

# ARTIN'S AXIOMS

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## 1. Introduction

In this chapter we discuss Artin's axioms for the representability of functors by algebraic spaces. As references we suggest the papers [Art69], [Art70], [Art74].

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 1 for an explanation.

## 2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 2. In this chapter the base scheme

$S$  will often be locally Noetherian (although we will always reiterate this condition when stating results).

### 3. Predeformation categories

Let  $S$  be a locally Noetherian base scheme. Let

$$p : \mathcal{X} \longrightarrow (Sch/S)_{fppf}$$

be a category fibred in groupoids. Let  $k$  be a field and let  $\text{Spec}(k) \rightarrow S$  be a morphism of finite type (see Morphisms, Lemma 17.1). We will sometimes simply say that  $k$  is a field of finite type over  $S$ . Let  $x_0$  be an object of  $\mathcal{X}$  lying over  $\text{Spec}(k)$ . Given  $S$ ,  $\mathcal{X}$ ,  $k$ , and  $x_0$  we will construct a predeformation category, as defined in Formal Deformation Theory, Definition 6.2. The construction will resemble to construction of Formal Deformation Theory, Remark 6.4.

First, by Morphisms, Lemma 17.1 we may pick an affine open  $\text{Spec}(\Lambda) \subset S$  such that  $\text{Spec}(k) \rightarrow S$  factors through  $\text{Spec}(\Lambda)$  and the associated ring map  $\Lambda \rightarrow k$  is finite. This provides us with the category  $\mathcal{C}_\Lambda$ , see Formal Deformation Theory, Definition 3.1. The category  $\mathcal{C}_\Lambda$ , up to canonical equivalence, does not depend on the choice of the affine open  $\text{Spec}(\Lambda)$  of  $S$ . Namely,  $\mathcal{C}_\Lambda$  is equivalent to the opposite of the category of factorizations

$$(3.0.1) \quad \text{Spec}(k) \rightarrow \text{Spec}(A) \rightarrow S$$

of the structure morphism such that  $A$  is an Artinian local ring and such that  $\text{Spec}(k) \rightarrow \text{Spec}(A)$  corresponds to a ring map  $A \rightarrow k$  which identifies  $k$  with the residue field of  $A$ .

We let  $\mathcal{F} = \mathcal{F}_{\mathcal{X}, k, x_0}$  be the category whose

- (1) objects are morphisms  $x_0 \rightarrow x$  of  $\mathcal{X}$  where  $p(x) = \text{Spec}(A)$  with  $A$  an Artinian local ring and  $p(x_0) \rightarrow p(x) \rightarrow S$  a factorization as in (3.0.1), and
- (2) morphisms  $(x_0 \rightarrow x) \rightarrow (x_0 \rightarrow x')$  are commutative diagrams

$$\begin{array}{ccc} x & \xleftarrow{\quad} & x' \\ & \nwarrow \quad \nearrow & \\ & x_0 & \end{array}$$

in  $\mathcal{X}$ . (Note the reversal of arrows.)

If  $x_0 \rightarrow x$  is an object of  $\mathcal{F}$  then writing  $p(x) = \text{Spec}(A)$  we obtain an object  $A$  of  $\mathcal{C}_\Lambda$ . We often say that  $x_0 \rightarrow x$  or  $x$  lies over  $A$ . A morphism of  $\mathcal{F}$  between objects  $x_0 \rightarrow x$  lying over  $A$  and  $x_0 \rightarrow x'$  lying over  $A'$  corresponds to a morphism  $x' \rightarrow x$  of  $\mathcal{X}$ , hence a morphism  $p(x' \rightarrow x) : \text{Spec}(A') \rightarrow \text{Spec}(A)$  which in turn corresponds to a ring map  $A \rightarrow A'$ . As  $\mathcal{X}$  is a category over the category of schemes over  $S$  we see that  $A \rightarrow A'$  is  $\Lambda$ -algebra homomorphism. Thus we obtain a functor

$$(3.0.2) \quad p : \mathcal{F} = \mathcal{F}_{\mathcal{X}, k, x_0} \longrightarrow \mathcal{C}_\Lambda.$$

We will use the notation  $\mathcal{F}(A)$  to denote the fibre category over an object  $A$  of  $\mathcal{C}_\Lambda$ . An object of  $\mathcal{F}(A)$  is simply a morphism  $x_0 \rightarrow x$  of  $\mathcal{X}$  such that  $x$  lies over  $\text{Spec}(A)$  and  $x_0 \rightarrow x$  lies over  $\text{Spec}(k) \rightarrow \text{Spec}(A)$ .

**Lemma 3.1.** *The functor  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  defined above is a predeformation category.*

**Proof.** We have to show that  $\mathcal{F}$  is (a) cofibred in groupoids over  $\mathcal{C}_\Lambda$  and (b) that  $\mathcal{F}(k)$  is a category equivalent to a category with a single object and a single morphism.

Proof of (a). The fibre categories of  $\mathcal{F}$  over  $\mathcal{C}_\Lambda$  are groupoids as the fibre categories of  $\mathcal{X}$  are groupoids. Let  $A \rightarrow A'$  be a morphism of  $\mathcal{C}_\Lambda$  and let  $x_0 \rightarrow x$  be an object of  $\mathcal{F}(A)$ . Because  $\mathcal{X}$  is fibred in groupoids, we can find a morphism  $x' \rightarrow x$  lying over  $\text{Spec}(A') \rightarrow \text{Spec}(A)$ . Since the composition  $A \rightarrow A' \rightarrow k$  is equal to the given map  $A \rightarrow k$  we see (by uniqueness of pullbacks up to isomorphism) that the pullback via  $\text{Spec}(k) \rightarrow \text{Spec}(A')$  of  $x'$  is  $x_0$ , i.e., that there exists a morphism  $x_0 \rightarrow x'$  lying over  $\text{Spec}(k) \rightarrow \text{Spec}(A')$  compatible with  $x_0 \rightarrow x$  and  $x' \rightarrow x$ . This proves that  $\mathcal{F}$  has pushforwards. We conclude by (the dual of) Categories, Lemma 33.2.

Proof of (b). If  $A = k$ , then  $\text{Spec}(k) = \text{Spec}(A)$  and since  $\mathcal{X}$  is fibred in groupoids over  $(\text{Sch}/S)_{fppf}$  we see that given any object  $x_0 \rightarrow x$  in  $\mathcal{F}(k)$  the morphism  $x_0 \rightarrow x$  is an isomorphism. Hence every object of  $\mathcal{F}(k)$  is isomorphic to  $x_0 \rightarrow x_0$ . Clearly the only self morphism of  $x_0 \rightarrow x_0$  in  $\mathcal{F}$  is the identity.  $\square$

Let  $S$  be a locally Noetherian base scheme. Let  $F : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism between categories fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . Let  $k$  be a field of finite type over  $S$ . Let  $x_0$  be an object of  $\mathcal{X}$  lying over  $\text{Spec}(k)$ . Set  $y_0 = F(x_0)$  which is an object of  $\mathcal{Y}$  lying over  $\text{Spec}(k)$ . Then  $F$  induces a functor

$$(3.1.1) \quad F : \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{F}_{\mathcal{Y},k,y_0}$$

of categories cofibred over  $\mathcal{C}_\Lambda$ . Namely, to the object  $x_0 \rightarrow x$  of  $\mathcal{F}_{\mathcal{X},k,x_0}(A)$  we associate the object  $F(x_0) \rightarrow F(x)$  of  $\mathcal{F}_{\mathcal{Y},k,y_0}(A)$ .

**Lemma 3.2.** *Let  $S$  be a locally Noetherian scheme. Let  $F : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(\text{Sch}/S)_{fppf}$  which is formally smooth on objects (see Criteria for Representability, Section 6). Then for every finite type field  $k$  over  $S$  and for every object  $x_0$  of  $\mathcal{X}$  over  $k$  the functor (3.1.1) is smooth in the sense of Formal Deformation Theory, Definition 8.1.*

**Proof.** This is a matter of unwinding the definitions. Details omitted.  $\square$

**Lemma 3.3.** *Let  $S$  be a locally Noetherian scheme. Let*

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

*be a 2-fibre product of categories fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . Let  $k$  be a finite type field over  $S$  and  $w_0$  an object of  $\mathcal{W}$  over  $k$ . Let  $x_0, z_0, y_0$  be the images of  $w_0$  under the morphisms in the diagram. Then*

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{W},k,w_0} & \longrightarrow & \mathcal{F}_{\mathcal{Z},k,z_0} \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathcal{X},k,x_0} & \longrightarrow & \mathcal{F}_{\mathcal{Y},k,y_0} \end{array}$$

*is a fibre product of predeformation categories.*

**Proof.** This is a matter of unwinding the definitions. Details omitted.  $\square$

#### 4. Pushouts and stacks

In this section we show that algebraic stacks behave well with respect to certain pushouts. The results in this section hold over any base scheme.

The following lemma is also correct when  $Y, X', X, Y'$  are algebraic spaces, see (insert future reference here).

**Lemma 4.1.** *Let  $S$  be a scheme. Let*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

*be a pushout in the category of schemes over  $S$  where  $X \rightarrow X'$  is a thickening and  $X \rightarrow Y$  is affine, see More on Morphisms, Lemma 11.1. Let  $\mathcal{Z}$  be an algebraic stack over  $S$ . Then the functor of fibre categories*

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

*is an equivalence of categories.*

**Proof.** Let  $y'$  be an object of left hand side. The sheaf  $\text{Isom}(y', y')$  on the category of schemes over  $Y'$  is representable by an algebraic space  $I$  over  $Y'$ , see Algebraic Stacks, Lemma 10.11. We conclude that the functor of the lemma is fully faithful as  $Y'$  is the pushout in the category of algebraic spaces as well as the category of schemes, see Pushouts of Spaces, Lemma 2.2.

Let  $(y, x', f)$  be an object of the right hand side. Here  $f : y|_X \rightarrow x'|_X$  is an isomorphism. To finish the proof we have to construct an object  $y'$  of  $\mathcal{Z}_{Y'}$  whose restrictions to  $Y$  and  $X'$  agree with  $y$  and  $x'$  in a manner compatible with  $\varphi$ . In fact, it suffices to construct  $y'$  fppf locally on  $Y'$ , see Stacks, Lemma 4.8. Choose a representable algebraic stack  $\mathcal{W}$  and a surjective smooth morphism  $\mathcal{W} \rightarrow \mathcal{Z}$ . Then

$$(Sch/Y)_{fppf} \times_{y, \mathcal{Z}} \mathcal{W} \quad \text{and} \quad (Sch/X')_{fppf} \times_{x', \mathcal{Z}} \mathcal{W}$$

are algebraic stacks representable by algebraic spaces  $V$  and  $U'$  smooth over  $Y$  and  $X'$ . The isomorphism  $f$  induces an isomorphism  $\varphi : V \times_Y X \rightarrow U' \times_{X'} X$  over  $X$ . By Pushouts of Spaces, Lemmas 2.3 and 2.8 we see that the pushout  $V' = V \amalg_{V \times_Y X} U'$  is an algebraic space smooth over  $Y'$  whose base change to  $Y$  and  $X'$  recovers  $V$  and  $U'$  in a manner compatible with  $\varphi$ .

Let  $W$  be the algebraic space representing  $\mathcal{W}$ . The projections  $V \rightarrow W$  and  $U' \rightarrow W$  agree as morphisms over  $V \times_Y X \cong U' \times_{X'} X$  hence the universal property of the pushout determines a morphism of algebraic spaces  $V' \rightarrow W$ . Choose a scheme  $Y'_1$  and a surjective étale morphism  $Y'_1 \rightarrow V'$ . Set  $Y_1 = Y \times_{Y'} Y'_1$ ,  $X'_1 = X' \times_{Y'} Y'_1$ ,  $X_1 = X \times_{Y'} Y'_1$ . The composition

$$(Sch/Y'_1) \rightarrow (Sch/V') \rightarrow (Sch/W) = \mathcal{W} \rightarrow \mathcal{Z}$$

corresponds by the 2-Yoneda lemma to an object  $y'_1$  of  $\mathcal{Z}$  over  $Y'_1$  whose restriction to  $Y_1$  and  $X'_1$  agrees with  $y|_{Y_1}$  and  $x'|_{X'_1}$  in a manner compatible with  $f|_{X_1}$ . Thus we have constructed our desired object smooth locally over  $Y'$  and we win.  $\square$

### 5. The Rim-Schlessinger condition

The motivation for the following definition comes from Lemma 4.1 and Formal Deformation Theory, Definition 15.1 and Lemma 15.4.

**Definition 5.1.** Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{Z}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . We say  $\mathcal{Z}$  satisfies *condition (RS)* if for every pushout

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' = Y \amalg_X X' \end{array}$$

in the category of schemes over  $S$  where

- (1)  $X, X', Y, Y'$  are spectra of local Artinian rings,
- (2)  $X, X', Y, Y'$  are of finite type over  $S$ , and
- (3)  $X \rightarrow X'$  (and hence  $Y \rightarrow Y'$ ) is a closed immersion

the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

If  $A$  is an Artinian local ring with residue field  $k$ , then any morphism  $\mathrm{Spec}(A) \rightarrow S$  is affine and of finite type if and only if the induced morphism  $\mathrm{Spec}(k) \rightarrow S$  is of finite type, see Morphisms, Lemmas 13.13 and 17.2.

**Lemma 5.2.** *Let  $\mathcal{X}$  be an algebraic stack over a locally Noetherian base  $S$ . Then  $\mathcal{X}$  satisfies (RS).*

**Proof.** Immediate from the definitions and Lemma 4.1.  $\square$

**Lemma 5.3.** *Let  $S$  be a scheme. Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  and  $q : \mathcal{Z} \rightarrow \mathcal{Y}$  be 1-morphisms of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . If  $\mathcal{X}$ ,  $\mathcal{Y}$ , and  $\mathcal{Z}$  satisfy (RS), then so does  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ .*

**Proof.** This is formal. Let

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' = Y \amalg_X X' \end{array}$$

be a diagram as in Definition 5.1. We have to show that

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{Y'} \longrightarrow (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_Y \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_X} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{X'}$$

is an equivalence. Using the definition of the 2-fibre product this becomes

$$(5.3.1) \quad \mathcal{X}_{Y'} \times_{\mathcal{Y}_{Y'}} \mathcal{Z}_{Y'} \longrightarrow (\mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y) \times_{(\mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X)} (\mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}).$$

We are given that each of the functors

$$\mathcal{X}_{Y'} \rightarrow \mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y, \quad \mathcal{Y}_{Y'} \rightarrow \mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X, \quad \mathcal{Z}_{Y'} \rightarrow \mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}$$

are equivalences. An object of the right hand side of (5.3.1) is a system

$$((x_Y, z_Y, \phi_Y), (x_{X'}, z_{X'}, \phi_{X'}), (\alpha, \beta)).$$

Then  $(x_Y, x_{Y'}, \alpha)$  is isomorphic to the image of an object  $x_{Y'}$  in  $\mathcal{X}_{Y'}$  and  $(z_Y, z_{Y'}, \beta)$  is isomorphic to the image of an object  $z_{Y'}$  of  $\mathcal{Z}_{Y'}$ . The pair of morphisms  $(\phi_Y, \phi_{X'})$  corresponds to a morphism  $\psi$  between the images of  $x_{Y'}$  and  $z_{Y'}$  in  $\mathcal{Y}_{Y'}$ . Then  $(x_{Y'}, z_{Y'}, \psi)$  is an object of the left hand side of (5.3.1) mapping to the given object of the right hand side. This proves that (5.3.1) is essentially surjective. We omit the proof that it is fully faithful.  $\square$

## 6. Deformation categories

We match the notation introduced above with the notation from the chapter “Formal Deformation Theory”.

**Lemma 6.1.** *Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$  satisfying (RS). For any field  $k$  of finite type over  $S$  and any object  $x_0$  of  $\mathcal{X}$  lying over  $k$  the predeformation category  $p : \mathcal{F}_{\mathcal{X}, k, x_0} \rightarrow \mathcal{C}_\Lambda$  (3.0.2) is a deformation category, see Formal Deformation Theory, Definition 15.8.*

**Proof.** Set  $\mathcal{F} = \mathcal{F}_{\mathcal{X}, k, x_0}$ . Let  $f_1 : A_1 \rightarrow A$  and  $f_2 : A_2 \rightarrow A$  be ring maps in  $\mathcal{C}_\Lambda$  with  $f_2$  surjective. We have to show that the functor

$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence, see Formal Deformation Theory, Lemma 15.4. Set  $X = \text{Spec}(A)$ ,  $X' = \text{Spec}(A_2)$ ,  $Y = \text{Spec}(A_1)$  and  $Y' = \text{Spec}(A_1 \times_A A_2)$ . Note that  $Y' = Y \amalg_X X'$  in the category of schemes, see More on Morphisms, Lemma 11.1. We know that in the diagram of functors of fibre categories

$$\begin{array}{ccc} \mathcal{X}_{Y'} & \longrightarrow & \mathcal{X}_Y \times_{\mathcal{X}_X} \mathcal{X}_{X'} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\text{Spec}(k)} & \xlongequal{\quad} & \mathcal{X}_{\text{Spec}(k)} \end{array}$$

the top horizontal arrow is an equivalence by Definition 5.1. Since  $\mathcal{F}(B)$  is the category of objects of  $\mathcal{X}_{\text{Spec}(B)}$  with an identification with  $x_0$  over  $k$  we win.  $\square$

**Remark 6.2.** Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $k$  be a field of finite type over  $S$  and  $x_0$  an object of  $\mathcal{X}$  over  $k$ . Let  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  be as in (3.0.2). If  $\mathcal{F}$  is a deformation category, i.e., if  $\mathcal{F}$  satisfies the Rim-Schlessinger condition (RS), then we see that  $\mathcal{F}$  satisfies Schlessingers conditions (S1) and (S2) by Formal Deformation Theory, Lemma 15.6. Let  $\overline{\mathcal{F}}$  be the functor of isomorphism classes, see Formal Deformation Theory, Remarks 5.2 (10). Then  $\overline{\mathcal{F}}$  satisfies (S1) and (S2) as well, see Formal Deformation Theory, Lemma 9.5. This holds in particular in the situation of Lemma 6.1.

## 7. Change of field

This section is the analogue of Formal Deformation Theory, Section 26. As pointed out there, to discuss what happens under change of field we need to write  $\mathcal{C}_{\Lambda, k}$  instead of  $\mathcal{C}_\Lambda$ . In the following lemma we use the notation  $\mathcal{F}_{l/k}$  introduced in Formal Deformation Theory, Situation 26.1.

**Lemma 7.1.** *Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $k$  be a field of finite type over  $S$  and let  $k \subset l$  be*

a finite extension. Let  $x_0$  be an object of  $\mathcal{F}$  lying over  $\mathrm{Spec}(k)$ . Denote  $x_{l,0}$  the restriction of  $x_0$  to  $\mathrm{Spec}(l)$ . Then there is a canonical functor

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

of categories cofibred in groupoids over  $\mathcal{C}_{\Lambda,l}$ . If  $\mathcal{X}$  satisfies (RS), then this functor is an equivalence.

**Proof.** Consider a factorization

$$\mathrm{Spec}(l) \rightarrow \mathrm{Spec}(B) \rightarrow S$$

as in (3.0.1). By definition we have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k}(B) = \mathcal{F}_{\mathcal{X},k,x_0}(B \times_l k)$$

see Formal Deformation Theory, Situation 26.1. Thus an object of this is a morphism  $x_0 \rightarrow x$  of  $\mathcal{X}$  lying over the morphism  $\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(B \times_l k)$ . Choosing pullback functor for  $\mathcal{X}$  we can associate to  $x_0 \rightarrow x$  the morphism  $x_{l,0} \rightarrow x_B$  where  $x_B$  is the restriction of  $x$  to  $\mathrm{Spec}(B)$  (via the morphism  $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(B \times_l k)$  coming from  $B \times_l k \subset B$ ). This construction is functorial in  $B$  and compatible with morphisms.

Next, assume  $\mathcal{X}$  satisfies (RS). Consider the diagrams

$$\begin{array}{ccc} l & \longleftarrow & B \\ \uparrow & & \uparrow \\ k & \longleftarrow & B \times_l k \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Spec}(l) & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(B \times_l k) \end{array}$$

The diagram on the left is a fibre product of rings. The diagram on the right is a pushout in the category of schemes, see More on Morphisms, Lemma 11.1. These schemes are all of finite type over  $S$  (see remarks following Definition 5.1). Hence (RS) kicks in to give an equivalence of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(B \times_l k)} \longrightarrow \mathcal{X}_{\mathrm{Spec}(k)} \times_{\mathcal{X}_{\mathrm{Spec}(l)}} \mathcal{X}_{\mathrm{Spec}(B)}$$

This implies that the functor defined above gives an equivalence of fibre categories. Hence the functor is an equivalence on categories cofibred in groupoids by (the dual of) Categories, Lemma 33.8.  $\square$

## 8. Tangent spaces

Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(\mathrm{Sch}/S)_{fppf}$ . Let  $k$  be a field of finite type over  $S$  and let  $x_0$  be an object of  $\mathcal{X}$  over  $k$ . In Formal Deformation Theory, Section 11 we have defined the *tangent space*

$$(8.0.1) \quad T\mathcal{F}_{\mathcal{X},k,x_0} = \left\{ \text{isomorphism classes of morphisms } \begin{array}{c} x_0 \rightarrow x \text{ over } \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k[\epsilon]) \end{array} \right\}$$

of the predeformation category  $\mathcal{F}_{\mathcal{X},k,x_0}$ . In Formal Deformation Theory, Section 18 we have defined

$$(8.0.2) \quad \mathrm{Inf}_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Ker}(\mathrm{Aut}_{\mathrm{Spec}(k[\epsilon])}(x'_0) \rightarrow \mathrm{Aut}_{\mathrm{Spec}(k)}(x_0))$$

where  $x'_0$  is the pullback of  $x_0$  to  $\mathrm{Spec}(k[\epsilon])$ . If  $\mathcal{X}$  satisfies the Rim-Schlessinger condition (RS), then  $T\mathcal{F}_{\mathcal{X},k,x_0}$  comes equipped with a natural  $k$ -vector space structure by Formal Deformation Theory, Lemma 11.2 (assumptions hold by Lemma 6.1 and Remark 6.2). Moreover, Formal Deformation Theory, Lemma 18.9 shows that

$\mathrm{Inf}_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0})$  has a natural  $k$ -vector space structure such that addition agrees with composition of automorphisms. A natural condition is to ask these vector spaces to have finite dimension.

The following lemma tells us this is true if  $\mathcal{X}$  is locally of finite type over  $S$  (see Morphisms of Stacks, Section 13).

**Lemma 8.1.** *Let  $S$  be a locally Noetherian scheme. Assume*

- (1)  $\mathcal{X}$  is an algebraic stack,
- (2)  $U$  is a scheme locally of finite type over  $S$ , and
- (3)  $(\mathrm{Sch}/U)_{fppf} \rightarrow \mathcal{X}$  is a smooth surjective morphism.

*Then, for any  $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$  as in Section 3 the tangent space  $T\mathcal{F}$  and infinitesimal automorphism space  $\mathrm{Inf}_{x_0}(\mathcal{F})$  have finite dimension over  $k$*

**Proof.** Let us write  $\mathcal{U} = (\mathrm{Sch}/U)_{fppf}$ . By our definition of algebraic stacks the 1-morphism  $\mathcal{U} \rightarrow \mathcal{X}$  is representable by algebraic spaces. Hence in particular the 2-fibre product

$$\mathcal{U}_{x_0} = (\mathrm{Sch}/\mathrm{Spec}(k))_{fppf} \times_{\mathcal{X}} \mathcal{U}$$

is representable by an algebraic space  $U_{x_0}$  over  $\mathrm{Spec}(k)$ . Then  $U_{x_0} \rightarrow \mathrm{Spec}(k)$  is smooth and surjective (in particular  $U_{x_0}$  is nonempty). By Spaces over Fields, Lemma 9.2 we can find a finite extension  $l \supset k$  and a point  $\mathrm{Spec}(l) \rightarrow U_{x_0}$  over  $k$ . We have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} = \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

by Lemma 7.1 and the fact that  $\mathcal{X}$  satisfies (RS). Thus we see that

$$T\mathcal{F} \otimes_k l \cong T\mathcal{F}_{\mathcal{X},l,x_{l,0}} \quad \text{and} \quad \mathrm{Inf}_{x_0}(\mathcal{F}) \otimes_k l \cong \mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$$

by Formal Deformation Theory, Lemmas 26.3 and 26.4 (these are applicable by Lemmas 5.2 and 6.1 and Remark 6.2). Hence it suffices to prove that  $T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$  and  $\mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$  have finite dimension over  $l$ . Note that  $x_{l,0}$  comes from a point  $u_0$  of  $\mathcal{U}$  over  $l$ .

We interrupt the flow of the argument to show that the lemma for infinitesimal automorphisms follows from the lemma for tangent spaces. Namely, let  $\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$ . Let  $r_0$  be the  $l$ -valued point  $(u_0, u_0, \mathrm{id}_{x_0})$  of  $\mathcal{R}$ . Combining Lemma 3.3 and Formal Deformation Theory, Lemma 24.2 we see that

$$\mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{\mathcal{X},l,x_{l,0}}) \subset T\mathcal{F}_{\mathcal{R},k,r_0}$$

Note that  $\mathcal{R}$  is an algebraic stack, see Algebraic Stacks, Lemma 14.2. Also,  $\mathcal{R}$  is representably by an algebraic space  $R$  smooth over  $U$  (via either projection, see Algebraic Stacks, Lemma 16.2). Hence, choose an scheme  $U'$  and a surjective étale morphism  $U' \rightarrow R$  we see that  $U'$  is smooth over  $U$ , hence locally of finite type over  $S$ . As  $(\mathrm{Sch}/U')_{fppf} \rightarrow \mathcal{R}$  is surjective and smooth, we have reduced the question to the case of tangent spaces.

The functor (3.1.1)

$$\mathcal{F}_{\mathcal{U},l,u_0} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is smooth by Lemma 3.2 and Criteria for Representability, Lemma 6.3. The induced map on tangent spaces

$$T\mathcal{F}_{\mathcal{U},l,u_0} \longrightarrow T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is  $l$ -linear (by Formal Deformation Theory, Lemma 11.4) and surjective (as smooth maps of predeformation categories induce surjective maps on tangent spaces by



Formal Deformation Theory, Lemma 8.8). Hence it suffices to prove that the tangent space of the deformation space associated to the representable algebraic stack  $\mathcal{U}$  at the point  $u_0$  is finite dimensional. Let  $\text{Spec}(R) \subset U$  be an affine open such that  $u_0 : \text{Spec}(l) \rightarrow U$  factors through  $\text{Spec}(R)$  and such that  $\text{Spec}(R) \rightarrow S$  factors through  $\text{Spec}(\Lambda) \subset S$ . Let  $\mathfrak{m}_R \subset R$  be the kernel of the  $\Lambda$ -algebra map  $\varphi_0 : R \rightarrow l$  corresponding to  $u_0$ . Note that  $R$ , being of finite type over the Noetherian ring  $\Lambda$ , is a Noetherian ring. Hence  $\mathfrak{m}_R = (f_1, \dots, f_n)$  is a finitely generated ideal. We have

$$T\mathcal{F}_{\mathcal{U}, l, u_0} = \{\varphi : R \rightarrow l[\epsilon] \mid \varphi \text{ is a } \Lambda\text{-algebra map and } \varphi \bmod \epsilon = \varphi_0\}$$

An element of the right hand side is determined by its values on  $f_1, \dots, f_n$  hence the dimension is at most  $n$  and we win. Some details omitted.  $\square$

**Lemma 8.2.** *Let  $S$  be a locally Noetherian scheme. Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  and  $q : \mathcal{Z} \rightarrow \mathcal{Y}$  be 1-morphisms of categories fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . Assume  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$  satisfy (RS). Let  $k$  be a field of finite type over  $S$  and let  $w_0$  be an object of  $\mathcal{W} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$  over  $k$ . Denote  $x_0, y_0, z_0$  the objects of  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$  you get from  $w_0$ . Then there is a 6-term exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Inf}_{w_0}(\mathcal{F}_{\mathcal{W}, k, w_0}) & \longrightarrow & \text{Inf}_{x_0}(\mathcal{F}_{\mathcal{X}, k, x_0}) \oplus \text{Inf}_{z_0}(\mathcal{F}_{\mathcal{Z}, k, z_0}) & \longrightarrow & \text{Inf}_{y_0}(\mathcal{F}_{\mathcal{Y}, k, y_0}) \\ & & & & \swarrow & & \\ & & T\mathcal{F}_{\mathcal{W}, k, w_0} & \xleftarrow{\quad} & T\mathcal{F}_{\mathcal{X}, k, x_0} \oplus T\mathcal{F}_{\mathcal{Z}, k, z_0} & \xrightarrow{\quad} & T\mathcal{F}_{\mathcal{Y}, k, y_0} \end{array}$$

of  $k$ -vector spaces.

**Proof.** Apply Lemmas 3.3 and 6.1 and Formal Deformation Theory, Lemma 18.14.  $\square$

## 9. Formal objects

In this section we transfer some of the notions already defined in the chapter “Formal Deformation Theory” to the current setting. In the following we will say “ $R$  is an  $S$ -algebra” to indicate that  $R$  is a ring endowed with a morphism of schemes  $\text{Spec}(R) \rightarrow S$ .

**Definition 9.1.** Let  $S$  be a locally Noetherian scheme. Let  $p : \mathcal{X} \rightarrow (\text{Sch}/S)_{fppf}$  be a category fibred in groupoids.

- (1) A *formal object*  $\xi = (R, \xi_n, f_n)$  of  $\mathcal{X}$  consists of a Noetherian complete local  $S$ -algebra  $R$ , objects  $\xi_n$  of  $\mathcal{X}$  lying over  $\text{Spec}(R/\mathfrak{m}_R^n)$ , and morphisms  $\xi_n \rightarrow \xi_{n+1}$  of  $\mathcal{X}$  lying over  $\text{Spec}(R/\mathfrak{m}^n) \rightarrow \text{Spec}(R/\mathfrak{m}^{n+1})$  such that  $R/\mathfrak{m}$  is a field of finite type over  $S$ .
- (2) A *morphism of formal objects*  $a : \xi = (R, \xi_n, f_n) \rightarrow \eta = (T, \eta_n, g_n)$  is given by morphisms  $a_n : \xi_n \rightarrow \eta_n$  such that for every  $n$  the diagram

$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow{f_n} & \xi_n \\ a_{n+1} \downarrow & & \downarrow a_n \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

is commutative. Applying the functor  $p$  we obtain a compatible collection of morphisms  $\text{Spec}(R/\mathfrak{m}_R^n) \rightarrow \text{Spec}(T/\mathfrak{m}_T^n)$  and hence a morphism  $a_0 : \text{Spec}(R) \rightarrow \text{Spec}(T)$  over  $S$ . We say that  $a$  *lies over*  $a_0$ .

Thus we obtain a category of formal object of  $\mathcal{X}$ . If  $F : \mathcal{X} \rightarrow \mathcal{Y}$  is a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ , then  $F$  induces a functor between categories of formal objects as well.

Given a formal object  $\xi = (R, \xi_n, f_n)$ , set  $k = R/\mathfrak{m}$  and  $x_0 = \xi_1$ . Then the formal object  $\xi$  defines a formal object  $\xi$  of  $\mathcal{F}_{\mathcal{X}, k, x_0}$  as defined in Formal Deformation Theory, Definition 7.1. We will use the terminology introduced in that chapter. In particular, we will say that  $\xi$  is *versal* if  $\xi$  (as a formal object of  $\mathcal{F}$ ) is versal in the sense of Formal Deformation Theory, Definition 8.13. We briefly spell out here what this means. Suppose given a morphism  $x_0 \rightarrow y$  lying over a closed immersion  $\text{Spec}(k) \rightarrow \text{Spec}(A)$  where  $A$  is an Artinian local ring with residue field  $k$ . Then versality implies there exists an  $n \geq 1$  and a commutative diagram

$$\begin{array}{ccc}
 & y & \text{Spec}(A) \\
 \swarrow & \uparrow & \swarrow \\
 \xi_n & \xleftarrow{\quad} \xi_1 = x_0 & \text{Spec}(R/\mathfrak{m}^n) \xleftarrow{\quad} \text{Spec}(k)
 \end{array}
 \quad \text{lying over}$$

Please compare with Formal Deformation Theory, Remark 8.14.

**Lemma 9.2.** *Let  $S$  be a locally Noetherian scheme. Let  $F : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\eta = (R, \eta_n, g_n)$  be a formal object of  $\mathcal{Y}$  and let  $\xi_1$  be an object of  $\mathcal{X}$  with  $F(\xi_1) \cong \eta_1$ . If  $F$  is formally smooth on objects (see Criteria for Representability, Section 6), then there exists a formal object  $\xi = (R, \xi_n, f_n)$  of  $\mathcal{X}$  such that  $F(\xi) \cong \eta$ .*

**Proof.** Note that each of the morphisms  $\text{Spec}(R/\mathfrak{m}^n) \rightarrow \text{Spec}(R/\mathfrak{m}^{n+1})$  is a first order thickening of affine schemes over  $S$ . Hence the assumption on  $F$  means that we can successively lift  $\xi_1$  to objects  $\xi_2, \xi_3, \dots$  of  $\mathcal{X}$  endowed with compatible isomorphisms  $\eta_n|_{\text{Spec}(R/\mathfrak{m}^{n-1})} \cong \eta_{n-1}$  and  $F(\eta_n) \cong \xi_n$ .  $\square$

Let  $S$  be a locally Noetherian scheme. Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Suppose that  $x$  is an object of  $\mathcal{X}$  over  $R$ , where  $R$  is a Noetherian complete local  $S$ -algebra with residue field of finite type over  $S$ . Then we can consider the system of restrictions  $\xi_n = x|_{\text{Spec}(R/\mathfrak{m}^n)}$  endowed with the natural morphisms  $\xi_1 \rightarrow \xi_2 \rightarrow \dots$  coming from transitivity of restriction. Thus  $\xi = (R, \xi_n, \xi_n \rightarrow \xi_{n+1})$  is a formal object of  $\mathcal{X}$ . This construction is functorial in the object  $x$ . Thus we obtain a functor

$$(9.2.1) \quad \left\{ \begin{array}{l} \text{objects } x \text{ of } \mathcal{X} \text{ such that } p(x) = \text{Spec}(R) \\ \text{where } R \text{ is Noetherian complete local} \\ \text{with } R/\mathfrak{m} \text{ of finite type over } S \end{array} \right\} \longrightarrow \{\text{formal objects of } \mathcal{X}\}$$

To be precise the left hand side is the full subcategory of  $\mathcal{X}$  consisting of objects as indicated and the right hand side is the category of formal objects of  $\mathcal{X}$  as in Definition 9.1.

**Definition 9.3.** Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . A formal object  $\xi = (R, \xi_n, f_n)$  of  $\mathcal{X}$  is called *effective* if it is in the essential image of the functor (9.2.1).

If the category fibred in groupoids is an algebraic stack, then every formal object is effective as follows from the next lemma.

**Lemma 9.4.** *Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be an algebraic stack over  $S$ . The functor (9.2.1) is an equivalence.*

**Proof.** Case I:  $\mathcal{X}$  is representable (by a scheme). Say  $\mathcal{X} = (Sch/X)_{fppf}$  for some scheme  $X$  over  $S$ . Unwinding the definitions we have to prove the following: Given a Noetherian complete local  $S$ -algebra  $R$  with  $R/\mathfrak{m}$  of finite type over  $S$  we have

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective. This follows from Formal Spaces, Lemma 24.2.

Case II.  $\mathcal{X}$  is representable by an algebraic space. Say  $\mathcal{X}$  is representable by  $X$ . Again we have to show that

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective for  $R$  as above. This is Formal Spaces, Lemma 24.3.

Case III: General case of an algebraic stack. A general remark is that the left and right hand side of (9.2.1) are categories fibred in groupoids over the category of affine schemes over  $S$  which are spectra of Noetherian complete local rings with residue field of finite type over  $S$ . We will also see in the proof below that they form stacks for a certain topology on this category.

We first prove fully faithfulness. Let  $R$  be a Noetherian complete local  $S$ -algebra with  $k = R/\mathfrak{m}$  of finite type over  $S$ . Let  $x, x'$  be objects of  $\mathcal{X}$  over  $R$ . As  $\mathcal{X}$  is an algebraic stack  $Isom(x, x')$  is representable by an algebraic space  $I$  over  $\mathrm{Spec}(R)$ , see Algebraic Stacks, Lemma 10.11. Applying Case II to  $I$  over  $\mathrm{Spec}(R)$  implies immediately that (9.2.1) is fully faithful on fibre categories over  $\mathrm{Spec}(R)$ . Hence the functor is fully faithful by Categories, Lemma 33.8.

Essential surjectivity. Let  $\xi = (R, \xi_n, f_n)$  be a formal object of  $\mathcal{X}$ . Choose a scheme  $U$  over  $S$  and a surjective smooth morphism  $f : (Sch/U)_{fppf} \rightarrow \mathcal{X}$ . For every  $n$  consider the fibre product

$$(Sch/\mathrm{Spec}(R/\mathfrak{m}^n))_{fppf} \times_{\xi_n, \mathcal{X}, f} (Sch/U)_{fppf}$$

By assumption this is representable by an algebraic space  $V_n$  surjective and smooth over  $\mathrm{Spec}(R/\mathfrak{m}^n)$ . The morphisms  $f_n : \xi_n \rightarrow \xi_{n+1}$  induce cartesian squares

$$\begin{array}{ccc} V_{n+1} & \longleftarrow & V_n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R/\mathfrak{m}^{n+1}) & \longleftarrow & \mathrm{Spec}(R/\mathfrak{m}^n) \end{array}$$

of algebraic spaces. By Spaces over Fields, Lemma 9.2 we can find a finite separable extension  $k \subset k'$  and a point  $v'_1 : \mathrm{Spec}(k') \rightarrow V_1$  over  $k$ . Let  $R \subset R'$  be the finite étale extension whose residue field extension is  $k \subset k'$  (exists and is unique by Algebra, Lemmas 145.8 and 145.10). By the infinitesimal lifting criterion of smoothness (see More on Morphisms of Spaces, Lemma 16.6) applied to  $V_n \rightarrow \mathrm{Spec}(R/\mathfrak{m}^n)$  for  $n = 2, 3, 4, \dots$  we can successively find morphisms  $v'_n : \mathrm{Spec}(R'/(\mathfrak{m}')^n) \rightarrow V_n$  over

$\mathrm{Spec}(R/\mathfrak{m}^n)$  fitting into commutative diagrams

$$\begin{array}{ccc} \mathrm{Spec}(R'/(\mathfrak{m}')^{n+1}) & \longleftarrow & \mathrm{Spec}(R'/(\mathfrak{m}')^n) \\ \downarrow v'_{n+1} & & \downarrow v'_n \\ V_{n+1} & \longleftarrow & V_n \end{array}$$

Composing with the projection morphisms  $V_n \rightarrow U$  we obtain a compatible system of morphisms  $u'_n : \mathrm{Spec}(R'/(\mathfrak{m}')^n) \rightarrow U$ . By Case I the family  $(u'_n)$  comes from a unique morphism  $u' : \mathrm{Spec}(R') \rightarrow U$ . Denote  $x'$  the object of  $\mathcal{X}$  over  $\mathrm{Spec}(R')$  we get by applying the 1-morphism  $f$  to  $u'$ . By construction, there exists a morphism of formal objects

$$(9.2.1)(x') = (R', x'|_{\mathrm{Spec}(R'/(\mathfrak{m}')^n)}, \dots) \longrightarrow (R, \xi_n, f_n)$$

lying over  $\mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)$ . Note that  $R' \otimes_R R'$  is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Denote  $p_0, p_1 : \mathrm{Spec}(R' \otimes_R R') \rightarrow \mathrm{Spec}(R')$  the two projections. By the fully faithfulness shown above there exists a canonical isomorphism  $\varphi : p_0^* x' \rightarrow p_1^* x'$  because we have such isomorphisms over  $\mathrm{Spec}((R' \otimes_R R')/\mathfrak{m}^n(R' \otimes_R R'))$ . We omit the proof that the isomorphism  $\varphi$  satisfies the cocycle condition (see Stacks, Definition 3.1). Since  $\{\mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)\}$  is an fppf covering we conclude that  $x'$  descends to an object  $x$  of  $\mathcal{X}$  over  $\mathrm{Spec}(R)$ . We omit the proof that  $x_n$  is the restriction of  $x$  to  $\mathrm{Spec}(R/\mathfrak{m}^n)$ .  $\square$

**Lemma 9.5.** *Let  $S$  be a scheme. Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  and  $q : \mathcal{Z} \rightarrow \mathcal{Y}$  be 1-morphisms of categories fibred in groupoids over  $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ . If the functor (9.2.1) is an equivalence for  $\mathcal{X}$ ,  $\mathcal{Y}$ , and  $\mathcal{Z}$ , then it is an equivalence for  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ .*

**Proof.** The left and the right hand side of (9.2.1) for  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$  are simply the 2-fibre products of the left and the right hand side of (9.2.1) for  $\mathcal{X}$ ,  $\mathcal{Z}$  over  $\mathcal{Y}$ . Hence the result follows as taking 2-fibre products is compatible with equivalences of categories, see Categories, Lemma 29.7.  $\square$

## 10. Approximation

A fundamental insight of Michael Artin is that you can approximate objects of a limit preserving stack. Namely, given an object  $x$  of the stack over a Noetherian complete local ring, you can find an object  $x_A$  over an algebraic ring which is “close to”  $x$ . Here an algebraic ring means a finite type  $S$ -algebra and close means adically close. In this section we present this in a simple, yet general form.

To formulate the result we need to pull together some definitions from different places in the stacks project. First, in Criteria for Representability, Section 5 we introduced *limit preserving on objects* for 1-morphisms of categories fibred in groupoids over the category of schemes. In More on Algebra, Definition 39.1 we defined the notion of a *G-ring*. Let  $S$  be a locally Noetherian scheme. Let  $A$  be an  $S$ -algebra. We say that  $A$  is *of finite type over  $S$*  or is a *finite type  $S$ -algebra* if  $\mathrm{Spec}(A) \rightarrow S$  is of finite type. In this case  $A$  is a Noetherian ring. Finally, given a ring  $A$  and ideal  $I$  we denote  $\mathrm{Gr}_I(A) = \bigoplus I^n/I^{n+1}$ .

**Lemma 10.1.** *Let  $S$  be a locally Noetherian scheme. Let  $p : \mathcal{X} \rightarrow (\mathrm{Sch}/S)_{\mathrm{fppf}}$  be a category fibred in groupoids. Let  $x$  be an object of  $\mathcal{X}$  lying over  $\mathrm{Spec}(R)$  where*

$R$  is a Noetherian complete local ring with residue field  $k$  of finite type over  $S$ . Let  $s \in S$  be the image of  $\text{Spec}(k) \rightarrow S$ . Assume that (a)  $\mathcal{O}_{S,s}$  is a G-ring and (b)  $p$  is limit preserving on objects. Then for every integer  $N \geq 1$  there exist

- (1) a finite type  $S$ -algebra  $A$ ,
- (2) a maximal ideal  $\mathfrak{m}_A \subset A$ ,
- (3) an object  $x_A$  of  $\mathcal{X}$  over  $\text{Spec}(A)$ ,
- (4) an  $S$ -isomorphism  $R/\mathfrak{m}_R^N \cong A/\mathfrak{m}_A^N$ ,
- (5) an isomorphism  $x|_{\text{Spec}(R/\mathfrak{m}_R^N)} \cong x_A|_{\text{Spec}(A/\mathfrak{m}_A^N)}$  compatible with (4), and
- (6) an isomorphism  $\text{Gr}_{\mathfrak{m}_R}(R) \cong \text{Gr}_{\mathfrak{m}_A}(A)$  of graded  $k$ -algebras.

**Proof.** Choose an affine open  $\text{Spec}(\Lambda) \subset S$  such that  $k$  is a finite  $\Lambda$ -algebra, see Morphisms, Lemma 17.1. We may and do replace  $S$  by  $\text{Spec}(\Lambda)$ .

We may write  $R$  as a directed colimit  $R = \text{colim } C_j$  where each  $C_j$  is a finite type  $\Lambda$ -algebra (see Algebra, Lemma 123.1). By assumption (b) the object  $x$  is isomorphic to the restriction of an object over one of the  $C_j$ . Hence we may choose a finite type  $\Lambda$ -algebra  $C$ , a  $\Lambda$ -algebra map  $C \rightarrow R$ , and an object  $x_C$  of  $\mathcal{X}$  over  $\text{Spec}(C)$  such that  $x = x_C|_{\text{Spec}(R)}$ . The choice of  $C$  is a bookkeeping device and could be avoided. For later use, let us write  $C = \Lambda[y_1, \dots, y_u]/(f_1, \dots, f_v)$  and we denote  $\bar{a}_i \in R$  the image of  $y_i$  under the map  $C \rightarrow R$ . Set  $\mathfrak{m}_C = C \cap \mathfrak{m}_R$ .

Choose a  $\Lambda$ -algebra surjection  $\Lambda[x_1, \dots, x_s] \rightarrow k$  and denote  $\mathfrak{m}'$  the kernel. By the universal property of polynomial rings we may lift this to a  $\Lambda$ -algebra map  $\Lambda[x_1, \dots, x_s] \rightarrow R$ . We add some variables (i.e., we increase  $s$  a bit) mapping to generators of  $\mathfrak{m}_R$ . Having done this we see that  $\Lambda[x_1, \dots, x_s] \rightarrow R/\mathfrak{m}_R^2$  is surjective. Then we see that

$$(10.1.1) \quad P = \Lambda[x_1, \dots, x_s]_{\mathfrak{m}'}^{\wedge} \longrightarrow R$$

is a surjective map of Noetherian complete local rings, see for example Formal Deformation Theory, Lemma 4.2.

Choose lifts  $a_i \in P$  of  $\bar{a}_i$  we found above. Choose generators  $b_1, \dots, b_r \in P$  for the kernel of (10.1.1). Choose  $c_{ji} \in P$  such that

$$f_j(a_1, \dots, a_u) = \sum c_{ji} b_i$$

in  $P$  which is possible by the choices made so far. Choose generators

$$k_1, \dots, k_t \in \text{Ker}(P^{\oplus r} \xrightarrow{(b_1, \dots, b_r)} P)$$

and write  $k_i = (k_{i1}, \dots, k_{ir})$  and  $K = (k_{ij})$  so that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1, \dots, b_r)} P \rightarrow R \rightarrow 0$$

is an exact sequence of  $P$ -modules. In particular we have  $\sum k_{ij} b_j = 0$ . After possibly increasing  $N$  we may assume  $N - 1$  works in the Artin-Rees lemma for the first two maps of this exact sequence (see More on Algebra, Section 3 for terminology).

By assumption  $\mathcal{O}_{S,s} = \Lambda_{\Lambda \cap \mathfrak{m}'}$  is a G-ring. Hence by More on Algebra, Proposition 39.10 the ring  $\Lambda[x_1, \dots, x_s]_{\mathfrak{m}'}$  is a G-ring. Hence by Smoothing Ring Maps, Theorem 13.2 there exist an étale ring map

$$\Lambda[x_1, \dots, x_s]_{\mathfrak{m}'} \rightarrow B,$$

a maximal ideal  $\mathfrak{m}_B$  of  $B$  lying over  $\mathfrak{m}'$ , and elements  $a'_i, b'_i, c'_{ij}, k'_{ij} \in B'$  such that

- (1)  $\kappa(\mathfrak{m}') = \kappa(\mathfrak{m}_B)$  which implies that  $\Lambda[x_1, \dots, x_s]_{\mathfrak{m}'} \subset B_{\mathfrak{m}_B} \subset P$  and  $P$  is identified with the completion of  $B$  at  $\mathfrak{m}_B$ , see remark preceding Smoothing Ring Maps, Theorem 13.2,
- (2)  $a_i - a'_i, b_i - b'_i, c_{ij} - c'_{ij}, k_{ij} - k'_{ij} \in (\mathfrak{m}')^N P$ , and
- (3)  $f_j(a'_1, \dots, a'_u) = \sum c'_{ji} b'_i$  and  $\sum k'_{ij} b'_j = 0$ .

Set  $A = B/(b'_1, \dots, b'_r)$  and denote  $\mathfrak{m}_A$  the image of  $\mathfrak{m}_B$  in  $A$ . (Note that  $A$  is essentially of finite type over  $\Lambda$ ; at the end of the proof we will show how to obtain an  $A$  which is of finite type over  $\Lambda$ .) There is a ring map  $C \rightarrow A$  sending  $y_i \mapsto a'_i$  because the  $a'_i$  satisfy the desired equations modulo  $(b'_1, \dots, b'_r)$ . Note that  $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$  as quotients of  $P = B^\wedge$  by property (2) above. Set  $x_A = x_C|_{\text{Spec}(A)}$ . Since the maps

$$C \rightarrow A \rightarrow A/\mathfrak{m}_A^N \cong R/\mathfrak{m}_R^N \quad \text{and} \quad C \rightarrow R \rightarrow R/\mathfrak{m}_R^N$$

are equal we see that  $x_A$  and  $x$  agree modulo  $\mathfrak{m}_R^N$  via the isomorphism  $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$ . At this point we have shown properties (1) – (5) of the statement of the lemma. To see (6) note that

$$P \oplus t \xrightarrow{K} P \oplus r \xrightarrow{(b_1, \dots, b_r)} P \quad \text{and} \quad P \oplus t \xrightarrow{K'} P \oplus r \xrightarrow{(b'_1, \dots, b'_r)} P$$

are two complexes of  $P$ -modules which are congruent modulo  $(\mathfrak{m}')^N$  with the first one being exact. By our choice of  $N$  above we see from More on Algebra, Lemma 3.2 that  $R = P/(b_1, \dots, b_r)$  and  $P/(b'_1, \dots, b'_r) = B^\wedge/(b'_1, \dots, b'_r) = A^\wedge$  have isomorphic associated graded algebras, which is what we wanted to show.

This last paragraph of the proof serves to clean up the issue that  $A$  is essentially of finite type over  $S$  and not yet of finite type. The construction above gives  $A = B/(b'_1, \dots, b'_r)$  and  $\mathfrak{m}_A \subset A$  with  $B$  étale over  $\Lambda[x_1, \dots, x_s]_{\mathfrak{m}'}$ . Hence  $A$  is of finite type over the Noetherian ring  $\Lambda[x_1, \dots, x_s]_{\mathfrak{m}'}$ . Thus we can write  $A = (A_0)_{\mathfrak{m}'}$  for some finite type  $\Lambda[x_1, \dots, x_n]$  algebra  $A_0$ . Then  $A = \text{colim}(A_0)_f$  where  $f \in \Lambda[x_1, \dots, x_n] \setminus \mathfrak{m}'$ , see Algebra, Lemma 9.9. Because  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  is limit preserving on objects, we see that  $x_A$  comes from some object  $x_{(A_0)_f}$  over  $\text{Spec}((A_0)_f)$  for an  $f$  as above. After replacing  $A$  by  $(A_0)_f$  and  $x_A$  by  $x_{(A_0)_f}$  and  $\mathfrak{m}_A$  by  $(A_0)_f \cap \mathfrak{m}_A$  the proof is finished.  $\square$

## 11. Versality

In the previous section we explained how to approximate objects over complete local rings by algebraic objects. But in order to show that a stack  $\mathcal{X}$  is an algebraic stack, we need to find smooth 1-morphisms from schemes towards  $\mathcal{X}$ . Since we are not going to assume a priori that  $\mathcal{X}$  has a representable diagonal, we cannot even speak about smooth morphisms towards  $\mathcal{X}$ . Instead, borrowing terminology from deformation theory, we will introduce versal objects.

Let  $S$  be a locally Noetherian scheme. Let  $U$  be a scheme over  $S$  with structure morphism  $U \rightarrow S$  locally of finite type. Let  $u_0 \in U$  be a finite type point of  $U$ , see Morphisms, Definition 17.3. Set  $k = \kappa(u_0)$ . Note that the composition  $\text{Spec}(k) \rightarrow S$  is also of finite type, see Morphisms, Lemma 16.3. Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $x$  be an object of  $\mathcal{X}$  which lies over  $U$ . Denote  $x_0$  the pullback of  $x$  by  $u_0$ . By the 2-Yoneda lemma  $x$  corresponds to a 1-morphism

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X},$$

see Algebraic Stacks, Section 5. We obtain a morphism of predeformation categories

$$(11.0.2) \quad \hat{x} : \mathcal{F}_{(Sch/U)_{fppf}, k, u_0} \longrightarrow \mathcal{F}_{\mathcal{X}, k, x_0},$$

over  $\mathcal{C}_\Lambda$  see (3.1.1).

**Definition 11.1.** Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $U$  be a scheme locally of finite type over  $S$ . Let  $x$  be an object of  $\mathcal{X}$  lying over  $U$ . Let  $u_0$  be finite type point of  $U$ . We say  $x$  is *versal* at  $u_0$  if the morphism  $\hat{x}$  (11.0.2) is smooth, see Formal Deformation Theory, Definition 8.1.

This definition matches our notion of versality for formal objects of  $\mathcal{X}$  in the following way. Observe that  $\mathcal{O}_{U, u_0}$  is a Noetherian local  $S$ -algebra with residue field  $k$ . Hence  $R = \mathcal{O}_{U, u_0}^\wedge$  is an object of  $\mathcal{C}_\Lambda^\wedge$ , see Formal Deformation Theory, Definition 4.1. There is an identification of predeformation categories

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf}, k, u_0},$$

see Formal Deformation Theory, Remark 7.12 for notation. Namely, given an Artinian local  $S$ -algebra  $A$  with residue field identified with  $k$  we have

$$\text{Mor}_{\mathcal{C}_\Lambda^\wedge}(R, A) = \{\varphi \in \text{Mor}_S(\text{Spec}(A), U) \mid \varphi|_{\text{Spec}(k)} = u_0\}$$

Let  $\xi$  be the formal object of  $\mathcal{X}$  over  $R$  associated to  $x|_{\text{Spec}(R)}$ , see (9.2.1). Then

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf}, k, u_0} \xrightarrow{\hat{x}} \mathcal{F}_{\mathcal{X}, k, x_0},$$

corresponds to  $\xi$  via the correspondence between formal objects and morphisms of Formal Deformation Theory, Equation (7.12.1). In other words, we see that

$$x \text{ is versal at } u_0 \Leftrightarrow \xi \text{ is versal}$$

It turns out that this notion is well behaved with respect to field extensions.

**Lemma 11.2.** *Let  $S, \mathcal{X}, U, x, u_0$  be as in Definition 11.1. Let  $l$  be a field and let  $u_{l,0} : \text{Spec}(l) \rightarrow U$  be a morphism with image  $u_0$  such that  $k = \kappa(u_0) \subset l$  is finite. Set  $x_{l,0} = x_0|_{\text{Spec}(l)}$ . If  $\mathcal{X}$  satisfies (RS) and  $x$  is versal at  $u_0$ , then*

$$\mathcal{F}_{(Sch/U)_{fppf}, l, u_{l,0}} \longrightarrow \mathcal{F}_{\mathcal{X}, l, x_{l,0}}$$

*is smooth.*

**Proof.** Note that  $(Sch/U)_{fppf}$  satisfies (RS) by Lemma 5.2. Hence the functor of the lemma is the functor

$$(\mathcal{F}_{(Sch/U)_{fppf}, k, u_0})_{l/k} \longrightarrow (\mathcal{F}_{\mathcal{X}, k, x_0})_{l/k}$$

associated to  $\hat{x}$ , see Lemma 7.1. Hence the lemma follows from Formal Deformation Theory, Lemma 26.5.  $\square$

We restate the approximation result in terms of versal objects.

**Lemma 11.3.** *Let  $S$  be a locally Noetherian scheme. Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $\xi = (R, \xi_n, f_n)$  be a formal object of  $\mathcal{X}$  with  $\xi_1$  lying over  $\text{Spec}(k) \rightarrow S$  with image  $s \in S$ . Assume*

- (1)  $\xi$  is versal,
- (2)  $\xi$  is effective,
- (3)  $\mathcal{O}_{S,s}$  is a G-ring, and

(4)  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  is limit preserving on objects.

Then there exist a morphism of finite type  $U \rightarrow S$ , a finite type point  $u_0 \in U$  with residue field  $k$ , and an object  $x$  of  $\mathcal{X}$  over  $U$  such that  $x$  is versal at  $u_0$  and such that  $x|_{\text{Spec}(k)} \cong \xi_1$ .

**Proof.** Choose an object  $x_R$  of  $\mathcal{X}$  lying over  $\text{Spec}(R)$  whose associated formal object is  $\xi$ . Let  $N = 2$  and apply Lemma 10.1. We obtain  $A, \mathfrak{m}_A, \xi_A, \dots$ . Let  $\eta = (A^\wedge, \eta_n, g_n)$  be the formal object associated to  $\xi_A|_{\text{Spec}(A^\wedge)}$ . We have a diagram

$$\begin{array}{ccc} & \eta & A^\wedge \\ & \downarrow & \downarrow \\ \xi \longrightarrow \xi_2 = \eta_2 & \text{lying over} & R \longrightarrow R/\mathfrak{m}_R^2 = A/\mathfrak{m}_A^2 \end{array}$$

The versality of  $\xi$  means exactly that we can find the dotted arrows in the diagrams, because we can successively find morphisms  $\xi \rightarrow \eta_3$ ,  $\xi \rightarrow \eta_4$ , and so on by Formal Deformation Theory, Remark 8.14. The corresponding ring map  $R \rightarrow A^\wedge$  is surjective by Formal Deformation Theory, Lemma 4.2. On the other hand, we have  $\dim_k \mathfrak{m}_R^n / \mathfrak{m}_R^{n+1} = \dim_k \mathfrak{m}_A^n / \mathfrak{m}_A^{n+1}$  for all  $n$  by construction. Hence  $R/\mathfrak{m}_R^n$  and  $A/\mathfrak{m}_A^n$  have the same (finite) length as  $\Lambda$ -modules by additivity of length and Formal Deformation Theory, Lemma 3.4. It follows that  $R/\mathfrak{m}_R^n \rightarrow A/\mathfrak{m}_A^n$  is an isomorphism for all  $n$ , hence  $R \rightarrow A^\wedge$  is an isomorphism. Thus  $\eta$  is isomorphic to a versal object, hence versal itself.  $\square$

**Example 11.4.** In this example we show that the local ring  $\mathcal{O}_{S,s}$  has to be a G-ring in order for the result of Lemma 11.3 to be true. Namely, let  $\Lambda$  be a Noetherian ring and let  $\mathfrak{m}$  be a maximal ideal of  $\Lambda$ . Set  $R = \Lambda_\mathfrak{m}^\wedge$ . Let  $\Lambda \rightarrow C \rightarrow R$  be a factorization with  $C$  of finite type over  $\Lambda$ . Set  $S = \text{Spec}(\Lambda)$ ,  $U = S \setminus \{\mathfrak{m}\}$ , and  $S' = U \amalg \text{Spec}(C)$ . Consider the functor  $F : (Sch/S)_{fppf}^{opp} \rightarrow \text{Sets}$  defined by the rule

$$F(T) = \begin{cases} * & \text{if } T \rightarrow S \text{ factors through } S' \\ \emptyset & \text{else} \end{cases}$$

Let  $\mathcal{X} = \mathcal{S}_F$  is the category fibred in sets associated to  $F$ , see Algebraic Stacks, Section 7. Then  $\mathcal{X} \rightarrow (Sch/S)_{fppf}$  is limit preserving on objects and there exists an effective, versal formal object  $\xi$  over  $R$ . Hence if the conclusion of Lemma 11.3 holds for  $\mathcal{X}$ , then there exists a finite type ring map  $\Lambda \rightarrow A$  and a maximal ideal  $\mathfrak{m}_A$  lying over  $\mathfrak{m}$  such that

- (1)  $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}_A)$ ,
- (2)  $\Lambda \rightarrow A$  and  $\mathfrak{m}_A$  satisfy condition (4) of Algebra, Lemma 136.2, and
- (3) there exists a  $\Lambda$ -algebra map  $C \rightarrow A$ .

Thus  $\Lambda \rightarrow A$  is smooth at  $\mathfrak{m}_A$  by the lemma cited. Slicing  $A$  we may assume that  $\Lambda \rightarrow A$  is étale at  $\mathfrak{m}_A$ , see for example More on Morphisms, Lemma 28.5 or argue directly. Write  $C = \Lambda[y_1, \dots, y_n]/(f_1, \dots, f_m)$ . Then  $C \rightarrow R$  corresponds to a solution in  $R$  of the system of equations  $f_1 = \dots = f_m = 0$ , see Smoothing Ring Maps, Section 13. Thus if the conclusion of Lemma 11.3 holds for every  $\mathcal{X}$  as above, then a system of equations which has a solution in  $R$  has a solution in the henselization of  $\Lambda_\mathfrak{m}$ . In other words, the approximation property holds for  $\Lambda_\mathfrak{m}^h$ . This implies that  $\Lambda_\mathfrak{m}^h$  is a G-ring (insert future reference here; see also discussion in Smoothing Ring Maps, Section 1) which in turn implies that  $\Lambda_\mathfrak{m}$  is a G-ring.



## 12. Axioms

Let  $S$  be a locally Noetherian scheme. Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Here are the axioms we will consider on  $\mathcal{X}$ .

- [-1] a set theoretic condition<sup>1</sup> to be ignored by readers who are not interested in set theoretical issues,
- [0]  $\mathcal{X}$  is a stack in groupoids for the étale topology,
- [1]  $\mathcal{X}$  is limit preserving,
- [2]  $\mathcal{X}$  satisfies the Rim-Schlessinger condition (RS),
- [3] the spaces  $T\mathcal{F}_{\mathcal{X},k,x_0}$  and  $\text{Inf}_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0})$  are finite dimensional for every  $k$  and  $x_0$ , see (8.0.1) and (8.0.2),
- [4] the functor (9.2.1) is an equivalence,
- [5]  $\mathcal{X}$  and  $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  satisfy openness of versality.

We still have to define the meaning of “limit preserving” and “openness of versality”.

## 13. Limit preserving

The morphism  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  is limit preserving on objects, as defined in Criteria for Representability, Section 5, if the functor of the definition below is essentially surjective. However, the example in Examples, Section 45 shows that this isn't equivalent to being limit preserving.

**Definition 13.1.** Let  $S$  be a scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . We say  $\mathcal{X}$  is *limit preserving* if for every affine scheme  $T$  over  $S$  which is a limit  $T = \lim T_i$  of a directed inverse system of affine schemes  $T_i$  over  $S$ , we have an equivalence

$$\text{colim } \mathcal{X}_{T_i} \longrightarrow \mathcal{X}_T$$

of fibre categories.

We spell out what this means. First, given objects  $x, y$  of  $\mathcal{X}$  over  $T_i$  we should have

$$\text{Mor}_{\mathcal{X}_T}(x|_T, y|_T) = \text{colim}_{i' \geq i} \text{Mor}_{\mathcal{X}_{T_{i'}}}(x|_{T_{i'}}, y|_{T_{i'}})$$

and second every object of  $\mathcal{X}_T$  is isomorphic to the restriction of an object over  $T_i$  for some  $i$ . Note that the first condition means that the presheaves  $\text{Isom}_{\mathcal{X}}(x, y)$  (see Stacks, Definition 2.2) are limit preserving.

**Lemma 13.2.** Let  $S$  be a scheme. Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  and  $q : \mathcal{Z} \rightarrow \mathcal{Y}$  be 1-morphisms of categories fibred in groupoids over  $(Sch/S)_{fppf}$ .

- (1) If  $\mathcal{X} \rightarrow (Sch/S)_{fppf}$  and  $\mathcal{Z} \rightarrow (Sch/S)_{fppf}$  are limit preserving on objects and  $\mathcal{Y}$  is limit preserving, then  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow (Sch/S)_{fppf}$  is limit preserving on objects.
- (2) If  $\mathcal{X}$ ,  $\mathcal{Y}$ , and  $\mathcal{Z}$  are limit preserving, then so is  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ .

**Proof.** This is formal. Proof of (1). Let  $T = \lim_{i \in I} T_i$  be the directed limit of affine schemes  $T_i$  over  $S$ . We will prove that the functor  $\text{colim } \mathcal{X}_{T_i} \rightarrow \mathcal{X}_T$  is essentially surjective. Recall that an object of the fibre product over  $T$  is a quadruple  $(T, x, z, \alpha)$  where  $x$  is an object of  $\mathcal{X}$  lying over  $T$ ,  $z$  is an object of  $\mathcal{Z}$  lying over  $T$ , and  $\alpha : p(x) \rightarrow q(z)$  is a morphism in the fibre category of  $\mathcal{Y}$  over  $T$ . By assumption

<sup>1</sup>The condition is the following: the supremum of all the cardinalities  $|\text{Ob}(\mathcal{X}_{\text{Spec}(k)})| \cong |\text{Arrows}(\mathcal{X}_{\text{Spec}(k)})|$  where  $k$  runs over the finite type fields over  $S$  is  $\leq$  than the size of some object of  $(Sch/S)_{fppf}$ .

on  $\mathcal{X}$  and  $\mathcal{Z}$  we can find an  $i$  and objects  $x_i$  and  $z_i$  over  $T_i$  such that  $x_i|_T \cong T$  and  $z_i|_T \cong z$ . Then  $\alpha$  corresponds to an isomorphism  $p(x_i)|_T \rightarrow q(z_i)|_T$  which comes from an isomorphism  $\alpha_{i'} : p(x_i)|_{T_{i'}} \rightarrow q(z_i)|_{T_{i'}}$  by our assumption on  $\mathcal{Y}$ . After replacing  $i$  by  $i'$ ,  $x_i$  by  $x_i|_{T_{i'}}$ , and  $z_i$  by  $z_i|_{T_{i'}}$ , we see that  $(T_i, x_i, z_i, \alpha_i)$  is an object of the fibre product over  $T_i$  which restricts to an object isomorphic to  $(T, x, z, \alpha)$  over  $T$  as desired.

We omit the arguments showing that  $\text{colim } \mathcal{X}_{T_i} \rightarrow \mathcal{X}_T$  is fully faithful in (2).  $\square$

**Lemma 13.3.** *Let  $S$  be a scheme. Let  $\mathcal{X}$  be an algebraic stack over  $S$ . Then the following are equivalent*

- (1)  $\mathcal{X}$  is a stack in setoids and  $\mathcal{X} \rightarrow (Sch/S)_{fppf}$  is limit preserving on objects,
- (2)  $\mathcal{X}$  is a stack in setoids and limit preserving,
- (3)  $\mathcal{X}$  is representable by an algebraic space locally of finite presentation.

**Proof.** Under each of the three assumptions  $\mathcal{X}$  is representable by an algebraic space  $X$  over  $S$ , see Algebraic Stacks, Proposition 13.3. It is clear that (1) and (2) are equivalent as a functor between setoids is an equivalence if and only if it is surjective on isomorphism classes. Finally, (1) and (3) are equivalent by Limits of Spaces, Proposition 3.9.  $\square$

#### 14. Openness of versality

Next, we come to openness of versality.

**Definition 14.1.** Let  $S$  be a locally Noetherian scheme.

- (1) Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . We say  $\mathcal{X}$  satisfies *openness of versality* if given a scheme  $U$  locally of finite type over  $S$ , an object  $x$  of  $\mathcal{X}$  over  $U$ , and a finite type point  $u_0 \in U$  such that  $x$  is versal at  $u_0$ , then there exists an open neighbourhood  $u_0 \in U' \subset U$  such that  $x$  is versal at every finite type point of  $U'$ .
- (2) Let  $f : \mathcal{Y} \rightarrow \mathcal{X}$  be a 1-morphism of categories fibred in groupoids. We say  $f$  satisfies *openness of versality* if given a morphism of schemes  $V \rightarrow U$  locally of finite type over  $S$ , an object  $x$  of  $\mathcal{X}$  over  $U$ , an object  $y$  of  $\mathcal{Y}$  over  $V$ , a morphism  $\alpha : f(y) \rightarrow x$  of  $\mathcal{X}$  over  $V \rightarrow U$ , and a finite type point  $v_0$  of  $V$  such that  $(y, \alpha)$  is versal at  $v_0$  as an object of  $\mathcal{Y} \times_{\mathcal{X}} (Sch/U)_{fppf}$ , then there exists an open neighbourhood  $v_0 \in V' \subset V$  such that  $(y, \alpha)$  is versal at every finite type point of  $V'$ .

Openness of versality is often the hardest to check. The following example shows that requiring this is necessary however.

**Example 14.2.** Let  $k$  be a field and set  $\Lambda = k[s, t]$ . Consider the functor  $F : \Lambda\text{-algebras} \rightarrow \text{Sets}$  defined by the rule

$$F(A) = \begin{cases} * & \text{if there exist } f_1, \dots, f_n \in A \text{ such that} \\ & A = (s, t, f_1, \dots, f_n) \text{ and } f_i s = 0 \ \forall i \\ \emptyset & \text{else} \end{cases}$$

Geometrically  $F(A) = *$  means there exists a quasi-compact open neighbourhood  $W$  of  $V(s, t) \subset \text{Spec}(A)$  such that  $s|_W = 0$ . Let  $\mathcal{X} \subset (Sch/\text{Spec}(\Lambda))_{fppf}$  be the full subcategory consisting of schemes  $T$  which have an affine open covering  $T = \bigcup \text{Spec}(A_j)$  with  $F(A_j) = *$  for all  $j$ . Then  $\mathcal{X}$  satisfies [0], [1], [2], [3], and [4]

but not [5]. Namely, over  $U = \text{Spec}(k[s, t]/(s))$  there exists an object  $x$  which is versal at  $u_0 = (s, t)$  but not at any other point. Details omitted.

Let  $S$  be a locally Noetherian scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . In the following two lemmas we will use the following property

(14.2.1) for all fields  $k$  of finite type over  $S$  and all  $x_0 \in \text{Ob}(\mathcal{X}_{\text{Spec}(k)})$  the map  $\mathcal{F}_{\mathcal{X}, k, x_0} \rightarrow \mathcal{F}_{\mathcal{Y}, k, f(x_0)}$  of predeformation categories is smooth

We formulate some lemmas around this concept. First we link it with (openness of) versality.

**Lemma 14.3.** *Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . Let  $U$  be a scheme locally of finite type over  $S$ . Let  $x$  be an object of  $\mathcal{X}$  over  $U$ . Assume that  $x$  is versal at every finite type point of  $U$  and that  $\mathcal{X}$  satisfies (RS). Then  $x : (\text{Sch}/U)_{fppf} \rightarrow \mathcal{X}$  satisfies (14.2.1).*

**Proof.** Let  $\text{Spec}(l) \rightarrow U$  be a morphism with  $l$  of finite type over  $S$ . Then the image  $u_0 \in U$  is a finite type point of  $U$  and  $\kappa(u_0) \subset l$  is a finite extension, see discussion in Morphisms, Section 17. Hence we see that  $\mathcal{F}_{\mathcal{X}, k, u_{l,0}} \rightarrow \mathcal{F}_{\mathcal{Y}, k, x_{l,0}}$  is smooth by Lemma 11.2.  $\square$

**Lemma 14.4.** *Let  $S$  be a locally Noetherian scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $g : \mathcal{Y} \rightarrow \mathcal{Z}$  be composable 1-morphisms of categories fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . If  $f$  and  $g$  satisfy (14.2.1) so does  $g \circ f$ .*

**Proof.** This is formal.  $\square$

**Lemma 14.5.** *Let  $S$  be a locally Noetherian scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $\mathcal{Z} \rightarrow \mathcal{Y}$  be 1-morphisms of categories fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . If  $f$  satisfies (14.2.1) so does the projection  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Z}$ .*

**Proof.** Follows immediately from Lemma 3.3 and Formal Deformation Theory, Lemma 8.7.  $\square$

**Lemma 14.6.** *Let  $S$  be a locally Noetherian scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . If  $f$  is formally smooth on objects, then  $f$  satisfies (14.2.1). If  $f$  is representable by algebraic spaces and smooth, then  $f$  satisfies (14.2.1).*

**Proof.** The first statement is equivalent to Lemma 3.2. The second follows from this and Criteria for Representability, Lemma 6.3.  $\square$

**Lemma 14.7.** *Let  $S$  be a locally Noetherian scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . Assume*

- (1)  $f$  is representable by algebraic spaces,
- (2)  $f$  satisfies (14.2.1),
- (3)  $\mathcal{X} \rightarrow (\text{Sch}/S)_{fppf}$  is limit preserving on objects, and
- (4)  $\mathcal{Y}$  is limit preserving.

*Then  $f$  is smooth.*

**Proof.** The key ingredient of the proof is More on Morphisms, Lemma 10.1 which (almost) says that a morphism of schemes of finite type over  $S$  satisfying (14.2.1) is a smooth morphism. The other arguments of the proof are essentially bookkeeping.

Let  $V$  be a scheme over  $S$  and let  $y$  be an object of  $\mathcal{Y}$  over  $V$ . Let  $Z$  be an algebraic space representing the 2-fibre product  $\mathcal{Z} = \mathcal{X} \times_{f, \mathcal{X}, y} (Sch/V)_{fppf}$ . We have to show that the projection morphism  $Z \rightarrow V$  is smooth, see Algebraic Stacks, Definition 10.1. In fact, it suffices to do this when  $V$  is an affine scheme locally of finite presentation over  $S$ , see Criteria for Representability, Lemma 5.6. Then  $(Sch/V)_{fppf}$  is limit preserving by Lemma 13.3. Hence  $Z \rightarrow S$  is locally of finite presentation by Lemmas 13.2 and 13.3. Choose a scheme  $W$  and a surjective étale morphism  $W \rightarrow Z$ . Then  $W$  is locally of finite presentation over  $S$ .

Since  $f$  satisfies (14.2.1) we see that so does  $\mathcal{Z} \rightarrow (Sch/V)_{fppf}$ , see Lemma 14.5. Next, we see that  $(Sch/W)_{fppf} \rightarrow \mathcal{Z}$  satisfies (14.2.1) by Lemma 14.6. Thus the composition

$$(Sch/W)_{fppf} \rightarrow \mathcal{Z} \rightarrow (Sch/V)_{fppf}$$

satisfies (14.2.1) by Lemma 14.4. More on Morphisms, Lemma 10.1 shows that the composition  $W \rightarrow Z \rightarrow V$  is smooth at every finite type point  $w_0$  of  $W$ . Since the smooth locus is open we conclude that  $W \rightarrow V$  is a smooth morphism of schemes by Morphisms, Lemma 17.7. Thus we conclude that  $Z \rightarrow V$  is a smooth morphism of algebraic spaces by definition.  $\square$

The lemma below is how we will use openness of versality.

**Lemma 14.8.** *Let  $S$  be a locally Noetherian scheme. Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $k$  be a finite type field over  $S$  and let  $x_0$  be an object of  $\mathcal{X}$  over  $\text{Spec}(k)$  with image  $s \in S$ . Assume*

- (1)  $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is representable by algebraic spaces,
- (2)  $\mathcal{X}$  satisfies axioms [1], [2], [3], [4], and openness of versality, and
- (3)  $\mathcal{O}_{S,s}$  is a  $G$ -ring.

*Then there exist a morphism of finite type  $U \rightarrow S$  and an object  $x$  of  $\mathcal{X}$  over  $U$  such that*

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

*is smooth and such that there exists a finite type point  $u_0 \in U$  whose residue field is  $k$  and such that  $x|_{u_0} \cong x_0$ .*

**Proof.** By axiom [2], Lemma 6.1, and Remark 6.2 we see that  $\mathcal{F}_{\mathcal{X}, k, x_0}$  satisfies (S1) and (S2). Since also the tangent space has finite dimension by axiom [3] we deduce from Formal Deformation Theory, Lemma 12.4 that  $\mathcal{F}_{\mathcal{X}, k, x_0}$  has a versal formal object  $\xi$ . By axiom [4]  $\xi$  is effective. By axiom [1] and Lemma 11.3 there exists a morphism of finite type  $U \rightarrow S$ , an object  $x$  of  $\mathcal{X}$  over  $U$ , and a finite type point  $u_0$  of  $U$  with residue field  $k$  such that  $x$  is versal at  $u_0$  and such that  $x|_{\text{Spec}(k)} \cong x_0$ . By openness of versality we may shrink  $U$  and assume that  $x$  is versal at every finite type point  $u_0$  of  $U$ . We claim that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth which proves the lemma. Namely, by Lemma 14.3  $x$  satisfies (14.2.1) whereupon Lemma 14.7 finishes the proof.  $\square$

## 15. Axioms for functors

Let  $S$  be a scheme. Let  $F : (Sch/S)_{fppf}^{opp} \rightarrow \text{Sets}$  be a functor. Denote  $\mathcal{X} = \mathcal{S}_F$  the category fibred in sets associated to  $F$ , see Algebraic Stacks, Section 7. In this

section we provide a translation between the material above as it applies to  $\mathcal{X}$ , to statements about  $F$ .

Let  $S$  be a locally Noetherian scheme. Let  $F : (Sch/S)_{fppf}^{opp} \rightarrow Sets$  be a functor. Let  $k$  be a field of finite type over  $S$ . Let  $x_0 \in F(\text{Spec}(k))$ . The associated predeformation category (3.0.2) corresponds to the functor

$$F_{k,x_0} : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \{x \in F(\text{Spec}(A)) \mid x|_{\text{Spec}(k)} = x_0\}.$$

Recall that we do not distinguish between categories cofibred in sets over  $\mathcal{C}_\Lambda$  and functor  $\mathcal{C}_\Lambda \rightarrow Sets$ , see Formal Deformation Theory, Remarks 5.2 (11). Given a transformation of functors  $a : F \rightarrow G$ , setting  $y_0 = a(x_0)$  we obtain a morphism

$$F_{k,x_0} \longrightarrow G_{k,y_0}$$

see (3.1.1). Lemma 3.2 tells us that if  $a : F \rightarrow G$  is formally smooth (in the sense of More on Morphisms of Spaces, Definition 10.1), then  $F_{k,x_0} \longrightarrow G_{k,y_0}$  is smooth as in Formal Deformation Theory, Remark 8.4.

Lemma 4.1 says that if  $Y' = Y \amalg_X X'$  in the category of schemes over  $S$  where  $X \rightarrow X'$  is a thickening and  $X \rightarrow Y$  is affine, then the map

$$F(Y \amalg_X X') \rightarrow F(Y) \times_{F(X)} F(X')$$

is a bijection, provided that  $F$  is an algebraic space. We say a general functor  $F$  satisfies the *Rim-Schlessinger condition* or we say  $F$  *satisfies (RS)* if given any pushout  $Y' = Y \amalg_X X'$  where  $Y, X, X'$  are spectra of Artinian local rings of finite type over  $S$ , then

$$F(Y \amalg_X X') \rightarrow F(Y) \times_{F(X)} F(X')$$

is a bijection. Thus every algebraic space satisfies (RS).

Lemma 6.1 says that given a functor  $F$  which satisfies (RS), then all  $F_{k,x_0}$  are deformation functors as in Formal Deformation Theory, Definition 15.8, i.e., they satisfy (RS) as in Formal Deformation Theory, Remark 15.5. In particular the tangent space

$$TF_{k,x_0} = \{x \in F(\text{Spec}(k[\epsilon])) \mid x|_{\text{Spec}(k)} = x_0\}$$

has the structure of a  $k$ -vector space by Formal Deformation Theory, Lemma 11.2.

Lemma 8.1 says that an algebraic space  $F$  locally of finite type over  $S$  gives rise to deformation functors  $F_{k,x_0}$  with finite dimensional tangent spaces  $TF_{k,x_0}$ .

A *formal object*<sup>2</sup>  $\xi = (R, \xi_n)$  of  $F$  consists of a Noetherian complete local  $S$ -algebra  $R$  whose residue field is of finite type over  $S$ , together with elements  $\xi_n \in F(\text{Spec}(R/\mathfrak{m}^n))$  such that  $\xi_{n+1}|_{\text{Spec}(R/\mathfrak{m}^n)} = \xi_n$ . A formal object  $\xi$  defines a formal object  $\xi$  of  $F_{R/\mathfrak{m}, \xi_1}$ . We say  $\xi$  is *versal* if and only if it is versal in the sense of Formal Deformation Theory, Definition 8.13. A formal object  $\xi = (R, \xi_n)$  is called *effective* if there exists an  $x \in F(\text{Spec}(R))$  such that  $\xi_n = x|_{\text{Spec}(R/\mathfrak{m}^n)}$  for all  $n \geq 1$ . Lemma 9.4 says that if  $F$  is an algebraic space, then every formal object is effective.

Let  $U$  be a scheme locally of finite type over  $S$  and let  $x \in F(U)$ . Let  $u_0 \in U$  be a finite type point. We say that  $x$  is *versal at  $u_0$*  if and only if  $\xi = (\mathcal{O}_{U, u_0}^\wedge, x|_{\text{Spec}(\mathcal{O}_{U, u_0}/\mathfrak{m}_{u_0}^n)})$  is a versal formal object in the sense described above.

<sup>2</sup>This is what Artin calls a formal deformation.

Let  $S$  be a locally Noetherian scheme. Let  $F : (Sch/S)_{fppf}^{opp} \rightarrow Sch$  be a functor. Here are the axioms we will consider on  $F$ .

- [-1] a set theoretic condition<sup>3</sup> to be ignored by readers who are not interested in set theoretical issues,
- [0]  $F$  is a sheaf for the étale topology,
- [1]  $F$  is limit preserving,
- [2]  $F$  satisfies the Rim-Schlessinger condition (RS),
- [3] every tangent space  $TF_{k,x_0}$  is finite dimensional,
- [4] every formal object is effective,
- [5]  $F$  satisfies openness of versality.

Here *limit preserving* is the notion defined in Limits of Spaces, Definition 3.1 and *openness of versality* means the following: Given a scheme  $U$  locally of finite type over  $S$ , given  $x \in F(U)$ , and given a finite type point  $u_0 \in U$  such that  $x$  is versal at  $u_0$ , then there exists an open neighbourhood  $u_0 \in U' \subset U$  such that  $x$  is versal at every finite type point of  $U'$ .

## 16. Algebraic spaces

The following is our first main result on algebraic spaces.

**Proposition 16.1.** *Let  $S$  be a locally Noetherian scheme. Let  $F : (Sch/S)_{fppf}^{opp} \rightarrow Sets$  be a functor. Assume that*

- (1)  $\Delta : F \rightarrow F \times F$  is representable by algebraic spaces,
- (2)  $F$  satisfies axioms [-1], [0], [1], [2], [3], [4], and [5],
- (3)  $\mathcal{O}_{S,s}$  is a  $G$ -ring for all finite type points  $s$  of  $S$ .

*Then  $F$  is an algebraic space.*

**Proof.** Lemma 14.8 applies to  $F$ . Using this we choose, for every finite type field  $k$  over  $S$  and  $x_0 \in F(\text{Spec}(k))$ , an affine scheme  $U_{k,x_0}$  of finite type over  $S$  and a smooth morphism  $U_{k,x_0} \rightarrow F$  such that there exists a finite type point  $u_{k,x_0} \in U_{k,x_0}$  with residue field  $k$  such that  $x_0$  is the image of  $u_{k,x_0}$ . Then

$$U = \coprod_{k,x_0} U_{k,x_0} \longrightarrow F$$

is smooth<sup>4</sup>. To finish the proof it suffices to show this map is surjective, see Bootstrap, Lemma 12.2 (this is where we use axiom [0]). By Criteria for Representability, Lemma 5.6 it suffices to show that  $U \times_F V \rightarrow V$  is surjective for those  $V \rightarrow F$  where  $V$  is an affine scheme locally of finite presentation over  $S$ . Since  $U \times_F V \rightarrow V$  is smooth the image is open. Hence it suffices to show that the image of  $U \times_F V \rightarrow V$  contains all finite type points of  $V$ , see Morphisms, Lemma 17.7. Let  $v_0 \in V$  be a finite type point. Then  $k = \kappa(v_0)$  is a finite type field over  $S$ . Denote  $x_0$  the composition  $\text{Spec}(k) \xrightarrow{v_0} V \rightarrow F$ . Then  $(u_{k,x_0}, v_0) : \text{Spec}(k) \rightarrow U \times_F V$  is a point mapping to  $v_0$  and we win.  $\square$

**Lemma 16.2.** *Let  $S$  be a locally Noetherian scheme. Let  $a : F \rightarrow G$  be a transformation of functors  $F, G : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ . Assume that*

<sup>3</sup>The condition is the following: the supremum of all the cardinalities  $|F(\text{Spec}(k))|$  where  $k$  runs over the finite type fields over  $S$  is  $\leq$  than the size of some object of  $(Sch/S)_{fppf}$ .

<sup>4</sup>Set theoretical remark: This coproduct is (isomorphic) to an object of  $(Sch/S)_{fppf}$  as we have a bound on the index set by axiom [-1], see Sets, Lemma 9.9.

- (1)  $a$  is injective,
- (2)  $F$  satisfies axioms [0], [1], [2], [4], and [5],
- (3)  $\mathcal{O}_{S,s}$  is a  $G$ -ring for all finite type points  $s$  of  $S$ ,
- (4)  $G$  is an algebraic space locally of finite type over  $S$ ,

Then  $F$  is an algebraic space.

**Proof.** By Lemma 8.1 the functor  $G$  satisfies [3]. As  $F \rightarrow G$  is injective, we conclude that  $F$  also satisfies [3]. Moreover, as  $F \rightarrow G$  is injective, we see that given schemes  $U, V$  and morphisms  $U \rightarrow F$  and  $V \rightarrow F$ , then  $U \times_F V = U \times_G V$ . Hence  $\Delta : F \rightarrow F \times F$  is representable (by schemes) as this holds for  $G$  by assumption. Thus Proposition 16.1 applies<sup>5</sup>.  $\square$

## 17. Algebraic stacks

Proposition 17.2 is our first main result on algebraic stacks.

**Lemma 17.1.** *Let  $S$  be a locally Noetherian scheme. Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}^{opp}$  be a category fibred in groupoids. Assume that*

- (1)  $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is representable by algebraic spaces,
- (2)  $\mathcal{X}$  satisfies axioms [-1], [0], [1], [2], [3], [4] (see Section 12),
- (3)  $\mathcal{X}$  satisfies openness of versality, and
- (4)  $\mathcal{O}_{S,s}$  is a  $G$ -ring for all finite type points  $s$  of  $S$ .

Then  $\mathcal{X}$  is an algebraic stack.

**Proof.** Lemma 14.8 applies to  $\mathcal{X}$ . Using this we choose, for every finite type field  $k$  over  $S$  and every isomorphism class of object  $x_0 \in \text{Ob}(\mathcal{X}_{\text{Spec}(k)})$ , an affine scheme  $U_{k,x_0}$  of finite type over  $S$  and a smooth morphism  $(Sch/U_{k,x_0})_{fppf} \rightarrow \mathcal{X}$  such that there exists a finite type point  $u_{k,x_0} \in U_{k,x_0}$  with residue field  $k$  such that  $x_0$  is the image of  $u_{k,x_0}$ . Then

$$(Sch/U)_{fppf} \rightarrow \mathcal{X}, \quad \text{with} \quad U = \coprod_{k,x_0} U_{k,x_0}$$

is smooth<sup>6</sup>. To finish the proof it suffices to show this map is surjective, see Criteria for Representability, Lemma 19.1 (this is where we use axiom [0]). By Criteria for Representability, Lemma 5.6 it suffices to show that  $(Sch/U)_{fppf} \times_{\mathcal{X}} (Sch/V)_{fppf} \rightarrow (Sch/V)_{fppf}$  is surjective for those  $y : (Sch/V)_{fppf} \rightarrow \mathcal{X}$  where  $V$  is an affine scheme locally of finite presentation over  $S$ . By assumption (1) the fibre product  $(Sch/U)_{fppf} \times_{\mathcal{X}} (Sch/V)_{fppf}$  is representable by an algebraic space  $W$ . Then  $W \rightarrow V$  is smooth, hence the image is open. Hence it suffices to show that the image of  $W \rightarrow V$  contains all finite type points of  $V$ , see Morphisms, Lemma 17.7. Let  $v_0 \in V$  be a finite type point. Then  $k = \kappa(v_0)$  is a finite type field over  $S$ . Denote  $x_0 = y|_{\text{Spec}(k)}$  the pullback of  $y$  by  $v_0$ . Then  $(u_{k,x_0}, v_0)$  will give a morphism  $\text{Spec}(k) \rightarrow W$  whose composition with  $W \rightarrow V$  is  $v_0$  and we win.  $\square$

**Proposition 17.2.** *Let  $S$  be a locally Noetherian scheme. Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}^{opp}$  be a category fibred in groupoids. Assume that*

- (1)  $\Delta_{\Delta} : \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$  is representable by algebraic spaces,
- (2)  $\mathcal{X}$  satisfies axioms [-1], [0], [1], [2], [3], [4], and [5] (see Section 12),

<sup>5</sup>The set theoretic condition [-1] holds for  $F$  as it holds for  $G$ . Details omitted.

<sup>6</sup>Set theoretical remark: This coproduct is (isomorphic) to an object of  $(Sch/S)_{fppf}$  as we have a bound on the index set by axiom [-1], see Sets, Lemma 9.9.

(3)  $\mathcal{O}_{S,s}$  is a  $G$ -ring for all finite type points  $s$  of  $S$ .

Then  $\mathcal{X}$  is an algebraic stack.

**Proof.** We first prove that  $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is representable by algebraic spaces. To do this it suffices to show that

$$\mathcal{Y} = \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, y} (Sch/V)_{fppf}$$

is representable by an algebraic space for any affine scheme  $V$  locally of finite presentation over  $S$  and object  $y$  of  $\mathcal{X} \times \mathcal{X}$  over  $V$ , see Criteria for Representability, Lemma 5.5<sup>7</sup>. Observe that  $\mathcal{Y}$  is fibred in setoids (Stacks, Lemma 2.5) and let  $Y : (Sch/S)_{fppf}^{opp} \rightarrow Sets$ ,  $T \mapsto \text{Ob}(\mathcal{Y}_T)/\cong$  be the functor of isomorphism classes. We will apply Proposition 16.1 to see that  $Y$  is an algebraic space.

Note that  $\Delta_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$  (and hence also  $Y \rightarrow Y \times Y$ ) is representable by algebraic spaces by condition (1) and Criteria for Representability, Lemma 4.4. Observe that  $Y$  is a sheaf for the étale topology by Stacks, Lemmas 6.3 and 6.7, i.e., property [0] holds. Also  $Y$  is limit preserving by Lemma 13.2, i.e., we have [1]. Note that  $Y$  has (RS), i.e., axiom [2] holds, by Lemmas 5.2 and 5.3. Axiom [3] for  $Y$  follows from Lemmas 8.1 and 8.2. Axiom [4] follows from Lemmas 9.4 and 9.5. Axiom [5] for  $Y$  follows directly from openness of versality for  $\Delta_{\mathcal{X}}$  which is part of axiom [5] for  $\mathcal{X}$ . Thus all the assumptions of Proposition 16.1 are satisfied and  $Y$  is an algebraic space.

At this point it follows from Lemma 17.1 that  $\mathcal{X}$  is an algebraic stack. □

## 18. Infinitesimal deformations

In this section we discuss a generalization of the notion of the tangent space introduced in Section 8. To do this intelligently, we borrow some notation from Formal Deformation Theory, Sections 10, 16, and 18.

Let  $S$  be a scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . Given a homomorphism  $A' \rightarrow A$  of  $S$ -algebras and an object  $x$  of  $\mathcal{X}$  over  $\text{Spec}(A)$  we write  $Lift(x, A')$  for the category of lifts of  $x$  to  $\text{Spec}(A')$ . An object of  $Lift(x, A')$  is a morphism  $x \rightarrow x'$  of  $\mathcal{X}$  lying over  $\text{Spec}(A) \rightarrow \text{Spec}(A')$  and morphisms of  $Lift(x, A')$  are defined as commutative diagrams. The set of isomorphism classes of  $Lift(x, A')$  is denoted  $\text{Lift}(x, A')$ . See Formal Deformation Theory, Definition 16.1 and Remark 16.2. If  $A' \rightarrow A$  is surjective with locally nilpotent kernel we call an element  $x'$  of  $\text{Lift}(x, A')$  a (*infinitesimal*) *deformation* of  $x$ . In this case the *group of infinitesimal automorphisms of  $x'$  over  $x$*  is the kernel

$$\text{Inf}(x'/x) = \text{Ker} \left( \text{Aut}_{\mathcal{X}_{\text{Spec}(A')}}(x') \rightarrow \text{Aut}_{\mathcal{X}_{\text{Spec}(A)}}(x) \right)$$

Note that an element of  $\text{Inf}(x'/x)$  is the same thing as a lift of  $\text{id}_x$  over  $\text{Spec}(A')$  for (the category fibred in sets associated to)  $\text{Aut}_{\mathcal{X}}(x)$ . Compare with Formal Deformation Theory, Definition 18.1 and Formal Deformation Theory, Remark 18.8.

If  $M$  is an  $A$ -module we denote  $A[M]$  the  $A$ -algebra whose underlying  $A$ -module is  $A \oplus M$  and whose multiplication is given by  $(a, m) \cdot (a', m') = (aa', am' + a'm)$ . When  $M = A$  this is the ring of dual numbers over  $A$ , which we denote  $A[\epsilon]$

<sup>7</sup>The set theoretic condition in Criteria for Representability, Lemma 5.5 will hold: the size of the algebraic space  $Y$  representing  $\mathcal{Y}$  is suitably bounded. Namely,  $Y \rightarrow S$  will be locally of finite type and  $Y$  will satisfy axiom [-1]. Details omitted.



as is customary. There is an  $A$ -algebra map  $A[M] \rightarrow A$ . The pullback of  $x$  to  $\mathrm{Spec}(A[M])$  is called the *trivial deformation* of  $x$  to  $\mathrm{Spec}(A[M])$ .

**Lemma 18.1.** *Let  $S$  be a scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(\mathrm{Sch}/S)_{fppf}$ . Let*

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' & \longrightarrow & A \end{array}$$

*be a commutative diagram of  $S$ -algebras. Let  $x$  be an object of  $\mathcal{X}$  over  $\mathrm{Spec}(A)$ , let  $y$  be an object of  $\mathcal{Y}$  over  $\mathrm{Spec}(B)$ , and let  $\phi : f(x)|_{\mathrm{Spec}(B)} \rightarrow y$  be a morphism of  $\mathcal{Y}$  over  $\mathrm{Spec}(B)$ . Then there is a canonical functor*

$$\mathrm{Lift}(x, A') \longrightarrow \mathrm{Lift}(y, B')$$

*of categories of lifts induced by  $f$  and  $\phi$ . The construction is compatible with compositions of 1-morphisms of categories fibred in groupoids in an obvious manner.*

**Proof.** This lemma proves itself.  $\square$

In the rest of this chapter the following strictly stronger version of the Rim-Schlessinger conditions will play an important role.

**Definition 18.2.** Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(\mathrm{Sch}/S)_{fppf}$ . We say  $\mathcal{X}$  satisfies *condition (RS\*)* if given an affine open  $\mathrm{Spec}(\Lambda) \subset S$  and a fibre product diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' = A \times_B B' & \longrightarrow & A \end{array}$$

of  $\Lambda$ -algebras, with  $B' \rightarrow B$  surjective with square zero kernel, the functor of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(A')} \longrightarrow \mathcal{X}_{\mathrm{Spec}(A)} \times_{\mathcal{X}_{\mathrm{Spec}(B)}} \mathcal{X}_{\mathrm{Spec}(B')}$$

is an equivalence of categories.

We make some observations: with  $A \rightarrow B \leftarrow B'$  as in Definition 18.2

- (1) if  $A, B, B'$  are of finite type over  $\Lambda$  and  $B$  is finite over  $A$ , then  $A'$  is of finite type over  $\Lambda$ , see More on Algebra, Lemma 4.1,
- (2) we have  $\mathrm{Spec}(A') = \mathrm{Spec}(A) \amalg_{\mathrm{Spec}(B)} \mathrm{Spec}(B')$  in the category of schemes, see More on Morphisms, Lemma 11.1,
- (3) if  $\mathcal{X}$  is an algebraic stack, then  $\mathcal{X}$  satisfies (RS\*) by Lemma 4.1, and
- (4) if  $\mathcal{X}$  satisfies (RS\*), then  $\mathcal{X}$  satisfies (RS) because (RS) covers exactly those cases of (RS\*) where  $A, B, B'$  are Artinian local.

Let  $S$  be a locally Noetherian base. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(\mathrm{Sch}/S)_{fppf}$ . We define a category whose objects are pairs  $(x, A' \rightarrow A)$  where

- (1)  $A' \rightarrow A$  is a surjection of  $S$ -algebras whose kernel is an ideal of square zero such that  $\mathrm{Spec}(A)$  maps into an affine open of  $S$ , and
- (2)  $x$  is an object of  $\mathcal{X}$  lying over  $\mathrm{Spec}(A)$ .

A morphism  $(y, B' \rightarrow B) \rightarrow (x, A' \rightarrow A)$  is given by a commutative diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' & \longrightarrow & A \end{array}$$

of  $S$ -algebras together with a morphism  $x|_{\text{Spec}(B)} \rightarrow y$  over  $\text{Spec}(B)$ . Let us call this the category of *deformation situations*.

**Lemma 18.3.** *Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(\text{Sch}/S)_{\text{fppf}}$ . Assume  $\mathcal{X}$  satisfies condition  $(RS^*)$ . Let  $A$  be an  $S$ -algebra such that  $\text{Spec}(A) \rightarrow S$  maps into an affine open and let  $x$  be an object of  $\mathcal{X}$  over  $\text{Spec}(A)$ .*

- (1) *There exists an  $A$ -linear functor  $\text{Inf}_x : \text{Mod}_A \rightarrow \text{Mod}_A$  such that given a deformation situation  $(x, A' \rightarrow A)$  and a lift  $x'$  there is an isomorphism  $\text{Inf}_x(I) \rightarrow \text{Inf}(x'/x)$  where  $I = \text{Ker}(A' \rightarrow A)$ .*
- (2) *There exists an  $A$ -linear functor  $T_x : \text{Mod}_A \rightarrow \text{Mod}_A$  such that*
  - (a) *given  $M$  in  $\text{Mod}_A$  there is a bijection  $T_x(M) \rightarrow \text{Lift}(x, A[M])$ ,*
  - (b) *given a deformation situation  $(x, A' \rightarrow A)$  there is an action*

$$T_x(I) \times \text{Lift}(x, A') \rightarrow \text{Lift}(x, A')$$

where  $I = \text{Ker}(A' \rightarrow A)$ . It is simply transitive if  $\text{Lift}(x, A') \neq \emptyset$ .

**Proof.** To define  $\text{Inf}_x$ , resp.  $T_x$  we consider the functors

$$\text{Mod}_A \longrightarrow \text{Sets}, \quad M \longrightarrow \text{Lift}(\text{id}_x, A[M]), \quad \text{resp.} \quad M \longrightarrow \text{Lift}(x, A[M])$$

(for the first consider lifts of  $\text{id}_x$  as automorphisms of the trivial deformation of  $x$  to  $A[M]$ ) and we apply Formal Deformation Theory, Lemma 10.4. This lemma is applicable, since  $(RS^*)$  tells us that

$$\text{Lift}(x, A[M \times N]) = \text{Lift}(x, A[M]) \times \text{Lift}(x, A[N])$$

as categories (and trivial deformations match up too).

Let  $(x, A' \rightarrow A)$  be a deformation situation. Consider the ring map  $g : A' \times_A A' \rightarrow A[I]$  defined by the rule  $g(a_1, a_2) = \bar{a}_1 \oplus a_2 - a_1$ . There is an isomorphism

$$A' \times_A A' \longrightarrow A' \times_A A[I]$$

given by  $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$ . This isomorphism commutes with the projections to  $A'$  on the first factor, and hence with the projections to  $A$ . Thus applying  $(RS^*)$  twice we find equivalences of categories

$$\begin{aligned} \text{Lift}(x, A') \times \text{Lift}(x, A') &= \text{Lift}(x, A' \times_A A') \\ &= \text{Lift}(x, A' \times_A A[I]) \\ &= \text{Lift}(x, A') \times \text{Lift}(x, A[I]) \end{aligned}$$

Using these maps and projection onto the last factor of the last product we see that we obtain “difference maps”

$$\text{Inf}(x'/x) \times \text{Inf}(x'/x) \longrightarrow \text{Inf}_x(I) \quad \text{and} \quad \text{Lift}(x, A') \times \text{Lift}(x, A') \longrightarrow T_x(I)$$

These difference maps satisfy the transitivity rule “ $(x'_1 - x'_2) + (x'_2 - x'_3) = x'_1 - x'_3$ ” because

$$\begin{array}{ccc} A' \times_A A' \times_A A' & \xrightarrow{(a_1, a_2, a_3) \mapsto (g(a_1, a_2), g(a_2, a_3))} & A[I] \times_A A[I] = A[I \times I] \\ & \searrow (a_1, a_2, a_3) \mapsto g(a_1, a_3) & \downarrow + \\ & & A[I] \end{array}$$

is commutative. Inverting the string of equivalences above we obtain an action which is free and transitive provided  $\text{Inf}(x'/x)$ , resp.  $\text{Lift}(x, A')$  is nonempty. Note that  $\text{Inf}(x'/x)$  is always nonempty as it is a group.  $\square$

**Remark 18.4** (Functoriality). Assumptions and notation as in Lemma 18.3. Suppose  $A \rightarrow B$  is a ring map and  $y = x|_{\text{Spec}(B)}$ . Let  $M \in \text{Mod}_A$ ,  $N \in \text{Mod}_B$  and let  $M \rightarrow N$  an  $A$ -linear map. Then there are canonical maps  $\text{Inf}_x(M) \rightarrow \text{Inf}_y(N)$  and  $T_x(M) \rightarrow T_y(N)$  simply because there is a pullback functor

$$\text{Lift}(x, A[M]) \rightarrow \text{Lift}(y, B[N])$$

coming from the ring map  $A[M] \rightarrow B[N]$ . Similarly, given a morphism of deformation situations  $(y, B' \rightarrow B) \rightarrow (x, A' \rightarrow A)$  we obtain a pullback functor  $\text{Lift}(x, A') \rightarrow \text{Lift}(y, B')$ . Since the construction of the action, the addition, and the scalar multiplication on  $\text{Inf}_x$  and  $T_x$  use only morphisms in the categories of lifts (see proof of Formal Deformation Theory, Lemma 10.4) we see that the constructions above are functorial. In other words we obtain  $A$ -linear maps

$$\text{Inf}_x(M) \rightarrow \text{Inf}_y(N) \quad \text{and} \quad T_x(M) \rightarrow T_y(N)$$

such that the diagrams

$$\begin{array}{ccc} \text{Inf}_y(J) & \longrightarrow & \text{Inf}(y'/y) \\ \uparrow & & \uparrow \\ \text{Inf}_x(I) & \longrightarrow & \text{Inf}(x'/x) \end{array} \quad \text{and} \quad \begin{array}{ccc} T_y(J) \times \text{Lift}(y, B') & \longrightarrow & \text{Lift}(y, B') \\ \uparrow & & \uparrow \\ T_x(I) \times \text{Lift}(x, A') & \longrightarrow & \text{Lift}(x, A') \end{array}$$

commute. Here  $I = \text{Ker}(A' \rightarrow A)$ ,  $J = \text{Ker}(B' \rightarrow B)$ ,  $x'$  is a lift of  $x$  to  $A'$  (which may not always exist) and  $y' = x'|_{\text{Spec}(B')}$ .

**Remark 18.5** (Automorphisms). Assumptions and notation as in Lemma 18.3. Let  $x', x''$  be lifts of  $x$  to  $A'$ . Then we have a composition map

$$\text{Inf}(x''/x) \times \text{Mor}_{\text{Lift}(x, A')}(x', x'') \times \text{Inf}(x'/x) \longrightarrow \text{Mor}_{\text{Lift}(x, A')}(x', x'').$$

Since  $\text{Lift}(x, A')$  is a groupoid, if  $\text{Mor}_{\text{Lift}(x, A')}(x', x'')$  is nonempty, then this defines a simply transitive left action of  $\text{Inf}(x'/x)$  on  $\text{Mor}_{\text{Lift}(x, A')}(x', x'')$  and a simply transitive right action by  $\text{Inf}(x'/x)$ . Now the lemma says that  $\text{Inf}(x'/x) = \text{Inf}_x(I) = \text{Inf}(x''/x)$ . We claim that the two actions described above agree via these identifications. Namely, either  $x' \not\cong x''$  in which the claim is clear, or  $x' \cong x''$  and in that case we may assume that  $x'' = x'$  in which case the result follows from the fact that  $\text{Inf}(x'/x)$  is commutative. In particular, we obtain a well defined action

$$\text{Inf}_x(I) \times \text{Mor}_{\text{Lift}(x, A')}(x', x'') \longrightarrow \text{Mor}_{\text{Lift}(x, A')}(x', x'')$$

which is simply transitive as soon as  $\text{Mor}_{\text{Lift}(x, A')}(x', x'')$  is nonempty.

**Remark 18.6** (Canonical element). Assumptions and notation as in Lemma 18.3. Choose an affine open  $\text{Spec}(\Lambda) \subset S$  such that  $\text{Spec}(A) \rightarrow S$  corresponds to a ring map  $\Lambda \rightarrow A$ . Consider the ring map

$$A \longrightarrow A[\Omega_{A/\Lambda}], \quad a \longmapsto (a, d_{A/\Lambda}(a))$$

Pulling back  $x$  along the corresponding morphism  $\text{Spec}(A[\Omega_{A/\Lambda}]) \rightarrow \text{Spec}(A)$  we obtain a deformation  $x_{can}$  of  $x$  over  $A[\Omega_{A/\Lambda}]$ . We call this the *canonical element*

$$x_{can} \in T_x(\Omega_{A/\Lambda}) = \text{Lift}(x, A[\Omega_{A/\Lambda}]).$$

Next, assume that  $\Lambda \rightarrow A$  is of finite type and let  $k = \kappa(\mathfrak{p})$  be a residue field at a finite type point  $u_0$  of  $U = \text{Spec}(A)$ . Let  $x_0 = x|_{u_0}$ . By (RS\*) and the fact that  $A[k] = A \times_k k[k]$  the space  $T_x(k)$  is the tangent space to the deformation functor  $\mathcal{F}_{\mathcal{X}, k, x_0}$ . Via

$$T\mathcal{F}_{U, k, u_0} = \text{Der}_\Lambda(A, k) = \text{Hom}_A(\Omega_{A/\Lambda}, k)$$

(see Formal Deformation Theory, Example 10.11) and functoriality of  $T_x$  the canonical element produces the map on tangent spaces induced by the object  $x$  over  $U$ . Namely,  $\theta \in T\mathcal{F}_{U, k, u_0}$  maps to  $T_x(\theta)(x_{can})$  in  $T_x(k) = T\mathcal{F}_{\mathcal{X}, k, x_0}$ .

**Remark 18.7** (Canonical automorphism). Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . Assume  $\mathcal{X}$  satisfies condition (RS\*). Let  $A$  be an  $S$ -algebra such that  $\text{Spec}(A) \rightarrow S$  maps into an affine open and let  $x, y$  be objects of  $\mathcal{X}$  over  $\text{Spec}(A)$ . Further, let  $A \rightarrow B$  be a ring map and let  $\alpha : x|_{\text{Spec}(B)} \rightarrow y|_{\text{Spec}(B)}$  be a morphism of  $\mathcal{X}$  over  $\text{Spec}(B)$ . Consider the ring map

$$B \longrightarrow B[\Omega_{B/A}], \quad b \longmapsto (b, d_{B/A}(b))$$

Pulling back  $\alpha$  along the corresponding morphism  $\text{Spec}(B[\Omega_{B/A}]) \rightarrow \text{Spec}(B)$  we obtain a morphism  $\alpha_{can}$  between the pullbacks of  $x$  and  $y$  over  $B[\Omega_{B/A}]$ . On the other hand, we can pullback  $\alpha$  by the morphism  $\text{Spec}(B[\Omega_{B/A}]) \rightarrow \text{Spec}(B)$  corresponding to the injection of  $B$  into the first summand of  $B[\Omega_{B/A}]$ . By the discussion of Remark 18.5 we can take the difference

$$\varphi(x, y, \alpha) = \alpha_{can} - \alpha|_{\text{Spec}(B[\Omega_{B/A}])} \in \text{Inf}_{x|_{\text{Spec}(B)}}(\Omega_{B/A}).$$

We will call this the *canonical automorphism*. It depends on all the ingredients  $A, x, y, A \rightarrow B$  and  $\alpha$ .

**Remark 18.8.** Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $A$  be an  $S$ -algebra such that  $\text{Spec}(A) \rightarrow S$  maps into an affine open. There is a notion of a *short exact sequence*

$$(x, A'_1 \rightarrow A) \rightarrow (x, A'_2 \rightarrow A) \rightarrow (x, A'_3 \rightarrow A)$$

of deformation situations: we ask the corresponding maps between the kernels  $I_i = \text{Ker}(A'_i \rightarrow A)$  give a short exact sequence

$$0 \rightarrow I_3 \rightarrow I_2 \rightarrow I_1 \rightarrow 0$$

of  $A$ -modules. Note that in this case the map  $A'_3 \rightarrow A'_1$  factors through  $A$ , hence there is a canonical isomorphism  $A'_1 = A[I_1]$ .

## 19. Obstruction theories

In this section we describe what an obstruction theory is. Contrary to the spaces of infinitesimal deformations and infinitesimal automorphisms, an obstruction theory is an additional piece of data. The formulation is motivated by the results of Lemma 18.3 and Remark 18.4.

**Definition 19.1.** Let  $S$  be a locally Noetherian base. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . An *obstruction theory* is given by the following data

- (1) for every  $S$ -algebra  $A$  such that  $\mathrm{Spec}(A) \rightarrow S$  maps into an affine open and every object  $x$  of  $\mathcal{X}$  over  $\mathrm{Spec}(A)$  an  $A$ -linear functor

$$\mathcal{O}_x : \mathrm{Mod}_A \rightarrow \mathrm{Mod}_A$$

of *obstruction modules*,

- (2) for  $(x, A)$  as in (1), a ring map  $A \rightarrow B$ ,  $M \in \mathrm{Mod}_A$ ,  $N \in \mathrm{Mod}_B$ , and an  $A$ -linear map  $M \rightarrow N$  an induced  $A$ -linear map  $\mathcal{O}_x(M) \rightarrow \mathcal{O}_y(N)$  where  $y = x|_{\mathrm{Spec}(B)}$ , and
- (3) for every deformation situation  $(x, A' \rightarrow A)$  an *obstruction* element  $o_x(A') \in \mathcal{O}_x(I)$  where  $I = \mathrm{Ker}(A' \rightarrow A)$ .

These data are subject to the following conditions

- (i) the functoriality maps turn the obstruction modules into a functor from the category of triples  $(x, A, M)$  to sets,
- (ii) for every morphism of deformation situations  $(y, B' \rightarrow B) \rightarrow (x, A' \rightarrow A)$  the element  $o_x(A')$  maps to  $o_y(B')$ , and
- (iii) we have

$$\mathrm{Lift}(x, A') \neq \emptyset \Leftrightarrow o_x(A') = 0$$

for every deformation situation  $(x, A' \rightarrow A)$ .

This last condition explains the terminology. The module  $\mathcal{O}_x(A')$  is called the *obstruction module*. The element  $o_x(A')$  is the *obstruction*. Most obstruction theories have additional properties, and in order to make them useful additional conditions are needed. Moreover, this is just a sample definition, for example in the definition we could consider only deformation situations of finite type over  $S$ .

One of the main reasons for introducing obstruction theories is to check openness of versality. The initial idea to do this is due to Artin, see the papers of Artin mentioned in the introduction. It has been taken up for example in the work by Flenner [Fle81], Hall [Hal12], Hall and Rydh [HR12], Olsson [Ols06], Olsson and Starr [OS03], and Lieblich [Lie06] (random order of references). Moreover, for particular categories fibred in groupoids, often authors develop a little bit of theory adapted to the problem at hand. We will develop this theory later (insert future reference here).

**Example 19.2.** Let  $S = \mathrm{Spec}(\Lambda)$  for some Noetherian ring  $\Lambda$ . Let  $W \rightarrow S$  be a morphism of schemes. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_W$ -module flat over  $S$ . Consider the functor

$$F : (Sch/S)_{fppf}^{opp} \longrightarrow \mathrm{Sets}, \quad T/S \longrightarrow H^0(W_T, \mathcal{F}_T)$$

where  $W_T = T \times_S W$  is the base change and  $\mathcal{F}_T$  is the pullback of  $\mathcal{F}$  to  $W_T$ . If  $T = \mathrm{Spec}(A)$  we will write  $W_T = W_A$ , etc. Let  $\mathcal{X} \rightarrow (Sch/S)_{fppf}$  be the category fibred in groupoids associated to  $F$ . Then  $\mathcal{X}$  has an obstruction theory. Namely,

- (1) given  $A$  over  $\Lambda$  and  $x \in H^0(W_A, \mathcal{F}_A)$  we set  $\mathcal{O}_x(M) = H^1(W_A, \mathcal{F}_A \otimes_A M)$ ,
- (2) given a deformation situation  $(x, A' \rightarrow A)$  we let  $o_x(A') \in \mathcal{O}_x(A)$  be the image of  $x$  under the boundary map

$$H^0(W_A, \mathcal{F}_A) \longrightarrow H^1(W_A, \mathcal{F}_A \otimes_A I)$$

coming from the short exact sequence of modules

$$0 \rightarrow \mathcal{F}_A \otimes_A I \rightarrow \mathcal{F}_{A'} \rightarrow \mathcal{F}_A \rightarrow 0.$$

We have omitted some details, in particular the construction of the short exact sequence above (it uses that  $W_A$  and  $W_{A'}$  have the same underlying topological space) and the explanation for why flatness of  $\mathcal{F}$  over  $S$  implies that the sequence above is short exact.

**Example 19.3** (Key example). Let  $S = \text{Spec}(\Lambda)$  for some Noetherian ring  $\Lambda$ . Say  $\mathcal{X} = (Sch/X)_{fppf}$  with  $X = \text{Spec}(R)$  and  $R = \Lambda[x_1, \dots, x_n]/J$ . The naive cotangent complex  $NL_{R/\Lambda}$  is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus_{i=1, \dots, n} R dx_i,$$

see Algebra, Lemma 129.2. Consider a deformation situation  $(x, A' \rightarrow A)$ . Denote  $I$  the kernel of  $A' \rightarrow A$ . The object  $x$  corresponds to  $(a_1, \dots, a_n)$  with  $a_i \in A$  such that  $f(a_1, \dots, a_n) = 0$  in  $A$  for all  $f \in J$ . Set

$$\begin{aligned} \mathcal{O}_x(A') &= \text{Hom}_R(J/J^2, I) / \text{Hom}_R(R^{\oplus n}, I) \\ &= \text{Ext}_R^1(NL_{R/\Lambda}, I) \\ &= \text{Ext}_A^1(NL_{R/\Lambda} \otimes_R A, I). \end{aligned}$$

Choose lifts  $a'_i \in A'$  of  $a_i$  in  $A$ . Then  $o_x(A')$  is the class of the map  $J/J^2 \rightarrow I$  defined by sending  $f \in J$  to  $f(a'_1, \dots, a'_n) \in I$ . We omit the verification that  $o_x(A')$  is independent of choices. It is clear that if  $o_x(A') = 0$  then the map lifts. Finally, functoriality is straightforward. Thus we obtain an obstruction theory. We observe that  $o_x(A')$  can be described a bit more canonically as the composition

$$NL_{R/\Lambda} \rightarrow NL_{A/\Lambda} \rightarrow NL_{A/A'} = I[1]$$

in  $D(A)$ , see Algebra, Lemma 129.6 for the last identification.

## 20. Naive obstruction theories

The title of this section refers to the fact that we will use the naive cotangent complex in this section. Let  $(x, A' \rightarrow A)$  be a deformation situation for a given category fibred in groupoids over a locally Noetherian scheme  $S$ . The key Example 19.3 suggests that any obstruction theory should be closely related to maps in  $D(A)$  with target the naive cotangent complex of  $A$ . Working this out we find a criterion for versality in Lemma 20.3 which leads to a criterion for openness of versality in Lemma 20.4. We introduce a notion of a naive obstruction theory in Definition 20.5 to try to formalize the notion a bit further.

In the following we will use the naive cotangent complex as defined in Algebra, Section 129. In particular, if  $A' \rightarrow A$  is a surjection of  $\Lambda$ -algebras with square zero kernel  $I$ , then there are maps

$$NL_{A'/\Lambda} \rightarrow NL_{A/\Lambda} \rightarrow NL_{A/A'}$$

whose composition is homotopy equivalent to zero (see Algebra, Remark 129.5). This doesn't form a distinguished triangle in general as we are using the naive cotangent complex and not the full one. There is a homotopy equivalence  $NL_{A/A'} \rightarrow I[1]$  (the complex consisting of  $I$  placed in degree  $-1$ , see Algebra, Lemma 129.6). Finally, note that there is a canonical map  $NL_{A/\Lambda} \rightarrow \Omega_{A/\Lambda}$ .

**Lemma 20.1.** *Let  $A \rightarrow k$  be a ring map with  $k$  a field. Let  $E \in D^-(A)$ . Then  $\text{Ext}_A^i(E, k) = \text{Hom}_k(H^{-i}(E \otimes^{\mathbf{L}} k), k)$ .*

**Proof.** Omitted. Hint: Replace  $E$  by a bounded above complex of free  $A$ -modules and compute both sides.  $\square$

**Lemma 20.2.** *Let  $\Lambda \rightarrow A \rightarrow k$  be finite type ring maps of Noetherian rings with  $k = \kappa(\mathfrak{p})$  for some prime  $\mathfrak{p}$  of  $A$ . Let  $\xi : E \rightarrow NL_{A/\Lambda}$  be morphism of  $D^-(A)$  such that  $H^{-1}(\xi \otimes^{\mathbf{L}} k)$  is not surjective. Then there exists a surjection  $A' \rightarrow A$  of  $\Lambda$ -algebras such that*

- (a)  $I = \text{Ker}(A' \rightarrow A)$  has square zero and is isomorphic to  $k$  as an  $A$ -module,
- (b)  $\Omega_{A'/\Lambda} \otimes k = \Omega_{A/\Lambda} \otimes k$ , and
- (c)  $E \rightarrow NL_{A/A'}$  is zero.

**Proof.** Let  $f \in A$ ,  $f \notin \mathfrak{p}$ . Suppose that  $A'' \rightarrow A_f$  satisfies (a), (b), (c) for the induced map  $E \otimes_A A_f \rightarrow NL_{A_f/\Lambda}$ , see Algebra, Lemma 129.13. Then we can set  $A' = A'' \times_{A_f} A$  and get a solution. Namely, it is clear that  $A' \rightarrow A$  satisfies (a) because  $\text{Ker}(A' \rightarrow A) = \text{Ker}(A'' \rightarrow A) = I$ . Pick  $f'' \in A''$  lifting  $f$ . Then the localization of  $A'$  at  $(f'', f)$  is isomorphic to  $A''$  (for example by More on Algebra, Lemma 4.3). Thus (b) and (c) are clear for  $A'$  too. In this way we see that we may replace  $A$  by the localization  $A_f$  (finitely many times). In particular (after such a replacement) we may assume that  $\mathfrak{p}$  is a maximal ideal of  $A$ , see Morphisms, Lemma 17.1.

Choose a presentation  $A = \Lambda[x_1, \dots, x_n]/J$ . Then  $NL_{A/\Lambda}$  is (canonically) homotopy equivalent to

$$J/J^2 \rightarrow \bigoplus_{i=1, \dots, n} \text{Ad}x_i,$$

see Algebra, Lemma 129.2. After localizing if necessary (using Nakayama's lemma) we can choose generators  $f_1, \dots, f_m$  of  $J$  such that  $f_j \otimes 1$  form a basis for  $J/J^2 \otimes_A k$ . Moreover, after renumbering, we can assume that the images of  $df_1, \dots, df_r$  form a basis for the image of  $J/J^2 \otimes k \rightarrow \bigoplus kdx_i$  and that  $df_{r+1}, \dots, df_m$  map to zero in  $\bigoplus kdx_i$ . With these choices the space

$$H^{-1}(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k) = H^{-1}(NL_{A/\Lambda} \otimes_A k)$$

has basis  $f_{r+1} \otimes 1, \dots, f_m \otimes 1$ . Changing basis once again we may assume that the image of  $H^{-1}(\xi \otimes^{\mathbf{L}} k)$  is contained in the  $k$ -span of  $f_{r+1} \otimes 1, \dots, f_{m-1} \otimes 1$ . Set

$$A' = \Lambda[x_1, \dots, x_n]/(f_1, \dots, f_{m-1}, \mathfrak{p}f_m)$$

By construction  $A' \rightarrow A$  satisfies (a). Since  $df_m$  maps to zero in  $\bigoplus kdx_i$  we see that (b) holds. Finally, by construction the induced map  $E \rightarrow NL_{A/A'} = I[1]$  induces the zero map  $H^{-1}(E \otimes_A^{\mathbf{L}} k) \rightarrow I \otimes_A k$ . By Lemma 20.1 we see that the composition is zero.  $\square$

The following lemma is our key technical result.

**Lemma 20.3.** *Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$  satisfying  $(RS^*)$ . Let  $U = \text{Spec}(A)$  be an affine scheme of finite type over  $S$  which maps into an affine open  $\text{Spec}(\Lambda)$ . Let  $x$  be an object of  $\mathcal{X}$  over  $U$ . Let  $\xi : E \rightarrow NL_{A/\Lambda}$  be a morphism of  $D^-(A)$ . Assume*

- (i) *for every deformation situation  $(x, A' \rightarrow A)$  we have:  $x$  lifts to  $\text{Spec}(A')$  if and only if  $E \rightarrow NL_{A/\Lambda} \rightarrow NL_{A/A'}$  is zero, and*
- (ii) *there is an isomorphism of functors  $T_x(-) \rightarrow \text{Ext}_A^0(E, -)$  such that  $E \rightarrow NL_{A/\Lambda} \rightarrow \Omega_{A/\Lambda}^1$  corresponds to the canonical element (see Remark 18.6).*

*Let  $u_0 \in U$  be a finite type point with residue field  $k = \kappa(u_0)$ . Consider the following statements*

- (1)  *$x$  is versal at  $u_0$ , and*
- (2)  *$\xi : E \rightarrow NL_{A/\Lambda}$  induces a surjection  $H^{-1}(E \otimes_A^{\mathbf{L}} k) \rightarrow H^{-1}(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$  and an injection  $H^0(E \otimes_A^{\mathbf{L}} k) \rightarrow H^0(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$ .*

*Then we always have (2)  $\Rightarrow$  (1) and we have (1)  $\Rightarrow$  (2) if  $u_0$  is a closed point.*

**Proof.** Let  $\mathfrak{p} = \text{Ker}(A \rightarrow k)$  be the prime corresponding to  $u_0$ .

Assume that  $x$  versal at  $u_0$  and that  $u_0$  is a closed point of  $U$ . If  $H^{-1}(\xi \otimes_A^{\mathbf{L}} k)$  is not surjective, then let  $A' \rightarrow A$  be an extension with kernel  $I$  as in Lemma 20.2. Because  $u_0$  is a closed point, we see that  $I$  is a finite  $A$ -module, hence that  $A'$  is a finite type  $\Lambda$ -algebra (this fails if  $u_0$  is not closed). In particular  $A'$  is Noetherian. By property (c) for  $A'$  and (i) for  $\xi$  we see that  $x$  lifts to an object  $x'$  over  $A'$ . Let  $\mathfrak{p}' \subset A'$  be kernel of the surjective map to  $k$ . By Artin-Rees (Algebra, Lemma 49.2) there exists an  $n > 1$  such that  $(\mathfrak{p}')^n \cap I = 0$ . Then we see that

$$B' = A'/(\mathfrak{p}')^n \longrightarrow A/\mathfrak{p}^n = B$$

is a small, essential extension of local Artinian rings, see Formal Deformation Theory, Lemma 3.12. On the other hand, as  $x$  is versal at  $u_0$  and as  $x'|_{\text{Spec}(B')}$  is a lift of  $x|_{\text{Spec}(B)}$ , there exists an integer  $m \geq n$  and a map  $q : A/\mathfrak{p}^m \rightarrow B'$  such that the composition  $A/\mathfrak{p}^m \rightarrow B' \rightarrow B$  is the quotient map. Since the maximal ideal of  $B'$  has  $n$ th power equal to zero, this  $q$  factors through  $B$  which contradicts the fact that  $B' \rightarrow B$  is an essential surjection. This contradiction shows that  $H^{-1}(\xi \otimes_A^{\mathbf{L}} k)$  is surjective.

Assume that  $x$  versal at  $u_0$ . By Lemma 20.1 the map  $H^0(\xi \otimes_A^{\mathbf{L}} k)$  is dual to the map  $\text{Ext}_A^0(NL_{A/\Lambda}, k) \rightarrow \text{Ext}_A^0(E, k)$ . Note that

$$\text{Ext}_A^0(NL_{A/\Lambda}, k) = \text{Der}_\Lambda(A, k) \quad \text{and} \quad T_x(k) = \text{Ext}_A^0(E, k)$$

Condition (ii) assures us the map  $\text{Ext}_A^0(NL_{A/\Lambda}, k) \rightarrow \text{Ext}_A^0(E, k)$  sends a tangent vector  $\theta$  to  $U$  at  $u_0$  to the corresponding infinitesimal deformation of  $x_0$ , see Remark 18.6. Hence if  $x$  is versal, then this map is surjective, see Formal Deformation Theory, Lemma 12.2. Hence  $H^0(\xi \otimes_A^{\mathbf{L}} k)$  is injective. This finishes the proof of (1)  $\Rightarrow$  (2) in case  $u_0$  is a closed point.

For the rest of the proof assume  $H^{-1}(E \otimes_A^{\mathbf{L}} k) \rightarrow H^{-1}(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$  is surjective and  $H^0(E \otimes_A^{\mathbf{L}} k) \rightarrow H^0(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$  injective. Set  $R = A_{\mathfrak{p}}^\wedge$  and let  $\eta$  be the formal object over  $R$  associated to  $x|_{\text{Spec}(R)}$ . The map  $d\eta$  on tangent spaces is surjective because it is identified with the dual of the injective map  $H^0(E \otimes_A^{\mathbf{L}} k) \rightarrow H^0(NL_{A/\Lambda} \otimes_A^{\mathbf{L}} k)$  (see previous paragraph). According to Formal Deformation Theory, Lemma 12.2 it suffices to prove the following: Let  $C' \rightarrow C$  be a small



extension of finite type Artinian local  $\Lambda$ -algebras with residue field  $k$ . Let  $R \rightarrow C$  be a  $\Lambda$ -algebra map compatible with identifications of residue fields. Let  $y = x|_{\text{Spec}(C)}$  and let  $y'$  be a lift of  $y$  to  $C'$ . To show: we can lift the  $\Lambda$ -algebra map  $R \rightarrow C$  to  $R \rightarrow C'$ .

Observe that it suffices to lift the  $\Lambda$ -algebra map  $A \rightarrow C$ . Let  $I = \text{Ker}(C' \rightarrow C)$ . Note that  $I$  is a 1-dimensional  $k$ -vector space. The obstruction  $ob$  to lifting  $A \rightarrow C$  is an element of  $\text{Ext}_A^1(NL_{A/\Lambda}, I)$ , see Example 19.3. By Lemma 20.1 and our assumption the map  $\xi$  induces an injection

$$\text{Ext}_A^1(NL_{A/\Lambda}, I) \longrightarrow \text{Ext}_A^1(E, I)$$

By the construction of  $ob$  and (i) the image of  $ob$  in  $\text{Ext}_A^1(E, I)$  is the obstruction to lifting  $x$  to  $A \times_C C'$ . By (RS\*) the fact that  $y/C$  lifts to  $y'/C'$  implies that  $x$  lifts to  $A \times_C C'$ . Hence  $ob = 0$  and we are done.  $\square$

The key lemma above allows us to conclude that we have openness of versality in some cases.

**Lemma 20.4.** *Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(\text{Sch}/S)_{fppf}$  satisfying (RS\*). Let  $U = \text{Spec}(A)$  be an affine scheme of finite type over  $S$  which maps into an affine open  $\text{Spec}(\Lambda)$ . Let  $x$  be an object of  $\mathcal{X}$  over  $U$ . Let  $\xi : E \rightarrow NL_{A/\Lambda}$  be a morphism of  $D^-(A)$ . Assume*

- (i) *for every deformation situation  $(x, A' \rightarrow A)$  we have:  $x$  lifts to  $\text{Spec}(A')$  if and only if  $E \rightarrow NL_{A/\Lambda} \rightarrow NL_{A/A'}$  is zero,*
- (ii) *there is an isomorphism of functors  $T_x(-) \rightarrow \text{Ext}_A^0(E, -)$  such that  $E \rightarrow NL_{A/\Lambda} \rightarrow \Omega_{A/\Lambda}^1$  corresponds to the canonical element (see Remark 18.6),*
- (iii) *the cohomology groups of  $E$  are finite  $A$ -modules.*

*If  $x$  is versal at a closed point  $u_0 \in U$ , then there exists an open neighbourhood  $u_0 \in U' \subset U$  such that  $x$  is versal at every finite type point of  $U'$ .*

**Proof.** Let  $C$  be the cone of  $\xi$  so that we have a distinguished triangle

$$E \rightarrow NL_{A/\Lambda} \rightarrow C \rightarrow E[1]$$

in  $D^-(A)$ . By Lemma 20.3 the assumption that  $x$  is versal at  $u_0$  implies that  $H^{-1}(C \otimes_A^{\mathbf{L}} k) = 0$ . By More on Algebra, Lemma 56.16 there exists an  $f \in A$  not contained in the prime corresponding to  $u_0$  such that  $H^{-1}(C \otimes_A^{\mathbf{L}} M) = 0$  for any  $A_f$ -module  $M$ . Using Lemma 20.3 again we see that we have versality for all finite type points of the open  $D(f) \subset U$ .  $\square$

The technical lemmas above suggest the following definition.

**Definition 20.5.** Let  $S$  be a locally Noetherian base. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . Assume that  $\mathcal{X}$  satisfies (RS\*). A *naive obstruction theory* is given by the following data

- (1) for every  $S$ -algebra  $A$  such that  $\text{Spec}(A) \rightarrow S$  maps into an affine open  $\text{Spec}(\Lambda) \subset S$  and every object  $x$  of  $\mathcal{X}$  over  $\text{Spec}(A)$  we are given an object  $E_x \in D^-(A)$  and a map  $\xi_x : E \rightarrow NL_{A/\Lambda}$ ,
- (2) given  $(x, A)$  as in (1) there are transformations of functors
$$\text{Inf}_x(-) \rightarrow \text{Ext}_A^{-1}(E_x, -) \quad \text{and} \quad T_x(-) \rightarrow \text{Ext}_A^0(E_x, -)$$
- (3) for  $(x, A)$  as in (1) and a ring map  $A \rightarrow B$  setting  $y = x|_{\text{Spec}(B)}$  there is a functoriality map  $E_x \rightarrow E_y$  in  $D(A)$ .

These data are subject to the following conditions

- (i) in the situation of (3) the diagram

$$\begin{array}{ccc} E_y & \xrightarrow{\xi_y} & NL_{B/\Lambda} \\ \uparrow & & \uparrow \\ E_x & \xrightarrow{\xi_x} & NL_{A/\Lambda} \end{array}$$

is commutative in  $D(A)$ ,

- (ii) given  $(x, A)$  as in (1) and  $A \rightarrow B \rightarrow C$  setting  $y = x|_{\text{Spec}(B)}$  and  $z = x|_{\text{Spec}(C)}$  the composition of the functoriality maps  $E_x \rightarrow E_y$  and  $E_y \rightarrow E_z$  is the functoriality map  $E_x \rightarrow E_z$ ,
- (iii) the maps of (2) are isomorphisms compatible with the functoriality maps and the maps of Remark 18.4,
- (iv) the composition  $E_x \rightarrow NL_{A/\Lambda} \rightarrow \Omega_{A/\Lambda}$  corresponds to the canonical element of  $T_x(\Omega_{A/\Lambda}) = \text{Ext}^0(E_x, \Omega_{A/\Lambda})$ , see Remark 18.6,
- (v) given a deformation situation  $(x, A' \rightarrow A)$  with  $I = \text{Ker}(A' \rightarrow A)$  the composition  $E_x \rightarrow NL_{A/\Lambda} \rightarrow NL_{A/A'}$  is zero in

$$\text{Hom}_A(E_x, NL_{A/\Lambda}) = \text{Ext}_A^0(E_x, NL_{A/A'}) = \text{Ext}_A^1(E_x, I)$$

if and only if  $x$  lifts to  $A'$ .

Thus we see in particular that we obtain an obstruction theory as in Section 19 by setting  $\mathcal{O}_x(-) = \text{Ext}_A^1(E_x, -)$ .

**Lemma 20.6.** *Let  $S$  and  $\mathcal{X}$  be as in Definition 20.5 and let  $\mathcal{X}$  be endowed with a naive obstruction theory. Let  $A \rightarrow B$  and  $y \rightarrow x$  be as in (3). Let  $k$  be a  $B$ -algebra which is a field. Then the functoriality map  $E_x \rightarrow E_y$  induces bijections*

$$H^i(E_x \otimes_A^{\mathbf{L}} k) \rightarrow H^i(E_y \otimes_A^{\mathbf{L}} k)$$

for  $i = 0, 1$ .

**Proof.** Let  $z = x|_{\text{Spec}(k)}$ . Then (RS\*) implies that

$$\text{Lift}(x, A[k]) = \text{Lift}(z, k[k]) \quad \text{and} \quad \text{Lift}(y, B[k]) = \text{Lift}(z, k[k])$$

because  $A[k] = A \times_k k[k]$  and  $B[k] = B \times_k k[k]$ . Hence the properties of a naive obstruction theory imply that the functoriality map  $E_x \rightarrow E_y$  induces bijections  $\text{Ext}_A^i(E_x, k) \rightarrow \text{Ext}_B^i(E_y, k)$  for  $i = -1, 0$ . By Lemma 20.1 our maps  $H^i(E_x \otimes_A^{\mathbf{L}} k) \rightarrow H^i(E_y \otimes_A^{\mathbf{L}} k)$ ,  $i = 0, 1$  induce isomorphisms on dual vector spaces hence are isomorphisms.  $\square$

**Lemma 20.7.** *Let  $S$  be a locally Noetherian scheme. Let  $p : \mathcal{X} \rightarrow (\text{Sch}/S)_{fppf}^{\text{opp}}$  be a category fibred in groupoids. Assume that  $\mathcal{X}$  satisfies (RS\*) and that  $\mathcal{X}$  has a naive obstruction theory. Then openness of versality holds for  $\mathcal{X}$  provided the complexes  $E_x$  of Definition 20.5 have finitely generated cohomology groups for pairs  $(A, x)$  where  $A$  is of finite type over  $S$ .*

**Proof.** Let  $U$  be a scheme locally of finite type over  $S$ , let  $x$  be an object of  $\mathcal{X}$  over  $U$ , and let  $u_0$  be a finite type point of  $U$  such that  $x$  is versal at  $u_0$ . We may first shrink  $U$  to an affine scheme such that  $u_0$  is a closed point and such that  $U \rightarrow S$  maps into an affine open  $\text{Spec}(\Lambda)$ . Say  $U = \text{Spec}(A)$ . Let  $\xi_x : E_x \rightarrow NL_{A/\Lambda}$  be the obstruction map. At this point we may apply Lemma 20.4 to conclude.  $\square$

## 21. A dual notion

Let  $(x, A' \rightarrow A)$  be a deformation situation for a given category  $\mathcal{X}$  fibred in groupoids over a locally Noetherian scheme  $S$ . Assume  $\mathcal{X}$  has an obstruction theory, see Definition 19.1. In practice one often has a complex  $K^\bullet$  of  $A$ -modules and isomorphisms of functors

$$\mathrm{Inf}_x(-) \rightarrow H^0(K^\bullet \otimes_A^{\mathbf{L}} -), \quad T_x(-) \rightarrow H^1(K^\bullet \otimes_A^{\mathbf{L}} -), \quad \mathcal{O}_x(-) \rightarrow H^2(K^\bullet \otimes_A^{\mathbf{L}} -)$$

In this section we formalize this a little bit and show how this leads to a verification of openness of versality in some cases.

**Example 21.1.** Let  $\Lambda, S, W, \mathcal{F}$  be as in Example 19.2. Assume that  $W \rightarrow S$  is proper and  $\mathcal{F}$  coherent. By Cohomology of Schemes, Remark 20.2 there exists a finite complex of finite projective  $\Lambda$ -modules  $N^\bullet$  which universally computes the cohomology of  $\mathcal{F}$ . In particular the obstruction spaces from Example 19.2 are  $\mathcal{O}_x(M) = H^1(N^\bullet \otimes_\Lambda M)$ . Hence with  $K^\bullet = N^\bullet \otimes_\Lambda A[-1]$  we see that  $\mathcal{O}_x(M) = H^2(K^\bullet \otimes_A^{\mathbf{L}} M)$ .

**Situation 21.2.** Let  $S$  be a locally Noetherian scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . Assume that  $\mathcal{X}$  has (RS\*) so that we can speak of the functor  $T_x(-)$ , see Lemma 18.3. Let  $U = \mathrm{Spec}(A)$  be an affine scheme of finite type over  $S$  which maps into an affine open  $\mathrm{Spec}(\Lambda)$ . Let  $x$  be an object of  $\mathcal{X}$  over  $U$ . Assume we are given

- (1) a complex of  $A$ -modules  $K^\bullet$ ,
- (2) a transformation of functors  $T_x(-) \rightarrow H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$ ,
- (3) for every deformation situation  $(x, A' \rightarrow A)$  with kernel  $I = \mathrm{Ker}(A' \rightarrow A)$  an element  $o_x(A') \in H^2(K^\bullet \otimes_A^{\mathbf{L}} I)$

satisfying the following (minimal) conditions

- (i) the transformation  $T_x(-) \rightarrow H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$  is an isomorphism,
- (ii) given a morphism  $(x, A'' \rightarrow A) \rightarrow (x, A' \rightarrow A)$  of deformation situations the element  $o_x(A')$  maps to the element  $o_x(A'')$  via the map  $H^2(K^\bullet \otimes_A^{\mathbf{L}} I) \rightarrow H^2(K^\bullet \otimes_A^{\mathbf{L}} I')$  where  $I' = \mathrm{Ker}(A'' \rightarrow A)$ , and
- (iii)  $x$  lifts to an object over  $\mathrm{Spec}(A')$  if and only if  $o_x(A') = 0$ .

It is possible to incorporate infinitesimal automorphisms as well, but we refrain from doing so in order to get the sharpest possible result.

In Situation 21.2 an important role will be played by  $K^\bullet \otimes_A^{\mathbf{L}} NL_{A/\Lambda}$ . Suppose we are given an element  $\xi \in H^1(K^\bullet \otimes_A^{\mathbf{L}} NL_{A/\Lambda})$ . Then (1) for any surjection  $A' \rightarrow A$  of  $\Lambda$ -algebras with kernel  $I$  of square zero the canonical map  $NL_{A/\Lambda} \rightarrow NL_{A/A'} = I[1]$  sends  $\xi$  to an element  $\xi_{A'} \in H^2(K^\bullet \otimes_A^{\mathbf{L}} I)$  and (2) the map  $NL_{A/\Lambda} \rightarrow \Omega_{A/\Lambda}$  sends  $\xi$  to an element  $\xi_{can}$  of  $H^1(K^\bullet \otimes_A^{\mathbf{L}} \Omega_{A/\Lambda})$ .

**Lemma 21.3.** *In Situation 21.2. Assume furthermore that*

- (iv) *given a short exact sequence of deformation situations as in Remark 18.8 and a lift  $x'_2 \in \mathrm{Lift}(x, A'_2)$  then  $o_x(A'_3) \in H^2(K^\bullet \otimes_A^{\mathbf{L}} I_3)$  equals  $\partial\theta$  where  $\theta \in H^1(K^\bullet \otimes_A^{\mathbf{L}} I_1)$  is the element corresponding to  $x'_2|_{\mathrm{Spec}(A'_1)}$  via  $A'_1 = A[I_1]$  and the given map  $T_x(-) \rightarrow H^1(K^\bullet \otimes_A^{\mathbf{L}} -)$ .*

*In this case there exists an element  $\xi \in H^1(K^\bullet \otimes_A^{\mathbf{L}} NL_{A/\Lambda})$  such that*

- (1) *for every deformation situation  $(x, A' \rightarrow A)$  we have  $\xi_{A'} = o_x(A')$ , and*

- (2)  $\xi_{can}$  matches the canonical element of Remark 18.6 via the given transformation  $T_x(-) \rightarrow H^1(K^\bullet \otimes_A^L -)$ .

**Proof.** Choose a  $\alpha : \Lambda[x_1, \dots, x_n] \rightarrow A$  with kernel  $J$ . Write  $P = \Lambda[x_1, \dots, x_n]$ . In the rest of this proof we work with

$$NL(\alpha) = (J/J^2 \longrightarrow \bigoplus \text{Ad}x_i)$$

which is permissible by Algebra, Lemma 129.2 and More on Algebra, Lemma 45.1. Consider the element  $o_x(P/J^2) \in H^2(K^\bullet \otimes_A^L J/J^2)$  and consider the quotient

$$C = (P/J^2 \times \bigoplus \text{Ad}x_i)/(J/J^2)$$

where  $J/J^2$  is embedded diagonally. Note that  $C \rightarrow A$  is a surjection with kernel  $\bigoplus \text{Ad}x_i$ . Moreover there is a section  $A \rightarrow C$  to  $C \rightarrow A$  given by mapping the class of  $f \in P$  to the class of  $(f, df)$  in the pushout. For later use, denote  $x_C$  the pullback of  $x$  along the corresponding morphism  $\text{Spec}(C) \rightarrow \text{Spec}(A)$ . Thus we see that  $o_x(C) = 0$ . We conclude that  $o_x(P/J^2)$  maps to zero in  $H^2(K^\bullet \otimes_A^L \bigoplus \text{Ad}x_i)$ . It follows that there exists some element  $\xi \in H^1(K^\bullet \otimes_A^L NL(\alpha))$  mapping to  $o_x(P/J^2)$ .

Note that for any deformation situation  $(x, A' \rightarrow A)$  there exists a  $\Lambda$ -algebra map  $P/J^2 \rightarrow A'$  compatible with the augmentations to  $A$ . Hence the element  $\xi$  satisfies the first property of the lemma by construction and property (ii) of Situation 21.2.

Note that our choice of  $\xi$  was well defined up to the choice of an element of  $H^1(K^\bullet \otimes_A^L \bigoplus \text{Ad}x_i)$ . We will show that after modifying  $\xi$  by an element of the aforementioned group we can arrange it so that the second assertion of the lemma is true. Let  $C' \subset C$  be the image of  $P/J^2$  under the  $\Lambda$ -algebra map  $P/J^2 \rightarrow C$  (inclusion of first factor). Observe that  $\text{Ker}(C' \rightarrow A) = \text{Im}(J/J^2 \rightarrow \bigoplus \text{Ad}x_i)$ . Set  $\overline{C} = A[\Omega_{A/\Lambda}]$ . The map  $P/J^2 \times \bigoplus \text{Ad}x_i \rightarrow \overline{C}$ ,  $(f, \sum f_i dx_i) \mapsto (f \bmod J, \sum f_i dx_i)$  factors through a surjective map  $C \rightarrow \overline{C}$ . Then

$$(x, \overline{C} \rightarrow A) \rightarrow (x, C \rightarrow A) \rightarrow (x, C' \rightarrow A)$$

is a short exact sequence of deformation situations. The associated splitting  $\overline{C} = A[\Omega_{A/\Lambda}]$  (from Remark 18.8) equals the given splitting above. Moreover, the section  $A \rightarrow C$  composed with the map  $C \rightarrow \overline{C}$  is the map  $(1, d) : A \rightarrow A[\Omega_{A/\Lambda}]$  of Remark 18.6. Thus  $x_C$  restricts to the canonical element  $x_{can}$  of  $T_x(\Omega_{A/\Lambda}) = \text{Lift}(x, A[\Omega_{A/\Lambda}])$ . By condition (iv) we conclude that  $o_x(P/J^2)$  maps to  $\partial x_{can}$  in

$$H^1(K^\bullet \otimes_A^L \text{Im}(J/J^2 \rightarrow \bigoplus \text{Ad}x_i))$$

By construction  $\xi$  maps to  $o_x(P/J^2)$ . It follows that  $x_{can}$  and  $\xi_{can}$  map to the same element in the displayed group which means (by the long exact cohomology sequence) that they differ by an element of  $H^1(K^\bullet \otimes_A^L \bigoplus \text{Ad}x_i)$  as desired.  $\square$

**Lemma 21.4.** *In Situation 21.2 assume that (iv) of Lemma 21.3 holds and that  $K^\bullet$  is a perfect object of  $D(A)$ . In this case, if  $x$  is versal at a closed point  $u_0 \in U$  then there exists an open neighbourhood  $u_0 \in U' \subset U$  such that  $x$  is versal at every finite type point of  $U'$ .*

**Proof.** We may assume that  $K^\bullet$  is a finite complex of finite projective  $A$ -modules. Thus the derived tensor product with  $K^\bullet$  is the same as simply tensoring with  $K^\bullet$ . Let  $E^\bullet$  be the dual perfect complex to  $K^\bullet$ , see More on Algebra, Lemma 56.21. (So  $E^n = \text{Hom}_A(K^{-n}, A)$  with differentials the transpose of the differentials of

$K^\bullet$ .) Let  $E \in D^-(A)$  denote the object represented by the complex  $E^\bullet[-1]$ . Let  $\xi \in H^1(\text{Tot}(K^\bullet \otimes_A NL_{A/\Lambda}))$  be the element constructed in Lemma 21.3 and denote  $\xi : E = E^\bullet[-1] \rightarrow NL_{A/\Lambda}$  the corresponding map (loc.cit.). We claim that the pair  $(E, \xi)$  satisfies all the assumptions of Lemma 20.4 which finishes the proof.

Namely, assumption (i) of Lemma 20.4 follows from conclusion (1) of Lemma 21.3 and the fact that  $H^2(K^\bullet \otimes_A^L -) = \text{Ext}^1(E, -)$  by loc.cit. Assumption (ii) of Lemma 20.4 follows from conclusion (2) of Lemma 21.3 and the fact that  $H^1(K^\bullet \otimes_A^L -) = \text{Ext}^0(E, -)$  by loc.cit. Assumption (iii) of Lemma 20.4 is clear.  $\square$

## 22. Examples of deformation problems

List of things that should go here:

- (1) Describe the general outline of an example.
- (2) Deformations of schemes:
  - (a) The Rim-Schlessinger condition.
  - (b) Computing the tangent space.
  - (c) Computing the infinitesimal deformations.
  - (d) The deformation category of an affine hypersurface.
- (3) Deformations of representations of abstract groups.
- (4) Deformations of representations of topological groups (e.g., profinite ones).
- (5) Deformations of sheaves (for example fix  $X/S$ , a finite type point  $s$  of  $S$ , and a quasi-coherent sheaf  $\mathcal{F}_s$  over  $X_s$ ).
- (6) Deformations of algebraic spaces (very similar to deformations of schemes; maybe even easier?).
- (7) Deformations of maps (eg morphisms between schemes; you can fix both or one of the target and/or source).
- (8) Add more here.

## 23. Other chapters

### Preliminaries

- (1) Introduction
- (2) Conventions
- (3) Set Theory
- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces
- (7) Sites and Sheaves
- (8) Stacks
- (9) Fields
- (10) Commutative Algebra
- (11) Brauer Groups
- (12) Homological Algebra
- (13) Derived Categories
- (14) Simplicial Methods
- (15) More on Algebra
- (16) Smoothing Ring Maps
- (17) Sheaves of Modules
- (18) Modules on Sites

### (19) Injectives

- (20) Cohomology of Sheaves
- (21) Cohomology on Sites
- (22) Differential Graded Algebra
- (23) Divided Power Algebra
- (24) Hypercoverings

### Schemes

- (25) Schemes
- (26) Constructions of Schemes
- (27) Properties of Schemes
- (28) Morphisms of Schemes
- (29) Cohomology of Schemes
- (30) Divisors
- (31) Limits of Schemes
- (32) Varieties
- (33) Topologies on Schemes
- (34) Descent
- (35) Derived Categories of Schemes

- |                                     |                                     |
|-------------------------------------|-------------------------------------|
| (36) More on Morphisms              | (63) Quotients of Groupoids         |
| (37) More on Flatness               | (64) Simplicial Spaces              |
| (38) Groupoid Schemes               | (65) Formal Algebraic Spaces        |
| (39) More on Groupoid Schemes       | (66) Restricted Power Series        |
| (40) Étale Morphisms of Schemes     | (67) Resolution of Surfaces         |
| Topics in Scheme Theory             | Deformation Theory                  |
| (41) Chow Homology                  | (68) Formal Deformation Theory      |
| (42) Adequate Modules               | (69) Deformation Theory             |
| (43) Dualizing Complexes            | (70) The Cotangent Complex          |
| (44) Étale Cohomology               | Algebraic Stacks                    |
| (45) Crystalline Cohomology         | (71) Algebraic Stacks               |
| (46) Pro-étale Cohomology           | (72) Examples of Stacks             |
| Algebraic Spaces                    | (73) Sheaves on Algebraic Stacks    |
| (47) Algebraic Spaces               | (74) Criteria for Representability  |
| (48) Properties of Algebraic Spaces | (75) Artin's Axioms                 |
| (49) Morphisms of Algebraic Spaces  | (76) Quot and Hilbert Spaces        |
| (50) Decent Algebraic Spaces        | (77) Properties of Algebraic Stacks |
| (51) Cohomology of Algebraic Spaces | (78) Morphisms of Algebraic Stacks  |
| (52) Limits of Algebraic Spaces     | (79) Cohomology of Algebraic Stacks |
| (53) Divisors on Algebraic Spaces   | (80) Derived Categories of Stacks   |
| (54) Algebraic Spaces over Fields   | (81) Introducing Algebraic Stacks   |
| (55) Topologies on Algebraic Spaces | Miscellany                          |
| (56) Descent and Algebraic Spaces   | (82) Examples                       |
| (57) Derived Categories of Spaces   | (83) Exercises                      |
| (58) More on Morphisms of Spaces    | (84) Guide to Literature            |
| (59) Pushouts of Algebraic Spaces   | (85) Desirables                     |
| (60) Groupoids in Algebraic Spaces  | (86) Coding Style                   |
| (61) More on Groupoids in Spaces    | (87) Obsolete                       |
| (62) Bootstrap                      | (88) GNU Free Documentation License |
| Topics in Geometry                  | (89) Auto Generated Index           |

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