RESTRICTED POWER SERIES

Contents

1. Introduction

In this chapter we discuss algebras topologically of finite type over pre-adic topological rings and their homomorphisms. Many of the results discussed here can be found in the paper [Elk73]. Other general references for this chapter are [DG67], [Abb10], and [FK].

2. Restricted power series

Let A be a topological ring complete with respect to a linear topology (More on Algebra, Definition 26.1). Let I_λ be a fundamental system of open ideals. Let $r \geq 0$ be an integer. In this setting one often denotes

$$A\{x_1, \ldots, x_r\} = \lim_\lambda A/I_\lambda[x_1, \ldots, x_r] = \lim_\lambda (A[x_1, \ldots, x_r]/I_\lambda A[x_1, \ldots, x_r])$$

endowed with the limit topology. In other words, this is the completion of the polynomial ring with respect to the ideals I_λ. We can think of elements of $A\{x_1, \ldots, x_r\}$ as power series

$$f = \sum\nolimits_{E=(e_1,\ldots,e_r)} a_E x_1^{e_1} \ldots x_r^{e_r}$$

in $x_1, \ldots, x_r$ with coefficients $a_E \in A$ which tend to zero in the topology of A. In other words, for any λ all but a finite number of a_E are in I_λ. For this reason elements of $A\{x_1, \ldots, x_r\}$ are sometimes called *restricted power series*. Sometimes this ring is denoted $A\langle x_1, \ldots, x_r\rangle$; we will refrain from using this notation.

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Remark 2.1 (Universal property restricted power series)**.** Let $A \to C$ be a continuous map of complete linearly topologized rings. Then any A-algebra map $A[x_1, \ldots x_r] \to C$ extends uniquely to a continuous map $A\{x_1, \ldots, x_r\} \to C$ on restricted power series.

Remark 2.2. Let A be a ring and let $I \subset A$ be an ideal. If A is I-adically complete, then the I-adic completion $A[x_1, \ldots, x_r]^\wedge$ of $A[x_1, \ldots, x_r]$ is the restricted power series ring over A as a ring. However, it is not clear that $A[x_1, \ldots, x_r]^\wedge$ is I-adically complete. We think of the topology on $A\{x_1, \ldots, x_r\}$ as the limit topology (which is always complete) whereas we often think of the topology on $A[x_1, \ldots, x_r]^\wedge$ as the I-adic topology (not always complete). If I is finitely generated, then $A\{x_1, \ldots, x_r\} = A[x_1, \ldots, x_r]^\wedge$ as topological rings, see Algebra, Lemmas 93.6 and 93.7.

3. Algebras topologically of finite type

Here is our definition. This definition is not generally agreed upon. Many authors impose further conditions, often because they are only interested in specific types of rings and not the most general case.

Definition 3.1. Let $A \to B$ be a continuous map of topological rings (More on Algebra, Definition 26.1). We say B is *topologically of finite type over* A if there exists an A-algebra map $A[x_1, \ldots, x_n] \to B$ whose image is dense in B.

If A is a complete, linearly topologized ring, then the restricted power series ring $A\{x_1, \ldots, x_r\}$ is topologically of finite type over A. For continuous taut maps of weakly admissible topological rings, this notion corresponds exactly to morphisms of finite type between the associated affine formal algebraic spaces.

Lemma 3.2. *Let S be a scheme. Let $\varphi : A \to B$ be a continuous map of weakly admissible topological rings over S. The following are equivalent*

(1) *Spf(φ) : Spf(B) $\to$ Spf(A) is of finite type,*
(2) *φ is taut and B is topologically of finite type over A.*

Proof. We can use Formal Spaces, Lemma 14.10 to relate tautness of φ to representability of $\mathrm{Spf}(\varphi)$. We will use this without further mention below. Note that $X = \mathrm{colim}\, \mathrm{Spec}(A/I)$ and $Y = \mathrm{colim}\, \mathrm{Spec}(B/J(I))$ where $I \subset A$ runs over the weak ideals of definition of A and $J(I)$ is the closure of IB in B.

Assume (2). Choose a ring map $A[x_1, \ldots, x_r] \to B$ whose image is dense. Then $A[x_1, \ldots, x_r] \to B \to B/J(I)$ has dense image too which means that it is surjective. Therefore $B/J(I)$ is of finite type over A/I. Let $T \to X$ be a morphism with T a quasi-compact scheme. Then $T \to X$ factors through $\mathrm{Spec}(B/I)$ for some I (Formal Spaces, Lemma 5.4). Then $T \times_X Y = T \times_{\mathrm{Spec}(A/I)} \mathrm{Spec}(B/J(I))$, see proof of Formal Spaces, Lemma 14.10. Henc $T \times_Y X \to T$ is of finite type as the base change of the morphism $\mathrm{Spec}(B/J(I)) \to \mathrm{Spec}(A/I)$ which is of finite type. Thus (1) is true.

Assume (1). Pick any $I \subset A$ as above. Since $\mathrm{Spec}(A/I) \times_X Y = \mathrm{Spec}(B/J(I))$ we see that $A/I \to B/J(I)$ is of finite type. Choose $b_1, \ldots, b_r \in B$ mapping to generators of $B/J(I)$ over A/I. We claim that the image of the ring map $A[x_1, \ldots, x_r] \to B$ sending x_i to b_i is dense. To prove this, let $I' \subset I$ be a second weak ideal of definition. Then we have

$$B/(J(I') + IB) = B/J(I)$$

because $J(I)$ is the closure of IB and because $J(I')$ is open. Hence we may apply Algebra, Lemma 122.8 to see that $A/I'[x_1, \ldots, x_r] \to B/J(I')$ is surjective Thus (2) is true, concluding the proof. □

Let A be a topological ring complete with respect to a linear topology. Let I_λ be a fundamental system of open ideals. Let $\mathcal{C}$ be the category of systems (B_λ) where

(1) B_λ is a finite type A/I_λ-algebra, and
(2) $B_\mu \to B_\lambda$ is an A/I_μ-algebra homomorphism which induces an isomorphism $B_\mu/I_\lambda B_\mu \to B_\lambda$.

Morphisms in $\mathcal{C}$ are given by systems of homomorphisms.

Lemma 3.3. *Let S be a scheme. Let X be an affine formal scheme over S. Assume X is McQuillan and let A be the weakly admissible topological ring associated to X. Then there is an anti-equivalence of categories between*

(1) *the category $\mathcal{C}$ introduced above, and*
(2) *the category of maps $Y \to X$ of finite type of affine formal algebraic spaces.*

Proof. Let I_λ be a fundamental system of weakly admissible ideals of definition in A. Then $Y \times_X \mathrm{Spec}(A/I_\lambda)$ is affine (Formal Spaces, Definition 18.1 and Lemma 14.7). Say $Y \times_X \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$. Then (B_λ) is an object of $\mathcal{C}$. Conversely, given a system (B_λ) we can set $Y = \mathrm{colim}\,\mathrm{Spec}(B_\lambda)$. Some details omitted. □

Remark 3.4. Let A be a weakly admissible topological ring and let I_λ be a fundamental system of weak ideals of definition. Let $X = \mathrm{Spf}(A)$, in other words, X is a McQuillan affine formal algebraic space. Let $f : Y \to X$ be a morphism of affine formal algebraic spaces. In general it will not be true that Y is McQuillan. More specifically, we can ask the following questions:

(1) Assume that $f : Y \to X$ is a closed immersion. Then Y is McQuillan and f corresponds to a continuous map $\varphi : A \to B$ of weakly admissible topological rings which is taut, whose kernel $K \subset A$ is a closed ideal, and whose image $\varphi(A)$ is dense in B, see Formal Spaces, Lemma 20.2. What conditions on A guarantee that $B = (A/K)^\wedge$ as in Formal Spaces, Example 20.3?
(2) What conditions on A guarantee that closed immersions $f : Y \to X$ correspond to quotients A/K of A by closed ideals, in other words, the corresponding continuous map φ is surjective and open?
(3) Suppose that $f : Y \to X$ is of finite type. Then we get $Y = \mathrm{colim}\,\mathrm{Spec}(B_\lambda)$ where (B_λ) is an object of $\mathcal{C}$ by Lemma 3.3. In this case it is true that there exists a fixed integer r such that B_λ is generated by r elements over A/I_λ for all λ (hint: use Algebra, Lemma 122.8). However, it is not clear that the projections $\lim B_\lambda \to B_\lambda$ are surjective, i.e., it is not clear that Y is McQuillan. Is there an example where Y is not McQuillan?
(4) Suppose that $f : Y \to X$ is of finite type and Y is McQuillan. Then f corresponds to a continuous map $\varphi : A \to B$ of weakly admissible topological rings. In fact φ is taut and B is topologically of finite type over A, see Lemma 3.2. In other words, f factors as

$$Y \longrightarrow \mathbf{A}^r_X \longrightarrow X$$

where the first arrow is a closed immersion of McQuillan affine formal algebraic spaces. However, then questions (1) and (2) are in force for $Y \to \mathbf{A}^r_X$.

Below we will answer these questions when X is countably indexed, i.e., when A has a countable fundamental system of open ideals. If you have answers to these questions in greater generality, or if you have counter examples, please email stacks.project@gmail.com.

Lemma 3.5. *Let S be a scheme. Let X be a countably indexed affine formal algebraic space over S. Let $f : Y \to X$ be a closed immersion of formal algebraic spaces over S. Then Y is a countably indexed affine formal algebraic space and f corresponds to $A \to A/K$ where A is an object of $\textit{WAdm}^{count}$ and $K \subset A$ is a closed ideal.*

Proof. we can use Formal Spaces, Lemmas 6.4, 14.8, and 20.2 to reduce to a morphism $A \to B$ of $\textit{WAdm}^{count}$ which is taut and has dense image. To finish the proof we apply Formal Spaces, Lemma 4.12. □

Lemma 3.6. *Let $B \to A$ be an arrow of $\textit{WAdm}^{count}$, see Formal Spaces, Section 16. The following are equivalent*

- (a) *$B \to A$ is taut and $B/J \to A/I$ is of finite type for every weak ideal of definition $J \subset B$ where $I \subset A$ is the closure of JA,*
- (b) *$B \to A$ is taut and $B/J \to A/I$ is of finite type for some weak ideal of definition $J \subset B$ with $I \subset A$ the closure of JA,*
- (c) *$B \to A$ is taut and A is topologically of finite type over B,*
- (d) *A is isomorphic to a quotient of $B\{x_1, \ldots, x_n\}$ by a closed ideal.*

Moreover, these equivalent conditions define a local property, i.e., they satisfy Formal Spaces, Axioms (1), (2), (3).

Proof. The implications (a) ⇒ (b), (c) ⇒ (a), (d) ⇒ (c) are straightforward from the definitions. Assume (b) holds and let $J \subset B$ and $I \subset A$ be as in (b). Choose a commutative diagram

$$\begin{tikzcd} A \arrow[r] & \ldots \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & A_1 \\ B \arrow[u] \arrow[r] & \ldots \arrow[r] & B/J_3 \arrow[u] \arrow[r] & B/J_2 \arrow[u] \arrow[r] & B/J_1 \arrow[u] \end{tikzcd}$$

such that $A_{n+1}/J_nA_{n+1} = A_n$ and such that $A = \lim A_n$ as in Formal Spaces, Lemma 16.5. We may assume $J = J_1$ by replacing J_1 by $J_1 + J$ if necessary. Let $\alpha_1, \ldots, \alpha_n \in A_1$ be generators of A_1 over $B/J_1 = B/J$. Since A is a countable limit of a system with surjective transition maps, we can find $a_1, \ldots, a_n \in A$ mapping to $\alpha_1, \ldots, \alpha_n$ in A_1. By Remark 2.1 we find a continuous map $B\{x_1, \ldots, x_n\} \to A$ mapping x_i to a_i. This map induces surjections $B/J_m[x_1, \ldots, x_n] \to A_m$ by Algebra, Lemma 122.8. For $m \geq 1$ we obtain a short exact sequence

$$0 \to K_m \to B/J_m[x_1, \ldots, x_n] \to A_m \to 0$$

The induced transition maps $K_{m+1} \to K_m$ are surjective because $A_{m+1}/J_mA_{m+1} = A_m$. Hence the inverse limit of these short exact sequences is exact, see Algebra, Lemma 83.4. Since $B\{x_1, \ldots, x_n\} = \lim B/J_m[x_1, \ldots, x_n]$ and $A = \lim A_m$ we conclude that $B\{x_1, \ldots, x_n\} \to A$ is surjective. As A is complete the kernel is a closed ideal. In this way we see that (a), (b), (c), and (d) are equivalent.

Let a diagram as in Formal Spaces, Diagram (16.1.1) be given. By Formal Spaces, Example 18.7 the maps $A \to (A')^\wedge$ and $B \to (B')^\wedge$ satisfy (a), (b), (c), and (d).

Moreover, by Formal Spaces, Lemma 16.5 in order to prove Formal Spaces, Axioms (1) and (2) we may assume both $A \to B$ and $(B')^\wedge \to (A')^\wedge$ are taut. Now pick a weak ideal of definition $J \subset B$. Let $J' \subset (B')^\wedge$, $I \subset A$, $I' \subset (A')^\wedge$ be the closure of $J(B')^\wedge$, JA, $J(A')^\wedge$. By what was said above, it suffices to consider the commutative diagram

$$\begin{tikzcd} A/I \arrow[r] & (A')^\wedge/I' \\ B/J \arrow[r] \arrow[u, "\overline{\varphi}"] & (B')^\wedge/J' \arrow[u, "\overline{\varphi}'"'] \end{tikzcd}$$

and to show (1) $\overline{\varphi}$ finite type $\Rightarrow$ $\overline{\varphi}'$ finite type, and (2) if $A \to A'$ is faithfully flat, then $\overline{\varphi}'$ finite type $\Rightarrow$ $\overline{\varphi}$ finite type. Note that $(B')^\wedge/J' = B'/JB'$ and $(A')^\wedge/I' = A'/IA'$ by the construction of the topologies on $(B')^\wedge$ and $(A')^\wedge$. In particular the horizontal maps in the diagram are étale. Part (1) now follows from Algebra, Lemma 6.2 and part (2) from Descent, Lemma 10.2 as the ring map $A/I \to (A')^\wedge/I' = A'/IA'$ is faithfully flat and étale.

We omit the proof of Formal Spaces, Axiom (3). □

Lemma 3.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of affine formal algebraic spaces. Assume Y countably indexed. The following are equivalent*

1. *f is locally of finite type,*
2. *f is of finite type,*
3. *f corresponds to a morphism $B \to A$ of WAdm^{count} satisfying the equivalent conditions of Lemma 3.6.*

Proof. Since X and Y are affine it is clear that conditions (1) and (2) are equivalent. In cases (1) and (2) we see that X is countably indexed as well by Formal Spaces, Lemma 14.8. Write $X = \mathrm{Spf}(A)$ and $Y = \mathrm{Spf}(B)$ for topological S-algebras A and B in WAdm^{count}, see Formal Spaces, Lemma 6.4. By Formal Spaces, Lemma 5.10 we see that f corresponds to a continuous map $B \to A$. Hence now the result follows from Lemma 3.2. □

Lemma 3.8. *Let P be the property of morphisms of WAdm^{count} defined by the equivalent conditions (a), (b), (c), and (d) of Lemma 3.6. Then under the assumptions of Formal Spaces, Lemma 16.2 the equivalent conditions (1), (2), and (3) are also equivalent to the condition*

(4) *f is locally of finite type.*

Proof. By Lemma 3.7 the condition on morphisms of WAdm^{count} translates into morphisms of countably indexed, affine formal algebraic spaces being of finite type. Thus the lemma follows from Formal Spaces, Lemma 18.6. □

4. Two categories

Let A be a ring and let $I \subset A$ be an ideal. In this section $^\wedge$ will mean I-adic completion. Set $A_n = A/I^n$ so that the I-adic completion of A is $A^\wedge = \lim A_n$. Let $\mathcal{C}$ be the category

$$\mathcal{C} = \left\{ \begin{matrix} \text{systems } (B_n, B_{n+1} \to B_n)_{n \in \mathbf{N}} \text{ where} \\ B_n \text{ is a finite type } A_n\text{-algebra}, \\ B_{n+1} \to B_n \text{ is an } A_{n+1}\text{-algebra map} \\ \text{which induces } B_{n+1}/I^n B_{n+1} \cong B_n \end{matrix} \right\} \tag{4.0.1}$$

Morphisms in $\mathcal{C}$ are given by systems of homomorphisms. Let $\mathcal{C}'$ be the category

$$\mathcal{C}' = \left\{ \begin{matrix} A\text{-algebras } B \text{ which are } I\text{-adically complete} \\ \text{such that } B/IB \text{ is of finite type over } A/I \end{matrix} \right\} \tag{4.0.2}$$

Morphisms in $\mathcal{C}'$ are A-algebra maps. There is a functor

$$\mathcal{C}' \longrightarrow \mathcal{C}, \quad B \longmapsto (B/I^nB) \tag{4.0.3}$$

Indeed, since B/IB is of finite type over A/I the ring maps $A_n = A/I^n \to B/I^nB$ are of finite type (apply Algebra, Lemma 19.1 to a ring map $A/I^n[x_1, \ldots, x_r] \to B/I^nB$ such that the images of $x_1, \ldots, x_r$ generate B/IB over A/I).

Lemma 4.1. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functor*

$$\mathcal{C} \longrightarrow \mathcal{C}', \quad (B_n) \longmapsto B = \lim B_n$$

is a quasi-inverse to (4.0.3). The completions $A[x_1, \ldots, x_r]^\wedge$ are in $\mathcal{C}'$ and any object of $\mathcal{C}'$ is of the form

$$B = A[x_1, \ldots, x_r]^\wedge / J$$

for some ideal $J \subset A[x_1, \ldots, x_r]^\wedge$.

Proof. Let (B_n) be an object of $\mathcal{C}$. By Algebra, Lemma 94.1 we see that $B = \lim B_n$ is I-adically complete and $B/I^nB = B_n$. Hence we see that B is an object of $\mathcal{C}'$ and that we can recover the object (B_n) by taking the quotients. Conversely, if B is an object of $\mathcal{C}'$, then $B = \lim B/I^nB$ by assumption. Thus $B \mapsto (B/I^nB)$ is a quasi-inverse to the functor of the lemma.

Since $A[x_1, \ldots, x_r]^\wedge = \lim A_n[x_1, \ldots, x_r]$ it is an object of $\mathcal{C}'$ by the first statement of the lemma. Finally, let B be an object of $\mathcal{C}'$. Choose $b_1, \ldots, b_r \in B$ whose images in B/IB generate B/IB as an algebra over A/I. Since B is I-adically complete, the A-algebra map $A[x_1, \ldots, x_r] \to B$, $x_i \mapsto b_i$ extends to an A-algebra map $A[x_1, \ldots, x_r]^\wedge \to B$. To finish the proof we have to show this map is surjective which follows from Algebra, Lemma 93.1 as our map $A[x_1, \ldots, x_r] \to B$ is surjective modulo I and as $B = B^\wedge$. □

We warn the reader that, in case A is not Noetherian, the quotient of an object of $\mathcal{C}'$ may not be an object of $\mathcal{C}'$. See Examples, Lemma 7.1. Next we show this does not happen when A is Noetherian.

Lemma 4.2. *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Then*

1. *every object of the category $\mathcal{C}'$, in particular the completion $A[x_1, \ldots, x_r]^\wedge$, is Noetherian,*
2. *if B is an object of $\mathcal{C}'$ and $J \subset B$ is an ideal, then B/J is an object of $\mathcal{C}'$.*

Proof. To see (1) by Lemma 4.1 we reduce to the case of the completion of the polynomial ring. This case follows from Algebra, Lemma 93.10 as $A[x_1, \ldots, x_r]$ is Noetherian (Algebra, Lemma 30.1). Part (2) follows from Algebra, Lemma 93.2 which tells us that ever finite B-module is IB-adically complete. □

Remark 4.3 (Base change)**.** Let $\varphi : A_1 \to A_2$ be a ring map and let $I_i \subset A_i$ be ideals such that $\varphi(I_1^c) \subset I_2$ for some $c \geq 1$. This induces ring maps $A_{1,cn} = A_1/I_1^{cn} \to A_2/I_2^n = A_{2,n}$ for all $n \geq 1$. Let $\mathcal{C}_i$ be the category (4.0.1) for (A_i, I_i). There is a base change functor

$$\mathcal{C}_1 \longrightarrow \mathcal{C}_2, \quad (B_n) \longmapsto (B_{cn} \otimes_{A_{1,cn}} A_{2,n}) \tag{4.3.1}$$

Let $\mathcal{C}'_i$ be the category (4.0.2) for (A_i, I_i). If I_2 is finitely generated, then there is a base change functor

$$\mathcal{C}'_1 \longrightarrow \mathcal{C}'_2, \quad B \longmapsto (B \otimes_{A_1} A_2)^\wedge \tag{4.3.2}$$

because in this case the completion is complete (Algebra, Lemma 93.7). If both I_1 and I_2 are finitely generated, then the two base change functors agree via the functors (4.0.3) which are equivalences by Lemma 4.1.

Remark 4.4 (Base change by closed immersion)**.** Let A be a Noetherian ring and $I \subset A$ an ideal. Let $\mathfrak{a} \subset A$ be an ideal. Denote $\bar{A} = A/\mathfrak{a}$. Let $\bar{I} \subset \bar{A}$ be an ideal such that $I^c\bar{A} \subset \bar{I}$ and $\bar{I}^d \subset I\bar{A}$ for some $c, d \geq 1$. In this case the base change functor (4.3.2) for (A, I) to $(\bar{A}, \bar{I})$ is given by $B \mapsto \bar{B} = B/\mathfrak{a}B$. Namely, we have

$$\bar{B} = (B \otimes_A \bar{A})^\wedge = (B/\mathfrak{a}B)^\wedge = B/\mathfrak{a}B \tag{4.4.1}$$

the last equality because any finite B-module is I-adically complete by Algebra, Lemma 93.2 and if annihilated by $\mathfrak{a}$ also $\bar{I}$-adically complete by Algebra, Lemma 93.14.

5. A naive cotangent complex

Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let B be an A-algebra which is I-adically complete such that $A/I \to B/IB$ is of finite type, i.e., an object of (4.0.2). By Lemma 4.2 we can write

$$B = A[x_1, \ldots, x_r]^\wedge/J$$

for some finitely generated ideal J. For a choice of presentation as above we define the naive cotangent complex in this setting by the formula

$$NL^\wedge_{B/A} = (J/J^2 \longrightarrow \bigoplus B\mathrm{d}x_i) \tag{5.0.2}$$

with terms sitting in degrees -1 and 0 where the map sends the residue class of $g \in J$ to the differential $\mathrm{d}g = \sum(\partial g/\partial x_i)\mathrm{d}x_i$. Here the partial derivative is taken by thinking of g as a power series. The following lemma shows that $NL^\wedge_{B/A}$ is well defined in $D(B)$, i.e., independent of the chosen presentation, although this could be shown directly by comparing presentations as in Algebra, Section 129.

Lemma 5.1. *Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let B be an object of (4.0.2). Then $NL^\wedge_{B/A} = R\lim NL_{B_n/A_n}$ in $D(B)$.*

Proof. In fact, the presentation $B = A[x_1, \ldots, x_r]^\wedge/J$ defines presentations

$$B_n = B/I^nB = A_n[x_1, \ldots, x_r]/J_n$$

where

$$J_n = JA_n[x_1, \ldots, x_r] = J/(J \cap I^nA[x_1, \ldots, x_r]^\wedge)$$

By Artin-Rees (Algebra, Lemma 49.2) in the Noetherian ring $A[x_1, \ldots, x_r]^\wedge$ (Lemma 4.2) we see that we have canonical surjections

$$J/I^nJ \to J_n \to J/I^{n-c}J, \quad n \geq c$$

for some $c \geq 0$. It follows that $\lim J_n/J_n^2 = J/J^2$ as any finite $A[x_1, \ldots, x_r]^\wedge$-module is I-adically complete (Algebra, Lemma 93.2). Thus

$$NL^\wedge_{B/A} = \lim(J_n/J_n^2 \longrightarrow \bigoplus B_n\mathrm{d}x_i)$$

(termwise limit) and the transition maps in the system are termwise surjective. The two term complex $J_n/J_n^2 \longrightarrow \bigoplus B_n \mathrm{d}x_i$ represents NL_{B_n/A_n} by Algebra, Section 129. It follows that $NL^\wedge_{B/A}$ represents $R\lim NL_{B_n/A_n}$ in the derived category by More on Algebra, Lemma 61.9. $\square$

Lemma 5.2. *Let A be a Noetherian ring and let $I \subset A$ be a ideal. Let $B \to C$ be morphism of (4.0.2). Then there is an exact sequence*

$$
\begin{tikzcd}
C \otimes_B H^0(NL^\wedge_{B/A}) \arrow[r] & H^0(NL^\wedge_{C/A}) \arrow[r] & H^0(NL^\wedge_{C/B}) \arrow[r] & 0 \\
H^{-1}(NL^\wedge_{B/A} \otimes_B C) \arrow[r] & H^{-1}(NL^\wedge_{C/A}) \arrow[r] & H^{-1}(NL^\wedge_{C/B}) \arrow[ull]
\end{tikzcd}
$$

Proof. Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Note that (B, IB) is a pair consisting of a Noetherian ring and an ideal, and C is in the corresponding category (4.0.2) for this pair. Hence we can choose a presentation $C = B[y_1, \ldots, y_s]^\wedge/J'$. Combinging these presentations gives a presentation

$$C = A[x_1, \ldots, x_r, y_1, \ldots, y_s]^\wedge/K$$

Then the reader verifies that we obtain a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \bigoplus C\mathrm{d}x_i \arrow[r] & \bigoplus C\mathrm{d}x_i \oplus \bigoplus C\mathrm{d}y_j \arrow[r] & \bigoplus C\mathrm{d}y_j \arrow[r] & 0 \\
 & J/J^2 \otimes_B C \arrow[r] \arrow[u] & K/K^2 \arrow[r] \arrow[u] & J'/(J')^2 \arrow[r] \arrow[u] & 0
\end{tikzcd}
$$

with exact rows. Note that the vertical arrow on the left hand side is the tensor product of the arrow defining $NL^\wedge_{B/A}$ with id_C. The lemma follows by applying the snake lemma (Algebra, Lemma 4.1). $\square$

Lemma 5.3. *With assumptions as in Lemma 5.2 assume that $B/I^nB \to C/I^nC$ is a local complete intersection homomorphism for all n. Then $H^{-1}(NL^\wedge_{B/A} \otimes_B C) \to H^{-1}(NL^\wedge_{C/A})$ is injective.*

Proof. By More on Algebra, Lemma 23.6 we see that this holds for the map between naive cotangent complexes of the situation modulo I^n for all n. In other words, we obtain a distinguished triangle in $D(C/I^nC)$ for every n. Using Lemma 5.1 this implies the lemma; details omitted. $\square$

Maps in the derived category out of a complex such as (5.0.2) are easy to understand by the result of the following lemma.

Lemma 5.4. *Let R be a ring. Let $M^\bullet$ be a complex of modules over R with $M^i = 0$ for $i > 0$ and M^0 a projective R-module. Let $K^\bullet$ be a second complex.*

1. *If $K^i = 0$ for $i \leq -2$, then $\mathrm{Hom}_{D(R)}(M^\bullet, K^\bullet) = \mathrm{Hom}_{K(R)}(M^\bullet, K^\bullet)$,*
2. *If $K^i = 0$ for $i \leq -3$ and $\alpha \in \mathrm{Hom}_{D(R)}(M^\bullet, K^\bullet)$ composed with $K^\bullet \to K^{-2}[2]$ comes from an R-module map $a : M^{-2} \to K^{-2}$ with $a \circ d_M^{-3} = 0$, then α can be represented by a map of complexes $a^\bullet : M^\bullet \to K^\bullet$ with $a^{-2} = a$.*

(3) *In (2) for any second map of complexes $(a')^\bullet : M^\bullet \to K^\bullet$ representing α with $a = (a')^{-2}$ there exist $h' : M^0 \to K^{-1}$ and $h : M^{-1} \to K^{-2}$ such that*

$$h \circ d_M^{-2} = 0, \quad (a')^{-1} = a^{-1} + d_K^{-2} \circ h + h' \circ d_M^{-1}, \quad (a')^0 = a^0 + d_K^{-1} \circ h'$$

Proof. Set $F^0 = M^0$. Choose a free R-module F^{-1} and a surjection $F^{-1} \to M^{-1}$. Choose a free R-module F^{-2} and a surjection $F^{-2} \to M^{-2} \times_{M^{-1}} F^{-1}$. Continuing in this way we obtain a quasi-isomorphism $p^\bullet : F^\bullet \to M^\bullet$ which is termwise surjective and with F^i free for all i.

Proof of (1). By Derived Categories, Lemma 19.8 we have

$$\mathrm{Hom}_{D(R)}(M^\bullet, K^\bullet) = \mathrm{Hom}_{K(R)}(F^\bullet, K^\bullet)$$

If $K^i = 0$ for $i \leq -2$, then any morphism of complexes $F^\bullet \to K^\bullet$ factors through $p^\bullet$. Similarly, any homotopy $\{h^i : F^i \to K^{i-1}\}$ factors through $p^\bullet$. Thus (1) holds.

Proof of (2). Choose $b^\bullet : F^\bullet \to K^\bullet$ representing α. The composition of α with $K^\bullet \to K^{-2}[2]$ is represented by $b^{-2} : F^{-2} \to K^{-2}$. As this is homotopic to $a \circ p^{-2} : F^{-2} \to M^{-2} \to K^{-2}$, there is a map $h : F^{-1} \to K^{-2}$ such that $b^{-2} = a \circ p^{-2} + h \circ d_F^{-2}$. Adjusting $b^\bullet$ by h viewed as a homotopy from $F^\bullet$ to $K^\bullet$, we find that $b^{-2} = a \circ p^{-2}$. Hence b^{-2} factors through p^{-2}. Since $F^0 = M^0$ the kernel of p^{-2} surjects onto the kernel of p^{-1} (for example because the kernel of $p^\bullet$ is an acyclic complex or by a diagram chase). Hence b^{-1} necessarily factors through p^{-1} as well and we see that (2) holds for these factorizations and $a^0 = b^0$.

Proof of (3) is omitted. Hint: There is a homotopy between $a^\bullet \circ p^\bullet$ and $(a')^\bullet \circ p^\bullet$ and we argue as before that this homotopy factors through $p^\bullet$. $\square$

Lemma 5.5. *Let R be a ring. Let $M^\bullet$ be a two term complex $M^{-1} \to M^0$ over R. If $\varphi, \psi \in End_{D(R)}(M^\bullet)$ are zero on $H^i(M^\bullet)$, then $\varphi \circ \psi = 0$.*

Proof. Apply Derived Categories, Lemma 12.5 to see that $\varphi \circ \psi$ factors through $\tau_{\leq -2} M^\bullet = 0$. $\square$

6. Rig-étale homomorphisms

In this and some of the later sections we will study ring maps as in Lemma 6.1. Condition (4) is one of the conditions used in [Art70] to define modifications. Ring maps like this are sometimes called rig-étale or rigid-étale ring maps in the literature. These and the analogously defined rig-smooth ring maps were studied in [Elk73]. A detailed exposition can also be found in [Abb10]. Our main goal will be to show that rig-étale ring maps are completions of finite type algebras, a result very similar to results found in Elkik's paper [Elk73].

Lemma 6.1. *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let B be an object of (4.0.2). The following are equivalent*

(1) *there exists a $c \geq 0$ such that multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$ for all $a \in I^c$,*
(2) *there exits a $c \geq 0$ such that $H^i(NL^\wedge_{B/A})$, $i = -1, 0$ is annihilated by I^c,*
(3) *there exists a $c \geq 0$ such that $H^i(NL_{B_n/A_n})$, $i = -1, 0$ is annihlated by I^c for all $n \geq 1$,*
(4) *$B = A[x_1, \ldots, x_r]^\wedge/J$ and for every $a \in I$ there exists a $c \geq 0$ such that*
    (a) *a^c annihilates $H^0(NL^\wedge_{B/A})$, and*
    (b) *there exist $f_1, \ldots, f_r \in J$ such that $a^c J \subset (f_1, \ldots, f_r) + J^2$.*

Proof. The equivalence of (1) and (2) follows from Lemma 5.5. The equivalence of (1) + (2) and (3) follows from Lemma 5.1. Some details omitted.

Assume the equivalent conditions (1), (2), (3) holds and let $B = A[x_1, \ldots, x_r]^\wedge/J$ be a presentation (see Lemma 4.1). Let $a \in I$. Let c be such that multplication by a^c is zero on $NL^\wedge_{B/A}$ which exists by (1). By Lemma 5.4 there exists a map $\alpha : \bigoplus B\text{d}x_i \to J/J^2$ such that $\text{d} \circ \alpha$ and $\alpha \circ \text{d}$ are both multiplication by a^c. Let $f_i \in J$ be an element whose class modulo J^2 is equal to $\alpha(\text{d}x_i)$. Then we see that (4)(a), (b) hold.

Assume (4) holds. Say $I = (a_1, \ldots, a_t)$. Let $c_i \geq 0$ be the integer such that (4)(a), (b) hold for $a_i^{c_i}$. Then we see that $I^{\sum c_i}$ annihilates $H^0(NL^\wedge_{B/A})$. Let $f_{i,1}, \ldots, f_{i,r} \in J$ be as in (4)(b) for a_i. Consider the composition

$$B^{\oplus r} \to J/J^2 \to \bigoplus B\text{d}x_i$$

where the jth basis vector is mapped to the class of $f_{i,j}$ in J/J^2. By (4)(a) and (b) the cokernel of the composition is annihilated by $a_i^{2c_i}$. Thus this map is surjective after inverting $a_i^{c_i}$, and hence an isomorphism (Algebra, Lemma 15.4). Thus the kernel of $B^{\oplus r} \to \bigoplus B\text{d}x_i$ is a_i-power torsion, and hence $H^{-1}(NL^\wedge_{B/A}) = \text{Ker}(J/J^2 \to \bigoplus B\text{d}x_i)$ is a_i-power torsion. Since B is Noetherian (Lemma 4.2), all modules including $H^{-1}(NL^\wedge_{B/A})$ are finite. Thus $a_i^{d_i}$ annihilates $H^{-1}(NL^\wedge_{B/A})$ for some $d_i \geq 0$. It follows that $I^{\sum d_i}$ annihilates $H^{-1}(NL^\wedge_{B/A})$ and we see that (2) holds. □

Lemma 6.2. *Let A be a Noetherian ring and let I be an ideal. Let B be a finite type A-algebra.*

(1) *If $\text{Spec}(B) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$, then $B^\wedge$ satisfies the equivalent conditions of Lemma 6.1.*
(2) *If $B^\wedge$ satisfies the equivalent conditions of Lemma 6.1, then there exists $g \in 1 + IB$ such that $\text{Spec}(B_g)$ is étale over $\text{Spec}(A) \setminus V(I)$.*

Proof. Assume $B^\wedge$ satisfies the equivalent conditions of Lemma 6.1. The naive cotangent complex $NL_{B/A}$ is a complex of finite type B-modules and hence H^{-1} and H^0 are finite B-modules. Completion is an exact functor on finite B-modules (Algebra, Lemma 93.3) and $NL^\wedge_{B^\wedge/A}$ is the completion of the complex $NL_{B/A}$ (this is easy to see by choosing presentations). Hence the assumption implies there exists a $c \geq 0$ such that $H^{-1}/I^n H^{-1}$ and $H^0/I^n H^0$ are annihilated by I^c for all n. By Nakayama's lemma (Algebra, Lemma 19.1) this means that $I^c H^{-1}$ and $I^c H^0$ are annihilated by an element of the form $g = 1 + x$ with $x \in IB$. After inverting g (which does not change the quotients $B/I^n B$) we see that $NL_{B/A}$ has cohomology annihilated by I^c. Thus $A \to B$ is étale at any prime of B not lying over $V(I)$ by the definition of étale ring maps, see Algebra, Definition 138.1.

Conversely, assume that $\text{Spec}(B) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Then for every $a \in I$ there exists a $c \geq 0$ such that multiplication by a^c is zero $NL_{B/A}$. Since $NL^\wedge_{B^\wedge/A}$ is the derived completion of $NL_{B/A}$ (see Lemma 5.1) it follows that $B^\wedge$ satisfies the equivalent conditions of Lemma 6.1. □

Lemma 6.3. *Assume the map $(A_1, I_1) \to (A_2, I_2)$ is as in Remark 4.3 with A_1 and A_2 Noetherian. Let B_1 be in (4.0.2) for (A_1, I_1). Let B_2 be the base change of B_1. If multiplication by $f_1 \in B_1$ on $NL^\wedge_{B_1/A_1}$ is zero in $D(B_1)$, then multiplication by the image $f_2 \in B_2$ on $NL^\wedge_{B_2/A_2}$ is zero in $D(B_2)$.*

Proof. Choose a presentation $B_1 = A_1[x_1, \ldots, x_r]^\wedge/J_1$. Since $A_2/I_2^n[x_1, \ldots, x_r] = A_1/I_1^{cn}[x_1, \ldots, x_r] \otimes_{A_1/I_1^{cn}} A_2/I_2^n$ we have

$$A_2[x_1, \ldots, x_r]^\wedge = (A_1[x_1, \ldots, x_r]^\wedge \otimes_{A_1} A_2)^\wedge$$

where we use I_2-adic completion on both sides (but of course I_1-adic completion for $A_1[x_1, \ldots, x_r]^\wedge$). Set $J_2 = J_1 A_2[x_1, \ldots, x_r]^\wedge$. Arguing similarly we get the presentation

$$\begin{aligned} B_2 &= (B_1 \otimes_{A_1} A_2)^\wedge \\ &= \lim \frac{A_1/I_1^{cn}[x_1, \ldots, x_r]}{J_1(A_1/I_1^{cn}[x_1, \ldots, x_r])} \otimes_{A_1/I_1^{cn}} A_2/I_2^n \\ &= \lim \frac{A_2/I_2^n[x_1, \ldots, x_r]}{J_2(A_2/I_2^n[x_1, \ldots, x_r])} \\ &= A_2[x_1, \ldots, x_r]^\wedge/J_2 \end{aligned}$$

for B_2 over A_2. Consider the commutative diagram

$$\begin{tikzcd} NL^\wedge_{B_1/A_1}: \arrow[d] & J_1/J_1^2 \arrow[r, "\mathrm{d}"'] \arrow[d] & \bigoplus B_1 \mathrm{d}x_i \arrow[d] \\ NL^\wedge_{B_2/A_2}: & J_2/J_2^2 \arrow[r] & \bigoplus B_2 \mathrm{d}x_i \end{tikzcd}$$

The induced arrow $J_1/J_1^2 \otimes_{B_1} B_2 \to J_2/J_2^2$ is surjective because J_2 is generated by the image of J_1. By Lemma 5.4 there is a map $\alpha_1 : \bigoplus B\mathrm{d}x_i \to J_1/J_1^2$ such that $f_1 \mathrm{id}_{\bigoplus B_1 \mathrm{d}x_i} = \mathrm{d} \circ \alpha_1$ and $f_1 \mathrm{id}_{J_1/J_1^2} = \alpha_1 \circ \mathrm{d}$. We define $\alpha_2 : \bigoplus B_1 \mathrm{d}x_i \to J_2/J_2^2$ by mapping $\mathrm{d}x_i$ to the image of $\alpha_1(\mathrm{d}x_i)$ in J_2/J_2^2. Because the image of the vertical arrows contains generators of the modules J_2/J_2^2 and $\bigoplus B_2 \mathrm{d}x_i$ it follows that α_2 also defines a homotopy between multiplication by f_2 and the zero map. $\square$

Lemma 6.4. *Let A be a Noetherian ring and I an ideal. Let B be a finite type A-algebra. Let $B^\wedge \to C$ be a surjective ring map with kernel J. If J/J^2 is annihilated by I^c for some $c \geq 0$, then C is isomorphic to the completion of a finite type A-algebra.*

Proof. Since $B^\wedge$ is Noetherian (Lemma 4.2), we see that J is a finitely generated ideal. Hence we conclude from Algebra, Lemma 20.5 that

$$\mathrm{Spec}(C) \setminus V(IC) \longrightarrow \mathrm{Spec}(B^\wedge) \setminus V(IB^\wedge)$$

is an open and closed immersion. Let $V \subset \mathrm{Spec}(B^\wedge) \setminus V(IB^\wedge)$ be the complement of the image viewed as an open and closed subscheme. Let $Z \subset \mathrm{Spec}(B^\wedge)$ be the scheme theoretic closure of V. Write $Z = \mathrm{Spec}(C')$. Then

$$\mathrm{Spec}(C \times C') = \mathrm{Spec}(C) \amalg Z \longrightarrow \mathrm{Spec}(B^\wedge)$$

is a finite morphism of schemes which is an isomorphism away from $V(IB^\wedge)$. Hence the corresponding ring map $B^\wedge \to C \times C'$ is finite and becomes an isomorphism on inverting any element of I. By More on Algebra, Proposition 63.15 and Remark 63.19 applied to $B \to B^\wedge$ and the finitely generated ideal IB, we conclude that $C \times C'$ is isomorphic to $D \otimes_B B^\wedge$ for some finite B-algebra D. (The reader can also prove this using Pushouts of Spaces, Lemma 4.1.) Then $D/ID \cong C/IC \times C'/IC'$. Let $\overline{e} \in D/ID$ be the idempotent corresponding to the factor C/IC. By More on Algebra, Lemma 6.9 there exists an étale ring map $B \to B'$ which induces an isomorphism $B/IB \to B'/IB'$ such that $D' = D \otimes_B B'$ contains an idempotent e lifting $\overline{e}$. Since $C \times C'$ is I-adically complete the pair $(C \times C', IC \times IC')$ is henselian (More on Algebra, Lemma 7.3). Thus we can factor the map $B \to C \times C'$ through B'. DOing so we may replace B by B' and D by D'. Then we find that $D = D_e \times D_{1-e} = D/(1-e) \times D/(e)$ is a product of finite type A-algebras and the completion of the first part is C and the completion of the second part is C'. □

Lemma 6.5. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be a finite type A-algebra such that* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is étale over* $\mathrm{Spec}(A) \setminus V(I)$. *Let C be a Noetherian A-algebra. Then any A-algebra map $B^\wedge \to C^\wedge$ of I-adic completions comes from a unique A-algebra map*

$$B \longrightarrow C^h$$

where C^h is the henselization of the pair (C, IC) as in More on Algebra, Lemma 7.12. Moreover, any A-algebra homomorphism $B \to C^h$ factors through some étale C-algebra C' such that $C/IC \to C'/IC'$ is an isomorphism.

Proof. Uniqueness follows from the fact that C^h is a subring of $C^\wedge$, see for example More on Algebra, Lemma 7.15. The final assertion follows from the fact that C^h is the filtered colimit of these C-algebras C', see proof of More on Algebra, Lemma 7.12. Having said this we now turn to the proof of existence.

Let $\varphi : B^\wedge \to C^\wedge$ be the given map. This defines a section

$$\sigma : (B \otimes_A C)^\wedge \longrightarrow C^\wedge$$

of the completion of the map $C \to B \otimes_A C$. We may replace (A, I, B, C, φ) by $(C, IC, B \otimes_A C, C, \sigma)$. In this way we see that we may assume that $A = C$.

Proof of existence in the case $A = C$. In this case the map $\varphi : B^\wedge \to A^\wedge$ is necessarily surjective. By Lemmas 6.2 and 5.2 we see that the cohomology groups of $NL^\wedge_{A^\wedge/_\varphi B^\wedge}$ are annihilated by a power of I. Since φ is surjective, this implies that $\mathrm{Ker}(\varphi)/\mathrm{Ker}(\varphi)^2$ is annihilated by a power of I. Hence $\varphi : B^\wedge \to A^\wedge$ is the completion of a finite type B-algebra $B \to D$, see Lemma 6.4. Hence $A \to D$ is a finite type algebra map which induces an isomorphism $A^\wedge \to D^\wedge$. By Lemma 6.2 we may replace D by a localization and assume that $A \to D$ is étale away from $V(I)$. Since $A^\wedge \to D^\wedge$ is an isomorphism, we see that $\mathrm{Spec}(D) \to \mathrm{Spec}(A)$ is also étale in a neighbourhood of $V(ID)$ (for example by More on Morphisms, Lemma 10.3). Thus $\mathrm{Spec}(D) \to \mathrm{Spec}(A)$ is étale. Therefore D maps to A^h and the lemma is proved. □

7. Rig-étale morphisms

We can use the notion introduced in the previous section to define a new type of morphism of locally Noetherian formal algebraic spaces. Before we do so, we have to check it is a local property.

Lemma 7.1. *For morphisms $A \to B$ of the category $WAdm^{Noeth}$ consider the condition P =“for some ideal of definition I of A the topology on B is the I-adic topology, the ring map $A/I \to B/IB$ is of finite type and $A \to B$ satisfies the equivalent conditions of Lemma 6.1”. Then P is a local property, see Formal Spaces, Remark 16.4.*

Proof. We have to show that Formal Spaces, Axioms (1), (2), and (3) hold for maps between Noetherian adic rings. For a Noetherian adic ring A with ideal of definition I we have $A\{x_1, \ldots, x_r\} = A[x_1, \ldots, x_r]^\wedge$ as topological A-algebras (see Remark 2.2). We will use without further mention that we know the axioms hold for the property "B is a quotient of $A[x_1, \ldots, x_r]^\wedge$", see Lemma 3.6.

Let a diagram as in Formal Spaces, Diagram (16.1.1) be given with A and B in the category $\textit{WAdm}^{Noeth}$. Pick an ideal of definition $I \subset A$. By the remarks above the topology on each ring in the diagram is the I-adic topology. Since $A \to A'$ and $B \to B'$ are étale we see that $NL^\wedge_{(A')^\wedge/A}$ and $NL^\wedge_{(B')^\wedge/B}$ are zero. By Lemmas 5.2 and 5.3 we get

$$H^i(NL^\wedge_{(B')^\wedge/(A')^\wedge}) \cong H^i(NL^\wedge_{(B')^\wedge/A}) \quad \text{and} \quad H^i(NL^\wedge_{B/A} \otimes_B (B')^\wedge) \cong H^i(NL^\wedge_{(B')^\wedge/A})$$

for $i = -1, 0$. Since B is Noetherian the ring map $B \to B' \to (B')^\wedge$ is flat (Algebra, Lemma 93.3) hence the tensor product comes out. Moreover, as B is I-adically complete, then if $B \to B'$ is faithfully flat, so is $B \to (B')^\wedge$. From these observations Formal Spaces, Axioms (1) and (2) follow immediately.

We omit the proof of Formal Spaces, Axiom (3). $\square$

Definition 7.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over S. We say f is *rig-étale* if f satisfies the equivalent conditions of Formal Spaces, Lemma 16.2 (in the setting of locally Noetherian formal algebraic spaces, see Formal Spaces, Remark 16.3) for the property P of Lemma 7.1.

To be sure, a rig-étale morphism is locally of finite type.

Lemma 7.3. *A rig-étale morphism of locally Noetherian formal algebraic spaces is locally of finite type.*

Proof. The property P in Lemma 7.1 implies the equivalent conditions (a), (b), (c), and (d) in Lemma 3.6. Hence this follows from Lemma 3.8. $\square$

8. Glueing rings along a principal ideal

In this situation we prove some results about the categories $\mathcal{C}$ and $\mathcal{C}'$ of Section 4 in case A is a Noetherian ring and $I = (a)$ is a principal ideal.

Remark 8.1 (Linear approximation)**.** Let A be a ring and $I \subset A$ be a finitely generated ideal. Let C be an I-adically complete A-algebra. Let $\psi : A[x_1, \ldots, x_r]^\wedge \to C$ be a continuous A-algebra map. Suppose given $\delta_i \in C$, $i = 1, \ldots, r$. Then we can consider

$$\psi' : A[x_1, \ldots, x_r]^\wedge \to C, \quad x_i \longmapsto \psi(x_i) + \delta_i$$

see Remark 2.1. Then we have

$$\psi'(g) = \psi(g) + \sum \psi(\partial g/\partial x_i)\delta_i + \xi$$

with error term $\xi \in (\delta_i\delta_j)$. This follows by writing g as a power series and working term by term. Convergence is automatic as the coefficients of g tend to zero. Details omitted.

Lemma 8.2. *Let A be a Noetherian ring and $I = (a)$ a principal ideal. Let B be an objects of (4.0.2). Assume given an integer $c \geq 0$ such that multiplication by a^c on $NL^\wedge_{B/A}$ is zero in $D(B)$. Let C be an I-adically complete A-algebra such that a is a nonzerodivisor on C. Let $n > 2c$. For any A_n-algebra map $\psi_n : B/a^nB \to C/a^nC$ there exists an A-algebra map $\varphi : B \to C$ such that $\psi_n \bmod a^{n-c} = \varphi \bmod a^{n-c}$.*

Proof. Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Choose a lift

$$\psi : A[x_1, \ldots, x_r]^\wedge \to C$$

of ψ_n. Then $\psi(J) \subset a^nC$ and $\psi(J^2) \subset a^{2n}C$ which determines a linear map

$$J/J^2 \longrightarrow a^nC/a^{2n}C, \quad g \longmapsto \psi(g)$$

By assumption and Lemma 5.4 there is a B-module map $\bigoplus B\mathrm{d}x_i \to a^nC/a^{2n}C$, $\mathrm{d}x_i \mapsto \delta_i$ such that $a^c\psi(g) = \sum \psi(\partial g/\partial x_i)\delta_i$ for all $g \in J$. Write $\delta_i = -a^c\delta'_i$ for some $\delta'_i \in a^{n-c}C$. Since a is a nonzerodivisor on C we see that $\psi(g) = -\sum \psi(\partial g/\partial x_i)\delta'_i$ in $C/a^{2n-c}C$. Then we look at the map

$$\psi' : A[x_1, \ldots, x_r]^\wedge \to C, \quad x_i \longmapsto \psi(x_i) + \delta'_i$$

A computation with power series (see Remark 8.1) shows that $\psi'(J) \subset a^{2n-2c}C$. Since $n > 2c$ we see that $n' = 2n - 2c = n + (n - 2c) > n$. Thus we obtain a morphism $\psi_{n'} : B/a^{n'}B \to C/a^{n'}C$ agreeing with ψ_n modulo a^{n-c}. Continuing in this fashion and taking the limit into $C = \lim C/a^tC$ we obtain the lemma. □

Lemma 8.3. *Let A be a Noetherian ring and $I = (a)$ a principal ideal. Let B be an object of (4.0.2). Assume given an integer $c \geq 0$ such that multiplication by a^c on $NL^\wedge_{B/A}$ is zero in $D(B)$. Let C be an I-adically complete A-algebra. Assume given an integer $d \geq 0$ such that $C[a^\infty] \cap a^dC = 0$. Let $n > \max(2c, c+d)$. For any A_n-algebra map $\psi_n : B/a^nB \to C/a^nC$ there exists an A-algebra map $\varphi : B \to C$ such that $\psi_n \bmod a^{n-c} = \varphi \bmod a^{n-c}$.*

If C is Noetherian we have $C[a^\infty] = C[a^e]$ for some $e \geq 0$. By Artin-Rees (Algebra, Lemma 49.2) there exists an integer f such that $a^nC \cap C[a^\infty] \subset a^{n-f}C[a^\infty]$ for all $n \geq f$. Then $d = e + f$ is an integer as in the lemma. This argument works in particular if C is an object of (4.0.2) by Lemma 4.2.

Proof. Let $C \to C'$ be the quotient of C by $C[a^\infty]$. The A-algebra C' is I-adically complete by Algebra, Lemma 93.15 and the fact that $\bigcap(C[a^\infty] + a^nC) = C[a^\infty]$ because for $n \geq d$ the sum $C[a^\infty] + a^nC$ is direct. For $m \geq d$ the diagram

$$\begin{tikzcd} 0 \arrow[r] & C[a^\infty] \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C[a^\infty] \arrow[r] & C/a^mC \arrow[r] & C'/a^mC' \arrow[r] & 0 \end{tikzcd}$$

has exact rows. Thus C is the fibre product of C' and C/a^mC over C'/a^mC'. Thus the lemma now follows formally from the lifting result of Lemma 8.2. □

Lemma 8.4. *Let A be a Noetherian ring and $I = (a)$ a principal ideal. Let B be an object of (4.0.2). Assume given an integer $c \geq 0$ such that multiplication by a^c on $NL^\wedge_{B/A}$ is zero in $D(B)$. Then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$.*

Proof. Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. By Lemma 5.4 we can find a map $\alpha : \bigoplus B\mathrm{d}x_i \to J/J^2$ such that $\mathrm{d} \circ \alpha$ and $\alpha \circ \mathrm{d}$ are both multiplication by a^c. Pick an element $f_i \in J$ whose class modulo J^2 is equal to $\alpha(\mathrm{d}x_i)$. Then we see that $\mathrm{d}f_i = a^c\mathrm{d}x_i$ in $\bigoplus \mathrm{d}x_i$. In particular we have a ring map

$$A[x_1, \ldots, x_r]^\wedge/(f_1, \ldots, f_r, \Delta(f_1, \ldots, f_r) - a^{rc}) \longrightarrow B$$

where $\Delta(f_1, \ldots, f_r) \in A[x_1, \ldots, x_r]^\wedge$ is the determinant of the matrix of partial derivatives of the f_i.

Pick a large integer N. Pick $F_1, \ldots, F_r \in A[x_1, \ldots, x_r]$ such that $F_i - f_i \in I^N A[x_1, \ldots, x_r]^\wedge$. Set

$$C = A[x_1, \ldots, x_r, z]/(F_1, \ldots, F_r, z\Delta(F_1, \ldots, F_r) - a^{rc})$$

We claim that multplication by a^{2rc} is zero on $NL_{C/A}$ in $D(C)$. Namely, the determinant of the matrix of the partial derivatives of the $r + 1$ generators of the ideal of C with respect to the variables $x_1, \ldots, x_{r+1}, z$ is $\Delta(F_1, \ldots, F_r)^2$. Since $\Delta(F_1, \ldots, F_r)$ divides a^{rc} we in C the claim follows for example from Algebra, Lemma 14.4. Let $C^\wedge$ be the I-adic completion of C. Since $NL^\wedge_{C^\wedge/A}$ is the I-adic completion of $NL_{C/A}$ we conclude that multiplication by a^{2rc} is zero on $NL^\wedge_{C^\wedge/A}$ as well.

By construction there is a (surjective) map $\psi_N : C/I^N C \to B/I^N B$ sending x_i to x_i and z to 1. By Lemma 8.3 (with the roles of B and C reversed) for N large enough we get a map $\varphi : C^\wedge \to B$ which agrees with ψ_N modulo I^{N-2rc}.

Since $\varphi : C^\wedge \to B$ is surjective modulo I we see that it is surjective (for example use Algebra, Lemma 93.1). By construction and assumption the naive cotangent complexes $NL^\wedge_{C^\wedge/A}$ and $NL^\wedge_{B/A}$ have cohomology annihilated by a fixed power of a. Thus the same thing is true for $NL^\wedge_{B/C^\wedge}$ by Lemma 5.2. Since φ is surjective we conclude that $\mathrm{Ker}(\varphi)/\mathrm{Ker}(\varphi)^2$ is annihilated by a power of a. The result of the lemma now follows from Lemma 6.4. □

9. Glueing rings along an ideal

Let A be a Noetherian ring. Let $I \subset A$ be an ideal. In this section we study I-adically complete A-algebras which are, in some vague sense, étale over the complement of $V(I)$ in $\mathrm{Spec}(A)$.

Lemma 9.1. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let t be the minimal number of generators for I. Let C be a Noetherian I-adically complete A-algebra. There exists an integer $d \geq 0$ depending only on $I \subset A \to C$ with the following property: given*

1. *$c \geq 0$ and B in (4.0.2) such that for $a \in I^c$ multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$,*
2. *an integer $n > 2t\max(c, d)$,*
3. *an A/I^n-algebra map $\psi_n : B/I^nB \to C/I^nC$,*

there exists a map $\varphi : B \to C$ of A-algebras such that $\psi_n \bmod I^{m-c} = \varphi \bmod I^{m-c}$ with $m = \lfloor \frac{n}{t} \rfloor$.

Proof. We prove this lemma by induction on the number of generators of I. Say $I = (a_1, \ldots, a_t)$. If $t = 0$, then $I = 0$ and there is nothing to prove. If $t = 1$, then the lemma follows from Lemma 8.3 because $2\max(c,d) \geq \max(2c, c+d)$. Assume $t > 1$.

Set $m = \lfloor \frac{n}{t} \rfloor$ as in the lemma. Set $\bar{A} = A/(a_t^m)$. Consider the ideal $\bar{I} = (\bar{a}_1, \ldots, \bar{a}_{t-1})$ in $\bar{A}$. Set $\bar{C} = C/(a_t^m)$. Note that $\bar{C}$ is a $\bar{I}$-adically complete Noetherian $\bar{A}$-algebra (use Algebra, Lemmas 93.2 and 93.14). Let $\bar{d}$ be the integer for $\bar{I} \subset \bar{A} \to \bar{C}$ which exists by induction hypothesis.

Let $d_1 \geq 0$ be an integer such that $C[a_t^\infty] \cap a_t^{d_1} C = 0$ as in Lemma 8.3 (see discussion following the lemma and before the proof).

We claim the lemma holds with $d = \max(\bar{d}, d_1)$. To see this, let c, B, n, ψ_n be as in the lemma.

Note that $\bar{I} \subset I\bar{A}$. Hence by Lemma 6.3 multiplication by an element of $\bar{I}^c$ on the cotangent complex of $\bar{B} = B/(a_t^m)$ is zero in $D(\bar{B})$. Also, we have

$$\bar{I}^{n-m+1} \supset I^n \bar{A}$$

Thus ψ_n gives rise to a map

$$\bar{\psi}_{n-m+1} : \bar{B}/\bar{I}^{n-m+1}\bar{B} \longrightarrow \bar{C}/\bar{I}^{n-m+1}\bar{C}$$

Since $n > 2t\max(c,d)$ and $d \geq \bar{d}$ we see that

$$n - m + 1 \geq (t-1)n/t > 2(t-1)\max(c,d) \geq 2(t-1)\max(c,\bar{d})$$

Hence we can find a morphism $\varphi_m : \bar{B} \to \bar{C}$ agreeing with $\bar{\psi}_{n-m+1}$ modulo the ideal $\bar{I}^{m'-c}$ where $m' = \lfloor \frac{n-m+1}{t-1} \rfloor$.

Since $m \geq n/t > 2\max(c,d) \geq 2\max(c,d_1) \geq \max(2c, c+d_1)$, we can apply Lemma 8.3 for the ring map $A \to B$ and the ideal (a_t) to find a morphism $\varphi : B \to C$ agreeing modulo a_t^{m-c} with φ_m.

All in all we find $\varphi : B \to C$ which agrees with ψ_n modulo

$$(a_t^{m-c}) + (a_1, \ldots, a_{t-1})^{m'-c} \subset I^{\min(m-c, m'-c)}$$

We leave it to the reader to see that $\min(m-c, m'-c) = m - c$. This concludes the proof. $\square$

Lemma 9.2. *Let A be a Noetherian ring and $I \subset A$ an ideal. Let $J \subset A$ be a nilpotent ideal. Consider a diagram*

$$\begin{tikzcd}
C \arrow[r] & C/JC \\
 & B_0 \arrow[u] \\
A \arrow[uu] \arrow[r] & A/J \arrow[u]
\end{tikzcd}$$

whose vertical arrows are of finite type such that

1. $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ *is étale over* $\mathrm{Spec}(A) \setminus V(I)$,
2. $\mathrm{Spec}(B_0) \to \mathrm{Spec}(A/J)$ *is étale over* $\mathrm{Spec}(A/J) \setminus V((I+J)/J)$, *and*
3. $B_0 \to C/JC$ *is étale and induces an isomorphism* $B_0/IB_0 = C/(I+J)C$.

Then we can fill in the diagram

$$
\begin{tikzcd}
C \arrow[r] & C/JC \\
B \arrow[u] \arrow[r] & B_0 \arrow[u] \\
A \arrow[u] \arrow[r] & A/J \arrow[u]
\end{tikzcd}
$$

with $A \to B$ *of finite type,* $B/JB = B_0$*,* $B \to C$ *étale, and* $\text{Spec}(B) \to \text{Spec}(A)$ *étale over* $\text{Spec}(A) \setminus V(I)$*.*

Proof. By induction on the smallest n such that $J^n = 0$ we reduce to the case $J^2 = 0$. Denote by a subscript zero the base change of objects to $A_0 = A/J$. Since $J^2 = 0$ we see that JC is a C_0-module.

Consider the canonical map

$$\gamma : J \otimes_{A_0} C_0 \longrightarrow JC$$

Since $\text{Spec}(C) \to \text{Spec}(A)$ is étale over the complement of $V(I)$ (and hence flat) we see that γ is an isomorphism away from $V(IC_0)$, see More on Morphisms, Lemma 8.1. In particular, the kernel and cokernel of γ are annihilated by a power of I (use that C_0 is Noetherian and that the modules in question are finite). Observe that $J \otimes_{A_0} C_0 = (J \otimes_{A_0} B_0) \otimes_{B_0} C_0$. Hence by More on Algebra, Lemma 63.16 there exists a unique B_0-module homomorphism

$$c : J \otimes_{A_0} B_0 \to N$$

with $c \otimes \text{id}_{C_0} = \gamma$ and $\text{Ker}(\gamma) = \text{Ker}(c)$ and $\text{Coker}(\gamma) = \text{Coker}(c)$. Moreover, N is a finite B_0-module, see More on Algebra, Remark 63.19.

Choose a presentation $B_0 = A[x_1, \ldots, x_r]/K$. To construct B we try to find the dotted arrow m fitting into the following pushout diagram

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[r, dotted] & B \arrow[r, dotted] & B_0 \arrow[r] & 0 \\
0 \arrow[r] & K/K^2 \arrow[u, dotted, "m"] \arrow[r] & A[x_1, \ldots, x_r]/K^2 \arrow[u, dotted] \arrow[r] & A[x_1, \ldots, x_r]/K \arrow[u, equal] \arrow[r] & 0 \\
& J \otimes_{A_0} B_0 \arrow[u] \arrow[uu, bend left, "c"] & & &
\end{tikzcd}
$$

where the curved arrow is the map c constructed above and the map $J \otimes_{A_0} B_0 \to K/K^2$ is the obvious one.

As $B_0 \to C_0$ is étale we can write $C_0 = B_0[y_1, \ldots, y_r]/(g_{0,1}, \ldots, g_{0,r})$ such that the determinant of the partial derivatives of the $g_{0,j}$ is invertible in C_0, see Algebra, Lemma 138.2. We combine this with the chosen presentation of B_0 to get a presentation $C_0 = A[x_1, \ldots, x_r, y_1, \ldots, y_s]/L$. Choose a lift $\psi : A[x_i, y_j] \to C$ of the map to C_0. Then it is the case that C fits into the diagram

$$\begin{tikzcd}
0 \arrow[r] & JC \arrow[r] & C \arrow[r] & C_0 \arrow[r] & 0 \\
0 \arrow[r] & L/L^2 \arrow[r] \arrow[u, "\mu"'] & A[x_i, y_j]/L^2 \arrow[r] \arrow[u] & A[x_i, y_j]/L \arrow[r] \arrow[u, equal] & 0 \\
& J \otimes_{A_0} C_0 \arrow[u] \arrow[uu, bend left] & & &
\end{tikzcd}$$

where the curved arrow is the map γ constructed above and the map $J \otimes_{A_0} C_0 \to L/L^2$ is the obvious one. By our choice of presentations and the fact that C_0 is a complete intersection over B_0 we have

$$L/L^2 = K/K^2 \otimes_{B_0} C_0 \oplus \bigoplus C_0 g_j$$

where $g_j \in L$ is any lift of $g_{0,j}$, see More on Algebra, Lemma 23.6.

Consider the three term complex

$$K^\bullet : J \otimes_{A_0} B_0 \to K/K^2 \to \bigoplus B_0 \mathrm{d}x_i$$

where the second arrow is the differential in the naive cotangent complex of B_0 over A for the given presentation and the last term is placed in degree 0. Since $\mathrm{Spec}(B_0) \to \mathrm{Spec}(A_0)$ is étale away from $V(I)$ the cohomology modules of this complex are supported on $V(IB_0)$. Namely, for $a \in I$ after inverting a we can apply More on Algebra, Lemma 23.6 for the ring maps $A_a \to A_{0,a} \to B_{0,a}$ and use that $NL_{A_{0,a}/A_a} = J_a$ and $NL_{B_{0,a}/A_{0,a}} = 0$ (some details omitted). Hence these cohomology groups are annihilated by a power of I.

Similarly, consider the three term complex

$$L^\bullet : J \otimes_{A_0} C_0 \to L/L^2 \to \bigoplus C_0 \mathrm{d}x_i \oplus \bigoplus C_0 \mathrm{d}y_j$$

By our direct sum decomposition of L/L^2 above and the fact that the the determinant of the partial derivatives of the $g_{0,j}$ is invertible in C_0 we see that the natural map $K^\bullet \to L^\bullet$ induces a quasi-isomorphism

$$K^\bullet \otimes_{B_0} C_0 \longrightarrow L^\bullet$$

Applying Dualizing Complexes, Lemma 8.14 we find that

$$\mathrm{Hom}_{D(B_0)}(K^\bullet, E) = \mathrm{Hom}_{D(C_0)}(L^\bullet, E \otimes_{B_0} C_0) \tag{9.2.1}$$

for any object $E \in D(B_0)$.

The maps $\mathrm{id}_{J \otimes_{A_0} C_0}$ and μ define an element in

$$\mathrm{Hom}_{D(C_0)}(L^\bullet, (J \otimes_{A_0} C_0 \to JC))$$

(the target two term complex is placed in degree -2 and -1) such that the composition with the map to $J \otimes_{A_0} C_0[2]$ is the element in $\mathrm{Hom}_{D(C_0)}(L^\bullet, J \otimes_{A_0} C_0[2])$ corresponding to $\mathrm{id}_{J \otimes_{A_0} C_0}$. Picture

$$\begin{tikzcd}
J \otimes_{A_0} C_0 \arrow[r] \arrow[d, "\mathrm{id}_{J \otimes_{A_0} C_0}"'] & L/L^2 \arrow[r] \arrow[d, "\mu"] & \bigoplus C_0 \mathrm{d}x_i \oplus \bigoplus C_0 \mathrm{d}y_j \\
J \otimes_{A_0} C_0 \arrow[r, "\gamma"] & JC &
\end{tikzcd}$$

Applying (9.2.1) we obtain a unique element

$$\xi \in \mathrm{Hom}_{D(B_0)}(K^\bullet, (J \otimes_{A_0} B_0 \to N))$$

Its composition with the map to $J \otimes_{A_0} B_0[2]$ is the element in $\mathrm{Hom}_{D(C_0)}(K^\bullet, J \otimes_{A_0} B_0[2])$ corresponding to $\mathrm{id}_{J \otimes_{A_0} B_0}$. By Lemma 5.4 we can find a map of complexes $K^\bullet \to (J \otimes_{A_0} B_0 \to N)$ representing ξ and equal to $\mathrm{id}_{J \otimes_{A_0} B_0}$ in degree -2. Denote $m : K/K^2 \to N$ the degree -1 part of this map. Picture

$$\begin{tikzcd}
J \otimes_{A_0} B_0 \arrow[r] \arrow[d, "\mathrm{id}_{J \otimes_{A_0} B_0}"'] & K/K^2 \arrow[r] \arrow[d, "m"] & \bigoplus B_0 \mathrm{d}x_i \\
J \otimes_{A_0} B_0 \arrow[r, "c"] & N &
\end{tikzcd}$$

Thus we can use m to create an algebra B by push out as explained above. However, we may still have to change m a bit to make sure that B maps to C in the correct manner.

Denote $m \otimes \mathrm{id}_{C_0} \oplus 0 : L/L^2 \to JC$ the map coming from the direct sum decomposition of L/L^2 (see above), using that $N \otimes_{B_0} C_0 = JC$, and using 0 on the second factor. By our choice of m above the maps of complexes $(\mathrm{id}_{J \otimes_{A_0} C_0}, \mu, 0)$ and $(\mathrm{id}_{J \otimes_{A_0} C_0}, m \otimes \mathrm{id}_{C_0} \oplus 0, 0)$ define the same element of $\mathrm{Hom}_{D(C_0)}(L^\bullet, (J \otimes_{A_0} C_0 \to JC))$. By Lemma 5.4 there exist maps $h : L^{-1} \to J \otimes_{A_0} C_0$ and $h' : L^0 \to JC$ which define a homotopy between $(\mathrm{id}_{J \otimes_{A_0} C_0}, \mu, 0)$ and $(\mathrm{id}_{J \otimes_{A_0} C_0}, m \otimes \mathrm{id}_{C_0} \oplus 0, 0)$. Picture

$$\begin{tikzcd}
J \otimes_{A_0} C_0 \arrow[r] \arrow[d, "\mathrm{id}_{J \otimes_{A_0} C_0}"'] & K/K^2 \otimes_{B_0} C_0 \oplus \bigoplus C_0 g_j \arrow[r] \arrow[dl, dotted, "h"'] \arrow[d, shift right, "\mu"'] \arrow[d, shift left, "m \otimes \mathrm{id}_{C_0} \oplus 0"] & \bigoplus C_0 \mathrm{d}x_i \oplus \bigoplus C_0 \mathrm{d}y_j \arrow[dl, dotted, "h'"'] \\
J \otimes_{A_0} C_0 \arrow[r, "\gamma"] & JC &
\end{tikzcd}$$

Since h precomposed with d_L^{-2} is zero it defines an element in $\mathrm{Hom}_{D(C_0)}(L^\bullet, J \otimes_{A_0} C_0[1])$ which comes from a unique element χ of $\mathrm{Hom}_{D(B_0)}(K^\bullet, J \otimes_{A_0} B_0[1])$ by (9.2.1). Applying Lemma 5.4 again we represent χ by a map $g : K/K^2 \to J \otimes_{A_0} B_0$. Then the base change $g \otimes \mathrm{id}_{C_0}$ and h differ by a homotopy $h'' : L^0 \to J \otimes_{A_0} C$. Hence if we modify m into $m + c \circ g$, then we find that $m \otimes \mathrm{id}_{C_0} \oplus 0$ and μ just differ by a map $h' : L^0 \to JC$.

Changing our choice of the map $\psi : A[x_i, y_j] \to C$ by sending x_i to $\psi(x_i) + h'(\mathrm{d}x_i)$ and sending y_j to $\psi(y_j) + h'(\mathrm{d}y_j)$, we find a commutative diagram

$$\begin{tikzcd}
N \arrow[r] & JC \\
K/K^2 \arrow[r] \arrow[u, "m"] & L/L^2 \arrow[u, "\mu"'] \\
J \otimes_{A_0} B_0 \arrow[r] \arrow[u] \arrow[uu, bend left, "c"] & J \otimes_{A_0} C_0 \arrow[u] \arrow[uu, bend right, "\gamma"']
\end{tikzcd}$$

At this point we can define B as the pushout in the first commutative diagram of the proof. The commutativity of the diagram just displayed, shows that there is an A-algebra map $B \to C$ compatible with the given map $N = JB \to JC$. As

$N \otimes_{B_0} C_0 = JC$ it follows from More on Morphisms, Lemma 8.1 that $B \to C$ is flat. From this it easily follows that it is étale. We omit the proof of the other properties as they are mostly self evident at this point. $\square$

Lemma 9.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be an object of (4.0.2). Assume there is an integer $c \geq 0$ such that for $a \in I^c$ multiplication by a on $NL^\wedge_{B/A}$ is zero in $D(B)$. Then there exists a finite type A-algebra C and an isomorphism $B \cong C^\wedge$.*

In Section 10 we will give a simpler proof of this result in case A is a G-ring.

Proof. We prove this lemma by induction on the number of generators of I. Say $I = (a_1, \ldots, a_t)$. If $t = 0$, then $I = 0$ and there is nothing to prove. If $t = 1$, then the lemma follows from Lemma 8.4. Assume $t > 1$.

For any $m \geq 1$ set $\bar{A}_m = A/(a_t^m)$. Consider the ideal $\bar{I}_m = (\bar{a}_1, \ldots, \bar{a}_{t-1})$ in $\bar{A}_m$. Let $B_m = B/(a_t^m)$ be the base change of B for the map $(A, I) \to (\bar{A}_m, \bar{I}_m)$, see (4.4.1). By Lemma 6.3 the assumption of the lemma holds for $\bar{I}_m \subset \bar{A}_m \to B_m$.

By induction hypothesis (on t) we can find a finite type $\bar{A}_m$-algebra C_m and a map $C_m \to B_m$ which induces an isomorphism $C_m^\wedge \cong B_m$ where the completion is with respect to $\bar{I}_m$. By Lemma 6.2 we may assume that $\mathrm{Spec}(C_m) \to \mathrm{Spec}(\bar{A}_m)$ is étale over $\mathrm{Spec}(\bar{A}_m) \setminus V(\bar{I}_m)$.

We claim that we may choose $A_m \to C_m \to B_m$ as in the previous paragraph such that moreover there are isomorphisms $C_m/(a_t^{m-1}) \to C_{m-1}$ compatible with the given A-algebra structure and the maps to $B_{m-1} = B_m/(a_t^{m-1})$. Namely, first fix a choice of $A_1 \to C_1 \to B_1$. Suppose we have found $C_{m-1} \to C_{m-2} \to \ldots \to C_1$ with the desired properties. Note that $C_m/(a_t^{m-1})$ is étale over $\mathrm{Spec}(\bar{A}_{m-1}) \setminus V(\bar{I}_{m-1})$. Hence by Lemma 6.5 there exists an étale extension $C_{m-1} \to C'_{m-1}$ which induces an isomorphism modulo $\bar{I}_{m-1}$ and an $\bar{A}_{m-1}$-algebra map $C_m/(a_t^{m-1}) \to C'_{m-1}$ inducing the isomorphism $B_m/(a_t^{m-1}) \to B_{m-1}$ on completions. Note that $C_m/(a_t^{m-1}) \to C'_{m-1}$ is étale over the complement of $V(\bar{I}_{m-1})$ by Morphisms, Lemma 37.18 and over $V(\bar{I}_{m-1})$ induces an isomorphism on completions hence is étale there too (for example by More on Morphisms, Lemma 10.3). Thus $C_m/(a_t^{m-1}) \to C'_{m-1}$ is étale. By the topological invariance of étale morphisms (Étale Morphisms, Theorem 15.2) there exists an étale ring map $C_m \to C'_m$ such that $C_m/(a_t^{m-1}) \to C'_{m-1}$ is isomorphic to $C_m/(a_t^{m-1}) \to C'_m/(a_t^{m-1})$. Observe that the $\bar{I}_m$-adic completion of C'_m is equal to the $\bar{I}_m$-adic completion of C_m, i.e., to B_m (details omitted). We apply Lemma 9.2 to the diagram

$$\begin{tikzcd}
& C'_m \arrow[r] & C'_m/(a_t^{m-1}) \\
C''_m \arrow[ur, dotted] \arrow[rr, dotted] & & C_{m-1} \arrow[u] \\
& \bar{A}_m \arrow[uu] \arrow[r] \arrow[ul, dotted] & \bar{A}_{m-1} \arrow[u]
\end{tikzcd}$$

to see that there exists a "lift" of C''_m of C_{m-1} to an algebra over $\bar{A}_m$ with all the desired properties.

By construction (C_m) is an object of the category (4.0.1) for the principal ideal (a_t). Thus the inverse limit $B' = \lim C_m$ is an (a_t)-adically complete A-algebra such that B'/a_tB' is of finite type over $A/(a_t)$, see Lemma 4.1. By construction the I-adic completion of B' is isomorphic to B (details omitted). Consider the complex $NL^\wedge_{B'/A}$ constructed using the (a_t)-adic topology. Choosing a presentation for B' (which induces a similar presentation for B) the reader immediately sees that $NL^\wedge_{B'/A} \otimes_{B'} B = NL^\wedge_{B/A}$. Since $a_t \in I$ and since the cohomology modules of $NL^\wedge_{B'/A}$ are finite B'-modules (hence complete for the a_t-adic topology), we conclude that a_t^c acts as zero on these cohomologies as the same thing is true by assumption for $NL^\wedge_{B/A}$. Thus multiplication by a_t^{2c} is zero on $NL^\wedge_{B'/A}$ by Lemma 5.5. Hence finally, we may apply Lemma 8.4 to $(a_t) \subset A \to B'$ to finish the proof. □

Lemma 9.4. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let B be an I-adically complete A-algebra with $A/I \to B/IB$ of finite type. The equivalent conditions of Lemma 6.1 are also equivalent to*

(5) *there exists a finite type A-algebra C with $\text{Spec}(C) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$ such that $B \cong C^\wedge$.*

Proof. First, assume conditions (1) – (4) hold. Then there exists a finite type A-algebra C with such that $B \cong C^\wedge$ by Lemma 9.3. In other words, $B_n = C/I^nC$. The naive cotangent complex $NL_{C/A}$ is a complex of finite type C-modules and hence H^{-1} and H^0 are finite C-modules. By assumption there exists a $c \geq 0$ such that H^{-1}/I^nH^{-1} and H^0/I^nH^0 are annihilated by I^c for some n. By Nakayama's lemma this means that I^cH^{-1} and I^cH^0 are annihilated by an element of the form $f = 1 + x$ with $x \in IC$. After inverting f (which does not change the quotients $B_n = C/I^nC$) we see that $NL_{C/A}$ has cohomology annihilated by I^c. Thus $A \to C$ is étale at any prime of C not lying over $V(I)$ by the definition of étale ring maps, see Algebra, Definition 138.1.

Conversely, assume that $A \to C$ of finite type is given such that $\text{Spec}(C) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Then for every $a \in I$ there exists an $c \geq 0$ such that multiplication by a^c is zero $NL_{C/A}$. Since $NL^\wedge_{C^\wedge/A}$ is the derived completion of $NL_{C/A}$ (see Lemma 5.1) it follows that $B = C^\wedge$ satisfies the equivalent conditions of Lemma 6.1. □

10. In case the base ring is a G-ring

If the base ring A is a Noetherian G-ring, then some of the material above simplifies somewhat and we obtain some additional results.

Proof of Lemma 9.3 in case A is a G-ring. This proof is easier in that it does not depend on the somewhat delicate deformation theory argument given in the proof of Lemma 9.2, but of course it requires a very strong assumption on the Noetherian ring A.

Choose a presentation $B = A[x_1, \ldots, x_r]^\wedge/J$. Choose generators $g_1, \ldots, g_m \in J$. Choose generators $k_1, \ldots, k_t$ of the module of relations between $g_1, \ldots, g_m$, i.e., such that

$$(A[x_1, \ldots, x_r]^\wedge)^{\oplus t} \xrightarrow{k_1, \ldots, k_t} (A[x_1, \ldots, x_r]^\wedge)^{\oplus m} \xrightarrow{g_1, \ldots, g_m} A[x_1, \ldots, x_r]^\wedge$$

is exact in the middle. Write $k_i = (k_{i1}, \ldots, k_{im})$ so that we have

$$\sum k_{ij} g_j = 0 \tag{10.0.1}$$

for $i = 1, \ldots, t$. Let $I^c = (a_1, \ldots, a_s)$. For each $l \in \{1, \ldots, s\}$ we know that multiplication by a_l on $NL^\wedge_{B/A}$ is zero in $D(B)$. By Lemma 5.4 we can find a map $\alpha_l : \bigoplus B\mathrm{d}x_i \to J/J^2$ such that $\mathrm{d} \circ \alpha_l$ and $\alpha_l \circ \mathrm{d}$ are both multiplication by a_l. Pick an element $f_{l,i} \in J$ whose class modulo J^2 is equal to $\alpha_l(\mathrm{d}x_i)$. Then we have for all $l = 1, \ldots, s$ and $i = 1, \ldots, r$ that

$$\sum\nolimits_{i'} (\partial f_{l,i}/\partial x_{i'})\mathrm{d}x_{i'} = a_l \mathrm{d}x_i + \sum h_{l,i}^{j',i'} g_{j'} \mathrm{d}x_{i'} \tag{10.0.2}$$

for some $h_{l,i}^{j',i'} \in A[x_1, \ldots, x_r]^\wedge$. We also have for $j = 1, \ldots, m$ and $l = 1, \ldots, s$ that

$$a_l g_j = \sum h_{l,j}^i f_{l,i} + \sum h_{l,j}^{j',j''} g_{j'} g_{j''} \tag{10.0.3}$$

for some $h_{l,j}^i$ and $h_{l,j}^{j',j''}$ in $A[x_1, \ldots, x_r]^\wedge$. Of course, since $f_{l,i} \in J$ we can write for $l = 1, \ldots, s$ and $i = 1, \ldots, r$

$$f_{l,i} = \sum h_{l,i}^j g_j \tag{10.0.4}$$

for some $h_{l,i}^j$ in $A[x_1, \ldots, x_r]^\wedge$.

Let $A[x_1, \ldots, x_r]^h$ be the henselization of the pair $(A[x_1, \ldots, x_r], IA[x_1, \ldots, x_r])$, see More on Algebra, Lemma 7.12. Since A is a Noetherian G-ring, so is $A[x_1, \ldots, x_r]$, see More on Algebra, Proposition 39.10. Hence we have approximation for the map $A[x_1, \ldots, x_r]^h \to A[x_1, \ldots, x_r]^\wedge$ with respect to the ideal generated by I, see Smoothing Ring Maps, Lemma 14.1. Choose a large integer M. Choose

$$G_j, K_{ij}, F_{l,i}, H_{l,j}^i, H_{l,j}^{j',j''}, H_{l,i}^j \in A[x_1, \ldots, x_r]^h$$

such that analogues of equations (10.0.1), (10.0.3), and (10.0.4) hold for these elements in $A[x_1, \ldots, x_r]^h$, i.e.,

$$\sum K_{ij} G_j = 0, \quad a_l G_j = \sum H_{l,j}^i F_{l,i} + \sum H_{l,j}^{j',j''} G_{j'} G_{j''}, \quad F_{l,i} = \sum H_{l,i}^j G_j$$

and such that we have

$$G_j - g_j, K_{ij} - k_{ij}, F_{l,i} - f_{l,i}, H_{l,j}^i - h_{l,j}^i, H_{l,j}^{j',j''} - h_{l,j}^{j',j''}, H_{l,i}^j - h_{l,i}^j \in I^M A[x_1, \ldots, x_r]^h$$

where we take liberty of thinking of $A[x_1, \ldots, x_r]^h$ as a subring of $A[x_1, \ldots, x_r]^\wedge$. Note that we cannot guarantee that the analogue of (10.0.2) holds in $A[x_1, \ldots, x_r]^h$, because it is not a polynomial equation. But since taking partial derivatives is A-linear, we do get the analogue modulo I^M. More precisely, we see that

$$\sum\nolimits_{i'} (\partial F_{l,i}/\partial x_{i'})\mathrm{d}x_{i'} - a_l \mathrm{d}x_i - \sum h_{l,i}^{j',i'} G_{j'} \mathrm{d}x_{i'} \in I^M A[x_1, \ldots, x_r]^\wedge \tag{10.0.5}$$

for $l = 1, \ldots, s$ and $i = 1, \ldots, r$.

With these choices, consider the ring

$$C^h = A[x_1, \ldots, x_r]^h / (G_1, \ldots, G_r)$$

and denote $C^\wedge$ its I-adic completion, namely

$$C^\wedge = A[x_1, \ldots, x_r]^\wedge / J', \quad J' = (G_1, \ldots, G_r) A[x_1, \ldots, x_r]^\wedge$$

In the following paragraphs we esthablish the fact that $C^\wedge$ is isomorphic to B. Then in the final paragraph we deal with show that C^h comes from a finite type algebra over A as in the statement of the lemma.

First consider the cokernel

$$\Omega = \mathrm{Coker}(J'/(J')^2 \longrightarrow \bigoplus C^\wedge \mathrm{d}x_i)$$

This $C^\wedge$ module is generated by the images of the elements $\mathrm{d}x_i$. Since $F_{l,i} \in J'$ by the analogue of (10.0.4) we see from (10.0.5) we see that $a_l \mathrm{d}x_i \in I^M\Omega$. As $I^c = (a_l)$ we see that $I^c\Omega \subset I^M\Omega$. Since $M > c$ we conclude that $I^c\Omega = 0$ by Algebra, Lemma 19.1.

Next, consider the kernel

$$H_1 = \mathrm{Ker}(J'/(J')^2 \longrightarrow \bigoplus C^\wedge \mathrm{d}x_i)$$

By the analogue of (10.0.3) we see that $a_l J' \subset (F_{l,i}) + (J')^2$. On the other hand, the determinant Δ_l of the matrix $(\partial F_{l,i}/\partial x_{i'})$ satisfies $\Delta_l = a_l^r \bmod I^M C^\wedge$ by (10.0.5). It follows that $a_l^{r+1}H_1 \subset I^M H_1$ (some details omitted; use Algebra, Lemma 14.4). Now $(a_1^{r+1}, \ldots, a_s^{r+1}) \supset I^{(sr+1)c}$. Hence $I^{(sr+1)c}H_1 \subset I^M H_1$ and since $M > (sr+1)c$ we conclude that $I^{(sr+1)c}H_1 = 0$.

By Lemma 5.5 we conclude that multiplication by an element of $I^{2(sr+1)c}$ on $NL^\wedge_{C^\wedge/A}$ is zero (note that the bound does not depend on M or the choice of the approximation, as long as M is large enough). Since $G_j - g_j$ is in the ideal generated by I^M we see that there is an isomorphism

$$\psi_M : C^\wedge/I^M C^\wedge \to B/I^M B$$

As M is large enough we can use Lemma 9.1 with $d = d(I \subset A \to B)$, with $C^\wedge$ playing the role of B, with $2(rs+1)c$ instead of c, to find a morphism

$$\psi : C^\wedge \longrightarrow B$$

which agrees with ψ_M modulo $I^{q-2(rs+1)c}$ where q is the quotent of M by the number of generators of I. We claim ψ is an isomorphism. Since $C^\wedge$ and B are I-adically complete the map ψ is surjective because it is surjective modulo I (see Algebra, Lemma 93.1). On the other hand, as M is large enough we see that

$$\mathrm{Gr}_I(C^\wedge) \cong \mathrm{Gr}_I(B)$$

as graded $\mathrm{Gr}_I(A[x_1, \ldots, x_r]^\wedge)$-modules by More on Algebra, Lemma 3.2. Since ψ is compatible with this isomorphism as it agrees with ψ_M modulo I, this means that $\mathrm{Gr}_I(\psi)$ is an isomorphism. As $C^\wedge$ and B are I-adically complete, it follows that ψ is an isomorphism.

This paragraph serves to deal with the issue that C^h is not of finite type over A. Namely, the ring $A[x_1, \ldots, x_r]^h$ is a filtered colimit of étale $A[x_1, \ldots, x_r]$ algebras A' such that $A/I[x_1, \ldots, x_r] \to A'/IA'$ is an isomorphism (see proof of More on Algebra, Lemma 7.12). Pick an A' such that $G_1, \ldots, G_m$ are the images of $G'_1, \ldots, G'_m \in A'$. Setting $C = A'/(G'_1, \ldots, G'_m)$ we get the finite type algebra we were looking for. $\square$

The following lemma isn't true in general if A is not a G-ring but just Noetherian. Namely, if $(A, \mathfrak{m})$ is local and $I = \mathfrak{m}$, then the lemma is equivalent to Artin approximation for A^h (as in Smoothing Ring Maps, Theorem 13.1) which does not hold for every Noetherian local ring.

Lemma 10.1. *Let A be a Noetherian G-ring. Let $I \subset A$ be an ideal. Let B, C be finite type A-algebras. For any A-algebra map $\varphi : B^\wedge \to C^\wedge$ of I-adic completions and any $N \geq 1$ there exist*

(1) *an étale ring map $C \to C'$ which induces an isomorphism $C/IC \to C'/IC'$,*
(2) *an A-algebra map $\varphi : B \to C'$*

such that φ and ψ agree modulo I^N into $C^\wedge = (C')^\wedge$.

Proof. The statement of the lemma makes sense as $C \to C'$ is flat (Algebra, Lemma 138.3) hence induces an isomorphism $C/I^nC \to C'/I^nC'$ for all n (More on Algebra, Lemma 63.2) and hence an isomorphism on completions. Let C^h be the henselization of the pair (C, IC), see More on Algebra, Lemma 7.12. Then C^h is the filtered colimit of the algebras C' and the maps $C \to C' \to C^h$ induce isomorphism on completions (More on Algebra, Lemma 7.15). Thus it suffices to prove there exists an A-algebra map $B \to C^h$ which is congruent to ψ modulo I^N. Write $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. The ring map ψ corresponds to elements $\hat{c}_1, \ldots, \hat{c}_n \in C^\wedge$ with $f_j(\hat{c}_1, \ldots, \hat{c}_n) = 0$ for $j = 1, \ldots, m$. Namely, as A is a Noetherian G-ring, so is C, see More on Algebra, Proposition 39.10. Thus Smoothing Ring Maps, Lemma 14.1 applies to give elements $c_1, \ldots, c_n \in C^h$ such that $f_j(c_1, \ldots, c_n) = 0$ for $j = 1, \ldots, m$ and such that $\hat{c}_i - c_i \in I^N C^h$. This determines the map $B \to C^h$ as desired. □

11. Rig-surjective morphisms

For morphisms locally of finite type between locally Noetherian formal algebraic spaces a definition borrowed from [Art70] can be used. See Remark 11.10 for a discussion of what to do in more general cases.

Definition 11.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces over S. Assume that X and Y are locally Noetherian and that f is locally of finite type. We say f is *rig-surjective* if for every solid diagram

$$\begin{tikzcd} \mathrm{Spf}(R') \arrow[r, dotted] \arrow[d, dotted] & X \arrow[d, "f"] \\ \mathrm{Spf}(R) \arrow[r, "p"] & Y \end{tikzcd}$$

where R is a complete discrete valuation ring and where p is an adic morphism there exists an extension of complete discrete valuation rings $R \subset R'$ and a morphism $\mathrm{Spf}(R') \to X$ making the displayed diagram commute.

We prove a few lemmas to explain what this means.

Lemma 11.2. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z are locally Noetherian and f and g locally of finite type. Then if f and g are rig-surjective, so is $g \circ f$.*

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemma 18.3). □

Lemma 11.3. *Let S be a scheme. Let $f : X \to Y$ and $Z \to Y$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z are locally Noetherian and f and g locally of finite type. If f is rig-surjective, then the base change $Z \times_Y X \to Z$ is too.*

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemmas 18.9 and 18.4). □

Lemma 11.4. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces over S. Assume X, Y, Z locally Noetherian and f and g locally of finite type. If $g \circ f : X \to Z$ is rig-surjective, so is $g : Y \to Z$.*

Proof. Immediate from the definition. □

Lemma 11.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces which is representable by algebraic spaces, étale, and surjective. Assume X and Y locally Noetherian. Then f is rig-surjective.*

Proof. Let $p : \text{Spf}(R) \to Y$ be an adic morphism where R is a complete discrete valuation ring. Let $Z = \text{Spf}(R) \times_Y X$. Then $Z \to \text{Spf}(R)$ is representable by algebraic spaces, étale, and surjective. Hence Z is nonempty. Pick a nonempty affine formal algebraic space V and an étale morphism $V \to Z$ (possible by our definitions). Then $V \to \text{Spf}(R)$ corresponds to $R \to A^\wedge$ where $R \to A$ is an étale ring map, see Formal Spaces, Lemma 14.13. Since $A^\wedge \neq 0$ (as $V \neq \emptyset$) we can find a maximal ideal $\mathfrak{m}$ of A lying over $\mathfrak{m}_R$. Then $A_\mathfrak{m}$ is a discrete valuation ring (More on Algebra, Lemma 33.4). Then $R' = A_\mathfrak{m}^\wedge$ is a complete discrete valuation ring (More on Algebra, Lemma 32.5). Applying Formal Spaces, Lemma 5.10. we find the desired morphism $\text{Spf}(R') \to V \to Z \to X$. □

Remark 11.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces which is locally of finite type. The upshot of the lemmas above is that we may check whether $f : X \to Y$ is rig-surjective, étale locally on Y. For example, suppose that $\{Y_i \to Y\}$ is a covering as in Formal Spaces, Definition 7.1. Then f is rig-surjective if and only if $f_i : X \times_Y Y_i \to Y_i$ is rig-surjective. Namely, if f is rig-surjective, so is any base change (Lemma 11.3). Conversely, if all f_i are rig-surjective, so is $\coprod f_i : \coprod X \times_Y Y_i \to \coprod Y_i$. By Lemma 11.5 the morphism $\coprod Y_i \to Y$ is rig-surjective. Hence $\coprod X \times_Y Y_i \to Y$ is rig-surjective (Lemma 11.2). Since this morphism factors through $X \to Y$ we see that $X \to Y$ is rig-surjective by Lemma 11.4.

Lemma 11.7. *Let S be a scheme. Let $f : X \to Y$ be a proper surjective morphism of locally Noetherian algebraic spaces over S. Let $T \subset |Y|$ be a closed subset and let $T' = |f|^{-1}(T) \subset |X|$. Then $X_{/T'} \to Y_{/T}$ is rig-surjective.*

Proof. The statement makes sense by Formal Spaces, Lemmas 15.6 and 18.10. Let $Y_j \to Y$ be a jointly surjective family of étale morphism where Y_j is an affine scheme for each j. Denote $T_j \subset Y_j$ the inverse image of T. Then $\{(Y_j)_{/T_j} \to Y_{/T}\}$ is a covering as in Formal Spaces, Definition 7.1. Moreover, setting $X_j = Y_j \times_Y X$ and $T'_j \subset |X_j|$ the inverse image of T, we have

$$(X_j)_{/T'_j} = (Y_j)_{/T_j} \times_{(Y_{/T})} X_{/T'}$$

By the discussion in Remark 11.6 we reduce to the case where Y is an affine Noetherian scheme treated in the next paragraph.

Assume $Y = \mathrm{Spec}(A)$ where A is a Noetherian ring. This implies that $Y_{/T} = \mathrm{Spf}(A^\wedge)$ where $A^\wedge$ is the I-adic completion of A for some ideal $I \subset A$. Let $p : \mathrm{Spf}(R) \to \mathrm{Spf}(A^\wedge)$ be an adic morphism where R is a complete discrete valuation ring. Let K be the field of fractions of R. Consider the composition $A \to A^\wedge \to R$. Since $X \to Y$ is surjective, the fibre $X_K = \mathrm{Spec}(K) \times_Y X$ is nonempty. Thus we may choose an affine scheme U and an étale morphism $U \to X$ such that U_K is nonempty. Let $u \in U_K$ be a closed point (possible as U_K is affine). By Morphisms, Lemma 21.3 the residue field $L = \kappa(u)$ is a finite extension of K. Let $R' \subset L$ be the integral closure of R in L. By More on Algebra, Remark 68.5 we see that R' is a discrete valuation ring. Because $X \to Y$ is proper we see that the given morphism $\mathrm{Spec}(L) = u \to U_K \to X_K \to X$ extends to a morphism $\mathrm{Spec}(R') \to X$ over the given morphism $\mathrm{Spec}(R) \to Y$ (Decent Spaces, Lemma 14.5). By commutativity of the diagram the induced morphisms $\mathrm{Spec}(R'/\mathfrak{m}_{R'}^n) \to X$ are points of $X_{/T'}$ and we find

$$\mathrm{Spf}((R')^\wedge) = \mathrm{colim}\,\mathrm{Spec}(R'/\mathfrak{m}_{R'}^n) \longrightarrow X_{/T'}$$

as desired (note that $(R')^\wedge$ is a complete discrete valuation ring by More on Algebra, Lemma 32.5; in fact in the current situation $R' = (R')^\wedge$ but we do not need this). ☐

Lemma 11.8. *Let A be a Noetherian ring complete with respect to an ideal I. Let B be an I-adically complete A-algebra. If $A/I^n \to B/I^nB$ is of finite type and flat for all n and faithfully flat for $n = 1$, then $Spf(B) \to Spf(A)$ is rig-surjective.*

Proof. We will use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 5.10. Let $\varphi : A \to R$ be a continuous map into a complete discrete valuation ring A. This implies that $\varphi(I) \subset \mathfrak{m}_R$. On the other hand, since we only need to produce the lift $\varphi' : B' \to R'$ in the case that φ corresponds to an adic morphism, we may assume that $\varphi(I) \neq 0$. Thus we may consider the base change $C = B\widehat{\otimes}_A R$, see Remark 4.3 for example. Then C is an $\mathfrak{m}_R$-adically complete R-algebra such that $C/\mathfrak{m}_R^nC$ is of finite type and flat over $R/\mathfrak{m}_R^n$ and such that $C/\mathfrak{m}_RC$ is nonzero. Pick any maximal ideal $\mathfrak{m} \subset C$ lying over $\mathfrak{m}_R$. By flatness (which implies going down) we see that $\mathrm{Spec}(C_\mathfrak{m}) \setminus V(\mathfrak{m}_RC_\mathfrak{m})$ is a nonempty open. Hence We can pick a prime $\mathfrak{q} \subset \mathfrak{m}$ such that $\mathfrak{q}$ defines a closed point of $\mathrm{Spec}(C_\mathfrak{m}) \setminus \{\mathfrak{m}\}$ and such that $\mathfrak{q} \notin V(IC_\mathfrak{m})$, see Properties, Lemma 6.4. Then $C/\mathfrak{q}$ is a dimension 1-local domain and we can find $C/\mathfrak{q} \subset R'$ with R' a discrete valuation ring (Algebra, Lemma 115.12). By construction $\mathfrak{m}_RR' \subset \mathfrak{m}_{R'}$ and we see that $C \to R'$ extends to a continuous map $C \to (R')^\wedge$ (in fact we can pick R' such that $R' = (R')^\wedge$ in our current situation but we do not need this). Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption gives a commutative diagram of rings

$$\begin{tikzcd} (R')^\wedge & C \arrow[l] & B \arrow[l] \\ R \arrow[u] & R \arrow[l] \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

which gives the desired lift. ☐

Lemma 11.9. *Let A be a Noetherian ring complete with respect to an ideal I. Let B be an I-adically complete A-algebra. Assume that*

(1) *the I-torsion in A is 0,*
(2) *$A/I^n \to B/I^nB$ is flat and of finite type for all n.*

Then Spf(B) $\to$ Spf(A) is rig-surjective if and only if $A/I \to B/IB$ is faithfully flat.

Proof. Faithful flatness implies rig-surjectivity by Lemma 11.8. To prove the converse we will use without further mention that the vanishing of I-torsion is equivalent to the vanising of I-power torsion (More on Algebra, Lemma 62.3). We will also use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 5.10.

Assume $\text{Spf}(B) \to \text{Spf}(A)$ is rig-surjective. Choose a maximal ideal $I \subset \mathfrak{m} \subset A$. The open $U = \text{Spec}(A_\mathfrak{m}) \setminus V(I_\mathfrak{m})$ of $\text{Spec}(A)$ is nonempty as the $I_\mathfrak{m}$-torsion of $A_\mathfrak{m}$ is zero (use Algebra, Lemma 61.4). Thus we can find a prime $\mathfrak{q} \subset \mathfrak{m}$ which defines a point of U (i.e., $I \not\subset \mathfrak{q}$) and which corresponds to a closed point of $\text{Spec}(A) \setminus \mathfrak{m}$, see Properties, Lemma 6.4. Then $A/\mathfrak{q}$ is a dimension 1-local domain and we can find $A/\mathfrak{q} \subset R$ with R a discrete valuation ring (Algebra, Lemma 115.12). By construction $IR \subset \mathfrak{m}_R$ and we see that $A \to R$ extends to a continuous map $A \to R^\wedge$ (in fact $R = R^\wedge$ in our situation but we do not need this). Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption gives a commutative diagram of rings

$$\begin{tikzcd} R' & B \arrow[l] \\ R^\wedge \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Thus we find a prime ideal of B lying over $\mathfrak{m}$. It follows that $\text{Spec}(B/IB) \to \text{Spec}(A/I)$ is surjective, whence $A/I \to B/IB$ is faithfully flat (Algebra, Lemma 38.15). □

Remark 11.10. The condition as formulated in Definition 11.1 is not right for morphisms of locally adic* formal algebraic spaces. For example, if $A = (\bigcup_{n \geq 1} k[t^{1/n}])^\wedge$ where the completion is the t-adic completion, then there are no adic morphisms $\text{Spf}(R) \to \text{Spf}(A)$ where R is a complete discrete valuation ring. Thus any morphism $X \to \text{Spf}(A)$ would be rig-surjective, but since A is a domain and $t \in A$ is not zero, we want to think of A as having at least one "rig-point", and we do not want to allow $X = \emptyset$. To cover this particular case, one can consider adic morphisms

$$\text{Spf}(R) \longrightarrow Y$$

where R is a valuation ring complete with respect to a principal ideal J whose radical is $\mathfrak{m}_R = \sqrt{J}$. In this case the value group of R can be embedded into $(\mathbf{R}, +)$ and one obtains the point of view used by Berkovich in defining an analytic space associated to Y, see [Ber90]. Another approach is championed by Huber. In his theory, one drops the hypothesis that $\text{Spec}(R/J)$ is a singleton, see [Hub93].

Lemma 11.11. *Let S be a scheme. Let $f : X \to Y$ be a morphism of formal algebraic spaces. Assume X and Y are locally Noetherian, f locally of finite type, and f a monomorphism. Then f is rig surjective if and only if every adic morphism Spf(R) $\to Y$ where R is a complete discrete valuation ring factors through X.*

Proof. One direction is trivial. For the other, suppose that $\text{Spf}(R) \to Y$ is an adic morphism such that there exists an extension of complete discrete valuation rings $R \subset R'$ with $\text{Spf}(R') \to \text{Spf}(R) \to X$ factoring through Y. Then $\text{Spec}(R'/\mathfrak{m}_R^n R') \to \text{Spec}(R/\mathfrak{m}_R^n)$ is surjective and flat, hence the morphisms $\text{Spec}(R/\mathfrak{m}_R^n) \to X$ factor through X as X satisfies the sheaf condition for fpqc coverings, see Formal Spaces, Lemma 23.1. In other words, $\text{Spf}(R) \to Y$ factors through X. $\square$

12. Algebraization

In this section we prove a generalization of the result on dilatations from the paper of Artin [Art70]. We first reformulate the algebra results proved above into the language of formal algebraic spaces.

Let S be a scheme. Let V be a locally Noetherian formal algebraic space over S. We denote $\mathcal{C}_V$ the category of formal algebraic spaces W over V such that the structure morphism $W \to V$ is rig-étale.

Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. Recall that $X_{/T}$ denotes the formal completion of X along T, see Formal Spaces, Section 9. More generally, for any algebraic space Y over X we denote $Y_{/T}$ the completion of Y along the inverse image of T in $|Y|$, so that $Y_{/T}$ is a formal algebraic space over $X_{/T}$.

Lemma 12.1. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. If $Y \to X$ is morphism of algebraic spaces which is locally of finite type and étale over $X \setminus T$, then $Y_{/T} \to X_{/T}$ is rig-étale, i.e., $Y_{/T}$ is an object of $\mathcal{C}_{X_{/T}}$ defined above.*

Proof. Choose a surjective étale morphism $U \to X$ with $U = \coprod U_i$ a disjoint union of affine schemes, see Properties of Spaces, Lemma 6.1. For each i choose a surjective étale morphism $V_i \to Y \times_X U_i$ where $V_i = \coprod V_{ij}$ is a disjoint union of affines. Write $U_i = \text{Spec}(A_i)$ and $V_{ij} = \text{Spec}(B_{ij})$. Let $I_i \subset A_i$ be an ideal cutting out the inverse image of T in U_i. Then we may apply Lemma 6.2 to see that the map of I_i-adic completions $A_i^\wedge \to B_{ij}^\wedge$ has the property P of Lemma 7.1. Since $\{\text{Spf}(A_i^\wedge) \to X_{/T}\}$ and $\{\text{Spf}(B_{ij}) \to Y_{/T}\}$ are coverings as in Formal Spaces, Definition 7.1 we see that $Y_{/T} \to X_{/T}$ is rig-étale by definition. $\square$

Lemma 12.2. *Let X be a Noetherian affine scheme. Let $T \subset X$ be a closed subset. Let U be an affine scheme and let $U \to X$ a finite type morphism étale over $X \setminus T$. Let V be a Noetherian affine scheme over X. For any morphism $c' : V_{/T} \to U_{/T}$ over $X_{/T}$ there exists an étale morphism $b : V' \to V$ of affine schemes which induces an isomorphism $b_{/T} : V'_{/T} \to V_{/T}$ and a morphism $a : V' \to U$ such that $c' = a_{/T} \circ b_{/T}^{-1}$.*

Proof. This is a reformulation of Lemma 6.5. $\square$

Lemma 12.3. *Let X be a Noetherian affine scheme. Let $T \subset X$ be a closed subset. Let $W \to X_{/T}$ be a rig-étale morphism of formal algebraic spaces with W an affine formal algebraic space. Then there exists an afffine scheme U, a finite type morphism $U \to X$ étale over $X \setminus T$ such that $W \cong U_{/T}$. Moreover, if $W \to X_{/T}$ is étale, then $U \to X$ is étale.*

Proof. The existence of U is a restatement of Lemma 9.4. The final statement follows from More on Morphisms, Lemma 10.3. $\square$

Let S be a scheme. Let X be a locally Noetherian algebraic space over S and let $T \subset |X|$ be a closed subset. Let us denote $\mathcal{C}_{X,T}$ the category of algebraic spaces Y over X such that the structure morphism $f : Y \to X$ is locally of finite type and an isomorphism over the complement of T. Formal completion defines a functor

$$F_{X,T} : \mathcal{C}_{X,T} \longrightarrow \mathcal{C}_{X_{/T}}, \quad (f : Y \to X) \longmapsto (f_{/T} : Y_{/T} \to X_{/T}) \tag{12.3.1}$$

see Lemma 12.1.

Lemma 12.4. *Let S be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of algebraic spaces. Let $T \subset |X|$ be closed. Assume that*

1. *X is locally Noetherian,*
2. *g is a monomorphism and locally of finite type,*
3. *$f|_{X \setminus T} : X \setminus T \to Y$ factors through g, and*
4. *$f_{/T} : X_{/T} \to Y$ factors trough g,*

then f factors through g.

Proof. Consider the fibre product $E = X \times_Y Z \to X$. By assumption the the open immersion $X \setminus T \to X$ factors through E and any morphism $\varphi : X' \to X$ with $|\varphi|(|X'|) \subset T$ factors through E as well, see Formal Spaces, Section 9. By More on Morphisms of Spaces, Lemma 17.3 this implies that $E \to X$ is étale at every point of E mapping to a point of T. Hence $E \to X$ is an étale monomorphism, hence an open immersion (Morphisms of Spaces, Lemma 45.2). Then it follows that $E = X$ since our assumptions imply that $|X| = |E|$. $\square$

Lemma 12.5. *Let S be a scheme. Let X, Y be locally Noetherian algebraic spaces over S. Let $T \subset |X|$ and $T' \subset |Y|$ be closed subsets. Let $a, b : X \to Y$ be morphisms of algebraic spaces over S such that $a|_{X \setminus T} = b|_{X \setminus T}$, such that $|a|(T) \subset T'$ and $|b|(T) \subset T'$, and such that $a_{/T} = b_{/T}$ as morphisms $X_{/T} \to Y_{/T'}$. Then $a = b$.*

Proof. Let E be the equalizer of a and b. Then E is an algebraic space and $E \to X$ is locally of finite type and a monomorphism, see Morphisms of Spaces, Lemma 4.1. Our assumptions imply we can apply Lemma 12.4 to the two morphisms $f = \mathrm{id} : X \to X$ and $g : E \to X$ and the closed subset T of $|X|$. $\square$

Lemma 12.6. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $s, t : R \to U$ be two morphisms of algebraic spaces over X. Assume*

1. *R, U are locally of finite type over X,*
2. *the base change of s and t to $X \setminus T$ is an étale equivalence relation, and*
3. *the formal completion $(t_{/T}, s_{/T}) : R_{/T} \to U_{/T} \times_{X_{/T}} U_{/T}$ is an equivalence relation too.*

Then $(t, s) : R \to U \times_X U$ is an étale equivalence relation.

Proof. The morphisms $s, t : R \to U$ are étale over $X \setminus T$ by assumption. Since the formal completions of the maps $s, t : R \to U$ are étale, we see that s and t are étale for example by More on Morphisms, Lemma 10.3. Applying Lemma 12.4 to the morphisms $\mathrm{id} : R \times_{U \times_X U} R \to R \times_{U \times_X U} R$ and $\Delta : R \to R \times_{U \times_X U} R$ we conclude that (t, s) is a monomorphism. Applying it again to $(t \circ \mathrm{pr}_0, s \circ \mathrm{pr}_1) : R \times_{s,U,t} R \to$

$U \times_X U$ and $(t, s) : R \to U \times_X U$ we find that "transitivity" holds. We omit the proof of the other two axioms of an equivalence relation. $\square$

Remark 12.7. Let S, X, and $T \subset |X|$ be as in (12.3.1). Let $U \to X$ be an algebraic space over X such that $U \to X$ is locally of finite type and étale outside of T. We will construct a factorization

$$U \longrightarrow Y \longrightarrow X$$

with Y in $\mathcal{C}_{X,T}$ such that $U_{/T} \to Y_{/T}$ is an isomorphism. We may assume the image of $U \to X$ contains $X \setminus T$, otherwise we replace U by $U \amalg (X \setminus T)$. For an algebraic space Z over X, let us denote Z° the open subspace which is the inverse image of $X \setminus T$. Let

$$R = U \amalg_{U^\circ} (U \times_X U)^\circ$$

be the pushout of $U^\circ \to U$ and the diagonal morphism $U^\circ \to U^\circ \times_X U^\circ = (U \times_X U)^\circ$. Since $U^\circ \to X$ is étale, the diagonal is an open immersion and we see that R is an algebraic space (this follows for example from Spaces, Lemma 8.4). The two projections $(U \times_X U)^\circ \to U$ extend to R and we obtain two étale morphisms $s, t : R \to U$. Checking on each piece separatedly we find that R is an étale equivalence relation on U. Set $Y = U/R$ which is an algebraic space by Bootstrap, Theorem 10.1. Since $U^\circ \to X \setminus T$ is a surjective étale morphism and since $R^\circ = U^\circ \times_{X \setminus T} U^\circ$ we see that $Y^\circ \to X \setminus T$ is an isomorphism. In other words, $Y \to X$ is an object of $\mathcal{C}_{X,T}$. On the other hand, the morphism $U \to Y$ induces an isomorphism $U_{/T} \to Y_{/T}$. Namely, the formal completion of R along the inverse image of T is equal to the formal completion of U along the inverse image of T by our choice of R. By our construction of the formal completion in Formal Spaces, Section 9 we conclude that $U_{/T} = Y_{/T}$.

Lemma 12.8. *Let S be a scheme. Let X be a Noetherian affine algebraic space over S. Let $T \subset |X|$ be a closed subset. Then the functor $F_{X,T}$ is an equivalence.*

Before we prove this lemma let us discuss an example. Suppose that $S = \mathrm{Spec}(k)$, $X = \mathbf{A}^1_k$, and $T = \{0\}$. Then $X_{/T} = \mathrm{Spf}(k[[x]])$. Let $W = \mathrm{Spf}(k[[x]] \times k[[x]])$. Then the corresponding Y is the affine line with zero doubled (Schemes, Example 14.3). Moreover, this is the output of the construction in Remark 12.7 starting with $U = X \amalg X$.

Proof. For any scheme or algebraic space Z over X, let us denote $Z_0 \subset Z$ the inverse image of T with the induced reduced closed subscheme or subspace structure. Note that $Z_0 = (Z_{/T})_{red}$ is the reduction of the formal completion.

The functor $F_{X,T}$ is faithful by Lemma 12.5.

Let Y, Y' be objects of $\mathcal{C}_{X,T}$ and let $a' : Y_{/T} \to Y'_{/T}$ be a morphism in $\mathcal{C}_{X_{/T}}$. To prove $F_{X,T}$ is fully faithful, we will construct a morphism $a : Y \to Y'$ in $\mathcal{C}_{X,T}$ such that $a' = a_{/T}$.

Let U be an affine scheme and let $U \to Y$ be an étale morphism. Because U is affine, U_0 is affine and the image of $U_0 \to Y_0 \to Y'_0$ is a quasi-compact subspace of $|Y'_0|$. Thus we can choose an affine scheme V and an étale morphism $V \to Y'$ such that the image of $|V_0| \to |Y'_0|$ contains this quasi-compact subset. Consider the formal algebraic space

$$W = U_{/T} \times_{Y'_{/T}} V_{/T}$$

By our choice of V the above, the map $W \to U_{/T}$ is surjective. Thus there exists an affine formal algebraic space W' and an étale morphism $W' \to W$ such that $W' \to W \to U_{/T}$ is surjective. Then $W' \to U_{/T}$ is étale. By Lemma 12.3 $W' = U'_{/T}$ for $U' \to U$ étale and U' affine. Write $V = \mathrm{Spec}(C)$. By Lemma 12.2 there exists an étale morphism $U'' \to U'$ of affines which is an isomorphism on completions and a morphism $U'' \to V$ whose completion is the composition $U''_{/T} \to U'_{/T} \to W \to V_{/T}$. Thus we get

$$Y \longleftarrow U'' \longrightarrow Y'$$

over X agreeing with the given map on formal completions such that the image of $U''_0 \to Y_0$ is the same as the image of $U_0 \to Y_0$.

Taking a disjoint union of U'' as constructed in the previous paragraph, we find a scheme U, an étale morphism $U \to Y$, and a morphism $b : U \to Y'$ over X, such that the diagram

$$\begin{tikzcd} U_{/T} \arrow[d] \arrow[dr, "b_{/T}"] & \\ Y_{/T} \arrow[r, "a'"] & Y'_{/T} \end{tikzcd}$$

is commutative and such that $U_0 \to Y_0$ is surjective. Taking a disjoint union with the open $X \setminus T$ (which is also open in Y and Y'), we find that we may even assume that $U \to Y$ is a surjective étale morphism. Let $R = U \times_Y U$. Then the two compositions $R \to U \to Y'$ agree both over $X \setminus T$ and after formal completion along T, whence are equal by Lemma 12.5. This means exactly that b factors as $U \to Y \to Y'$ to give us our desired morphism $a : Y \to Y'$.

Essential surjectivity. Let W be an object of $\mathcal{C}_{X_{/T}}$. We prove W is in the essential image in a number of steps.

Step 1: W is an affine formal algebraic space. Then we can find $U \to X$ of finite type and étale over $X \setminus T$ such that $U_{/T}$ is isomorphic to W, see Lemma 12.3. Thus we see that W is in the essential image by the construction in Remark 12.7.

Step 2: W is separated. Choose $\{W_i \to W\}$ as in Formal Spaces, Definition 7.1. By Step 1 the formal algebraic spaces W_i and $W_i \times_W W_j$ are in the essential image. Say $W_i = (Y_i)_{/T}$ and $W_i \times_W W_j = (Y_{ij})_{/T}$. By fully faithfulness we obtain morphisms $t_{ij} : Y_{ij} \to Y_i$ and $s_{ij} : Y_{ij} \to Y_j$ matching the projections $W_i \times_W W_j \to W_i$ and $W_i \times_W W_j \to W_j$. Set $R = \coprod Y_{ij}$ and $U = \coprod Y_i$ and denote $s = \coprod s_{ij} : R \to U$ and $t = \coprod t_{ij} : R \to U$. Applying Lemma 12.6 we find that $(t, s) : R \to U \times_X U$ is an étale equivalence relation. Thus we can take the quotient $Y = U/R$ and it is an algebraic space, see Bootstrap, Theorem 10.1. Since completion commutes with fibre products and taking quotient sheaves, we find that $Y_{/T} \cong W$ in $\mathcal{C}_{X_{/T}}$.

Step 3: W is general. Choose $\{W_i \to W\}$ as in Formal Spaces, Definition 7.1. The formal algebraic spaces W_i and $W_i \times_W W_j$ are separated. Hence by Step 2 the formal algebraic spaces W_i and $W_i \times_W W_j$ are in the essential image. Then we argue exactly as in the previous paragraph to see that W is in the essential image as well. This concludes the proof. $\square$

Theorem 12.9. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. The functor $F_{X,T}$ (12.3.1) is an equivalence.*

Proof. The theorem is essentially a formal consequence of Lemma 12.8. We give the details but we encourage the reader to think it through for themselves. Let $g : U \to X$ be a surjective étale morphism with $U = \coprod U_i$ and each U_i affine. Denote $F_{U,T}$ the functor for U and the inverse image of T in $|U|$.

Since $U = \coprod U_i$ both the category $\mathcal{C}_{U,T}$ and the category $\mathcal{C}_{U_{/T}}$ decompose as a product of categories, one for each i. Since the functors $F_{U_i,T}$ are equivalences for all i by the lemma we find that the same is true for $F_{U,T}$.

Since $F_{U,T}$ is faithful, it follows that $F_{X,T}$ is faithful too. Namely, if $a, b : Y \to Y'$ are morphisms in $\mathcal{C}_{X,T}$ such that $a_{/T} = b_{/T}$, then we find on pulling back that the base changes $a_U, b_U : U \times_X Y \to U \times_X Y'$ are equal. Since $U \times_X Y \to Y$ is surjective étale, this implies that $a = b$.

At this point we know that $F_{X,T}$ is faithful for every situation as in the theorem. Let $R = U \times_X U$ where U is as above. Let $t, s : R \to U$ be the projections. Since X is Noetherian, so is R. Thus the functor $F_{R,T}$ (defined in the obvious manner) is faithful. Let $Y \to X$ and $Y' \to X$ be objects of $\mathcal{C}_{X,T}$. Let $a' : Y_{/T} \to Y'_{/T}$ be a morphism in the category $\mathcal{C}_{X_{/T}}$. Taking the base change to U we obtain a morphism $a'_U : (U \times_X Y)_{/T} \to (U \times_X Y')_{/T}$ in the category $\mathcal{C}_{U_{/T}}$. Since the functor $F_{U,T}$ is fully faithful we obtain a morphism $a_U : U \times_X Y \to U \times_X Y'$ with $F_{U,T}(a_U) = a'_U$. Since $s^*(a'_U) = t^*(a'_U)$ and since $F_{R,T}$ is faithful, we find that $s^*(a_U) = t^*(a_U)$. Since

$$R \times_X Y \rightrightarrows U \times_X Y \longrightarrow Y$$

is an equalizer diagram of sheaves, we find that a_U descends to a morphism $a : Y \to Y'$. We omit the proof that $F_{X,T}(a) = a'$.

At this point we know that $F_{X,T}$ is faithful for every situation as in the theorem. To finish the proof we show that $F_{X,T}$ is essentially surjective. Let $W \to X_{/T}$ be an object of $\mathcal{C}_{X_{/T}}$. Then $U \times_X W$ is an object of $\mathcal{C}_{U_{/T}}$. By the affine case we find an object $V \to U$ of $\mathcal{C}_{U,T}$ and an isomorphism $\alpha : F_{U,T}(V) \to U \times_X W$ in $\mathcal{C}_{U_{/T}}$. By fully faithfulness of $F_{R,T}$ we find a unique morphism $h : s^*V \to t^*V$ in the category $\mathcal{C}_{R,T}$ such that $F_{R,T}(h)$ corresponds, via the isomorphism α, to the canonical descent datum on $U \times_X W$ in the category $\mathcal{C}_{R_{/T}}$. Using faithfulness of our functor on $R \times_{s,U,t} R$ we see that h satisfies the cocycle condition. We conclude, for example by the much more general Bootstrap, Lemma 11.2, that there exists an object $Y \to X$ of $\mathcal{C}_{X,T}$ and an isomorphism $\beta : U \times_X Y \to V$ such that the descent datum h corresponds, via β, to the canonical descent datum on $U \times_X Y$. We omit the verification that $F_{X,T}(Y)$ is isomorphic to W; hint: in the category of formal algebraic spaces there is descent for morphisms along étale coverings. $\square$

We are often interested as to whether the output of the construction of Theorem 12.9 is a separated algebraic space. In the next few lemmas we match properties of $Y \to X$ and the corresponding completion $Y_{/T} \to X_{/T}$.

Lemma 12.10. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $W \to X_{/T}$ be an object of the category $\mathcal{C}_{X_{/T}}$ and let $Y \to X$ be the object corresponding to W via Theorem 12.9. Then $Y \to X$ is quasi-compact if and only if $W \to X_{/T}$ is so.*

Proof. These conditions may be checked after base change to an affine scheme étale over X, resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms of Spaces, Lemma 8.7 as well as Formal Spaces, Lemma 12.3. If $U \to X$ ranges over étale morphisms wtih U affine, then the formal completions $U_{/T} \to X_{/T}$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume X is affine.

Let $V \to Y$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. Then $V_{/T} \to Y_{/T} = W$ is a surjective étale morphism. Thus if Y is quasi-compact, we can choose J is finite, and we conclude that W is quasi-compact. Conversely, if W is quasi-compact, then we can find a finite subset $J' \subset J$ such that $\coprod_{j \in J'} (V_j)_{/T} \to W$ is surjective. Then it follows that

$$(X \setminus T) \amalg \coprod\nolimits_{j \in J'} V_j \longrightarrow Y$$

is surjective. This either follows from the construction of Y in the proof of Lemma 12.8 or it follows since we have

$$|Y| = |X \setminus T| \amalg |W_{red}|$$

as $Y_{/T} = W$. □

Lemma 12.11. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $W \to X_{/T}$ be an object of the category $\mathcal{C}_{X_{/T}}$ and let $Y \to X$ be the object corresponding to W via Theorem 12.9. Then $Y \to X$ is quasi-separated if and only if $W \to X_{/T}$ is so.*

Proof. These conditions may be checked after base change to an affine scheme étale over X, resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms of Spaces, Lemma 4.12 as well as Formal Spaces, Lemma 21.5, If $U \to X$ ranges over étale morphisms wtih U affine, then the formal completions $U_{/T} \to X_{/T}$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume X is affine.

Let $V \to Y$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. Then Y is quasi-separated if and only if $V_j \times_Y V_{j'}$ is quasi-compact for all $j, j' \in J$. Similarly, W is quasi-separated if and only if $(V_j \times_Y V_{j'})_{/T} = (V_j)_{/T} \times_{Y_{/T}} (V_{j'})_{/T}$ is quasi-compact for all $j, j' \in J$. Since X is Noetherian affine, we see that

$$(V_j \times_Y V_{j'}) \times_X (X \setminus T)$$

is quasi-compact. Hence we conclude the equvalence holds by the equality

$$|V_j \times_Y V_{j'}| = |(V_j \times_Y V_{j'}) \times_X (X \setminus T)| \amalg |(V_j \times_Y V_{j'})_{/T}|$$

and the fact that the second summand is closed in the left hand side. □

Lemma 12.12. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $W \to X_{/T}$ be an object of the category $\mathcal{C}_{X_{/T}}$ and let $Y \to X$ be the object corresponding to W via Theorem 12.9. Then $Y \to X$ is separated if and only if $W \to X_{/T}$ is separated and $\Delta : W \to W \times_{X_{/T}} W$ is rig-surjective.*

Proof. These conditions may be checked after base change to an affine scheme étale over X, resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms of Spaces, Lemma 4.12 as well as Formal Spaces, Lemma 21.5. If $U \to X$ ranges over étale morphisms wtih U affine, then the formal completions $U_{/T} \to X_{/T}$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume X is affine. In the proof of both directions we may assume that $Y \to X$ and $W \to X_{/T}$ are quasi-separated by Lemma 12.11.

Proof of easy direction. Assume $Y \to X$ is separated. Then $Y \to Y \times_X Y$ is a closed immersion and it follows that $W \to W \times_{X_{/T}} W$ is a closed immersion too, i.e., we see that $W \to X_{/T}$ is separated. Let

$$p : \mathrm{Spf}(R) \longrightarrow W \times_{X_{/T}} W = (Y \times_X Y)_{/T}$$

be an adic morphism where R is a complete discrete valuation ring with fraction field K. The composition into $Y \times_X Y$ corresponds to a morphism $g : \mathrm{Spec}(R) \to Y \times_X Y$, see Formal Spaces, Lemma 24.3. Since p is an adic morphism, so is the composition $\mathrm{Spf}(R) \to X$. Thus we see that $g(\mathrm{Spec}(K))$ is a point of

$$(Y \times_X Y) \times_X (X \setminus T) \cong X \setminus T \cong Y \times_X (X \setminus T)$$

(small detail omitted). Hence this lifts to a K-point of Y and we obtain a commutaive diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & Y \times_X Y \end{tikzcd}$$

Since $Y \to X$ was assumed separated we find the dotted arrow exists (Cohomology of Spaces, Lemma 18.1). Applying the functor completion along T we find that p can be lifted to a morphism into W, i.e., $W \to W \times_{X_{/T}} W$ is rig-surjective.

Proof of hard direction. Assume $W \to X_{/T}$ separated and $W \to W \times_{X_{/T}} W$ rig-surjective. By Cohomology of Spaces, Lemma 18.1 and Remark 18.3 it suffices to show that given any commtutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "g"] \arrow[ur, dashed] & Y \times_X Y \end{tikzcd}$$

where R is a complete discrete valuation ring with fraction field K, there is at most one dotted arrow making the diagram commute. Let $h : \mathrm{Spec}(R) \to X$ be the composition of g with the morphism $Y \times_X Y \to X$. There are three cases: Case I: $h(\mathrm{Spec}(R)) \subset (X \setminus T)$. This case is trivial because $Y \times_X (X \setminus T) = X \setminus T$. Case II: h maps $\mathrm{Spec}(R)$ into T. This case follows from our assumption that $W \to X_{/T}$ is separated. Namely, if T denotes the reduced induced closed subspace structure on T, then h factors through T and

$$W \times_{X_{/T}} T = Y \times_X T \longrightarrow T$$

is separated by assumption (and for example Formal Spaces, Lemma 21.5) which implies we get the lifting property by Cohomology of Spaces, Lemma 18.1 applied to the displayed arrow. Case III: $h(\mathrm{Spec}(K))$ is not in T but h maps the closed point of $\mathrm{Spec}(R)$ into T. In this case the corresponding morphism

$$g_{/T} : \mathrm{Spf}(R) \longrightarrow (Y \times_X Y)_{/T} = W \times_{X_{/T}} W$$

is an adic morphism (detail omitted). Hence our assumption that $W \to W \times_{X_{/T}} W$ be rig-surjective implies we can lift $g_{/T}$ to a morphism $e : \mathrm{Spf}(R) \to W = Y_{/T}$ (see Lemma 11.11 for why we do not need to extend R). Algebraizing the composition

$\mathrm{Spf}(R) \to Y$ using Formal Spaces, Lemma 24.3 we find a morphism $\mathrm{Spec}(R) \to Y$ lifing g as desired. ☐

Lemma 12.13. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $T \subset |X|$ be a closed subset. Let $W \to X_{/T}$ be an object of the category $\mathcal{C}_{X_{/T}}$ and let $Y \to X$ be the object corresponding to W via Theorem 12.9. Then $Y \to X$ is proper if and only if the following conditions hold*

1. *$W \to X_{/T}$ is proper,*
2. *$W \to X_{/T}$ is rig-surjective, and*
3. *$\Delta : W \to W \times_{X_{/T}} W$ is rig-surjective.*

Proof. These conditions may be checked after base change to an affine scheme étale over X, resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms of Spaces, Lemma 37.2 as well as Formal Spaces, Lemma 22.2. If $U \to X$ ranges over étale morphisms wtih U affine, then the formal completions $U_{/T} \to X_{/T}$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume X is affine. In the proof of both directions we may assume that $Y \to X$ and $W \to X_{/T}$ are separated and quasi-compact and that $W \to W \times_{X_{/T}} W$ is rig-surjective by Lemmas 12.10 and 12.12.

Proof of the easy direction. Assume $Y \to X$ is proper. Then $Y_{/T} = Y \times_X X_{/T} \to X_{/T}$ is proper too. Let

$$p : \mathrm{Spf}(R) \longrightarrow X_{/T}$$

be an adic morphism where R is a complete discrete valuation ring with fraction field K. Then p corresponds to a morphism $g : \mathrm{Spec}(R) \to X$, see Formal Spaces, Lemma 24.3. Since p is an adic morphism, we have $p(\mathrm{Spec}(K)) \not\in T$. Since $Y \to X$ is an isomorphism over $X \setminus T$ we can lift to X and obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

Since $Y \to X$ was assumed proper we find the dotted arrow exists. (Cohomology of Spaces, Lemma 18.2). Applying the functor completion along T we find that p can be lifted to a morphism into W, i.e., $W \to X_{/T}$ is rig-surjective.

Proof of hard direction. Assume $W \to X_{/T}$ proper, $W \to W \times_{X_{/T}} W$ rig-surjective, and $W \to X_{/T}$ rig-surjective. By Cohomology of Spaces, Lemma 18.2 and Remark 18.3 it suffices to show that given any commtutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "g"] \arrow[ur, dashed] & X \end{tikzcd}$$

where R is a complete discrete valuation ring with fraction field K, there is a dotted arrow making the diagram commute. Let $h : \mathrm{Spec}(R) \to X$ be the composition of g with the morphism $Y \times_X Y \to X$. There are three cases: Case I: $h(\mathrm{Spec}(R)) \subset (X \setminus T)$. This case is trivial because $Y \times_X (X \setminus T) = X \setminus T$. Case II: h maps $\mathrm{Spec}(R)$ into T. This case follows from our assumption that $W \to X_{/T}$ is proper.

Namely, if T denotes the reduced induced closed subspace structure on T, then h factors through T and

$$W \times_{X_{/T}} T = Y \times_X T \longrightarrow T$$

is proper by assumption which implies we get the lifting property by Cohomology of Spaces, Lemma 18.2 applied to the displayed arrow. Case III: $h(\mathrm{Spec}(K))$ is not in T but h maps the closed point of $\mathrm{Spec}(R)$ into T. In this case the corresponding morphism

$$g_{/T} : \mathrm{Spf}(R) \longrightarrow Y_{/T} = W$$

is an adic morphism (detail omitted). Hence our assumption that $W \to X_{/T}$ be rig-surjective implies we can lift $g_{/T}$ to a morphism $e : \mathrm{Spf}(R') \to W = Y_{/T}$ for some extension of complete discrete valuation rings $R \subset R'$. Algebraizing the composition $\mathrm{Spf}(R') \to Y$ using Formal Spaces, Lemma 24.3 we find a morphism $\mathrm{Spec}(R') \to Y$ lifing g. By the discussion in Cohomology of Spaces, Remark 18.3 this is sufficient to conclude that $Y \to X$ is proper. $\square$

13. Application to modifications

Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. We set $S = \mathrm{Spec}(A)$ and $U = S \setminus \{\mathfrak{m}\}$. In this section we will consider the category

$$\left\{ f : X \longrightarrow S \;\middle|\; \begin{array}{c} X \text{ is an algebraic space} \\ f \text{ is a proper morphism} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array} \right\} \tag{13.0.1}$$

A morphism from X/S to X'/S will be a morphism of algebraic spaces $X \to X'$ compatible with the structure morphisms over S.

Let $A \to B$ be a local homomorphism of local Noetherian rings such that $\mathfrak{m}_B = \sqrt{\mathfrak{m}_A B}$. Then base change along the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ gives a functor from the category (13.0.1) for A to the category (13.0.1) for B.

Lemma 13.1. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with $\mathfrak{m}$-adic completion $A^\wedge$. Then base change defines an equivalence of categories between the category (13.0.1) for A with the category (13.0.1) for the completion $A^\wedge$.*

Proof. Set $S = \mathrm{Spec}(A)$ as in (13.0.1) and $T = V(\mathfrak{m})$. Similarly, Write $S' = \mathrm{Spec}(A^\wedge)$ and $T' = V(\mathfrak{m}^\wedge)$. The morphism $S' \to S$ defines an isomorphism $S'_{/T'} \to S_{/T}$ of formal completions. Let $\mathcal{C}_{S,T}$, $\mathcal{C}_{S_{/T}}$, $\mathcal{C}_{S'_{/T'}}$, and $\mathcal{C}_{S',T'}$ be the corresponding categories as used in (12.3.1). By Theorem 12.9 (in fact we only need the affine case treated in Lemma 12.8) we see that

$$\mathcal{C}_{S,T} = \mathcal{C}_{S_{/T}} = \mathcal{C}_{S'_{/T'}} = \mathcal{C}_{S',T'}$$

Note that $f : X \to S$ is an object of (13.0.1) if and only if $f : X \to S$ is an object of $\mathcal{C}_{S,T}$ and f is proper. Hence, to finish the proof we have to show that an object $f : X \to S$ of $\mathcal{C}_{S,T}$ is proper over S if and only if the base change $f' : X' \to S'$ is proper over S'. This you can deduce from Lemma 12.13 (translating the properness into properties of the formal completion which lives in $\mathcal{C}_{S_{/T}} = \mathcal{C}_{S'_{/T'}}$), or you can deduce it from Descent on Spaces, Lemma 10.17. $\square$

Lemma 13.2. *Let $A \to B$ be a local map of local Noetherian rings such that*

1. *$A \to B$ is flat,*
2. *$\mathfrak{m}_B = \mathfrak{m}_A B$, and*

(3) $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$

(equivalently, $A \to B$ induces an isomorphism on completions, see More on Algebra, Lemma 32.7). Then the base change functor from the category (13.0.1) for A to the category (13.0.1) for B is an equivalence.

Proof. This follows immediately from Lemma 13.1. □

Lemma 13.3. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $f : X \to S$ be an object of (13.0.1). Then there exists a U-admissible blowup $S' \to S$ which dominates X.*

Proof. Special case of More on Morphisms of Spaces, Lemma 28.3. □

Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $A^\wedge$ be the completion of A. Set $S^\wedge = \mathrm{Spec}(A^\wedge)$, $S = \mathrm{Spec}(A)$ and let $U^\wedge \subset S^\wedge$, $U \subset S$ be the complement of the closed point. Picture

$$\begin{tikzcd} U^\wedge \arrow[r] \arrow[d] & S^\wedge \arrow[d] \\ U \arrow[r] & S \end{tikzcd}$$

This is a cartesian square of schemes.

Lemma 13.4. *With assumption and notation as above. If $Y \to S^\wedge$ is a $U^\wedge$-admissible blowup, then there exists a U-admissible blowup $X \to S$ such that $Y = X \times_S S^\wedge$.*

Proof. By definition there exists an ideal $J \subset A^\wedge$ such that $V(J) = \{\mathfrak{m}A^\wedge\}$ and such that Y is the blowup of $S^\wedge$ in the closed subscheme defined by J, see Divisors, Definition 20.1. Since $A^\wedge$ is Noetherian this implies $\mathfrak{m}^n A^\wedge \subset J$ for some n. Since $A^\wedge/\mathfrak{m}^n A^\wedge = A/\mathfrak{m}^n$ we find an ideal $\mathfrak{m}^n \subset I \subset A$ such that $J = IA^\wedge$. Let $X \to S$ be the blowup in I. Since $A \to A^\wedge$ is flat we conclude that the base change of X is Y by Divisors, Lemma 18.3. □

14. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Hypercoverings

Schemes

(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
(32) Varieties

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