} \to \mathcal{C}$ the colimit $\mathrm{colim}_\mathcal{J} M$ exists if and only if $\mathrm{colim}_{\mathcal{I} \times \mathcal{J}} M \circ p$ exists and if so these colimits are equal.*

Proof. This is a special case of Lemma 17.5. □

18. Finite limits and colimits

A *finite* (co)limit is a (co)limit whose diagram category is finite, i.e., the diagram category has finitely many objects and finitely many morphisms. A (co)limit is called *nonempty* if the index category is nonempty. A (co)limit is called *connected* if the index category is connected, see Definition 16.1. It turns out that there are "enough" finite diagram categories.

Lemma 18.1. *Let $\mathcal{I}$ be a category with*

1. $\mathrm{Ob}(\mathcal{I})$ *is finite, and*
2. *there exist finitely many morphisms $f_1, \ldots, f_m \in Arrows(\mathcal{I})$ such that every morphism of $\mathcal{I}$ is a composition $f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_k}$.*

Then there exists a functor $F : \mathcal{J} \to \mathcal{I}$ such that

(a) *$\mathcal{J}$ is a finite category, and*
(b) *for any diagram $M : \mathcal{I} \to \mathcal{C}$ the (co)limit of M over $\mathcal{I}$ exists if and only if the (co)limit of $M \circ F$ over $\mathcal{J}$ exists and in this case the (co)limits are canonically isomorphic.*

Moreover, $\mathcal{J}$ is connected (resp. nonempty) if and only if $\mathcal{I}$ is so.

Proof. Say $\mathrm{Ob}(\mathcal{I}) = \{x_1, \ldots, x_n\}$. Denote $s, t : \{1, \ldots, m\} \to \{1, \ldots, n\}$ the functions such that $f_j : x_{s(j)} \to x_{t(j)}$. We set $\mathrm{Ob}(\mathcal{J}) = \{y_1, \ldots, y_n, z_1, \ldots, z_n\}$ Besides the identity morphisms we introduce morphisms $g_j : y_{s(j)} \to z_{t(j)}$, $j = 1, \ldots, m$ and morphisms $h_i : y_i \to z_i$, $i = 1, \ldots, n$. Since all of the nonidentity morphisms in $\mathcal{J}$ go from a y to a z there are no compositions to define and no associativity to check. Set $F(y_i) = F(z_i) = x_i$. Set $F(g_j) = f_j$ and $F(h_i) = \mathrm{id}_{x_i}$. It is clear that F is a functor. It is clear that $\mathcal{J}$ is finite. It is clear that $\mathcal{J}$ is connected, resp. nonempty if and only if $\mathcal{I}$ is so.

Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Consider an object W of $\mathcal{C}$ and morphisms $q_i : W \to M(x_i)$ as in Definition 14.1. Then by taking $q_i : W \to M(F(y_i)) = M(F(z_i)) = M(x_i)$ we obtain a family of maps as in Definition 14.1 for the diagram $M \circ F$. Conversely, suppose we are given maps $qy_i : W \to M(F(y_i))$ and $qz_i : W \to M(F(z_i))$ as in Definition 14.1 for the diagram $M \circ F$. Since

$$M(F(h_i)) = \mathrm{id} : M(F(y_i)) = M(x_i) \longrightarrow M(x_i) = M(F(z_i))$$

we conclude that $qy_i = qz_i$ for all i. Set q_i equal to this common value. The compatibility of $q_{s(j)} = qy_{s(j)}$ and $q_{t(j)} = qz_{t(j)}$ with the morphism $M(f_j)$ guarantees that the family q_i is compatible with all morphisms in $\mathcal{I}$ as by assumption every such morphism is a composition of the morphisms f_j. Thus we have found a canonical bijection

$$\lim_{B \in \mathrm{Ob}(\mathcal{J})} \mathrm{Mor}_\mathcal{C}(W, M(F(B))) = \lim_{A \in \mathrm{Ob}(\mathcal{I})} \mathrm{Mor}_\mathcal{C}(W, M(A))$$

which implies the statement on limits in the lemma. The statement on colimits is proved in the same way (proof omitted). □

Lemma 18.2. *Let $\mathcal{C}$ be a category. The following are equivalent:*

1. *Connected finite limits exist in $\mathcal{C}$.*
2. *Equalizers and fibre products exist in $\mathcal{C}$.*

Proof. Since equalizers and fibre products are finite connected limits we see that (1) implies (2). For the converse, let $\mathcal{I}$ be a finite connected diagram category. Let

$F : \mathcal{J} \to \mathcal{I}$ be the functor of diagram categories constructed in the proof of Lemma 18.1. Then we see that we may replace $\mathcal{I}$ by $\mathcal{J}$. The result is that we may assume that $\mathrm{Ob}(\mathcal{I}) = \{x_1, \ldots, x_n\} \amalg \{y_1, \ldots, y_m\}$ with $n, m \geq 1$ such that all nonidentity morphisms in $\mathcal{I}$ are morphisms $f : x_i \to y_j$ for some i and j.

Suppose that $n > 1$. Since $\mathcal{I}$ is connected there exist indices i_1, i_2 and j_0 and morphisms $a : x_{i_1} \to y_{j_0}$ and $b : x_{i_2} \to y_{j_0}$. Consider the category

$$\mathcal{I}' = \{x\} \amalg \{x_1, \ldots, \hat{x}_{i_1}, \ldots, \hat{x}_{i_2}, \ldots x_n\} \amalg \{y_1, \ldots, y_m\}$$

with

$$\mathrm{Mor}_{\mathcal{I}'}(x, y_j) = \mathrm{Mor}_{\mathcal{I}}(x_{i_1}, y_j) \amalg \mathrm{Mor}_{\mathcal{I}}(x_{i_2}, y_j)$$

and all other morphism sets the same as in $\mathcal{I}$. For any functor $M : \mathcal{I} \to \mathcal{C}$ we can construct a functor $M' : \mathcal{I}' \to \mathcal{C}$ by setting

$$M'(x) = M(x_{i_1}) \times_{M(a), M(y_j), M(b)} M(x_{i_2})$$

and for a morphism $f' : x \to y_j$ corresponding to, say, $f : x_{i_1} \to y_j$ we set $M'(f) = M(f) \circ \mathrm{pr}_1$. Then the functor M has a limit if and only if the functor M' has a limit (proof omitted). Hence by induction we reduce to the case $n = 1$.

If $n = 1$, then the limit of any $M : \mathcal{I} \to \mathcal{C}$ is the successive equalizer of pairs of maps $x_1 \to y_j$ hence exists by assumption. $\square$

Lemma 18.3. *Let $\mathcal{C}$ be a category. The following are equivalent:*

1. *Nonempty finite limits exist in $\mathcal{C}$.*
2. *Products of pairs and equalizers exist in $\mathcal{C}$.*
3. *Products of pairs and fibre products exist in $\mathcal{C}$.*

Proof. Since products of pairs, fibre products, and equalizers are limits with nonempty index categories we see that (1) implies both (2) and (3). Assume (2). Then finite nonempty products and equalizers exist. Hence by Lemma 14.10 we see that finite nonempty limits exist, i.e., (1) holds. Assume (3). If $a, b : A \to B$ are morphisms of $\mathcal{C}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(pr_1, pr_2), A \times A, \Delta} A.$$

Thus (3) implies (2), and the lemma is proved. $\square$

Lemma 18.4. *Let $\mathcal{C}$ be a category. The following are equivalent:*

1. *Finite limits exist in $\mathcal{C}$.*
2. *Finite products and equalizers exist.*
3. *The category has a final object and fibred products exist.*

Proof. Since products of pairs, fibre products, equalizers, and final objects are limits over finite index categories we see that (1) implies both (2) and (3). By Lemma 14.10 above we see that (2) implies (1). Assume (3). Note that the product $A \times A$ is the fibre product over the final object. If $a, b : A \to B$ are morphisms of $\mathcal{C}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(pr_1, pr_2), A \times A, \Delta} A.$$

Thus (3) implies (2) and the lemma is proved. $\square$

Lemma 18.5. *Let $\mathcal{C}$ be a category. The following are equivalent:*

1. *Connected finite colimits exist in $\mathcal{C}$.*
2. *Coequalizers and pushouts exist in $\mathcal{C}$.*

Proof. Omitted. Hint: This is dual to Lemma 18.2. □

Lemma 18.6. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Nonempty finite colimits exist in $\mathcal{C}$.*
(2) *Coproducts of pairs and coequalizers exist in $\mathcal{C}$.*
(3) *Coproducts of pairs and pushouts exist in $\mathcal{C}$.*

Proof. Omitted. Hint: This is the dual of Lemma 18.3. □

Lemma 18.7. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *finite colimits exist in $\mathcal{C}$,*
(2) *finite coproducts and coequalizers exist in $\mathcal{C}$, and*
(3) *$\mathcal{C}$ has an initial object and pushouts exist.*

Proof. Omitted. Hint: This is dual to Lemma 18.4. □

19. Filtered colimits

Colimits are easier to compute or describe when they are over a filtered diagram. Here is the definition.

Definition 19.1. We say that a diagram $M : \mathcal{I} \to \mathcal{C}$ is *directed*, or *filtered* if the following conditions hold:

(1) the category $\mathcal{I}$ has at least one object,
(2) for every pair of objects x, y of $\mathcal{I}$ there exists an object z and morphisms $x \to z$, $y \to z$, and
(3) for every pair of objects x, y of $\mathcal{I}$ and every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$ there exists a morphism $c : y \to z$ of $\mathcal{I}$ such that $M(c \circ a) = M(c \circ b)$ as morphisms in $\mathcal{C}$.

We say that an index category $\mathcal{I}$ is *directed*, or *filtered* if $\text{id} : \mathcal{I} \to \mathcal{I}$ is filtered (in other words you erase the M in part (3) above.)

We observe that any diagram with filtered index category is filtered, and this is how filtered colimits usually come about. In fact, if $M : \mathcal{I} \to \mathcal{C}$ is a filtered diagram, then we can factor M as $\mathcal{I} \to \mathcal{I}' \to \mathcal{C}$ where $\mathcal{I}'$ is a filtered index category[1] such that $\text{colim}_\mathcal{I} M$ exists if and only if $\text{colim}_{\mathcal{I}'} M'$ exists in which case the colimits are canonically isomorphic.

Suppose that $M : \mathcal{I} \to \textit{Sets}$ is a filtered diagram. In this case we may describe the equivalence relation in the formula

$$\text{colim}_\mathcal{I} M = (\textstyle\coprod_{i \in I} M_i)/\sim$$

simply as follows

$$m_i \sim m_{i'} \Leftrightarrow \exists i'', \phi : i \to i'', \phi' : i' \to i'', M(\phi)(m_i) = M(\phi')(m_{i'}).$$

In other words, two elements are equal in the colimit if and only if they "eventually become equal".

[1] Namely, let $\mathcal{I}'$ have the same objects as $\mathcal{I}$ but where $\text{Mor}_{\mathcal{I}'}(x, y)$ is the quotient of $\text{Mor}_\mathcal{I}(x, y)$ by the equivalence relation which identifies $a, b : x \to y$ if $M(a) = M(b)$.

Lemma 19.2. *Let $\mathcal{I}$ and $\mathcal{J}$ be index categories. Assume that $\mathcal{I}$ is filtered and $\mathcal{J}$ is finite. Let $M : \mathcal{I} \times \mathcal{J} \to Sets$, $(i,j) \mapsto M_{i,j}$ be a diagram of diagrams of sets. In this case*

$$\mathrm{colim}_i \lim_j M_{i,j} = \lim_j \mathrm{colim}_i M_{i,j}.$$

In particular, colimits over $\mathcal{I}$ commute with finite products, fibre products, and equalizers of sets.

Proof. Omitted. In fact, it is a fun exercise to prove that a category is filtered if and only if colimits over the category commute with finite limits (into the category of sets). □

We give a counter example to the lemma in the case where $\mathcal{J}$ is infinite. Namely, let $\mathcal{I}$ consist of $\mathbf{N} = \{1, 2, 3, \ldots\}$ with a unique morphism $i \to i'$ whenever $i \leq i'$. Let $\mathcal{J}$ consist of the discrete category $\mathbf{N} = \{1, 2, 3, \ldots\}$ (only morphisms are identities). Let $M_{i,j} = \{1, 2, \ldots, i\}$ with obvious inclusion maps $M_{i,j} \to M_{i',j}$ when $i \leq i'$. In this case $\mathrm{colim}_i M_{i,j} = \mathbf{N}$ and hence

$$\lim_j \mathrm{colim}_i M_{i,j} = \prod\nolimits_j \mathbf{N} = \mathbf{N}^{\mathbf{N}}$$

On the other hand $\lim_j M_{i,j} = \prod_j M_{i,j}$ and hence

$$\mathrm{colim}_i \lim_j M_{i,j} = \bigcup\nolimits_i \{1, 2, \ldots, i\}^{\mathbf{N}}$$

which is smaller than the other limit.

It turns out we sometimes need a more finegrained control over the possible conditions one can impose on index categories. Thus we add some lemmas on the possible things one can require.

Lemma 19.3. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every pair of objects x, y of $\mathcal{I}$ there exists an object z and morphisms $x \to z$ and $y \to z$. Then colimits of diagrams of sets over $\mathcal{I}$ commute with finite nonempty products.*

Proof. Let M and N be diagrams of sets over $\mathcal{I}$. To prove the lemma we have to show that the canonical map

$$\mathrm{colim}(M_i \times N_i) \longrightarrow \mathrm{colim}\, M_i \times \mathrm{colim}\, N_i$$

is an isomorphism. If $\mathcal{I}$ is empty, then this is true because the colimit of sets over the empty category is the empty set. If $\mathcal{I}$ is nonempty, then we construct a map $\mathrm{colim}\, M_i \times \mathrm{colim}\, N_i \to \mathrm{colim}(M_i \times N_i)$ as follows. Suppose that $m \in M_i$ and $n \in N_j$ give rise to elements s and t of the respective colimits. By assumption we can find $a : i \to k$ and $b : j \to k$ in $\mathcal{I}$. Then $(M(a)(m), N(b)(n))$ is an element of $M_k \times N_k$ and we map (s, t) to the corresponding element of $\mathrm{colim}\, M_i \times N_i$. We omit the verification that this map is well defined and that it is an inverse of the map displayed above. □

Lemma 19.4. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every pair of objects x, y of $\mathcal{I}$ there exists an object z and morphisms $x \to z$ and $y \to z$. Let $M : \mathcal{I} \to Ab$ be a diagram of abelian groups over $\mathcal{I}$. Then the set underlying $\mathrm{colim}_i M_i$ is the colimit of M viewed as a diagram of sets over $\mathcal{I}$.*

Proof. In this proof all colimits are taken in the category of sets. By Lemma 19.3 we have $\operatorname{colim} M_i \times \operatorname{colim} M_i = \operatorname{colim}(M_i \times M_i)$ hence we can use the maps $+ : M_i \times M_i \to M_i$ to define an addition map on $\operatorname{colim} M_i$. A straightforward argument, which we omit, shows that the set $\operatorname{colim} M_i$ with this addition is the colimit in the category of abelian groups. $\square$

Lemma 19.5. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every solid diagram*

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d, dotted] \\ z \arrow[r, dotted] & w \end{tikzcd}$$

in $\mathcal{I}$ there exists an object w and dotted arrows making the diagram commute. Then $\mathcal{I}$ is a (possibly empty) disjoint union of categories satisfying the condition above and the condition of Lemma 19.3.

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Otherwise, we define a relation on objects of $\mathcal{I}$ by saying that $x \sim y$ if there exists a z and morphisms $x \to z$ and $y \to z$. This is an equivalence relation by the assumption of the lemma. Hence $\mathrm{Ob}(\mathcal{I})$ is a disjoint union of equivalence classes. Let $\mathcal{I}_j$ be the full subcategories corresponding to these equivalence classes. Then $\mathcal{I} = \coprod \mathcal{I}_j$ as desired. $\square$

Lemma 19.6. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every solid diagram*

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d, dotted] \\ z \arrow[r, dotted] & w \end{tikzcd}$$

in $\mathcal{I}$ there exists an object w and dotted arrows making the diagram commute. Then an injective morphism $M \to N$ of diagrams of sets (resp. abelian groups) over $\mathcal{I}$ gives rise to an injective map $\operatorname{colim} M_i \to \operatorname{colim} N_i$ of sets (resp. abelian groups).

Proof. We first show that it suffices to prove the lemma for the case of a diagram of sets. Namely, by Lemma 19.5 we can write $\mathcal{I} = \coprod \mathcal{I}_j$ where each $\mathcal{I}_j$ satisfies the condition of the lemma as well as the condition of Lemma 19.3. Thus, if M is a diagram of abelian groups over $\mathcal{I}$, then

$$\operatorname{colim}_{\mathcal{I}} M = \bigoplus\nolimits_j \operatorname{colim}_{\mathcal{I}_j} M|_{\mathcal{I}_j}$$

It follows that it suffices to prove the result for the categories $\mathcal{I}_j$. However, colimits of abelian groups over these categories are computed by the colimits of the underlying sets (Lemma 19.4) hence we reduce to the case of an injective map of diagrams of sets.

Here we say that $M \to N$ is injective if all the maps $M_i \to N_i$ are injective. In fact, we will identify M_i with the image of $M_i \to N_i$, i.e., we will think of M_i as a subset of N_i. We will use the description of the colimits given in Section 15 without further mention. Let $s, s' \in \operatorname{colim} M_i$ map to the same element of $\operatorname{colim} N_i$. Say s comes from an element m of M_i and s' comes from an element m' of $M_{i'}$. Then we can find a sequence $i = i_0, i_1, \ldots, i_n = i'$ of objects of $\mathcal{I}$ and morphisms

$$\begin{tikzcd}
& i_1 \arrow[dl] \arrow[dr] & & i_3 \arrow[dl] & & i_{2n-1} \arrow[dr] & \\
i = i_0 & & i_2 & & \ldots & & i_{2n} = i'
\end{tikzcd}$$

and elements $n_{i_j} \in N_{i_j}$ mapping to each other under the maps $N_{i_{2k-1}} \to N_{i_{2k-2}}$ and $N_{i_{2k-1}} \to N_{i_{2k}}$ induced from the maps in $\mathcal{I}$ above with $n_{i_0} = m$ and $n_{i_{2n}} = m'$. We will prove by induction on n that this implies $s = s'$. The base case $n = 0$ is trivial. Assume $n \geq 1$. Using the assumption on $\mathcal{I}$ we find a commutative diagram

$$\begin{tikzcd}
& i_1 \arrow[dl] \arrow[dr] & \\
i_0 \arrow[dr] & & i_2 \arrow[dl] \\
& w &
\end{tikzcd}$$

We conclude that m and n_{i_2} map to the same element of N_w because both are the image of the element n_{i_1}. In particular, this element is an element $m'' \in M_w$ which gives rise to the same element as s in $\operatorname{colim} M_i$. Then we find the chain

$$\begin{tikzcd}
& i_3 \arrow[dl] \arrow[dr] & & i_5 \arrow[dl] & & i_{2n-1} \arrow[dr] & \\
w & & i_4 & & \ldots & & i_{2n} = i'
\end{tikzcd}$$

and the elements n_{i_j} for $j \geq 3$ which has a smaller length than the chain we started with. This proves the induction step and the proof of the lemma is complete. $\square$

Lemma 19.7. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume*

1. *for every pair of morphisms $a : w \to x$ and $b : w \to y$ in $\mathcal{I}$ there exists an object z and morphisms $c : x \to z$ and $d : y \to z$ such that $c \circ a = d \circ b$, and*
2. *for every pair of morphisms $a, b : x \to y$ there exists a morphism $c : y \to z$ such that $c \circ a = c \circ b$.*

Then $\mathcal{I}$ is a (possibly empty) union of disjoint filtered index categories $\mathcal{I}_j$.

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Otherwise, we define a relation on objects of $\mathcal{I}$ by saying that $x \sim y$ if there exists a z and morphisms $x \to z$ and $y \to z$. This is an equivalence relation by the first assumption of the lemma. Hence $\mathrm{Ob}(\mathcal{I})$ is a disjoint union of equivalence classes. Let $\mathcal{I}_j$ be the full subcategories corresponding to these equivalence classes. The rest is clear from the definitions. $\square$

Lemma 19.8. *Let $\mathcal{I}$ be an index category satisfying the hypotheses of Lemma 19.7 above. Then colimits over $\mathcal{I}$ commute with fibre products and equalizers in sets (and more generally with finite connected limits).*

Proof. By Lemma 19.7 we may write $\mathcal{I} = \coprod \mathcal{I}_j$ with each $\mathcal{I}_j$ filtered. By Lemma 19.2 we see that colimits of $\mathcal{I}_j$ commute with equalizers and fibred products. Thus it suffices to show that equalizers and fibre products commute with coproducts in the category of sets (including empty coproducts). In other words, given a set J and sets A_j, B_j, C_j and set maps $A_j \to B_j$, $C_j \to B_j$ for $j \in J$ we have to show that

$$(\coprod\nolimits_{j\in J} A_j) \times_{(\coprod_{j\in J} B_j)} (\coprod\nolimits_{j\in J} C_j) = \coprod\nolimits_{j\in J} A_j \times_{B_j} C_j$$

and given $a_j, a'_j : A_j \to B_j$ that

$$\text{Equalizer}(\coprod\nolimits_{j\in J} a_j, \coprod\nolimits_{j\in J} a'_j) = \coprod\nolimits_{j\in J} \text{Equalizer}(a_j, a'_j)$$

This is true even if $J = \emptyset$. Details omitted. □

20. Cofiltered limits

Limits are easier to compute or describe when they are over a cofiltered diagram. Here is the definition.

Definition 20.1. We say that a diagram $M : \mathcal{I} \to \mathcal{C}$ is *codirected* or *cofiltered* if the following conditions hold:

1. the category $\mathcal{I}$ has at least one object,
2. for every pair of objects x, y of $\mathcal{I}$ there exists an object z and morphisms $z \to x$, $z \to y$, and
3. for every pair of objects x, y of $\mathcal{I}$ and every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$ there exists a morphism $c : w \to x$ of $\mathcal{I}$ such that $M(a \circ c) = M(b \circ c)$ as morphisms in $\mathcal{C}$.

We say that an index category $\mathcal{I}$ is *codirected*, or *cofiltered* if $\text{id} : \mathcal{I} \to \mathcal{I}$ is cofiltered (in other words you erase the M in part (3) above.)

We observe that any diagram with cofiltered index category is cofiltered, and this is how this situation usually occurs.

As an example of why cofiltered limits of sets are "easier" than general ones, we mention the fact that a cofiltered diagram of finite nonempty sets has nonempty limit (Lemma 21.5). This result does not hold for a general limit of finite nonempty sets.

21. Limits and colimits over partially ordered sets

A special case of diagrams is given by systems over partially ordered sets.

Definition 21.1. Let $(I, \geq)$ be a partially ordered set. Let $\mathcal{C}$ be a category.

1. A *system over I in $\mathcal{C}$*, sometimes called a *inductive system over I in $\mathcal{C}$* is given by objects M_i of $\mathcal{C}$ and for every $i \leq i'$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \text{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i \leq i' \leq i''$.
2. An *inverse system over I in $\mathcal{C}$*, sometimes called a *projective system over I in $\mathcal{C}$* is given by objects M_i of $\mathcal{C}$ and for every $i \geq i'$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \text{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i \geq i' \geq i''$. (Note reversal of inequalities.)

We will say $(M_i, f_{ii'})$ is a (inverse) system over I to denote this. The maps $f_{ii'}$ are sometimes called the *transition maps*.

In other words a system over I is just a diagram $M : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is the category with objects I and a unique arrow $i \to i'$ if and only $i \leq i'$. And an inverse system is a diagram $M : \mathcal{I}^{opp} \to \mathcal{C}$. From this point of view we could take (co)limits of any (inverse) system over I. However, it is customary to take *only colimits of systems over I* and *only limits of inverse systems over I*. More precisely: Given a system $(M_i, f_{ii'})$ over I the colimit of the system $(M_i, f_{ii'})$ is defined as

$$\operatorname{colim}_{i\in I} M_i = \operatorname{colim}_{\mathcal{I}} M,$$

i.e., as the colimit of the corresponding diagram. Given a inverse system $(M_i, f_{ii'})$ over I the limit of the inverse system $(M_i, f_{ii'})$ is defined as

$$\lim_{i\in I} M_i = \lim_{\mathcal{I}^{opp}} M,$$

i.e., as the limit of the corresponding diagram.

Definition 21.2. With notation as above. We say the system (resp. inverse system) $(M_i, f_{ii'})$ is a *directed system* (resp. *directed inverse system*) if the partially ordered set I is *directed*: I is nonempty and for all $i_1, i_2 \in I$ there exists $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$.

In this case the colimit is sometimes (unfortunately) called the "direct limit". We will not use this last terminology. It turns out that diagrams over a filtered category are no more general than directed systems in the following sense.

Lemma 21.3. *Let $\mathcal{I}$ be a filtered index category. There exists a directed partially ordered set $(I, \geq)$ and a system $(x_i, \varphi_{ii'})$ over I in $\mathcal{I}$ with the following properties:*

1. *For every category $\mathcal{C}$ and every diagram $M : \mathcal{I} \to \mathcal{C}$ with values in $\mathcal{C}$, denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding system over I. If $\operatorname{colim}_{i\in I} M(x_i)$ exists then so does $\operatorname{colim}_{\mathcal{I}} M$ and the transformation*
$$\theta : \operatorname{colim}_{i\in I} M(x_i) \longrightarrow \operatorname{colim}_{\mathcal{I}} M$$
of Lemma 14.7 is an isomorphism.
2. *For every category $\mathcal{C}$ and every diagram $M : \mathcal{I}^{opp} \to \mathcal{C}$ in $\mathcal{C}$, denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding inverse system over I. If $\lim_{i\in I} M(x_i)$ exists then so does $\lim_{\mathcal{I}} M$ and the transformation*
$$\theta : \lim_{\mathcal{I}^{opp}} M \longrightarrow \lim_{i\in I} M(x_i)$$
of Lemma 14.8 is an isomorphism.

Proof. As mentioned in the beginning of the section, we may view partially ordered sets as categories and systems as functors. Throughout the proof, we will freely shift between these two points of view. We prove the first statement by constructing a category $\mathcal{I}_0$, corresponding to a directed set, and a cofinal functor $M_0 : \mathcal{I}_0 \to \mathcal{I}$. Then, by Lemma 17.2, the colimit of a diagram $M : \mathcal{I} \to \mathcal{C}$ coincides with the colimit of the diagram $M \circ M_0|\mathcal{I}_0 \to \mathcal{C}$, from which the statement follows. The second statement is dual to the first and may be proved by interpreting a limit in $\mathcal{C}$ as a colimit in $\mathcal{C}^{opp}$. We omit the details.

A category $\mathcal{F}$ is called *finitely generated* if there exists a finite set F of arrows in $\mathcal{F}$, such that each arrow in $\mathcal{F}$ may be obtained by composing arrows from F. In particular, this implies that $\mathcal{F}$ has finitely many objects. We start the proof by reducing to the case when $\mathcal{I}$ has the property that every finitely generated subcategory of $\mathcal{I}$ may be extended to a finitely generated subcategory with a unique final object.

Let ω denote the directed set of finite ordinals, which we view as a filtered category. It is easy to verify that the product category $\mathcal{I}\times\omega$ is also filtered, and the projection $\Pi : \mathcal{I}\times\omega \to \mathcal{I}$ is cofinal.

Now let $\mathcal{F}$ be any finitely generated subcategory of $\mathcal{I}\times\omega$. By using the axioms of a filtered category and a simple induction argument on a finite set of generators of $\mathcal{F}$, we may construct a cocone $(\{f_i\}, i_\infty)$ in $\mathcal{I}$ for the diagram $\mathcal{F}\to\mathcal{I}$. That is, a morphism $f_i : i \to i_\infty$ for every object i in $\mathcal{F}$ such that for each arrow $f : i \to i'$ in $\mathcal{F}$ we have $f_i = f \circ f_{i'}$. We also choose i_∞ such that it is not contained in $\mathcal{F}$. This is possible since we may always post-compose the arrows f_i with an arrow which is the identity on the $\mathcal{I}$-component and strictly increasing on the ω-component. Now let $\mathcal{F}^+$ denote the category consisting of all objects and arrows in $\mathcal{F}$ together with the object i_∞, the identity arrow id_{i_∞} and the arrows f_i. Since there are no arrows from i_∞ in $\mathcal{F}^+$ to any object of $\mathcal{F}$, the arrow set in $\mathcal{F}^+$ is closed under composition, so $\mathcal{F}^+$ is indeed a category. By construction, it is a finitely generated subcategory of $\mathcal{I}$ which has i_∞ as unique final object. Since, by Lemma 17.2, the colimit of any diagram $M : \mathcal{I}\to\mathcal{C}$ coincides with the colimit of $M\circ\Pi$, this gives the desired reduction.

The set of all finitely generated subcategories of $\mathcal{I}$ with a unique final object is naturally ordered by inclusion. We take $\mathcal{I}_0$ to be the category corresponding to this set. We also have a functor $M_0 : \mathcal{I}_0 \to \mathcal{I}$, which takes an arrow $\mathcal{F}\subset\mathcal{F}'$ in $\mathcal{I}_0$ to the unique map from the final object of $\mathcal{F}$ to the final object of $\mathcal{F}'$. Given any two finitely generated subcategories of $\mathcal{I}$, the category generated by these two categories is also finitely generated. By our assumption on $\mathcal{I}$, it is also contained in a finitely generated subcategory of $\mathcal{I}$ with a unique final object. This shows that $\mathcal{I}_0$ is directed.

Finally, we verify that M_0 is cofinal. Since any object of $\mathcal{I}$ is the final object in the subcategory consisting of only that object and its identity arrow, the functor M_0 is surjective on objects. In particular, Condition (1) of Definition 17.1 is satisfied. Given an object i of $\mathcal{I}$, $\mathcal{F}_1, \mathcal{F}_2$ in $\mathcal{I}_0$ and maps $\varphi_1 : i \to M_0(\mathcal{F}_1)$ and $\varphi_2 : i \to M_0(\mathcal{F}_2)$ in $\mathcal{I}$, we can take $\mathcal{F}_{12}$ to be a finitely generated category with a unique final object containg $\mathcal{F}_1$, $\mathcal{F}_2$ and the morphisms φ_1, φ_2. The resulting diagram commutes

$$\begin{tikzcd}
& M_0(\mathcal{F}_{12}) & \\
M_0(\mathcal{F}_1) \arrow[ur] & & M_0(\mathcal{F}_2) \arrow[ul] \\
& i \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

since it lives in the category $\mathcal{F}_{12}$ and $M_0(\mathcal{F}_{12})$ is final in this category. Hence also Condition (2) is satisfied, which concludes the proof. □

Remark 21.4. Note that a finite directed set $(I, \geq)$ always has a greatest object i_∞. Hence any colimit of a system $(M_i, f_{ii'})$ over such a set is trivial in the sense that the colimit equals M_{i_∞}. In contrast, a colimit indexed by a finite filtered category need not be trival. For instance, let $\mathcal{I}$ be the category with a single object i and a single non-trivial morphism e satisfying $e = e \circ e$. The colimit of a diagram

$M : \mathcal{I} \to Sets$ is the image of the idempotent $M(e)$. This illustrates that something like the trick of passing to $\mathcal{I} \times \omega$ in the proof of Lemma 21.3 is essential.

Lemma 21.5. *If $S : \mathcal{I} \to Sets$ is a cofiltered diagram of sets and all the S_i are finite nonempty, then $\lim_i S_i$ is nonempty. In other words, the limit of a directed inverse system of finite nonempty sets is nonempty.*

Proof. The two statements are equivalent by Lemma 21.3. Let I be a directed partially ordered set and let $(S_i)_{i \in I}$ be an inverse system of finite nonempty sets over I. Let us say that a *subsystem* T is a family $T = (T_i)_{i \in I}$ of nonempty subsets $T_i \subset S_i$ such that $T_{i'}$ is mapped into T_i by the transition map $S_{i'} \to S_i$ for all $i' \geq i$. Denote $\mathcal{T}$ the set of subsystems. We order $\mathcal{T}$ by inclusion. Suppose T_α, $\alpha \in A$ is a totally ordered family of elements of $\mathcal{T}$. Say $T_\alpha = (T_{\alpha,i})_{i \in I}$. Then we can find a lower bound $T = (T_i)_{i \in I}$ by setting $T_i = \bigcap_{\alpha \in A} T_{\alpha,i}$ which is manifestly a finite nonempty subset of S_i as all the $T_{\alpha,i}$ are nonempty and as the T_α form a totally ordered family. Thus we may apply Zorn's lemma to see that $\mathcal{T}$ has minimal elements.

Let's analyze what a minimal element $T \in \mathcal{T}$ looks like. First observe that the maps $T_{i'} \to T_i$ are all surjective. Namely, as I is a directed partially ordered set and T_i is finite, the intersection $T'_i = \bigcap_{i' \geq i} \mathrm{Im}(T_{i'} \to T_i)$ is nonempty. Thus $T' = (T'_i)$ is a subsystem contained in T and by minimality $T' = T$. Finally, we claim that T_i is a singleton for each i. Namely, if $x \in T_i$, then we can define $T'_{i'} = (T_{i'} \to T_i)^{-1}(\{x\})$ for $i' \geq i$ and $T'_j = T_j$ if $j \not\geq i$. This is another subsystem as we've seen above that the transition maps of the subsystem T are surjective. By minimality we see that $T = T'$ which indeed implies that T_i is a singleton. This holds for every $i \in I$, hence we see that $T_i = \{x_i\}$ for some $x_i \in S_i$ with $x_{i'} \mapsto x_i$ under the map $S_{i'} \to S_i$ for every $i' \geq i$. In other words, $(x_i) \in \lim S_i$ and the lemma is proved. □

22. Essentially constant systems

Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$. Assume the index category $\mathcal{I}$ is filtered. In this case there are three successively stronger notions which pick out an object X of $\mathcal{C}$. The first is just

$$X = \mathrm{colim}_{i \in \mathcal{I}} M_i.$$

Then X comes equipped with the coprojections $M_i \to X$. A stronger condition would be to require that X is the colimit and that there exists an $i \in \mathcal{I}$ and a morphism $X \to M_i$ such that the composition $X \to M_i \to X$ is id_X. A stronger condition is the following.

Definition 22.1. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$.

(1) Assume the index category $\mathcal{I}$ is filtered. We say M *is essentially constant* with *value* X if $X = \mathrm{colim}_i M_i$ and there exists an $i \in \mathcal{I}$ and a morphism $X \to M_i$ such that
   (a) $X \to M_i \to X$ is id_X, and
   (b) for all j there exist k and morphisms $i \to k$ and $j \to k$ such that the morphism $M_j \to M_k$ equals the composition $M_j \to X \to M_i \to M_k$.
(2) Assume the index category $\mathcal{I}$ is cofiltered. We say M is *essentially constant* with *value* X if $X = \lim_i M_i$ and there exists an $i \in \mathcal{I}$ and a morphism $M_i \to X$ such that
   (a) $X \to M_i \to X$ is id_X, and

(b) for all j there exist k and morphisms $k \to i$ and $k \to j$ such that the morphism $M_k \to M_j$ equals the composition $M_k \to M_i \to X \to M_j$.

Which of the two versions is meant will be clear from context. If there is any confusion we will distinguish between these by saying that the first version means M is essentially constant as an *ind-object*, and in the second case we will say it is essentially constant as a *pro-object*. This terminology is further explained in Remarks 22.3 and 22.4. In fact we will often use the terminology "essentially constant system" which formally speaking is only defined for systems over directed partially ordered sets.

Definition 22.2. Let $\mathcal{C}$ be a category. A directed system $(M_i, f_{ii'})$ is an *essentially constant system* if M viewed as a functor $I \to \mathcal{C}$ defines an essentially constant diagram. A directed inverse system $(M_i, f_{ii'})$ is an *essentially constant inverse system* if M viewed as a functor $I^{opp} \to \mathcal{C}$ defines an essentially constant inverse diagram.

If $(M_i, f_{ii'})$ is an essentially constant system and the morphisms $f_{ii'}$ are monomorphisms, then for all $i \leq i'$ sufficiently large the morphisms $f_{ii'}$ are isomorphisms. In general this need not be the case however. An example is the system

$$\mathbf{Z}^2 \to \mathbf{Z}^2 \to \mathbf{Z}^2 \to \ldots$$

with maps given by $(a, b) \mapsto (a + b, 0)$. This system is essentially constant with value $\mathbf{Z}$. A non-example is to let $M = \bigoplus_{n \geq 0} \mathbf{Z}$ and to let $S : M \to M$ be the shift operator $(a_0, a_1, \ldots) \mapsto (a_1, a_2, \ldots)$. In this case the system $M \to M \to M \to \ldots$ with transition maps S has colimit 0, and a map $0 \to M$ but the system is not essentially constant.

Remark 22.3. Let $\mathcal{C}$ be a category. There exists a big category Ind-$\mathcal{C}$ of *ind-objects of* $\mathcal{C}$. Namely, if $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are filtered diagrams in $\mathcal{C}$, then we can define

$$\mathrm{Mor}_{\text{Ind-}\mathcal{C}}(F, G) = \lim_i \mathrm{colim}_j \mathrm{Mor}_{\mathcal{C}}(F(i), G(j)).$$

There is a canonical functor $\mathcal{C} \to$ Ind-$\mathcal{C}$ which maps X to the *constant system* on X. This is a fully faithful embedding. In this language one sees that a diagram F is essentially constant if and only F is isomorphic to a constant system. If we ever need this material, then we will formulate this into a lemma and prove it here.

Remark 22.4. Let $\mathcal{C}$ be a category. There exists a big category Pro-$\mathcal{C}$ of *pro-objects* of $\mathcal{C}$. Namely, if $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are cofiltered diagrams in $\mathcal{C}$, then we can define

$$\mathrm{Mor}_{\text{Pro-}\mathcal{C}}(F, G) = \lim_j \mathrm{colim}_i \mathrm{Mor}_{\mathcal{C}}(F(i), G(j)).$$

There is a canonical functor $\mathcal{C} \to$ Pro-$\mathcal{C}$ which maps X to the *constant system* on X. This is a fully faithful embedding. In this language one sees that a diagram F is essentially constant if and only F is isomorphic to a constant system. If we ever need this material, then we will formulate this into a lemma and prove it here.

Lemma 22.5. *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with filtered (resp. cofiltered) index category $\mathcal{I}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If M is essentially constant as an ind-object (resp. pro-object), then so is $F \circ M : \mathcal{I} \to \mathcal{D}$.*

Proof. If X is a value for M, then it follows immediately from the definition that $F(X)$ is a value for $F \circ M$. $\square$

Lemma 22.6. *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with filtered index category $\mathcal{I}$. The following are equivalent*

1. *M is an essentially constant ind-object, and*
2. *$X = \operatorname{colim}_i M_i$ exists and for any W in $\mathcal{C}$ the map*
$$\operatorname{colim}_i \operatorname{Mor}_\mathcal{C}(W, M_i) \longrightarrow \operatorname{Mor}_\mathcal{C}(W, X)$$
is bijective.

Proof. Assume (2) holds. Then $\operatorname{id}_X \in \operatorname{Mor}_\mathcal{C}(X, X)$ comes from a morphism $X \to M_i$ for some i, i.e., $X \to M_i \to X$ is the identity. Then both maps

$$\operatorname{Mor}_\mathcal{C}(W, X) \longrightarrow \operatorname{colim}_i \operatorname{Mor}_\mathcal{C}(W, M_i) \longrightarrow \operatorname{Mor}_\mathcal{C}(W, X)$$

are bijective for all W where the first one is induced by the morphism $X \to M_i$ we found above, and the composition is the identity. This means that the composition

$$\operatorname{colim}_i \operatorname{Mor}_\mathcal{C}(W, M_i) \longrightarrow \operatorname{Mor}_\mathcal{C}(W, X) \longrightarrow \operatorname{colim}_i \operatorname{Mor}_\mathcal{C}(W, M_i)$$

is the identity too. Setting $W = M_j$ and starting with id_{M_j} in the colimit, we see that $M_j \to X \to M_i \to M_k$ is equal to $M_j \to M_k$ for some k large enough. This proves (1) holds. The proof of (1) $\Rightarrow$ (2) is omitted. $\square$

Lemma 22.7. *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with cofiltered index category $\mathcal{I}$. The following are equivalent*

1. *M is an essentially constant pro-object, and*
2. *$X = \lim_i M_i$ exists and for any W in $\mathcal{C}$ the map*
$$\operatorname{colim}_{i \in \mathcal{I}^{opp}} \operatorname{Mor}_\mathcal{C}(M_i, W) \longrightarrow \operatorname{Mor}_\mathcal{C}(X, W)$$
is bijective.

Proof. Assume (2) holds. Then $\operatorname{id}_X \in \operatorname{Mor}_\mathcal{C}(X, X)$ comes from a morphism $M_i \to X$ for some i, i.e., $X \to M_i \to X$ is the identity. Then both maps

$$\operatorname{Mor}_\mathcal{C}(X, W) \longrightarrow \operatorname{colim}_i \operatorname{Mor}_\mathcal{C}(M_i, W) \longrightarrow \operatorname{Mor}_\mathcal{C}(X, W)$$

are bijective for all W where the first one is induced by the morphism $M_i \to X$ we found above, and the composition is the identity. This means that the composition

$$\operatorname{colim}_i \operatorname{Mor}_\mathcal{C}(M_i, W) \longrightarrow \operatorname{Mor}_\mathcal{C}(X, W) \longrightarrow \operatorname{colim}_i \operatorname{Mor}_\mathcal{C}(M_i, W)$$

is the identity too. Setting $W = M_j$ and starting with id_{M_j} in the colimit, we see that $M_k \to M_i \to X \to M_j$ is equal to $M_k \to M_j$ for some k large enough. This proves (1) holds. The proof of (1) $\Rightarrow$ (2) is omitted. $\square$

Lemma 22.8. *Let $\mathcal{C}$ be a category. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of filtered index categories. If H is cofinal, then any diagram $M : \mathcal{J} \to \mathcal{C}$ is essentially constant if and only if $M \circ H$ is essentially constant.*

Proof. This follows formally from Lemmas 22.6 and 17.2. $\square$

Lemma 22.9. *Let $\mathcal{I}$ and $\mathcal{J}$ be filtered categories and denote $p : \mathcal{I} \times \mathcal{J} \to \mathcal{J}$ the projection. Then $\mathcal{I} \times \mathcal{J}$ is filtered and a diagram $M : \mathcal{J} \to \mathcal{C}$ is essentially constant if and only if $M \circ p : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ is essentially constant.*

Proof. We omit the verification that $\mathcal{I} \times \mathcal{J}$ is filtered. The equivalence follows from Lemma 22.8 because p is cofinal (verification omitted). $\square$

Lemma 22.10. *Let $\mathcal{C}$ be a category. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of cofiltered index categories. If H is initial, then any diagram $M : \mathcal{J} \to \mathcal{C}$ is essentially constant if and only if $M \circ H$ is essentially constant.*

Proof. This follows formally from Lemmas 22.7, 17.4, 17.2, and the fact that if $\mathcal{I}$ is initial in $\mathcal{J}$, then $\mathcal{I}^{opp}$ is cofinal in $\mathcal{J}^{opp}$. $\square$

23. Exact functors

Definition 23.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) Suppose all finite limits exist in $\mathcal{A}$. We say F is *left exact* if it commutes with all finite limits.
(2) Suppose all finite colimits exist in $\mathcal{A}$. We say F is *right exact* if it commutes with all finite colimits.
(3) We say F is *exact* if it is both left and right exact.

Lemma 23.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose all finite limits exist in $\mathcal{A}$, see Lemma 18.4. The following are equivalent:*

(1) *F is left exact,*
(2) *F commutes with finite products and equalizers, and*
(3) *F transforms a final object of $\mathcal{A}$ into a final object of $\mathcal{B}$, and commutes with fibre products.*

Proof. Lemma 14.10 shows that (2) implies (1). Suppose (3) holds. The fibre product over the final object is the product. If $a, b : A \to B$ are morphisms of $\mathcal{A}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(pr_1, pr_2), A \times A, \Delta} A.$$

Thus (3) implies (2). Finally (1) implies (3) because the empty limit is a final object, and fibre products are limits. $\square$

24. Adjoint functors

Definition 24.1. Let $\mathcal{C}$, $\mathcal{D}$ be categories. Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors. We say that u is a *left adjoint* of v, or that v is a *right adjoint* to u if there are bijections

$$\mathrm{Mor}_{\mathcal{D}}(u(X), Y) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, v(Y))$$

functorial in $X \in \mathrm{Ob}(\mathcal{C})$, and $Y \in \mathrm{Ob}(\mathcal{D})$.

In other words, this means that there is a *given* isomorphism of functors $\mathcal{C}^{opp} \times \mathcal{D} \to$ *Sets* from $\mathrm{Mor}_{\mathcal{D}}(u(-), -)$ to $\mathrm{Mor}_{\mathcal{C}}(-, v(-))$. For any object X of $\mathcal{C}$ we obtain a morphism $X \to v(u(X))$ corresponding to $\mathrm{id}_{u(X)}$. Similarly, for any object Y of $\mathcal{D}$ we obtain a morphism $u(v(Y)) \to Y$ corresponding to $\mathrm{id}_{v(Y)}$. These maps are called the *adjunction maps*. The adjunction maps are functorial in X and Y, hence we obtain morphisms of functors $\mathrm{id}_{\mathcal{C}} \to v \circ u$ and $u \circ v \to \mathrm{id}_{\mathcal{D}}$. Moreover, if $\alpha : u(X) \to Y$ and $\beta : X \to v(Y)$ are morphisms, then the following are equivalent

(1) α and β correspond to each other via the bijection of the definition,
(2) β is the composition $X \to v(u(X)) \xrightarrow{v(\alpha)} v(Y)$, and
(3) α is the composition $u(X) \xrightarrow{u(\beta)} u(v(Y)) \to Y$.

In this way one can reformulate the notion of adjoint functors in terms of adjunction maps.

Lemma 24.2. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. If for each $y \in \mathrm{Ob}(\mathcal{D})$ the functor $x \mapsto \mathrm{Mor}_\mathcal{D}(u(x), y)$ is representable, then u has a right adjoint.*

Proof. For each y choose an object $v(y)$ and an isomorphism $\mathrm{Mor}_\mathcal{C}(-, v(y)) \to \mathrm{Mor}_\mathcal{D}(u(-), y)$ of functors. By Yoneda's lemma (Lemma 3.5) for any morphism $g : y \to y'$ the transformation of functors

$$\mathrm{Mor}_\mathcal{C}(-, v(y)) \to \mathrm{Mor}_\mathcal{D}(u(-), y) \to \mathrm{Mor}_\mathcal{D}(u(-), y') \to \mathrm{Mor}_\mathcal{C}(-, v(y'))$$

corresponds to a unique morphism $v(g) : v(y) \to v(y')$. We omit the verification that v is a functor and that it is right adjoint to u. □

Lemma 24.3. *Let u be a left adjoint to v as in Definition 24.1. Then*

(1) *u is fully faithful $\Leftrightarrow$ $id \cong v \circ u$.*
(2) *v is fully faithful $\Leftrightarrow$ $u \circ v \cong id$.*

Proof. Assume u is fully faithful. We have to show the adjunction map $X \to v(u(X))$ is an isomorphism. Let $X' \to v(u(X))$ be any morphism. By adjointness this corresponds to a morphism $u(X') \to u(X)$. By fully faithfulness of u this corresponds to a morphism $X' \to X$. Thus we see that $X \to v(u(X))$ defines a bijection $\mathrm{Mor}(X', X) \to \mathrm{Mor}(X', v(u(X)))$. Hence it is an isomorphism. Conversely, if $\mathrm{id} \cong v \circ u$ then u has to be fully faithful, as v defines an inverse on morphism sets.

Part (2) is dual to part (1). □

Lemma 24.4. *Let u be a left adjoint to v as in Definition 24.1.*

(1) *Suppose that $M : \mathcal{I} \to \mathcal{C}$ is a diagram, and suppose that $\mathrm{colim}_\mathcal{I} M$ exists in $\mathcal{C}$. Then $u(\mathrm{colim}_\mathcal{I} M) = \mathrm{colim}_\mathcal{I} u \circ M$. In other words, u commutes with (representable) colimits.*
(2) *Suppose that $M : \mathcal{I} \to \mathcal{D}$ is a diagram, and suppose that $\lim_\mathcal{I} M$ exists in $\mathcal{D}$. Then $v(\lim_\mathcal{I} M) = \lim_\mathcal{I} v \circ M$. In other words v commutes with representable limits.*

Proof. A morphism from a colimit into an object is the same as a compatible system of morphisms from the constituents of the limit into the object, see Remark 14.4. So

$$\begin{aligned} \mathrm{Mor}_\mathcal{D}(u(\mathrm{colim}_{i \in \mathcal{I}} M_i), Y) &= \mathrm{Mor}_\mathcal{C}(\mathrm{colim}_{i \in \mathcal{I}} M_i, v(Y)) \\ &= \lim_{i \in \mathcal{I}^{opp}} \mathrm{Mor}_\mathcal{C}(M_i, v(Y)) \\ &= \lim_{i \in \mathcal{I}^{opp}} \mathrm{Mor}_\mathcal{D}(u(M_i), Y) \end{aligned}$$

proves that $u(\mathrm{colim}_{i \in \mathcal{I}} M_i)$ is the colimit we are looking for. A similar argument works for the other statement. □

Lemma 24.5. *Let u be a left adjoint of v as in Definition 24.1.*

(1) *If $\mathcal{C}$ has finite colimits, then u is right exact.*
(2) *If $\mathcal{D}$ has finite limits, then v is left exact.*

Proof. Obvious from the definitions and Lemma 24.4. □

25. Localization in categories

The basic idea of this section is given a category $\mathcal{C}$ and a set of arrows S to construct a functor $F : \mathcal{C} \to S^{-1}\mathcal{C}$ such that all elements of S become invertible in $S^{-1}\mathcal{C}$ and such that F is universal among all functors with this property. References for this section are [GZ67, Chapter I, Section 2] and [Ver96, Chapter II, Section 2].

Definition 25.1. Let $\mathcal{C}$ be a category. A set of arrows S of $\mathcal{C}$ is called a *left multiplicative system* if it has the following properties:

LMS1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements of S is in S.

LMS2 Every solid diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "t"'] & Y \arrow[d, dotted, "s"] \\ Z \arrow[r, dotted, "f"] & W \end{tikzcd}$$

with $t \in S$ can be completed to a commutative dotted square with $s \in S$.

LMS3 For every pair of morphisms $f, g : X \to Y$ and $t \in S$ with target X such that $f \circ t = g \circ t$ there exists a $s \in S$ with source Y such that $s \circ f = s \circ g$.

A set of arrows S of $\mathcal{C}$ is called a *right multiplicative system* if it has the following properties:

RMS1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements of S is in S.

RMS2 Every solid diagram

$$\begin{tikzcd} X \arrow[r, dotted, "g"'] \arrow[d, dotted, "t"'] & Y \arrow[d, "s"] \\ Z \arrow[r, "f"] & W \end{tikzcd}$$

with $s \in S$ can be completed to a commutative dotted square with $t \in S$.

RMS3 For every pair of morphisms $f, g : X \to Y$ and $s \in S$ with source Y such that $s \circ f = s \circ g$ there exists a $t \in S$ with target X such that $f \circ t = g \circ t$.

A set of arrows S of $\mathcal{C}$ is called a *multiplicative system* if it is both a left multiplicative system and a right multiplicative system. In other words, this means that MS1, MS2, MS3 hold, where MS1 = LMS1 + RMS1, MS2 = LMS2 + RMS2, and MS3 = LMS3 + RMS3.

These conditions are useful to construct the categories $S^{-1}\mathcal{C}$ as follows.

Left calculus of fractions. Let $\mathcal{C}$ be a category and let S be a left multiplicative system. We define a new category $S^{-1}\mathcal{C}$ as follows (we verify this works in the proof of Lemma 25.2):

(1) We set $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
(2) Morphisms $X \to Y$ of $S^{-1}\mathcal{C}$ are given by pairs $(f : X \to Y', s : Y \to Y')$ with $s \in S$ up to equivalence. (Think of this as $s^{-1}f : X \to Y$.)
(3) Two pairs $(f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $(f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are said to be equivalent if there exists a third pair $(f_3 : X \to Y_3, s_3 : Y \to Y_3)$ and morphisms $u : Y_1 \to Y_3$ and $v : Y_2 \to Y_3$ of $\mathcal{C}$ fitting into the

commutative diagram

$$\begin{tikzcd}
 & Y_1 \arrow[d, "u"] & \\
X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\
 & Y_2 \arrow[u, "v"'] &
\end{tikzcd}$$

(4) The composition of the equivalence classes of the pairs $(f : X \to Y', s : Y \to Y')$ and $(g : Y \to Z', t : Z \to Z')$ is defined as the equivalence class of a pair $(h \circ f : X \to Z'', u \circ t : Z \to Z'')$ where h and $u \in S$ are chosen to fit into a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "s"'] & Z' \arrow[d, "u"] \\
Y' \arrow[r, "h"] & Z''
\end{tikzcd}$$

which exists by assumption.

Lemma 25.2. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system.*

(1) *The relation on pairs defined above is an equivalence relation.*
(2) *The composition rule given above is well defined on equivalence classes.*
(3) *Composition is associative and hence $S^{-1}\mathcal{C}$ is a category.*

Proof. Proof of (1). Let us say two pairs $p_1 = (f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $p_2 = (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are elementary equivalent if there exists a morphism $a : Y_1 \to Y_2$ of $\mathcal{C}$ such that $a \circ f_1 = f_2$ and $a \circ s_1 = s_2$. Diagram:

$$\begin{tikzcd}
X \arrow[r, "f_1"'] \arrow[d, equal] & Y_1 \arrow[d, "a"] & Y \arrow[l, "s_1"] \arrow[d, equal] \\
X \arrow[r, "f_2"] & Y_2 & Y \arrow[l, "s_2"']
\end{tikzcd}$$

Let us denote this property by saying $p_1 E p_2$. Note that pEp and $aEb, bEc \Rightarrow aEc$. Part (1) claims that the relation $p \sim p' \Leftrightarrow \exists q : pEq \wedge p'Eq$ is an equivalence relation. A simple formal argument, using the properties of E above shows that it suffices to prove $p_3Ep_1, p_3Ep_2 \Rightarrow p_1 \sim p_2$. Thus suppose that we are given a commutative diagram

$$\begin{tikzcd}
 & Y_1 & \\
X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 \arrow[u, "a_{31}"'] \arrow[d, "a_{32}"] & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\
 & Y_2 &
\end{tikzcd}$$

with $s_i \in S$. First we apply LMS2 to get a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "s_3"'] \arrow[d, "s_1"'] & Y_3 \arrow[d, dotted, "s_{34}"] \\ Y_1 \arrow[r, dotted, "a_{14}"] & Y_4 \end{tikzcd}$$

with $s_{34} \in S$. Then we have $s_{34} \circ s_3 = a_{14} \circ a_{31} \circ s_3$. Hence by LMS3 there exists a morphism $s_{44} : Y_4 \to Y_4'$, $s_{44} \in S$ such that $s_{44} \circ s_{34} = s_{44} \circ a_{14} \circ a_{31}$. Hence after replacing Y_4 by Y_4', a_{14} by $s_{44} \circ a_{14}$, and s_{34} by $s_{44} \circ s_{34}$ we may assume that $s_{34} = a_{14} \circ a_{31}$. Next, we apply LMS2 to get a commutative diagram

$$\begin{tikzcd} Y_3 \arrow[r, "s_{34}"'] \arrow[d, "a_{32}"'] & Y_4 \arrow[d, dotted, "s_{45}"] \\ Y_2 \arrow[r, dotted, "a_{25}"] & Y_5 \end{tikzcd}$$

with $s_{45} \in S$. Thus we obtain a pair $p_5 = (s_{45} \circ s_{34} \circ f_3 : X \to Y_5, s_{45} \circ s_{34} \circ s_3 : Y \to Y_5)$ and the morphisms $s_{45} \circ a_{14} : Y_1 \to Y_5$ and $a_{25} : Y_2 \to Y_5$ show that indeed $p_1 E p_5$ and $p_2 E p_5$ as desired.

Proof of (2). Let $p = (f : X \to Y', s : Y \to Y')$ and $q = (g : Y \to Z', t : Z \to Z')$ be pairs as in the definition of composition above. To compose we have to choose a diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s"'] & Z' \arrow[d, "u_2"] \\ Y' \arrow[r, "h_2"] & Z_2 \end{tikzcd}$$

We first show that the equivalence class of the pair $r_2 = (h_2 \circ f : X \to Z_2, u_2 \circ t : Z \to Z_2)$ is independent of the choice of (Z_2, h_2, u_2). Namely, suppose that (Z_3, h_3, u_3) is another choice with corresponding composition $r_3 = (h_3 \circ f : X \to Z_3, u_3 \circ t : Z \to Z_3)$. Then by LMS2 we can choose a diagram

$$\begin{tikzcd} Z' \arrow[r, "u_3"'] \arrow[d, "u_2"'] & Z_3 \arrow[d, "u_{34}"] \\ Z_2 \arrow[r, "h_{24}"] & Z_4 \end{tikzcd}$$

with $u_{34} \in S$. Hence we obtain a pair $r_4 = (h_{24} \circ h_2 \circ f : X \to Z_4, u_{34} \circ u_3 \circ t : Z \to Z_4)$ and the morphisms $h_{24} : Z_2 \to Z_4$ and $u_{34} : Z_3 \to Z_4$ show that we have $r_2 E r_4$ and $r_3 E r_4$ as desired. Thus it now makes sense to define $p \circ q$ as the equivalence class of all possible pairs r obtained as above.

To finish the proof of (2) we have to show that given pairs p_1, p_2, q such that $p_1 E p_2$ then $p_1 \circ q = p_2 \circ q$ and $q \circ p_1 = q \circ p_2$ whenever the compositions make sense. To do this, write $p_1 = (f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $p_2 = (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ and let $a : Y_1 \to Y_2$ be a morphism of $\mathcal{C}$ such that $f_2 = a \circ f_1$ and $s_2 = a \circ s_1$. First assume that $q = (g : Y \to Z', t : Z \to Z')$. In this case choose a commutative diagram as the one on the left

$$
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "s_2"'] & Z' \arrow[d, "u"] \\
Y_2 \arrow[r, "h"] & Z''
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "s_1"'] & Z' \arrow[d, "u"] \\
Y_1 \arrow[r, "h\circ a"] & Z''
\end{tikzcd}
$$

which implies the diagram on the right is commutative as well. Using these diagrams we see that both compositions are the equivalence class of $(h\circ a\circ f_1 : X \to Z'', u\circ t : Z \to Z'')$. Thus $p_1 \circ q = p_2 \circ q$. The proof of the other case, in which we have to show $q \circ p_1 = q \circ p_2$, is omitted.

Proof of (3). We have to prove associativity of composition. Consider a solid diagram

$$
\begin{tikzcd}
 & & & Z \arrow[d] \\
 & & Y \arrow[r] \arrow[d] & Z' \arrow[d, dotted] \\
 & X \arrow[r] \arrow[d] & Y' \arrow[r, dotted] \arrow[d, dotted] & Z'' \arrow[d, dotted] \\
W \arrow[r] & X' \arrow[r, dotted] & Y'' \arrow[r, dotted] & Z'''
\end{tikzcd}
$$

which gives rise to three composable pairs. Using LMS2 we can choose the dotted arrows making the squares commutative and such that the vertical arrows are in S. Then it is clear that the composition of the three pairs is the equivalence class of the pair $(W \to Z''', Z \to Z''')$ gotten by composing the horizontal arrows on the bottom row and the vertical arrows on the right column. $\square$

We can "write any finite collection of morphisms with the same target as fractions with common denominator".

Lemma 25.3. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Given any finite collection $g_i : X_i \to Y$ of morphisms of $S^{-1}\mathcal{C}$ we can find an element $s : Y \to Y'$ of S and $f_i : X_i \to Y'$ such that g_i is the equivalence class of the pair $(f_i : X_i \to Y', s : Y \to Y')$.*

Proof. For each i choose a representative $(X_i \to Y_i, s_i : Y \to Y_i)$. The lemma follows if we can find a morphism $s : Y \to Y'$ in S such that for each i there is a morphism $a_i : Y_i \to Y'$ with $a_i \circ s_i = s$. If we have two indices $i = 1, 2$, then we can do this by completing the square

$$
\begin{tikzcd}
Y \arrow[r, "s_2"'] \arrow[d, "s_1"'] & Y_2 \arrow[d, "t_2"] \\
Y_1 \arrow[r, "a_1"] & Y'
\end{tikzcd}
$$

with $t_2 \in S$ as is possible by Definition 25.1. Then $s = t_2 \circ s_2 \in S$ works. If we have $n > 2$ morphisms, then we use the above trick to reduce to the case of $n - 1$ morphisms, and we win by induction. $\square$

There is an easy characterization of equality of morphisms if they have the same denominator.

Lemma 25.4. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Let $A, B : X \to Y$ be morphisms of $S^{-1}\mathcal{C}$ which are the equivalence classes of $(f : X \to Y', s : Y \to Y')$ and $(g : X \to Y', s : Y \to Y')$. Then $A = B$ if and only if there exists a morphism $a : Y' \to Y''$ with $a \circ s \in S$ and such that $a \circ f = a \circ g$.*

Proof. The equality of A and B means that there exists a commutative diagram

$$\begin{tikzcd} & Y' \arrow[d, "u"] & \\ X \arrow[ru, "f"] \arrow[r, "h"] \arrow[rd, "g"'] & Z & Y \arrow[lu, "s"'] \arrow[l, "t"'] \arrow[ld, "s"] \\ & Y' \arrow[u, "v"'] & \end{tikzcd}$$

with $t \in S$. In particular $u \circ s = v \circ s$. Hence by LMS3 there exists a $s' : Z \to Y''$ in S such that $s' \circ u = s' \circ v$. Setting a equal to this common value does the job. □

Remark 25.5. Let $\mathcal{C}$ be a category. Let S be a left multiplicative system. Given an object Y of $\mathcal{C}$ we denote Y/S the category whose objects are $s : Y \to Y'$ with $s \in S$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} & Y \arrow[ld, "s"'] \arrow[rd, "t"] & \\ Y' \arrow[rr, "a"] & & Y'' \end{tikzcd}$$

where $a : Y' \to Y''$ is arbitrary. We claim that the category Y/S is filtered (see Definition 19.1). Namely, LMS1 implies that $\mathrm{id}_Y : Y \to Y$ is in Y/S hence Y/S is nonempty. LMS2 implies that given $s_1 : Y \to Y_1$ and $s_2 : Y \to Y_2$ we can find a diagram

$$\begin{tikzcd} Y \arrow[r, "s_2"'] \arrow[d, "s_1"'] & Y_2 \arrow[d, "t"] \\ Y_1 \arrow[r, "a"] & Y_3 \end{tikzcd}$$

with $t \in S$. Hence $s_1 : Y \to Y_1$ and $s_2 : Y \to Y_2$ both map to $t \circ s_2 : Y \to Y_3$ in Y/S. Finally, given two morphisms a, b from $s_1 : Y \to Y_1$ to $s_2 : Y \to Y_2$ in Y/S we see that $a \circ s_1 = b \circ s_1$ hence by LMS3 there exists a $t : Y_2 \to Y_3$ such that $t \circ a = t \circ b$. Now the combined results of Lemmas 25.3 and 25.4 tell us that

$$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X, Y) = \mathrm{colim}_{(s:Y \to Y') \in Y/S} \mathrm{Mor}_{\mathcal{C}}(X, Y') \tag{25.5.1}$$

This formula expressing morphism sets in $S^{-1}\mathcal{C}$ as a filtered colimit of morphism sets in $\mathcal{C}$ is occasionally useful.

Lemma 25.6. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$.*

(1) *The rules $X \mapsto X$ and $(f : X \to Y) \mapsto (f : X \to Y, id_Y : Y \to Y)$ define a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.*

(2) *For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$.*
(3) *If $G : \mathcal{C} \to \mathcal{D}$ is any functor such that $G(s)$ is invertible for every $s \in S$, then there exists a unique functor $H : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $H \circ Q = G$.*

Proof. Parts (1) and (2) are clear. To see (3) just set $H(X) = G(X)$ and set $H((f : X \to Y', s : Y \to Y')) = G(s)^{-1} \circ G(f)$. Details omitted. $\square$

Lemma 25.7. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. The localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite colimits.*

Proof. This is clear from (25.5.1), Remark 14.4, and Lemma 14.9. $\square$

Lemma 25.8. *Let $\mathcal{C}$ be a category. Let S be a left multiplicative system. If $f : X \to Y$, $f' : X' \to Y'$ are two morphisms of $\mathcal{C}$ and if*

$$\begin{tikzcd} Q(X) \arrow[r, "a"'] \arrow[d, "Q(f)"'] & Q(X') \arrow[d, "Q(f')"] \\ Q(Y) \arrow[r, "b"] & Q(Y') \end{tikzcd}$$

is a commutative diagram in $S^{-1}\mathcal{C}$, then there exists a morphism $f'' : X'' \to Y''$ in $\mathcal{C}$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "f"'] & X'' \arrow[d, "f''"] & X' \arrow[l, "s"] \arrow[d, "f'"] \\ Y \arrow[r, "h"] & Y'' & Y' \arrow[l, "t"'] \end{tikzcd}$$

in $\mathcal{C}$ with $s, t \in S$ and $a = s^{-1}g$, $b = t^{-1}h$.

Proof. We choose maps and objects in the following way: First write $a = s^{-1}g$ for some $s : X' \to X''$ in S and $g : X \to X''$. By LMS2 we can find $t : Y' \to Y''$ in S and $f'' : X'' \to Y''$ such that

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "f"'] & X'' \arrow[d, "f''"] & X' \arrow[l, "s"] \arrow[d, "f'"] \\ Y & Y'' & Y' \arrow[l, "t"'] \end{tikzcd}$$

commutes. Now in this diagram we are going to repeatedly change our choice of

$$X'' \xrightarrow{f''} Y'' \xleftarrow{t} Y'$$

by postcomposing both t and f'' by a morphism $d : Y'' \to Y'''$ with the property that $d \circ t \in S$. According to Remark 25.5 we may after such a replacement assume that there exists a morphism $h : Y \to Y''$ such that $b = t^{-1}h$. At this point we have everything as in the lemma except that we don't know that the left square of the diagram commutes. However, we do know that $Q(f''g) = Q(hf)$ in $S^{-1}\mathcal{D}$ because the right square commutes, the outer square commutes in $S^{-1}\mathcal{D}$ by assumption, and because $Q(s), Q(t)$ are isomorphisms. Hence using Lemma 25.4 we can find a morphism $d : Y'' \to Y'''$ in S (!) such that $df''g = dhf$. Hence we make one more replacement of the kind described above and we win. $\square$

Right calculus of fractions. Let $\mathcal{C}$ be a category and let S be a right multiplicative system. We define a new category $S^{-1}\mathcal{C}$ as follows (we verify this works in the proof of Lemma 25.9):

(1) We set $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
(2) Morphisms $X \to Y$ of $S^{-1}\mathcal{C}$ are given by pairs $(f : X' \to Y, s : X' \to X)$ with $s \in S$ up to equivalence. (Think of this as $fs^{-1} : X \to Y$.)
(3) Two pairs $(f_1 : X_1 \to Y, s_1 : X_1 \to X)$ and $(f_2 : X_2 \to Y, s_2 : X_2 \to X)$ are said to be equivalent if there exists a third pair $(f_3 : X_3 \to Y, s_3 : X_3 \to X)$ and morphisms $u : X_3 \to X_1$ and $v : X_3 \to X_2$ of $\mathcal{C}$ fitting into the commutative diagram

$$\begin{tikzcd} & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & X_3 \arrow[l, "s_3"'] \arrow[r, "f_3"] \arrow[u, "u"'] \arrow[d, "v"] & Y \\ & X_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

(4) The composition of the equivalence classes of the pairs $(f : X' \to Y, s : X' \to X)$ and $(g : Y' \to Z, t : Y' \to Y)$ is defined as the equivalence class of a pair $(g \circ h : X'' \to Z, s \circ u : X'' \to X)$ where h and $u \in S$ are chosen to fit into a commutative diagram

$$\begin{tikzcd} X'' \arrow[r, "h"] \arrow[d, "u"'] & Y' \arrow[d, "t"] \\ X' \arrow[r, "f"] & Y \end{tikzcd}$$

which exists by assumption.

Lemma 25.9. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system.*

(1) *The relation on pairs defined above is an equivalence relation.*
(2) *The composition rule given above is well defined on equivalence classes.*
(3) *Composition is associative and hence $S^{-1}\mathcal{C}$ is a category.*

Proof. This lemma is dual to Lemma 25.2. It follows formally from that lemma by replacing $\mathcal{C}$ by its opposite category in which S is a left multiplicative system. $\square$

We can "write any finite collection of morphisms with the same source as fractions with common denominator".

Lemma 25.10. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Given any finite collection $g_i : X \to Y_i$ of morphisms of $S^{-1}\mathcal{C}$ we can find an element $s : X' \to X$ of S and $f_i : X' \to Y_i$ such that g_i is the equivalence class of the pair $(f_i : X' \to Y_i, s : X' \to X)$.*

Proof. This lemma is the dual of Lemma 25.3 and follows formally from that lemma by replacing all categories in sight by their opposites. $\square$

There is an easy characterization of equality of morphisms if they have the same denominator.

Lemma 25.11. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Let $A, B : X \to Y$ be morphisms of $S^{-1}\mathcal{C}$ which are the equivalence classes of $(f : X' \to Y, s : X' \to X)$ and $(g : X' \to Y, s : X' \to X)$. Then $A = B$ if and only if there exists a morphism $a : X'' \to X'$ with $s \circ a \in S$ and such that $f \circ a = g \circ a$.*

Proof. This is dual to Lemma 25.4. $\square$

Remark 25.12. Let $\mathcal{C}$ be a category. Let S be a right multiplicative system. Given an object X of $\mathcal{C}$ we denote S/X the category whose objects are $s : X' \to X$ with $s \in S$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} X' \arrow[rr, "a"'] \arrow[dr, "s"'] & & X'' \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

where $a : X' \to X''$ is arbitrary. The category S/X is cofiltered (see Definition 20.1). (This is dual to the corresponding statement in Remark 25.5.) Now the combined results of Lemmas 25.10 and 25.11 tell us that

$$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X, Y) = \mathrm{colim}_{(s:X' \to X) \in (S/X)^{opp}} \mathrm{Mor}_{\mathcal{C}}(X', Y) \tag{25.12.1}$$

This formula expressing morphisms in $S^{-1}\mathcal{C}$ as a filtered colimit of morphisms in $\mathcal{C}$ is occasionally useful.

Lemma 25.13. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$.*

1. *The rules $X \mapsto X$ and $(f : X \to Y) \mapsto (f : X \to Y, id_X : X \to X)$ define a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.*
2. *For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$.*
3. *If $G : \mathcal{C} \to \mathcal{D}$ is any functor such that $G(s)$ is invertible for every $s \in S$, then there exists a unique functor $H : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $H \circ Q = G$.*

Proof. This lemma is the dual of Lemma 25.6 and follows formally from that lemma by replacing all categories in sight by their opposites. $\square$

Lemma 25.14. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. The localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite limits.*

Proof. This is clear from (25.12.1), Remark 14.4, and Lemma 14.9. $\square$

Lemma 25.15. *Let $\mathcal{C}$ be a category. Let S be a right multiplicative system. If $f : X \to Y$, $f' : X' \to Y'$ are two morphisms of $\mathcal{C}$ and if*

$$\begin{tikzcd} Q(X) \arrow[r, "a"'] \arrow[d, "Q(f)"'] & Q(X') \arrow[d, "Q(f')"] \\ Q(Y) \arrow[r, "b"] & Q(Y') \end{tikzcd}$$

is a commutative diagram in $S^{-1}\mathcal{C}$, then there exists a morphism $f'' : X'' \to Y''$ in $\mathcal{C}$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X'' \arrow[l, "s"'] \arrow[r, "g"] \arrow[d, "f''"] & X' \arrow[d, "f'"] \\ Y & Y'' \arrow[l, "t"] \arrow[r, "h"] & Y' \end{tikzcd}$$

in $\mathcal{C}$ with $s, t \in S$ and $a = gs^{-1}$, $b = ht^{-1}$.

Proof. This lemma is dual to Lemma 25.8 but we can also prove it directly as follows. We choose maps and objects in the following way: First write $b = ht^{-1}$ for some $t : Y'' \to Y$ in S and $h : Y'' \to Y'$. By RMS2 we can find $s : X'' \to X$ in S and $f'' : X'' \to Y''$ such that

$$\begin{tikzcd} X \arrow[d, "f"'] & X'' \arrow[l, "s"'] \arrow[d, "f''"] & X' \arrow[d, "f'"] \\ Y & Y'' \arrow[l, "t"] \arrow[r, "h"] & Y' \end{tikzcd}$$

commutes. Now in this diagram we are going to repeatedly change our choice of

$$X \xleftarrow{s} X'' \xrightarrow{f''} Y''$$

by precomposing both s and f'' by a morphism $d : X''' \to X''$ with the property that $s \circ d \in S$. According to Remark 25.12 we may after such a replacement assume that there exists a morphism $g : X'' \to X'$ such that $a = gs^{-1}$. At this point we have everything as in the lemma except that we don't know that the right square of the diagram commutes. However, we do know that $Q(f'g) = Q(hf'')$ in $S^{-1}\mathcal{D}$ because the left square commutes, the outer square commutes in $S^{-1}\mathcal{D}$ by assumption, and because $Q(s), Q(t)$ are isomorphisms. Hence using Lemma 25.11 we can find a morphism $d : X''' \to X''$ in S (!) such that $f'gd = hf''d$. Hence we make one more replacement of the kind described above and we win. $\square$

Multiplicative systems and two sided calculus of fractions. If S is a multiplicative system then left and right calculus of fractions given canonically isomorphic categories.

Lemma 25.16. *Let $\mathcal{C}$ be a category and let S be a multiplicative system. The category of left fractions and the category of right fractions $S^{-1}\mathcal{C}$ are canonically isomorphic.*

Proof. Denote $\mathcal{C}_{left}, \mathcal{C}_{right}$ the two categories of fractions. By the universal properties of Lemmas 25.6 and 25.13 we obtain functors $\mathcal{C}_{left} \to \mathcal{C}_{right}$ and $\mathcal{C}_{right} \to \mathcal{C}_{left}$. By the uniqueness of these functors they are each others inverse. $\square$

Definition 25.17. Let $\mathcal{C}$ be a category and let S be a multiplicative system. We say S is *saturated* if, in addition to MS1, MS2, MS3 we also have

MS4 Given three composable morphisms f, g, h, if $fg, gh \in S$, then $g \in S$.

Note that a saturated multiplicative system contains all isomorphisms. Moreover, if f, g, h are composable morphisms in a category and fg, gh are isomorphisms, then g is an isomorphism (because then g has both a left and a right inverse, hence is invertible).

Lemma 25.18. *Let $\mathcal{C}$ be a category and let S be a multiplicative system. Denote $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ the localization functor. The set*

$$\hat{S} = \{f \in Arrows(\mathcal{C}) \mid Q(f) \text{ is an isomorphism}\}$$

is equal to

$$S' = \{f \in Arrows(\mathcal{C}) \mid \text{there exist } g, h \text{ such that } gf, fh \in S\}$$

and is the smallest saturated multiplicative system containing S. In particular, if S is saturated, then $\hat{S} = S$.

Proof. It is clear that $S \subset S' \subset \hat{S}$ because elements of S' map to morphisms in $S^{-1}\mathcal{C}$ which have both left and right inverses. Note that S' satisfies MS4, and that $\hat{S}$ satisfies MS1. Next, we prove that $S' = \hat{S}$.

Let $f \in \hat{S}$. Let $s^{-1}g = ht^{-1}$ be the inverse morphism in $S^{-1}\mathcal{C}$. (We may use both left fractions and right fractions to describe morphisms in $S^{-1}\mathcal{C}$, see Lemma 25.16.) The relation $\text{id}_X = s^{-1}gf$ in $S^{-1}\mathcal{C}$ means there exists a commutative diagram

$$\begin{tikzcd}
 & X' \arrow[d, "u"] & \\
X \arrow[ur, "gf"] \arrow[r, "f'"] \arrow[dr, "\text{id}_X"'] & X'' & X \arrow[ul, "s"'] \arrow[l, "s'"'] \arrow[dl, "\text{id}_X"] \\
 & X \arrow[u, "v"'] &
\end{tikzcd}$$

for some morphisms f', u, v and $s' \in S$. Hence $ugf = s' \in S$. Similarly, using that $\text{id}_Y = fht^{-1}$ one proves that $fhw \in S$ for some w. We conclude that $f \in S'$. Thus $S' = \hat{S}$. Provided we prove that $S' = \hat{S}$ is a multiplicative system it is now clear that this implies that $S' = \hat{S}$ is the smallest saturated system containing S.

Our remarks above take care of MS1 and MS4, so to finish the proof of the lemma we have to show that LMS2, RMS2, LMS3, RMS3 hold for $\hat{S}$. Let us check that LMS2 holds for $\hat{S}$. Suppose we have a solid diagram

$$\begin{tikzcd}
X \arrow[r, "g"'] \arrow[d, "t"'] & Y \arrow[d, dotted, "s"] \\
Z \arrow[r, dotted, "f"] & W
\end{tikzcd}$$

with $t \in \hat{S}$. Pick a morphism $a : Z \to Z'$ such that $at \in S$. Then we can use LMS2 for S to find a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "g"'] \arrow[d, "t"'] & Y \arrow[dd, "s"] \\
Z \arrow[d, "a"'] & \\
Z' \arrow[r, "f'"] & W
\end{tikzcd}$$

and setting $f = f' \circ a$ we win. The proof of RMS2 is dual to this. Finally, suppose given a pair of morphisms $f, g : X \to Y$ and $t \in \hat{S}$ with target X such that $ft = gt$. Then we pick a morphism b such that $tb \in S$. Then $ftb = gtb$ which implies by

LMS3 for S that there exists an $s \in S$ with source Y such that $sf = sg$ as desired. The proof of RMS3 is dual to this. $\square$

26. Formal properties

In this section we discuss some formal properties of the 2-category of categories. This will lead us to the definition of a (strict) 2-category later.

Let us denote $\mathrm{Ob}(\mathit{Cat})$ the class of all categories. For every pair of categories $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathit{Cat})$ we have the "small" category of functors $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$. Composition of transformation of functors such as

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F''"{name=U}] \arrow[r, "F'"'{name=M}] \arrow[r, bend right=50, "F"'{name=D}] & \mathcal{B} \arrow[from=U, to=M, phantom, "\Downarrow t'"] \arrow[from=M, to=D, phantom, "\Downarrow t"] \end{tikzcd} \text{ composes to } \begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F''"{name=U}] \arrow[r, bend right, "F"'{name=D}] & \mathcal{B} \arrow[from=U, to=D, phantom, "\Downarrow t \circ t'"] \end{tikzcd}$$

is called *vertical* composition. We will use the usual symbol $\circ$ for this. Next, we will define *horizontal* composition. In order to do this we explain a bit more of the structure at hand.

Namely for every triple of categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ there is a composition law

$$\circ : \mathrm{Ob}(\mathrm{Fun}(\mathcal{B}, \mathcal{C})) \times \mathrm{Ob}(\mathrm{Fun}(\mathcal{A}, \mathcal{B})) \longrightarrow \mathrm{Ob}(\mathrm{Fun}(\mathcal{A}, \mathcal{C}))$$

coming from composition of functors. This composition law is associative, and identity functors act as units. In other words – forgetting about transformations of functors – we see that *Cat* forms a category. How does this structure interact with the morphisms between functors?

Well, given $t : F \to F'$ a transformation of functors $F, F' : \mathcal{A} \to \mathcal{B}$ and a functor $G : \mathcal{B} \to \mathcal{C}$ we can define a transformation of functors $G \circ F \to G \circ F'$. We will denote this transformation ${}_Gt$. It is given by the formula $({}_Gt)_x = G(t_x) : G(F(x)) \to G(F'(x))$ for all $x \in \mathcal{A}$. In this way composition with G becomes a functor

$$\mathrm{Fun}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C}).$$

To see this you just have to check that ${}_G(\mathrm{id}_F) = \mathrm{id}_{G \circ F}$ and that ${}_G(t_1 \circ t_2) = {}_Gt_1 \circ {}_Gt_2$. Of course we also have that ${}_{\mathrm{id}_\mathcal{A}}t = t$.

Similarly, given $s : G \to G'$ a transformation of functors $G, G' : \mathcal{B} \to \mathcal{C}$ and $F : \mathcal{A} \to \mathcal{B}$ a functor we can define s_F to be the transformation of functors $G \circ F \to G' \circ F$ given by $(s_F)_x = s_{F(x)} : G(F(x)) \to G'(F(x))$ for all $x \in \mathcal{A}$. In this way composition with F becomes a functor

$$\mathrm{Fun}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C}).$$

To see this you just have to check that $(\mathrm{id}_G)_F = \mathrm{id}_{G \circ F}$ and that $(s_1 \circ s_2)_F = s_{1,F} \circ s_{2,F}$. Of course we also have that $s_{\mathrm{id}_\mathcal{B}} = s$.

These constructions satisfy the additional properties

$${}_{G_1}({}_{G_2}t) = {}_{G_1 \circ G_2}t, \ (s_{F_1})_{F_2} = s_{F_1 \circ F_2}, \text{ and } {}_H(s_F) = ({}_Hs)_F$$

whenever these make sense. Finally, given functors $F, F' : \mathcal{A} \to \mathcal{B}$, and $G, G' : \mathcal{B} \to \mathcal{C}$ and transformations $t : F \to F'$, and $s : G \to G'$ the following diagram is commutative

$$\begin{tikzcd} G \circ F \arrow[r, "{}_G t"] \arrow[d, "s_F"'] & G \circ F' \arrow[d, "s_{F'}"] \\ G' \circ F \arrow[r, "{}_{G'} t"'] & G' \circ F' \end{tikzcd}$$

in other words ${}_{G'}t \circ s_F = s_{F'} \circ {}_G t$. To prove this we just consider what happens on any object $x \in \mathrm{Ob}(\mathcal{A})$:

$$\begin{tikzcd} G(F(x)) \arrow[r, "G(t_x)"] \arrow[d, "s_{F(x)}"'] & G(F'(x)) \arrow[d, "s_{F'(x)}"] \\ G'(F(x)) \arrow[r, "G'(t_x)"'] & G'(F'(x)) \end{tikzcd}$$

which is commutative because s is a transformation of functors. This compatibility relation allows us to define horizontal composition.

Definition 26.1. Given a diagram as in the left hand side of:

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow t"] & \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow s"] & \mathcal{C} \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} \mathcal{A} \arrow[rr, bend left, "G\circ F"] \arrow[rr, bend right, "G'\circ F'"'] \arrow[rr, phantom, "\Downarrow s\star t"] & & \mathcal{C} \end{tikzcd}$$

we define the *horizontal* composition $s \star t$ to be the transformation of functors ${}_{G'}t \circ s_F = s_{F'} \circ {}_G t$.

Now we see that we may recover our previously constructed transformations ${}_G t$ and s_F as ${}_G t = \mathrm{id}_G \star t$ and $s_F = s \star \mathrm{id}_F$. Furthermore, all of the rules we found above are consequences of the properties stated in the lemma that follows.

Lemma 26.2. *The horizontal and vertical compositions have the following properties*

(1) $\circ$ *and* $\star$ *are associative,*
(2) *the identity transformations* id_F *are units for* $\circ$,
(3) *the identity transformations of the identity functors* $id_{id_\mathcal{A}}$ *are units for* $\star$ *and* $\circ$, *and*
(4) *given a diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"] \arrow[r, "F'" description] \arrow[r, bend right=50, "F''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow t"] \arrow[r, phantom, shift right=3ex, "\Downarrow t'"] & \mathcal{B} \arrow[r, bend left=50, "G"] \arrow[r, "G'" description] \arrow[r, bend right=50, "G''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow s"] \arrow[r, phantom, shift right=3ex, "\Downarrow s'"] & \mathcal{C} \end{tikzcd}$$

we have $(s' \circ s) \star (t' \circ t) = (s' \star t') \circ (s \star t)$.

Proof. The last statement turns using our previous notation into the following equation

$$s'_{F''} \circ {}_{G'}t' \circ s_{F'} \circ {}_G t = (s' \circ s)_{F''} \circ {}_G(t' \circ t).$$

According to our result above applied to the middle composition we may rewrite the left hand side as $s'_{F''} \circ s_{F''} \circ_G t' \circ_G t$ which is easily shown to be equal to the right hand side. $\square$

Another way of formulating condition (4) of the lemma is that composition of functors and horizontal composition of transformation of functors gives rise to a functor

$$(\circ, \star) : \mathrm{Fun}(\mathcal{B}, \mathcal{C}) \times \mathrm{Fun}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C})$$

whose source is the product category, see Definition 2.20.

27. 2-categories

We will give a definition of (strict) 2-categories as they appear in the setting of stacks. Before you read this take a look at Section 26 and Example 28.2. Basically, you take this example and you write out all the rules satisfied by the objects, 1-morphisms and 2-morphisms in that example.

Definition 27.1. A (strict) *2-category* $\mathcal{C}$ consists of the following data

1. A set of objects $\mathrm{Ob}(\mathcal{C})$.
2. For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a category $\mathrm{Mor}_\mathcal{C}(x, y)$. The objects of $\mathrm{Mor}_\mathcal{C}(x, y)$ will be called *1-morphisms* and denoted $F : x \to y$. The morphisms between these 1-morphisms will be called *2-morphisms* and denoted $t : F' \to F$. The composition of 2-morphisms in $\mathrm{Mor}_\mathcal{C}(x, y)$ will be called *vertical* composition and will be denoted $t \circ t'$ for $t : F' \to F$ and $t' : F'' \to F'$.
3. For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a functor
$$(\circ, \star) : \mathrm{Mor}_\mathcal{C}(y, z) \times \mathrm{Mor}_\mathcal{C}(x, y) \longrightarrow \mathrm{Mor}_\mathcal{C}(x, z).$$
The image of the pair of 1-morphisms (F, G) on the left hand side will be called the *composition* of F and G, and denoted $F \circ G$. The image of the pair of 2-morphisms (t, s) will be called the *horizontal* composition and denoted $t \star s$.

These data are to satisfy the following rules:

1. The set of objects together with the set of 1-morphisms endowed with composition of 1-morphisms forms a category.
2. Horizontal composition of 2-morphisms is associative.
3. The identity 2-morphism $\mathrm{id}_{\mathrm{id}_x}$ of the identity 1-morphism id_x is a unit for horizontal composition.

This is obviously not a very pleasant type of object to work with. On the other hand, there are lots of examples where it is quite clear how you work with it. The only example we have so far is that of the 2-category whose objects are a given collection of categories, 1-morphisms are functors between these categories, and 2-morphisms are natural transformations of functors, see Section 26. As far as this text is concerned all 2-categories will be sub 2-categories of this example. Here is what it means to be a sub 2-category.

Definition 27.2. Let $\mathcal{C}$ be a 2-category. A *sub 2-category* $\mathcal{C}'$ of $\mathcal{C}$, is given by a subset $\mathrm{Ob}(\mathcal{C}')$ of $\mathrm{Ob}(\mathcal{C})$ and sub categories $\mathrm{Mor}_{\mathcal{C}'}(x, y)$ of the categories $\mathrm{Mor}_\mathcal{C}(x, y)$ for all $x, y \in \mathrm{Ob}(\mathcal{C}')$ such that these, together with the operations $\circ$ (composition 1-morphisms), $\circ$ (vertical composition 2-morphisms), and $\star$ (horizontal composition) form a 2-category.

Remark 27.3. Big 2-categories. In many texts a 2-category is allowed to have a class of objects (but hopefully a "class of classes" is not allowed). We will allow these "big" 2-categories as well, but only in the following list of cases (to be updated as we go along):

(1) The 2-category of categories *Cat*.
(2) The $(2,1)$-category of categories *Cat*.
(3) The 2-category of groupoids *Groupoids*.
(4) The $(2,1)$-category of groupoids *Groupoids*.
(5) The 2-category of fibred categories over a fixed category.
(6) The $(2,1)$-category of fibred categories over a fixed category.

See Definition 28.1. Note that in each case the class of objects of the 2-category $\mathcal{C}$ is a proper class, but for all objects $x, y \in \mathrm{Ob}(C)$ the category $\mathrm{Mor}_{\mathcal{C}}(x, y)$ is "small" (according to our conventions).

The notion of equivalence of categories that we defined in Section 2 extends to the more general setting of 2-categories as follows.

Definition 27.4. Two objects x, y of a 2-category are *equivalent* if there exist 1-morphisms $F : x \to y$ and $G : y \to x$ such that $F \circ G$ is 2-isomorphic to id_y and $G \circ F$ is 2-isomorphic to id_x.

Sometimes we need to say what it means to have a functor from a category into a 2-category.

Definition 27.5. Let $\mathcal{A}$ be a category and let $\mathcal{C}$ be a 2-category.

(1) A *functor* from an ordinary category into a 2-category will ignore the 2-morphisms unless mentioned otherwise. In other words, it will be a "usual" functor into the category formed out of 2-category by forgetting all the 2-morphisms.
(2) A *weak functor*, or a *pseudo functor* φ from $\mathcal{A}$ into the 2-category $\mathcal{C}$ is given by the following data
   (a) a map $\varphi : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{C})$,
   (b) for every pair $x, y \in \mathrm{Ob}(\mathcal{A})$, and every morphism $f : x \to y$ a 1-morphism $\varphi(f) : \varphi(x) \to \varphi(y)$,
   (c) for every $x \in \mathrm{Ob}(A)$ a 2-morphism $\alpha_x : \mathrm{id}_{\varphi(x)} \to \varphi(\mathrm{id}_x)$, and
   (d) for every pair of composable morphisms $f : x \to y$, $g : y \to z$ of $\mathcal{A}$ a 2-morphism $\alpha_{g,f} : \varphi(g \circ f) \to \varphi(g) \circ \varphi(f)$.

   These data are subject to the following conditions:
   (a) the 2-morphisms α_x and $\alpha_{g,f}$ are all isomorphisms,
   (b) for any morphism $f : x \to y$ in $\mathcal{A}$ we have $\alpha_{\mathrm{id}_y, f} = \alpha_y \star \mathrm{id}_{\varphi(f)}$:

$$\begin{tikzcd} \varphi(x) \arrow[r, bend left, "\varphi(f)"] \arrow[r, bend right, "\varphi(f)"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_{\varphi(f)}"] & \varphi(y) \arrow[r, bend left, "\mathrm{id}_y"] \arrow[r, bend right, "\varphi(\mathrm{id}_y)"'] \arrow[r, phantom, "\Downarrow \alpha_y"] & \varphi(y) \end{tikzcd} = \begin{tikzcd} \varphi(x) \arrow[r, bend left, "\varphi(f)"] \arrow[r, bend right, "\varphi(\mathrm{id}_y)\circ\varphi(f)"'] \arrow[r, phantom, "\Downarrow \alpha_{\mathrm{id}_y, f}"] & \varphi(y) \end{tikzcd}$$

   (c) for any morphism $f : x \to y$ in $\mathcal{A}$ we have $\alpha_{f, \mathrm{id}_x} = \mathrm{id}_{\varphi(f)} \star \alpha_x$,
   (d) for any triple of composable morphisms $f : w \to x$, $g : x \to y$, and $h : y \to z$ of $\mathcal{A}$ we have

$$(\mathrm{id}_{\varphi(h)} \star \alpha_{g,f}) \circ \alpha_{h, g \circ f} = (\alpha_{h,g} \star \mathrm{id}_{\varphi(f)}) \circ \alpha_{h \circ g, f}$$

in other words the following diagram with objects 1-morphisms and arrows 2-morphisms commutes

$$\begin{tikzcd}
\varphi(h \circ g \circ f) \arrow[r, "\alpha_{h\circ g, f}"'] \arrow[d, "\alpha_{h, g\circ f}"'] & \varphi(h \circ g) \circ \varphi(f) \arrow[d, "\alpha_{h,g} \star \mathrm{id}_{\varphi(f)}"] \\
\varphi(h) \circ \varphi(g \circ f) \arrow[r, "\mathrm{id}_{\varphi(h)} \star \alpha_{g,f}"] & \varphi(h) \circ \varphi(g) \circ \varphi(f)
\end{tikzcd}$$

Again this is not a very workable notion, but it does sometimes come up. There is a theorem that says that any pseudo-functor is isomorphic to a functor. Finally, there are the notions of *functor between* 2*-categories*, and *pseudo functor between* 2*-categories*. This last notion leads us into 3-category territory. We would like to avoid having to define this at almost any cost!

28. (2, 1)-categories

Some 2-categories have the property that all 2-morphisms are isomorphisms. These will play an important role in the following, and they are easier to work with.

Definition 28.1. A (strict) $(2,1)$*-category* is a 2-category in which all 2-morphisms are isomorphisms.

Example 28.2. The 2-category *Cat*, see Remark 27.3, can be turned into a $(2,1)$-category by only allowing isomorphisms of functors as 2-morphisms.

In fact, more generally any 2-category $\mathcal{C}$ produces a $(2,1)$-category by considering the sub 2-category $\mathcal{C}'$ with the same objects and 1-morphisms but whose 2-morphisms are the invertible 2-morphisms of $\mathcal{C}$. In this situation we will say "*let* $\mathcal{C}'$ *be the* $(2,1)$*-category associated to* $\mathcal{C}$" or similar. For example, the $(2,1)$-category of groupoids means the 2-category whose objects are groupoids, whose 1-morphisms are functors and whose 2-morphisms are isomorphisms of functors. Except that this is a bad example as a transformation between functors between groupoids is automatically an isomorphism!

Remark 28.3. Thus there are variants of the construction of Example 28.2 above where we look at the 2-category of groupoids, or categories fibred in groupoids over a fixed category, or stacks. And so on.

29. 2-fibre products

In this section we introduce 2-fibre products. Suppose that $\mathcal{C}$ is a 2-category. We say that a diagram

$$\begin{tikzcd}
w \arrow[r] \arrow[d] & y \arrow[d] \\
x \arrow[r] & z
\end{tikzcd}$$

2-commutes if the two 1-morphisms $w \to y \to z$ and $w \to x \to z$ are 2-isomorphic. In a 2-category it is more natural to ask for 2-commutativity of diagrams than for actually commuting diagrams. (Indeed, some may say that we should not work with strict 2-categories at all, and in a "weak" 2-category the notion of a commutative diagram of 1-morphisms does not even make sense.) Correspondingly the notion of a fibre product has to be adjusted.

Let $\mathcal{C}$ be a 2-category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_\mathcal{C}(x, z)$ and $g \in \mathrm{Mor}_\mathcal{C}(y, z)$. In order to define the 2-fibre product of f and g we are going to look at 2-commutative diagrams

$$\begin{tikzcd} w \arrow[r, "a"'] \arrow[d, "b"'] & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z. \end{tikzcd}$$

Now in the case of categories, the fibre product is a final object in the category of such diagrams. Correspondingly a 2-fibre product is a final object in a 2-category (see definition below). The *2-category of 2-commutative diagrams* is the 2-category defined as follows:

(1) Objects are quadruples (w, a, b, ϕ) as above where ϕ is an invertible 2-morphism $\phi : f \circ a \to g \circ b$,
(2) 1-morphisms from (w', a', b', ϕ') to (w, a, b, ϕ) are given by $(k : w' \to w, \alpha : a' \to a \circ k, \beta : b' \to b \circ k)$ such that

$$\begin{tikzcd} f \circ a' \arrow[r, "\mathrm{id}_f \star \alpha"'] \arrow[d, "\phi'"'] & f \circ a \circ k \arrow[d, "\phi \star \mathrm{id}_k"] \\ f \circ b' \arrow[r, "\mathrm{id}_f \star \beta"] & f \circ b \circ k \end{tikzcd}$$

is commutative,
(3) given a second 1-morphism $(k', \alpha', \beta') : (w'', a'', b'', \phi'') \to (w', \alpha', \beta', \phi')$ the composition of 1-morphisms is given by the rule

$$(k, \alpha, \beta) \circ (k', \alpha', \beta') = (k \circ k', (\alpha \star \mathrm{id}_{k'}) \circ \alpha', (\beta \star \mathrm{id}_{k'}) \circ \beta'),$$

(4) a 2-morphism between 1-morphisms (k_i, α_i, β_i), $i = 1, 2$ with the same is given by a 2-morphism $\delta : k_1 \to k_2$ such that

$$\begin{tikzcd} a' \arrow[r, "\alpha_1"'] \arrow[dr, "\alpha_2"'] & a \circ k_1 \arrow[d, "\mathrm{id}_a \star \delta"] \\ & a \circ k_2 \end{tikzcd} \qquad \begin{tikzcd} b \circ k_1 \arrow[d, "\mathrm{id}_b \star \delta"'] & b' \arrow[l, "\beta_1"] \arrow[dl, "\beta_2"] \\ b \circ k_2 & \end{tikzcd}$$

commute,
(5) vertical composition of 2-morphisms is given by vertical composition of the morphisms δ in $\mathcal{C}$, and
(6) horizontal composition of the diagram

$$\begin{tikzcd}[column sep=huge] (w'', a'', b'', \phi'') \arrow[r, bend left, "{(k'_1, \alpha'_1, \beta'_1)}"] \arrow[r, bend right, "{(k'_2, \alpha'_2, \beta'_2)}"'] \arrow[r, phantom, "\Downarrow\delta'"] & (w', a', b', \phi') \arrow[r, bend left, "{(k_1, \alpha_1, \beta_1)}"] \arrow[r, bend right, "{(k_2, \alpha_2, \beta_2)}"'] \arrow[r, phantom, "\Downarrow\delta"] & (w, a, b, \phi) \end{tikzcd}$$

is given by the diagram

$$\begin{tikzcd}[column sep=10em] (w'', a'', b'', \phi'') \arrow[r, bend left, "{(k_1 \circ k'_1, (\alpha_1 \star \mathrm{id}_{k'_1}) \circ \alpha'_1, (\beta_1 \star \mathrm{id}_{k'_1}) \circ \beta'_1)}"] \arrow[r, bend right, "{(k_2 \circ k'_2, (\alpha_2 \star \mathrm{id}_{k'_2}) \circ \alpha'_2, (\beta_2 \star \mathrm{id}_{k'_2}) \circ \beta'_2)}"'] \arrow[r, phantom, "\Downarrow\delta \star \delta'"] & (w, a, b, \phi) \end{tikzcd}$$

Note that if $\mathcal{C}$ is actually a $(2,1)$-category, the morphisms α and β in (2) above are automatically also isomorphisms[2]. In addition the 2-category of 2-commutative diagrams is also a $(2,1)$-category if $\mathcal{C}$ is a $(2,1)$-category.

Definition 29.1. A *final object* of a $(2,1)$-category $\mathcal{C}$ is an object x such that

(1) for every $y \in \mathrm{Ob}(\mathcal{C})$ there is a morphism $y \to x$, and
(2) every two morphisms $y \to x$ are isomorphic by a unique 2-morphism.

Likely, in the more general case of 2-categories there are different flavours of final objects. We do not want to get into this and hence we only define 2-fibre products in the $(2,1)$-case.

Definition 29.2. Let $\mathcal{C}$ be a $(2,1)$-category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_\mathcal{C}(x, z)$ and $g \in \mathrm{Mor}_\mathcal{C}(y, z)$. A *2-fibre product of f and g* is a final object in the category of 2-commutative diagrams described above. If a 2-fibre product exists we will denote it $x \times_z y \in \mathrm{Ob}(\mathcal{C})$, and denote the required morphisms $p \in \mathrm{Mor}_\mathcal{C}(x \times_z y, x)$ and $q \in \mathrm{Mor}_\mathcal{C}(x \times_z y, y)$ making the diagram

$$\begin{tikzcd} x \times_z y \arrow[r, "p"] \arrow[d, "q"'] & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z \end{tikzcd}$$

2-commute and we will denote the given invertible 2-morphism exhibiting this by $\psi : f \circ p \to g \circ q$.

Thus the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $a \in \mathrm{Mor}_\mathcal{C}(w, x)$ and $b \in \mathrm{Mor}_\mathcal{C}(w, y)$ with a given 2-isomorphism $\phi : f \circ a \to g \circ b$ there is a $\gamma \in \mathrm{Mor}_\mathcal{C}(w, x \times_z y)$ making the diagram

$$\begin{tikzcd} w \arrow[rrd, "a"] \arrow[rd, dashed, "\gamma"] \arrow[rdd, "b"'] & & \\ & x \times_z y \arrow[r, "p"'] \arrow[d, "q"'] & x \arrow[d, "f"] \\ & y \arrow[r, "g"] & z \end{tikzcd}$$

2-commute such that for suitable choices of $a \to p \circ \gamma$ and $b \to q \circ \gamma$ the diagram

$$\begin{tikzcd} f \circ a \arrow[r] \arrow[d, "\phi"'] & f \circ p \circ \gamma \arrow[d, "\psi \star \mathrm{id}_\gamma"] \\ g \circ b \arrow[r] & g \circ q \circ \gamma \end{tikzcd}$$

commutes. Moreover γ is unique up to isomorphism. Of course the exact properties are finer than this. All of the cases of 2-fibre products that we will need later on come from the following example of 2-fibre products in the 2-category of categories.

Example 29.3. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. We define a category $\mathcal{A} \times_\mathcal{C} \mathcal{B}$ as follows:

[2] In fact it seems in the 2-category case that one could define another 2-category of 2-commutative diagrams where the direction of the arrows α, β is reversed, or even where the direction of only one of them is reversed. This is why we restrict to $(2,1)$-categories later on.

(1) an object of $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is a triple (A, B, f), where $A \in \mathrm{Ob}(\mathcal{A})$, $B \in \mathrm{Ob}(\mathcal{B})$, and $f : F(A) \to G(B)$ is an isomorphism in $\mathcal{C}$,
(2) a morphism $(A, B, f) \to (A', B', f')$ is given by a pair (a, b), where $a : A \to A'$ is a morphism in $\mathcal{A}$, and $b : B \to B'$ is a morphism in $\mathcal{B}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "f"] \arrow[d, "F(a)"] & G(B) \arrow[d, "G(b)"] \\ F(A') \arrow[r, "f'"] & G(B') \end{tikzcd}$$

is commutative.

Moreover, we define functors $p : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{A}$ and $q : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{B}$ by setting

$$p(A, B, f) = A, \quad q(A, B, f) = B,$$

in other words, these are the forgetful functors. We define a transformation of functors $\psi : F \circ p \to G \circ q$. On the object $\xi = (A, B, f)$ it is given by $\psi_\xi = f : F(p(\xi)) = F(A) \to G(B) = G(q(\xi))$.

Lemma 29.4. *In the $(2, 1)$-category of categories 2-fibre products exist and are given by the construction of Example 29.3.*

Proof. Let us check the universal property: let $\mathcal{W}$ be a category, let $a : \mathcal{W} \to \mathcal{A}$ and $b : \mathcal{W} \to \mathcal{B}$ be functors, and let $t : F \circ a \to G \circ b$ be an isomorphism of functors.

Consider the functor $\gamma : \mathcal{W} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ given by $W \mapsto (a(W), b(W), t_W)$. (Check this is a functor omitted.) Moreover, consider $\alpha : a \to p \circ \gamma$ and $\beta : b \to q \circ \gamma$ obtained from the identities $p \circ \gamma = a$ and $q \circ \gamma = b$. Then it is clear that (γ, α, β) is a morphism from (W, a, b, t) to $(\mathcal{A} \times_{\mathcal{C}} \mathcal{B}, p, q, \psi)$.

Let $(k, \alpha', \beta') : (W, a, b, t) \to (\mathcal{A} \times_{\mathcal{C}} \mathcal{B}, p, q, \psi)$ be a second such morphism. For an object W of $\mathcal{W}$ let us write $k(W) = (a_k(W), b_k(W), t_{k,W})$. Hence $p(k(W)) = a_k(W)$ and so on. The map α' corresponds to functorial maps $\alpha' : a(W) \to a_k(W)$. Since we are working in the $(2, 1)$-category of categories, in fact each of the maps $a(W) \to a_k(W)$ is an isomorphism. We can use these (and their counterparts $b(W) \to b_k(W)$) to get isomorphisms

$$\delta_W : \gamma(W) = (a(W), b(W), t_W) \longrightarrow (a_k(W), b_k(W), t_{k,W}) = k(W).$$

It is straightforward to show that δ defines a 2-isomorphism between γ and k in the 2-category of 2-commutative diagrams as desired. □

Remark 29.5. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. Another, slightly more symmetrical, construction of a 2-fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is as follows. An object is a quintuple (A, B, C, a, b) where A, B, C are objects of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and where $a : F(A) \to C$ and $b : G(B) \to C$ are isomorphisms. A morphism $(A, B, C, a, b) \to (A', B', C', a', b')$ is given by a triple of morphisms $A \to A', B \to B', C \to C'$ compatible with the morphisms a, b, a', b'. We can prove directly that this leads to a 2-fibre product. However, it is easier to observe that the functor $(A, B, C, a, b) \mapsto (A, B, b^{-1} \circ a)$ gives an equivalence from the category of quintuples to the category constructed in Example 29.3.

Lemma 29.6. *Let*

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "I"] \arrow[rd, "K"] & \\ \mathcal{X} \arrow[r, "H"] \arrow[rd, "L"] & \mathcal{Z} \arrow[rd, "M"] & \mathcal{B} \arrow[d, "G"] \\ & \mathcal{A} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

be a 2-commutative diagram of categories. A choice of isomorphisms $\alpha : G \circ K \to M \circ I$ and $\beta : M \circ H \to F \circ L$ determines a morphism

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$$

of 2-fibre products associated to this situation.

Proof. Just use the functor

$$(X, Y, \phi) \longmapsto (L(X), K(Y), \alpha_Y^{-1} \circ M(\phi) \circ \beta_X^{-1})$$

on objects and

$$(a, b) \longmapsto (L(a), K(b))$$

on morphisms. □

Lemma 29.7. *Assumptions as in Lemma 29.6.*

1. *If K and L are faithful then the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is faithful.*
2. *If K and L are fully faithful and M is faithful then the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is fully faithful.*
3. *If K and L are equivalences and M is fully faithful then the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is an equivalence.*

Proof. Let (X, Y, ϕ) and (X', Y', ϕ') be objects of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. Set $Z = H(X)$ and identify it with $I(Y)$ via ϕ. Also, identify $M(Z)$ with $F(L(X))$ via α_X and identify $M(Z)$ with $G(K(Y))$ via β_Y. Similarly for $Z' = H(X')$ and $M(Z')$. The map on morphisms is the map

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{X}}(X, X') \times_{\mathrm{Mor}_{\mathcal{Z}}(Z, Z')} \mathrm{Mor}_{\mathcal{Y}}(Y, Y') \arrow[d] \\ \mathrm{Mor}_{\mathcal{A}}(L(X), L(X')) \times_{\mathrm{Mor}_{\mathcal{C}}(M(Z), M(Z'))} \mathrm{Mor}_{\mathcal{B}}(K(Y), K(Y')) \end{tikzcd}$$

Hence parts (1) and (2) follow. Moreover, if K and L are equivalences and M is fully faithful, then any object (A, B, ϕ) is in the essential image for the following reasons: Pick X, Y such that $L(X) \cong A$ and $K(Y) \cong B$. Then the fully faithfulness of M guarantees that we can find an isomorphism $H(X) \cong I(Y)$. Some details omitted. □

Lemma 29.8. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[rd] & & \mathcal{C} \arrow[ld] \arrow[rd] & & \mathcal{E} \arrow[ld] \\ & \mathcal{B} & & \mathcal{D} & \end{tikzcd}$$

be a diagram of categories and functors. Then there is a canonical isomorphism

$$(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \times_{\mathcal{D}} \mathcal{E} \cong \mathcal{A} \times_{\mathcal{B}} (\mathcal{C} \times_{\mathcal{D}} \mathcal{E})$$

of categories.

Proof. Just use the functor

$$((A, C, \phi), E, \psi) \longmapsto (A, (C, E, \psi), \phi)$$

if you know what I mean. □

Henceforth we do not write the parentheses when dealing with fibred products of more than 2 categories.

Lemma 29.9. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[dr] & & \mathcal{C} \arrow[dl] \arrow[dr] & & \mathcal{E} \arrow[dl] \\ & \mathcal{B} \arrow[dr, "F"'] & & \mathcal{D} \arrow[dl, "G"] & \\ & & \mathcal{F} & & \end{tikzcd}$$

be a commutative diagram of categories and functors. Then there is a canonical functor

$$pr_{02} : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \longrightarrow \mathcal{A} \times_{\mathcal{F}} \mathcal{E}$$

of categories.

Proof. If we write $\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ as $(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \times_{\mathcal{D}} \mathcal{E}$ then we can just use the functor

$$((A, C, \phi), E, \psi) \longmapsto (A, E, G(\psi) \circ F(\phi))$$

if you know what I mean. □

Lemma 29.10. *Let*

$$\mathcal{A} \to \mathcal{B} \leftarrow \mathcal{C} \leftarrow \mathcal{D}$$

be a diagram of categories and functors. Then there is a canonical isomorphism

$$\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{C}} \mathcal{D} \cong \mathcal{A} \times_{\mathcal{B}} \mathcal{D}$$

of categories.

Proof. Omitted. □

We claim that this means you can work with these 2-fibre products just like with ordinary fibre products. Here are some further lemmas that actually come up later.

Lemma 29.11. *Let*

$$\begin{tikzcd} \mathcal{C}_3 \arrow[r] \arrow[d] & \mathcal{S} \arrow[d, "\Delta"] \\ \mathcal{C}_1 \times \mathcal{C}_2 \arrow[r, "G_1 \times G_2"] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

be a 2-fibre product of categories. Then there is a canonical isomorphism $\mathcal{C}_3 \cong \mathcal{C}_1 \times_{G_1, \mathcal{S}, G_2} \mathcal{C}_2$.

Proof. We may assume that $\mathcal{C}_3$ is the category $(\mathcal{C}_1 \times \mathcal{C}_2) \times_{\mathcal{S} \times \mathcal{S}} \mathcal{S}$ constructed in Example 29.3. Hence an object is a triple $((X_1, X_2), S, \phi)$ where $\phi = (\phi_1, \phi_2) : (G_1(X_1), G_2(X_2)) \to (S, S)$ is an isomorphism. Thus we can associate to this the triple $(X_1, X_2, \phi_2 \circ \phi_1^{-1})$. Conversely, if (X_1, X_2, ψ) is an object of $\mathcal{C}_1 \times_{G_1, \mathcal{S}, G_2} \mathcal{C}_2$, then we can associate to this the triple $((X_1, X_2), G_1(X_1), (\mathrm{id}_{G_1(X_1)}, \psi))$. We claim these constructions given mutually inverse functors. We omit describing how to deal with morphisms and show they are mutually inverse. □

Lemma 29.12. *Let*

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{S} \arrow[d, "\Delta"] \\ \mathcal{C} \arrow[r, "G_1 \times G_2"] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

be a 2-fibre product of categories. Then there is a canonical isomorphism

$$\mathcal{C}' \cong (\mathcal{C} \times_{G_1, \mathcal{S}, G_2} \mathcal{C}) \times_{(p,q), \mathcal{C} \times \mathcal{C}, \Delta} \mathcal{C}.$$

Proof. An object of the right hand side is given by $((C_1, C_2, \phi), C_3, \psi)$ where $\phi : G_1(C_1) \to G_2(C_2)$ is an isomorphism and $\psi = (\psi_1, \psi_2) : (C_1, C_2) \to (C_3, C_3)$ is an isomorphism. Hence we can associate to this the triple $(C_3, G_1(C_1), (G_1(\psi_1^{-1}), \varphi^{-1} \circ G_2(\psi_2^{-1})))$ which is an object of $\mathcal{C}'$. Details omitted. □

Lemma 29.13. *Let $\mathcal{A} \to \mathcal{C}$, $\mathcal{B} \to \mathcal{C}$ and $\mathcal{C} \to \mathcal{D}$ be functors between categories. Then the diagram*

$$\begin{tikzcd} \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{A} \times_{\mathcal{D}} \mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r, "\Delta_{\mathcal{C}/\mathcal{D}}"] & \mathcal{C} \times_{\mathcal{D}} \mathcal{C} \end{tikzcd}$$

is a 2-fibre product diagram.

Proof. Omitted. □

Lemma 29.14. *Let*

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a 2-fibre product. Then the diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{tikzcd}$$

is 2-cartesian.

Proof. This is a purely 2-category theoretic statement, valid in any $(2,1)$-category with 2-fibre products. Explicitly, it follows from the following chain of equivalences:

$$\begin{aligned} \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} (\mathcal{U} \times_{\mathcal{V}} \mathcal{U}) &= \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} ((\mathcal{X} \times_{\mathcal{Y}} \mathcal{V}) \times_{\mathcal{V}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{V})) \\ &= \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \times_{\mathcal{Y}} \mathcal{V}) \\ &= \mathcal{X} \times_{\mathcal{Y}} \mathcal{V} = \mathcal{U} \end{aligned}$$

see Lemmas 29.8 and 29.10. □

30. Categories over categories

In this section we have a functor $p : \mathcal{S} \to \mathcal{C}$. We think of $\mathcal{S}$ as being on top and of $\mathcal{C}$ as being at the bottom. To make sure that everybody knows what we are talking about we define the 2-category of categories over $\mathcal{C}$.

Definition 30.1. Let $\mathcal{C}$ be a category. The *2-category of categories over* $\mathcal{C}$ is the sub 2-category of *Cat* defined as follows:

(1) Its objects will be functors $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

In this situation we will denote

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

Since we have defined this as a sub 2-category of *Cat* we do not have to check any of the axioms. Rather we just have to check things such as "vertical composition of 2-morphisms over $\mathcal{C}$ gives another 2-morphism over $\mathcal{C}$". This is clear.

Analogously to the fibre of a map of spaces, we have the notion of a fibre category, and some notions of lifting associated to this situation.

Definition 30.2. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$.

(1) The *fibre category* over an object $U \in \mathrm{Ob}(\mathcal{C})$ is the category $\mathcal{S}_U$ with objects
$$\mathrm{Ob}(\mathcal{S}_U) = \{x \in \mathrm{Ob}(\mathcal{S}) : p(x) = U\}$$
and morphisms
$$\mathrm{Mor}_{\mathcal{S}_U}(x, y) = \{\phi \in \mathrm{Mor}_{\mathcal{S}}(x, y) : p(\phi) = \mathrm{id}_U\}.$$
(2) A *lift* of an object $U \in \mathrm{Ob}(\mathcal{C})$ is an object $x \in \mathrm{Ob}(\mathcal{S})$ such that $p(x) = U$, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. We will also sometime say that x *lies over* U.
(3) Similarly, a *lift* of a morphism $f : V \to U$ in $\mathcal{C}$ is a morphism $\phi : y \to x$ in $\mathcal{S}$ such that $p(\phi) = f$. We sometimes say that ϕ *lies over* f.

There are some observations we could make here. For example if $F : (\mathcal{S}, p) \to (\mathcal{S}', p')$ is a 1-morphism of categories over $\mathcal{C}$, then F induces functors of fibre categories $F : \mathcal{S}_U \to \mathcal{S}'_U$. Similarly for 2-morphisms.

Here is the obligatory lemma describing the 2-fibre product in the $(2, 1)$-category of categories over $\mathcal{C}$.

Lemma 30.3. *Let $\mathcal{C}$ be a category. The $(2, 1)$-category of categories over $\mathcal{C}$ has 2-fibre products. Suppose that $F : \mathcal{X} \to \mathcal{S}$ and $G : \mathcal{Y} \to \mathcal{S}$ are morphisms of categories over $\mathcal{C}$. An explicit 2-fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by the following description*

(1) *an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is a quadruple (U, x, y, f), where $U \in \mathrm{Ob}(\mathcal{C})$, $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$, and $f : F(x) \to G(y)$ is an isomorphism in $\mathcal{S}_U$,*
(2) *a morphism $(U, x, y, f) \to (U', x', y', f')$ is given by a pair (a, b), where $a : x \to x'$ is a morphism in $\mathcal{X}$, and $b : y \to y'$ is a morphism in $\mathcal{Y}$ such that*

(a) *a and b induce the same morphism $U \to U'$, and*
(b) *the diagram*

$$\begin{tikzcd} F(x) \arrow[r, "f"] \arrow[d, "F(a)"] & G(y) \arrow[d, "G(b)"] \\ F(x') \arrow[r, "f'"] & G(y') \end{tikzcd}$$

is commutative.

The functors $p : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{X}$ and $q : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{Y}$ are the forgetful functors in this case. The transformation $\psi : F \circ p \to G \circ q$ is given on the object $\xi = (U, x, y, f)$ by $\psi_\xi = f : F(p(\xi)) = F(x) \to G(y) = G(q(\xi))$.

Proof. Let us check the universal property: let $p_{\mathcal{W}} : \mathcal{W} \to \mathcal{C}$ be a category over $\mathcal{C}$, let $X : \mathcal{W} \to \mathcal{X}$ and $Y : \mathcal{W} \to \mathcal{Y}$ be functors over $\mathcal{C}$, and let $t : F \circ X \to G \circ Y$ be an isomorphism of functors over $\mathcal{C}$. The desired functor $\gamma : \mathcal{W} \to \mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by $W \mapsto (p_{\mathcal{W}}(W), X(W), Y(W), t_W)$. Details omitted; compare with Lemma 29.4. □

Lemma 30.4. *Let $\mathcal{C}$ be a category. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be morphisms of categories over $\mathcal{C}$. For any object U of $\mathcal{C}$ we have the following identity of fibre categories*

$$(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})_U = \mathcal{X}_U \times_{\mathcal{S}_U} \mathcal{Y}_U$$

Proof. Omitted. □

31. Fibred categories

A very brief discussion of fibred categories is warranted.

Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Given an object $x \in \mathcal{S}$ with $p(x) = U$, and given a morphism $f : V \to U$, we can try to take some kind of "fibre product $V \times_U x$" (or a *base change* of x via $V \to U$). Namely, a morphism from an object $z \in \mathcal{S}$ into "$V \times_U x$" should be given by a pair (φ, g), where $\varphi : z \to x$, $g : p(z) \to V$ such that $p(\varphi) = f \circ g$. Pictorially:

$$\begin{tikzcd} z \arrow[r, dash] \arrow[d, squiggly, "p"] & ? \arrow[r] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\ p(z) \arrow[r] & V \arrow[r, "f"] & U \end{tikzcd}$$

If such a morphism $V \times_U x \to x$ exists then it is called a strongly cartesian morphism.

Definition 31.1. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. A *strongly cartesian morphism*, or more precisely a *strongly $\mathcal{C}$-cartesian morphism* is a morphism $\varphi : y \to x$ of $\mathcal{S}$ such that for every $z \in \mathrm{Ob}(\mathcal{S})$ the map

$$\mathrm{Mor}_{\mathcal{S}}(z, y) \longrightarrow \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(z), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(z), p(y)),$$

given by $\psi \longmapsto (\varphi \circ \psi, p(\psi))$ is bijective.

Note that by the Yoneda Lemma 3.5, given $x \in \mathrm{Ob}(\mathcal{S})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ and the morphism $f : V \to U$ of $\mathcal{C}$, if there is a strongly cartesian morphism $\varphi : y \to x$

with $p(\varphi) = f$, then (y, φ) is unique up to unique isomorphism. This is clear from the definition above, as the functor

$$z \longmapsto \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(z), U)} \mathrm{Mor}_{\mathcal{C}}(p(z), V)$$

only depends on the data $(x, U, f : V \to U)$. Hence we will sometimes use $V \times_U x \to x$ or $f^*x \to x$ to denote a strongly cartesian morphism which is a lift of f.

Lemma 31.2. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$.*

(1) *The composition of two strongly cartesian morphisms is strongly cartesian.*
(2) *Any isomorphism of $\mathcal{S}$ is strongly cartesian.*
(3) *Any strongly cartesian morphism φ such that $p(\varphi)$ is an isomorphism, is an isomorphism.*

Proof. Proof of (1). Let $\varphi : y \to x$ and $\psi : z \to y$ be strongly cartesian. Let t be an arbitrary object of $\mathcal{S}$. Then we have

$$\begin{aligned}
&\mathrm{Mor}_{\mathcal{S}}(t, z) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, y) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z)) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(y)) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z)) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z))
\end{aligned}$$

hence $z \to x$ is strongly cartesian.

Proof of (2). Let $y \to x$ be an isomorphism. Then $p(y) \to p(x)$ is an isomorphism too. Hence $\mathrm{Mor}_{\mathcal{C}}(p(z), p(y)) \to \mathrm{Mor}_{\mathcal{C}}(p(z), p(x))$ is a bijection. Hence $\mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(z), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(z), p(y))$ is bijective to $\mathrm{Mor}_{\mathcal{S}}(z, x)$. Hence the displayed map of Definition 31.1 is a bijection as $y \to x$ is an isomorphism, and we conclude that $y \to x$ is strongly cartesian.

Proof of (3). Assume $\varphi : y \to x$ is strongly cartesian with $p(\varphi) : p(y) \to p(x)$ an isomorphism. Applying the definition with $z = x$ shows that $(\mathrm{id}_x, p(\varphi)^{-1})$ comes from a unique morphism $\chi : x \to y$. We omit the verification that χ is the inverse of φ. $\square$

Lemma 31.3. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be composable functors between categories. Let $x \to y$ be a morphism of $\mathcal{A}$. If $x \to y$ is strongly $\mathcal{B}$-cartesian and $F(x) \to F(y)$ is strongly $\mathcal{C}$-cartesian, then $x \to y$ is strongly $\mathcal{C}$-cartesian.*

Proof. This follows directly from the definition. $\square$

Lemma 31.4. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$. Assume*

(1) *$x \to y$ is strongly cartesian,*
(2) *$p(x) \times_{p(y)} p(z)$ exists, and*
(3) *there exists a strongly cartesian morphism $a : w \to z$ in $\mathcal{S}$ with $p(w) = p(x) \times_{p(y)} p(z)$ and $p(a) = pr_2 : p(x) \times_{p(y)} p(z) \to p(z)$.*

Then the fibre product $x \times_y z$ exists and is isomorphic to w.

Proof. Since $x \to y$ is strongly cartesian there exists a unique morphism $b : w \to x$ such that $p(b) = \text{pr}_1$. To see that w is the fibre product we compute

$$\begin{aligned}
&\text{Mor}_\mathcal{S}(t, w) \\
&= \text{Mor}_\mathcal{S}(t, z) \times_{\text{Mor}_\mathcal{C}(p(t), p(z))} \text{Mor}_\mathcal{C}(p(t), p(w)) \\
&= \text{Mor}_\mathcal{S}(t, z) \times_{\text{Mor}_\mathcal{C}(p(t), p(z))} (\text{Mor}_\mathcal{C}(p(t), p(x)) \times_{\text{Mor}_\mathcal{C}(p(t), p(y))} \text{Mor}_\mathcal{C}(p(t), p(z))) \\
&= \text{Mor}_\mathcal{S}(t, z) \times_{\text{Mor}_\mathcal{C}(p(t), p(y))} \text{Mor}_\mathcal{C}(p(t), p(x)) \\
&= \text{Mor}_\mathcal{S}(t, z) \times_{\text{Mor}_\mathcal{S}(t, y)} \text{Mor}_\mathcal{S}(t, y) \times_{\text{Mor}_\mathcal{C}(p(t), p(y))} \text{Mor}_\mathcal{C}(p(t), p(x)) \\
&= \text{Mor}_\mathcal{S}(t, z) \times_{\text{Mor}_\mathcal{S}(t, y)} \text{Mor}_\mathcal{S}(t, x)
\end{aligned}$$

as desired. The first equality holds because $a : w \to z$ is strongly cartesian and the last equality holds because $x \to y$ is strongly cartesian. $\square$

Definition 31.5. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. We say $\mathcal{S}$ is a *fibred category over* $\mathcal{C}$ if given any $x \in \text{Ob}(\mathcal{S})$ lying over $U \in \text{Ob}(\mathcal{C})$ and any morphism $f : V \to U$ of $\mathcal{C}$, there exists a strongly cartesian morphism $f^*x \to x$ lying over f.

Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. For every $f : V \to U$ and $x \in \text{Ob}(\mathcal{S}_U)$ as in the definition we may choose a strongly cartesian morphism $f^*x \to x$ lying over f. By the axiom of choice we may choose $f^*x \to x$ for all $f : V \to U = p(x)$ simultaneously. We claim that for every morphism $\phi : x \to x'$ in $\mathcal{S}_U$ and $f : V \to U$ there is a unique morphism $f^*\phi : f^*x \to f^*x'$ in $\mathcal{S}_V$ such that

$$\begin{tikzcd}
f^*x \arrow[r, "f^*\phi"'] \arrow[d] & f^*x' \arrow[d] \\
x \arrow[r, "\phi"] & x'
\end{tikzcd}$$

commutes. Namely, the arrow exists and is unique because $f^*x' \to x'$ is strongly cartesian. The uniqueness of this arrow guarantees that f^* (now also defined on morphisms) is a functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$.

Definition 31.6. Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category.

(1) A *choice of pullbacks*[3] for $p : \mathcal{S} \to \mathcal{C}$ is given by a choice of a strongly cartesian morphism $f^*x \to x$ lying over f for any morphism $f : V \to U$ of $\mathcal{C}$ and any $x \in \text{Ob}(\mathcal{S}_U)$.
(2) Given a choice of pullbacks, for any morphism $f : V \to U$ of $\mathcal{C}$ the functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$ described above is called a *pullback functor* (associated to the choices $f^*x \to x$ made above).

Of course we may always assume our choice of pullbacks has the property that $\text{id}_U^*x = x$, although in practice this is a useless property without imposing further assumptions on the pullbacks.

Lemma 31.7. *Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. Assume given a choice of pullbacks for $p : \mathcal{S} \to \mathcal{C}$.*

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[3]This is probably nonstandard terminology. In some texts this is called a "cleavage" but it conjures up the wrong image. Maybe a "cleaving" would be a better word. A related notion is that of a "splitting", but in many texts a "splitting" means a choice of pullbacks such that $g^*f^* = (f \circ g)^*$ for any composable pair of morphisms. Compare also with Definition 34.2.

(1) *For any pair of composable morphisms $f : V \to U$, $g : W \to V$ there is a unique isomorphism*

$$\alpha_{g,f} : (f \circ g)^* \longrightarrow g^* \circ f^*$$

as functors $\mathcal{S}_U \to \mathcal{S}_W$ such that for every $y \in \mathrm{Ob}(\mathcal{S}_U)$ the following diagram commutes

$$\begin{tikzcd} g^*f^*y \arrow[r] & f^*y \arrow[d] \\ (f \circ g)^*y \arrow[u, "(\alpha_{g,f})_y"] \arrow[r] & y \end{tikzcd}$$

(2) *If $f = id_U$, then there is a canonical isomorphism $\alpha_U : id \to (id_U)^*$ as functors $\mathcal{S}_U \to \mathcal{S}_U$.*

(3) *The quadruple $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{g,f}, \alpha_U)$ defines a pseudo functor from $\mathcal{C}^{opp}$ to the $(2,1)$-category of categories, see Definition 27.5.*

Proof. In fact, it is clear that the commutative diagram of part (1) uniquely determines the morphism $(\alpha_{g,f})_y$ in the fibre category $\mathcal{S}_W$. It is an isomorphism since both the morphism $(f \circ g)^*y \to y$ and the composition $g^*f^*y \to f^*y \to y$ are strongly cartesian morphisms lifting $f \circ g$ (see discussion following Definition 31.1 and Lemma 31.2). In the same way, since $\mathrm{id}_x : x \to x$ is clearly strongly cartesian over id_U (with $U = p(x)$) we see that there exists an isomorphism $(\alpha_U)_x : x \to (\mathrm{id}_U)^*x$. (Of course we could have assumed beforehand that $f^*x = x$ whenever f is an identity morphism, but it is better for the sake of generality not to assume this.) We omit the verification that $\alpha_{g,f}$ and α_U so obtained are transformations of functors. We also omit the verification of (3). $\square$

Lemma 31.8. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is fibred over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is fibred over $\mathcal{C}$.*

Proof. Denote $p_i : \mathcal{S}_i \to \mathcal{C}$ the given functors. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be functors over $\mathcal{C}$, and let $i : F \circ G \to \mathrm{id}_{\mathcal{S}_2}$, $j : G \circ F \to \mathrm{id}_{\mathcal{S}_1}$ be isomorphisms of functors over $\mathcal{C}$. We claim that in this case F maps strongly cartesian morphisms to strongly cartesian morphisms. Namely, suppose that $\varphi : y \to x$ is strongly cartesian in $\mathcal{S}_1$. Set $f : V \to U$ equal to $p_1(\varphi)$. Suppose that $z' \in \mathrm{Ob}(\mathcal{S}_2)$, with $W = p_2(z')$, and we are given $g : W \to V$ and $\psi' : z' \to F(x)$ such that $p_2(\psi') = f \circ g$. Then

$$\psi = j \circ G(\psi') : G(z') \to G(F(x)) \to x$$

is a morphism in $\mathcal{S}_1$ with $p_1(\psi) = f \circ g$. Hence by assumption there exists a unique morphism $\xi : G(z') \to y$ lying over g such that $\psi = \varphi \circ \xi$. This in turn gives a morphism

$$\xi' = F(\xi) \circ i^{-1} : z' \to F(G(z')) \to F(y)$$

lying over g with $\psi' = F(\varphi) \circ \xi'$. We omit the verification that ξ' is unique. $\square$

The conclusion from Lemma 31.8 is that equivalences map strongly cartesian morphisms to strongly cartesian morphisms. But this may not be the case for an arbitrary functor between fibred categories over $\mathcal{C}$. Hence we define the 2-category of fibred categories as follows.

Definition 31.9. Let $\mathcal{C}$ be a category. The *2-category of fibred categories over $\mathcal{C}$* is the sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 30.1) defined as follows:

(1) Its objects will be fibred categories $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

In this situation we will denote

$$\mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

Note the condition on 1-morphisms. Note also that this is a true 2-category and not a $(2,1)$-category. Hence when taking 2-fibre products we first pass to the associated $(2,1)$-category.

Lemma 31.10. *Let $\mathcal{C}$ be a category. The $(2,1)$-category of fibred categories over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 30.3.*

Proof. Basically what one has to show here is that given $F : \mathcal{X} \to \mathcal{S}$ and $G : \mathcal{Y} \to \mathcal{S}$ morphisms of fibred categories over $\mathcal{C}$, then the category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ described in Lemma 30.3 is fibred. Let us show that $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ has plenty of strongly cartesian morphisms. Namely, suppose we have (U, x, y, ϕ) an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$. And suppose $f : V \to U$ is a morphism in $\mathcal{C}$. Choose strongly cartesian morphisms $a : f^*x \to x$ in $\mathcal{X}$ lying over f and $b : f^*y \to y$ in $\mathcal{Y}$ lying over f. By assumption $F(a)$ and $G(b)$ are strongly cartesian. Since $\phi : F(x) \to G(y)$ is an isomorphism, by the uniqueness of strongly cartesian morphisms we find a unique isomorphism $f^*\phi : F(f^*x) \to G(f^*y)$ such that $G(b) \circ f^*\phi = \phi \circ F(a)$. In other words $(G(a), G(b)) : (V, f^*x, f^*y, f^*\phi) \to (U, x, y, \phi)$ is a morphism in $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$. We omit the verification that this is a strongly cartesian morphism (and that these are in fact the only strongly cartesian morphisms). □

Lemma 31.11. *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. If $p : \mathcal{S} \to \mathcal{C}$ is a fibred category and p factors through $p' : \mathcal{S} \to \mathcal{C}/U$ then $p' : \mathcal{S} \to \mathcal{C}/U$ is a fibred category.*

Proof. Suppose that $\varphi : x' \to x$ is strongly cartesian with respect to p. We claim that φ is strongly cartesian with respect to p' also. Set $g = p'(\varphi)$, so that $g : V'/U \to V/U$ for some morphisms $f : V \to U$ and $f' : V' \to U$. Let $z \in \mathrm{Ob}(\mathcal{S})$. Set $p'(z) = (W \to U)$. To show that φ is strongly cartesian for p' we have to show

$$\mathrm{Mor}_{\mathcal{S}}(z, x') \longrightarrow \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}/U}(W/U, V/U)} \mathrm{Mor}_{\mathcal{C}/U}(W/U, V'/U),$$

given by $\psi' \longmapsto (\varphi \circ \psi', p'(\psi'))$ is bijective. Suppose given an element (ψ, h) of the right hand side, then in particular $g \circ h = p(\psi)$, and by the condition that φ is strongly cartesian we get a unique morphism $\psi' : z \to x'$ with $\psi = \varphi \circ \psi'$ and $p(\psi') = h$. OK, and now $p'(\psi') : W/U \to V/U$ is a morphism whose corresponding map $W \to V$ is h, hence equal to h as a morphism in $\mathcal{C}/U$. Thus ψ' is a unique morphism $z \to x'$ which maps to the given pair (ψ, h). This proves the claim.

Finally, suppose given $g : V'/U \to V/U$ and x with $p'(x) = V/U$. Since $p : \mathcal{S} \to \mathcal{C}$ is a fibred category we see there exists a strongly cartesian morphism $\varphi : x' \to x$ with

$p(\varphi) = g$. By the same argument as above it follows that $p'(\varphi) = g : V'/U \to V/U$. And as seen above the morphism φ is strongly cartesian. Thus the conditions of Definition 31.5 are satisfied and we win. $\square$

Lemma 31.12. *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be functors between categories. If $\mathcal{A}$ is fibred over $\mathcal{B}$ and $\mathcal{B}$ is fibred over $\mathcal{C}$, then $\mathcal{A}$ is fibred over $\mathcal{C}$.*

Proof. This follows from the definitions and Lemma 31.3. $\square$

Lemma 31.13. *Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$ with $x \to y$ strongly cartesian. If $p(x) \times_{p(y)} p(z)$ exists, then $x \times_y z$ exists, $p(x \times_y z) = p(x) \times_{p(y)} p(z)$, and $x \times_y z \to z$ is strongly cartesian.*

Proof. Pick a strongly cartesian morphism $\mathrm{pr}_2^*z \to z$ lying over $\mathrm{pr}_2 : p(x) \times_{p(y)} p(z) \to p(z)$. Then $\mathrm{pr}_2^*z = x \times_y z$ by Lemma 31.4. $\square$

Lemma 31.14. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of fibred categories over $\mathcal{C}$. There exist 1-morphisms of fibred categories over $\mathcal{C}$*

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left, "u"] & \mathcal{X}' \arrow[l, shift left, "w"] \arrow[r, "v"] & \mathcal{Y} \end{tikzcd}$$

such that $F = v \circ u$ and such that

1. *$u : \mathcal{X} \to \mathcal{X}'$ is fully faithful,*
2. *w is left adjoint to u, and*
3. *$v : \mathcal{X}' \to \mathcal{Y}$ is a fibred category.*

Proof. Denote $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ the structure functors. We construct $\mathcal{X}'$ explicitly as follows. An object of $\mathcal{X}'$ is a quadruple (U, x, y, f) where $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$ and $f : y \to F(x)$ is a morphism in $\mathcal{Y}_U$. A morphism $(a, b) : (U, x, y, f) \to (U', x', y', f')$ is given by $a : x \to x'$ and $b : y \to y'$ with $p(a) = q(b) : U \to U'$ and such that $f' \circ b = F(a) \circ f$.

Let us make a choice of pullbacks for both p and q and let us use the same notation to indicate them. Let (U, x, y, f) be an object and let $h : V \to U$ be a morphism. Consider the morphism $c : (V, h^*x, h^*y, h^*f) \to (U, x, y, f)$ coming from the given strongly cartesian maps $h^*x \to x$ and $h^*y \to y$. We claim c is strongly cartesian in $\mathcal{X}'$ over $\mathcal{C}$. Namely, suppose we are given an object (W, x', y', f') of $\mathcal{X}'$, a morphism $(a, b) : (W, x', y', f') \to (U, x, y, f)$ lying over $W \to U$, and a factorization $W \to V \to U$ of $W \to U$ through h. As $h^*x \to x$ and $h^*y \to y$ are strongly cartesian we obtain morphisms $a' : x' \to h^*x$ and $b' : y' \to h^*y$ lying over the given morphism $W \to V$. Consider the diagram

$$\begin{tikzcd} y' \arrow[r] \arrow[d, "f'"] & h^*y \arrow[r] \arrow[d, "h^*f"] & y \arrow[d, "f"] \\ F(x') \arrow[r] & F(h^*x) \arrow[r] & F(x) \end{tikzcd}$$

The outer rectangle and the right square commute. Since F is a 1-morphism of fibred categories the morphism $F(h^*x) \to F(x)$ is strongly cartesian. Hence the left square commutes by the universal property of strongly cartesian morphisms. This proves that $\mathcal{X}'$ is fibred over $\mathcal{C}$.

The functor $u : \mathcal{X} \to \mathcal{X}'$ is given by $x \mapsto (p(x), x, F(x), \mathrm{id})$. This is fully faithful. The functor $\mathcal{X}' \to \mathcal{Y}$ is given by $(U, x, y, f) \mapsto y$. The functor $w : \mathcal{X}' \to \mathcal{X}$ is given by $(U, x, y, f) \mapsto x$. Each of these functors is a 1-morphism of fibred categories over $\mathcal{C}$ by our description of strongly cartesian morphisms of $\mathcal{X}'$ over $\mathcal{C}$. Adjointness of w and u means that

$$\mathrm{Mor}_{\mathcal{X}}(x, x') = \mathrm{Mor}_{\mathcal{X}'}((U, x, y, f), (p(x'), x', F(x'), \mathrm{id})),$$

which follows immediately from the definitions.

Finally, we have to show that $\mathcal{X}' \to \mathcal{Y}$ is a fibred category. Let $c : y' \to y$ be a morphism in $\mathcal{Y}$ and let (U, x, y, f) be an object of $\mathcal{X}'$ lying over y. Set $V = q(y')$ and let $h = q(c) : V \to U$. Let $a : h^*x \to x$ and $b : h^*y \to y$ be the strongly cartesian morphisms covering h. Since F is a 1-morphism of fibred categories we may identify $h^*F(x) = F(h^*x)$ with strongly cartesian morphism $F(a) : F(h^*x) \to F(x)$. By the universal property of $b : h^*y \to y$ there is a morphism $c' : y' \to h^*y$ in $\mathcal{Y}_V$ such that $c = b \circ c'$. We claim that

$$(a, c) : (V, h^*x, y', h^*f \circ b') \longrightarrow (U, x, y, f)$$

is strongly cartesian in $\mathcal{X}'$ over $\mathcal{Y}$. To see this let (W, x_1, y_1, f_1) be an object of $\mathcal{X}'$, let $(a_1, b_1) : (W, x_1, y_1, f_1) \to (U, x, y, f)$ be a morphism and let $b_1 = c \circ c_1$ for some morphism $c_1 : y_1 \to y'$. Then

$$(a_1', c_1) : (W, x_1, y_1, f_1) \longrightarrow (V, h^*x, y', h^*f \circ b')$$

(where $a_1' : x_1 \to h^*x$ is the unique morphism lying over the given morphism $p(a_1) = q(b_1) : W \to V$ such that $a_1 = a \circ a_1'$) is the desired morphism. $\square$

32. Inertia

Given fibred categories $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ over a category $\mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{S}'$ we have the diagonal morphism

$$\Delta = \Delta_{\mathcal{S}/\mathcal{S}'} : \mathcal{S} \longrightarrow \mathcal{S} \times_{\mathcal{S}'} \mathcal{S}$$

in the $(2,1)$-category of fibred categories over $\mathcal{C}$.

Lemma 32.1. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be fibred categories. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. Consider the category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ over $\mathcal{C}$ whose*

1. *objects are pairs (x, α) where $x \in \mathrm{Ob}(\mathcal{S})$ and $\alpha : x \to x$ is an automorphism with $F(\alpha) = id$,*
2. *morphisms $(x, \alpha) \to (y, \beta)$ are given by morphisms $\phi : x \to y$ such that*

$$\begin{tikzcd} x \arrow[r, "\phi"] \arrow[d, "\alpha"'] & y \arrow[d, "\beta"] \\ x \arrow[r, "\phi"] & y \end{tikzcd}$$

   commutes, and
3. *the functor $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ is given by $(x, \alpha) \mapsto p(x)$.*

Then

1. *there is an equivalence*

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$$

   in the $(2,1)$-category of categories over $\mathcal{C}$, and
2. *$\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ is a fibred category over $\mathcal{C}$.*

Proof. Note that (2) follows from (1) by Lemma 31.10. Thus it suffices to prove (1). We will use without further mention the construction of the 2-fibre product from Lemma 31.10. In particular an object of $\mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$ is a triple $(x, y, (\iota, \kappa))$ where x and y are objects of $\mathcal{S}$, and $(\iota, \kappa) : (x, x, \mathrm{id}_{F(x)}) \to (y, y, \mathrm{id}_{F(y)})$ is an isomorphism in $\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}$. This just means that $\iota, \kappa : x \to y$ are isomorphisms and that $F(\iota) = F(\kappa)$. Consider the functor

$$I_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$$

which to an object (x, α) of the left hand side assigns the object $(x, x, (\alpha, \mathrm{id}_x))$ of the right hand side and to a morphism ϕ of the left hand side assigns the morphism (ϕ, ϕ) of the right hand side. We claim that a quasi-inverse to that morphism is given by the functor

$$\mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S} \longrightarrow I_{\mathcal{S}/\mathcal{S}'}$$

which to an object $(x, y, (\iota, \kappa))$ of the left hand side assigns the object $(x, \kappa^{-1} \circ \iota)$ of the right hand side and to a morphism $(\phi, \phi') : (x, y, (\iota, \kappa)) \to (z, w, (\lambda, \mu))$ of the left hand side assigns the morphism ϕ. Indeed, the endo-functor of $I_{\mathcal{S}/\mathcal{S}'}$ induced by composing the two functors above is the identity on the nose, and the endo-functor induced on $\mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$ is isomorphic to the identity via the natural isomorphism

$$(\iota^{-1} \circ \kappa, \kappa \circ \iota^{-1} \circ \kappa) : (x, x, (\kappa^{-1} \circ \iota, \mathrm{id}_x)) \longrightarrow (x, y, (\iota, \kappa)).$$

Some details omitted. $\square$

Definition 32.2. Let $\mathcal{C}$ be a category.

(1) Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. The *relative inertia of $\mathcal{S}$ over $\mathcal{S}'$* is the fibred category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ of Lemma 32.1.
(2) By the *inertia fibred category $\mathcal{I}_\mathcal{S}$ of $\mathcal{S}$* we mean $\mathcal{I}_\mathcal{S} = \mathcal{I}_{\mathcal{S}/\mathcal{C}}$.

Note that there are canonical 1-morphisms

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \quad \text{and} \quad \mathcal{I}_\mathcal{S} \longrightarrow \mathcal{S} \tag{32.2.1}$$

of fibred categories over $\mathcal{C}$. In terms of the description of Lemma 32.1 these simply map the object (x, α) to the object x and the morphism $\phi : (x, \alpha) \to (y, \beta)$ to the morphism $\phi : x \to y$. There is also a *neutral section*

$$e : \mathcal{S} \to \mathcal{I}_{\mathcal{S}/\mathcal{S}'} \quad \text{and} \quad e : \mathcal{S} \to \mathcal{I}_\mathcal{S} \tag{32.2.2}$$

defined by the rules $x \mapsto (x, \mathrm{id}_x)$ and $(\phi : x \to y) \mapsto \phi$. This is a right inverse to (32.2.1). Given a 2-commutative square

$$\begin{tikzcd} \mathcal{S}_1 \arrow[r, "G"'] \arrow[d, "F_1"'] & \mathcal{S}_2 \arrow[d, "F_2"] \\ \mathcal{S}'_1 \arrow[r, "G'"] & \mathcal{S}'_2 \end{tikzcd}$$

there is a *functoriality map*

$$\mathcal{I}_{\mathcal{S}_1/\mathcal{S}'_1} \longrightarrow \mathcal{I}_{\mathcal{S}_2/\mathcal{S}'_2} \quad \text{and} \quad \mathcal{I}_{\mathcal{S}_1} \longrightarrow \mathcal{I}_{\mathcal{S}_2} \tag{32.2.3}$$

defined by the rules $(x, \alpha) \mapsto (G(x), G(\alpha))$ and $\phi \mapsto G(\phi)$. In particular there is always a comparison map

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{I}_\mathcal{S} \tag{32.2.4}$$

and all the maps above are compatible with this.

Lemma 32.3. *Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of categories fibred over a category $\mathcal{C}$. Then the diagram*

$$\begin{tikzcd} \mathcal{I}_{\mathcal{S}/\mathcal{S}'} \arrow[r, "(32.2.4)"'] \arrow[d, "F\circ(32.2.1)"'] & \mathcal{I}_{\mathcal{S}} \arrow[d, "(32.2.3)"] \\ \mathcal{S}' \arrow[r, "e"] & \mathcal{I}_{\mathcal{S}'} \end{tikzcd}$$

is a 2-fibre product.

Proof. Omitted. □

33. Categories fibred in groupoids

In this section we explain how to think about categories in groupoids and we see how they are basically the same as functors with values in the $(2,1)$-category of groupoids.

Definition 33.1. We say that $\mathcal{S}$ is *fibred in groupoids* over $\mathcal{C}$ if the following two conditions hold:

(1) For every morphism $f : V \to U$ in $\mathcal{C}$ and every lift x of U there is a lift $\phi : y \to x$ of f with target x.
(2) For every pair of morphisms $\phi : y \to x$ and $\psi : z \to x$ and any morphism $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique lift $\chi : z \to y$ of f such that $\phi \circ \chi = \psi$.

Condition (2) phrased differently says that applying the functor p gives a bijection between the sets of dotted arrows in the following commutative diagram below:

$$\begin{tikzcd} y \arrow[r] & x \\ z \arrow[u, dashed] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} p(y) \arrow[r] & p(x) \\ p(z) \arrow[u, dashed] \arrow[ur] & \end{tikzcd}$$

Another way to think about the second condition is the following. Suppose that $g : W \to V$ and $f : V \to U$ are morphisms in $\mathcal{C}$. Let $x \in \mathrm{Ob}(\mathcal{S}_U)$. By the first condition we can lift f to $\phi : y \to x$ and then we can lift g to $\psi : z \to y$. Instead of doing this two step process we can directly lift $g \circ f$ to $\gamma : z' \to x$. This gives the solid arrows in the diagram

(33.1.1)
$$\begin{tikzcd} z' \arrow[d, dashed, <->] \arrow[rrd, "\gamma"] & & \\ z \arrow[r, "\psi"] \arrow[d, squiggly, "p"] & y \arrow[r, "\phi"] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\ W \arrow[r, "g"] & V \arrow[r, "f"] & U \end{tikzcd}$$

where the squiggly arrows represent not morphisms but the functor p. Applying the second condition to the arrows $\phi \circ \psi$, γ and id_W we conclude that there is a unique morphism $\chi : z \to z'$ in $\mathcal{S}_W$ such that $\gamma \circ \chi = \phi \circ \psi$. Similarly there is a unique morphism $z' \to z$. The uniqueness implies that the morphisms $z' \to z$ and $z \to z'$ are mutually inverse, in other words isomorphisms.

It should be clear from this discussion that a category fibred in groupoids is very closely related to a fibred category. Here is the result.

Lemma 33.2. *Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. The following are equivalent*

(1) *$p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids, and*
(2) *all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$.*

Moreover, in this case every morphism of $\mathcal{S}$ is strongly cartesian. In addition, given $f^*x \to x$ lying over f for all $f : V \to U = p(x)$ the data $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{f,g}, \alpha_U)$ constructed in Lemma 31.7 defines a pseudo functor from $\mathcal{C}^{opp}$ in to the $(2,1)$-category of groupoids.

Proof. Assume $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids. To show all fibre categories $\mathcal{S}_U$ for $U \in \mathrm{Ob}(\mathcal{C})$ are groupoids, we must exhibit for every $f : y \to x$ in $\mathcal{S}_U$ an inverse morphism. The diagram on the left (in $\mathcal{S}_U$) is mapped by p to the diagram on the right:

$$\begin{tikzcd} y \arrow[r, "f"] & x \\ x \arrow[u, dashed] \arrow[ur, "\mathrm{id}_x"'] \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\mathrm{id}_U"] & U \\ U \arrow[u, dashed] \arrow[ur, "\mathrm{id}_U"'] \end{tikzcd}$$

Since only id_U makes the diagram on the right commute, there is a unique $g : x \to y$ making the diagram on the left commute, so $fg = \mathrm{id}_x$. By a similar argument there is a unique $h : y \to x$ so that $gh = \mathrm{id}_y$. Then $fgh = f : y \to x$. We have $fg = \mathrm{id}_x$, so $h = f$. Condition (2) of Definition 33.1 says exactly that every morphism of $\mathcal{S}$ is strongly cartesian. Hence condition (1) of Definition 33.1 implies that $\mathcal{S}$ is a fibred category over $\mathcal{C}$.

Conversely, assume all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$. We have to check conditions (1) and (2) of Definition 33.1. The first condition follows trivially. Let $\phi : y \to x$, $\psi : z \to x$ and $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ be as in condition (2) of Definition 33.1. Write $U = p(x)$, $V = p(y)$, $W = p(z)$, $p(\phi) = g : V \to U$, $p(\psi) = h : W \to U$. Choose a strongly cartesian $g^*x \to x$ lying over g. Then we get a morphism $i : y \to g^*x$ in $\mathcal{S}_V$, which is therefore an isomorphism. We also get a morphism $j : z \to g^*x$ corresponding to the pair (ψ, f) as $g^*x \to x$ is strongly cartesian. Then one checks that $\chi = i^{-1} \circ j$ is a solution.

We have seen in the proof of (1) $\Rightarrow$ (2) that every morphism of $\mathcal{S}$ is strongly cartesian. The final statement follows directly from Lemma 31.7. □

Lemma 33.3. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $\mathcal{S}'$ be the subcategory of $\mathcal{S}$ defined as follows*

(1) $\mathrm{Ob}(\mathcal{S}') = \mathrm{Ob}(\mathcal{S})$, *and*
(2) *for $x, y \in \mathrm{Ob}(\mathcal{S}')$ the set of morphisms between x and y in $\mathcal{S}'$ is the set of strongly cartesian morphisms between x and y in $\mathcal{S}$.*

Let $p' : \mathcal{S}' \to \mathcal{C}$ be the restriction of p to $\mathcal{S}'$. Then $p' : \mathcal{S}' \to \mathcal{C}$ is fibred in groupoids.

Proof. Note that the construction makes sense since by Lemma 31.2 the identity morphism of any object of $\mathcal{S}$ is strongly cartesian, and the composition of strongly cartesian morphisms is strongly cartesian. The first lifting property of Definition 33.1 follows from the condition that in a fibred category given any morphism f :

$V \to U$ and x lying over U there exists a strongly cartesian morphism $\varphi : y \to x$ lying over f. Let us check the second lifting property of Definition 33.1 for the category $p' : \mathcal{S}' \to \mathcal{C}$ over $\mathcal{C}$. To do this we argue as in the discussion following Definition 33.1. Thus in Diagram 33.1.1 the morphisms ϕ, ψ and γ are strongly cartesian morphisms of $\mathcal{S}$. Hence γ and $\phi \circ \psi$ are strongly cartesian morphisms of $\mathcal{S}$ lying over the same arrow of $\mathcal{C}$ and having the same target in $\mathcal{S}$. By the discussion following Definition 31.1 this means these two arrows are isomorphic as desired (here we use also that any isomorphism in $\mathcal{S}$ is strongly cartesian, by Lemma 31.2 again). □

Example 33.4. A homomorphism of groups $p : G \to H$ gives rise to a functor $p : \mathcal{S} \to \mathcal{C}$ as in Example 2.12. This functor $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids if and only if p is surjective. The fibre category $\mathcal{S}_U$ over the (unique) object $U \in \mathrm{Ob}(\mathcal{C})$ is the category associated to the kernel of p as in Example 2.6.

Given $p : \mathcal{S} \to \mathcal{C}$, we can ask: if the fibre category $\mathcal{S}_U$ is a groupoid for all $U \in \mathrm{Ob}(\mathcal{C})$, must $\mathcal{S}$ be fibred in groupoids over $\mathcal{C}$? We can see the answer is no as follows. Start with a category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$. Altering the morphisms in $\mathcal{S}$ which do not map to the identity morphism on some object does not alter the categories $\mathcal{S}_U$. Hence we can violate the existence and uniqueness conditions on lifts. One example is the functor from Example 33.4 when $G \to H$ is not surjective. Here is another example.

Example 33.5. Let $\mathrm{Ob}(\mathcal{C}) = \{A, B, T\}$ and $\mathrm{Mor}_{\mathcal{C}}(A, B) = \{f\}$, $\mathrm{Mor}_{\mathcal{C}}(B, T) = \{g\}$, $\mathrm{Mor}_{\mathcal{C}}(A, T) = \{h\} = \{gf\}$, plus the identity morphism for each object. See the diagram below for a picture of this category. Now let $\mathrm{Ob}(\mathcal{S}) = \{A', B', T'\}$ and $\mathrm{Mor}_{\mathcal{S}}(A', B') = \emptyset$, $\mathrm{Mor}_{\mathcal{S}}(B', T') = \{g'\}$, $\mathrm{Mor}_{\mathcal{S}}(A', T') = \{h'\}$, plus the identity morphisms. The functor $p : \mathcal{S} \to \mathcal{C}$ is obvious. Then for every $U \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{S}_U$ is the category with one object and the identity morphism on that object, so a groupoid, but the morphism $f : A \to B$ cannot be lifted. Similarly, if we declare $\mathrm{Mor}_{\mathcal{S}}(A', B') = \{f'_1, f'_2\}$ and $\mathrm{Mor}_{\mathcal{S}}(A', T') = \{h'\} = \{g'f'_1\} = \{g'f'_2\}$, then the fibre categories are the same and $f : A \to B$ in the diagram below has two lifts.

$$\begin{tikzcd}
B' \arrow[r, "g'"] & T' & & B \arrow[r, "g"] & T \\
A' \arrow[u, dashed, "??"] \arrow[ur, "h'"'] & & \textit{above} & A \arrow[u, "f"] \arrow[ur, "gf=h"'] &
\end{tikzcd}$$

Later we would like to make assertions such as "any category fibred in groupoids over $\mathcal{C}$ is equivalent to a split one", or "any category fibred in groupoids whose fibre categories are setlike is equivalent to a category fibred in sets". The notion of equivalence depends on the 2-category we are working with.

Definition 33.6. Let $\mathcal{C}$ be a category. The 2-*category of categories fibred in groupoids over* $\mathcal{C}$ is the sub 2-category of the 2-category of fibred categories over $\mathcal{C}$ (see Definition 31.9) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in groupoids.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).

(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism! Hence this is actually a $(2,1)$-category and not just a 2-category. Here is the obligatory lemma on 2-fibre products.

Lemma 33.7. *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in groupoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 30.3.*

Proof. By Lemma 31.10 the fibre product as described in Lemma 30.3 is a fibred category. Hence it suffices to prove that the fibre categories are groupoids, see Lemma 33.2. By Lemma 30.4 it is enough to show that the 2-fibre product of groupoids is a groupoid, which is clear (from the construction in Lemma 29.4 for example). □

Lemma 33.8. *Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids, and suppose that $G : \mathcal{S} \to \mathcal{S}'$ is a functor over $\mathcal{C}$.*

1. *Then G is faithful (resp. fully faithful, resp. an equivalence) if and only if for each $U \in \mathrm{Ob}(\mathcal{C})$ the induced functor $G_U : \mathcal{S}_U \to \mathcal{S}'_U$ is faithful (resp. fully faithful, resp. an equivalence).*
2. *If G is an equivalence, then G is an equivalence in the 2-category of categories fibred in groupoids over $\mathcal{C}$.*

Proof. Let x, y be objects of $\mathcal{S}$ lying over the same object U. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{S}}(x,y) \arrow[rr, "G"'] \arrow[dr, "p"'] & & \mathrm{Mor}_{\mathcal{S}'}(G(x),G(y)) \arrow[dl, "p'"] \\ & \mathrm{Mor}_{\mathcal{C}}(U,U) & \end{tikzcd}$$

From this diagram it is clear that if G is faithful (resp. fully faithful) then so is each G_U.

Suppose G is an equivalence. For every object x' of $\mathcal{S}'$ there exists an object x of $\mathcal{S}$ such that $G(x)$ is isomorphic to x'. Suppose that x' lies over U' and x lies over U. Then there is an isomorphism $f : U' \to U$ in $\mathcal{C}$, namely, p' applied to the isomorphism $x' \to G(x)$. By the axioms of a category fibred in groupoids there exists an arrow $f^*x \to x$ of $\mathcal{S}$ lying over f. Hence there exists an isomorphism $\alpha : x' \to G(f^*x)$ such that $p'(\alpha) = \mathrm{id}_{U'}$ (this time by the axioms for $\mathcal{S}'$). All in all we conclude that for every object x' of $\mathcal{S}'$ we can choose a pair $(o_{x'}, \alpha_{x'})$ consisting of an object $o_{x'}$ of $\mathcal{S}$ and an isomorphism $\alpha_{x'} : x' \to G(o_{x'})$ with $p(\alpha_{x'}) = \mathrm{id}_{p'(x')}$. From this point on we proceed as usual (see proof of Lemma 2.19) to produce an inverse functor $F : \mathcal{S}' \to \mathcal{S}$, by taking $x' \mapsto o_{x'}$ and $\varphi' : x' \to y'$ to the unique arrow $\varphi_{\varphi'} : o_{x'} \to o_{y'}$ with $\alpha_{x'}^{-1} \circ G(\varphi_{\varphi'}) \circ \alpha_{y'} = \varphi'$. With these choices F is a functor over $\mathcal{C}$. We omit the verification that $G \circ F$ and $F \circ G$ are 2-isomorphic (in the 2-category of categories fibred in groupoids over $\mathcal{C}$).

Suppose that G_U is faithful (resp. fully faithful) for all $U \in \mathrm{Ob}(\mathcal{C})$. To show that G is faithful (resp. fully faithful) we have to show for any objects $x, y \in \mathrm{Ob}(\mathcal{S})$ that G induces an injection (resp. bijection) between $\mathrm{Mor}_{\mathcal{S}}(x,y)$ and $\mathrm{Mor}_{\mathcal{S}'}(G(x),G(y))$. Set $U = p(x)$ and $V = p(y)$. It suffices to prove that G induces an injection (resp.

bijection) between morphism $x \to y$ lying over f to morphisms $G(x) \to G(y)$ lying over f for any morphism $f : U \to V$. Now fix $f : U \to V$. Denote $f^*y \to y$ a pullback. Then also $G(f^*y) \to G(y)$ is a pullback. The set of morphisms from x to y lying over f is bijective to the set of morphisms between x and f^*y lying over id_U. (By the second axiom of a category fibred in groupoids.) Similarly the set of morphisms from $G(x)$ to $G(y)$ lying over f is bijective to the set of morphisms between $G(x)$ and $G(f^*y)$ lying over id_U. Hence the fact that G_U is faithful (resp. fully faithful) gives the desired result.

Finally suppose for all G_U is an equivalence for all U, so it is fully faithful and essentially surjective. We have seen this implies G is fully faithful, and thus to prove it is an equivalence we have to prove that it is essentially surjective. This is clear, for if $z' \in \mathrm{Ob}(\mathcal{S}')$ then $z' \in \mathrm{Ob}(\mathcal{S}'_U)$ where $U = p'(z')$. Since G_U is essentially surjective we know that z' is isomorphic, in $\mathcal{S}'_U$, to an object of the form $G_U(z)$ for some $z \in \mathrm{Ob}(\mathcal{S}_U)$. But morphisms in $\mathcal{S}'_U$ are morphisms in $\mathcal{S}'$ and hence z' is isomorphic to $G(z)$ in $\mathcal{S}'$. $\square$

Lemma 33.9. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids. Let $G : \mathcal{S} \to \mathcal{S}'$ be a functor over $\mathcal{C}$. Then G is fully faithful if and only if the diagonal*

$$\Delta_G : \mathcal{S} \longrightarrow \mathcal{S} \times_{G,\mathcal{S}',G} \mathcal{S}$$

is an equivalence.

Proof. By Lemma 33.8 it suffices to look at fibre categories over an object U of $\mathcal{C}$. An object of the right hand side is a triple (x, x', α) where $\alpha : G(x) \to G(x')$ is a morphism in $\mathcal{S}'_U$. The functor Δ_G maps the object x of $\mathcal{S}_U$ to the triple $(x, x, \mathrm{id}_{G(x)})$. Note that (x, x', α) is in the essential image of Δ_G if and only if $\alpha = G(\beta)$ for some morphism $\beta : x \to x'$ in $\mathcal{S}_U$ (details omitted). Hence in order for Δ_G to be an equivalence, every α has to be the image of a morphism $\beta : x \to x'$, and also every two distinct morphisms $\beta, \beta' : x \to x'$ have to given distinct morphisms $G(\beta), G(\beta')$. This proves one direction of the lemma. We omit the proof of the other direction. $\square$

Lemma 33.10. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}_i$, $i = 1, 2, 3, 4$ be categories fibred in groupoids over $\mathcal{C}$. Suppose that $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ and $\psi : \mathcal{S}_3 \to \mathcal{S}_4$ are equivalences over $\mathcal{C}$. Then*

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}_2, \mathcal{S}_3) \longrightarrow \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_4), \quad \alpha \longmapsto \psi \circ \alpha \circ \varphi$$

is an equivalence of categories.

Proof. This is a generality and holds in any 2-category. $\square$

Lemma 33.11. *Let $\mathcal{C}$ be a category. If $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids, then so is the inertia fibred category $\mathcal{I}_\mathcal{S} \to \mathcal{C}$.*

Proof. Clear from the construction in Lemma 32.1 or by using (from the same lemma) that $I_\mathcal{S} \to \mathcal{S} \times_{\Delta, \mathcal{S} \times_\mathcal{C} \mathcal{S}, \Delta} \mathcal{S}$ is an equivalence and appealing to Lemma 33.7. $\square$

Lemma 33.12. *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids and p factors through $p' : \mathcal{S} \to \mathcal{C}/U$ then $p' : \mathcal{S} \to \mathcal{C}/U$ is fibred in groupoids.*

Proof. We have already seen in Lemma 31.11 that p' is a fibred category. Hence it suffices to prove the fibre categories are groupoids, see Lemma 33.2. For $V \in \text{Ob}(\mathcal{C})$ we have

$$\mathcal{S}_V = \coprod\nolimits_{f : V \to U} \mathcal{S}_{(f : V \to U)}$$

where the left hand side is the fibre category of p and the right hand side is the disjoint union of the fibre categories of p'. Hence the result. □

Lemma 33.13. *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be functors between categories. If $\mathcal{A}$ is fibred in groupoids over $\mathcal{B}$ and $\mathcal{B}$ is fibred in groupoids over $\mathcal{C}$, then $\mathcal{A}$ is fibred in groupoids over $\mathcal{C}$.*

Proof. One can prove this directly from the definition. However, we will argue using the criterion of Lemma 33.2. By Lemma 31.12 we see that $\mathcal{A}$ is fibred over $\mathcal{C}$. To finish the proof we show that the fibre category $\mathcal{A}_U$ is a groupoid for U in $\mathcal{C}$. Namely, if $x \to y$ is a morphism of $\mathcal{A}_U$, then its image in $\mathcal{B}$ is an isomorphism as $\mathcal{B}_U$ is a groupoid. But then $x \to y$ is an isomorphism, for example by Lemma 31.2 and the fact that every morphism of $\mathcal{A}$ is strongly $\mathcal{B}$-cartesian (see Lemma 33.2). □

Lemma 33.14. *Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$. If $p(x) \times_{p(y)} p(z)$ exists, then $x \times_y z$ exists and $p(x \times_y z) = p(x) \times_{p(y)} p(z)$.*

Proof. Follows from Lemma 31.13. □

Lemma 33.15. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. There exists a factorization $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ by 1-morphisms of categories fibred in groupoids over $\mathcal{C}$ such that $\mathcal{X} \to \mathcal{X}'$ is an equivalence over $\mathcal{C}$ and such that $\mathcal{X}'$ is a category fibred in groupoids over $\mathcal{Y}$.*

Proof. Denote $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ the structure functors. We construct $\mathcal{X}'$ explicitly as follows. An object of $\mathcal{X}'$ is a quadruple (U, x, y, f) where $x \in \text{Ob}(\mathcal{X}_U)$, $y \in \text{Ob}(\mathcal{Y}_U)$ and $f : F(x) \to y$ is an isomorphism in $\mathcal{Y}_U$. A morphism $(a, b) : (U, x, y, f) \to (U', x', y', f')$ is given by $a : x \to x'$ and $b : y \to y'$ with $p(a) = q(b)$ and such that $f' \circ F(a) = b \circ f$. In other words $\mathcal{X}' = \mathcal{X} \times_{F, \mathcal{Y}, \text{id}} \mathcal{Y}$ with the construction of the 2-fibre product from Lemma 30.3. By Lemma 33.7 we see that $\mathcal{X}'$ is a category fibred in groupoids over $\mathcal{C}$ and that $\mathcal{X}' \to \mathcal{Y}$ is a morphism of categories over $\mathcal{C}$. As functor $\mathcal{X} \to \mathcal{X}'$ we take $x \mapsto (p(x), x, F(x), \text{id}_{F(x)})$ on objects and $(a : x \to x') \mapsto (a, F(a))$ on morphisms. It is clear that the composition $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ equals F. We omit the verification that $\mathcal{X} \to \mathcal{X}'$ is an equivalence of fibred categories over $\mathcal{C}$.

Finally, we have to show that $\mathcal{X}' \to \mathcal{Y}$ is a category fibred in groupoids. Let $b : y' \to y$ be a morphism in $\mathcal{Y}$ and let (U, x, y, f) be an object of $\mathcal{X}'$ lying over y. Because $\mathcal{X}$ is fibred in groupoids over $\mathcal{C}$ we can find a morphism $a : x' \to x$ lying over $U' = q(y') \to q(y) = U$. Since $\mathcal{Y}$ is fibred in groupoids over $\mathcal{C}$ and since both $F(x') \to F(x)$ and $y' \to y$ lie over the same morphism $U' \to U$ we can find $f' : F(x') \to y'$ lying over $\text{id}_{U'}$ such that $f \circ F(a) = b \circ f'$. Hence we obtain $(a, b) : (U', x', y', f') \to (U, x, y, f)$. This verifies the first condition (1) of Definition 33.1. To see (2) let $(a, b) : (U', x', y', f') \to (U, x, y, f)$ and $(a', b') : (U'', x'', y'', f'') \to (U, x, y, f)$ be morphisms of $\mathcal{X}'$ and let $b'' : y' \to y''$ be a morphism of $\mathcal{Y}$ such that $b' \circ b'' = b$. We have to show that there exists a unique morphism $a'' : x' \to x''$

such that $f'' \circ F(a'') = b'' \circ f'$ and such that $(a', b') \circ (a'', b'') = (a, b)$. Because $\mathcal{X}$ is fibred in groupoids we know there exists a unique morphism $a'' : x' \to x''$ such that $a' \circ a'' = a$ and $p(a'') = q(b'')$. Because $\mathcal{Y}$ is fibred in groupoids we see that $F(a'')$ is the unique morphism $F(x') \to F(x'')$ such that $F(a') \circ F(a'') = F(a)$ and $q(F(a'')) = q(b'')$. The relation $f'' \circ F(a'') = b'' \circ f'$ follows from this and the given relations $f \circ F(a) = b \circ f'$ and $f \circ F(a') = b' \circ f''$. $\square$

Lemma 33.16. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. Assume we have a 2-commutative diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[dr, "f"'] & \mathcal{X} \arrow[l, "a"] \arrow[r, "b"'] \arrow[d, "F"] & \mathcal{X}'' \arrow[dl, "g"] \\ & \mathcal{Y} & \end{tikzcd}$$

where a and b are equivalences of categories over $\mathcal{C}$ and f and g are categories fibred in groupoids. Then there exists an equivalence $h : \mathcal{X}'' \to \mathcal{X}'$ of categories over $\mathcal{Y}$ such that $h \circ b$ is 2-isomorphic to a as 1-morphisms of categories over $\mathcal{C}$. If the diagram above actually commutes, then we can arrange it so that $h \circ b$ is 2-isomorphic to a as 1-morphisms of categories over $\mathcal{Y}$.

Proof. We will show that both $\mathcal{X}'$ and $\mathcal{X}''$ over $\mathcal{Y}$ are equivalent to the category fibred in groupoids $\mathcal{X} \times_{F, \mathcal{Y}, \mathrm{id}} \mathcal{Y}$ over $\mathcal{Y}$, see proof of Lemma 33.15. Choose a quasi-inverse $b^{-1} : \mathcal{X}'' \to \mathcal{X}$ in the 2-category of categories over $\mathcal{C}$. Since the right triangle of the diagram is 2-commutative we see that

$$\begin{tikzcd} \mathcal{X} \arrow[d, "F"'] & \mathcal{X}'' \arrow[l, "b^{-1}"] \arrow[d, "g"] \\ \mathcal{Y} & \mathcal{Y} \arrow[l] \end{tikzcd}$$

is 2-commutative. Hence we obtain a 1-morphism $c : \mathcal{X}'' \to \mathcal{X} \times_{F, \mathcal{Y}, \mathrm{id}} \mathcal{Y}$ by the universal property of the 2-fibre product. Moreover c is a morphism of categories over $\mathcal{Y}$ (!) and an equivalence (by the assumption that b is an equivalence, see Lemma 29.7). Hence c is an equivalence in the 2-category of categories fibred in groupoids over $\mathcal{Y}$ by Lemma 33.8.

We still have to construct a 2-isomorphism between $c \circ b$ and the functor $d : \mathcal{X} \to \mathcal{X} \times_{F, \mathcal{Y}, \mathrm{id}} \mathcal{Y}$, $x \mapsto (p(x), x, F(x), \mathrm{id}_{F(x)})$ constructed in the proof of Lemma 33.15. Let $\alpha : F \to g \circ b$ and $\beta : b^{-1} \circ b \to \mathrm{id}$ be 2-isomorphisms between 1-morphisms of categories over $\mathcal{C}$. Note that $c \circ b$ is given by the rule

$$x \mapsto (p(x), b^{-1}(b(x)), g(b(x)), \alpha_x \circ F(\beta_x))$$

on objects. Then we see that

$$(\beta_x, \alpha_x) : (p(x), x, F(x), \mathrm{id}_{F(x)}) \longrightarrow (p(x), b^{-1}(b(x)), g(b(x)), \alpha_x \circ F(\beta_x))$$

is a functorial isomorphism which gives our 2-morphism $d \to b \circ c$. Finally, if the diagram commutes then α_x is the identity for all x and we see that this 2-morphism is a 2-morphism in the 2-category of categories over $\mathcal{Y}$. $\square$

34. Presheaves of categories

In this section we compare the notion of fibred categories with the closely related notion of a "presheaf of categories". The basic construction is explained in the following example.

Example 34.1. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Cat}$ is a functor to the 2-category of categories, see Definition 27.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. From this we can construct a fibred category $\mathcal{S}_F$ over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\begin{aligned} \mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in \mathrm{Mor}_\mathcal{C}(V, U), \phi \in \mathrm{Mor}_{F(V)}(y, f^*x)\} \\ &= \coprod\nolimits_{f \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathrm{Mor}_{F(V)}(y, f^*x) \end{aligned}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^*y$ and $\phi : y \to f^*x$ to be $g^*(\phi) \circ \psi$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. Let us check that this is indeed a fibred category. Given $f : V \to U$ in $\mathcal{C}$ and (U, x) a lift of U, then we claim $(f, \mathrm{id}_{f^*x}) : (V, f^*x) \to (U, x)$ is a strongly cartesian lift of f. We have to show a h in the diagram on the left determines (h, ν) on the right:

$$\begin{tikzcd} V \arrow[r, "f"] & U \\ W \arrow[u, dashed, "h"] \arrow[ru, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} (V, f^*x) \arrow[r, "{(f, \mathrm{id}_{f^*x})}"] & (U, x) \\ (W, z) \arrow[u, dashed, "{(h,\nu)}"] \arrow[ru, "{(g,\psi)}"'] & \end{tikzcd}$$

Just take $\nu = \psi$ which works because $f \circ h = g$ and hence $g^*x = h^*f^*x$. Moreover, this is the only lift making the diagram (on the right) commute.

Definition 34.2. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Cat}$ is a functor to the 2-category of categories. We will write $p_F : \mathcal{S}_F \to \mathcal{C}$ for the fibred category constructed in Example 34.1. A *split fibred category* is a fibred category isomorphic (!) over $\mathcal{C}$ to one of these categories $\mathcal{S}_F$.

Lemma 34.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}$ be a fibred category over $\mathcal{C}$. Then $\mathcal{S}$ is split if and only if for some choice of pullbacks (see Definition 31.6) the pullback functors $(f \circ g)^*$ and $g^* \circ f^*$ are equal.*

Proof. This is immediate from the definitions. □

Lemma 34.4. *Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. There exists a functor $F : \mathcal{C} \to \textit{Cat}$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ in the 2-category of fibred categories over $\mathcal{C}$. In other words, every fibred category is equivalent to a split one.*

Proof. Let us make a choice of pullbacks (see Definition 31.6). By Lemma 31.7 we get pullback functors f^* for every morphism f of $\mathcal{C}$.

We construct a new category $\mathcal{S}'$ as follows. The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^*x_1 \to f_2^*x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_x).

At this point we need to check that p' makes $\mathcal{S}'$ into a fibred category over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$ which maps strongly cartesian morphisms to strongly cartesian morphisms. We omit the verifications.

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \mathrm{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \mathrm{id}_V) = (g^*\varphi, \mathrm{id}_{V'})$ where we use the unique identifications $g^*f_i^*x_i = (f_i \circ g)^*x_i$ from Lemma 31.7 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^*x_1$ to $(f_2 \circ g)^*x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. □

35. Presheaves of groupoids

In this section we compare the notion of categories fibred in groupoids with the closely related notion of a "presheaf of groupoids". The basic construction is explained in the following example.

Example 35.1. This example is the analogue of Example 34.1, for "presheaves of groupoids" instead of "presheaves of categories". The output will be a category fibred in groupoids instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to \text{Groupoids}$ is a functor to the category of groupoids, see Definition 27.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. We construct a category $\mathcal{S}_F$ fibred in groupoids over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\begin{aligned} \mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in \mathrm{Mor}_{\mathcal{C}}(V, U), \phi \in \mathrm{Mor}_{F(V)}(y, f^*x)\} \\ &= \coprod\nolimits_{f \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathrm{Mor}_{F(V)}(y, f^*x) \end{aligned}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^*y$ and $\phi : y \to f^*x$ to be $g^*(\phi) \circ \psi$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. The condition that $F(U)$ is a groupoid for every U guarantees that $\mathcal{S}_F$ is fibred in groupoids over $\mathcal{C}$, as we have already seen in Example 34.1 that $\mathcal{S}_F$ is a fibred category, see Lemma 33.2. But we can also prove conditions (1), (2) of Definition 33.1 directly as follows: (1) Lifts of morphisms exist since given $f : V \to U$ in $\mathcal{C}$ and (U, x) an object of $\mathcal{S}_F$ over U, then $(f, \mathrm{id}_{f^*x}) : (V, f^*x) \to (U, x)$ is a lift of f. (2) Suppose given solid diagrams as follows

$$
\begin{tikzcd}
V \arrow[r, "f"] & U \\
W \arrow[u, dashed, "h"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(V,y) \arrow[r, "{(f,\phi)}"] & (U,x) \\
(W,z) \arrow[u, dashed, "{(h,\nu)}"] \arrow[ur, "{(g,\psi)}"'] &
\end{tikzcd}
$$

Then for the dotted arrows we have $\nu = (h^*\phi)^{-1} \circ \psi$ so given h there exists a ν which is unique by uniqueness of inverses.

Definition 35.2. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to$ *Groupoids* is a functor to the 2-category of groupoids. We will write $p_F : \mathcal{S}_F \to \mathcal{C}$ for the category fibred in groupoids constructed in Example 35.1. A *split category fibred in groupoids* is a category fibred in groupoids isomorphic (!) over $\mathcal{C}$ to one of these categories $\mathcal{S}_F$.

Lemma 35.3. *Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. There exists a functor $F : \mathcal{C} \to$ Groupoids such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ over $\mathcal{C}$. In other words, every category fibred in groupoids is equivalent to a split one.*

Proof. Make a choice of pullbacks (see Definition 31.6). By Lemmas 31.7 and 33.2 we get pullback functors f^* for every morphism f of $\mathcal{C}$.

We construct a new category $\mathcal{S}'$ as follows. The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^*x_1 \to f_2^*x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_x).

At this point we need to check that p' makes $\mathcal{S}'$ into a category fibred in groupoids over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$. We omit the verifications.

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \mathrm{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \mathrm{id}_V) = (g^*\varphi, \mathrm{id}_{V'})$ where we use the unique identifications $g^*f_i^*x_i = (f_i \circ g)^*x_i$ from Lemma 33.2 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^*x_1$ to $(f_2 \circ g)^*x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. $\square$

We will see an alternative proof of this lemma in Section 39.

36. Categories fibred in sets

Definition 36.1. A category is called *discrete* if the only morphisms are the identity morphisms.

A discrete category has only one interesting piece of information: its set of objects. Thus we sometime confuse discrete categories with sets.

Definition 36.2. Let $\mathcal{C}$ be a category. A *category fibred in sets*, or a *category fibred in discrete categories* is a category fibred in groupoids all of whose fibre categories are discrete.

We want to clarify the relationship between categories fibred in sets and presheaves (see Definition 3.3). To do this it makes sense to first make the following definition.

Definition 36.3. Let $\mathcal{C}$ be a category. The 2-*category of categories fibred in sets over* $\mathcal{C}$ is the sub 2-category of the category of categories fibred in groupoids over $\mathcal{C}$ (see Definition 33.6) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in sets.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism. Hence this 2-category is actually a $(2, 1)$-category. Here is the obligatory lemma on the existence of 2-fibre products.

Lemma 36.4. *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in sets over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 30.3 returns a category fibred in sets if one starts out with such.*

Proof. Omitted. □

Example 36.5. This example is the analogue of Examples 34.1 and 35.1 for presheaves instead of "presheaves of categories". The output will be a category fibred in sets instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Sets}$ is a presheaf. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^* : F(U) \to F(V)$. We construct a category $\mathcal{S}_F$ fibred in sets over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) = \{f \in \mathrm{Mor}_{\mathcal{C}}(V, U) \mid f^*x = y\}$$

Composition is inherited from composition in $\mathcal{C}$ which works as $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. As every fibre category $\mathcal{S}_{F,U}$ is discrete with underlying set $F(U)$ and we have already see in Example 35.1 that $\mathcal{S}_F$ is a category fibred in groupoids, we conclude that $\mathcal{S}_F$ is fibred in sets.

Lemma 36.6. *Let $\mathcal{C}$ be a category. The only 2-morphisms between categories fibred in sets are identities. In other words, the 2-category of categories fibred in sets is a category. Moreover, there is an equivalence of categories*

$$\left\{\begin{matrix}\textit{the category of presheaves}\\ \textit{of sets over } \mathcal{C}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{the category of categories}\\ \textit{fibred in sets over } \mathcal{C}\end{matrix}\right\}$$

The functor from left to right is the construction $F \to \mathcal{S}_F$ discussed in Example 36.5. The functor from right to left assigns to $p : \mathcal{S} \to \mathcal{C}$ the presheaf of objects $U \mapsto \mathrm{Ob}(\mathcal{S}_U)$.

Proof. The first assertion is clear, as the only morphisms in the fibre categories are identities.

Suppose that $p : \mathcal{S} \to \mathcal{C}$ is fibred in sets. Let $f : V \to U$ be a morphism in $\mathcal{C}$ and let $x \in \mathrm{Ob}(\mathcal{S}_U)$. Then there is exactly one choice for the object f^*x. Thus we see that $(f \circ g)^*x = g^*(f^*x)$ for f, g as in Lemma 33.2. It follows that we may think of the assignments $U \mapsto \mathrm{Ob}(\mathcal{S}_U)$ and $f \mapsto f^*$ as a presheaf on $\mathcal{C}$. $\square$

Here is an important example of a category fibred in sets.

Example 36.7. Let $\mathcal{C}$ be a category. Let $X \in \mathrm{Ob}(\mathcal{C})$. Consider the representable presheaf $h_X = \mathrm{Mor}_\mathcal{C}(-, X)$ (see Example 3.4). On the other hand, consider the category $p : \mathcal{C}/X \to \mathcal{C}$ from Example 2.13. The fibre category $(\mathcal{C}/X)_U$ has as objects morphisms $h : U \to X$, and only identities as morphisms. Hence we see that under the correspondence of Lemma 36.6 we have

$$h_X \longleftrightarrow \mathcal{C}/X.$$

In other words, the category $\mathcal{C}/X$ is canonically equivalent to the category $\mathcal{S}_{h_X}$ associated to h_X in Example 36.5.

For this reason it is tempting to define a “representable” object in the 2-category of categories fibred in groupoids to be a category fibred in sets whose associated presheaf is representable. However, this is would not be a good definition for use since we prefer to have a notion which is invariant under equivalences. To make this precise we study exactly which categories fibred in groupoids are equivalent to categories fibred in sets.

37. Categories fibred in setoids

Definition 37.1. Let us call a category a *setoid*[4] if it is a groupoid where every object has exactly one automorphism: the identity.

If C is a set with an equivalence relation $\sim$, then we can make a setoid $\mathcal{C}$ as follows: $\mathrm{Ob}(\mathcal{C}) = C$ and $\mathrm{Mor}_\mathcal{C}(x, y) = \emptyset$ unless $x \sim y$ in which case we set $\mathrm{Mor}_\mathcal{C}(x, y) = \{1\}$. Transitivity of $\sim$ means that we can compose morphisms. Conversely any setoid category defines an equivalence relation on its objects (isomorphism) such that you recover the category (up to unique isomorphism – not equivalence) from the procedure just described.

Discrete categories are setoids. For any setoid $\mathcal{C}$ there is a canonical procedure to make a discrete category equivalent to it, namely one replaces $\mathrm{Ob}(\mathcal{C})$ by the set of isomorphism classes (and adds identity morphisms). In terms of sets endowed with an equivalence relation this corresponds to taking the quotient by the equivalence relation.

Definition 37.2. Let $\mathcal{C}$ be a category. A *category fibred in setoids* is a category fibred in groupoids all of whose fibre categories are setoids.

Below we will clarify the relationship between categories fibred in setoids and categories fibred in sets.

[4]A set on steroids!?

Definition 37.3. Let $\mathcal{C}$ be a category. The *2-category of categories fibred in setoids over $\mathcal{C}$* is the sub 2-category of the category of categories fibred in groupoids over $\mathcal{C}$ (see Definition 33.6) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in setoids.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism. Hence this 2-category is actually a $(2, 1)$-category.
Here is the obligatory lemma on the existence of 2-fibre products.

Lemma 37.4. *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in setoids over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 30.3 returns a category fibred in setoids if one starts out with such.*

Proof. Omitted. $\square$

Lemma 37.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}$ be a category over $\mathcal{C}$.*

(1) *If $\mathcal{S} \to \mathcal{S}'$ is an equivalence over $\mathcal{C}$ with $\mathcal{S}'$ fibred in sets over $\mathcal{C}$, then*
   (a) *$\mathcal{S}$ is fibred in setoids over $\mathcal{C}$, and*
   (b) *for each $U \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Ob}(\mathcal{S}_U) \to \mathrm{Ob}(\mathcal{S}'_U)$ identifies the target as the set of isomorphism classes of the source.*
(2) *If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in setoids, then there exists a category fibred in sets $p' : \mathcal{S}' \to \mathcal{C}$ and an equivalence $can : \mathcal{S} \to \mathcal{S}'$ over $\mathcal{C}$.*

Proof. Let us prove (2). An object of the category $\mathcal{S}'$ will be a pair (U, ξ), where $U \in \mathrm{Ob}(\mathcal{C})$ and ξ is an isomorphism class of objects of $\mathcal{S}_U$. A morphism $(U, \xi) \to (V, \psi)$ is given by a morphism $x \to y$, where $x \in \xi$ and $y \in \psi$. Here we identify two morphisms $x \to y$ and $x' \to y'$ if they induce the same morphism $U \to V$, and if for some choices of isomorphisms $x \to x'$ in $\mathcal{S}_U$ and $y \to y'$ in $\mathcal{S}_V$ the compositions $x \to x' \to y'$ and $x \to y \to y'$ agree. By construction there are surjective maps on objects and morphisms from $\mathcal{S} \to \mathcal{S}'$. We define composition of morphisms in $\mathcal{S}'$ to be the unique law that turns $\mathcal{S} \to \mathcal{S}'$ into a functor. Some details omitted. $\square$

Thus categories fibred in setoids are exactly the categories fibred in groupoids which are equivalent to categories fibred in sets. Moreover, an equivalence of categories fibred in sets is an isomorphism by Lemma 36.6.

Lemma 37.6. *Let $\mathcal{C}$ be a category. The construction of Lemma 37.5 part (2) gives a functor*

$$F : \left\{\begin{matrix}\textit{the 2-category of categories}\\ \textit{fibred in setoids over } \mathcal{C}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\textit{the category of categories}\\ \textit{fibred in sets over } \mathcal{C}\end{matrix}\right\}$$

(see Definition 27.5). This functor is an equivalence in the following sense:

(1) *for any two 1-morphisms $f, g : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = F(g)$ there exists a unique 2-isomorphism $f \to g$,*
(2) *for any morphism $h : F(\mathcal{S}_1) \to F(\mathcal{S}_2)$ there exists a 1-morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = h$, and*
(3) *any category fibred in sets $\mathcal{S}$ is equal to $F(\mathcal{S})$.*

In particular, defining $F_i \in PSh(\mathcal{C})$ by the rule $F_i(U) = \text{Ob}(\mathcal{S}_{i,U})/\cong$, we have

$$\text{Mor}_{Cat/\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_2)\Big/\textit{2-isomorphism} = \text{Mor}_{PSh(\mathcal{C})}(F_1, F_2)$$

More precisely, given any map $\phi : F_1 \to F_2$ there exists a 1-morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ which induces ϕ on isomorphism classes of objects and which is unique up to unique 2-isomorphism.

Proof. By Lemma 36.6 the target of F is a category hence the assertion makes sense. The construction of Lemma 37.5 part (2) assigns to $\mathcal{S}$ the category fibred in sets whose value over U is the set of isomorphism classes in $\mathcal{S}_U$. Hence it is clear that it defines a functor as indicated. Let $f, g : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = F(g)$ be as in (1). For each object U of $\mathcal{C}$ and each object x of $\mathcal{S}_{1,U}$ we see that $f(x) \cong g(x)$ by assumption. As $\mathcal{S}_2$ is fibred in setoids there exists a unique isomorphism $t_x : f(x) \to g(x)$ in $\mathcal{S}_{2,U}$. Clearly the rule $x \mapsto t_x$ gives the desired 2-isomorphism $f \to g$. We omit the proofs of (2) and (3). To see the final assertion use Lemma 36.6 to see that the right hand side is equal to $\text{Mor}_{Cat(\mathcal{C})}(F(\mathcal{S}_1), F(\mathcal{S}_2))$ and apply (1) and (2) above. $\square$

Here is another characterization of categories fibred in setoids among all categories fibred in groupoids.

Lemma 37.7. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. The following are equivalent:*

1. *$p : \mathcal{S} \to \mathcal{C}$ be a category fibred in setoids, and*
2. *the canonical 1-morphism $\mathcal{I}_\mathcal{S} \to \mathcal{S}$, see (32.2.1), is an equivalence (of categories over $\mathcal{C}$).*

Proof. Assume (2). The category $\mathcal{I}_\mathcal{S}$ has objects (x, α) where $x \in \mathcal{S}$, say with $p(x) = U$, and $\alpha : x \to x$ is a morphism in $\mathcal{S}_U$. Hence if $\mathcal{I}_\mathcal{S} \to \mathcal{S}$ is an equivalence over $\mathcal{C}$ then every pair of objects (x, α), (x, α') are isomorphic in the fibre category of $\mathcal{I}_\mathcal{S}$ over U. Looking at the definition of morphisms in $\mathcal{I}_\mathcal{S}$ we conclude that α, α' are conjugate in the group of automorphisms of x. Hence taking $\alpha' = \text{id}_x$ we conclude that every automorphism of x is equal to the identity. Since $\mathcal{S} \to \mathcal{C}$ is fibred in groupoids this implies that $\mathcal{S} \to \mathcal{C}$ is fibred in setoids. We omit the proof of (1) $\Rightarrow$ (2). $\square$

Lemma 37.8. *Let $\mathcal{C}$ be a category. The construction of Lemma 37.6 which associates to a category fibred in setoids a presheaf is compatible with products, in the sense that the presheaf associated to a 2-fibre product $\mathcal{X} \times_\mathcal{Y} \mathcal{Z}$ is the fibre product of the presheaves associated to $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$.*

Proof. Let $U \in \text{Ob}(\mathcal{C})$. The lemma just says that

$$\text{Ob}((\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_U)/\cong \quad \text{equals} \quad \text{Ob}(\mathcal{X}_U)/\cong \times_{\text{Ob}(\mathcal{Y}_U)/\cong} \text{Ob}(\mathcal{Z}_U)/\cong$$

the proof of which we omit. (But note that this would not be true in general if the category $\mathcal{Y}_U$ is not a setoid.) $\square$

38. Representable categories fibred in groupoids

Here is our definition of a representable category fibred in groupoids. As promised this is invariant under equivalences.

Definition 38.1. Let $\mathcal{C}$ be a category. A category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$ is called *representable* if there exists an object X of $\mathcal{C}$ and an equivalence $j : \mathcal{S} \to \mathcal{C}/X$ (in the 2-category of groupoids over $\mathcal{C}$).

The usual abuse of notation is to say that X *represents* $\mathcal{S}$ and not mention the equivalence j. We spell out what this entails.

Lemma 38.2. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids.*

(1) *$\mathcal{S}$ is representable if and only if the following conditions are satisfied:*
  (a) *$\mathcal{S}$ is fibred in setoids, and*
  (b) *the presheaf $U \mapsto \mathrm{Ob}(\mathcal{S}_U)/\cong$ is representable.*

(2) *If $\mathcal{S}$ is representable the pair (X, j), where j is the equivalence $j : \mathcal{S} \to \mathcal{C}/X$ is uniquely determined up to isomorphism.*

Proof. The first assertion follows immediately from Lemma 37.5. For the second, suppose that $j' : \mathcal{S} \to \mathcal{C}/X'$ is a second such pair. Choose a 1-morphism $t' : \mathcal{C}/X' \to \mathcal{S}$ such that $j' \circ t' \cong \mathrm{id}_{\mathcal{C}/X'}$ and $t' \circ j' \cong \mathrm{id}_{\mathcal{S}}$. Then $j \circ t' : \mathcal{C}/X' \to \mathcal{C}/X$ is an equivalence. Hence it is an isomorphism, see Lemma 36.6. Hence by the Yoneda Lemma 3.5 (via Example 36.7 for example) it is given by an isomorphism $X' \to X$. $\square$

Lemma 38.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Assume that $\mathcal{X}$, $\mathcal{Y}$ are representable by objects X, Y of $\mathcal{C}$. Then*

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{X}, \mathcal{Y})\Big/\text{2-isomorphism} = \mathrm{Mor}_{\mathcal{C}}(X, Y)$$

More precisely, given $\phi : X \to Y$ there exists a 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ which induces ϕ on isomorphism classes of objects and which is unique up to unique 2-isomorphism.

Proof. By Example 36.7 we have $\mathcal{C}/X = \mathcal{S}_{h_X}$ and $\mathcal{C}/Y = \mathcal{S}_{h_Y}$. By Lemma 37.6 we have

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{X}, \mathcal{Y})\Big/\text{2-isomorphism} = \mathrm{Mor}_{PSh(\mathcal{C})}(h_X, h_Y)$$

By the Yoneda Lemma 3.5 we have $\mathrm{Mor}_{PSh(\mathcal{C})}(h_X, h_Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$. $\square$

39. Representable 1-morphisms

Let $\mathcal{C}$ be a category. In this section we explain what it means for a 1-morphism between categories fibred in groupoids over $\mathcal{C}$ to be representable. Note that the 2-category of categories fibred in groupoids over $\mathcal{C}$ is a "full" sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 33.6). Hence if $\mathcal{S}$, $\mathcal{S}'$ are fibred in groupoids over $\mathcal{C}$ then

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

denotes the category of 1-morphisms in this 2-category (see Definition 30.1). These are all groupoids, see remarks following Definition 33.6. Here is the 2-category analogue of the Yoneda lemma.

Lemma 39.1 (2-Yoneda lemma)**.** *Let $\mathcal{S} \to \mathcal{C}$ be fibred in groupoids. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor*

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \longrightarrow \mathcal{S}_U$$

given by $G \mapsto G(id_U)$ is an equivalence.

Proof. Make a choice of pullbacks for $\mathcal{S}$ (see Definition 31.6). We define a functor

$$\mathcal{S}_U \longrightarrow \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S})$$

as follows. Given $x \in \mathrm{Ob}(\mathcal{S}_U)$ the associated functor is

(1) on objects: $(f : V \to U) \mapsto f^*x$, and
(2) on morphisms: the arrow $(g : V'/U \to V/U)$ maps to the composition

$$(f \circ g)^*x \xrightarrow{(\alpha_{g,f})_x} g^*f^*x \to f^*x$$

where $\alpha_{g,f}$ is as in Lemma 33.2.

We omit the verification that this is an inverse to the functor of the lemma. □

Remark 39.2. We can use the 2-Yoneda lemma to give an alternative proof of Lemma 35.3. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. We define a contravariant functor F from $\mathcal{C}$ to the category of groupoids as follows: for $U \in \mathrm{Ob}(\mathcal{C})$ let

$$F(U) = \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}).$$

If $f : U \to V$ the induced functor $\mathcal{C}/U \to \mathcal{C}/V$ induces the morphism $F(f) : F(V) \to F(U)$. Clearly F is a functor. Let $\mathcal{S}'$ be the associated category fibred in groupoids from Example 35.1. There is an obvious functor $G : \mathcal{S}' \to \mathcal{S}$ over $\mathcal{C}$ given by taking the pair (U, x), where $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(U)$, to $x(\mathrm{id}_U) \in \mathcal{S}$. Now Lemma 39.1 implies that for each U,

$$G_U : \mathcal{S}'_U = F(U) = \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \to \mathcal{S}_U$$

is an equivalence, and thus G equivalence between $\mathcal{S}$ and $\mathcal{S}'$ by Lemma 33.8.

Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{C}/U \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $\mathcal{C}$. We want to describe the 2-fibre product

$$\begin{tikzcd}
(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C}/U \arrow[r, "G"] & \mathcal{Y}
\end{tikzcd}$$

Let $y = G(\mathrm{id}_U) \in \mathcal{Y}_U$. Make a choice of pullbacks for $\mathcal{Y}$ (see Definition 31.6). Then G is isomorphic to the functor $(f : V \to U) \mapsto f^*y$, see Lemma 39.1 and its proof. We may think of an object of $(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ as a quadruple $(V, f : V \to U, x, \phi)$, see Lemma 30.3. Using the description of G above we may think of ϕ as an isomorphism $\phi : f^*y \to F(x)$ in $\mathcal{Y}_V$.

Lemma 39.3. *In the situation above the fibre category of $(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ over an object $f : V \to U$ of $\mathcal{C}/U$ is the category described as follows:*

(1) *objects are pairs (x, ϕ), where $x \in \mathrm{Ob}(\mathcal{X}_V)$, and $\phi : f^*y \to F(x)$ is a morphism in $\mathcal{Y}_V$,*
(2) *the set of morphisms between (x, ϕ) and (x', ϕ') is the set of morphisms $\psi : x \to x'$ in $\mathcal{X}_V$ such that $F(\psi) = \phi' \circ \phi^{-1}$.*

Proof. See discussion above. □

Lemma 39.4. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. Let $G : \mathcal{C}/U \to \mathcal{Y}$ be a 1-morphism. Then*

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U$$

is a category fibred in groupoids.

Proof. We have already seen in Lemma 33.7 that the composition

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U \longrightarrow \mathcal{C}$$

is a category fibred in groupoids. Then the lemma follows from Lemma 33.12. □

Definition 39.5. Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. We say F is *representable*, or that $\mathcal{X}$ *is relatively representable over* $\mathcal{Y}$, if for every $U \in \mathrm{Ob}(\mathcal{C})$ and any $G : \mathcal{C}/U \to \mathcal{Y}$ the category fibred in groupoids

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U$$

is representable over $\mathcal{C}/U$.

Lemma 39.6. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. If F is representable then every one of the functors*

$$F_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$$

between fibre categories is faithful.

Proof. Clear from the description of fibre categories in Lemma 39.3 and the characterization of representable fibred categories in Lemma 38.2. □

Lemma 39.7. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. Make a choice of pullbacks for $\mathcal{Y}$. Assume*

(1) *each functor $F_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$ between fibre categories is faithful, and*
(2) *for each U and each $y \in \mathcal{Y}_U$ the presheaf*

$$(f : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathcal{X}_V, \phi : f^*y \to F(x)\}/\cong$$

is a representable presheaf on $\mathcal{C}/U$.

Then F is representable.

Proof. Clear from the description of fibre categories in Lemma 39.3 and the characterization of representable fibred categories in Lemma 38.2. □

Before we state the next lemma we point out that the 2-category of categories fibred in groupoids is a $(2, 1)$-category, and hence we know what it means to say that it has a final object (see Definition 29.1). And it has a final object namely $\mathrm{id} : \mathcal{C} \to \mathcal{C}$. Thus we define 2-*products* of categories fibred in groupoids over $\mathcal{C}$ as the 2-fibred products

$$\mathcal{X} \times \mathcal{Y} := \mathcal{X} \times_{\mathcal{C}} \mathcal{Y}.$$

With this definition in place the following lemma makes sense.

Lemma 39.8. *Let $\mathcal{C}$ be a category. Let $\mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:*

(1) *The diagonal $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable.*
(2) *For every U in $\mathcal{C}$, any $G : \mathcal{C}/U \to \mathcal{S}$ is representable.*

Proof. Suppose the diagonal is representable, and let U, G be given. Consider any $V \in \mathrm{Ob}(\mathcal{C})$ and any $G' : \mathcal{C}/V \to \mathcal{S}$. Note that $\mathcal{C}/U \times \mathcal{C}/V = \mathcal{C}/U \times V$ is representable. Hence the fibre product

$$\begin{tikzcd} (\mathcal{C}/U \times V) \times_{(\mathcal{S}\times\mathcal{S})} \mathcal{S} \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\ \mathcal{C}/U \times V \arrow[r, "{(G,G')}"] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

is representable by assumption. This means there exists $W \to U \times V$ in $\mathcal{C}$, such that

$$\begin{tikzcd} \mathcal{C}/W \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\ \mathcal{C}/U \times \mathcal{C}/V \arrow[r] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

is cartesian. This implies that $\mathcal{C}/W \cong \mathcal{C}/U \times_{\mathcal{S}} \mathcal{C}/V$ (see Lemma 29.11) as desired.

Assume (2) holds. Consider any $V \in \mathrm{Ob}(\mathcal{C})$ and any $(G, G') : \mathcal{C}/V \to \mathcal{S} \times \mathcal{S}$. We have to show that $\mathcal{C}/V \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$ is representable. What we know is that $\mathcal{C}/V \times_{G,\mathcal{S},G'} \mathcal{C}/V$ is representable, say by $a : W \to V$ in $\mathcal{C}/V$. The equivalence

$$\mathcal{C}/W \to \mathcal{C}/V \times_{G,\mathcal{S},G'} \mathcal{C}/V$$

followed by the second projection to $\mathcal{C}/V$ gives a second morphism $a' : W \to V$. Consider $W' = W \times_{(a,a'),V\times V} V$. There exists an equivalence

$$\mathcal{C}/W' \cong \mathcal{C}/V \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$$

namely

$$\begin{aligned} \mathcal{C}/W' &\cong \mathcal{C}/W \times_{(\mathcal{C}/V \times \mathcal{C}/V)} \mathcal{C}/V \\ &\cong \big(\mathcal{C}/V \times_{(G,\mathcal{S},G')} \mathcal{C}/V\big) \times_{(\mathcal{C}/V \times \mathcal{C}/V)} \mathcal{C}/V \\ &\cong \mathcal{C}/V \times_{(\mathcal{S}\times\mathcal{S})} \mathcal{S} \end{aligned}$$

(for the last isomorphism see Lemma 29.12) which proves the lemma. □

Biographical notes: Parts of this have been taken from Vistoli's notes [Vis04].

40. A criterion for representability

The following lemma is often useful to prove the existence of universal objects in big categories, please see the discussion in Remark 40.2.

Lemma 40.1. *Let $\mathcal{C}$ be a big*[5] *category which has limits. Let $F : \mathcal{C} \to Sets$ be a functor. Assume that*

(1) *F commutes with limits,*
(2) *there exists a family $\{x_i\}_{i\in I}$ of objects of $\mathcal{C}$ and for each $i \in I$ an element $f_i \in F(x_i)$ such that for $y \in \mathrm{Ob}(\mathcal{C})$ and $g \in F(y)$ there exists an i and a morphism $\varphi : x_i \to y$ with $F(\varphi(f_i)) = g$.*

Then F is representable, i.e., there exists an object x of $\mathcal{C}$ such that

$$F(y) = \mathrm{Mor}_{\mathcal{C}}(x, y)$$

functorially in y.

[5]See Remark 2.2.

Proof. Let $\mathcal{I}$ be the category whose objects are the pairs (x_i, f_i) and whose morphisms $(x_i, f_i) \to (x_{i'}, f_{i'})$ are maps $\varphi : x_i \to x_{i'}$ in $\mathcal{C}$ such that $F(\varphi)(f_i) = f_{i'}$. Set

$$x = \lim_{(x_i, f_i) \in \mathcal{I}} x_i$$

(this will not be the x we are looking for, see below). The limit exists by assumption. As F commutes with limits we have

$$F(x) = \lim_{(x_i, f_i) \in \mathcal{I}} F(x_i).$$

Hence there is a universal element $f \in F(x)$ which maps to $f_i \in F(x_i)$ under F applied to the projection map $x \to x_i$. Using f we obtain a transformation of functors

$$\xi : \mathrm{Mor}_{\mathcal{C}}(x, -) \longrightarrow F(-)$$

see Section 3. Let y be an arbitrary object of $\mathcal{C}$ and let $g \in F(y)$. Choose $x_i \to y$ such that f_i maps to g which is possible by assumption. Then F applied to the maps

$$x \longrightarrow x_i \longrightarrow y$$

(the first being the projection map of the limit defining x) sends f to g. Hence the transformation ξ is surjective.

In order to find the object representing F we let $e : x' \to x$ be the equalizer of all self maps $\varphi : x \to x$ with $F(\varphi)(f) = f$. Since F commutes with limits, it commutes with equalizers, and we see there exists an $f' \in F(x')$ mapping to f in $F(x)$. Since ξ is surjective and since f' maps to x we see that also $\xi' : \mathrm{Mor}_{\mathcal{C}}(x', -) \to F(-)$ is surjective. Finally, suppose that $a, b : x' \to y$ are two maps such that $F(a)(f) = F(b)(f)$. We have to show $a = b$. Consider the equalizer $e' : x'' \to x'$. Again we find $f'' \in F(x'')$ mapping to f'. Choose a map $\psi : x \to x''$ such that $F(\psi)(f) = f''$. Then we see that $e \circ e' \circ \psi : x \to x$ is a morphism with $F(e \circ e' \circ \psi)(f) = f$. Hence $e \circ e' \circ \psi \circ e = e$. This means that $e : x' \to x$ factors through $e'' \circ e : x'' \to x$ and since e and e' are monomorphisms this implies $x'' = x'$, i.e., $a = b$ as desired. □

Remark 40.2. The lemma above is often used to construct the free something on something. For example the free abelian group on a set, the free group on a set, etc. The idea, say in the case of the free group on a set E is to consider the functor

$$F : \mathit{Groups} \to \mathit{Sets}, \quad G \longmapsto \mathrm{Map}(E, G)$$

This functor commutes with limits. As our family of objects we can take a family $E \to G_i$ consisting of groups G_i of cardinality at most $\max(\aleph_0, |E|)$ and set maps $E \to G_i$ such that every isomorphism class of such a structure occurs at least once. Namely, if $E \to G$ is a map from E to a group G, then the subgroup G' generated by the image has cardinality at most $\max(\aleph_0, |G|)$. The lemma tells us the functor is representable, hence there exists a group F_E such that $\mathrm{Mor}_{\mathit{Groups}}(F_E, G) = \mathrm{Map}(E, G)$. In particular, the identity morphism of F_E corresponds to a map $E \to F_E$ and one can show that F_E is generated by the image without imposing any relations.

Another typical application is that we can use the lemma to construct colimits once it is know that limits exist. We illustrate it using the category of topological spaces which has limits by Topology, Lemma 13.1. Namely, suppose that $\mathcal{I} \to \mathit{Top}$, $i \mapsto X_i$ is a functor. Then we can consider

$$F : \mathit{Top} \longrightarrow \mathit{Sets}, \quad Y \longmapsto \lim_{\mathcal{I}} \mathrm{Mor}_{\mathit{Top}}(X_i, Y)$$

This functor commutes with limits. Moreover, given any topological space Y and an element $(\varphi_i : X_i \to Y)$ of $F(Y)$, there is a subspace $Y' \subset Y$ of cardinality at most $|\coprod X_i|$ such that the morphisms φ_i map into Y'. Namely, we can take the induced topology on the union of the images of the φ_i. Thus it is clear that the hypotheses of the lemma are satisfied and we find a topological space X representing the functor F, which precisely means that X is the colimit of the diagram $i \mapsto X_i$.

Theorem 40.3 (Adjoint functor theorem)**.** *Let $G : \mathcal{C} \to \mathcal{D}$ be a functor of big categories. Assume $\mathcal{C}$ has limits, G commutes with them, and for every object y of $\mathcal{D}$ there exists a set of pairs $(x_i, f_i)_{i\in I}$ with $x_i \in \mathrm{Ob}(\mathcal{C})$, $f_i \in \mathrm{Mor}_\mathcal{C}(y, G(x_i))$ such that for any pair (x, f) with $x \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_\mathcal{C}(y, G(x))$ there is an i and a morphism $h : x_i \to x$ such that $f = G(h) \circ f_i$. Then G has a left adjoint F.*

Proof. The assumptions imply that for every object y of $\mathcal{D}$ the functor $x \mapsto \mathrm{Mor}_\mathcal{D}(y, G(x))$ satisfies the assumptions of Lemma 40.1. Thus it is representable by an object, let's call it $F(y)$. An application of Yoneda's lemma (Lemma 3.5) turns the rule $y \mapsto F(y)$ into a functor which by construction is an adjoint to G. We omit the details. □

41. Other chapters

References

[EM45] Samuel Eilenberg Eilenberg and Saunders Mac Lane, *General theory of natural equivalences*, Transactions of the American Mathematical Society **58** (1945), 231–294.

[GZ67] Pierre Gabriel and Michel Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967.

[Ver96] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253, With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.

[Vis04] Angelo Vistoli, *Notes on Grothendieck topologies, fibered categories and descent theory*, 2004.