TOPOLOGIES ON SCHEMES

Contents

1. Introduction

In this document we explain what the different topologies on the category of schemes are. Some references are [Gro71] and [BLR90]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

2. The general procedure

In this section we explain a general procedure for producing the sites we will be working with. Suppose we want to study sheaves over schemes with respect to some topology τ. In order to get a site, as in Sites, Definition 6.2, of schemes with that topology we have to do some work. Namely, we cannot simply say "consider all schemes with the Zariski topology" since that would give a "big" category. Instead, in each section of this chapter we will proceed as follows:

(1) We define a class Cov_τ of coverings of schemes satisfying the axioms of Sites, Definition 6.2. It will always be the case that a Zariski open covering of a scheme is a covering for τ.
(2) We single out a notion of standard τ-covering within the category of affine schemes.
(3) We define what is an "absolute" big τ-site Sch_τ. These are the sites one gets by appropriately choosing a set of schemes and a set of coverings.

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(4) For any object S of Sch_τ we define the big τ-site $(Sch/S)_\tau$ and for suitable τ the small[1] τ-site S_τ.
(5) In addition there is a site $(Aff/S)_\tau$ using the notion of standard τ-covering of affines whose category of sheaves is equivalent to the category of sheaves on $(Sch/S)_\tau$.

The above is a little clumsy in that we do not end up with a canonical choice for the big τ-site of a scheme, or even the small τ-site of a scheme. If you are willing to ignore set theoretic difficulties, then you can work with classes and end up with canonical big and small sites...

3. The Zariski topology

Definition 3.1. Let T be a scheme. A *Zariski covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is an open immersion and such that $T = \bigcup f_i(T_i)$.

This defines a (proper) class of coverings. Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 3.2. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a Zariski covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering and for each i we have a Zariski covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a Zariski covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a Zariski covering.*

Proof. Omitted. ☐

Lemma 3.3. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a Zariski covering of T. Then there exists a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is a standard open of T, see Schemes, Definition 5.2. Moreover, we may choose each U_j to be an open of one of the T_i.*

Proof. Follows as T is quasi-compact and standard opens form a basis for its topology. This is also proved in Schemes, Lemma 5.1. ☐

Thus we define the corresponding standard coverings of affines as follows.

Definition 3.4. Compare Schemes, Definition 5.2. Let T be an affine scheme. A *standard Zariski covering* of T is a a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ with each $U_j \to T$ inducing an isomorphism with a standard affine open of T.

Definition 3.5. A *big Zariski site* is any site Sch_{Zar} as in Sites, Definition 6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of Zariski coverings Cov_0 among these schemes.
(2) As underlying category of Sch_{Zar} take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
(3) As coverings of Sch_{Zar} choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of Zariski coverings, and the set Cov_0 chosen above.

[1]The words big and small here do not relate to bigness/smallness of the corresponding categories.

It is shown in Sites, Lemma 8.6 that, after having chosen the category Sch_α, the category of sheaves on Sch_α does not depend on the choice of coverings chosen in (3) above. In other words, the topos $Sh(Sch_{Zar})$ only depends on the choice of the category Sch_α. It is shown in Sets, Lemma 9.9 that these categories are closed under many constructions of algebraic geometry, e.g., fibre products and taking open and closed subschemes. We can also show that the exact choice of Sch_α does not matter too much, see Section 10.

Another approach would be to assume the existence of a strongly inaccessible cardinal and to define Sch_{Zar} to be the category of schemes contained in a chosen universe with set of coverings the Zariski coverings contained in that same universe.

Before we continue with the introduction of the big Zariski site of a scheme S, let us point out that the topology on a big Zariski site Sch_{Zar} is in some sense induced from the Zariski topology on the category of all schemes.

Lemma 3.6. *Let Sch_{Zar} be a big Zariski site as in Definition 3.5. Let $T \in \mathrm{Ob}(Sch_{Zar})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary Zariski covering of T. There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} which is tautologically equivalent (see Sites, Definition 8.2) to $\{T_i \to T\}_{i\in I}$.*

Proof. Since each $T_i \to T$ is an open immersion, we see by Sets, Lemma 9.9 that each T_i is isomorphic to an object V_i of Sch_{Zar}. The covering $\{V_i \to T\}_{i\in I}$ is tautologically equivalent to $\{T_i \to T\}_{i\in I}$ (using the identity map on I both ways). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 11.1. $\square$

Definition 3.7. Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S.

(1) The *big Zariski site of S*, denoted $(Sch/S)_{Zar}$, is the site Sch_{Zar}/S introduced in Sites, Section 24.
(2) The *small Zariski site of S*, which we denote S_{Zar}, is the full subcategory of $(Sch/S)_{Zar}$ whose objects are those U/S such that $U \to S$ is an open immersion. A covering of S_{Zar} is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}(S_{Zar})$.
(3) The *big affine Zariski site of S*, denoted $(Aff/S)_{Zar}$, is the full subcategory of $(Sch/S)_{Zar}$ whose objects are affine U/S. A covering of $(Aff/S)_{Zar}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ which is a standard Zariski covering.

It is not completely clear that the small Zariski site and the big affine Zariski site are sites. We check this now.

Lemma 3.8. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. Both S_{Zar} and $(Aff/S)_{Zar}$ are sites.*

Proof. Let us show that S_{Zar} is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 6.2. Since $(Sch/S)_{Zar}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}(S_{Zar})$ we also have $U_i \in \mathrm{Ob}(S_{Zar})$. This follows from the definitions as the composition of open immersions is an open immersion.

Let us show that $(Aff/S)_{Zar}$ is a site. Reasoning as above, it suffices to show that the collection of standard Zariski coverings of affines satisfies properties (1), (2) and

(3) of Sites, Definition 6.2. Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. For each $i \in \{1, \ldots, n\}$ let $g_{i1}, \ldots, g_{in_i} \in R_{f_i}$ be elements generating the unit ideal of R_{f_i}. Write $g_{ij} = f_{ij}/f_i^{e_{ij}}$ which is possible. After replacing f_{ij} by $f_i f_{ij}$ if necessary, we have that $D(f_{ij}) \subset D(f_i) \cong \mathrm{Spec}(R_{f_i})$ is equal to $D(g_{ij}) \subset \mathrm{Spec}(R_{f_i})$. Hence we see that the family of morphisms $\{D(g_{ij}) \to \mathrm{Spec}(R)\}$ is a standard Zariski covering. From these considerations it follows that (2) holds for standard Zariski coverings. We omit the verification of (1) and (3). $\square$

Lemma 3.9. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The underlying categories of the sites Sch_{Zar}, $(Sch/S)_{Zar}$, S_{Zar}, and $(Aff/S)_{Zar}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{Zar}$, and S_{Zar} both have a final object, namely S/S.*

Proof. For Sch_{Zar} it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{Zar})$. The fibre product $V \times_U W$ in Sch_{Zar} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{Zar}$. This proves the result for $(Sch/S)_{Zar}$. If $U \to S$, $V \to U$ and $W \to U$ are open immersions then so is $V \times_U W \to S$ and hence we get the result for S_{Zar}. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{Zar}$. $\square$

Next, we check that the big affine site defines the same topos as the big site.

Lemma 3.10. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The functor $(Aff/S)_{Zar} \to (Sch/S)_{Zar}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((Aff/S)_{Zar})$ to $Sh((Sch/S)_{Zar})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (Aff/S)_{Zar} \to (Sch/S)_{Zar}$. Being cocontinuous just means that any Zariski covering of T/S, T affine, can be refined by a standard Zariski covering of T. This is the content of Lemma 3.3. Hence (1) holds. We see u is continuous simply because a standard Zariski covering is a Zariski covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. $\square$

Let us check that the notion of a sheaf on the small Zariski site corresponds to notion of a sheaf on S.

Lemma 3.11. *The category of sheaves on S_{Zar} is equivalent to the category of sheaves on the underlying topological space of S.*

Proof. We will use repeatedly that for any object U/S of S_{Zar} the morphism $U \to S$ is an isomorphism onto an open subscheme. Let $\mathcal{F}$ be a sheaf on S. Then we define a sheaf on S_{Zar} by the rule $\mathcal{F}'(U/S) = \mathcal{F}(\mathrm{Im}(U \to S))$. For the converse, we choose for every open subscheme $U \subset S$ an object $U'/S \in \mathrm{Ob}(S_{Zar})$ with $\mathrm{Im}(U' \to S) = U$ (here you have to use Sets, Lemma 9.9). Given a sheaf $\mathcal{G}$ on S_{Zar} we define a sheaf on S by setting $\mathcal{G}(U) = \mathcal{G}(U'/S)$. To see that $\mathcal{G}'$ is a sheaf we use that for any open covering $U = \bigcup_{i \in I} U_i$ the covering $\{U_i \to U\}_{i \in I}$ is combinatorially equivalent to a covering $\{U'_j \to U'\}_{j \in J}$ in S_{Zar} by Sets, Lemma 11.1, and we use Sites, Lemma 8.4. Details omitted. $\square$

From now on we will not make any distinction between a sheaf on S_{Zar} or a sheaf on S. We will always use the procedures of the proof of the lemma to go between the two notions. Next, we establish some relationships between the topoi associated to these sites.

Lemma 3.12. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor $T_{Zar} \to (Sch/S)_{Zar}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{Zar}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{Zar} \to (Sch/S)_{Zar}$. In other words, given and open immersion $j : U \to T$ corresponding to an object of T_{Zar} we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 3.9. Moreover, T_{Zar} has equalizers (as any two morphisms with the same source and target are the same) and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 20.5 and 20.6. $\square$

Lemma 3.13. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The inclusion functor $S_{Zar} \to (Sch/S)_{Zar}$ satisfies the hypotheses of Sites, Lemma 20.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{Zar} \longrightarrow S_{Zar}$$

and a morphism of topoi

$$i_S : Sh(S_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 3.12. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{Zar} \to (Sch/S)_{Zar}$, in addition to the properties seen in the proof of Lemma 3.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows. $\square$

Definition 3.14. In the situation of Lemma 3.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small Zariski site*, and for a sheaf $\mathcal{F}$ on the big Zariski site we denote $\mathcal{F}|_{S_{Zar}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\begin{aligned}
\mathrm{Mor}_{Sh(S_{Zar})}(\mathcal{F}|_{S_{Zar}}, \mathcal{G}) &= \mathrm{Mor}_{Sh((Sch/S)_{Zar})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\
\mathrm{Mor}_{Sh(S_{Zar})}(\mathcal{G}, \mathcal{F}|_{S_{Zar}}) &= \mathrm{Mor}_{Sh((Sch/S)_{Zar})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})
\end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$.

Lemma 3.15. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor*

$$u : (Sch/T)_{Zar} \longrightarrow (Sch/S)_{Zar}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{Zar} \longrightarrow (Sch/T)_{Zar}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 3.12). Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. □

Lemma 3.16. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}.*

(1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 3.12 and i_T as in Lemma 3.13.*
(2) *The functor $S_{Zar} \to T_{Zar}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi*

$$f_{small} : Sh(T_{Zar}) \longrightarrow Sh(S_{Zar}).$$

The functors f_{small}^{-1} and $f_{small,*}$ agree with the usual notions f^{-1} and f_* is we identify sheaves on T_{Zar}, resp. S_{Zar} with sheaves on T, resp. S via Lemma 3.11.

(3) *We have a commutative diagram of morphisms of sites*

$$\begin{tikzcd} T_{Zar} \arrow[d, "f_{small}"'] & (Sch/T)_{Zar} \arrow[l, "\pi_T"'] \arrow[d, "f_{big}"] \\ S_{Zar} & (Sch/S)_{Zar} \arrow[l, "\pi_S"] \end{tikzcd}$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.

(4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

Statement (2): See Sites, Example 15.2.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functor $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 3.14 we have: for $\mathcal{F}$ a sheaf on the big Zariski site of T

$$(f_{big,*}\mathcal{F})|_{S_{Zar}} = f_{small,*}(\mathcal{F}|_{T_{Zar}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small Zariski site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 3.17. *Given schemes X, Y, Y in $(Sch/S)_{Zar}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 3.15. For the functors on the small sites this is Sheaves, Lemma 21.2 via the identification of Lemma 3.11. $\square$

We can think about a sheaf on the big Zariski site of S as a collection of "usual" sheaves on all schemes over S.

Lemma 3.18. *Let S be a scheme contained in a big Zariski site Sch_{Zar}. A sheaf $\mathcal{F}$ on the big Zariski site $(Sch/S)_{Zar}$ is given by the following data:*

1. *for every $T/S \in \mathrm{Ob}((Sch/S)_{Zar})$ a sheaf $\mathcal{F}_T$ on T,*
2. *for every $f : T' \to T$ in $(Sch/S)_{Zar}$ a map $c_f : f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

- (i) *given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{Zar}$ the composition $g^{-1}c_f \circ c_g$ is equal to $c_{f\circ g}$, and*
- (ii) *if $f : T' \to T$ in $(Sch/S)_{Zar}$ is an open immersion then c_f is an isomorphism.*

Proof. Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U'/S)$ for any open $U \subset T$, and $U' \to T$ an open immersion in $(Sch/T)_{Zar}$ with image U, see Lemmas 3.11 and 3.12. Hence given $f : T' \to T$ over S and $U, U' \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U'/S) \to \mathcal{F}(U' \times_T T'/S) = \mathcal{F}_{T'}(f^{-1}(U))$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U' \times_T T' \to U'$ over S. The collection of these maps are compatible with restrictions, and hence define an f-map c_f from $\mathcal{F}_T$ to $\mathcal{F}_{T'}$, see Sheaves, Definition 21.7 and the discussion surrounding it. It is clear that $c_{f\circ g}$ is the composition of c_f and c_g, since composition of restriction maps of $\mathcal{F}$ gives restriction maps.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ (where we think of c_f as an f-map again). The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. $\square$

4. The étale topology

Let S be a scheme. We would like to define the étale-topology on the category of schemes over S. According to our general principle we first introduce the notion of an étale covering.

Definition 4.1. Let T be a scheme. An *étale covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is étale and such that $T = \bigcup f_i(T_i)$.

Lemma 4.2. *Any Zariski covering is an étale covering.*

Proof. This is clear from the definitions and the fact that an open immersion is an étale morphism, see Morphisms, Lemma 37.9. $\square$

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 4.3. *Let T be a scheme.*

1. *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an étale covering of T.*
2. *If $\{T_i \to T\}_{i \in I}$ is an étale covering and for each i we have an étale covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an étale covering.*
3. *If $\{T_i \to T\}_{i \in I}$ is an étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an étale covering.*

Proof. Omitted. $\square$

Lemma 4.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an étale covering of T. Then there exists an étale covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted. $\square$

Thus we define the corresponding standard coverings of affines as follows.

Definition 4.5. Let T be an affine scheme. A *standard étale covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine and étale over T and $T = \bigcup f_j(U_j)$.

In the definition above we do **not** assume the morphisms f_j are standard étale. The reason is that if we did then the standard étale coverings would not define a site on Aff/S, for example because of Algebra, Lemma 138.15 part (4). On the other hand, an étale morphism of affines is automatically standard smooth, see Algebra, Lemma 138.2. Hence a standard étale covering is a standard smooth covering and a standard syntomic covering.

Definition 4.6. A *big étale site* is any site $\mathit{Sch}_{\acute{e}tale}$ as in Sites, Definition 6.2 constructed as follows:

1. Choose any set of schemes S_0, and any set of étale coverings Cov_0 among these schemes.
2. As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
3. Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of étale coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big étale site of a scheme S, let us point out that the topology on a big étale site $\mathit{Sch}_{\acute{e}tale}$ is in some sense induced from the étale topology on the category of all schemes.

Lemma 4.7. *Let $Sch_{étale}$ be a big étale site as in Definition 4.6. Let $T \in \mathrm{Ob}(Sch_{étale})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary étale covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site $Sch_{étale}$ which refines $\{T_i \to T\}_{i \in I}$.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a standard étale covering, then it is tautologically equivalent to a covering in $Sch_{étale}$.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{étale}$.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 4.3 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an étale covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of $Sch_{étale}$ by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. $\square$

Definition 4.8. Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S.

(1) The *big étale site of S*, denoted $(Sch/S)_{étale}$, is the site $Sch_{étale}/S$ introduced in Sites, Section 24.
(2) The *small étale site of S*, which we denote $S_{étale}$, is the full subcategory of $(Sch/S)_{étale}$ whose objects are those U/S such that $U \to S$ is étale. A covering of $S_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in \mathrm{Ob}(S_{étale})$.
(3) The *big affine étale site of S*, denoted $(Aff/S)_{étale}$, is the full subcategory of $(Sch/S)_{étale}$ whose objects are affine U/S. A covering of $(Aff/S)_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ which is a standard étale covering.

It is not completely clear that the big affine étale site or the small étale site are sites. We check this now.

Lemma 4.9. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. Both $S_{étale}$ and $(Aff/S)_{étale}$ are sites.*

Proof. Let us show that $S_{étale}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 6.2. Since $(Sch/S)_{étale}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in \mathrm{Ob}(S_{étale})$ we also have $U_i \in \mathrm{Ob}(S_{étale})$. This follows from the definitions as the composition of étale morphisms is an étale morphism.

Let us show that $(Aff/S)_{étale}$ is a site. Reasoning as above, it suffices to show that the collection of standard étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This is clear since for example, given a standard étale covering $\{T_i \to T\}_{i \in I}$ and for each i we have a standard étale covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard étale covering because $\bigcup_{i \in I} J_i$ is finite and each T_{ij} is affine. $\square$

Lemma 4.10. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The underlying categories of the sites $Sch_{étale}$, $(Sch/S)_{étale}$, $S_{étale}$, and $(Aff/S)_{étale}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{étale}$, and $S_{étale}$ both have a final object, namely S/S.*

Proof. For $Sch_{étale}$ it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{étale})$. The fibre product $V \times_U W$ in $Sch_{étale}$ is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{étale}$. This proves the result for $(Sch/S)_{étale}$. If $U \to S$, $V \to U$ and $W \to U$ are étale then so is $V \times_U W \to S$ and hence we get the result for $S_{étale}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{étale}$. □

Next, we check that the big affine site defines the same topos as the big site.

Lemma 4.11. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The functor $(Aff/S)_{étale} \to (Sch/S)_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{étale})$ to $Sh((Sch/S)_{étale})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (Aff/S)_{étale} \to (Sch/S)_{étale}$. Being cocontinuous just means that any étale covering of T/S, T affine, can be refined by a standard étale covering of T. This is the content of Lemma 4.4. Hence (1) holds. We see u is continuous simply because a standard étale covering is a étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

Next, we establish some relationships between the topoi associated to these sites.

Lemma 4.12. *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$. The functor $T_{étale} \to (Sch/S)_{étale}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{étale} \to (Sch/S)_{étale}$. In other words, given an étale morphism $j : U \to T$ corresponding to an object of $T_{étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 4.10. Let $a, b : U \to V$ be two morphisms in $T_{étale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/T}, V \times_T V, (a,b)} U \times_T U$$

which is a fibre product of schemes étale over T, hence étale over T. Thus $T_{étale}$ has equalizers and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 20.5 and 20.6. □

Lemma 4.13. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The inclusion functor $S_{étale} \to (Sch/S)_{étale}$ satisfies the hypotheses of Sites, Lemma 20.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

and a morphism of topoi

$$i_S : Sh(S_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 4.12. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{étale} \to (Sch/S)_{étale}$, in addition to the properties seen in the proof of Lemma 4.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 20.8. $\square$

Definition 4.14. In the situation of Lemma 4.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small étale site*, and for a sheaf $\mathcal{F}$ on the big étale site we denote $\mathcal{F}|_{S_{étale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\begin{aligned} \mathrm{Mor}_{Sh(S_{étale})}(\mathcal{F}|_{S_{étale}}, \mathcal{G}) &= \mathrm{Mor}_{Sh((Sch/S)_{étale})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\ \mathrm{Mor}_{Sh(S_{étale})}(\mathcal{G}, \mathcal{F}|_{S_{étale}}) &= \mathrm{Mor}_{Sh((Sch/S)_{étale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F}) \end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{étale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{étale}} = \mathcal{G}$.

Lemma 4.15. *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$. The functor*

$$u : (Sch/T)_{étale} \longrightarrow (Sch/S)_{étale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{étale} \longrightarrow (Sch/T)_{étale}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 4.12). Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. $\square$

Lemma 4.16. *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$.*

1. *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 4.12 and i_T as in Lemma 4.13.*

(2) *The functor $S_{étale} \to T_{étale}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi*
$$f_{small} : Sh(T_{étale}) \longrightarrow Sh(S_{étale}).$$
We have $f_{small,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.

(3) *We have a commutative diagram of morphisms of sites*
$$\begin{tikzcd} T_{étale} \arrow[d, "f_{small}"'] & (Sch/T)_{étale} \arrow[l, "\pi_T"] \arrow[d, "f_{big}"] \\ S_{étale} & (Sch/S)_{étale} \arrow[l, "\pi_S"'] \end{tikzcd}$$
so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.

(4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{étale} \to T_{étale}$, $u(U \to S) = (U \times_S T \to T)$ transforms coverings into coverings and commutes with fibre products, see Lemma 4.3 (3) and 4.10. Moreover, both $S_{étale}$, $T_{étale}$ have final objects, namely S/S and T/T and $u(S/S) = T/T$. Hence by Sites, Proposition 15.6 the functor u corresponds to a morphism of sites $T_{étale} \to S_{étale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 16.2. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. $\square$

In the situation of the lemma, using the terminology of Definition 4.14 we have: for $\mathcal{F}$ a sheaf on the big étale site of T
$$(f_{big,*}\mathcal{F})|_{S_{étale}} = f_{small,*}(\mathcal{F}|_{T_{étale}}),$$
This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 4.17. *Given schemes X, Y, Y in $Sch_{étale}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 4.15. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 4.16. $\square$

We can think about a sheaf on the big étale site of S as a collection of "usual" sheaves on all schemes over S.

Lemma 4.18. *Let S be a scheme contained in a big étale site $Sch_{étale}$. A sheaf $\mathcal{F}$ on the big étale site $(Sch/S)_{étale}$ is given by the following data:*

(1) *for every $T/S \in \mathrm{Ob}((Sch/S)_{étale})$ a sheaf $\mathcal{F}_T$ on $T_{étale}$,*
(2) *for every $f : T' \to T$ in $(Sch/S)_{étale}$ a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

(i) *given any* $f : T' \to T$ *and* $g : T'' \to T'$ *in* $(Sch/S)_{\acute{e}tale}$ *the composition* $g_{small}^{-1}c_f \circ c_g$ *is equal to* $c_{f\circ g}$*, and*
(ii) *if* $f : T' \to T$ *in* $(Sch/S)_{\acute{e}tale}$ *is étale then* c_f *is an isomorphism.*

Proof. Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{\acute{e}tale})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U/S)$ for any $U \to T$ in $T_{\acute{e}tale}$ see Lemma 4.12. Hence given $f : T' \to T$ over S and $U \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U/S) \to \mathcal{F}(U \times_T T'/S) = \mathcal{F}_{T'}(U \times_T T')$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U \times_T T' \to U$ over S. The collection of these maps are compatible with restrictions, and hence define a map $c'_f : \mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ where $u : T_{\acute{e}tale} \to T'_{\acute{e}tale}$ is the base change functor associated to f. By adjunction of $f_{small,*}$ (see Sites, Section 14) with f_{small}^{-1} this is the same as a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$. It is clear that $c'_{f\circ g}$ is the composition of c'_f and $f_{small,*}c'_g$, since composition of restriction maps of $\mathcal{F}$ gives restriction maps, and this gives the desired relationship among c_f, c_g and $c_{f\circ g}$.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{\acute{e}tale})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ where we think of c_f as a map $\mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ again. The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. □

5. The smooth topology

In this section we define the smooth topology. This is a bit pointless as it will turn out later (see More on Morphisms, Section 28) that this topology defines the same topos as the étale topology. But still it makes sense and it is used occasionally.

Definition 5.1. Let T be a scheme. An *smooth covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is smooth and such that $T = \bigcup f_i(T_i)$.

Lemma 5.2. *Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.*

Proof. This is clear from the definitions, the fact that an étale morphism is smooth see Morphisms, Definition 37.1 and Lemma 4.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 5.3. *Let* T *be a scheme.*

(1) *If* $T' \to T$ *is an isomorphism then* $\{T' \to T\}$ *is an smooth covering of* T*.*
(2) *If* $\{T_i \to T\}_{i\in I}$ *is a smooth covering and for each* i *we have a smooth covering* $\{T_{ij} \to T_i\}_{j\in J_i}$*, then* $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ *is a smooth covering.*
(3) *If* $\{T_i \to T\}_{i\in I}$ *is a smooth covering and* $T' \to T$ *is a morphism of schemes then* $\{T' \times_T T_i \to T'\}_{i\in I}$ *is a smooth covering.*

Proof. Omitted. □

Lemma 5.4. *Let* T *be an affine scheme. Let* $\{T_i \to T\}_{i\in I}$ *be a smooth covering of* T*. Then there exists a smooth covering* $\{U_j \to T\}_{j=1,\ldots,m}$ *which is a refinement of* $\{T_i \to T\}_{i\in I}$ *such that each* U_j *is an affine scheme, and such that each morphism*

$U_j \to T$ is standard smooth, see Morphisms, Definition 35.1. Moreover, we may choose each U_j to be open affine in one of the T_i.

Proof. Omitted, but see Algebra, Lemma 132.10. $\square$

Thus we define the corresponding standard coverings of affines as follows.

Definition 5.5. Let T be an affine scheme. A *standard smooth covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard smooth and $T = \bigcup f_j(U_j)$.

Definition 5.6. A *big smooth site* is any site Sch_{smooth} as in Sites, Definition 6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of smooth coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of smooth coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big smooth site of a scheme S, let us point out that the topology on a big smooth site Sch_{smooth} is in some sense induced from the smooth topology on the category of all schemes.

Lemma 5.7. *Let Sch_{smooth} be a big smooth site as in Definition 5.6. Let $T \in \text{Ob}(Sch_{smooth})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary smooth covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{smooth} which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard smooth covering, then it is tautologically equivalent to a covering of Sch_{smooth}.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{smooth}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 5.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an smooth covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{smooth} by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. $\square$

Definition 5.8. Let S be a scheme. Let Sch_{smooth} be a big smooth site containing S.

(1) The *big smooth site of S*, denoted $(Sch/S)_{smooth}$, is the site Sch_{smooth}/S introduced in Sites, Section 24.
(2) The *big affine smooth site of S*, denoted $(Aff/S)_{smooth}$, is the full subcategory of $(Sch/S)_{smooth}$ whose objects are affine U/S. A covering of $(Aff/S)_{smooth}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{smooth}$ which is a standard smooth covering.

Next, we check that the big affine site defines the same topos as the big site.

Lemma 5.9. *Let S be a scheme. Let $Sch_{étale}$ be a big smooth site containing S. The functor $(Aff/S)_{smooth} \to (Sch/S)_{smooth}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{smooth})$ to $Sh((Sch/S)_{smooth})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (Aff/S)_{smooth} \to (Sch/S)_{smooth}$. Being cocontinuous just means that any smooth covering of T/S, T affine, can be refined by a standard smooth covering of T. This is the content of Lemma 5.4. Hence (1) holds. We see u is continuous simply because a standard smooth covering is a smooth covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

Lemma 5.10. *Let Sch_{smooth} be a big smooth site. Let $f : T \to S$ be a morphism in Sch_{smooth}. The functor*

$$u : (Sch/T)_{smooth} \longrightarrow (Sch/S)_{smooth}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{smooth} \longrightarrow (Sch/T)_{smooth}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{smooth}) \longrightarrow Sh((Sch/S)_{smooth})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. □

6. The syntomic topology

In this section we define the syntomic topology. This topology is quite interesting in that it often has the same cohomology groups as the fppf topology but is technically easier to deal with.

Definition 6.1. Let T be a scheme. An *syntomic covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is syntomic and such that $T = \bigcup f_i(T_i)$.

Lemma 6.2. *Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.*

Proof. This is clear from the definitions and the fact that a smooth morphism is syntomic, see Morphisms, Lemma 35.7 and Lemma 5.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 6.3. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an syntomic covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a syntomic covering and for each i we have a syntomic covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a syntomic covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a syntomic covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a syntomic covering.*

Proof. Omitted. □

Lemma 6.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a syntomic covering of T. Then there exists a syntomic covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \to T$ is standard syntomic, see Morphisms, Definition 32.1. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted, but see Algebra, Lemma 131.15. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 6.5. Let T be an affine scheme. A *standard syntomic covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard syntomic and $T = \bigcup f_j(U_j)$.

Definition 6.6. A *big syntomic site* is any site $Sch_{syntomic}$ as in Sites, Definition 6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of syntomic coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of syntomic coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big syntomic site of a scheme S, let us point out that the topology on a big syntomic site $Sch_{syntomic}$ is in some sense induced from the syntomic topology on the category of all schemes.

Lemma 6.7. *Let $Sch_{syntomic}$ be a big syntomic site as in Definition 6.6. Let $T \in \text{Ob}(Sch_{syntomic})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary syntomic covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{syntomic}$ which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard syntomic covering, then it is tautologically equivalent to a covering in $Sch_{syntomic}$.*

(3) *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{syntomic}$.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 6.3 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an syntomic covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j \in J}$ is a covering as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{Zar} by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. □

Definition 6.8. Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S.

(1) The *big syntomic site of* S, denoted $(Sch/S)_{syntomic}$, is the site $Sch_{syntomic}/S$ introduced in Sites, Section 24.
(2) The *big affine syntomic site of* S, denoted $(Aff/S)_{syntomic}$, is the full subcategory of $(Sch/S)_{syntomic}$ whose objects are affine U/S. A covering of $(Aff/S)_{syntomic}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{syntomic}$ which is a standard syntomic covering.

Next, we check that the big affine site defines the same topos as the big site.

Lemma 6.9. *Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S. The functor $(Aff/S)_{syntomic} \to (Sch/S)_{syntomic}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{syntomic})$ to $Sh((Sch/S)_{syntomic})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (Aff/S)_{syntomic} \to (Sch/S)_{syntomic}$. Being cocontinuous just means that any syntomic covering of T/S, T affine, can be refined by a standard syntomic covering of T. This is the content of Lemma 6.4. Hence (1) holds. We see u is continuous simply because a standard syntomic covering is a syntomic covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

Lemma 6.10. *Let $Sch_{syntomic}$ be a big syntomic site. Let $f : T \to S$ be a morphism in $Sch_{syntomic}$. The functor*

$$u : (Sch/T)_{syntomic} \longrightarrow (Sch/S)_{syntomic}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{syntomic} \longrightarrow (Sch/T)_{syntomic}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{syntomic}) \longrightarrow Sh((Sch/S)_{syntomic})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. $\square$

7. The fppf topology

Let S be a scheme. We would like to define the fppf-topology[2] on the category of schemes over S. According to our general principle we first introduce the notion of an fppf-covering.

Definition 7.1. Let T be a scheme. An *fppf covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.

Lemma 7.2. *Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.*

Proof. This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation, see Morphisms, Lemmas 32.6 and 32.7, and Lemma 6.2. $\square$

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 7.3. *Let T be a scheme.*

1. *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fppf covering of T.*
2. *If $\{T_i \to T\}_{i \in I}$ is an fppf covering and for each i we have an fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fppf covering.*
3. *If $\{T_i \to T\}_{i \in I}$ is an fppf covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fppf covering.*

Proof. The first assertion is clear. The second follows as the composition of flat morphisms is flat (see Morphisms, Lemma 26.5) and the composition of morphisms of finite presentation is of finite presentation (see Morphisms, Lemma 22.3). The third follows as the base change of a flat morphism is flat (see Morphisms, Lemma 26.7) and the base change of a morphism of finite presentation is of finite presentation (see Morphisms, Lemma 22.4). Moreover, the base change of a surjective family of morphisms is surjective (proof omitted). $\square$

Lemma 7.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fppf covering of T. Then there exists an fppf covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definitions using that a morphism which is flat and locally of finite presentation is open, see Morphisms, Lemma 26.9. $\square$

Thus we define the corresponding standard coverings of affines as follows.

[2] The letters fppf stand for "fidèlement plat de présentation finie".

Definition 7.5. Let T be an affine scheme. A *standard fppf covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, flat and of finite presentation over T and $T = \bigcup f_j(U_j)$.

Definition 7.6. A *big fppf site* is any site Sch_{fppf} as in Sites, Definition 6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of fppf coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of fppf coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big fppf site of a scheme S, let us point out that the topology on a big fppf site Sch_{fppf} is in some sense induced from the fppf topology on the category of all schemes.

Lemma 7.7. *Let Sch_{fppf} be a big fppf site as in Definition 7.6. Let $T \in \text{Ob}(Sch_{fppf})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary fppf covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{fppf} which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard fppf covering, then it is tautologically equivalent to a covering of Sch_{fppf}.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{fppf}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 7.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an fppf covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{fppf} by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. $\square$

Definition 7.8. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S.

(1) The *big fppf site of S*, denoted $(Sch/S)_{fppf}$, is the site Sch_{fppf}/S introduced in Sites, Section 24.
(2) The *big affine fppf site of S*, denoted $(Aff/S)_{fppf}$, is the full subcategory of $(Sch/S)_{fppf}$ whose objects are affine U/S. A covering of $(Aff/S)_{fppf}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{fppf}$ which is a standard fppf covering.

It is not completely clear that the big affine fppf site is a site. We check this now.

Lemma 7.9. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. Then $(Aff/S)_{fppf}$ is a site.*

Proof. Let us show that $(\mathit{Aff}/S)_{fppf}$ is a site. Reasoning as in the proof of Lemma 4.9 it suffices to show that the collection of standard fppf coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This is clear since for example, given a standard fppf covering $\{T_i \to T\}_{i \in I}$ and for each i we have a standard fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard fppf covering because $\bigcup_{i \in I} J_i$ is finite and each T_{ij} is affine. $\square$

Lemma 7.10. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The underlying categories of the sites Sch_{fppf}, $(\mathit{Sch}/S)_{fppf}$, and $(\mathit{Aff}/S)_{fppf}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(\mathit{Sch}/S)_{fppf}$ has a final object, namely S/S.*

Proof. For Sch_{fppf} it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(\mathit{Sch}_{fppf})$. The fibre product $V \times_U W$ in Sch_{fppf} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(\mathit{Sch}/S)_{fppf}$. This proves the result for $(\mathit{Sch}/S)_{fppf}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(\mathit{Aff}/S)_{fppf}$. $\square$

Next, we check that the big affine site defines the same topos as the big site.

Lemma 7.11. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The functor $(\mathit{Aff}/S)_{fppf} \to (\mathit{Sch}/S)_{fppf}$ is cocontinuous and induces an equivalence of topoi from $\mathit{Sh}((\mathit{Aff}/S)_{fppf})$ to $\mathit{Sh}((\mathit{Sch}/S)_{fppf})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (\mathit{Aff}/S)_{fppf} \to (\mathit{Sch}/S)_{fppf}$. Being cocontinuous just means that any fppf covering of T/S, T affine, can be refined by a standard fppf covering of T. This is the content of Lemma 7.4. Hence (1) holds. We see u is continuous simply because a standard fppf covering is a fppf covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. $\square$

Next, we establish some relationships between the topoi associated to these sites.

Lemma 7.12. *Let Sch_{fppf} be a big fppf site. Let $f : T \to S$ be a morphism in Sch_{fppf}. The functor*

$$u : (\mathit{Sch}/T)_{fppf} \longrightarrow (\mathit{Sch}/S)_{fppf}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (\mathit{Sch}/S)_{fppf} \longrightarrow (\mathit{Sch}/T)_{fppf}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : \mathit{Sh}((\mathit{Sch}/T)_{fppf}) \longrightarrow \mathit{Sh}((\mathit{Sch}/S)_{fppf})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. $\square$

Lemma 7.13. *Given schemes X, Y, Y in $(Sch/S)_{fppf}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 7.12. $\square$

8. The fpqc topology

Definition 8.1. Let T be a scheme. An *fpqc covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \ldots, n$ with $\bigcup_{j=1}^{n} f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. It is slightly harder to recognize an fpqc covering, hence we provide some lemmas to do so.

Lemma 8.2. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

1. *$\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering,*
2. *each f_i is flat and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,*
3. *each f_i is flat and there exists an affine open covering $T = \bigcup_{\alpha \in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \ldots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U_\alpha = \bigcup_{j=1,\ldots,n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$.*

If T is quasi-separated, these are also equivalent to

4. *each f_i is flat, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T.*

Proof. We omit the proof of the equivalence of (1), (2), and (3). From now on assume T is quasi-separated. We prove (4) implies (2). Let $U \subset T$ be an affine open. To prove (2) it suffices to show that for every $t \in U$ there exist finitely many quasi-compact opens $U_j \subset T_{i_j}$ such that $f_{i_j}(U_j) \subset U$ and such that $\bigcup f_{i_j}(U_j)$ is a neighbourhood of t in U. By assumption there do exist finitely many quasi-compact opens $U'_j \subset T_{i_j}$ such that such that $\bigcup f_{i_j}(U'_j)$ is a neighbourhood of t in T. Since T is quasi-separated we see that $U_j = U'_j \cap f_j^{-1}(U)$ is quasi-compact open as desired. Since it is clear that (2) implies (4) the proof is finished. $\square$

Lemma 8.3. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

1. *$\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering, and*
2. *setting $T' = \coprod_{i \in I} T_i$, and $f = \coprod_{i \in I} f_i$ the family $\{f : T' \to T\}$ is an fpqc covering.*

Proof. Suppose that $U \subset T$ is an affine open. If (1) holds, then we find $i_1, \ldots, i_n \in I$ and affine opens $U_j \subset T_{i_j}$ such that $U = \bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$. Then $U_1 \amalg \ldots \amalg U_n \subset T'$ is a quasi-compact open surjecting onto U. Thus $\{f : T' \to T\}$ is an fpqc covering by Lemma 8.2. Conversely, if (2) holds then there exists a quasi-compact open $U' \subset T'$ with $U = f(U')$. Then $U_j = U' \cap T_j$ is quasi-compact open in T_j and empty for almost all j. By Lemma 8.2 we see that (1) holds. □

Lemma 8.4. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *the family $\{f_i : T_i \to T\}_{i \in I}$ can be refined by a fpqc covering of T.*

Then $\{f_i : T_i \to T\}_{i \in I}$ is a fpqc covering of T.

Proof. Let $\{g_j : X_j \to T\}_{j \in J}$ be an fpqc covering refining $\{f_i : T_i \to T\}$. Suppose that $U \subset T$ is affine open. Choose $j_1, \ldots, j_m \in J$ and $V_k \subset X_{j_k}$ affine open such that $U = \bigcup g_{j_k}(V_k)$. For each j pick $i_j \in I$ and a morphism $h_j : X_j \to T_{i_j}$ such that $g_j = f_{i_j} \circ h_j$. Since $h_{j_k}(V_k)$ is quasi-compact we can find a quasi-compact open $h_{j_k}(V_k) \subset U_k \subset f^{-1}_{i_{j_k}}(U)$. Then $U = \bigcup f_{i_{j_k}}(U_k)$. We conclude that $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering by Lemma 8.2. □

Lemma 8.5. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *there exists an fpqc covering $\{g_j : S_j \to T\}_{j \in J}$ such that each $\{S_j \times_T T_i \to S_j\}_{i \in I}$ is an fpqc covering.*

Then $\{f_i : T_i \to T\}_{i \in I}$ is a fpqc covering of T.

Proof. We will use Lemma 8.2 without further mention. Let $U \subset T$ be an affine open. By (2) we can find quasi-compact opens $V_j \subset S_j$ for $j \in J$, almost all empty, such that $U = \bigcup g_j(V_j)$. Then for each j we can choose quasi-compact opens $W_{ij} \subset S_j \times_T T_i$ for $i \in I$, almost all empty, with $V_j = \bigcup_i \mathrm{pr}_1(W_{ij})$. Thus $\{S_j \times_T T_i \to T\}$ is an fpqc covering. Since this covering refines $\{f_i : T_i \to T\}$ we conclude by Lemma 8.4. □

Lemma 8.6. *Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.*

Proof. We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 7.2. Let $\{f_i : U_i \to U\}_{i \in I}$ be an fppf covering. By definition this means that the f_i are flat which checks the first condition of Definition 8.1. To check the second, let $V \subset U$ be an affine open subset. Write $f_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since each f_i is open (Morphisms, Lemma 26.9), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} f_i(V_{ij})$ is an open covering of V. Since V is quasi-compact, this covering has a finite refinement. This finishes the proof. □

The fpqc[3] topology cannot be treated in the same way as the fppf topology[4]. Namely, suppose that R is a nonzero ring. For any faithfully flat ring map $R \to R'$ the morphism $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is an fpqc-covering. We claim that there does

[3] The letters fpqc stand for "fidèlement plat quasi-compacte".

[4] A more precise statement would be that the analogue of Lemma 7.7 for the fpqc topology does not hold.

not exist a set A of fpqc-coverings of $\mathrm{Spec}(R)$ such that every fpqc-covering can be refined by an element of A. For example, if $R = k$ is a field, then for any set I we can consider the purely transcendental field extension $k \subset k(\{t_i\}_{i\in I})$. We leave it to the reader to show that there does not exist a set of morphisms of schemes $\{S_j \to \mathrm{Spec}(k)\}_{j\in J}$ such that every morphism $\mathrm{Spec}(k(\{t_i\}_{i\in I}))$ is dominated by one of the schemes S_j.

A mildly interesting option is to consider only those faithfully flat ring extensions $R \to R'$ where the cardinality of R' is suitably bounded. (And if you consider all schemes in a fixed universe as in SGA4 then you are bounding the cardinality by a strongly inaccessible cardinal.) However, it is not so clear what happens if you change the cardinal to a bigger one.

For these reasons we do not introduce fpqc sites and we will not consider cohomology with respect to the fpqc-topology.

On the other hand, given a contravariant functor $F : Sch^{opp} \to Sets$ it does make sense to ask whether F satisfies the sheaf property for the fpqc topology, see below. Moreover, we can wonder about descent of object in the fpqc topology, etc. Simply put, for certain results the correct generality is to work with fpqc coverings.

Lemma 8.7. *Let T be a scheme.*

1. *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fpqc covering of T.*
2. *If $\{T_i \to T\}_{i\in I}$ is an fpqc covering and for each i we have an fpqc covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an fpqc covering.*
3. *If $\{T_i \to T\}_{i\in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is an fpqc covering.*

Proof. Part (1) is immediate. Recall that the composition of flat morphisms is flat and that the base change of a flat morphism is flat (Morphisms, Lemmas 26.7 and 26.5). Thus we can apply Lemma 8.2 in each case to check that our families of morphisms are fpqc coverings.

Proof of (2). Assume $\{T_i \to T\}_{i\in I}$ is an fpqc covering and for each i we have an fpqc covering $\{f_{ij} : T_{ij} \to T_i\}_{j\in J_i}$. Let $U \subset T$ be an affine open. We can find quasi-compact opens $U_i \subset T_i$ for $i \in I$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then for each i we can choose quasi-compact opens $W_{ij} \subset T_{ij}$ for $j \in J_i$, almost all empty, with $U_i = \bigcup_j f_{ij}(U_{ij})$. Thus $\{T_{ij} \to T\}$ is an fpqc covering.

Proof of (3). Assume $\{T_i \to T\}_{i\in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes. Let $U' \subset T'$ be an affine open which maps into the affine open $U \subset T$. Choose quasi-compact opens $U_i \subset T_i$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then $U' \times_U U_i$ is a quasi-compact open of $T' \times_T T_i$ and $U' = \bigcup \mathrm{pr}_1(U' \times_U U_i)$. Since T' can be covered by such affine opens $U' \subset T'$ we see that $\{T' \times_T T_i \to T'\}_{i\in I}$ is an fpqc covering by Lemma 8.2 □

Lemma 8.8. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be an fpqc covering of T. Then there exists an fpqc covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definition. □

Definition 8.9. Let T be an affine scheme. A *standard fpqc covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,n}$ with each U_j is affine, flat over T and $T = \bigcup f_j(U_j)$.

Since we do not introduce the affine site we have to show directly that the collection of all standard fpqc coverings satisfies the axioms.

Lemma 8.10. *Let T be an affine scheme.*

1. *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a standard fpqc covering of T.*
2. *If $\{T_i \to T\}_{i\in I}$ is a standard fpqc covering and for each i we have a standard fpqc covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a standard fpqc covering.*
3. *If $\{T_i \to T\}_{i\in I}$ is a standard fpqc covering and $T' \to T$ is a morphism of affine schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a standard fpqc covering.*

Proof. This follows formally from the fact that compositions and base changes of flat morphisms are flat (Morphisms, Lemmas 26.7 and 26.5) and that fibre products of affine schemes are affine (Schemes, Lemma 17.2). $\square$

Lemma 8.11. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. Assume that*

1. *each f_i is flat, and*
2. *every affine scheme Z and morphism $h : Z \to T$ there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ which refines the family $\{T_i \times_T Z \to Z\}_{i\in I}$.*

Then $\{f_i : T_i \to T\}_{i\in I}$ is a fpqc covering of T.

Proof. Let $T = \bigcup U_\alpha$ be an affine open covering. For each α the pullback family $\{T_i \times_T U_\alpha \to U_\alpha\}$ can be refined by a standard fpqc covering, hence is an fpqc covering by Lemma 8.4. As $\{U_\alpha \to T\}$ is an fpqc covering we conclude that $\{T_i \to T\}$ is an fpqc covering by Lemma 8.5. $\square$

Definition 8.12. Let F be a contravariant functor on the category of schemes with values in sets.

1. Let $\{U_i \to T\}_{i\in I}$ be a family of morphisms of schemes with fixed target. We say that F *satisfies the sheaf property for the given family* if for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i\times_T U_j} = \xi_j|_{U_i\times_T U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
2. We say that F *satisfies the sheaf property for the fpqc topology* if it satisfies the sheaf property for any fpqc covering.

We try to avoid using the terminology "F is a sheaf" in this situation since we are not defining a category of fpqc sheaves as we explained above.

Lemma 8.13. *Let F be a contravariant functor on the category of schemes with values in sets. Then F satisfies the sheaf property for the fpqc topology if and only if it satisfies*

1. *the sheaf property for every Zariski covering, and*
2. *the sheaf property for any standard fpqc covering.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for $\{V \to U\}$ with V, U affine and $V \to U$ faithfully flat.*

Proof. Assume (1) and (2) hold. Let $\{f_i : T_i \to T\}_{i\in I}$ be an fpqc covering. Let $s_i \in F(T_i)$ be a family of elements such that s_i and s_j map to the same element of $F(T_i \times_T T_j)$. Let $W \subset T$ be the maximal open subset such that there exists a unique $s \in F(W)$ with $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$ for all i. Such a maximal open exists because F satisfies the sheaf property for Zariski coverings; in fact W is the union of all opens with this property. Let $t \in T$. We will show $t \in W$. To do this we pick an affine open $t \in U \subset T$ and we will show there is a unique $s \in F(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all i.

By Lemma 8.8 we can find a standard fpqc covering $\{U_j \to U\}_{j=1,\ldots,n}$ refining $\{U \times_T T_i \to U\}$, say by morphisms $h_j : U_j \to T_{i_j}$. By (2) we obtain a unique element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{i_j})$. Note that for any scheme $V \to U$ over U there is a unique section $s_V \in F(V)$ which restricts to $F(h_j \circ \mathrm{pr}_2)(s_{i_j})$ on $V \times_U U_j$ for $j = 1,\ldots,n$. Namely, this is true if V is affine by (2) as $\{V \times_U U_j \to V\}$ is a standard fpqc covering and in general this follows from (1) and the affine case by choosing an affine open covering of V. In particular, $s_V = s|_V$. Now, taking $V = U \times_T T_i$ and using that $s_{i_j}|_{T_{i_j}\times_T T_i} = s_i|_{T_{i_j}\times_T T_i}$ we conclude that $s|_{U\times_T T_i} = s_V = s_i|_{U\times_T T_i}$ which is what we had to show.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $\{T_i \to T\}$ is a standard fpqc covering, then $\coprod T_i \to T$ is a faithfully flat morphism of affine schemes. In the presence of (1) we have $F(\coprod T_i) = \prod F(T_i)$ and similarly $F((\coprod T_i) \times_T (\coprod T_i)) = \prod F(T_i \times_T T_{i'})$. Thus the sheaf condition for $\{T_i \to T\}$ and $\{\coprod T_i \to T\}$ is the same. $\square$

9. Change of topologies

Let $f : X \to Y$ be a morphism of schemes over a base scheme S. In this case we have the following morphisms of sites (with suitable choices of sites as in Remark 9.1 below):

(1) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$,
(2) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{syntomic}$,
(3) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{smooth}$,
(4) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{étale}$,
(5) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{Zar}$,
(6) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{syntomic}$,
(7) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{smooth}$,
(8) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{étale}$,
(9) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{Zar}$,
(10) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{smooth}$,
(11) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{étale}$,
(12) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{Zar}$,
(13) $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{étale}$,
(14) $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{Zar}$,
(15) $(Sch/X)_{Zar} \longrightarrow (Sch/Y)_{Zar}$,
(16) $(Sch/X)_{fppf} \longrightarrow Y_{étale}$,
(17) $(Sch/X)_{syntomic} \longrightarrow Y_{étale}$,
(18) $(Sch/X)_{smooth} \longrightarrow Y_{étale}$,
(19) $(Sch/X)_{étale} \longrightarrow Y_{étale}$,
(20) $(Sch/X)_{fppf} \longrightarrow Y_{Zar}$,

(21) $(Sch/X)_{syntomic} \longrightarrow Y_{Zar}$,
(22) $(Sch/X)_{smooth} \longrightarrow Y_{Zar}$,
(23) $(Sch/X)_{étale} \longrightarrow Y_{Zar}$,
(24) $(Sch/X)_{Zar} \longrightarrow Y_{Zar}$,
(25) $X_{étale} \longrightarrow Y_{étale}$,
(26) $X_{étale} \longrightarrow Y_{Zar}$,
(27) $X_{Zar} \longrightarrow Y_{Zar}$,

In each case the underlying continuous functor $Sch/Y \to Sch/X$, or $Y_\tau \to Sch/X$ is the functor $Y'/Y \mapsto X \times_Y Y'/X$. Namely, in the sections above we have seen the morphisms $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ and $f_{small} : X_\tau \to Y_\tau$ for τ as above. We also have seen the morphisms of sites $\pi_Y : (Sch/Y)_\tau \to Y_\tau$ for $\tau \in \{étale, Zariski\}$. On the other hand, it is clear that the identity functor $(Sch/X)_\tau \to (Sch/X)_{\tau'}$ defines a morphism of sites when τ is a stronger topology than τ'. Hence composing these gives the list of possible morphisms above.

Because of the simple description of the underlying functor it is clear that given morphisms of schemes $X \to Y \to Z$ the composition of two of the morphisms of sites above, e.g.,

$$(Sch/X)_{\tau_0} \longrightarrow (Sch/Y)_{\tau_1} \longrightarrow (Sch/Z)_{\tau_2}$$

is the corresponding morphism of sites associated to the morphism of schemes $X \to Z$.

Remark 9.1. Take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set of schemes $\{X, Y, S\}$. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained. Next, for $\tau \in \{Zariski, étale, smooth, syntomic\}$ let Sch_τ have the same underlying category as Sch_{fppf} with coverings $\text{Cov}_\tau \subset \text{Cov}_{fppf}$ simply the subset of τ-coverings. It is straightforward to check that this gives rise to a big site Sch_τ.

10. Change of big sites

In this section we explain what happens on changing the big Zariski/fppf/étale sites.

Let $\tau, \tau' \in \{Zariski, étale, smooth, syntomic, fppf\}$. Given two big sites Sch_τ and $Sch'_{\tau'}$ we say that *Sch_τ is contained in $Sch'_{\tau'}$* if $\text{Ob}(Sch_\tau) \subset \text{Ob}(Sch'_{\tau'})$ and $\text{Cov}(Sch_\tau) \subset \text{Cov}(Sch'_{\tau'})$. In this case τ is stronger than τ', for example, no fppf site can be contained in an étale site.

Lemma 10.1. *Any set of big Zariski sites is contained in a common big Zariski site. The same is true, mutatis mutandis, for big fppf and big étale sites.*

Proof. This is true because the union of a set of sets is a set, and the constructions in Sets, Lemmas 9.2 and 11.1 allow one to start with any initially given set of schemes and coverings. $\square$

Lemma 10.2. *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Suppose given big sites Sch_τ and Sch'_τ. Assume that Sch_τ is contained in Sch'_τ. The inclusion functor $Sch_\tau \to Sch'_\tau$ satisfies the assumptions of Sites, Lemma 20.8. There are* *morphisms of topoi*

$$\begin{aligned} g : Sh(Sch_\tau) & \longrightarrow Sh(Sch'_\tau) \\ f : Sh(Sch'_\tau) & \longrightarrow Sh(Sch_\tau) \end{aligned}$$

such that $f \circ g \cong id$. For any object S of Sch_τ the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 20.8 also. Hence similarly we obtain morphisms

$$\begin{aligned} g : Sh((Sch/S)_\tau) & \longrightarrow Sh((Sch'/S)_\tau) \\ f : Sh((Sch'/S)_\tau) & \longrightarrow Sh((Sch/S)_\tau) \end{aligned}$$

with $f \circ g \cong id$.

Proof. Assumptions (b), (c), and (e) of Sites, Lemma 20.8 are immediate for the functors $Sch_\tau \to Sch'_\tau$ and $(Sch/S)_\tau \to (Sch'/S)_\tau$. Property (a) holds by Lemma 3.6, 4.7, 5.7, 6.7, or 7.7. Property (d) holds because fibre products in the categories Sch_τ, Sch'_τ exist and are compatible with fibre products in the category of schemes. ☐

Discussion: The functor $g^{-1} = f_*$ is simply the restriction functor which associates to a sheaf $\mathcal{G}$ on Sch'_τ the restriction $\mathcal{G}|_{Sch_\tau}$. Hence this lemma simply says that given any sheaf of sets $\mathcal{F}$ on Sch_τ there exists a canonical sheaf $\mathcal{F}'$ on Sch'_τ such that $\mathcal{F}|_{Sch'_\tau} = \mathcal{F}'$. In fact the sheaf $\mathcal{F}'$ has the following description: it is the sheafification of the presheaf

$$Sch'_\tau \longrightarrow Sets, \quad V \longmapsto \operatorname{colim}_{V \to U} \mathcal{F}(U)$$

where U is an object of Sch_τ. This is true because $\mathcal{F}' = f^{-1}\mathcal{F} = (u_p\mathcal{F})^\#$ according to Sites, Lemmas 20.5 and 20.8.

11. Other chapters

References

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