HOMOLOGICAL ALGEBRA

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1. Introduction

Basic homological algebra will be explained in this document. We add as needed in the other parts, since there is clearly an infinite amount of this stuff around. A reference is [ML63].

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2. Basic notions

The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) Nothing yet.

3. Preadditive and additive categories

Here is the definition of a preadditive category.

Definition 3.1. A category $\mathcal{A}$ is called *preadditive* if each morphism set $\mathrm{Mor}_{\mathcal{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \longrightarrow \mathrm{Mor}(x, z)$$

are bilinear. A functor $F : \mathcal{A} \to \mathcal{B}$ of preadditive categories is called *additive* if and only if $F : \mathrm{Mor}(x, y) \to \mathrm{Mor}(F(x), F(y))$ is a homomorphism of abelian groups for all $x, y \in \mathrm{Ob}(\mathcal{A})$.

In particular for every x, y there exists at least one morphism $x \to y$, namely the zero map.

Lemma 3.2. *Let $\mathcal{A}$ be a preadditive category. Let x be an object of $\mathcal{A}$. The following are equivalent*

(1) *x is an initial object,*
(2) *x is a final object, and*
(3) *$id_x = 0$ in $\mathrm{Mor}_{\mathcal{A}}(x, x)$.*

Furthermore, if such an object 0 exists, then a morphism $\alpha : x \to y$ factors through 0 if and only if $\alpha = 0$.

Proof. Omitted. □

Definition 3.3. In a preadditive category $\mathcal{A}$ we call *zero object*, and we denote it 0 any final and initial object as in Lemma 3.2 above.

Lemma 3.4. *Let $\mathcal{A}$ be a preadditive category. Let $x, y \in \mathrm{Ob}(\mathcal{A})$. If the product $x \times y$ exists, then so does the coproduct $x \coprod y$. If the coproduct $x \coprod y$ exists, then so does the product $x \times y$. In this case also $x \coprod y \cong x \times y$.*

Proof. Suppose that $z = x \times y$ with projections $p : z \to x$ and $q : z \to y$. Denote $i : x \to z$ the morphism corresponding to $(1, 0)$. Denote $j : y \to z$ the morphism corresponding to $(0, 1)$. Thus we have the commutative diagram

$$\begin{tikzcd}
x \arrow[rr, "1"] \arrow[dr, "i"] & & x \\
& z \arrow[ur, "p"] \arrow[dr, "q"] & \\
y \arrow[ur, "j"] \arrow[rr, "1"] & & y
\end{tikzcd}$$

where the diagonal compositions are zero. It follows that $i \circ p + j \circ q : z \to z$ is the identity since it is a morphism which upon composing with p gives p and upon composing with q gives q. Suppose given morphisms $a : x \to w$ and $b : y \to w$.

Then we can form the map $a \circ p + b \circ q : z \to w$. In this way we get a bijection $\mathrm{Mor}(z, w) = \mathrm{Mor}(x, w) \times \mathrm{Mor}(y, w)$ which show that $z = x \coprod y$.

We leave it to the reader to construct the morphisms p, q given a coproduct $x \coprod y$ instead of a product. $\square$

Definition 3.5. Given a pair of objects x, y in a preadditive category $\mathcal{A}$ we call *direct sum*, and we denote it $x \oplus y$ the product $x \times y$ endowed with the morphisms i, j, p, q as in Lemma 3.4 above.

Remark 3.6. Note that the proof of Lemma 3.4 shows that given p and q the morphisms i, j are uniquely determined by the rules $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$. Moreover, we automatically have $i \circ p + j \circ q = \mathrm{id}_{x \oplus y}$. Similarly, given i, j the morphisms p and q are uniquely determined. Finally, given objects x, y, z and morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, then z is the direct sum of x and y with the four morphisms equal to i, j, p, q.

Lemma 3.7. *Let $\mathcal{A}$, $\mathcal{B}$ be preadditive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.*

Proof. Suppose F is additive. A direct sum z of x and y is characterized by having morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, according to Remark 3.6. Clearly $F(x), F(y), F(z)$ and the morphisms $F(i), F(j), F(p), F(q)$ satisfy exactly the same relations (by additivity) and we see that $F(z)$ is a direct sum of $F(x)$ and $F(y)$. $\square$

Definition 3.8. A category $\mathcal{A}$ is called *additive* if it is preadditive and finite products exist, in other words it has a zero object and direct sums.

Namely the empty product is a finite product and if it exists, then it is a final object.

Definition 3.9. Let $\mathcal{A}$ be a preadditive category. Let $f : x \to y$ be a morphism.

(1) A *kernel* of f is a morphism $i : z \to x$ such that (a) $f \circ i = 0$ and (b) for any $i' : z' \to x$ such that $f \circ i' = 0$ there exists a unique morphism $g : z' \to z$ such that $i' = i \circ g$.
(2) If the kernel of f exists, then we denote this $\mathrm{Ker}(f) \to x$.
(3) A *cokernel* of f is a morphism $p : y \to z$ such that (a) $p \circ f = 0$ and (b) for any $p' : y \to z'$ such that $p' \circ f = 0$ there exists a unique morphism $g : z \to z'$ such that $p' = g \circ p$.
(4) If a cokernel of f exists we denote this $y \to \mathrm{Coker}(f)$.
(5) If a kernel of f exists, then a *coimage of* f is a cokernel for the morphism $\mathrm{Ker}(f) \to x$.
(6) If a kernel and coimage exist then we denote this $x \to \mathrm{Coim}(f)$.
(7) If a cokernel of f exists, then the *image of* f is a kernel of the morphism $y \to \mathrm{Coker}(f)$.
(8) If a cokernel and image of f exist then we denote this $\mathrm{Im}(f) \to y$.

We first relate the direct sum to kernels as follows.

Lemma 3.10. *Let $\mathcal{C}$ be a preadditive category. Let $x \oplus y$ with morphisms i, j, p, q as in Lemma 3.4 be a direct sum in $\mathcal{C}$. Then $i : x \to x \oplus y$ is a kernel of $q : x \oplus y \to y$. Dually, p is a cokernel for j.*

Proof. Let $f : z \to x \oplus y$ be a morphism such that $q \circ f = 0$. We have to show that there exists a unique morphism $g : z \to x$ such that $f = i \circ g$. Since $i \circ p + j \circ q$ is the identity on $x \oplus y$ we see that

$$f = (i \circ p + j \circ q) \circ f = i \circ p \circ f$$

and hence $g = p \circ f$ works. Uniquess holds because $p \circ i$ is the identity on x. The proof of the second statement is dual. $\square$

Lemma 3.11. *Let $f : x \to y$ be a morphism in a preadditive category such that the kernel, cokernel, image and coimage all exist. Then f can be factored uniquely as $x \to Coim(f) \to Im(f) \to y$.*

Proof. There is a canonical morphism $\mathrm{Coim}(f) \to y$ because $\mathrm{Ker}(f) \to x \to y$ is zero. The composition $\mathrm{Coim}(f) \to y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $x \to y \to \mathrm{Coker}(f)$ which is zero. Hence $\mathrm{Coim}(f) \to y$ factors uniquely through $\mathrm{Im}(f) \to y$, which gives us the desired map. $\square$

Example 3.12. Let k be a field. Consider the category of filtered vector spaces over k. (See Definition 16.1.) Consider the filtered vector spaces (V, F) and (W, F) with $V = W = k$ and

$$F^iV = \begin{cases} V & \text{if} \quad i < 0 \\ 0 & \text{if} \quad i \geq 0 \end{cases} \text{ and } F^iW = \begin{cases} W & \text{if} \quad i \leq 0 \\ 0 & \text{if} \quad i > 0 \end{cases}$$

The map $f : V \to W$ corresponding to id_k on the underlying vector spaces has trivial kernel and cokernel but is not an isomorphism. Note also that $\mathrm{Coim}(f) = V$ and $\mathrm{Im}(f) = W$. This means that the category of filtered vector spaces over k is not abelian.

4. Karoubian categories

Skip this section on a first reading.

Definition 4.1. Let $\mathcal{C}$ be a preadditive category. We say $\mathcal{C}$ is *Karoubian* if every idempotent endomorphism of an object of $\mathcal{C}$ has a kernel.

The dual notion would be that every idempotent endomorphism of an object has a cokernel. However, in view of the (dual of the) following lemma that would be an equivalent notion.

Lemma 4.2. *Let $\mathcal{C}$ be a preadditive category. The following are equivalent*

1. *$\mathcal{C}$ is Karoubian,*
2. *every idempotent endomorphism of an object of $\mathcal{C}$ has a cokernel, and*
3. *given an idempotent endomorphism $p : z \to z$ of $\mathcal{C}$ there exists a direct sum decomposition $z = x \oplus y$ such that p corresponds to the projection onto y.*

Proof. Assume (1) and let $p : z \to z$ be as in (3). Let $x = \mathrm{Ker}(p)$ and $y = \mathrm{Ker}(1 - p)$. There are maps $x \to z$ and $y \to z$. Since $(1 - p)p = 0$ we see that $p : z \to z$ factors through y, hence we obtain a morphism $z \to y$. Similarly we obtain a morphism $z \to x$. We omit the verification that these four morphisms induce an isomorphism $x = y \oplus z$ as in Remark 3.6. Thus (1) $\Rightarrow$ (3). The implication (2) $\Rightarrow$ (3) is dual. Finally, condition (3) implies (1) and (2) by Lemma 3.10. $\square$

Lemma 4.3. *Let $\mathcal{D}$ be a preadditive category.*

1. *If $\mathcal{D}$ has countable products and kernels of maps which have a right inverse, then $\mathcal{D}$ is Karoubian.*
2. *If $\mathcal{D}$ has countable coproducts and cokernels of maps which have a left inverse, then $\mathcal{D}$ is Karoubian.*

Proof. Let X be an object of $\mathcal{D}$ and let $e : X \to X$ be an idempotent. The functor

$$W \longmapsto \mathrm{Ker}(\mathrm{Mor}_{\mathcal{D}}(W, X) \xrightarrow{e} \mathrm{Mor}_{\mathcal{D}}(W, X))$$

if representable if and only if e has a kernel. Note that for any abelian group A and idempotent endomorphism $e : A \to A$ we have

$$\mathrm{Ker}(e : A \to A) = \mathrm{Ker}(\Phi : \prod\nolimits_{n \in \mathbf{N}} A \to \prod\nolimits_{n \in \mathbf{N}} A)$$

where

$$\Phi(a_1, a_2, a_3, \ldots) = (ea_1 + (1 - e)a_2, ea_2 + (1 - e)a_3, \ldots)$$

Moreover, Φ has the right inverse

$$\Psi(a_1, a_2, a_3, \ldots) = (a_1, (1 - e)a_1 + ea_2, (1 - e)a_2 + ea_3, \ldots).$$

Hence (1) holds. The proof of (2) is dual (using the dual definition of a Karoubian category, namely condition (2) of Lemma 4.2). $\square$

5. Abelian categories

An abelian category is a category satisfying just enough axioms so the snake lemma holds. An axiom (that is sometimes forgotten) is that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ of Lemma 3.11 is always an isomorphism. Example 3.12 shows that it is necessary.

Definition 5.1. A category $\mathcal{A}$ is *abelian* if it is additive, if all kernels and cokernels exist, and if the natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism for all morphisms f of $\mathcal{A}$.

Lemma 5.2. *Let $\mathcal{A}$ be a preadditive category. The additions on sets of morphisms make $\mathcal{A}^{opp}$ into a preadditive category. Furthermore, $\mathcal{A}$ is additive if and only if $\mathcal{A}^{opp}$ is additive, and $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{opp}$ is abelian.*

Proof. Omitted. $\square$

Definition 5.3. Let $f : x \to y$ be a morphism in an abelian category.

1. We say f is *injective* if $\mathrm{Ker}(f) = 0$.
2. We say f is *surjective* if $\mathrm{Coker}(f) = 0$.

If $x \to y$ is injective, then we say that x is a *subobject* of y and we use the notation $x \subset y$. If $x \to y$ is surjective, then we say that y is a *quotient* of x.

Lemma 5.4. *Let $f : x \to y$ be a morphism in an abelian category. Then*

1. *f is injective if and only if f is a monomorphism, and*
2. *f is surjective if and only if f is an epimorphism.*

Proof. Omitted. $\square$

In an abelian category, if $x \subset y$ is a subobject, then we denote

$$x/y = \mathrm{Coker}(x \to y).$$

Lemma 5.5. *Let $\mathcal{A}$ be an abelian category. All finite limits and finite colimits exist in $\mathcal{A}$.*

Proof. To show that finite limits exist it suffices to show that finite products and equalizers exist, see Categories, Lemma 18.4. Finite products exist by definition and the equalizer of $a, b : x \to y$ is the kernel of $a - b$. The argument for finite colimits is similar but dual to this. $\square$

Example 5.6. Let $\mathcal{A}$ be an abelian category. Pushouts and fibre products in $\mathcal{A}$ have the following simple descriptions:

(1) If $a : x \to y$, $b : z \to y$ are morphisms in $\mathcal{A}$, then we have the fibre product: $x \times_y z = \mathrm{Ker}((a, -b) : x \oplus z \to y)$.
(2) If $a : y \to x$, $b : y \to z$ are morphisms in $\mathcal{A}$, then we have the pushout: $x \amalg_y z = \mathrm{Coker}((a, -b) : y \to x \oplus z)$.

Definition 5.7. Let $\mathcal{A}$ be an additive category. We say a sequence of morphisms

$$\ldots \to x \to y \to z \to \ldots$$

in $\mathcal{A}$ is a *complex* if the composition of any two (drawn) arrows is zero. If $\mathcal{A}$ is abelian then we say a sequence as above is *exact at y* if $\mathrm{Im}(x \to y) = \mathrm{Ker}(y \to z)$. We say it is *exact* if it is exact at every object. A *short exact sequence* is an exact complex of the form

$$0 \to A \to B \to C \to 0.$$

In the following lemma we assume the reader knows what it means for a sequence of abelian groups to be exact.

Lemma 5.8. *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a complex of $\mathcal{A}$.*

(1) *$M_1 \to M_2 \to M_3 \to 0$ is exact if and only if*
$$0 \to \mathrm{Hom}_{\mathcal{A}}(M_3, N) \to \mathrm{Hom}_{\mathcal{A}}(M_2, N) \to \mathrm{Hom}_{\mathcal{A}}(M_1, N)$$
is an exact sequence of abelian groups for all objects N of $\mathcal{A}$, and
(2) *$0 \to M_1 \to M_2 \to M_3$ is exact if and only if*
$$0 \to \mathrm{Hom}_{\mathcal{A}}(N, M_1) \to \mathrm{Hom}_{\mathcal{A}}(N, M_2) \to \mathrm{Hom}_{\mathcal{A}}(N, M_1)$$
is an exact sequence of abelian groups for all objects N of $\mathcal{A}$.

Proof. Omitted. Hint: See Algebra, Lemma 10.1. $\square$

Definition 5.9. Let $\mathcal{A}$ be an abelian category. Let $i : A \to B$ and $q : B \to C$ be morphisms of $\mathcal{A}$ such that $0 \to A \to B \to C \to 0$ is a short exact sequence. We say the short exact sequence is *split* if there exist morphisms $j : C \to B$ and $p : B \to A$ such that (B, i, j, p, q) is the direct sum of A and C.

Lemma 5.10. *Let $\mathcal{A}$ be an abelian category. Let $0 \to A \to B \to C \to 0$ be a short exact sequence.*

(1) *Given a morphism $s : C \to B$ left inverse to $B \to C$, there exists a unique $\pi : B \to A$ such that (s, π) splits the short exact sequence as in Definition 5.9.*

(2) *Given a morphism $\pi : B \to A$ right inverse to $A \to B$, there exists a unique $s : C \to B$ such that (s, π) splits the short exact sequence as in Definition 5.9.*

Proof. Omitted. $\square$

Lemma 5.11. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ x \arrow[r, "k"] & z \end{tikzcd}$$

be a commutative diagram.

(1) *The diagram is cartesian if and only if*

$$0 \to w \xrightarrow{(g,f)} x \oplus y \xrightarrow{(k,-h)} z$$

is exact.

(2) *The diagram is cocartesian if and only if*

$$w \xrightarrow{(g,-f)} x \oplus y \xrightarrow{(k,h)} z \to 0$$

is exact.

Proof. Let $u = (g, f) : w \to x \oplus y$ and $v = (k, -h) : x \oplus y \to z$. Let $p : x \oplus y \to x$ and $q : x \oplus y \to y$ be the canonical projections. Let $i : \mathrm{Ker}(v) \to x \oplus y$ be the canonical injection. By Example 5.6, the diagram is cartesian if and only if there exists an isomorphism $r : \mathrm{Ker}(v) \to w$ with $f \circ r = q \circ i$ and $g \circ r = p \circ i$. The sequence $0 \to w \overset{u}{\to} x \oplus y \overset{v}{\to} z$ is exact if and only if there exists an isomorphism $r : \mathrm{Ker}(v) \to w$ with $u \circ r = i$. But given $r : \mathrm{Ker}(v) \to w$, we have $f \circ r = q \circ i$ and $g \circ r = p \circ i$ if and only if $q \circ u \circ r = f \circ r = q \circ i$ and $p \circ u \circ r = g \circ r = p \circ i$, hence if and only if $u \circ r = i$. This proves (1), and then (2) follows by duality. $\square$

Lemma 5.12. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ x \arrow[r, "k"] & z \end{tikzcd}$$

be a commutative diagram.

(1) *If the diagram is cartesian, then the morphism Ker$(f) \to$ Ker(k) induced by g is an isomorphism.*
(2) *If the diagram is cocartesian, then the morphism Coker$(f) \to$ Coker(k) induced by h is an isomorphism.*

Proof. Suppose the diagram is cartesian. Let $e : \mathrm{Ker}(f) \to \mathrm{Ker}(k)$ be induced by g. Let $i : \mathrm{Ker}(f) \to w$ and $j : \mathrm{Ker}(k) \to x$ be the canonical injections. There exists $t : \mathrm{Ker}(k) \to w$ with $f \circ t = 0$ and $g \circ t = j$. Hence, there exists $u : \mathrm{Ker}(k) \to \mathrm{Ker}(f)$ with $i \circ u = t$. It follows $g \circ i \circ u \circ e = g \circ t \circ e = j \circ e = g \circ i$ and $f \circ i \circ u \circ e = 0 = f \circ i$, hence $i \circ u \circ e = i$. Since i is a monomorphism this implies $u \circ e = \mathrm{id}_{\mathrm{Ker}(f)}$. Furthermore, we have $j \circ e \circ u = g \circ i \circ u = g \circ t = j$. Since j is a monomorphism this implies $e \circ u = \mathrm{id}_{\mathrm{Ker}(k)}$. This proves (1). Now, (2) follows by duality. $\square$

Lemma 5.13. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ x \arrow[r, "k"] & z \end{tikzcd}$$

be a commutative diagram.

(1) *If the diagram is cartesian and k is an epimorphism, then the diagram is cocartesian and f is an epimorphism.*
(2) *If the diagram is cocartesian and g is a monomorphism, then the diagram is cartesian and h is a monomorphism.*

Proof. Suppose the diagram is cartesian and k is an epimorphism. Let $u = (g, f) : w \to x \oplus y$ and let $v = (k, -h) : x \oplus y \to z$. As k is an epimorphism, v is an epimorphism, too. Therefore and by Lemma 5.11, the sequence $0 \to w \xrightarrow{u} x \oplus y \xrightarrow{v} z \to 0$ is exact. Thus, the diagram is cocartesian by Lemma 5.11. Finally, f is an epimorphism by Lemma 5.12 and Lemma 5.4. This proves (1), and (2) follows by duality. $\square$

Lemma 5.14. *Let $\mathcal{A}$ be an abelian category.*

(1) *If $x \to y$ is surjective, then for every $z \to y$ the projection $x \times_y z \to z$ is surjective.*
(2) *If $x \to y$ is injective, then for every $x \to z$ the morphism $z \to z \amalg_x y$ is injective.*

Proof. Immediately from Lemma 5.4 and Lemma 5.13. $\square$

Lemma 5.15. *Let $\mathcal{A}$ be an abelian category. Let $f : x \to y$ and $g : y \to z$ be morphisms with $g \circ f = 0$. Then, the following statements are equivalent:*

(1) *The sequence $x \xrightarrow{f} y \xrightarrow{g} z$ is exact.*
(2) *For every $h : w \to y$ with $g \circ h = 0$ there exist an object v, an epimorphism $k : v \to w$ and a morphism $l : v \to x$ with $h \circ k = f \circ l$.*

Proof. Let $i : \mathrm{Ker}(g) \to y$ be the canonical injection. Let $p : x \to \mathrm{Coim}(f)$ be the canonical projection. Let $j : \mathrm{Im}(f) \to \mathrm{Ker}(g)$ be the canonical injection.

Suppose (1) holds. Let $h : w \to y$ with $g \circ h = 0$. There exists $c : w \to \mathrm{Ker}(g)$ with $i \circ c = h$. Let $v = x \times_{\mathrm{Ker}(g)} w$ with canonical projections $k : v \to w$ and $l : v \to x$, so that $c \circ k = p \circ l$. Then, $h \circ k = i \circ c \circ k = i \circ j \circ p \circ l = f \circ l$. As $j \circ p$ is an epimorphism by hypothesis, k is an epimorphism by Lemma 5.13. This implies (2).

Suppose (2) holds. Then, $g \circ i = 0$. So, there are an object w, an epimorphism $k : w \to \mathrm{Ker}(g)$ and a morphism $l : w \to x$ with $f \circ l = i \circ k$. It follows $i \circ j \circ p \circ l = f \circ l = i \circ k$. Since i is a monomorphism we see that $j \circ p \circ l = k$ is an epimorphism. So, j is an epimorphisms and thus an isomorphism. This implies (1). $\square$

Lemma 5.16. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\alpha"] & y \arrow[r, "g"] \arrow[d, "\beta"] & z \arrow[d, "\gamma"] \\ u \arrow[r, "k"] & v \arrow[r, "l"] & w \end{tikzcd}$$

be a commutative diagram.

(1) *If the first row is exact and k is a monomorphism, then the induced sequence $Ker(\alpha) \to Ker(\beta) \to Ker(\gamma)$ is exact.*
(2) *If the second row is exact and g is an epimorphism, then the induced sequence $Coker(\alpha) \to Coker(\beta) \to Coker(\gamma)$ is exact.*

Proof. Suppose the first row is exact and k is a monomorphism. Let $a : \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta)$ and $b : \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma)$ be the induced morphisms. Let $h : \mathrm{Ker}(\alpha) \to x$, $i : \mathrm{Ker}(\beta) \to y$ and $j : \mathrm{Ker}(\gamma) \to z$ be the canonical injections. As j is a monomorphism we have $b \circ a = 0$. Let $c : s \to \mathrm{Ker}(\beta)$ with $b \circ c = 0$. Then, $g \circ i \circ c = j \circ b \circ c = 0$. By Lemma 5.15 there are an object t, an epimorphism $d : t \to s$ and a morphism $e : t \to x$ with $i \circ c \circ d = f \circ e$. Then, $k \circ \alpha \circ e = \beta \circ f \circ e = \beta \circ i \circ c \circ d = 0$. As k is a monomorphism we get $\alpha \circ e = 0$. So, there exists $m : t \to \mathrm{Ker}(\alpha)$ with $h \circ m = e$. It follows $i \circ a \circ m = f \circ h \circ m = f \circ e = i \circ c \circ d$. As i is a monomorphism we get $a \circ m = c \circ d$. Thus, Lemma 5.15 implies (1), and then (2) follows by duality. $\square$

Lemma 5.17. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd}
 & x \arrow[r, "f"] \arrow[d, "\alpha"] & y \arrow[r, "g"] \arrow[d, "\beta"] & z \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & u \arrow[r, "k"] & v \arrow[r, "l"] & w &
\end{tikzcd}$$

be a commutative diagram with exact rows.

(1) *There exists a unique morphism $\delta : Ker(\gamma) \to Coker(\alpha)$ such that the diagram*

$$\begin{tikzcd}
y \arrow[d, "\beta"'] & y \times_z Ker(\gamma) \arrow[l, "\pi'"'] \arrow[r, "\pi"] & Ker(\gamma) \arrow[d, "\delta"] \\
v \arrow[r, "\iota'"] & Coker(\alpha) \amalg_u v & Coker(\alpha) \arrow[l, "\iota"']
\end{tikzcd}$$

commutes, where π and π' are the canonical projections and ι and ι' are the canonical coprojections.
(2) *The induced sequence*

$$Ker(\alpha) \xrightarrow{f'} Ker(\beta) \xrightarrow{g'} Ker(\gamma) \xrightarrow{\delta} Coker(\alpha) \xrightarrow{k'} Coker(\beta) \xrightarrow{l'} Coker(\gamma)$$

is exact. If f is injective then so is f', and if l is surjective then so is l'.

Proof. As π is an epimorphism and ι is a monomorphism by Lemma 5.13, uniqueness of δ is clear. Let $p = y \times_z \mathrm{Ker}(\gamma)$ and $q = \mathrm{Coker}(\alpha) \amalg_u v$. Let $h : \mathrm{Ker}(\beta) \to y$, $i : \mathrm{Ker}(\gamma) \to z$ and $j : \mathrm{Ker}(\pi) \to p$ be the canonical injections. Let $p : u \to \mathrm{Coker}(\alpha)$ be the canonical projection. Keeping in mind Lemma 5.13 we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\pi) \arrow[r, "j"] & p \arrow[r, "\pi"] \arrow[d, "\pi'"'] & \mathrm{Ker}(\gamma) \arrow[r] \arrow[d, "i"'] & 0 \\
 & x \arrow[r, "f"] \arrow[d, "\alpha"'] & y \arrow[r, "g"] \arrow[d, "\beta"'] & z \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & u \arrow[r, "k"] \arrow[d, "p"'] & v \arrow[r, "l"] \arrow[d, "\iota'"'] & w & \\
0 \arrow[r] & \mathrm{Coker}(\alpha) \arrow[r, "\iota"] & q & &
\end{tikzcd}$$

As $l \circ \beta \circ \pi' = \gamma \circ i \circ \pi = 0$ and as the third row of the diagram above is exact, there is an $a : p \to u$ with $k \circ a = \beta \circ \pi'$. As the upper right quadrangle of the diagram above is cartesian, Lemma 5.12 yields an epimorphism $b : x \to \mathrm{Ker}(\pi)$ with $\pi' \circ j \circ b = f$. It follows $k \circ a \circ j \circ b = \beta \circ \pi' \circ j \circ b = \beta \circ f = k \circ \alpha$. As k is a monomorphism this implies $a \circ j \circ b = \alpha$. It follows $p \circ a \circ j \circ b = p \circ \alpha = 0$. As b is an epimorphism this implies $p \circ a \circ j = 0$. Therefore, as the top row of the diagram above is exact, there exists $\delta : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ with $\delta \circ \pi = p \circ a$. It follows $\iota \circ \delta \circ \pi = \iota \circ p \circ a = \iota' \circ k \circ a = \iota' \circ \beta \circ \pi'$ as desired.

As the upper right quadrangle in the diagram above is cartesian there is a $c : \mathrm{Ker}(\beta) \to p$ with $\pi' \circ c = h$ and $\pi \circ c = g'$. It follows $\iota \circ \delta \circ g' = \iota \circ \delta \circ \pi \circ c = \iota' \circ \beta \circ \pi' \circ c = \iota' \circ \beta \circ h = 0$. As ι is a monomorphism this implies $\delta \circ g' = 0$.

Next, let $d : r \to \mathrm{Ker}(\gamma)$ with $\delta \circ d = 0$. Applying Lemma 5.15 to the exact sequence $p \xrightarrow{\pi} \mathrm{Ker}(\gamma) \to 0$ and d yields an object s, an epimorphism $m : s \to r$ and a morphism $n : s \to p$ with $\pi \circ n = d \circ m$. As $p \circ a \circ n = \delta \circ d \circ m = 0$, applying Lemma 5.15 to the exact sequence $x \xrightarrow{\alpha} u \xrightarrow{p} \mathrm{Coker}(\alpha)$ and $a \circ n$ yields an object t, an epimorphism $\varepsilon : t \to s$ and a morphism $\zeta : t \to x$ with $a \circ n \circ \varepsilon = \alpha \circ \zeta$. It holds $\beta \circ \pi' \circ n \circ \varepsilon = k \circ \alpha \circ \zeta = \beta \circ f \circ \zeta$. Let $\eta = \pi' \circ n \circ \varepsilon - f \circ \zeta : t \to y$. Then, $\beta \circ \eta = 0$. It follows that there is a $\vartheta : t \to \mathrm{Ker}(\beta)$ with $\eta = h \circ \vartheta$. It holds $i \circ g' \circ \vartheta = g \circ h \circ \vartheta = g \circ \pi' \circ n \circ \varepsilon - g \circ f \circ \zeta = i \circ \pi \circ n \circ \varepsilon = i \circ d \circ m \circ \varepsilon$. As i is a monomorphism we get $g' \circ \vartheta = d \circ m \circ \varepsilon$. Thus, as $m \circ \varepsilon$ is an epimorphism, Lemma 5.15 implies that $\mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha)$ is exact. Then, the claim follows by Lemma 5.16 and duality. $\square$

Lemma 5.18. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd}
 & & & x \arrow[rr] \arrow[dl] \arrow[dd, "\alpha"] & & y \arrow[rr] \arrow[dl] \arrow[dd, "\beta"] & & z \arrow[rr] \arrow[dl] \arrow[dd, "\gamma"] & & 0 \\
 & & x' \arrow[rr] \arrow[dd, "\alpha'"] & & y' \arrow[rr] \arrow[dd, "\beta'"] & & z' \arrow[rr] \arrow[dd, "\gamma'"] & & 0 & \\
 & 0 \arrow[rr] & & u \arrow[rr] \arrow[dl] & & v \arrow[rr] \arrow[dl] & & w \arrow[dl] & & \\
0 \arrow[rr] & & u' \arrow[rr] & & v' \arrow[rr] & & w' & & &
\end{tikzcd}$$

be a commutative diagram with exact rows. Then, the induced diagram

$$\begin{tikzcd}
Ker(\alpha) \arrow[r] \arrow[d] & Ker(\beta) \arrow[r] \arrow[d] & Ker(\gamma) \arrow[r, "\delta"] \arrow[d] & Coker(\alpha) \arrow[r] \arrow[d] & Coker(\beta) \arrow[r] \arrow[d] & Coker(\gamma) \arrow[d] \\
Ker(\alpha') \arrow[r] & Ker(\beta') \arrow[r] & Ker(\gamma') \arrow[r, "\delta'"] & Coker(\alpha') \arrow[r] & Coker(\beta') \arrow[r] & Coker(\gamma')
\end{tikzcd}$$

commutes.

Proof. Omitted. □

Lemma 5.19. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd}
w \arrow[r] \arrow[d, "\alpha"] & x \arrow[r] \arrow[d, "\beta"] & y \arrow[r] \arrow[d, "\gamma"] & z \arrow[d, "\delta"] \\
w' \arrow[r] & x' \arrow[r] & y' \arrow[r] & z'
\end{tikzcd}$$

be a commutative diagram with exact rows.

(1) *If α, γ are surjective and δ is injective, then β is surjective.*
(2) *If β, δ are injective and α is surjective, then γ is injective.*

Proof. Assume α, γ are surjective and δ is injective. We may replace w' by $\mathrm{Im}(w' \to x')$, i.e., we may assume that $w' \to x'$ is injective. We may replace z by $\mathrm{Im}(y \to z)$, i.e., we may assume that $y \to z$ is surjective. Then we may apply Lemma 5.17 to

$$\begin{tikzcd}
 & \mathrm{Ker}(y \to z) \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(y' \to z') \arrow[r] & y' \arrow[r] & z' &
\end{tikzcd}$$

to conclude that $\mathrm{Ker}(y \to z) \to \mathrm{Ker}(y' \to z')$ is surjective. Finally, we apply Lemma 5.17 to

$$\begin{tikzcd}
 & w \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & \mathrm{Ker}(y \to z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & w' \arrow[r] & x' \arrow[r] & \mathrm{Ker}(y' \to z') &
\end{tikzcd}$$

to conclude that $x \to x'$ is surjective. This proves (1). The proof of (2) is dual to this. □

Lemma 5.20. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd}
v \arrow[r] \arrow[d, "\alpha"] & w \arrow[r] \arrow[d, "\beta"] & x \arrow[r] \arrow[d, "\gamma"] & y \arrow[r] \arrow[d, "\delta"] & z \arrow[d, "\epsilon"] \\
v' \arrow[r] & w' \arrow[r] & x' \arrow[r] & y' \arrow[r] & z'
\end{tikzcd}$$

be a commutative diagram with exact rows. If β, δ are isomorphisms, ϵ is injective, and α is surjective then γ is an isomorphism.

Proof. Immediate consequence of Lemma 5.19. □

6. Extensions

Definition 6.1. Let $\mathcal{A}$ be an abelian category. Let $A, C \in \mathrm{Ob}(\mathcal{A})$. An *extension E of B by A* is a short exact sequence

$$0 \to A \to E \to B \to 0.$$

By abuse of language we often omit mention of the morphisms $A \to E$ and $E \to B$, although they are definitively part of the structure of an extension.

Definition 6.2. Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. The set of isomorphism classes of extensions of B by A is denoted

$$\mathrm{Ext}_{\mathcal{A}}(B, A).$$

This is called the *Ext-group*.

This definition works, because by our conventions $\mathcal{A}$ is a set, and hence $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set. In any of the cases of "big" abelian categories listed in Categories, Remark 2.2. one can check by hand that $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set as well. Also, we will see later that this is always the case when $\mathcal{A}$ has either enough projectives or enough injectives. Insert future reference here.

Actually we can turn $\mathrm{Ext}_{\mathcal{A}}(-,-)$ into a functor

$$\mathcal{A}^{opp} \times \mathcal{A} \longrightarrow Sets, \quad (A, B) \longmapsto \mathrm{Ext}_{\mathcal{A}}(A, B)$$

as follows:

(1) Given a morphism $B' \to B$ and an extension E of B by A we define $E' = E \times_B B'$ so that we have the following commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & E \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

The extension E' is called the *pullback of E via $B' \to B$*.

(2) Given a morphism $A \to A'$ and an extension E of B by A we define $E' = A' \coprod_A E$ so that we have the following commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & E' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

The extension E' is called the *pushout of E via $A \to A'$*.

To see that this defines a functor as indicated above there are several things to verify. First of all functoriality in the variable B requires that $(E \times_B B') \times_{B'} B'' = E \times_B B''$ which is a general property of fibre products. Dually one deals with functoriality in the variable A. Finally, given $A \to A'$ and $B' \to B$ we have to show that

$$A' \coprod\nolimits_A (E \times_B B') \cong (A' \coprod\nolimits_A E) \times_B B'$$

as extensions of B' by A'. Recall that $A' \coprod_A E$ is a quotient of $A' \oplus E$. Thus the right hand side is a quotient of $A' \oplus E \times_B B'$, and it is straightforward to see that the kernel is exactly what you need in order to get the left hand side.

Note that if E_1 and E_2 are extensions of B by A, then $E_1 \oplus E_2$ is an extension of $B \oplus B$ by $A \oplus A$. We pull back by the diagonal map $B \to B \oplus B$ and we push out by the sum map $A \oplus A \to A$ to get an extension $E_1 + E_2$ of B by A.

$$\begin{tikzcd}
0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "\Sigma"] & E_1 \oplus E_2 \arrow[r] \arrow[d] & B \oplus B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \oplus B \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u] & E_1 + E_2 \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, "\Delta"'] & 0
\end{tikzcd}$$

The extension $E_1 + E_2$ is called the *Baer sum* of the given extensions.

Lemma 6.3. *The construction $(E_1, E_2) \mapsto E_1 + E_2$ above defines a commutative group law on $Ext_{\mathcal{A}}(B, A)$ which is functorial in both variables.*

Proof. Omitted. □

Lemma 6.4. *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathcal{A}$.*

(1) *There is a canonical six term exact sequence of abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M_3, N) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M_2, N) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M_1, N) \arrow[dll] \\
 & Ext_{\mathcal{A}}(M_3, N) \arrow[r] & Ext_{\mathcal{A}}(M_2, N) \arrow[r] & Ext_{\mathcal{A}}(M_1, N)
\end{tikzcd}$$

for all objects N of $\mathcal{A}$, and

(2) *there is a canonical six term exact sequence of abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(N, M_1) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(N, M_2) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(N, M_3) \arrow[dll] \\
 & Ext_{\mathcal{A}}(N, M_1) \arrow[r] & Ext_{\mathcal{A}}(N, M_2) \arrow[r] & Ext_{\mathcal{A}}(N, M_3)
\end{tikzcd}$$

for all objects N of $\mathcal{A}$.

Proof. Omitted. Hint: The boundary maps are defined using either the pushout or pullback of the given short exact sequence. □

7. Additive functors

Recall that we defined, in Categories, Definition 23.1 the notion of a "right exact", "left exact" and "exact" functor in the setting of a functor between categories that have finite (co)limits. Thus this applies in particular to functors between abelian categories.

Lemma 7.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.*

(1) *If F is either left or right exact, then it is additive.*
(2) *If F is additive then it is left exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact.*

(3) *If F is additive then it is right exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact.*
(4) *If F is additive then it is exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.*

Proof. Let us first note that if F commutes with the empty limit or the empty colimit, then $F(0) = 0$. In particular F applied to the zero morphism is zero. We will use this below without mention.

Suppose that F is left exact, i.e., commutes with finite limits. Then $F(A \times A) = F(A) \times F(A)$ with projections $F(p)$ and $F(q)$. Hence $F(A \oplus A) = F(A) \oplus F(A)$ with all four morphisms $F(i), F(j), F(p), F(q)$ equal to their counterparts in $\mathcal{B}$ as they satisfy the same relations, see Remark 3.6. Then $f = F(p+q)$ is a morphism $f : F(A) \oplus F(A) \to F(A)$ such that $f \circ F(i) = F(p \circ i + q \circ i) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. And similarly $f \circ F(j) = \mathrm{id}_A$. We conclude that $F(p+q) = F(p) + F(q)$. For any pair of morphisms $a, b : B \to A$ the map $g = F(i \circ a + j \circ b) : F(B) \to F(A) \oplus F(A)$ is a morphism such that $F(p) \circ g = F(p \circ (i \circ a + j \circ b)) = F(a)$ and similarly $F(q) \circ g = F(b)$. Hence $g = F(i) \circ F(a) + F(j) \circ F(b)$. The sum of a and b is the composition

$$B \xrightarrow{i \circ a + j \circ b} A \oplus A \xrightarrow{p+q} A.$$

Applying F we get

$$F(B) \xrightarrow{F(i) \circ F(a) + F(j) \circ F(b)} F(A) \oplus F(A) \xrightarrow{F(p)+F(q)} A.$$

where we used the expressions for f and g obtained above. Hence F is additive.[1]

Denote $f : B \to C$ a map from B to C. Exactness of $0 \to A \to B \to C$ just means that $A = \mathrm{Ker}(f)$. Clearly the kernel of f is the equalizer of the two maps f and 0 from B to C. Hence if F commutes with limits, then $F(\mathrm{Ker}(f)) = \mathrm{Ker}(F(f))$ which exactly means that $0 \to F(A) \to F(B) \to F(C)$ is exact.

Conversely, suppose that F is additive and transforms any short exact sequence $0 \to A \to B \to C$ into an exact sequence $0 \to F(A) \to F(B) \to F(C)$. Because it is additive it commutes with direct sums and hence finite products in $\mathcal{A}$. To show it commutes with finite limits it therefore suffices to show that it commutes with equalizers. But equalizers in an abelian category are the same as the kernel of the difference map, hence it suffices to show that F commutes with taking kernels. Let $f : A \to B$ be a morphism. Factor f as $A \to I \to B$ with $f' : A \to I$ surjective and $i : I \to B$ injective. (This is possible by the definition of an abelian category.) Then it is clear that $\mathrm{Ker}(f) = \mathrm{Ker}(f')$. Also $0 \to \mathrm{Ker}(f') \to A \to I \to 0$ and $0 \to I \to B \to B/I \to 0$ are short exact. By the condition imposed on F we see that $0 \to F(\mathrm{Ker}(f')) \to F(A) \to F(I)$ and $0 \to F(I) \to F(B) \to F(B/I)$ are exact. Hence it is also the case that $F(\mathrm{Ker}(f'))$ is the kernel of the map $F(A) \to F(B)$, and we win.

The proof of (3) is similar to the proof of (2). Statement (4) is a combination of (2) and (3). □

[1] I'm sure there is an infinitely slicker proof of this.

Lemma 7.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. For every pair of objects A, B of $\mathcal{A}$ the functor F induces an abelian group homomorphism*

$$Ext_{\mathcal{A}}(B, A) \longrightarrow Ext_{\mathcal{B}}(F(B), F(A))$$

which maps the extension E to $F(E)$.

Proof. Omitted. □

The following lemma is used in the proof that the category of abelian sheaves on a site is abelian, where the functor b is sheafification.

Lemma 7.3. *Let $a : \mathcal{A} \to \mathcal{B}$ and $b : \mathcal{B} \to \mathcal{A}$ be functors. Assume that*

1. *$\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b,*
2. *$\mathcal{B}$ is abelian and b is left exact, and*
3. *$ba \cong id_{\mathcal{A}}$.*

Then $\mathcal{A}$ is abelian.

Proof. As $\mathcal{B}$ is abelian we see that all finite limits and colimits exist in $\mathcal{B}$ by Lemma 5.5. Since b is a left adjoint we see that b is also right exact and hence exact, see Categories, Lemma 24.5. Let $\varphi : B_1 \to B_2$ be a morphism of $\mathcal{B}$. In particular, if $K = \text{Ker}(B_1 \to B_2)$, then K is the equalizer of 0 and φ and hence bK is the equalizer of 0 and $b\varphi$, hence bK is the kernel of $b\varphi$. Similarly, if $Q = \text{Coker}(B_1 \to B_2)$, then Q is the coequalizer of 0 and φ and hence bQ is the coequalizer of 0 and $b\varphi$, hence bQ is the cokernel of $b\varphi$. Thus we see that every morphism of the form $b\varphi$ in $\mathcal{A}$ has a kernel and a cokernel. However, since $ba \cong \text{id}$ we see that every morphism of $\mathcal{A}$ is of this form, and we conclude that kernels and cokernels exist in $\mathcal{A}$. In fact, the argument shows that if $\psi : A_1 \to A_2$ is a morphism then

$$\text{Ker}(\psi) = b\text{Ker}(a\psi), \quad \text{and} \quad \text{Coker}(\psi) = b\text{Coker}(a\psi).$$

Now we still have to show that $\text{Coim}(\psi) = \text{Im}(\psi)$. We do this as follows. First note that since $\mathcal{A}$ has kernels and cokernels it has all finite limits and colimits (see proof of Lemma 5.5). Hence we see by Categories, Lemma 24.5 that a is left exact and hence transforms kernels (=equalizers) into kernels.

$$\begin{aligned}
\text{Coim}(\psi) &= \text{Coker}(\text{Ker}(\psi) \to A_1) && \text{by definition} \\
&= b\text{Coker}(a(\text{Ker}(\psi) \to A_1)) && \text{by formula above} \\
&= b\text{Coker}(\text{Ker}(a\psi) \to aA_1)) && a \text{ preserves kernels} \\
&= b\text{Coim}(a\psi) && \text{by definition} \\
&= b\text{Im}(a\psi) && \mathcal{B} \text{ is abelian} \\
&= b\text{Ker}(aA_2 \to \text{Coker}(a\psi)) && \text{by definition} \\
&= \text{Ker}(baA_2 \to b\text{Coker}(a\psi)) && b \text{ preserves kernels} \\
&= \text{Ker}(A_2 \to b\text{Coker}(a\psi)) && ba = \text{id}_{\mathcal{A}} \\
&= \text{Ker}(A_2 \to \text{Coker}(\psi)) && \text{by formula above} \\
&= \text{Im}(\psi) && \text{by definition}
\end{aligned}$$

Thus the lemma holds. □

8. Localization

In this section we note how Gabriel-Zisman localization interacts with the additive structure on a category.

Lemma 8.1. *Let $\mathcal{C}$ be a preadditive category. Let S be a left or right multiplicative system. There exists a canonical preadditive structure on $S^{-1}\mathcal{C}$ such that the localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is additive.*

Proof. We will prove this in the case S is a left multiplicative system. The case where S is a right multiplicative system is dual. Suppose that X, Y are objects of $\mathcal{C}$ and that $\alpha, \beta : X \to Y$ are morphisms in $S^{-1}\mathcal{C}$. According to Categories, Lemma 25.3 we may represent these by pairs $s^{-1}f, s^{-1}g$ with common denominator s. In this case we define $\alpha + \beta$ to be the equivalence class of $s^{-1}(f + g)$. In the rest of the proof we show that this is well defined and that composition is bilinear. Once this is done it is clear that Q is an additive functor.

Let us show construction above is well defined. An abstract way of saying this is that filtered colimits of abelian groups agree with filtered colimits of sets and to use Categories, Equation (25.5.1). We can work this out in a bit more detail as follows. Say $s : Y \to Y_1$ and $f, g : X \to Y_1$. Suppose we have a second representation of α, β as $(s')^{-1}f', (s')^{-1}g'$ with $s' : Y \to Y_2$ and $f', g' : X \to Y_2$. By Categories, Remark 25.5 we can find a morphism $s_3 : Y \to Y_3$ and morphisms $a_1 : Y_1 \to Y_3$, $a_2 : Y_2 \to Y_3$ such that $a_1 \circ s = s_3 = a_2 \circ s'$ and also $a_1 \circ f = a_2 \circ f'$ and $a_1 \circ g = a_2 \circ g'$. Hence we see that $s^{-1}(f + g)$ is equivalent to

$$\begin{aligned} s_3^{-1}(a_1 \circ (f + g)) &= s_3^{-1}(a_1 \circ f + a_1 \circ g) \\ &= s_3^{-1}(a_2 \circ f' + a_2 \circ g') \\ &= s_3^{-1}(a_2 \circ (f' + g')) \end{aligned}$$

which is equivalent to $(s')^{-1}(f' + g')$.

Fix $s : Y \to Y'$ and $f, g : X \to Y'$ with $\alpha = s^{-1}f$ and $\beta = s^{-1}g$ as morphisms $X \to Y$ in $S^{-1}\mathcal{C}$. To show that composition is bilinear first consider the case of a morphism $\gamma : Y \to Z$ in $S^{-1}\mathcal{C}$. Say $\gamma = t^{-1}h$ for some $h : Y \to Z'$ and $t : Z \to Z'$ in S. Using LMS2 we choose morphisms $a : Y' \to Z''$ and $t' : Z' \to Z''$ in S such that $a \circ s = t' \circ h$. Picture

$$\begin{tikzcd} & & Z \arrow[d, "t"] \\ & Y \arrow[r, "h"] \arrow[d, "s"] & Z' \arrow[d, "t'"] \\ X \arrow[r, "{f,g}"] & Y' \arrow[r, "a"] & Z'' \end{tikzcd}$$

Then $\gamma \circ \alpha = (t' \circ t)^{-1}(a \circ f)$ and $\gamma \circ \beta = (t' \circ t)^{-1}(a \circ g)$. Hence we see that $\gamma \circ (\alpha + \beta)$ is represented by $(t' \circ t)^{-1}(a \circ (f + g)) = (t' \circ t)^{-1}(a \circ f + a \circ g)$ which represents $\gamma \circ \alpha + \gamma \circ \beta$.

Finally, assume that $\delta : W \to X$ is another morphism of $S^{-1}\mathcal{C}$. Say $\delta = r^{-1}i$ for some $i : W \to X'$ and $r : X \to X'$ in S. We claim that we can find a morphism

$s : Y' \to Y''$ in S and morphisms $a'', b'' : X' \to Y''$ such that the following diagram commutes

$$\begin{tikzcd} & & Y \arrow[d, "s"] \\ & X \arrow[r, "{f,g,f+g}"] \arrow[d, "s"] & Y' \arrow[d, "s'"] \\ W \arrow[r, "i"] & X' \arrow[r, "{a'',b'',a''+b''}"] & Y'' \end{tikzcd}$$

Namely, using LMS2 we can first choose $s_1 : Y' \to Y_1$, $s_2 : Y' \to Y_2$ in S and $a : X' \to Y_1$, $b : X' \to Y_2$ such that $a \circ s = s_1 \circ f$ and $b \circ s = s_2 \circ f$. Then using that the category Y'/S is filtered (see Categories, Remark 25.5), we can find a $s' : Y' \to Y''$ and morphisms $a' : Y_1 \to Y''$, $b' : Y_2 \to Y''$ such that $s' = a' \circ s_1$ and $s' = b' \circ s_2$. Setting $a'' = a' \circ a$ and $b'' = b' \circ b$ works. At this point we see that the compositions $\alpha \circ \delta$ and $\beta \circ \delta$ are represented by $(s' \circ s)^{-1}a''$ and $(s' \circ s)^{-1}b''$. Hence $\alpha \circ \delta + \beta \circ \delta$ is represented by $(s' \circ s)^{-1}(a'' + b'')$ which by the diagram again is a representative of $(\alpha + \beta) \circ \delta$. $\square$

Lemma 8.2. *Let $\mathcal{C}$ be an additive category. Let S be a left or right multiplicative system. Then $S^{-1}\mathcal{C}$ is an additive category and the localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is additive.*

Proof. By Lemma 8.1 we see that $S^{-1}\mathcal{C}$ is preadditive and that Q is additive. Recall that the functor Q commutes with finite colimits (resp. finite limits), see Categories, Lemmas 25.7 and 25.14. We conclude that $S^{-1}\mathcal{C}$ has a zero object and direct sums, see Lemmas 3.2 and 3.4. $\square$

The following lemma describes the kernel (see Definition 9.5) of the localization functor in case we invert a multiplicative system.

Lemma 8.3. *Let $\mathcal{C}$ be an additive category. Let S be a multiplicative system. Let X be an object of $\mathcal{C}$. The following are equivalent*

1. *$Q(X) = 0$ in $S^{-1}\mathcal{C}$,*
2. *there exists $Y \in \mathrm{Ob}(\mathcal{C})$ such that $0 : X \to Y$ is an element of S, and*
3. *there exists $Z \in \mathrm{Ob}(\mathcal{C})$ such that $0 : Z \to X$ is an element of S.*

Proof. If (2) holds we see that $0 = Q(0) : Q(X) \to Q(Y)$ is an isomorphism. In the additive category $S^{-1}\mathcal{C}$ this implies that $Q(X) = 0$. Hence (2) $\Rightarrow$ (1). Similarly, (3) $\Rightarrow$ (1). Suppose that $Q(X) = 0$. This implies that the morphism $f : 0 \to X$ is transformed into an isomorphism in $S^{-1}\mathcal{C}$. Hence by Categories, Lemma 25.18 there exists a morphism $g : Z \to 0$ such that $fg \in S$. This proves (1) $\Rightarrow$ (3). Similarly, (1) $\Rightarrow$ (2). $\square$

Lemma 8.4. *Let $\mathcal{A}$ be an abelian category.*

1. *If S is a left multiplicative system, then the category $S^{-1}\mathcal{A}$ has cokernels and the functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ commutes with them.*
2. *If S is a right multiplicative system, then the category $S^{-1}\mathcal{A}$ has kernels and the functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ commutes with them.*
3. *If S is a multiplicative system, then the category $S^{-1}\mathcal{A}$ is abelian and the functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ is exact.*

Proof. Assume S is a left multiplicative system. Let $a : X \to Y$ be a morphism of $S^{-1}\mathcal{A}$. Then $a = s^{-1}f$ for some $s : Y \to Y'$ in S and $f : X \to Y'$. Since $Q(s)$ is an isomorphism we see that the existence of $\text{Coker}(a : X \to Y)$ is equivalent to the existence of $\text{Coker}(Q(f) : X \to Y')$. Since $\text{Coker}(Q(f))$ is the coequalizer of 0 and $Q(f)$ we see that $\text{Coker}(Q(f))$ is represented by $Q(\text{Coker}(f))$ by Categories, Lemma 25.7. This proves (1).

Part (2) is dual to part (1).

If S is a multiplicative system, then S is both a left and a right multiplicative system. Thus we see that $S^{-1}\mathcal{A}$ has kernels and cokernels and Q commutes with kernels and cokernels. To finish the proof of (3) we have to show that $\text{Coim} = \text{Im}$ in $S^{-1}\mathcal{A}$. Again using that any arrow in $S^{-1}\mathcal{A}$ is isomorphic to an arrow $Q(f)$ we see that the result follows from the result for $\mathcal{A}$. $\square$

9. Serre subcategories

In [Ser53, Chapter I, Section 1] a notion of a "class" of abelian groups is defined. This notion has been extended to abelian categories by many authors (in slightly different ways). We will use the following variant which is virtually identical to Serre's original definition.

Definition 9.1. Let $\mathcal{A}$ be an abelian category.

(1) A *Serre subcategory* of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that given an exact sequence
$$A \to B \to C$$
with $A, C \in \text{Ob}(\mathcal{C})$, then also $B \in \text{Ob}(\mathcal{C})$.

(2) A *weak Serre subcategory* of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that given an exact sequence
$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$
with A_0, A_1, A_3, A_4 in $\mathcal{C}$, then also A_2 in $\mathcal{C}$.

In some references the second notion is called a "thick" subcategory and in other references the first notion is called a "thick" subcategory. However, it seems that the notion of a Serre subcategory is universally accepted to be the one defined above. Note that in both cases the category $\mathcal{C}$ is abelian and that the inclusion functor $\mathcal{C} \to \mathcal{A}$ is a fully faithful exact functor. Let's characterize these types of subcategories in more detail.

Lemma 9.2. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a Serre subcategory if and only if the following conditions are satisfied:*

(1) $0 \in \text{Ob}(\mathcal{C})$,
(2) *$\mathcal{C}$ is a strictly full subcategory of $\mathcal{A}$,*
(3) *any subobject or quotient of an object of $\mathcal{C}$ is an object of $\mathcal{C}$,*
(4) *if $A \in \text{Ob}(\mathcal{A})$ is an extension of objects of $\mathcal{C}$ then also $A \in \text{Ob}(\mathcal{C})$.*

Moreover, a Serre subcategory is an abelian category and the inclusion functor is exact.

Proof. Omitted. $\square$

Lemma 9.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a weak Serre subcategory if and only if the following conditions are satisfied:*

(1) $0 \in \mathrm{Ob}(\mathcal{C})$,
(2) *$\mathcal{C}$ is a strictly full subcategory of $\mathcal{A}$,*
(3) *kernels and cokernels in $\mathcal{A}$ of morphisms between objects of $\mathcal{C}$ are in $\mathcal{C}$,*
(4) *if $A \in \mathrm{Ob}(\mathcal{A})$ is an extension of objects of $\mathcal{C}$ then also $A \in \mathrm{Ob}(\mathcal{C})$.*

Moreover, a weak Serre subcategory is an abelian category and the inclusion functor is exact.

Proof. Omitted. □

Lemma 9.4. *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ forms a Serre subcategory of $\mathcal{A}$.*

Proof. Omitted. □

Definition 9.5. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ is called the *kernel of the functor F*, and is sometimes denoted $\mathrm{Ker}(F)$.

Lemma 9.6. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory. There exists an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor*

$$F : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$$

which is essentially surjective and whose kernel is $\mathcal{C}$. The category $\mathcal{A}/\mathcal{C}$ and the functor F are characterized by the following universal property: For any exact functor $G : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{C} \subset Ker(G)$ there exists a factorization $G = H \circ F$ for a unique exact functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$.

Proof. Consider the set of arrows of $\mathcal{A}$ defined by the following formula

$$S = \{f \in \mathrm{Arrows}(\mathcal{A}) \mid \mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathrm{Ob}(\mathcal{C})\}.$$

We claim that S is a multiplicative system. To prove this we have to check MS1, MS2, MS3, see Categories, Definition 25.1.

It is clear that identities are elements of S. Suppose that $f : A \to B$ and $g : B \to C$ are elements of S. There are exact sequences

$$\begin{gathered} 0 \to \mathrm{Ker}(f) \to \mathrm{Ker}(gf) \to \mathrm{Ker}(g) \\ \mathrm{Coker}(f) \to \mathrm{Coker}(gf) \to \mathrm{Coker}(g) \to 0 \end{gathered}$$

Hence it follows that $gf \in S$. This proves MS1. (In fact, a similar argument will show that S is a saturated multiplicative system, see Categories, Definition 25.17.)

Consider a solid diagram

$$\begin{tikzcd} A \arrow[r, "g"'] \arrow[d, "t"'] & B \arrow[d, dotted, "s"] \\ C \arrow[r, dotted, "f"] & C \amalg_A B \end{tikzcd}$$

with $t \in S$. Set $W = C \amalg_A B = \mathrm{Coker}((t, -g) : A \to C \oplus B)$. Then $\mathrm{Ker}(t) \to \mathrm{Ker}(s)$ is surjective and $\mathrm{Coker}(t) \to \mathrm{Coker}(s)$ is an isomorphism. Hence s is an element of S. This proves LMS2 and the proof of RMS2 is dual.

Finally, consider morphisms $f, g : B \to C$ and a morphism $s : A \to B$ in S such that $f \circ s = g \circ s$. This means that $(f - g) \circ s = 0$. In turn this means that $I = \mathrm{Im}(f - g) \subset C$ is a quotient of $\mathrm{Coker}(s)$ hence an object of $\mathcal{C}$. Thus $t : C \to C' = C/I$ is an element of S such that $t \circ (f - g) = 0$, i.e., such that $t \circ f = t \circ g$. This proves LMS3 and the proof of RMS3 is dual.

Having proved that S is a multiplicative system we set $\mathcal{A}/\mathcal{C} = S^{-1}\mathcal{A}$, and we set F equal to the localization functor Q. By Lemma 8.4 the category $\mathcal{A}/\mathcal{C}$ is abelian and F is exact. If X is in the kernel of $F = Q$, then by Lemma 8.3 we see that $0 : X \to Z$ is an element of S and hence X is an object of $\mathcal{C}$, i.e., the kernel of F is $\mathcal{C}$. Finally, if G is as in the statement of the lemma, then G turns every element of S into an isomorphism. Hence we obtain the functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ from the universal property of localization, see Categories, Lemma 25.6. □

Lemma 9.7. *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Let $\mathcal{C} = Ker(F)$. Then the induced functor $\overline{F} : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ is faithful.*

Proof. This is true because the kernel of $\overline{F}$ is zero by construction. Namely, if $f : X \to Y$ is a morphism in $\mathcal{A}/\mathcal{C}$ such that $\overline{F}(f) = 0$, then $\mathrm{Ker}(f) \to X$ and $Y \to \mathrm{Coker}(f)$ are transformed into isomorphisms by $\overline{F}$, hence are isomorphisms by the remark on the kernel of $\overline{F}$. Thus $f = 0$. □

10. K-groups

Definition 10.1. Let $\mathcal{A}$ be an abelian category. We denote $K_0(\mathcal{A})$ the *zeroth K-group of $\mathcal{A}$*. It is the abelian group constructed as follows. Take the free abelian group on the objects on $\mathcal{A}$ and for every short exact sequence $0 \to A \to B \to C \to 0$ impose the relation $[B] - [A] - [C] = 0$.

Another way to say this is that there is a presentation

$$\bigoplus_{A \to B \to C \text{ ses}} \mathbf{Z}[A \to B \to C] \longrightarrow \bigoplus_{A \in \mathrm{Ob}(\mathcal{A})} \mathbf{Z}[A] \longrightarrow K_0(\mathcal{A}) \longrightarrow 0$$

with $[A \to B \to C] \mapsto [B] - [A] - [C]$ of $K_0(\mathcal{A})$. The short exact sequence $0 \to 0 \to 0 \to 0 \to 0$ leads to the relation $[0] = 0$ in $K_0(\mathcal{A})$. There are no set-theoretical issues as all of our categories are "small" if not mentioned otherwise. Some examples of K-groups for categories of modules over rings where computed in Algebra, Section 53.

Lemma 10.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Then F induces a homomorphism of K-groups $K_0(F) : K_0(\mathcal{A}) \to K_0(\mathcal{B})$ by simply setting $K_0(F)([A]) = [F(A)]$.*

Proof. Proves itself. □

Suppose we are given an object M of an abelian category $\mathcal{A}$ and a complex of the form

$$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots \tag{10.2.1}$$

In this situation we define

$$H^0(M, \varphi, \psi) = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi), \quad \text{and} \quad H^1(M, \varphi, \psi) = \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi).$$

Lemma 10.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory and set $\mathcal{B} = \mathcal{A}/\mathcal{C}$.*

1. *The exact functors $\mathcal{C} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{B}$ induce an exact sequence*
$$K_0(\mathcal{C}) \to K_0(\mathcal{A}) \to K_0(\mathcal{B}) \to 0$$
of K-groups, and
2. *the kernel of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ is equal to the collection of elements of the form*
$$[H^0(M, \varphi, \psi)] - [H^1(M, \varphi, \psi)]$$
where (M, φ, ψ) is a complex as in (10.2.1) with the property that it becomes exact in $\mathcal{B}$; in other words that $H^0(M, \varphi, \psi)$ and $H^1(M, \varphi, \psi)$ are objects of $\mathcal{C}$.

Proof. We omit the proof of (1). The proof of (2) is in a sense completely combinatorial. First we remark that any class of the type $[H^0(M, \varphi, \psi)] - [H^1(M, \varphi, \psi)]$ is zero in $K_0(\mathcal{A})$ by the following calculation
$$\begin{aligned}
0 &= [M] - [M] \\
&= [\mathrm{Ker}(\varphi)] + [\mathrm{Im}(\varphi)] - [\mathrm{Ker}(\psi)] - [\mathrm{Im}(\psi)] \\
&= [\mathrm{Ker}(\varphi)/\mathrm{Im}(\psi)] - [\mathrm{Ker}(\psi)/\mathrm{Im}(\varphi)] \\
&= [H^1(M, \varphi, \psi)] - [H^0(M, \varphi, \psi)]
\end{aligned}$$
as desired. Hence it suffices to show that any element in the kernel of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ is of this form.

Any element x in $K_0(\mathcal{C})$ can be represented as the difference $x = [P] - [Q]$ of two objects of $\mathcal{C}$ (fun exercise). Suppose that this element maps to zero in $K_0(\mathcal{A})$. This means that there exist

1. a finite set $I = I^+ \coprod I^-$,
2. for each $i \in I$ a short exact sequence
$$0 \to A_i \to B_i \to C_i \to 0$$
in the abelian category $\mathcal{A}$

such that
$$[P] - [Q] = \sum\nolimits_{i \in I^+} ([B_i] - [A_i] - [C_i]) - \sum\nolimits_{i \in I^-} ([B_i] - [A_i] - [C_i])$$
in the free abelian group on the objects of $\mathcal{A}$. We can rewrite this as
$$[P] + \sum\nolimits_{i \in I^+} ([A_i] + [C_i]) + \sum\nolimits_{i \in I^-} [B_i] = [Q] + \sum\nolimits_{i \in I^-} ([A_i] + [C_i]) + \sum\nolimits_{i \in I^+} [B_i].$$
Since the right and left hand side should contain the same objects of $\mathcal{A}$ counted with multiplicity, this means there should be a bijection τ between the terms which occur above. Set
$$T^+ = \{p\} \coprod \{a, c\} \times I^+ \coprod \{b\} \times I^-$$
and
$$T^- = \{q\} \coprod \{a, c\} \times I^- \coprod \{b\} \times I^+.$$

Set $T = T^+ \coprod T^- = \{p, q\} \coprod \{a, b, c\} \times I$. For $t \in T$ define

$$O(t) = \begin{cases} P & \text{if} \quad t = p \\ Q & \text{if} \quad t = q \\ A_i & \text{if} \quad t = (a, i) \\ B_i & \text{if} \quad t = (b, i) \\ C_i & \text{if} \quad t = (c, i) \end{cases}$$

Hence we can view $\tau : T^+ \to T^-$ as a bijection such that $O(t) = O(\tau(t))$ for all $t \in T^+$. Let $t_0^- = \tau(p)$ and let $t_0^+ \in T^+$ be the unique element such that $\tau(t_0^+) = q$. Consider the object

$$M^+ = \bigoplus\nolimits_{t \in T^+} O(t)$$

By using τ we see that it is equal to the object

$$M^- = \bigoplus\nolimits_{t \in T^-} O(t)$$

Consider the map

$$\varphi : M^+ \longrightarrow M^-$$

which on the summand $O(t) = A_i$ corresponding to $t = (a, i)$, $i \in I^+$ uses the map $A_i \to B_i$ into the summand $O((b, i)) = B_i$ of M^- and on the summand $O(t) = B_i$ corresponding to (b, i), $i \in I^-$ uses the map $B_i \to C_i$ into the summand $O((c, i)) = C_i$ of M^-. The map is zero on the summands corresponding to p and (c, i), $i \in I^+$. Similarly, consider the map

$$\psi : M^- \longrightarrow M^+$$

which on the summand $O(t) = A_i$ corresponding to $t = (a, i)$, $i \in I^-$ uses the map $A_i \to B_i$ into the summand $O((b, i)) = B_i$ of M^+ and on the summand $O(t) = B_i$ corresponding to (b, i), $i \in I^+$ uses the map $B_i \to C_i$ into the summand $O((c, i)) = C_i$ of M^+. The map is zero on the summands corresponding to q and (c, i), $i \in I^-$.

Note that the kernel of φ is equal to the direct sum of the summand P and the summands $O((c, i)) = C_i$, $i \in I^+$ and the subobjects A_i inside the summands $O((b, i)) = B_i$, $i \in I^-$. The image of ψ is equal to the direct sum of the summands $O((c, i)) = C_i$, $i \in I^+$ and the subobjects A_i inside the summands $O((b, i)) = B_i$, $i \in I^-$. In other words we see that

$$P \cong \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi).$$

In exactly the same way we see that

$$Q \cong \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi).$$

Since as we remarked above the existence of the bijection τ shows that $M^+ = M^-$ we see that the lemma follows. $\square$

11. Cohomological delta-functors

Definition 11.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A *cohomological δ-functor* or simply a *δ-functor* from $\mathcal{A}$ to $\mathcal{B}$ is given by the following data:

(1) a collection $F^n : \mathcal{A} \to \mathcal{B}$, $n \geq 0$ of additive functors, and
(2) for every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{A}$ a collection $\delta_{A \to B \to C} : F^n(C) \to F^{n+1}(A)$, $n \geq 0$ of morphisms of $\mathcal{B}$.

These data are assumed to satisfy the following axioms

(1) for every short exact sequence as above the sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^0(A) & \longrightarrow & F^0(B) & \longrightarrow & F^0(C) \\ & & & & & \xrightarrow{\delta_{A\to B\to C}} & \\ & & F^1(A) & \longrightarrow & F^1(B) & \longrightarrow & F^1(C) \\ & & & & & \xrightarrow{\delta_{A\to B\to C}} & \\ & & F^2(A) & \longrightarrow & F^2(B) & \longrightarrow & \ldots \end{array}$$

is exact, and

(2) for every morphism $(A \to B \to C) \to (A' \to B' \to C')$ of short exact sequences of $\mathcal{A}$ the diagrams

$$\begin{array}{ccc} F^n(C) & \xrightarrow[\delta_{A\to B\to C}]{} & F^{n+1}(A) \\ \downarrow & & \downarrow \\ F^n(C') & \xrightarrow{\delta_{A'\to B'\to C'}} & F^{n+1}(A') \end{array}$$

are commutative.

Note that this in particular implies that F^0 is left exact.

Definition 11.2. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let (F^n, δ_F) and (G^n, δ_G) be δ-functors from $\mathcal{A}$ to $\mathcal{B}$. A *morphism of δ-functors from F to G* is a collection of transformation of functors $t^n : F^n \to G^n$, $n \geq 0$ such that for every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{A}$ the diagrams

$$\begin{array}{ccc} F^n(C) & \xrightarrow[\delta_{F,A\to B\to C}]{} & F^{n+1}(A) \\ {\scriptstyle t^n}\downarrow & & \downarrow{\scriptstyle t^{n+1}} \\ G^n(C) & \xrightarrow{\delta_{G,A\to B\to C}} & G^{n+1}(A) \end{array}$$

are commutative.

Definition 11.3. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F = (F^n, \delta_F)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$. We say F is a *universal δ-functor* if an only if for every δ-functor $G = (G^n, \delta_G)$ and any morphism of functors $t : F^0 \to G^0$ there exists a unique morphism of δ-functors $\{t^n\}_{n\geq 0} : F \to G$ such that $t = t^0$.

Lemma 11.4. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F = (F^n, \delta_F)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$. Suppose that for every $n > 0$ and any $A \in \mathrm{Ob}(\mathcal{A})$ there exists an injective morphism $u : A \to B$ (depending on A and n) such that $F^n(u) : F^n(A) \to F^n(B)$ is zero. Then F is a universal δ-functor.*

Proof. Let $G = (G^n, \delta_G)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$ and let $t : F^0 \to G^0$ be a morphism of functors. We have to show there exists a unique morphism of δ-functors $\{t^n\}_{n\geq 0} : F \to G$ such that $t = t^0$. We construct t^n by induction on n. For $n = 0$ we set $t^0 = t$. Suppose we have already constructed a unique sequence of transformation of functors t^i for $i \leq n$ compatible with the maps δ in degrees $\leq n$.

Let $A \in \mathrm{Ob}(\mathcal{A})$. By assumption we may choose a embedding $u : A \to B$ such that $F^{n+1}(u) = 0$. Let $C = B/u(A)$. The long exact cohomology sequence for the short exact sequence $0 \to A \to B \to C \to 0$ and the δ-functor F gives that $F^{n+1}(A) = \mathrm{Coker}(F^n(B) \to F^n(C))$ by our choice of u. Since we have already defined t^n we can set

$$t_A^{n+1} : F^{n+1}(A) \to G^{n+1}(A)$$

equal to the unique map such that

$$\begin{tikzcd}
\mathrm{Coker}(F^n(B) \to F^n(C)) \arrow[r, "t^n"'] \arrow[d, "\delta_{F,A\to B\to C}"'] & \mathrm{Coker}(G^n(B) \to G^n(C)) \arrow[d, "\delta_{G,A\to B\to C}"] \\
F^{n+1}(A) \arrow[r, "t_A^{n+1}"] & G^{n+1}(A)
\end{tikzcd}$$

commutes. This is clearly uniquely determined by the requirements imposed. We omit the verification that this defines a transformation of functors. □

Lemma 11.5. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. If there exists a universal δ-functor (F^n, δ_F) from $\mathcal{A}$ to $\mathcal{B}$ with $F^0 = F$, then it is determined up to unique isomorphism of δ-functors.*

Proof. Immediate from the definitions. □

12. Complexes

Of course the notions of a chain complex and a cochain complex are dual and you only have to read one of the two parts of this section. So pick the one you like. (Actually, this doesn't quite work right since the conventions on numbering things are not adapted to an easy transition between chain and cochain complexes.)

A *chain complex* $A_\bullet$ in an additive category $\mathcal{A}$ is a complex

$$\ldots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \ldots$$

of $\mathcal{A}$. In other words, we are given an object A_i of $\mathcal{A}$ for all $i \in \mathbf{Z}$ and for all $i \in \mathbf{Z}$ a morphism $d_i : A_i \to A_{i-1}$ such that $d_{i-1} \circ d_i = 0$ for all i. A *morphism of chain complexes* $f : A_\bullet \to B_\bullet$ is given by a family of morphisms $f_i : A_i \to B_i$ such that all the diagrams

$$\begin{tikzcd}
A_i \arrow[r, "d_i"'] \arrow[d, "f_i"'] & A_{i-1} \arrow[d, "f_{i-1}"] \\
B_i \arrow[r, "d_i"] & B_{i-1}
\end{tikzcd}$$

commute. The *category of chain complexes of* $\mathcal{A}$ is denoted $\mathrm{Ch}(\mathcal{A})$. The full subcategory consisting of objects of the form

$$\ldots \to A_2 \to A_1 \to A_0 \to 0 \to 0 \to \ldots$$

is denoted $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. In other words, a chain complex $A_\bullet$ belongs to $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ if and only if $A_i = 0$ for all $i < 0$. A *homotopy* h between a pair of morphisms of chain complexes $f, g : A_\bullet \to B_\bullet$ is is a collection of morphisms $h_i : A_i \to B_{i+1}$ such that we have

$$f_i - g_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$$

for all i. Clearly, the notions of chain complex, morphism of chain complexes, and homotopies between morphisms of chain complexes makes sense even in a preadditive category.

Lemma 12.1. *Let $\mathcal{A}$ be an additive category. Let $f, g : B_\bullet \to C_\bullet$ be morphisms of chain complexes. Suppose given morphisms of chain complexes $a : A_\bullet \to B_\bullet$, and $c : C_\bullet \to D_\bullet$. If $\{h_i : B_i \to C_{i+1}\}$ defines a homotopy between f and g, then $\{c_{i+1} \circ h_i \circ a_i\}$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Omitted. □

In particular this means that it makes sense to define the category of chain complexes with maps up to homotopy. We'll return to this later.

Definition 12.2. Let $\mathcal{A}$ be an additive category. We say a morphism $a : A_\bullet \to B_\bullet$ is a *homotopy equivalence* if there exists a morphism $b : B_\bullet \to A_\bullet$ such that there exists a homotopy between $a \circ b$ and id_A and there exists a homotopy between $b \circ a$ and id_B. If there exists such a morphism between $A_\bullet$ and $B_\bullet$, then we say that $A_\bullet$ and $B_\bullet$ are *homotopy equivalent*.

In other words, two complexes are homotopy equivalent if they become isomorphic in the category of complexes up to homotopy.

Lemma 12.3. *Let $\mathcal{A}$ be an abelian category.*

(1) *The category of chain complexes in $\mathcal{A}$ is abelian.*
(2) *A morphism of complexes $f : A_\bullet \to B_\bullet$ is injective if and only if each $f_n : A_n \to B_n$ is injective.*
(3) *A morphism of complexes $f : A_\bullet \to B_\bullet$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.*
(4) *A sequence of chain complexes*

$$A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$$

is exact at $B_\bullet$ if and only if each sequence

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

is exact at B_i.

Proof. Omitted. □

For any $i \in \mathbf{Z}$ the ith *homology group* of a chain complex $A_\bullet$ in an abelian category is defined by the following formula

$$H_i(A_\bullet) = \text{Ker}(d_i)/\text{Im}(d_{i+1}).$$

If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes of $\mathcal{A}$ then we get an induced morphism $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ because clearly $f_i(\text{Ker}(d_i : A_i \to A_{i-1})) \subset \text{Ker}(d_i : B_i \to B_{i-1})$, and similarly for $\text{Im}(d_{i+1})$. Thus we obtain a functor

$$H_i : \text{Ch}(\mathcal{A}) \longrightarrow \mathcal{A}.$$

Definition 12.4. Let $\mathcal{A}$ be an abelian category.

(1) A morphism of chain complexes $f : A_\bullet \to B_\bullet$ is called a *quasi-isomorphism* if the induced maps $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.
(2) A chain complex $A_\bullet$ is called *acyclic* if all of its homology objects $H_i(A_\bullet)$ are zero.

Lemma 12.5. *Let $\mathcal{A}$ be an abelian category.*

(1) *If the maps $f, g : A_\bullet \to B_\bullet$ are homotopic, then the induced maps $H_i(f)$ and $H_i(g)$ are equal.*
(2) *If the map $f : A_\bullet \to B_\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.*

Proof. Omitted. □

Lemma 12.6. *Let $\mathcal{A}$ be an abelian category. Suppose that*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of chain complexes of $\mathcal{A}$. Then there is a canonical long exact homology sequence

$$\begin{tikzcd}
\ldots & \ldots & \ldots \arrow[dll] \\
H_i(A_\bullet) \arrow[r] & H_i(B_\bullet) \arrow[r] & H_i(C_\bullet) \arrow[dll] \\
H_{i-1}(A_\bullet) \arrow[r] & H_{i-1}(B_\bullet) \arrow[r] & H_{i-1}(C_\bullet) \arrow[dll] \\
\ldots & \ldots & \ldots
\end{tikzcd}$$

Proof. Omitted. The maps come from the Snake Lemma 5.17 applied to the diagrams

$$\begin{tikzcd}
& A_i/\mathrm{Im}(d_{A,i+1}) \arrow[r] \arrow[d, "d_{A,i}"] & B_i/\mathrm{Im}(d_{B,i+1}) \arrow[r] \arrow[d, "d_{B,i}"] & C_i/\mathrm{Im}(d_{C,i+1}) \arrow[r] \arrow[d, "d_{C,i}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(d_{A,i-1}) \arrow[r] & \mathrm{Ker}(d_{B,i-1}) \arrow[r] & \mathrm{Ker}(d_{C,i-1}) &
\end{tikzcd}$$

□

A *cochain complex* $A^\bullet$ in an additive category $\mathcal{A}$ is a complex

$$\ldots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \ldots$$

of $\mathcal{A}$. In other words, we are given an object A^i of $\mathcal{A}$ for all $i \in \mathbf{Z}$ and for all $i \in \mathbf{Z}$ a morphism $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all i. A *morphism of cochain complexes* $f : A^\bullet \to B^\bullet$ is given by a family of morphisms $f^i : A^i \to B^i$ such that all the diagrams

$$\begin{tikzcd}
A^i \arrow[r, "d^i"'] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\
B^i \arrow[r, "d^i"] & B^{i+1}
\end{tikzcd}$$

commute. The *category of cochain complexes of* $\mathcal{A}$ is denoted $\mathrm{CoCh}(\mathcal{A})$. The full subcategory consisting of objects of the form

$$\ldots \to 0 \to 0 \to A^0 \to A^1 \to A^2 \to \ldots$$

is denoted $\mathrm{CoCh}_{\geq 0}(\mathcal{A})$. In other words, a cochain complex $A^\bullet$ belongs to the subcategory $\mathrm{CoCh}_{\geq 0}(\mathcal{A})$ if and only if $A^i = 0$ for all $i < 0$. A *homotopy* h between a pair of morphisms of cochain complexes $f, g : A^\bullet \to B^\bullet$ is is a collection of morphisms $h^i : A^i \to B^{i-1}$ such that we have

$$f^i - g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$$

for all i. Clearly, the notions of cochain complex, morphism of cochain complexes, and homotopies between morphisms of cochain complexes makes sense even in a preadditive category.

Lemma 12.7. *Let $\mathcal{A}$ be an additive category. Let $f, g : B^\bullet \to C^\bullet$ be morphisms of cochain complexes. Suppose given morphisms of cochain complexes $a : A^\bullet \to B^\bullet$, and $c : C^\bullet \to D^\bullet$. If $\{h^i : B^i \to C^{i-1}\}$ defines a homotopy between f and g, then $\{c^{i-1} \circ h^i \circ a^i\}$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Omitted. □

In particular this means that it makes sense to define the category of cochain complexes with maps up to homotopy. We'll return to this later.

Definition 12.8. Let $\mathcal{A}$ be an additive category. We say a morphism $a : A^\bullet \to B^\bullet$ is a *homotopy equivalence* if there exists a morphism $b : B^\bullet \to A^\bullet$ such that there exists a homotopy between $a \circ b$ and id_A and there exists a homotopy between $b \circ a$ and id_B. If there exists such a morphism between $A^\bullet$ and $B^\bullet$, then we say that $A^\bullet$ and $B^\bullet$ are *homotopy equivalent.*

In other words, two complexes are homotopy equivalent if they become isomorphic in the category of complexes up to homotopy.

Lemma 12.9. *Let $\mathcal{A}$ be an abelian category.*

(1) *The category of cochain complexes in $\mathcal{A}$ is abelian.*
(2) *A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is injective if and only if each $f^n : A^n \to B^n$ is injective.*
(3) *A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is surjective if and only if each $f^n : A^n \to B^n$ is surjective.*
(4) *A sequence of cochain complexes*

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet$$

is exact at $B^\bullet$ if and only if each sequence

$$A^i \xrightarrow{f^i} B^i \xrightarrow{g^i} C^i$$

is exact at B^i.

Proof. Omitted. □

For any $i \in \mathbf{Z}$ the ith *cohomology group* of a cochain complex $A^\bullet$ is defined by the following formula

$$H^i(A^\bullet) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1}).$$

If $f : A^\bullet \to B^\bullet$ is a morphism of cochain complexes of $\mathcal{A}$ then we get an induced morphism $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ because clearly $f^i(\mathrm{Ker}(d^i : A^i \to A^{i+1})) \subset \mathrm{Ker}(d^i : B^i \to B^{i+1})$, and similarly for $\mathrm{Im}(d^{i-1})$. Thus we obtain a functor

$$H^i : \mathrm{CoCh}(\mathcal{A}) \longrightarrow \mathcal{A}.$$

Definition 12.10. Let $\mathcal{A}$ be an abelian category.

(1) A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ of $\mathcal{A}$ is called a *quasi-isomorphism* if the induced maps $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.
(2) A cochain complex $A^\bullet$ is called *acyclic* if all of its cohomology objects $H^i(A^\bullet)$ are zero.

Lemma 12.11. *Let $\mathcal{A}$ be an abelian category.*
(1) *If the maps $f, g : A^\bullet \to B^\bullet$ are homotopic, then the induced maps $H^i(f)$ and $H^i(g)$ are equal.*
(2) *If $f : A^\bullet \to B^\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.*

Proof. Omitted. □

Lemma 12.12. *Let $\mathcal{A}$ be an abelian category. Suppose that*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

is a short exact sequence of chain complexes of $\mathcal{A}$. Then there is a canonical long exact homology sequence

$$\begin{tikzcd}
\ldots & \ldots & \ldots \arrow[dll] \\
H^i(A^\bullet) \arrow[r] & H^i(B^\bullet) \arrow[r] & H^i(C^\bullet) \arrow[dll] \\
H^{i+1}(A^\bullet) \arrow[r] & H^{i+1}(B^\bullet) \arrow[r] & H^{i+1}(C^\bullet) \arrow[dll] \\
\ldots & \ldots & \ldots
\end{tikzcd}$$

Proof. Omitted. The maps come from the Snake Lemma 5.17 applied to the diagrams

$$\begin{tikzcd}
 & A^i/\mathrm{Im}(d_A^{i-1}) \arrow[r] \arrow[d, "d_A^i"] & B^i/\mathrm{Im}(d_B^{i-1}) \arrow[r] \arrow[d, "d_B^i"] & C^i/\mathrm{Im}(d_C^{i-1}) \arrow[r] \arrow[d, "d_C^i"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(d_A^{i+1}) \arrow[r] & \mathrm{Ker}(d_B^{i+1}) \arrow[r] & \mathrm{Ker}(d_C^{i+1}) &
\end{tikzcd}$$

□

13. Truncation of complexes

Let $\mathcal{A}$ be an abelian category. Let $A_\bullet$ be a chain complex. There are several ways to *truncate* the complex $A_\bullet$.

(1) The *"stupid" truncation* $\sigma_{\leq n}$ is the subcomplex $\sigma_{\leq n} A_\bullet$ defined by the rule $(\sigma_{\leq n} A_\bullet)_i = 0$ if $i > n$ and $(\sigma_{\leq n} A_\bullet)_i = A_i$ if $i \leq n$. In a picture

$$\begin{tikzcd}
\sigma_{\leq n} A_\bullet \arrow[d] & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \ldots \\
A_\bullet & \ldots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r] & A_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Note the property $\sigma_{\leq n} A_\bullet / \sigma_{\leq n-1} A_\bullet = A_n[-n]$.

(2) The *"stupid" truncation* $\sigma_{\geq n}$ is the quotient complex $\sigma_{\geq n}A_\bullet$ defined by the rule $(\sigma_{\geq n}A_\bullet)_i = A_i$ if $i \geq n$ and $(\sigma_{\geq n}A_\bullet)_i = 0$ if $i < n$. In a picture

$$\begin{tikzcd}
A_\bullet \arrow[d] & \ldots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \ldots \\
\sigma_{\geq n}A_\bullet & \ldots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

The map of complexes $\sigma_{\geq n}A_\bullet \to \sigma_{\geq n+1}A_\bullet$ is surjective with kernel $A_n[-n]$.

(3) The *canonical truncation* $\tau_{\geq n}A_\bullet$ is defined by the picture

$$\begin{tikzcd}
\tau_{\geq n}A_\bullet \arrow[d] & \ldots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & \mathrm{Ker}(d_n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
A_\bullet & \ldots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r] & A_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Note that these complexes have the property that

$$H_i(\tau_{\geq n}A_\bullet) = \begin{cases} H_i(A_\bullet) & \text{if} \quad i \geq n \\ 0 & \text{if} \quad i < n \end{cases}$$

(4) The *canonical truncation* $\tau_{\leq n}A_\bullet$ is defined by the picture

$$\begin{tikzcd}
A_\bullet \arrow[d] & \ldots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \ldots \\
\tau_{\leq n}A_\bullet & \ldots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}(d_{n+1}) \arrow[r] & A_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Note that these complexes have the property that

$$H_i(\tau_{\leq n}A_\bullet) = \begin{cases} H_i(A_\bullet) & \text{if} \quad i \leq n \\ 0 & \text{if} \quad i > n \end{cases}$$

Let $\mathcal{A}$ be an abelian category. Let $A^\bullet$ be a cochain complex. There are four ways to truncate the complex $A^\bullet$.

(1) The *"stupid" truncation* $\sigma_{\geq n}$ is the subcomplex $\sigma_{\geq n}A^\bullet$ defined by the rule $(\sigma_{\geq n}A^\bullet)^i = 0$ if $i < n$ and $(\sigma_{\geq n}A^\bullet)^i = A_i$ if $i \geq n$. In a picture

$$\begin{tikzcd}
\sigma_{\geq n}A^\bullet \arrow[d] & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \ldots \\
A^\bullet & \ldots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

Note the property $\sigma_{\geq n}A^\bullet/\sigma_{\geq n+1}A^\bullet = A^n[-n]$.

(2) The *"stupid" truncation* $\sigma_{\leq n}$ is the quotient complex $\sigma_{\leq n}A^\bullet$ defined by the rule $(\sigma_{\geq n}A^\bullet)^i = 0$ if $i > n$ and $(\sigma_{\geq n}A^\bullet)^i = A^i$ if $i \leq n$. In a picture

$$\begin{tikzcd}
A^\bullet \arrow[d] & \ldots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \ldots \\
\sigma_{\leq n}A^\bullet & \ldots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

The map of complexes $\sigma_{\leq n}A^\bullet \to \sigma_{\leq n-1}A^\bullet$ is surjective with kernel $A^n[-n]$.

(3) The *canonical truncation* $\tau_{\leq n}A^\bullet$ is defined by the picture

$$\begin{tikzcd}
\tau_{\leq n}A^\bullet \arrow[d] & \ldots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & \mathrm{Ker}(d^n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
A^\bullet & \ldots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

Note that these complexes have the property that

$$H^i(\tau_{\leq n}A^\bullet) = \begin{cases} H^i(A^\bullet) & \text{if} \quad i \leq n \\ 0 & \text{if} \quad i > n \end{cases}$$

(4) The *canonical truncation* $\tau_{\geq n}A^\bullet$ is defined by the picture

$$\begin{tikzcd}
A^\bullet \arrow[d] & \ldots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \ldots \\
\tau_{\geq n}A^\bullet & \ldots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}(d^{n-1}) \arrow[r] & A^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

Note that these complexes have the property that

$$H^i(\tau_{\leq n}A^\bullet) = \begin{cases} 0 & \text{if} \quad i < n \\ H^i(A^\bullet) & \text{if} \quad i \geq n \end{cases}$$

14. Homotopy and the shift functor

It is an annoying feature that signs and indices have to be part of any discussion of homological algebra[2].

Definition 14.1. Let $\mathcal{A}$ be an additive category. Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For any $k \in \mathbf{Z}$ we define the *k-shifted chain complex* $A[k]_\bullet$ as follows:

(1) we set $A[k]_n = A_{n+k}$, and
(2) we set $d_{A[k],n} : A[k]_n \to A[k]_{n-1}$ equal to $d_{A[k],n} = (-1)^k d_{A,n+k}$.

If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes, then we let $f[k] : A[k]_\bullet \to B[k]_\bullet$ be the morphism of chain complexes with $f[k]_n = f_{k+n}$.

Of course this means we have functors $[k] : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ which mutually commute (on the nose, without any intervening isomorphisms of functors), such that $A[k][l]_\bullet = A[k+l]_\bullet$ and with $[0] = \mathrm{id}_{\mathrm{Ch}(\mathcal{A})}$.

Definition 14.2. Let $\mathcal{A}$ be an abelian category. Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For any $k \in \mathbf{Z}$ we identify $H_{i+k}(A_\bullet) \to H_i(A[k]_\bullet)$ via the identification $A_{i+k} = A[k]_i$.

This identification is functorial in $A_\bullet$. Note that since no signs are involved in this definition we actually get a compatible system of identifications of all the homology objects $H_{i-k}(A[k]_\bullet)$, which are further compatible with the identifications $A[k][l]_\bullet = A[k+l]_\bullet$ and with $[0] = \mathrm{id}_{\mathrm{Ch}(\mathcal{A})}$.

Let $\mathcal{A}$ be an additive category. Suppose that $A_\bullet$ and $B_\bullet$ are chain complexes, $a, b : A_\bullet \to B_\bullet$ are morphisms of chain complexes, and $\{h_i : A_i \to B_{i+1}\}$ is a homotopy between a and b. Recall that this means that $a_i - b_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$.

[2] I am sure you think that my conventions are wrong. If so and if you feel strongly about it then drop me an email with an explanation.

What if $a = b$? Then we obtain the formula $0 = d_{i+1} \circ h_i + h_{i-1} \circ d_i$, in other words, $-d_{i+1} \circ h_i = h_{i-1} \circ d_i$. By definition above this means the collection $\{h_i\}$ above defines a morphism of chain complexes

$$A_\bullet \longrightarrow B[1]_\bullet.$$

Such a thing is the same as a morphism $A[-1]_\bullet \to B_\bullet$ by our remarks above. This proves the following lemma.

Lemma 14.3. *Let $\mathcal{A}$ be an additive category. Suppose that $A_\bullet$ and $B_\bullet$ are chain complexes. Given any morphism of chain complexes $a : A_\bullet \to B_\bullet$ there is a bijection between the set of homotopies from a to a and $\mathrm{Mor}_{Ch(\mathcal{A})}(A_\bullet, B[1]_\bullet)$. More generally, the set of homotopies between a and b is either empty or a principal homogeneous space under the group $\mathrm{Mor}_{Ch(\mathcal{A})}(A_\bullet, B[1]_\bullet)$.*

Proof. See above. ☐

Lemma 14.4. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

be a sort exact sequence of complexes. Suppose that $\{s_n : C_n \to B_n\}$ is a family of morphisms which split the short exact sequences $0 \to A_n \to B_n \to C_n \to 0$. Let $\pi_n : B_n \to A_n$ be the associated projections, see Lemma 5.10. Then the family of morphisms

$$\pi_{n-1} \circ d_{B,n} \circ s_n : C_n \to A_{n-1}$$

define a morphism of complexes $\delta(s) : C_\bullet \to A[-1]_\bullet$.

Proof. Denote $i : A_\bullet \to B_\bullet$ and $q : B_\bullet \to C_\bullet$ the maps of complexes in the short exact sequence. Then $i_{n-1} \circ \pi_{n-1} \circ d_{B,n} \circ s_n = d_{B,n} \circ s_n - s_{n-1} \circ d_{C,n}$. Hence $i_{n-2} \circ d_{A,n-1} \circ \pi_{n-1} \circ d_{B,n} \circ s_n = d_{B,n-1} \circ (d_{B,n} \circ s_n - s_{n-1} \circ d_{C,n}) = -d_{B,n-1} \circ s_{n-1} \circ d_{C,n}$ as desired. ☐

Lemma 14.5. *Notation and assumptions as in Lemma 14.4 above. The morphism of complexes $\delta(s) : C_\bullet \to A[-1]_\bullet$ induces the maps*

$$H_i(\delta(s)) : H_i(C_\bullet) \longrightarrow H_i(A[-1]_\bullet) = H_{i-1}(A_\bullet)$$

which occur in the long exact homology sequence associated to the short exact sequence of chain complexes by Lemma 12.6.

Proof. Omitted. ☐

Lemma 14.6. *Notation and assumptions as in Lemma 14.4 above. Suppose $\{s'_n : C_n \to B_n\}$ is a second choice of splittings. Write $s'_n = s_n + i_n \circ h_n$ for some unique morphisms $h_n : C_n \to A_n$. The family of maps $\{h_n : C_n \to A[-1]_{n+1}\}$ is a homotopy between the associated morphisms $\delta(s), \delta(s') : C_\bullet \to A[-1]_\bullet$.*

Proof. Omitted. ☐

Definition 14.7. Let $\mathcal{A}$ be an additive category. Let $A^\bullet$ be a cochain complex with boundary maps $d_A^n : A^n \to A^{n+1}$. For any $k \in \mathbf{Z}$ we define the *k-shifted cochain complex* $A[k]^\bullet$ as follows:

1. we set $A[k]^n = A^{n+k}$, and
2. we set $d_{A[k]}^n : A[k]^n \to A[k]^{n+1}$ equal to $d_{A[k]}^n = (-1)^k d_A^{n+k}$.

If $f : A^\bullet \to B^\bullet$ is a morphism of cochain complexes, then we let $f[k] : A[k]^\bullet \to B[k]^\bullet$ be the morphism of cochain complexes with $f[k]^n = f^{k+n}$.

Of course this means we have functors $[k] : \mathrm{CoCh}(\mathcal{A}) \to \mathrm{CoCh}(\mathcal{A})$ which mutually commute (on the nose, without any intervening isomorphisms of functors) and such that $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \mathrm{id}_{\mathrm{CoCh}(\mathcal{A})}$.

Definition 14.8. Let $\mathcal{A}$ be an abelian category. Let $A^\bullet$ be a cochain complex with boundary maps $d_A^n : A^n \to A^{n+1}$. For any $k \in \mathbf{Z}$ we identify $H^{i+k}(A^\bullet) \longrightarrow H^i(A[k]^\bullet)$ via the identification $A^{i+k} = A[k]^i$.

This identification is functorial in $A^\bullet$. Note that since no signs are involved in this definition we actually get a compatible system of identifications of all the homology objects $H^{i-k}(A[k]^\bullet)$, which are further compatible with the identifications $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \mathrm{id}_{\mathrm{CoCh}(\mathcal{A})}$.

Let $\mathcal{A}$ be an additive category. Suppose that $A^\bullet$ and $B^\bullet$ are cochain complexes, $a, b : A^\bullet \to B^\bullet$ are morphisms of cochain complexes, and $\{h^i : A^i \to B^{i-1}\}$ is a homotopy between a and b. Recall that this means that $a^i - b^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$. What if $a = b$? Then we obtain the formula $0 = d^{i-1} \circ h^i + h^{i+1} \circ d^i$, in other words, $-d^{i-1} \circ h^i = h^{i+1} \circ d^i$. By definition above this means the collection $\{h^i\}$ above defines a morphism of cochain complexes

$$A^\bullet \longrightarrow B[-1]^\bullet.$$

Such a thing is the same as a morphism $A[1]^\bullet \to B^\bullet$ by our remarks above. This proves the following lemma.

Lemma 14.9. *Let $\mathcal{A}$ be an additive category. Suppose that $A^\bullet$ and $B^\bullet$ are cochain complexes. Given any morphism of cochain complexes $a : A^\bullet \to B^\bullet$ there is a bijection between the set of homotopies from a to a and $\mathrm{Mor}_{CoCh(\mathcal{A})}(A^\bullet, B[-1]^\bullet)$. More generally, the set of homotopies between a and b is either empty or a principal homogeneous space under the group $\mathrm{Mor}_{CoCh(\mathcal{A})}(A^\bullet, B[-1]^\bullet)$.*

Proof. See above. $\square$

Lemma 14.10. *Let $\mathcal{A}$ be an additive category. Let*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be a complex (!) of complexes. Suppose that we are given splittings $B^n = A^n \oplus C^n$ compatible with the maps in the displayed sequence. Let $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$ be the corresponding maps. Then the family of morphisms

$$\pi^{n+1} \circ d_B^n \circ s^n : C^n \to A^{n+1}$$

define a morphism of complexes $\delta : C^\bullet \to A[1]^\bullet$.

Proof. Denote $i : A^\bullet \to B^\bullet$ and $q : B^\bullet \to C^\bullet$ the maps of complexes in the short exact sequence. Then $i^{n+1} \circ \pi^{n+1} \circ d_B^n \circ s^n = d_B^n \circ s^n - s^{n+1} \circ d_C^n$. Hence $i^{n+2} \circ d_A^{n+1} \circ \pi^{n+1} \circ d_B^n \circ s^n = d_B^{n+1} \circ (d_B^n \circ s^n - s^{n+1} \circ d_C^n) = -d_B^{n+1} \circ s^{n+1} \circ d_C^n$ as desired. $\square$

Lemma 14.11. *Notation and assumptions as in Lemma 14.10 above. Assume in addition that $\mathcal{A}$ is abelian. The morphism of complexes $\delta : C^\bullet \to A[1]^\bullet$ induces the maps*

$$H^i(\delta) : H^i(C^\bullet) \longrightarrow H^i(A[1]^\bullet) = H^{i+1}(A^\bullet)$$

which occur in the long exact homology sequence associated to the short exact sequence of cochain complexes by Lemma 12.12.

Proof. Omitted. ☐

Lemma 14.12. *Notation and assumptions as in Lemma 14.10. Let $\alpha : A^\bullet \to B^\bullet$, $\beta : B^\bullet \to C^\bullet$ be the given morphisms of complexes. Suppose $(s')^n : C^n \to B^n$ and $(\pi')^n : B^n \to A^n$ is a second choice of splittings. Write $(s')^n = s^n + \alpha^n \circ h^n$ and $(\pi')^n = \pi^n + g^n \circ \beta^n$ for some unique morphisms $h^n : C^n \to A^n$ and $g^n : C^n \to A^n$. Then*

1. *$g^n = -h^n$, and*
2. *the family of maps $\{g^n : C^n \to A[1]^{n-1}\}$ is a homotopy between $\delta, \delta' : C^\bullet \to A[1]^\bullet$, more precisely $(\delta')^n = \delta^n + g^{n+1} \circ d_C^n + d_{A[1]}^{n-1} \circ g^n$.*

Proof. As $(s')^n$ and $(\pi')^n$ are splittings we have $(\pi')^n \circ (s')^n = 0$. Hence

$$0 = (\pi^n + g^n \circ \beta^n) \circ (s^n + \alpha^n \circ h^n) = g^n \circ \beta^n \circ s^n + \pi^n \circ \alpha^n \circ h^n = g^n + h^n$$

which proves (1). We compute $(\delta')^n$ as follows

$$(\pi^{n+1} + g^{n+1} \circ \beta^{n+1}) \circ d_B^n \circ (s^n + \alpha^n \circ h^n) = \delta^n + g^{n+1} \circ d_C^n + d_A^n \circ h^n$$

Since $h^n = -g^n$ and since $d_{A[1]}^{n-1} = -d_A^n$ we conclude that (2) holds. ☐

15. Graded objects

We make the following definition.

Definition 15.1. Let $\mathcal{A}$ be an additive category. The *category of graded objects of* $\mathcal{A}$, denoted $\text{Gr}(\mathcal{A})$, is the category with

1. objects $A = (A^i)$ are families of objects A^i, $i \in \mathbf{Z}$ of objects of $\mathcal{A}$, and
2. morphisms $f : A = (A^i) \to B = (B^i)$ are families of morphisms $f^i : A^i \to B^i$ of $\mathcal{A}$.

If $\mathcal{A}$ has countable direct sums, then we can asssociated to an object $A = (A^i)$ of $\text{Gr}(\mathcal{A})$ the object

$$A = \bigoplus\nolimits_{i \in \mathbf{Z}} A^i$$

and set $k^iA = A^i$. In this case $\text{Gr}(\mathcal{A})$ is equivalent to the category of pairs (A, k) consisting of an object A of $\mathcal{A}$ and a direct sum decomposition

$$A = \bigoplus\nolimits_{i \in \mathbf{Z}} k^iA$$

by direct summands indexed by $\mathbf{Z}$ and a morphism $(A, k) \to (B, k)$ of such objects is given by a morphism $\varphi : A \to B$ of $\mathcal{A}$ such that $\varphi(k^iA) \subset k^iB$ for all $i \in \mathbf{Z}$. Whenever our additive category $\mathcal{A}$ has countable direct sums we will use this equivalence without further mention.

However, with our definitions an additive or abelian category does not necessarily have all (countable) direct sums. In this case our definition still makes sense. For example, if $\mathcal{A} = \text{Vect}_k$ is the category of finite dimensional vector spaces over a field k, then $\text{Gr}(\text{Vect}_k)$ is the category of vector spaces with a given gradation all of whose graded pieces are finite dimensional, and not the category of finite dimensional vector spaces with a given graduation.

Lemma 15.2. *Let $\mathcal{A}$ be an abelian category. The category of graded objects $Gr(\mathcal{A})$ is abelian.*

Proof. Let $f : A = (A^i) \to B = (B^i)$ be a morphism of graded objects of $\mathcal{A}$ given by morphisms $f^i : A^i \to B^i$ of $\mathcal{A}$. Then we have $\text{Ker}(f) = (\text{Ker}(f^i))$ and $\text{Coker}(f) = (\text{Coker}(f^i))$ in the category $\text{Gr}(\mathcal{A})$. Since we have $\text{Im} = \text{Coim}$ in $\mathcal{A}$ we see the same thing holds in $\text{Gr}(\mathcal{A})$. $\square$

Remark 15.3 (Warning)**.** There are abelian categories $\mathcal{A}$ having countable direct sums but where countable direct sums are not exact. An example is the opposite of the category of abelian sheaves on $\mathbf{R}$. Namely, the category of abelian sheaves on $\mathbf{R}$ has countable products, but countable products are not exact. For such a category the functor $\text{Gr}(\mathcal{A}) \to \mathcal{A}$, $(A^i) \mapsto \bigoplus A^i$ described above is not exact. It is still true that $\text{Gr}(\mathcal{A})$ is equivalent to the category of graded objects (A, k) of $\mathcal{A}$, but the kernel in the category of graded objects of a map $\varphi : (A, k) \to (B, k)$ is not equal to $\text{Ker}(\varphi)$ endowed with a direct sum decomposition, but rather it is the direct sum of the kernels of the maps $k^iA \to k^iB$.

Definition 15.4. Let $\mathcal{A}$ be an additive category. If $A = (A^i)$ is a graded object, then the kth *shift* $A[k]$ is the graded object with $A[k]^i = A^{k+i}$.

If A and B are graded objects of $\mathcal{A}$, then we have

$$\text{Hom}_{\text{Gr}(\mathcal{A})}(A, B[k]) = \text{Hom}_{\text{Gr}(\mathcal{A})}(A[-k], B) \tag{15.4.1}$$

and an element of this group is sometimes called a map of graded objects *homogeneous of degree k*.

Given any set G we can define G-graded objects of $\mathcal{A}$ as the category whose objects are $A = (A^g)_{g \in G}$ families of objects parametrized by elements of G. Morphisms $f : A \to B$ are defined as families of maps $f^g : A^g \to B^g$ where g runs over the elements of G. If G is an abelian group, then we can (unambiguously) define shift functors $[g]$ on the category of G-graded objects by the rule $(A[g])^{g_0} = A^{g+g_0}$. A particular case of this type of construction is when $G = \mathbf{Z} \times \mathbf{Z}$. In this case the objects of the category are called *bigraded* objects of $\mathcal{A}$. The (p, q) component of a bigraded object A is usually denoted $A^{p,q}$. For $(a, b) \in \mathbf{Z} \times \mathbf{Z}$ we write $A[a, b]$ in stead of $A[(a, b)]$. A morphism $A \to A[a, b]$ is sometimes called a *map of bidegree* (a, b).

16. Filtrations

A nice reference for this material is [Del71, Section 1]. (Note that our conventions regarding abelian categories are different.)

Definition 16.1. Let $\mathcal{A}$ be an abelian category.

(1) A *decreasing filtration* F on an object A is a family $(F^nA)_{n \in \mathbf{Z}}$ of subobjects of A such that
$$A \supset \ldots \supset F^nA \supset F^{n+1}A \supset \ldots \supset 0$$
(2) A *filtered object of* $\mathcal{A}$ is pair (A, F) consisting of an object A of $\mathcal{A}$ and a decreasing filtration F on A.
(3) A *morphism* $(A, F) \to (B, F)$ *of filtered objects* is given by a morphism $\varphi : A \to B$ of $\mathcal{A}$ such that $\varphi(F^iA) \subset F^iB$ for all $i \in \mathbf{Z}$.
(4) The category of filtered objects is denoted $\text{Fil}(\mathcal{A})$.

(5) Given a filtered object (A, F) and a subobject $X \subset A$ the *induced filtration* on X is the filtration with $F^n X = X \cap F^n A$.
(6) Given a filtered object (A, F) and a surjection $\pi : A \to Y$ the *quotient filtration* is the filtration with $F^n Y = \pi(F^n A)$.
(7) A filtration F on an object A is said to be *finite* if there exist n, m such that $F^n A = A$ and $F^m A = 0$.
(8) Given a filtered object (A, F) we say $\bigcap F^i A$ exists if there exists a biggest subobject of A contained in all $F^i A$. We say $\bigcup F^i A$ exists if there exists a smallest subobject of A containing all $F^i A$.
(9) The filtration on a filtered object (A, F) is said to be *separated* if $\bigcap_i F^i A = 0$ and *exhaustive* if $\bigcup F^i A = A$.

By abuse of notation we say that a morphism $f : (A, F) \to (B, F)$ of filtered objects is *injective* if $f : A \to B$ is injective in the abelian category $\mathcal{A}$. Similarly we say f is *surjective* if $f : A \to B$ is surjective in the category $\mathcal{A}$. Being injective (resp. surjective) is equivalent to being a monomorphism (resp. epimorphism) in $\text{Fil}(\mathcal{A})$. By Lemma 16.2 this is also equivalent to having zero kernel (resp. cokernel).

Lemma 16.2. *Let $\mathcal{A}$ be an abelian category. The category of filtered objects Fil($\mathcal{A}$) has the following properties:*

(1) *It is an additive category.*
(2) *It has a zero object.*
(3) *It has kernels and cokernels, images and coimages.*
(4) *In general it is not an abelian category.*

Proof. It is clear that $\text{Fil}(\mathcal{A})$ is additive with direct sum given by $(A, F) \oplus (B, F) = (A \oplus B, F)$ where $F^p(A \oplus B) = F^p A \oplus F^p B$. The kernel of a morphism $f : (A, F) \to (B, F)$ of filtered objects is the injection $\text{Ker}(f) \subset A$ where $\text{Ker}(f)$ is endowed with the induced filtration. The cokernel of a morphism $f : A \to B$ of filtered objects is the surjection $B \to \text{Coker}(f)$ where $\text{Coker}(f)$ is endowed with the quotient filtration. Since all kernels and cokernels exist, so do all coimages and images. See Example 3.12 for the last statement. □

Definition 16.3. Let $\mathcal{A}$ be an abelian category. A morphism $f : A \to B$ of filtered objects of $\mathcal{A}$ is said to be *strict* if $f(F^i A) = f(A) \cap F^i B$ for all $i \in \mathbf{Z}$.

This also equivalent to requiring that $f^{-1}(F^i B) = F^i A + \text{Ker}(f)$ for all $i \in \mathbf{Z}$. We characterize strict morphisms as follows.

Lemma 16.4. *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a morphism of filtered objects of $\mathcal{A}$. The following are equivalent*

(1) *f is strict,*
(2) *the morphism Coim(f) $\to$ Im(f) of Lemma 3.11 is an isomorphism.*

Proof. Note that $\text{Coim}(f) \to \text{Im}(f)$ is an isomorphism of objects of $\mathcal{A}$, and that part (2) signifies that it is an isomorphism of filtered objects. By the description of kernels and cokernels in the proof of Lemma 16.2 we see that the filtration on $\text{Coim}(f)$ is the quotient filtration coming from $A \to \text{Coim}(f)$. Similarly, the filtration on $\text{Im}(f)$ is the induced filtration coming from the injection $\text{Im}(f) \to B$. The definition of strict is exactly that the quotient filtration is the induced filtration. □

Lemma 16.5. *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a strict monomorphism of filtered objects. Let $g : A \to C$ be a morphism of filtered objects. Then $f \oplus g : A \to B \oplus C$ is a strict monomorphism.*

Proof. Clear from the definitions. $\square$

Lemma 16.6. *Let $\mathcal{A}$ be an abelian category. Let $f : B \to A$ be a strict epimorphism of filtered objects. Let $g : C \to A$ be a morphism of filtered objects. Then $f \oplus g : B \oplus C \to A$ is a strict epimorphism.*

Proof. Clear from the definitions. $\square$

Lemma 16.7. *Let $\mathcal{A}$ be an abelian category. Let (A, F), (B, F) be filtered objects. Let $u : A \to B$ be a morphism of filtered objects. If u is injective then u is strict if and only if the filtration on A is the induced filtration. If u is surjective then u is strict if and only if the filtration on B is the quotient filtration.*

Proof. This is immediate from the definition. $\square$

Lemma 16.8. *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$, $g : B \to C$ be strict morphisms of filtered objects.*

1. *In general the composition $g \circ f$ is not strict.*
2. *If g is injective, then $g \circ f$ is strict.*
3. *If f is surjective, then $g \circ f$ is strict.*

Proof. Let B a vector space over a field k with basis e_1, e_2, with the filtration $F^nB = B$ for $n < 0$, with $F^0B = ke_1$, and $F^nB = 0$ for $n > 0$. Now take $A = k(e_1 + e_2)$ and $C = B/ke_2$ with filtrations induced by B, i.e., such that $A \to B$ and $B \to C$ are strict (Lemma 16.7). Then $F^n(A) = A$ for $n < 0$ and $F^n(A) = 0$ for $n \geq 0$. Also $F^n(C) = C$ for $n \leq 0$ and $F^n(C) = 0$ for $n > 0$. So the (nonzero) composition $A \to C$ is not strict.

Assume g is injective. Then

$$\begin{aligned} g(f(F^pA)) &= g(f(A) \cap F^pB) \\ &= g(f(A)) \cap g(F^p(B)) \\ &= (g \circ f)(A) \cap (g(B) \cap F^pC) \\ &= (g \circ f)(A) \cap F^pC. \end{aligned}$$

The first equality as f is strict, the second because g is injective, the third because g is strict, and the fourth because $(g \circ f)(A) \subset g(B)$.

Assume f is surjective. Then

$$\begin{aligned} (g \circ f)^{-1}(F^iC) &= f^{-1}(F^iB + \mathrm{Ker}(g)) \\ &= f^{-1}(F^iB) + f^{-1}(\mathrm{Ker}(g)) \\ &= F^iA + \mathrm{Ker}(f) + \mathrm{Ker}(g \circ f) \\ &= F^iA + \mathrm{Ker}(g \circ f) \end{aligned}$$

The first equality because g is strict, the second because f is surjective, the third because f is strict, and the last because $\mathrm{Ker}(f) \subset \mathrm{Ker}(g \circ f)$. $\square$

The following lemma says that subobjects of a filtered object have a well defined filtration independent of a choice of writing the object as a cokernel.

Lemma 16.9. *Let $\mathcal{A}$ be an abelian category. Let (A, F) be a filtered object of $\mathcal{A}$. Let $X \subset Y \subset A$ be subobjects of A. On the object*

$$Y/X = Ker(A/X \to A/Y)$$

the quotient filtration coming from the induced filtration on Y and the induced filtration coming from the quotient filtration on A/X agree. Any of the morphisms $X \to Y$, $X \to A$, $Y \to A$, $Y \to A/X$, $Y \to Y/X$, $Y/X \to A/X$ are strict (with induced/quotient filtrations).

Proof. The quotient filtration Y/X is given by $F^p(Y/X) = F^pY/(X \cap F^pY) = F^pY/F^pX$ because $F^pY = Y \cap F^pA$ and $F^pX = X \cap F^pA$. The induced filtration from the injection $Y/X \to A/X$ is given by

$$\begin{aligned} F^p(Y/X) &= Y/X \cap F^p(A/X) \\ &= Y/X \cap (F^pA + X)/X \\ &= (Y \cap F^pA)/(X \cap F^pA) \\ &= F^pY/F^pX. \end{aligned}$$

Hence the first statement of the lemma. The proof of the other cases is similar. $\square$

Lemma 16.10. *Let $\mathcal{A}$ be an abelian category. Let $A, B, C \in Fil(\mathcal{A})$. Let $f : A \to B$ and $g : A \to C$ be morphisms Then there exists a pushout*

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & C \amalg_A B \end{tikzcd}$$

in $Fil(\mathcal{A})$. If f is strict, so is f'.

Proof. Set $C \amalg_A B$ equal to $\text{Coker}((1, -1) : A \to C \oplus B)$ in $\text{Fil}(\mathcal{A})$. This cokernel exists, by Lemma 16.2. It is a pushout, see Example 5.6. Note that $F^p(C \times_A B)$ is the image of $F^pC \oplus F^pB$. Hence

$$(f')^{-1}(F^p(C \times_A B)) = g(f^{-1}(F^pB))) + F^pC$$

Whence the last statement. $\square$

Lemma 16.11. *Let $\mathcal{A}$ be an abelian category. Let $A, B, C \in Fil(\mathcal{A})$. Let $f : B \to A$ and $g : C \to A$ be morphisms Then there exists a pushout*

$$\begin{tikzcd} B \times_A C \arrow[r, "f'"'] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ C \arrow[r, "f"] & A \end{tikzcd}$$

in $Fil(\mathcal{A})$. If f is strict, so is f'.

Proof. This lemma is dual to Lemma 16.10. $\square$

Let $\mathcal{A}$ be an abelian category. Let (A, F) be a filtered object of $\mathcal{A}$. We denote $\text{gr}^p_F(A) = \text{gr}^p(A)$ the object $F^pA/F^{p+1}A$ of $\mathcal{A}$. This defines an additive functor

$$\text{gr}^p : \text{Fil}(\mathcal{A}) \longrightarrow \mathcal{A}, \quad (A, F) \longmapsto \text{gr}^p(A).$$

Recall that we have defined the category $\text{Gr}(\mathcal{A})$ of graded objects of $\mathcal{A}$ in Section 15. For (A, F) in $\text{Fil}(\mathcal{A})$ we may set

$$\text{gr}(A) = \text{the graded object of } \mathcal{A} \text{ whose } p\text{th graded piece is } \text{gr}^p(A)$$

and if $\mathcal{A}$ has countable direct sums, then we simply have

$$\text{gr}(A) = \bigoplus \text{gr}^p(A)$$

This defines an additive functor

$$\text{gr} : \text{Fil}(\mathcal{A}) \longrightarrow \text{Gr}(\mathcal{A}), \quad (A, F) \longmapsto \text{gr}(A).$$

Lemma 16.12. *Let $\mathcal{A}$ be an abelian category.*

(1) *Let A be a filtered object and $X \subset A$. Then for each p the sequence*

$$0 \to gr^p(X) \to gr^p(A) \to gr^p(A/X) \to 0$$

is exact (with induced filtration on X and quotient filtration on A/X).

(2) *Let $f : A \to B$ be a morphism of filtered objects of $\mathcal{A}$. Then for each p the sequences*

$$0 \to gr^p(Ker(f)) \to gr^p(A) \to gr^p(Coim(f)) \to 0$$

and

$$0 \to gr^p(Im(f)) \to gr^p(B) \to gr^p(Coker(f)) \to 0$$

are exact.

Proof. We have $F^{p+1}X = X \cap F^{p+1}A$, hence map $\text{gr}^p(X) \to \text{gr}^p(A)$ is injective. Dually the map $\text{gr}^p(A) \to \text{gr}^p(A/X)$ is surjective. The kernel of $F^pA/F^{p+1}A \to A/X + F^{p+1}A$ is clearly $F^{p+1}A + X \cap F^pA/F^{p+1}A = F^pX/F^{p+1}X$ hence exactness in the middle. The two short exact sequence of (2) are special cases of the short exact sequence of (1). □

Lemma 16.13. *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a morphism of finite filtered objects of $\mathcal{A}$. The following are equivalent*

(1) *f is strict,*
(2) *the morphism $Coim(f) \to Im(f)$ is an isomorphism,*
(3) *$gr(Coim(f)) \to gr(Im(f))$ is an isomorphism,*
(4) *the sequence $gr(Ker(f)) \to gr(A) \to gr(B)$ is exact,*
(5) *the sequence $gr(A) \to gr(B) \to gr(Coker(f))$ is exact, and*
(6) *the sequence*

$$0 \to gr(Ker(f)) \to gr(A) \to gr(B) \to gr(Coker(f)) \to 0$$

is exact.

Proof. The equivalence of (1) and (2) is Lemma 16.4. By Lemma 16.12 we see that (4), (5), (6) imply (3) and that (3) implies (4), (5), (6). Hence it suffices to show that (3) implies (2). Thus we have to show that if $f : A \to B$ is an injective and surjective map of finite filtered objects which induces and isomorphism $\text{gr}(A) \to \text{gr}(B)$, then f induces an isomorphism of filtered objects. In other words, we have to show that $f(F^pA) = F^pB$ for all p. As the filtrations are finite we may prove this by descending induction on p. Suppose that $f(F^{p+1}A) = F^{p+1}B$. Then commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^{p+1}A \arrow[r] \arrow[d, "f"] & F^pA \arrow[r] \arrow[d, "f"] & \mathrm{gr}^p(A) \arrow[r] \arrow[d, "\mathrm{gr}^p(f)"] & 0 \\
0 \arrow[r] & F^{p+1}B \arrow[r] & F^pB \arrow[r] & \mathrm{gr}^p(B) \arrow[r] & 0
\end{tikzcd}$$

and the five lemma imply that $f(F^pA) = F^pB$. ☐

Lemma 16.14. *Let $\mathcal{A}$ be an abelian category. Let $A \to B \to C$ be a complex of filtered objects of $\mathcal{A}$. Assume $\alpha : A \to B$ and $\beta : B \to C$ are strict morphisms of filtered objects. Then $gr(Ker(\beta)/Im(\alpha)) = Ker(gr(\beta))/Im(gr(\alpha)))$.*

Proof. This follows formally from Lemma 16.12 and the fact that $\mathrm{Coim}(\alpha) \cong \mathrm{Im}(\alpha)$ and $\mathrm{Coim}(\beta) \cong \mathrm{Im}(\beta)$ by Lemma 16.4. ☐

Lemma 16.15. *Let $\mathcal{A}$ be an abelian category. Let $A \to B \to C$ be a complex of filtered objects of $\mathcal{A}$. Assume A, B, C have finite filtrations and that $gr(A) \to gr(B) \to gr(C)$ is exact. Then*

(1) *for each $p \in \mathbf{Z}$ the sequence $gr^p(A) \to gr^p(B) \to gr^p(C)$ is exact,*
(2) *for each $p \in \mathbf{Z}$ the sequence $F^p(A) \to F^p(B) \to F^p(C)$ is exact,*
(3) *for each $p \in \mathbf{Z}$ the sequence $A/F^p(A) \to B/F^p(B) \to C/F^p(C)$ is exact,*
(4) *the maps $A \to B$ and $B \to C$ are strict, and*
(5) *$A \to B \to C$ is exact (as a sequence in $\mathcal{A}$).*

Proof. Part (1) is immediate from the definitions. We will prove (3) by induction on the length of the filtrations. If each of A, B, C has only one nonzero graded part, then (3) holds as $\mathrm{gr}(A) = A$, etc. Let n be the largest integer such that at least one of F^nA, F^nB, F^nC is nonzero. Set $A' = A/F^nA$, $B' = B/F^nB$, $C' = C/F^nC$ with induced filtrations. Note that $\mathrm{gr}(A) = F^nA \oplus \mathrm{gr}(A')$ and similarly for B and C. The induction hypothesis applies to $A' \to B' \to C'$, which implies that $A/F^p(A) \to B/F^p(B) \to C/F^p(C)$ is exact for $p \geq n$. To conclude the same for $p = n+1$, i.e., to prove that $A \to B \to C$ is exact we use the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^nA \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^nB \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^nC \arrow[r] & C \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

whose rows are short exact sequences of objects of $\mathcal{A}$. The proof of (2) is dual. Of course (5) follows from (2).

To prove (4) denote $f : A \to B$ and $g : B \to C$ the given morphisms. We know that $f(F^p(A)) = \mathrm{Ker}(F^p(B) \to F^p(C))$ by (2) and $f(A) = \mathrm{Ker}(g)$ by (5). Hence $f(F^p(A)) = \mathrm{Ker}(F^p(B) \to F^p(C)) = \mathrm{Ker}(g) \cap F^p(B) = f(A) \cap F^p(B)$ which proves that f is strict. The proof that g is strict is dual to this. ☐

17. Spectral sequences

A nice discussion of spectral sequences may be found in [Eis95]. See also [McC01], [Lan02], etc.

Definition 17.1. Let $\mathcal{A}$ be an abelian category.

(1) A *spectral sequence in* $\mathcal{A}$ is given by a system $(E_r, d_r)_{r\geq 1}$ where each E_r is an object of $\mathcal{A}$, each $d_r : E_r \to E_r$ is a morphism such that $d_r \circ d_r = 0$ and $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ for $r \geq 1$.
(2) A *morphism of spectral sequences* $f : (E_r, d_r)_{r\geq 1} \to (E'_r, d'_r)_{r\geq 1}$ is given by a family of morphisms $f_r : E_r \to E'_r$ such that $f_r \circ d_r = d'_r \circ f_r$ and such that f_{r+1} is the morphism induced by f_r via the identifications $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ and $E'_{r+1} = \mathrm{Ker}(d'_r)/\mathrm{Im}(d'_r)$.

We will sometimes loosen this definition somewhat and allow E_{r+1} to be an object with a given isomorphism $E_{r+1} \to \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$. In addition we sometimes have a system $(E_r, d_r)_{r\geq r_0}$ for some r_0 satisfying the properties of the definition above for indices $\geq r$. We will also call this a spectral sequence since by a simple renumbering it falls under the definition anyway. In fact, sometimes it makes sense to allow $r_0 = 0$ or even $r_0 = -1$ due to conventions in the literature.

Given a spectral sequence $(E_r, d_r)_{r\geq 1}$ we define

$$0 = B_1 \subset B_2 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_2 \subset Z_1 = E_1$$

by the following simple procedure. Set $B_2 = \mathrm{Im}(d_1)$ and $Z_2 = \mathrm{Ker}(d_1)$. Then it is clear that $d_2 : Z_2/B_2 \to Z_2/B_2$. Hence we can define B_3 as the unique subobject of E_1 containing B_2 such that B_3/B_2 is the image of d_2. Similarly we can define Z_3 as the unique subobject of E_1 containing B_2 such that Z_3/B_2 is the kernel of d_2. And so on and so forth. In particular we have

$$E_r = Z_r/B_r$$

for all $r \geq 1$. I case the spectral sequence starts at $r = r_0$ then we can similarly construct B_i, Z_i as subobjects in E_{r_0}.

Definition 17.2. Let $\mathcal{A}$ be an abelian category. Let $(E_r, d_r)_{r\geq 1}$ be a spectral sequence.

(1) If the subobjects $Z_\infty = \bigcap Z_r$ and $B_\infty = \bigcup B_r$ of E_1 exist then we define the *limit* of the spectral sequence to be the object
$$E_\infty = Z_\infty/B_\infty.$$
(2) We say that the spectral sequence *collapses at* E_r, or *degenerates at* E_r if the differentials $d_r, d_{r+1}, \ldots$ are all zero.

Note that if the spectral sequence collapses at E_r, then we have $E_r = E_{r+1} = \ldots = E_\infty$ (and the limit exists of course). Also, almost any abelian category we will encounter has countable sums and intersections.

Remark 17.3 (Variant)**.** It is often the case that the terms of a spectral sequence have additional structure, for example a grading or a bigrading. To accomodate this (and to get around certain technical issues) we introduce the following notion. Let $\mathcal{A}$ be an abelian category. Let $(T_r)_{r\geq 1}$ be a sequence of *translation* or *shift* functors, i.e., $T_r : \mathcal{A} \to \mathcal{A}$ is an isomorphism of categories. In this setting a *spectral* *sequence* is given by a system $(E_r, d_r)_{r\geq 1}$ where each E_r is an object of $\mathcal{A}$, each $d_r : E_r \to T_rE_r$ is a morphism such that $T_rd_r \circ d_r = 0$ so that

$$\ldots \longrightarrow T_r^{-1}E_r \xrightarrow{T_r^{-1}d_r} E_r \xrightarrow{d_r} T_rE_r \xrightarrow{T_rd_r} T_r^2E_r \longrightarrow \ldots$$

is a complex and $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(T_r^{-1}d_r)$ for $r \geq 1$. It is clear what a *morphism of spectral sequences* means in this setting. In this setting we can still define

$$0 = B_1 \subset B_2 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_2 \subset Z_1 = E_1$$

and Z_∞ and B_∞ (if they exist) as above.

18. Spectral sequences: exact couples

Definition 18.1. Let $\mathcal{A}$ be an abelian category.

(1) An *exact couple* is a datum (A, E, α, f, g) where A, E are objects of $\mathcal{A}$ and α, f, g are morphisms as in the following diagram

$$\begin{tikzcd} A \arrow[rr, "\alpha"'] & & A \arrow[dl, "g"] \\ & E \arrow[ul, "f"] & \end{tikzcd}$$

with the property that the kernel of each arrow is the image of its predecessor. So $\mathrm{Ker}(\alpha) = \mathrm{Im}(f)$, $\mathrm{Ker}(f) = \mathrm{Im}(g)$, and $\mathrm{Ker}(g) = \mathrm{Im}(\alpha)$.

(2) A *morphism of exact couples* $t : (A, E, \alpha, f, g) \to (A', E', \alpha', f', g')$ is given by morphisms $t_A : A \to A'$ and $t_E : E \to E'$ such that $\alpha' \circ t_A = t_A \circ \alpha$, $f' \circ t_E = t_A \circ f$, and $g' \circ t_A = t_E \circ g$.

Lemma 18.2. *Let (A, E, α, f, g) be an exact couple in an abelian category $\mathcal{A}$. Set*

(1) $d = g \circ f : E \to E$ *so that* $d \circ d = 0$,
(2) $E' = Ker(d)/Im(d)$,
(3) $A' = Im(\alpha)$,
(4) $\alpha' : A' \to A'$ *induced by* α,
(5) $f' : E' \to A'$ *induced by* f,
(6) $g' : A' \to E'$ *induced by* "$g \circ \alpha^{-1}$".

Then we have

(1) $Ker(d) = f^{-1}(Ker(g)) = f^{-1}(Im(\alpha))$,
(2) $Im(d) = g(Im(f)) = g(Ker(\alpha))$,
(3) $(A', E', \alpha', f', g')$ *is an exact couple.*

Proof. Omitted. □

Hence it is clear that given an exact couple (A, E, α, f, g) we get a spectral sequence by setting $E_1 = E$, $d_1 = d$, $E_2 = E'$, $d_2 = d' = g' \circ f'$, $E_3 = E''$, $d_3 = d'' = g'' \circ f''$, and so on.

Definition 18.3. Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. The *spectral sequence associated to the exact couple* is the spectral sequence $(E_r, d_r)_{r\geq 1}$ with $E_1 = E$, $d_1 = d$, $E_2 = E'$, $d_2 = d' = g' \circ f'$, $E_3 = E''$, $d_3 = d'' = g'' \circ f''$, and so on.

Lemma 18.4. *Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. Let $(E_r, d_r)_{r\geq 1}$ be the spectral sequence associated to the exact couple. In this case we have*

$$0 = B_1 \subset \ldots \subset B_{r+1} = g(Ker(\alpha^r)) \subset \ldots \subset Z_{r+1} = f^{-1}(Im(\alpha^r)) \subset \ldots \subset Z_1 = E$$

and the map $d_{r+1} : E_{r+1} \to E_{r+1}$ is described by the following rule: For any (test) object T of $\mathcal{A}$ and any elements $x : T \to Z_{r+1}$ and $y : T \to A$ such that $f \circ x = \alpha^r \circ y$ we have

$$d_{r+1} \circ \overline{x} = \overline{g \circ y}$$

where $\overline{x} : T \to E_{r+1}$ is the induced morphism.

Proof. Omitted. □

Note that in the situation of the lemma we obviously have

$$B_\infty = g\left(\bigcup\nolimits_r \mathrm{Ker}(\alpha^r)\right) \subset Z_\infty = f^{-1}\left(\bigcap\nolimits_r \mathrm{Im}(\alpha^r)\right)$$

provided these exist and in this case $E_\infty = Z_\infty / B_\infty$.

Remark 18.5 (Variant)**.** Let $\mathcal{A}$ be an abelian category. Let $S, T : \mathcal{A} \to \mathcal{A}$ be shift functors, i.e., isomorphisms of categories. We will indicate the n-fold compositions by $S^n A$ and $T^n A$ for $A \in \mathrm{Ob}(\mathcal{A})$ and $n \in \mathbf{Z}$. In this situation an *exact couple* is a datum (A, E, α, f, g) where A, E are objects of $\mathcal{A}$ and $\alpha : A \to T^{-1}A$, $f : E \to A$, $g : A \to SE$ are morphisms such that

$$TE \xrightarrow{Tf} TA \xrightarrow{T\alpha} A \xrightarrow{g} SE \xrightarrow{Sf} SA$$

is an exact complex. Let's visualize this as follows

$$\begin{tikzcd}
TA \arrow[rrr, "T\alpha"'] & & & A \arrow[dl, "g"] \arrow[rrr, "\alpha"'] & & & T^{-1}A \arrow[dl, "T^{-1}g"] \\
& TE \arrow[ul, "Tf"] \arrow[r, dotted, no head] & SE & & E \arrow[ul, "f"'] \arrow[r, dotted, no head] & T^{-1}SE &
\end{tikzcd}$$

We set $d = g \circ f : E \to SE$. Then $d \circ S^{-1}d = g \circ f \circ S^{-1}g \circ S^{-1}f = 0$ because $f \circ S^{-1}g = 0$. Set $E' = \mathrm{Ker}(d)/\mathrm{Im}(S^{-1}d)$. Set $A' = \mathrm{Im}(T\alpha)$. Let $\alpha' : A' \to T^{-1}A'$ induced by α. Let $f' : E' \to A'$ be induced by f which works because $f(\mathrm{Ker}(d)) \subset \mathrm{Ker}(g) = \mathrm{Im}(T\alpha)$. Finally, let $g' : A' \to TSE'$ induced by "$Tg \circ (T\alpha)^{-1}$"[3].

In exactly the same way as above we find

(1) $\mathrm{Ker}(d) = f^{-1}(\mathrm{Ker}(g)) = f^{-1}(\mathrm{Im}(T\alpha))$,
(2) $\mathrm{Im}(d) = g(\mathrm{Im}(f)) = g(\mathrm{Ker}(\alpha))$,
(3) $(A', E', \alpha', f', g')$ is an exact couple for the shift functors TS and T.

We obtain a spectral sequence (as in Remark 17.3) with $E_1 = E$, $E_2 = E'$, etc, with $d_r : E_r \to T^{r-1}SE_r$ for all $r \geq 1$. Lemma 18.4 tells us that

$$SB_{r+1} = g(\mathrm{Ker}(T^{-r+1}\alpha \circ \ldots \circ T^{-1}\alpha \circ \alpha))$$

and

$$Z_{r+1} = f^{-1}(\mathrm{Im}(T\alpha \circ T^2\alpha \circ \ldots \circ T^r\alpha))$$

---

[3] This works because $TSE' = \mathrm{Ker}(TSd)/\mathrm{Im}(Td)$ and $Tg(\mathrm{Ker}(T\alpha)) = Tg(\mathrm{Im}(Tf)) = \mathrm{Im}(T(d))$ and $TS(d)(\mathrm{Im}(Tg)) = \mathrm{Im}(TSg \circ TSf \circ Tg) = 0$.

in this situation. The description of the map d_{r+1} is similar to that given in the lemma. (It may be easier to use these explicit descriptions to prove one gets a spectral sequence from such an exact couple.)

19. Spectral sequences: differential objects

Definition 19.1. Let $\mathcal{A}$ be an abelian category. A *differential object* of $\mathcal{A}$ is a pair (A, d) consisting of an object A of $\mathcal{A}$ endowed with a selfmap d such that $d \circ d = 0$. A *morphism of differential objects* $(A, d) \to (B, d)$ is given by a morphism $\alpha : A \to B$ such that $d \circ \alpha = \alpha \circ d$.

Lemma 19.2. *Let $\mathcal{A}$ be an abelian category. The category of differential objects of $\mathcal{A}$ is abelian.*

Proof. Omitted. □

Definition 19.3. For a differential object (A, d) we denote

$$H(A, d) = \mathrm{Ker}(d)/\mathrm{Im}(d)$$

its *homology.*

Lemma 19.4. *Let $\mathcal{A}$ be an abelian category. Let $0 \to (A, d) \to (B, d) \to (C, d) \to 0$ be a short exact sequence of differential objects. Then we get an exact homology sequence*

$$\ldots \to H(C, d) \to H(A, d) \to H(B, d) \to H(C, d) \to \ldots$$

Proof. Apply Lemma 12.12 to the short exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

where the vertical arrows are d. □

We come to an important example of a spectral sequence. Let $\mathcal{A}$ be an abelian category. Let (A, d) be a differential object of $\mathcal{A}$. Let $\alpha : (A, d) \to (A, d)$ be an endomorphism of this differential object. If we assume α injective, then we get a short exact sequence

$$0 \to (A, d) \to (A, d) \to (A/\alpha A, d) \to 0$$

of differential objects. By the Lemma 19.4 we get an exact couple

$$\begin{tikzcd} H(A,d) \arrow[rr, "\overline{\alpha}"'] & & H(A,d) \arrow[dl, "g"] \\ & H(A/\alpha A, d) \arrow[ul, "f"] & \end{tikzcd}$$

where g is the canonical map and f is the map defined in the snake lemma. Thus we get an associated spectral sequence! Since in this case we have $E_1 = H(A/\alpha A, d)$ we see that it makes sense to define $E_0 = A/\alpha A$ and $d_0 = d$. In other words, we start the spectral sequence with $r = 0$. According to our conventions in Section 17 we define a sequence of subobjects

$$0 = B_0 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_0 = E_0$$

with the property that $E_r = Z_r/B_r$. Namely we have for $r \geq 1$ that

1. B_r is the image of $(\alpha^{r-1})^{-1}(dA)$ under the natural map $A \to A/\alpha A$,
2. Z_r is the image of $d^{-1}(\alpha^r A)$ under the natural map $A \to A/\alpha A$, and
3. $d_r : E_r \to E_r$ is given as follows: given an element $z \in Z_r$ choose an element $y \in A$ such that $d(z) = \alpha^r(y)$. Then $d_r(z + B_r + \alpha A) = y + B_r + \alpha A$.

Warning: It is not necessarily the case that $\alpha A \subset (\alpha^{r-1})^{-1}(dA)$, nor $\alpha A \subset d^{-1}(\alpha^r A)$. It is true that $(\alpha^{r-1})^{-1}(dA) \subset d^{-1}(\alpha^r A)$. We have

$$E_r = \frac{d^{-1}(\alpha^r A) + \alpha A}{(\alpha^{r-1})^{-1}(dA) + \alpha A}.$$

It is not hard to verify directly that (1) – (3) give a spectral sequence.

Definition 19.5. Let $\mathcal{A}$ be an abelian category. Let (A, d) be a differential object of $\mathcal{A}$. Let $\alpha : A \to A$ be an injective selfmap of A which commutes with d. The *spectral sequence associated to* (A, d, α) is the spectral sequence $(E_r, d_r)_{r \geq 0}$ described above.

Remark 19.6 (Variant)**.** Let $\mathcal{A}$ be an abelian category and let $S, T : \mathcal{A} \to \mathcal{A}$ be shift functors, i.e., isomorphisms of categories. Assume that $TS = ST$ as functors. Consider pairs (A, d) consisting of an object A of $\mathcal{A}$ and a morphism $d : A \to SA$ such that $d \circ S^{-1}d = 0$. The category of these objects is abelian. We define $H(A, d) = \text{Ker}(d)/\text{Im}(S^{-1}d)$ and we observe that $H(SA, Sd) = SH(A, d)$ (canonical isomorphism). Given a short exact sequence

$$0 \to (A, d) \to (B, d) \to (C, d) \to 0$$

we obtain a long exact homology sequence

$$\ldots \to S^{-1}H(C, d) \to H(A, d) \to H(B, d) \to H(C, d) \to SH(A, d) \to \ldots$$

(note the shifts in the boundary maps). Since $ST = TS$ the functor T defines a shift functor on pairs by setting $T(A, d) = (TA, Td)$. Next, let $\alpha : (A, d) \to T^{-1}(A, d)$ be injective with cokernel (Q, d). Then we get an exact couple as in Remark 18.5 with shift functors TS and T given by

$$(H(A, d), S^{-1}H(Q, d), \overline{\alpha}, f, g)$$

where $\overline{\alpha} : H(A, d) \to T^{-1}H(A, d)$ is induced by α, the map $f : S^{-1}H(Q, d) \to H(A, d)$ is the boundary map and $g : H(A, d) \to TH(Q, d) = TS(S^{-1}H(Q, d))$ is induced by the quotient map $A \to TQ$. Thus we get a spectral sequence as above with $E_1 = S^{-1}H(Q, d)$ and differentials $d_r : E_r \to T^r SE_r$. As above we set $E_0 = S^{-1}Q$ and $d_0 : E_0 \to SE_0$ given by $S^{-1}d : S^{-1}Q \to Q$. If according to our conventions we define $B_r \subset Z_r \subset E_0$, then we have for $r \geq 1$ that

1. SB_r is the image of
$$(T^{-r+1}\alpha \circ \ldots \circ T^{-1}\alpha)^{-1}\text{Im}(T^{-r}S^{-1}d)$$
under the natural map $T^{-1}A \to Q$,
2. Z_r is the image of
$$(S^{-1}T^{-1}d)^{-1}\text{Im}(\alpha \circ \ldots \circ T^{r-1}\alpha)$$
under the natural map $S^{-1}T^{-1}A \to S^{-1}Q$.

The differentials can be described as follows: if $x \in Z_r$, then pick $x' \in S^{-1}T^{-1}A$ mapping to x. Then $S^{-1}T^{-1}d(x')$ is $(\alpha \circ \ldots \circ T^{r-1}\alpha)(y)$ for some $y \in T^{r-1}A$. Then $d_r(x) \in T^rSE_r$ is represented by the class of the image of y in $T^rSE_0 = T^rQ$ modulo T^rSB_r.

20. Spectral sequences: filtered differential objects

We can build a spectral sequence starting with a filtered differential object.

Definition 20.1. Let $\mathcal{A}$ be an abelian category. A *filtered differential object* (K, F, d) is a filtered object (K, F) of $\mathcal{A}$ endowed with an endomorphism $d : (K, F) \to (K, F)$ whose square is zero: $d \circ d = 0$.

To describe the spectral sequence associated to such an object we assume, for the moment, that $\mathcal{A}$ is an abelian category which has countable direct sums and countable direct sums are exact (this is not automatic, see Remark 15.3). Let (K, F, d) be a filtered differential object of $\mathcal{A}$. Note that each F^nK is a differential object by itself. Consider the object $A = \bigoplus F^nK$ and endow it with a differential d by using d on each summand. Then (A, d) is a differential object of $\mathcal{A}$ which comes equipped with a grading. Consider the map

$$\alpha : A \to A$$

which is given by the inclusions $F^nA \to F^{n-1}A$. This is clearly an injective morphism of differential objects $\alpha : (A, d) \to (A, d)$. Hence, by Definition 19.5 we get a spectral sequence. We will call this *the spectral sequence associated to the filtered differential object* (K, F, d).

Let us figure out the terms of this spectral sequence. First, note that $A/\alpha A = \mathrm{gr}(K)$ endowed with its differential $d = \mathrm{gr}(d)$. Hence we see that

$$E_0 = \mathrm{gr}(K), \quad d_0 = \mathrm{gr}(d).$$

Hence the homology of the graded differential object $\mathrm{gr}(K)$ is the next term:

$$E_1 = H(\mathrm{gr}(K), \mathrm{gr}(d)).$$

In addition we see that E_0 is a graded object of $\mathcal{A}$ and that d_0 is compatible with the grading. Hence clearly E_1 is a graded object as well. But it turns out that the differential d_1 does not preserve this grading; instead it shifts the degree by 1.

To work this out precisely, we define

$$Z_r^p = \frac{F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K}{F^{p+1}K}$$

and

$$B_r^p = \frac{F^pK \cap d(F^{p-r+1}K) + F^{p+1}K}{F^{p+1}K}.$$

This notation, although quite natural, seems to be different from the notation in most places in the literature. Perhaps it does not matter, since the literature does not seem to have a consistent choice of notation either. With these choices we see that $B_r \subset E_0$, resp. $Z_r \subset E_0$ (as defined in Section 19) is equal to $\bigoplus_p B_r^p$, resp. $\bigoplus_p Z_r^p$. Hence if we define

$$E_r^p = Z_r^p / B_r^p$$

for $r \geq 0$ and $p \in \mathbf{Z}$, then we have $E_r = \bigoplus_p E_r^p$. We can define a differential $d_r^p : E_r^p \to E_r^{p+r}$ by the rule

$$z + F^{p+1}K \longmapsto dz + F^{p+r+1}K$$

where $z \in F^pK \cap d^{-1}(F^{p+r}K)$.

Lemma 20.2. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ in $Gr(\mathcal{A})$ associated to (K, F, d) such that $d_r : E_r \to E_r[1]$ for all r and such that the graded pieces E_r^p and maps $d_r^p : E_r^p \to E_r^{p+r}$ are as given above. Furthermore, $E_0^p = gr^pK$, $d_0^p = gr^p(d)$, and $E_1^p = H(gr^pK, d)$.*

Proof. If $\mathcal{A}$ has countable direct sums and if countable direct sums are exact, then this follows from the discussion above. In general, we proceed as follows; we strongly suggest the reader skip this proof. Consider the object $A = (F^{p+1}K)$ of $\mathrm{Gr}(\mathcal{A})$, i.e., we put $F^{p+1}K$ in degree p (the funny shift in numbering to get numbering correct later on). We endow it with a differential d by using d on each component. Then (A, d) is a differential object of $\mathrm{Gr}(\mathcal{A})$. Consider the map

$$\alpha : A \to A[-1]$$

which is given in degree p by the inclusions $F^{p+1}A \to F^pA$. This is clearly an injective morphism of differential objects $\alpha : (A, d) \to (A, d)[-1]$. Hence, we can apply Remark 19.6 with $S = \mathrm{id}$ and $T = [1]$. The corresponding spectral sequence $(E_r, d_r)_{r\geq 0}$ in $\mathrm{Gr}(\mathcal{A})$ is the spectral sequence we are looking for. Let us unwind the definitions a bit. First of all we have $E_r = (E_r^p)$ is an object of $\mathrm{Gr}(\mathcal{A})$. Then, since $T^rS = [r]$ we have $d_r : E_r \to E_r[r]$ which means that $d_r^p : E_r^p \to E_r^{p+r}$.

To see that the description of the graded pieces hold, we argue as above. Namely, first we have $E_0 = \mathrm{Coker}(\alpha : A \to A[-1])$ and by our choice of numbering above this gives $E_0^p = \mathrm{gr}^pK$. The first differential is given by $d_0^p = \mathrm{gr}^pd : E_0^p \to E_0^p$. Next, the description of the boundaries B_r and the cocycles Z_r in Remark 19.6 translates into a straightforward manner into the formulae for Z_r^p and B_r^p given above. $\square$

Lemma 20.3. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The spectral sequence $(E_r, d_r)_{r\geq 0}$ associated to (K, F, d) has*

$$d_1^p : E_1^p = H(gr^pK) \longrightarrow H(gr^{p+1}K) = E_1^{p+1}$$

equal to the boundary map in homology associated to the short exact sequence of differential objects

$$0 \to gr^{p+1}K \to F^pK/F^{p+2}K \to gr^{p+1}K \to 0.$$

Proof. Omitted. $\square$

Definition 20.4. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The *induced filtration* on $H(K, d)$ is the filtration defined by $F^pH(K, d) = \mathrm{Im}(H(F^pK, d) \to H(K, d))$.

Lemma 20.5. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. If Z_∞^p and B_∞^p exist (see proof), then associated graded $gr(H(K))$ of the cohomology of K is a graded subquotient of the graded object E_∞ having $E_\infty^p = Z_\infty^p / B_\infty^p$ in degree p.*

Proof. Here we have

$$Z_\infty^p = \bigcap\nolimits_r Z_r^p = \frac{\bigcap_r (F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K)}{F^{p+1}K}$$

and

$$B_\infty^p = \bigcup\nolimits_r B_r^p = \frac{\bigcup_r (F^pK \cap d(F^{p-r+1}K) + F^{p+1}K)}{F^{p+1}K}.$$

Thus

$$E_\infty^p = \frac{\bigcap_r (F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K)}{\bigcup_r (F^pK \cap d(F^{p-r+1}K) + F^{p+1}K)}.$$

and the top and bottom exist. On the other hand, we have

$$\mathrm{gr}^p H(K) = \frac{\mathrm{Ker}(d) \cap F^pK + F^{p+1}K}{\mathrm{Im}(d) \cap F^pK + F^{p+1}K}$$

The result follows since

$$\mathrm{Ker}(d) \cap F^pK + F^{p+1}K \subset \bigcup\nolimits_r \left(F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K\right) \tag{20.5.1}$$

and

$$\bigcap\nolimits_r \left(F^pK \cap d(F^{p-r+1}K) + F^{p+1}K\right) \subset \mathrm{Im}(d) \cap F^pK + F^{p+1}K. \tag{20.5.2}$$

□

Definition 20.6. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. We say the spectral sequence associated to (K, F, d) *converges* if $\mathrm{gr}(H(K)) = E_\infty$ via Lemma 20.5. In this case we also say that $(E_r, d_r)_{r \geq 0}$ *abuts to* or *converges to* $H(K)$.

In the literature one finds more refined notions distinguishing between "weakly converging", "abutting" and "converging". Namely, one can require the filtration on $H(K)$ to be either "arbitrary", or "exhaustive and separated", or "exhaustive and complete" in addition to the condition that $\mathrm{gr}(H(K)) = E_\infty$. We try to avoid introducing this notation by simply adding the relevant information in the statements of the results.

Lemma 20.7. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The associated spectral sequence converges if and only if for every $p \in \mathbf{Z}$ we have equality in equations (20.5.2) and (20.5.1).*

Proof. Immediate from the discussions above. □

21. Spectral sequences: filtered complexes

Definition 21.1. Let $\mathcal{A}$ be an abelian category. A *filtered complex $K^\bullet$ of $\mathcal{A}$* is a complex of $\mathrm{Fil}(\mathcal{A})$ (see Definition 16.1).

We will denote the filtration on the objects by F. Thus F^pK^n denotes the pth step in the filtration of the nth term of the complex. Note that each $F^pK^\bullet$ is a complex of $\mathcal{A}$. Hence we could also have defined a filtered complex as a filtered object in the (abelian) category of complexes of $\mathcal{A}$. In particular $\mathrm{gr}K^\bullet$ is a graded object of the category of complexes of $\mathcal{A}$.

To describe the spectral sequence associated to such an object we assume, for the moment, that $\mathcal{A}$ is an abelian category which has countable direct sums and countable direct sums are exact (this is not automatic, see Remark 15.3). Let us denote d the differential of K. Forgetting the grading we can think of $\bigoplus K^n$ as a filtered differential object of $\mathcal{A}$. Hence according to Section 20 we obtain a spectral sequence $(E_r, d_r)_{r\geq 0}$. In this section we work out the terms of this spectral sequence, and we endow the terms of this spectral sequence with additional structure coming from the grading of K.

First we point out that $E_0^p = \text{gr}^p K^\bullet$ is a complex and hence is graded. Thus E_0 is bigraded in a natural way. It is customary to use the bigrading

$$E_0 = \bigoplus\nolimits_{p,q} E_0^{p,q}, \quad E_0^{p,q} = \text{gr}^p K^{p+q}$$

The idea is that $p+q$ should be thought of as the *total degree* of the (co)homology classes. Also, p is called the *filtration degree*, and q is called the *complementary degree*. The differential d_0 is compatible with this bigrading in the following way

$$d_0 = \bigoplus d_0^{p,q}, \quad d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}.$$

Namely, d_0^p is just the differential on the complex $\text{gr}^p K^\bullet$ (which occurs as $\text{gr}^p E_0$ just shifted a bit).

To go further we identify the objects B_r^p and Z_r^p introduced in Section 20 as graded objects and we work out the corresponding decompositions of the differentials. We do this in a completely straightforward manner, but again we warn the reader that our notation is not the same as notation found elsewhere. We define

$$Z_r^{p,q} = \frac{F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and

$$B_r^{p,q} = \frac{F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}.$$

and of course $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$. With these definitions it is completely clear that $Z_r^p = \bigoplus_q Z_r^{p,q}$, $B_r^p = \bigoplus_q B_r^{p,q}$, and $E_r^p = \bigoplus_q E_r^{p,q}$. Moreover,

$$0 \subset \ldots \subset B_r^{p,q} \subset \ldots \subset Z_r^{p,q} \subset \ldots \subset E_0^{p,q}$$

and hence it makes sense to define $Z_\infty^{p,q} = \bigcap_r Z_r^{p,q}$ and $B_\infty^{p,q} = \bigcup_r B_r^{p,q}$ and $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$ provided these exist. Also, the map d_r^p decomposes as the direct sum of the maps

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}, \quad z + F^{p+1}K^{p+q} \mapsto dz + F^{p+r+1}K^{p+q+1}$$

where $z \in F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1})$.

Lemma 21.2. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ in the category of bigraded objects of $\mathcal{A}$ associated to $(K^\bullet, F)$ such that d_r has bidegree $(r, -r+1)$ and such that E_r has bigraded pieces $E_r^{p,q}$ and maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ as given above. Furthermore, we have $E_0^{p,q} = gr^p(K^{p+q})$, $d_0^{p,q} = gr^p(d^{p+q})$, and $E_1^{p,q} = H^{p+q}(gr^p(K^\bullet))$.*

Proof. If $\mathcal{A}$ has countable direct sums and if countable direct sums are exact, then this follows from the discussion above. In general, we proceed as follows; we strongly suggest the reader skip this proof. Consider the bigraded object $A = (F^{p+1}K^{p+1+q})$ of $\mathcal{A}$, i.e., we put $F^{p+1}K^{p+1+q}$ in degree (p, q) (the funny shift in numbering to get numbering correct later on). We endow it with a differential $d : A \to A[0, 1]$ by using d on each component. Then (A, d) is a differential bigraded object. Consider the map

$$\alpha : A \to A[-1, 1]$$

which is given in degree (p, q) by the inclusion $F^{p+1}K^{p+q} \to F^pK^{p+q}$. This is an injective morphism of differential objects $\alpha : (A, d) \to (A, d)[-1, 1]$. Hence, we can apply Remark 19.6 with $S = [0, 1]$ and $T = [1, -1]$. The corresponding spectral sequence $(E_r, d_r)_{r\geq 0}$ of bigraded objects is the spectral sequence we are looking for. Let us unwind the definitions a bit. First of all we have $E_r = (E_r^{p,q})$. Then, since $T^rS = [r, -r+1$ we have $d_r : E_r \to E_r[r, -r+1]$ which means that $d_r^p : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

To see that the description of the graded pieces hold, we argue as above. Namely, first we have

$$E_0 = \mathrm{Coker}(\alpha : A \to A[-1, 1])[0, -1] = \mathrm{Coker}(\alpha[0, -1] : A[0, -1] \to A[-1, 0])$$

and by our choice of numbering above this gives

$$E_0^{p,q} = \mathrm{Coker}(F^{p+1}K^{p+q} \to F^pK^{p+q}) = \mathrm{gr}^pK^{p+q}$$

The first differential is given by $d_0^{p,q} = \mathrm{gr}^p d^{p+q} : E_0^{p,q} \to E_0^{p,q+1}$. Next, the description of the boundaries B_r and the cocycles Z_r in Remark 19.6 translates into a straightforward manner into the formulae for $Z_r^{p,q}$ and $B_r^{p,q}$ given above. ☐

Lemma 21.3. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. Let $(E_r, d_r)_{r\geq 0}$ be the spectral sequence associated to $(K^\bullet, F)$.*

(1) *The map*

$$d_1^{p,q} : E_1^{p,q} = H^{p+q}(gr^p(K^\bullet)) \longrightarrow E_1^{p+1,q} = H^{p+q+1}(gr^{p+1}(K^\bullet))$$

is equal to the boundary map in cohomology associated to the short exact sequence of complexes

$$0 \to gr^{p+1}(K^\bullet) \to F^pK^\bullet/F^{p+2}K^\bullet \to gr^{p+1}(K^\bullet) \to 0.$$

(2) *Assume that $d(F^pK) \subset F^{p+1}K$ for all $p \in \mathbf{Z}$. Then d induces the zero differential on $gr^p(K^\bullet)$ and hence $E_1^{p,q} = gr^p(K^\bullet)^{p+q}$. Furthermore, in this case*

$$d_1^{p,q} : E_1^{p,q} = gr^p(K^\bullet)^{p+q} \longrightarrow E_1^{p,q} = gr^{p+1}(K^\bullet)^{p+q+1}$$

is the morphism induced by d.

Proof. Omitted. But compare Lemma 20.3. ☐

Lemma 21.4. *Let $\mathcal{A}$ be an abelian category. Let $\alpha : (K^\bullet, F) \to (L^\bullet, F)$ be a morphism of filtered complexes of $\mathcal{A}$. Let $(E_r(K), d_r)_{r\geq 0}$, resp. $(E_r(L), d_r)_{r\geq 0}$ be the spectral sequence associated to $(K^\bullet, F)$, resp. $(L^\bullet, F)$. The morphism α induces a canonical morphism of spectral sequences $\{\alpha_r : E_r(K) \to E_r(L)\}_{r\geq 0}$ compatible with the bigradings.*

Proof. Obvious from the explicit representation of the terms of the spectral sequences. ☐

Definition 21.5. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The *induced filtration* on $H^n(K^\bullet)$ is the filtration defined by $F^pH^n(K^\bullet) = \mathrm{Im}(H^n(F^pK^\bullet) \to H^n(K^\bullet))$.

Lemma 21.6. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. If $Z_\infty^{p,q}$ and $B_\infty^{p,q}$ exist (see above), then the associated graded $gr(H^n(K^\bullet))$ of the cohomology of $K^\bullet$ is a graded subquotient of the graded object $\bigoplus_{p+q=n} E_\infty^{p,q}$.*

Proof. Let $q = n - p$. As in the proof of Lemma 20.5 we see that

$$E_\infty^{p,q} = \frac{\bigcap_r (F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) + F^{p+1}K^n)}{\bigcup_r (F^pK^n \cap d(F^{p-r+1}K^{n-1}) + F^{p+1}K^n)}.$$

On the other hand, we have

$$\mathrm{gr}^p H^n(K) = \frac{\mathrm{Ker}(d) \cap F^pK^n + F^{p+1}K^n}{\mathrm{Im}(d) \cap F^pK^n + F^{p+1}K^n} \tag{21.6.1}$$

The result follows since

$$\mathrm{Ker}(d) \cap F^pK^n + F^{p+1}K^n \subset \bigcup\nolimits_r \left(F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) + F^{p+1}K^n\right) \tag{21.6.2}$$

and

$$\bigcap\nolimits_r \left(F^pK^n \cap d(F^{p-r+1}K^{n-1}) + F^{p+1}K^n\right) \subset \mathrm{Im}(d) \cap F^pK^n + F^{p+1}K^n. \tag{21.6.3}$$

□

Definition 21.7. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. We say the spectral sequence associated to $(K^\bullet, F)$ *converges* if $\mathrm{gr}H^n(K^\bullet) = \bigoplus_{p+q=n} E_\infty^{p,q}$ for every $n \in \mathbf{Z}$.

This is often symbolized by the notation $E_r^{p,q} \Rightarrow H^{p+q}(K^\bullet)$. Please read the remarks following Definition 20.6.

Lemma 21.8. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The associated spectral sequence converges if and only if for every $p, q \in \mathbf{Z}$ we have equality in equations (21.6.3) and (21.6.2).*

Proof. Immediate from the discussions above. □

Lemma 21.9. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume that the filtration on each K^n is finite (see Definition 16.1). Then*

1. *the filtration on each $H^n(K^\bullet)$ is finite, and*
2. *the spectral sequence associated to $(K^\bullet, F)$ converges.*

Proof. Part (1) is clear from Equation (21.6.1). We will use Lemma 21.8 to prove part (2). Fix $p, n \in \mathbf{Z}$. Look at the left hand side of Equation (21.6.3). The expression is equal to the right hand side since $F^mK^{n-1} = 0$ for $m \ll 0$. Similarly, use $F^mK^{n+1} = K^{n+1}$ for $m \gg 0$ to prove equality in Equation (21.6.2). □

22. Spectral sequences: double complexes

Definition 22.1. Let $\mathcal{A}$ be an additive category. A *double complex* in $\mathcal{A}$ is given by a system $(\{A^{p,q}, d_1^{p,q}, d_2^{p,q}\}_{p,q \in \mathbf{Z}})$, where each $A^{p,q}$ is an object of $\mathcal{A}$ and $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ and $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ are morphisms of $\mathcal{A}$ such that the following rules hold:

1. $d_1^{p+1,q} \circ d_1^{p,q} = 0$
2. $d_2^{p,q+1} \circ d_2^{p,q} = 0$
3. $d_1^{p,q+1} \circ d_2^{p,q} = d_2^{p+1,q} \circ d_1^{p,q}$

for all $p, q \in \mathbf{Z}$.

This is just the cochain version of the definition. It says that each $A^{p,\bullet}$ is a cochain complex and that each $d_1^{p,\bullet}$ is a morphism of complexes $A^{p,\bullet} \to A^{p+1,\bullet}$ such that $d_1^{p+1,\bullet} \circ d_1^{p,\bullet} = 0$ as morphisms of complexes. In other words a double complex can be seen as a complex of complexes. So in the diagram

$$\begin{tikzcd}
\ldots & \ldots & \ldots & \ldots \\
\ldots \arrow[r] & A^{p,q+1} \arrow[u] \arrow[r, "d_1^{p,q+1}"] & A^{p+1,q+1} \arrow[u] \arrow[r] & \ldots \\
\ldots \arrow[r] & A^{p,q} \arrow[u, "d_2^{p,q}"] \arrow[r, "d_1^{p,q}"] & A^{p+1,q} \arrow[u, "d_2^{p+1,q}"'] \arrow[r] & \ldots \\
\ldots & \ldots \arrow[u] & \ldots \arrow[u] & \ldots
\end{tikzcd}$$

any square commutes. Warning: In the literature one encounters a different definition where a "bicomplex" or a "double complex" has the property that the squares in the diagram anti-commute.

Example 22.2. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be abelian categories. Suppose that

$$\otimes : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}, \quad (X, Y) \longmapsto X \otimes Y$$

is a functor which is bilinear on morphisms, see Categories, Definition 2.20 for the definition of $\mathcal{A} \times \mathcal{B}$. Given a complexes $X^\bullet$ of $\mathcal{A}$ and $Y^\bullet$ of $\mathcal{B}$ we obtain a double complex

$$K^{\bullet,\bullet} = X^\bullet \otimes Y^\bullet$$

in $\mathcal{C}$. Here the first differential $K^{p,q} \to K^{p+1,q}$ is the morphism $X^p \otimes Y^q \to X^{p+1} \otimes Y^q$ induced by the morphism $X^p \to X^{p+1}$ and the identity on Y^q. Similarly for the second differential.

Let $A^{\bullet,\bullet}$ be a double complex. It is customary to denote $H_I^p(A^{\bullet,\bullet})$ the complex with terms $\mathrm{Ker}(d_1^{p,q})/\mathrm{Im}(d_1^{p-1,q})$ (varying q) and differential induced by d_2. Then $H_{II}^q(H_I^p(A^{\bullet,\bullet}))$ denotes its cohomology in degree q. It is also customary to denote $H_{II}^q(A^{\bullet,\bullet})$ the complex with terms $\mathrm{Ker}(d_2^{p,q})/\mathrm{Im}(d_2^{p,q-1})$ (varying p) and differential induced by d_1. Then $H_I^p(H_{II}^q(A^{\bullet,\bullet}))$ denotes its cohomology in degree q. It will turn out that these cohomology groups show up as the terms in the spectral sequence for a filtration on the associated to total complex.

Definition 22.3. Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet}$ be a double complex. The *associated simple complex* $sA^\bullet$, also sometimes called the *associated total complex* is given by

$$sA^n = \bigoplus\nolimits_{n=p+q} A^{p,q}$$

(if it exists) with differential

$$d_{sA}^n = \sum\nolimits_{n=p+q} (d_1^{p,q} + (-1)^p d_2^{p,q})$$

Alternatively, we sometimes write $\mathrm{Tot}(A^{\bullet,\bullet})$ to denote this complex.

If countable direct sums exist in $\mathcal{A}$ or if for each n at most finitely many $A^{p,n-p}$ are nonzero, then $sA^\bullet$ exists. Note that the definition is *not* symmetric in the indices (p, q).

There are two natural filtrations on the simple complex $sA^\bullet$ associated to the double complex $A^{\bullet,\bullet}$. Namely, we define

$$F_I^p(sA^n) = \bigoplus\nolimits_{i+j=n,\ i\geq p} A^{i,j} \quad \text{and} \quad F_{II}^p(sA^n) = \bigoplus\nolimits_{i+j=n,\ j\geq p} A^{i,j}.$$

It is immediately verified that $(sA^\bullet, F_I)$ and $(sA^\bullet, F_{II})$ are filtered complexes. By Section 21 we obtain two spectral sequences. It is customary to denote $({}'E_r, {}'d_r)_{r\geq 0}$ the spectral sequence associated to the filtration F_I and to denote $({}''E_r, {}''d_r)_{r\geq 0}$ the spectral sequence associated to the filtration F_{II}. Here is a description of these spectral sequences.

Lemma 22.4. *Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. The spectral sequences associated to $K^{\bullet,\bullet}$ have the following terms:*

1. ${}'E_0^{p,q} = K^{p,q}$ *with* ${}'d_0^{p,q} = (-1)^p d_2^{p,q} : K^{p,q} \to K^{p,q+1}$,
2. ${}''E_0^{p,q} = K^{q,p}$ *with* ${}''d_0^{p,q} = d_1^{q,p} : K^{q,p} \to K^{q+1,p}$,
3. ${}'E_1^{p,q} = H^q(K^{p,\bullet})$ *with* ${}'d_1^{p,q} = H^q(d_1^{p,\bullet})$,
4. ${}''E_1^{p,q} = H^q(K^{\bullet,p})$ *with* ${}''d_1^{p,q} = (-1)^q H^q(d_2^{\bullet,p})$,
5. ${}'E_2^{p,q} = H_I^p(H_{II}^q(K^{\bullet,\bullet}))$,
6. ${}''E_2^{p,q} = H_{II}^p(H_I^q(K^{\bullet,\bullet}))$.

Proof. Omitted. □

These spectral sequences define two filtrations on $H^n(sK^\bullet)$. We will denote these F_I and F_{II}.

Definition 22.5. Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. We say the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ *converges* if Definition 21.7 applies. In other words, for all n

$$\mathrm{gr}_{F_I}(H^n(sK^\bullet)) = \oplus_{p+q=n} {}'E_\infty^{p,q}$$

via the canonical comparison of Lemma 21.6. Similarly we say the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$ *converges* if Definition 21.7 applies. In other words for all n

$$\mathrm{gr}_{F_{II}}(H^n(sK^\bullet)) = \oplus_{p+q=n} {}''E_\infty^{p,q}$$

via the canonical comparison of Lemma 21.6.

Same caveats as those following Definition 20.6.

Lemma 22.6 (First quadrant spectral sequence)**.** *Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. Assume that for every $n \in \mathbf{Z}$ there are only finitely many nonzero $K^{p,q}$ with $p+q=n$. Then*

1. *the filtrations F_I, F_{II} on each $H^n(K^\bullet)$ are finite,*
2. *the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ converges, and*
3. *the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$ converges.*

Proof. Follows immediately from Lemma 21.9. □

Here is our first application of spectral sequences.

Lemma 22.7. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex. Let $A^{\bullet,\bullet}$ be a double complex. Let $\alpha^p : K^p \to A^{p,0}$ be morphisms. Assume that*

1. *For every $n \in \mathbf{Z}$ there are only finitely many nonzero $A^{p,q}$ with $p+q=n$.*
2. *We have $A^{p,q} = 0$ if $q < 0$.*
3. *The morphisms α^p give rise to a morphism of complexes $\alpha : K^\bullet \to A^{\bullet,0}$.*

(4) *The complex $A^{p,\bullet}$ is exact in all degrees $q \neq 0$ and the morphism $K^p \to A^{p,0}$ induces an isomorphism $K^p \to Ker(d_2^{p,0})$.*

Then α induces a quasi-isomorphism

$$K^\bullet \longrightarrow sA^\bullet$$

of complexes. Moreover, there is a variant of this lemma involving the second variable q instead of p.

Proof. The map is simply the map given by the morphisms $K^n \to A^{n,0} \to sA^n$, which are easily seen to define a morphism of complexes. Consider the spectral sequence $('E_r, 'd_r)_{r \geq 0}$ associated to the double complex $A^{\bullet,\bullet}$. By Lemma 22.6 this spectral sequence converges and the induced filtration on $H^n(sA^\bullet)$ is finite for each n. By Lemma 22.4 and assumption (4) we have $'E_1^{p,q} = 0$ unless $q = 0$ and $'E_1^{p,0} = K^p$ with differential $'d_1^{p,0}$ identified with d_K^p. Hence $'E_2^{p,0} = H^p(K^\bullet)$ and zero otherwise. This clearly implies $d_2^{p,q} = d_3^{p,q} = \ldots = 0$ for degree reasons. Hence we conclude that $H^n(sA^\bullet) = H^n(K^\bullet)$. We omit the verification that this identification is given by the morphism of complexes $K^\bullet \to sA^\bullet$ introduced above. □

Remark 22.8. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a weak Serre subcategory (see Definition 9.1). Suppose that $K^{\bullet,\bullet}$ is a double complex to which Lemma 22.6 applies such that for some $r \geq 0$ all the objects $'E_r^{p,q}$ belong to $\mathcal{C}$. We claim all the cohomology groups $H^n(sK^\bullet)$ belong to $\mathcal{C}$. Namely, the assumptions imply that the kernels and images of $'d_r^{p,q}$ are in $\mathcal{C}$. Whereupon we see that each $'E_{r+1}^{p,q}$ is in $\mathcal{C}$. By induction we see that each $'E_\infty^{p,q}$ is in $\mathcal{C}$. Hence each $H^n(sK^\bullet)$ has a finite filtration whose subquotients are in $\mathcal{C}$. Using that $\mathcal{C}$ is closed under extensions we conclude that $H^n(sK^\bullet)$ is in $\mathcal{C}$ as claimed.

The same result holds for the second spectral sequence associated to $K^{\bullet,\bullet}$. Similarly, if $(K^\bullet, F)$ is a filtered complex to which Lemma 21.9 applies and for some $r \geq 0$ all the objects $E_r^{p,q}$ belong to $\mathcal{C}$, then each $H^n(K^\bullet)$ is an object of $\mathcal{C}$.

Remark 22.9. Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet,\bullet}$ be a triple complex. The associated total complex is the complex with terms

$$\mathrm{Tot}^n(A^{\bullet,\bullet,\bullet}) = \bigoplus\nolimits_{p+q+r=n} A^{p,q,r}$$

and differential

$$d^n_{\mathrm{Tot}(A^{\bullet,\bullet,\bullet})} = \sum\nolimits_{p+q+r=n} d_1^{p,q,r} + (-1)^p d_2^{p,q,r} + (-1)^{p+q} d_3^{p,q,r}$$

With this definition a simple calculation shows that the associated total complex is equal to

$$\mathrm{Tot}(A^{\bullet,\bullet,\bullet}) = \mathrm{Tot}(\mathrm{Tot}_{12}(A^{\bullet,\bullet,\bullet})) = \mathrm{Tot}(\mathrm{Tot}_{23}(A^{\bullet,\bullet,\bullet}))$$

In other words, we can either first combine the first two of the variables and then combine sum of those with the last, or we can first combine the last two variables and then combine the first with the sum of the last two.

Lemma 22.10. *Let $M^\bullet$ be a complex of abelian groups. Let*

$$\ldots \to A_2^\bullet \to A_1^\bullet \to A_0^\bullet \to M^\bullet \to 0$$

be an exact complex of complexes of abelian groups such that for all $p \in \mathbf{Z}$ the complexes

$$\ldots \to Ker(d^p_{A_2^\bullet}) \to Ker(d^p_{A_1^\bullet}) \to Ker(d^p_{A_0^\bullet}) \to Ker(d^p_{M^\bullet}) \to 0$$

are exact as well. Set $A^{p,q} = A^q_{-p}$ to obtain a double complex. Then $Tot(A^{\bullet,\bullet}) \to M^\bullet$ induced by $A_0^\bullet \to M^\bullet$ is a quasi-isomorphism.

Proof. Write $T^\bullet = \mathrm{Tot}(A^{\bullet,\bullet})$. Let $x \in \mathrm{Ker}(\mathrm{d}^0_{T^\bullet})$ represent a cohomology class ξ. Write $x = \sum_{i=n,\ldots,0} x_i$ with $x_i \in A^i_i$. Assume $n > 0$. Then x_n is in the kernel of $d^n_{A_n^\bullet}$ and maps to zero in the cohomology of $A^\bullet_{n-1}$ (because it maps to an element which is the boundary of x_{n-1} up to sign). The condition on exactness of kernels of differentials implies that the cohomology class of x_n is in the image of $H^n(A^\bullet_{n+1}) \to H^n(A^\bullet_n)$ (details omitted). Thus we can modify x by a boundary and reach the situation where x_n is a boundary. Modifying x once more we see that we may assume $x_n = 0$. By induction we see that every cohomology class ξ is represented by a cocycle $x = x_0$. Finally, the condition on exactness of kernels tells us two such cocycles x_0 and x'_0 are cohomologous if and only if their image in $H^0(M^\bullet)$ are the same. □

Lemma 22.11. *Let $M^\bullet$ be a complex of abelian groups. Let*

$$0 \to M^\bullet \to A_0^\bullet \to A_1^\bullet \to A_2^\bullet \to \ldots$$

be an exact complex of complexes of abelian groups such that for all $p \in \mathbf{Z}$ the complexes

$$0 \to Coker(d^p_{M^\bullet}) \to Coker(d^p_{A_0^\bullet}) \to Coker(d^p_{A_1^\bullet}) \to Coker(d^p_{A_2^\bullet}) \to \ldots$$

are exact as well. Set $A^{p,q} = A^q_p$ to obtain a double complex. Let $Tot_\pi(A^{\bullet,\bullet})$ be the product total complex associated to the double complex (see proof). Then the map $M^\bullet \to Tot_\pi(A^{\bullet,\bullet})$ induced by $M^\bullet \to A_0^\bullet$ is a quasi-isomorphism.

Proof. Abbreviating $T^\bullet = \mathrm{Tot}_\pi(A^{\bullet,\bullet})$ we define

$$T^n = \prod\nolimits_{p+q=n} A^{p,q} = \prod\nolimits_{p+q=n} A^q_p$$

As differential we use

$$\mathrm{d}((x_{p,q})) = (f_p(x_{p-1,q}) + (-1)^p \mathrm{d}_{A_p^\bullet}(x_{p,q-1}))$$

Let $x \in \mathrm{Ker}(\mathrm{d}^0_{T^\bullet})$ represent a cohomology class $\xi \in H^0(T^\bullet)$. Write $x = (x_i)$ with $x_i \in A^{-i}_i$. Note that x_0 maps to zero in $\mathrm{Coker}(A^{-1}_1 \to A^0_1)$. Hence we see that $x_0 = m_0 + \mathrm{d}(y)$ for some $m_0 \in M^0$. Then $\mathrm{d}(m_0) = 0$ because $\mathrm{d}(x_0) = 0$ as x is a cocycle. Thus, replacing ξ by something in the image of $H^0(M^\bullet) \to H^0(T^\bullet)$ we may assume that x_0 is in the image of $\mathrm{d} : A^{-1}_0 \to A^0_0$.

Assume $x_0 \in \mathrm{Im}(A^{-1}_0 \to A^0_0)$. We claim that in this case $\xi = 0$. To prove this we find, by induction on n elements $y_1, \ldots, y_n$ with $y_i \in A^{-i-1}_i$ such that $x_0 = \mathrm{d}(y_0)$ and $x_j = f_{j-1}(y_{j-1}) + (-1)^j \mathrm{d}(y_j)$. This is clear for $n = 0$. Proof of induction step is omitted. Taking $y = (y_i)$ we find that $\mathrm{d}(y) = \xi$.

This shows that $H^0(M^\bullet) \to H^0(T^\bullet)$ is surjective. We omit the proof of injectivity. □

23. Injectives

Definition 23.1. Let $\mathcal{A}$ be an abelian category. An object $J \in \mathrm{Ob}(\mathcal{A})$ is called *injective* if for every injection $A \hookrightarrow B$ and every morphism $A \to J$ there exists a morphism $B \to J$ making the following diagram commute

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[dl, dashed] \\ J & \end{tikzcd}$$

Here is the obligatory characterization of injective objects.

Lemma 23.2. *Let $\mathcal{A}$ be an abelian category. Let I be an object of $\mathcal{A}$. The following are equivalent:*

1. *The object I is injective.*
2. *The functor $B \mapsto \mathrm{Hom}_{\mathcal{A}}(B, I)$ is exact.*
3. *Any short exact sequence*
$$0 \to I \to A \to B \to 0$$
in $\mathcal{A}$ is split.
4. *We have $Ext_{\mathcal{A}}(B, I) = 0$ for all $B \in \mathrm{Ob}(\mathcal{A})$.*

Proof. Omitted. □

Lemma 23.3. *Let $\mathcal{A}$ be an abelian category. Suppose I_ω, $\omega \in \Omega$ is a set of injective objects of $\mathcal{A}$. If $\prod_{\omega \in \Omega} I_\omega$ exists then it is injective.*

Proof. Omitted. □

Definition 23.4. Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ has *enough injectives* if every object A has an injective morphism $A \to J$ into an injective object J.

Definition 23.5. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *functorial injective embeddings* if there exists a functor

$$J : \mathcal{A} \longrightarrow \mathrm{Arrows}(\mathcal{A})$$

such that

1. $s \circ J = \mathrm{id}_{\mathcal{A}}$,
2. for any object $A \in \mathrm{Ob}(\mathcal{A})$ the morphism $J(A)$ is injective, and
3. for any object $A \in \mathrm{Ob}(\mathcal{A})$ the object $t(J(A))$ is an injective object of $\mathcal{A}$.

We will denote such a functor by $A \mapsto (A \to J(A))$.

24. Projectives

Definition 24.1. Let $\mathcal{A}$ be an abelian category. An object $P \in \mathrm{Ob}(\mathcal{A})$ is called *projective* if for every surjection $A \to B$ and every morphism $P \to B$ there exists a morphism $P \to A$ making the following diagram commute

$$\begin{tikzcd} A \arrow[r] & B \\ P \arrow[u, dashed] \arrow[ur] & \end{tikzcd}$$

Here is the obligatory characterization of projective objects.

Lemma 24.2. *Let $\mathcal{A}$ be an abelian category. Let P be an object of $\mathcal{A}$. The following are equivalent:*

1. *The object P is projective.*
2. *The functor $B \mapsto \mathrm{Hom}_{\mathcal{A}}(P, B)$ is exact.*
3. *Any short exact sequence*
$$0 \to A \to B \to P \to 0$$
in $\mathcal{A}$ is split.
4. *We have $Ext_{\mathcal{A}}(P, A) = 0$ for all $A \in \mathrm{Ob}(\mathcal{A})$.*

Proof. Omitted. $\square$

Lemma 24.3. *Let $\mathcal{A}$ be an abelian category. Suppose P_ω, $\omega \in \Omega$ is a set of projective objects of $\mathcal{A}$. If $\coprod_{\omega \in \Omega} P_\omega$ exists then it is projective.*

Proof. Omitted. $\square$

Definition 24.4. Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ has *enough projectives* if every object A has an surjective morphism $P \to A$ from an projective object P onto it.

Definition 24.5. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *functorial projective surjections* if there exists a functor
$$P : \mathcal{A} \longrightarrow \mathrm{Arrows}(\mathcal{A})$$
such that

1. $t \circ J = \mathrm{id}_{\mathcal{A}}$,
2. for any object $A \in \mathrm{Ob}(\mathcal{A})$ the morphism $P(A)$ is surjective, and
3. for any object $A \in \mathrm{Ob}(\mathcal{A})$ the object $s(P(A))$ is an projective object of $\mathcal{A}$.

We will denote such a functor by $A \mapsto (P(A) \to A)$.

25. Injectives and adjoint functors

Here are some lemmas on adjoint functors and their relationship with injectives. See also Lemma 7.3.

Lemma 25.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

1. *u is right adjoint to v, and*
2. *v transforms injective maps into injective maps.*

Then u transforms injectives into injectives.

Proof. Let I be an injective object of $\mathcal{A}$. Let $\varphi : N \to M$ be an injective map in $\mathcal{B}$ and let $\alpha : N \to uI$ be a morphism. By adjointness we get a morphism $\alpha : vN \to I$ and by assumption $v\varphi : vN \to vM$ is injective. Hence as I is an injective object we get a morphism $\beta : vM \to I$ extending α. By adjointness again this corresponds to a morphism $\beta : M \to uI$ as desired. $\square$

Remark 25.2. Let $\mathcal{A}$, $\mathcal{B}$, $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be as in Lemma 25.1. In the presence of assumption (1) assumption (2) is equivalent to requiring that v is exact. Moreover, condition (2) is necessary. Here is an example. Let $A \to B$ be a ring map. Let $u : \mathrm{Mod}_B \to \mathrm{Mod}_A$ be $u(N) = N_A$ and let $v : \mathrm{Mod}_A \to \mathrm{Mod}_B$ be $v(M) = M \otimes_A B$. Then u is right adjoint to v, and u is exact and v is right exact, but v does not transform injective maps into injective maps in general (i.e., v is not left exact). Moreover, it is **not** the case that u transforms injective B-modules into injective A-modules. For example, if $A = \mathbf{Z}$ and $B = \mathbf{Z}/p\mathbf{Z}$, then the injective B-module $\mathbf{Z}/p\mathbf{Z}$ is not an injective $\mathbf{Z}$-module. In fact, the lemma applies to this example if and only if the ring map $A \to B$ is flat.

Lemma 25.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

(1) *u is right adjoint to v,*
(2) *v transforms injective maps into injective maps,*
(3) *$\mathcal{A}$ has enough injectives, and*
(4) *$vB = 0$ implies $B = 0$ for any $B \in \mathrm{Ob}(\mathcal{B})$.*

Then $\mathcal{B}$ has enough injectives.

Proof. Pick $B \in \mathrm{Ob}(\mathcal{B})$. Pick an injection $vB \to I$ for I an injective object of $\mathcal{A}$. According to Lemma 25.1 and the assumptions the corresponding map $B \to uI$ is the injection of B into an injective object. $\square$

Remark 25.4. Let $\mathcal{A}$, $\mathcal{B}$, $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be as In Lemma 25.3. In the presence of conditions (1) and (2) condition (4) is equivalent to v being faithful. Moreover, condition (4) is needed. An example is to consider the case where the functors u and v are both the zero functor.

Lemma 25.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

(1) *u is right adjoint to v,*
(2) *v transforms injective maps into injective maps,*
(3) *$\mathcal{A}$ has enough injectives,*
(4) *$vB = 0$ implies $B = 0$ for any $B \in \mathrm{Ob}(\mathcal{B})$, and*
(5) *$\mathcal{A}$ has functorial injective hulls.*

Then $\mathcal{B}$ has functorial injective hulls.

Proof. Let $A \mapsto (A \to J(A))$ be a functorial injective hull on $\mathcal{A}$. Then $B \mapsto (B \to uJ(vB))$ is a functorial injective hull on $\mathcal{B}$. Compare with the proof of Lemma 25.3. $\square$

Lemma 25.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ be a functor. If there exists a subset $\mathcal{P} \subset \mathrm{Ob}(\mathcal{B})$ such that*

(1) *every object of $\mathcal{B}$ is a quotient of an element of $\mathcal{P}$, and*
(2) *for every $P \in \mathcal{P}$ there exists an object Q of $\mathcal{A}$ such that $\mathrm{Hom}_\mathcal{A}(Q, A) = \mathrm{Hom}_\mathcal{B}(P, u(A))$ functorially in A,*

then there exists a left adjoint v of u.

Proof. By the Yoneda lemma (Categories, Lemma 3.5) the object Q of $\mathcal{A}$ corresponding to P is defined up to unique isomorphism by the formula $\mathrm{Hom}_\mathcal{A}(Q, A) = \mathrm{Hom}_\mathcal{B}(P, u(A))$. Let us write $Q = v(P)$. Denote $i_P : P \to u(v(P))$ the map corresponding to $\mathrm{id}_{v(P)}$ in $\mathrm{Hom}_\mathcal{A}(v(P), v(P))$. Functoriality in (2) implies that the bijection is given by

$$\mathrm{Hom}_\mathcal{A}(v(P), A) \to \mathrm{Hom}_\mathcal{B}(P, u(A)), \quad \varphi \mapsto u(\varphi) \circ i_P$$

For any pair of elements $P_1, P_2 \in \mathcal{P}$ there is a canonical map

$$\mathrm{Hom}_{\mathcal{B}}(P_2, P_1) \to \mathrm{Hom}_{\mathcal{A}}(v(P_2), v(P_1)), \quad \varphi \mapsto v(\varphi)$$

which is characterized by the rule $u(v(\varphi)) \circ i_{P_2} = i_{P_1} \circ \varphi$ in $\mathrm{Hom}_{\mathcal{B}}(P_2, u(v(P_1)))$. Note that $\varphi \mapsto v(\varphi)$ is compatible with composition; this can be seen directly from the characterization. Hence $P \mapsto v(P)$ is a functor from the full subcategory of $\mathcal{B}$ whose objects are the elements of $\mathcal{P}$.

Given an arbitrary object B of $\mathcal{B}$ choose an exact sequence

$$P_2 \to P_1 \to B \to 0$$

which is possible by assumption (1). Define $v(B)$ to be the object of $\mathcal{A}$ fitting into the exact sequence

$$v(P_2) \to v(P_1) \to v(B) \to 0$$

Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(v(B), A) &= \mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(v(P_1), A) \to \mathrm{Hom}_{\mathcal{A}}(v(P_2), A)) \\ &= \mathrm{Ker}(\mathrm{Hom}_{\mathcal{B}}(P_1, u(A)) \to \mathrm{Hom}_{\mathcal{B}}(P_2, u(A))) \\ &= \mathrm{Hom}_{\mathcal{B}}(B, u(A)) \end{aligned}$$

Hence we see that we may take $\mathcal{P} = \mathrm{Ob}(\mathcal{B})$, i.e., we see that v is everywhere defined. □

26. Essentially constant systems

In this section we discuss essentially constant systems with values in additive categories.

Lemma 26.1. *Let $\mathcal{I}$ be a category, let $\mathcal{A}$ be a pre-additive Karoubian category, and let $M : \mathcal{I} \to \mathcal{A}$ be a diagram.*

1. *Assume $\mathcal{I}$ is filtered. The following are equivalent*
   - (a) *M is essentially constant,*
   - (b) *$X = \mathrm{colim}\, M$ exists and there exists a cofinal filtered subcategory $\mathcal{I}' \subset \mathcal{I}$ and for $i' \in \mathrm{Ob}(\mathcal{I}')$ a direct sum decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$ such that $X_{i'}$ maps isomorphically to X and $Z_{i'}$ to zero in $M_{i''}$ for some $i' \to i''$ in $\mathcal{I}'$.*
2. *Assume $\mathcal{I}$ is cofiltered. The following are equivalent*
   - (a) *M is essentially constant,*
   - (b) *$X = \lim M$ exists and there exists an initial cofiltered subcategory $\mathcal{I}' \subset \mathcal{I}$ and for $i' \in \mathrm{Ob}(\mathcal{I}')$ a direct sum decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$ such that X maps isomorphically to $X_{i'}$ and $M_{i''} \to Z_{i'}$ is zero for some $i'' \to i'$ in $\mathcal{I}'$.*

Proof. Assume (1)(a), i.e., $\mathcal{I}$ is filtered and M is essentially constant. Let $X = \mathrm{colim}\, M_i$. Choose i and $X \to M_i$ as in Categories, Definition 22.1. Let $\mathcal{I}'$ be the full subcategory consisting of objects which are the target of a morphism with source i. Suppose $i' \in \mathrm{Ob}(\mathcal{I}')$ and choose a morphism $i \to i'$. Then $X \to M_i \to M_{i'}$ composed with $M_{i'} \to X$ is the identity on X. As $\mathcal{A}$ is Karoubian, we find a direct summand decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$, where $Z_{i'} = \mathrm{Ker}(M_{i'} \to X)$ and $X_{i'}$ maps isomorphically to X. Pick $i \to k$ and $i' \to k$ such that $M_{i'} \to X \to M_i \to M_k$ equals $M_{i'} \to M_k$ as in Categories, Definition 22.1. Then we see that $M_{i'} \to M_k$ annihilates $Z_{i'}$. Thus (1)(b) holds.

Assume (1)(b), i.e., $\mathcal{I}$ is filtered and we have $\mathcal{I}' \subset \mathcal{I}$ and for $i' \in \mathrm{Ob}(\mathcal{I}')$ a direct sum decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$ as stated in the lemma. To see that M is essentially constant we can replace $\mathcal{I}$ by $\mathcal{I}'$, see Categories, Lemmas 22.8 and 17.2. Pick any $i \in \mathrm{Ob}(\mathcal{I})$ and denote $X \to M_i$ the inverse of the isomorphism $X_i \to X$ followed by the inclusion map $X_i \to M_i$. If j is a second object, then choose $j \to k$ such that $Z_j \to M_k$ is zero. Since $\mathcal{I}$ is filtered we may also assume there is a morphism $i \to k$ (after possibly increasing k). Then $M_j \to X \to M_i \to M_k$ and $M_j \to M_k$ both annihilate Z_j. Thus after postcomposing by a morphism $M_k \to M_l$ which annihilates the summand Z_k, we find that $M_j \to X \to M_i \to M_l$ and $M_j \to M_l$ are equal, i.e., M is essentially constant.

The proof of (2) is dual. $\square$

Lemma 26.2. *Let $\mathcal{I}$ be a category. Let $\mathcal{A}$ be an additive, Karoubian category. Let $F : \mathcal{I} \to \mathcal{A}$ and $G : \mathcal{I} \to \mathcal{A}$ be functors. The following are equivalent*

1. $\mathrm{colim}_\mathcal{I} F \oplus G$ *exists, and*
2. $\mathrm{colim}_\mathcal{I} F$ *and* $\mathrm{colim}_\mathcal{I} G$ *exist.*

In this case $\mathrm{colim}_\mathcal{I} F \oplus G = \mathrm{colim}_\mathcal{I} F \oplus \mathrm{colim}_\mathcal{I} G$.

Proof. Assume (1) holds. Set $W = \mathrm{colim}_\mathcal{I} F \oplus G$. Note that the projection onto F defines natural tranformation $F \oplus G \to F \oplus G$ which is idempotent. Hence we obtain an idempotent endomorphism $W \to W$ by Categories, Lemma 14.7. Since $\mathcal{A}$ is Karoubian we get a corresponding direct sum decomposition $W = X \oplus Y$, see Lemma 4.2. A straightforward argument (omitted) shows that $X = \mathrm{colim}_\mathcal{I} F$ and $Y = \mathrm{colim}_\mathcal{I} G$. Thus (2) holds. We omit the proof that (2) implies (1). $\square$

Lemma 26.3. *Let $\mathcal{I}$ be a filtered category. Let $\mathcal{A}$ be an additive, Karoubian category. Let $F : \mathcal{I} \to \mathcal{A}$ and $G : \mathcal{I} \to \mathcal{A}$ be functors. The following are equivalent*

1. *$F \oplus G : \mathcal{I} \to \mathcal{A}$ is essentially constant, and*
2. *F and G are essentially constant.*

Proof. Assume (1) holds. In particular $W = \mathrm{colim}_\mathcal{I} F \oplus G$ exists and hence by Lemma 26.2 we have $W = X \oplus Y$ with $X = \mathrm{colim}_\mathcal{I} F$ and $Y = \mathrm{colim}_\mathcal{I} G$. A straightforward argument (omitted) using for example the characterization of Categories, Lemma 22.6 shows that F is essentially constant with value X and G is essentially constant with value Y. Thus (2) holds. The proof that (2) implies (1) is omitted. $\square$

27. Inverse systems

Let $\mathcal{C}$ be a category. In Categories, Section 21 we defined the notion of an inverse system over a partially ordered set (with values in the category $\mathcal{C}$). If the partially ordered set is $\mathbf{N} = \{1, 2, 3, \ldots\}$ with the usual ordering such an inverse system over $\mathbf{N}$ is often simply called an *inverse system*. It consists quite simply of a pair $(M_i, f_{ii'})$ where each M_i, $i \in \mathbf{N}$ is an object of $\mathcal{C}$, and for each $i > i'$, $i, i' \in \mathbf{N}$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that moreover $f_{i'i''} \circ f_{ii'} = f_{ii''}$ whenever this makes sense. It is clear that in fact it suffices to give the morphisms $M_2 \to M_1$, $M_3 \to M_2$, and so on. Hence an inverse system is frequently pictured as follows

$$M_1 \xleftarrow{\varphi_2} M_2 \xleftarrow{\varphi_3} M_3 \leftarrow \ldots$$

Moreover, we often omit the transition maps φ_i from the notation and we simply say "let (M_i) be an inverse system".

The collection of all inverse systems with values in $\mathcal{C}$ forms a category with the obvious notion of morphism.

Lemma 27.1. *Let $\mathcal{C}$ be a category.*

(1) *If $\mathcal{C}$ is an additive category, then the category of inverse systems with values in $\mathcal{C}$ is an additive cateogry.*
(2) *If $\mathcal{C}$ is an abelian category, then the category of inverse systems with values in $\mathcal{C}$ is an abelian cateogry. A sequence $(K_i) \to (L_i) \to (M_i)$ of inverse systems is exact if and only if each $K_i \to L_i \to N_i$ is exact.*

Proof. Omitted. □

The limit (see Categories, Section 21) of such an inverse system is denoted $\lim M_i$, or $\lim_i M_i$. If $\mathcal{C}$ is the category of abelian groups (or sets), then the limit always exists and in fact can be described as follows

$$\lim_i M_i = \{(x_i) \in \prod M_i \mid \varphi_i(x_i) = x_{i-1},\ i = 2, 3, \ldots\}$$

see Categories, Section 15. However, given a short exact sequence

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

of inverse systems of abelian groups it is not always the case that the associated system of limits is exact. In order to discuss this further we introduce the following notion.

Definition 27.2. Let $\mathcal{C}$ be an abelian category. We say the inverse system (A_i) satisfies the *Mittag-Leffler condition*, or for short is *ML*, if for every i there exists a $c = c(i) \geq i$ such that

$$\mathrm{Im}(A_k \to A_i) = \mathrm{Im}(A_c \to A_i)$$

for all $k \geq c$.

It turns out that the Mittag-Leffler condition is good enough to ensure that the lim-functor is exact, provided one works within the abelian category of abelian groups, or abelian sheaves, etc. It is shown in a paper by A. Neeman (see [Nee02]) that this condition is not strong enough in a general abelian category (where limits of inverse systems exist).

Lemma 27.3. *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups.

(1) *In any case the sequence*
$$0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i$$
is exact.
(2) *If (B_i) is ML, then also (C_i) is ML.*
(3) *If (A_i) is ML, then*
$$0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i \to 0$$
is exact.

Proof. Nice exercise. See Algebra, Lemma 84.1 for part (3). □

Lemma 27.4. *Let*
$$(A_i) \to (B_i) \to (C_i) \to (D_i)$$
be an exact sequence of inverse systems of abelian groups. If the system (A_i) is ML, then the sequence
$$\lim_i B_i \to \lim_i C_i \to \lim_i D_i$$
is exact.

Proof. Let $Z_i = \mathrm{Ker}(C_i \to D_i)$ and $I_i = \mathrm{Im}(A_i \to B_i)$. Then $\lim Z_i = \mathrm{Ker}(\lim C_i \to \lim D_i)$ and we get a short exact sequence of systems
$$0 \to (I_i) \to (B_i) \to (Z_i) \to 0$$
Moreover, by Lemma 27.3 we see that (I_i) has (ML), thus another application of Lemma 27.3 shows that $\lim B_i \to \lim Z_i$ is surjective which proves the lemma. $\square$

The following characterization of essentially constant inverse systems shows in particular that they have ML.

Lemma 27.5. *Let $\mathcal{A}$ be an abelian category. Let (A_i) be an inverse system in $\mathcal{A}$ with limit $A = \lim A_i$. Then (A_i) is essentially constant (see Categories, Definition 22.1) if and only if there exists an i and for all $j \geq i$ a direct sum decomposition $A_j = A \oplus Z_j$ such that (a) the maps $A_{j'} \to A_j$ are compatible with the direct sum decompositions, (b) for all j there exists some $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero.*

Proof. Assume (A_i) is essentially constant. Then there exists an i and a morphism $A_i \to A$ such that for all $j \geq i$ there exists a $j' \geq j$ such that $A_{j'} \to A_j$ factors as $A_{j'} \to A_i \to A \to A_j$ (the last map comes from $A = \lim A_i$). Hence setting $Z_j = \mathrm{Ker}(A_j \to A)$ for all $j \geq i$ works. Proof of the converse is omitted. $\square$

Lemma 27.6. *Let*
$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$
be an exact sequence of inverse systems of abelian groups. If (A_i) has ML and (C_i) is essentially constant, then (B_i) has ML.

Proof. After renumbering we may assume that $C_i = C \oplus Z_i$ compatible with transition maps and that for all i there exists an $i' \geq i$ such that $Z_{i'} \to Z_i$ is zero, see Lemma 27.5. Pick i. Let $c \geq i$ by an integer such that $\mathrm{Im}(A_c \to A) = \mathrm{Im}(A_{i'} \to A_i)$ for all $i' \geq c$. Let $c' \geq c$ be an integer such that $Z_{c'} \to Z_c$ is zero. For $i' \geq c'$ consider the maps
$$\begin{tikzcd}
0 \arrow[r] & A_{i'} \arrow[r] \arrow[d] & B_{i'} \arrow[r] \arrow[d] & C \oplus Z_{i'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{c'} \arrow[r] \arrow[d] & B_{c'} \arrow[r] \arrow[d] & C \oplus Z_{c'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{c} \arrow[r] \arrow[d] & B_{c} \arrow[r] \arrow[d] & C \oplus Z_{c} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{i} \arrow[r] & B_{i} \arrow[r] & C \oplus Z_{i} \arrow[r] & 0
\end{tikzcd}$$
Because $Z_{c'} \to Z_c$ is zero the image $\mathrm{Im}(B_{c'} \to B_c)$ is an extension C by a subgroup $A' \subset A_c$ which contains the image of $A_{c'} \to A_c$. Hence $\mathrm{Im}(B_{c'} \to B_i)$ is an

extension of C by the image of A' which is the image of $A_c \to A_i$ by our choice of c. In exactly the same way one shows that $\mathrm{Im}(B_{i'} \to B_i)$ is an extension of C by the image of $A_c \to A_i$. Hence $\mathrm{Im}(B_{c'} \to B_i) = \mathrm{Im}(B_{i'} \to B_i)$ and we win. $\square$

The "correct" version of the following lemma is More on Algebra, Lemma 61.2.

Lemma 27.7. *Let*
$$(A_i^{-2} \to A_i^{-1} \to A_i^0 \to A_i^1)$$
be an inverse system of complexes of abelian groups and denote $A^{-2} \to A^{-1} \to A^0 \to A^1$ its limit. Denote (H_i^{-1}), (H_i^0) the inverse systems of cohomologies, and denote H^{-1}, H^0 the cohomologies of $A^{-2} \to A^{-1} \to A^0 \to A^1$. If (A_i^{-2}) and (A_i^{-1}) are ML and (H_i^{-1}) is essentially constant, then $H^0 = \lim H_i^0$.

Proof. Let $Z_i^j = \mathrm{Ker}(A_i^j \to A_i^{j+1})$ and $I_i^j = \mathrm{Im}(A_i^{j-1} \to A_i^j)$. Note that $\lim Z_i^0 = \mathrm{Ker}(\lim A_i^0 \to \lim A_i^1)$ as taking kernels commutes with limits. The systems (I_i^{-1}) and (I_i^0) have ML as quotients of the systems (A_i^{-2}) and (A_i^{-1}), see Lemma 27.3. Thus an exact sequence
$$0 \to (I_i^{-1}) \to (Z_i^{-1}) \to (H_i^{-1}) \to 0$$
of inverse systems where (I_i^{-1}) has ML and where (H_i^{-1}) is essentially constant by assumption. Hence (Z_i^{-1}) has ML by Lemma 27.6. The exact sequence
$$0 \to (Z_i^{-1}) \to (A_i^{-1}) \to (I_i^0) \to 0$$
and an application of Lemma 27.3 shows that $\lim A_i^{-1} \to \lim I_i^0$ is surjective. Finally, the exact sequence
$$0 \to (I_i^0) \to (Z_i^0) \to (H_i^0) \to 0$$
and Lemma 27.3 show that $\lim I_i^0 \to \lim Z_i^0 \to \lim H_i^0 \to 0$ is exact. Putting everything together we win. $\square$

Sometimes we need a version of the lemma above where we take limits over big ordinals.

Lemma 27.8. *Let α be an ordinal. Let $K_\beta^\bullet$, $\beta < \alpha$ be an inverse system of complexes of abelian groups over α. If for all $\beta < \alpha$ the complex $K_\beta^\bullet$ is acyclic and the map*
$$K_\beta^n \longrightarrow \lim_{\gamma < \beta} K_\gamma^n$$
is surjective, then the complex $\lim_{\beta < \alpha} K_\beta^\bullet$ is acyclic.

Proof. By transfinite induction we prove this holds for every ordinal α and every system as in the lemma. In particular, whilst proving the result for α we may assume the complexes $\lim_{\gamma < \beta} K_\gamma^n$ are acyclic.

Let $x \in \lim_{\beta < \alpha} K_\alpha^0$ with $\mathrm{d}(x) = 0$. We will find a $y \in K_\alpha^{-1}$ with $\mathrm{d}(y) = x$. Write $x = (x_\beta)$ where $x_\beta \in K_\beta^0$ is the image of x for $\beta < \alpha$. We will construct $y = (y_\beta)$ by transfinite induction.

For $\beta = 0$ let $y_0 \in K_0^{-1}$ be any element with $\mathrm{d}(y_0) = x_0$.

For $\beta = \gamma + 1$ a successor, we have to find an element y_β which maps both to y_γ by the transition map $f : K_\beta^\bullet \to K_\gamma^\bullet$ and to x_β under the differential. As a first approximation we choose y'_β with $\mathrm{d}(y'_\beta) = x_\beta$. Then the difference $y_\gamma - f(y'_\beta)$ is in the kernel of the differential, hence equal to $\mathrm{d}(z_\gamma)$ for some $z_\gamma \in K_\gamma^{-2}$. By assumption, the map $f^{-2} : K_\beta^{-2} \to K_\gamma^{-2}$ is surjective. Hence we write $z_\gamma = f(z_\beta)$ and change y'_β into $y_\beta = y'_\beta + \mathrm{d}(z_\beta)$ which works.

If β is a limit ordinal, then we have the element $(y_\gamma)_{\gamma<\beta}$ in $\lim_{\gamma<\beta} K_\gamma^{-1}$ whose differential is the image of x_β. Thus we can argue in exactly the same manner as above using the termwise surjective map of complexes $f : K_\beta^\bullet \to \lim_{\gamma<\beta} K_\gamma^\bullet$ and the fact (see first paragraph of proof) that we may assume $\lim_{\gamma<\beta} K_\gamma^\bullet$ is acyclic by induction. □

28. Exactness of products

Lemma 28.1. *Let I be a set. For $i \in I$ let $L_i \to M_i \to N_i$ be a complex of abelian groups. Let $H_i = Ker(M_i \to N_i)/Im(L_i \to M_i)$ be the cohomology. Then*

$$\prod L_i \to \prod M_i \to \prod N_i$$

is a complex of abelian groups with homology $\prod H_i$.

Proof. Omitted. □

29. Other chapters

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References

[Del71] Pierre Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57.

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, 1995.

[Lan02] Serge Lang, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.

[McC01] John McCleary, *A user's guide to spectral sequences*, second ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001.

[ML63] Saunders Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press Inc., Publishers, New York, 1963.

[Nee02] Amnon Neeman, *A counterexample to a 1961 "theorem" in homological algebra*, Invent. Math. **148** (2002), no. 2, 397–420, With an appendix by P. Deligne.

[Ser53] Jean-Pierre Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. (2) **58** (1953), 258–294.