

# INJECTIVES

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## 1. Introduction

In future chapters we will use the existence of injectives and K-injective complexes to do cohomology of sheaves of modules on ringed sites. In this chapter we explain how to produce injectives and K-injective complexes first for modules on sites and later more generally for Grothendieck abelian categories.

We observe that we already know that the category of abelian groups and the category of modules over a ring have enough injectives, see More on Algebra, Sections 41 and 42

## 2. Baer's argument for modules

There is another, more set-theoretic approach to showing that any  $R$ -module  $M$  can be imbedded in an injective module. This approach constructs the injective module by a transfinite colimit of push-outs. While this method is somewhat abstract and more complicated than the one of More on Algebra, Section 42, it is also more general. Apparently this method originates with Baer, and was revisited by Cartan and Eilenberg in [CE56] and by Grothendieck in [Gro57]. There Grothendieck uses it to show that many other abelian categories have enough injectives. We will get back to the general case later (insert future reference here).

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We begin with a few set theoretic remarks. Let  $\{B_\beta\}_{\beta \in \alpha}$  be an inductive system of objects in some category  $\mathcal{C}$ , indexed by an ordinal  $\alpha$ . Assume that  $\text{colim}_{\beta \in \alpha} B_\beta$  exists in  $\mathcal{C}$ . If  $A$  is an object of  $\mathcal{C}$ , then there is a natural map

$$(2.0.1) \quad \text{colim}_{\beta \in \alpha} \text{Mor}_{\mathcal{C}}(A, B_\beta) \longrightarrow \text{Mor}_{\mathcal{C}}(A, \text{colim}_{\beta \in \alpha} B_\beta).$$

because if one is given a map  $A \rightarrow B_\beta$  for some  $\beta$ , one naturally gets a map from  $A$  into the colimit by composing with  $B_\beta \rightarrow \text{colim}_{\beta \in \alpha} B_\alpha$ . Note that the left colimit is one of sets! In general, (2.0.1) is neither injective or surjective.

**Example 2.1.** Consider the category of sets. Let  $A = \mathbf{N}$  and  $B_n = \{1, \dots, n\}$  be the inductive system indexed by the natural numbers where  $B_n \rightarrow B_m$  for  $n \leq m$  is the obvious map. Then  $\text{colim } B_n = \mathbf{N}$ , so there is a map  $A \rightarrow \text{colim } B_n$ , which does not factor as  $A \rightarrow B_m$  for any  $m$ . Consequently,  $\text{colim } \text{Mor}(A, B_n) \rightarrow \text{Mor}(A, \text{colim } B_n)$  is not surjective.

**Example 2.2.** Next we give an example where the map fails to be injective. Let  $B_n = \mathbf{N}/\{1, 2, \dots, n\}$ , that is, the quotient set of  $\mathbf{N}$  with the first  $n$  elements collapsed to one element. There are natural maps  $B_n \rightarrow B_m$  for  $n \leq m$ , so the  $\{B_n\}$  form a system of sets over  $\mathbf{N}$ . It is easy to see that  $\text{colim } B_n = \{*\}$ : it is the one-point set. So it follows that  $\text{Mor}(A, \text{colim } B_n)$  is a one-element set for every set  $A$ . However,  $\text{colim } \text{Mor}(A, B_n)$  is **not** a one-element set. Consider the family of maps  $A \rightarrow B_n$  which are just the natural projections  $\mathbf{N} \rightarrow \mathbf{N}/\{1, 2, \dots, n\}$  and the family of maps  $A \rightarrow B_n$  which map the whole of  $A$  to the class of 1. These two families of maps are distinct at each step and thus are distinct in  $\text{colim } \text{Mor}(A, B_n)$ , but they induce the same map  $A \rightarrow \text{colim } B_n$ .

Nonetheless, if we map out of a finite set then (2.0.1) is an isomorphism always.

**Lemma 2.3.** *Suppose that, in (2.0.1),  $\mathcal{C}$  is the category of sets and  $A$  is a finite set, then the map is a bijection.*

**Proof.** Let  $f : A \rightarrow \text{colim } B_\beta$ . The range of  $f$  is finite, containing say elements  $c_1, \dots, c_r \in \text{colim } B_\beta$ . These all come from some elements in  $B_\beta$  for  $\beta \in \alpha$  large by definition of the colimit. Thus we can define  $\tilde{f} : A \rightarrow B_\beta$  lifting  $f$  at a finite stage. This proves that (2.0.1) is surjective. Next, suppose two maps  $f : A \rightarrow B_\gamma, f' : A \rightarrow B_{\gamma'}$  define the same map  $A \rightarrow \text{colim } B_\beta$ . Then each of the finitely many elements of  $A$  gets sent to the same point in the colimit. By definition of the colimit for sets, there is  $\beta \geq \gamma, \gamma'$  such that the finitely many elements of  $A$  get sent to the same points in  $B_\beta$  under  $f$  and  $f'$ . This proves that (2.0.1) is injective.  $\square$

The most interesting case of the lemma is when  $\alpha = \omega$ , i.e., when the system  $\{B_\beta\}$  is a system  $\{B_n\}_{n \in \mathbf{N}}$  over the natural numbers as in Examples 2.1 and 2.2. The essential idea is that  $A$  is “small” relative to the long chain of compositions  $B_1 \rightarrow B_2 \rightarrow \dots$ , so that it has to factor through a finite step. A more general version of this lemma can be found in Sets, Lemma 7.1. Next, we generalize this to the category of modules.

**Definition 2.4.** Let  $\mathcal{C}$  be a category, let  $I \subset \text{Arrow}(\mathcal{C})$ , and let  $\alpha$  be an ordinal. An object  $A$  of  $\mathcal{C}$  is said to be  $\alpha$ -small with respect to  $I$  if whenever  $\{B_\beta\}$  is a system over  $\alpha$  with transition maps in  $I$ , then the map (2.0.1) is an isomorphism.

In the rest of this section we shall restrict ourselves to the category of  $R$ -modules for a fixed commutative ring  $R$ . We shall also take  $I$  to be the collection of injective

maps, i.e., the *monomorphisms* in the category of modules over  $R$ . In this case, for any system  $\{B_\beta\}$  as in the definition each of the maps

$$B_\beta \rightarrow \operatorname{colim}_{\beta \in \alpha} B_\beta$$

is an injection. It follows that the map (2.0.1) is an *injection*. We can in fact interpret the  $B_\beta$ 's as submodules of the module  $B = \operatorname{colim}_{\beta \in \alpha} B_\beta$ , and then we have  $B = \bigcup_{\beta \in \alpha} B_\beta$ . This is not an abuse of notation if we identify  $B_\alpha$  with the image in the colimit. We now want to show that modules are always small for “large” ordinals  $\alpha$ .

**Proposition 2.5.** *Let  $R$  be a ring. Let  $M$  be an  $R$ -module. Let  $\kappa$  the cardinality of the set of submodules of  $M$ . If  $\alpha$  is an ordinal whose cofinality is bigger than  $\kappa$ , then  $M$  is  $\alpha$ -small with respect to injections.*

**Proof.** The proof is straightforward, but let us first think about a special case. If  $M$  is finite, then the claim is that for any inductive system  $\{B_\beta\}$  with injections between them, parametrized by a limit ordinal, any map  $M \rightarrow \operatorname{colim} B_\beta$  factors through one of the  $B_\beta$ . And this we proved in Lemma 2.3.

Now we start the proof in the general case. We need only show that the map (2.0.1) is a surjection. Let  $f : M \rightarrow \operatorname{colim} B_\beta$  be a map. Consider the subobjects  $\{f^{-1}(B_\beta)\}$  of  $M$ , where  $B_\beta$  is considered as a subobject of the colimit  $B = \bigcup_{\beta} B_\beta$ . If one of these, say  $f^{-1}(B_\beta)$ , fills  $M$ , then the map factors through  $B_\beta$ .

So suppose to the contrary that all of the  $f^{-1}(B_\beta)$  were proper subobjects of  $M$ . However, we know that

$$\bigcup f^{-1}(B_\beta) = f^{-1}\left(\bigcup B_\beta\right) = M.$$

Now there are at most  $\kappa$  different subobjects of  $M$  that occur among the  $f^{-1}(B_\alpha)$ , by hypothesis. Thus we can find a subset  $S \subset \alpha$  of cardinality at most  $\kappa$  such that as  $\beta'$  ranges over  $S$ , the  $f^{-1}(B_{\beta'})$  range over *all* the  $f^{-1}(B_\alpha)$ .

However,  $S$  has an upper bound  $\tilde{\alpha} < \alpha$  as  $\alpha$  has cofinality bigger than  $\kappa$ . In particular, all the  $f^{-1}(B_{\beta'})$ ,  $\beta' \in S$  are contained in  $f^{-1}(B_{\tilde{\alpha}})$ . It follows that  $f^{-1}(B_{\tilde{\alpha}}) = M$ . In particular, the map  $f$  factors through  $B_{\tilde{\alpha}}$ .  $\square$

From this lemma we will be able to deduce the existence of lots of injectives. Let us recall the criterion of Baer.

**Lemma 2.6.** *Let  $R$  be a ring. An  $R$ -module  $Q$  is injective if and only if in every commutative diagram*

$$\begin{array}{ccc} \mathfrak{a} & \longrightarrow & Q \\ \downarrow & \nearrow & \\ R & & \end{array}$$

for  $\mathfrak{a} \subset R$  an ideal, the dotted arrow exists.

**Proof.** Assume  $Q$  satisfies the assumption of the lemma. Let  $M \subset N$  be  $R$ -modules, and let  $\varphi : M \rightarrow Q$  be an  $R$ -module map. Arguing as in the proof of More on Algebra, Lemma 41.1 we see that it suffices to prove that if  $M \neq N$ , then we can find an  $R$ -module  $M'$ ,  $M \subset M' \subset N$  such that (a) the inclusion  $M \subset M'$

is strict, and (b)  $\varphi$  can be extended to  $M'$ . To find  $M'$ , let  $x \in N$ ,  $x \notin M$ . Let  $\psi : R \rightarrow N$ ,  $r \mapsto rx$ . Set  $\mathfrak{a} = \psi^{-1}(M)$ . By assumption the morphism

$$\mathfrak{a} \xrightarrow{\psi} M \xrightarrow{\varphi} Q$$

can be extended to a morphism  $\varphi' : R \rightarrow Q$ . Note that  $\varphi'$  annihilates the kernel of  $\psi$  (as this is true for  $\varphi$ ). Thus  $\varphi'$  gives rise to a morphism  $\varphi'' : \text{Im}(\psi) \rightarrow Q$  which agrees with  $\varphi$  on the intersection  $M \cap \text{Im}(\psi)$  by construction. Thus  $\varphi$  and  $\varphi''$  glue to give an extension of  $\varphi$  to the strictly bigger module  $M' = \mathcal{F} + \text{Im}(\psi)$ .  $\square$

If  $M$  is an  $R$ -module, then in general we may have a semi-complete diagram as in Lemma 2.6. In it, we can form the *push-out*

$$\begin{array}{ccc} \mathfrak{a} & \longrightarrow & Q \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \oplus_{\mathfrak{a}} Q. \end{array}$$

Here the vertical map is injective, and the diagram commutes. The point is that we can extend  $\mathfrak{a} \rightarrow Q$  to  $R$  if we extend  $Q$  to the larger module  $R \oplus_{\mathfrak{a}} Q$ .

The key point of Baer's argument is to repeat this procedure transfinitely many times. To do this we first define, given an  $R$ -module  $M$  the following (huge) pushout

$$(2.6.1) \quad \begin{array}{ccc} \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \text{Hom}_R(\mathfrak{a}, M)} \mathfrak{a} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \text{Hom}_R(\mathfrak{a}, M)} R & \longrightarrow & \mathbf{M}(M). \end{array}$$

Here the top horizontal arrow maps the element  $a \in \mathfrak{a}$  in the summand corresponding to  $\varphi$  to the element  $\varphi(a) \in M$ . The left vertical arrow maps  $a \in \mathfrak{a}$  in the summand corresponding to  $\varphi$  simply to the element  $a \in R$  in the summand corresponding to  $\varphi$ . The fundamental properties of this construction are formulated in the following lemma.

**Lemma 2.7.** *Let  $R$  be a ring.*

- (1) *The construction  $M \mapsto (M \rightarrow \mathbf{M}(M))$  is functorial in  $M$ .*
- (2) *The map  $M \rightarrow \mathbf{M}(M)$  is injective.*
- (3) *For any ideal  $\mathfrak{a}$  and any  $R$ -module map  $\varphi : \mathfrak{a} \rightarrow M$  there is an  $R$ -module map  $\varphi' : R \rightarrow \mathbf{M}(M)$  such that*

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{\varphi} & M \\ \downarrow & & \downarrow \\ R & \xrightarrow{\varphi'} & \mathbf{M}(M) \end{array}$$

*commutes.*

**Proof.** Parts (2) and (3) are immediate from the construction. To see (1), let  $\chi : M \rightarrow N$  be an  $R$ -module map. We claim there exists a canonical commutative

diagram

$$\begin{array}{ccccc}
 \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \text{Hom}_R(\mathfrak{a}, M)} \mathfrak{a} & \longrightarrow & M & \xrightarrow{\chi} & N \\
 \downarrow & \searrow & \searrow & & \downarrow \\
 \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \text{Hom}_R(\mathfrak{a}, M)} R & & \bigoplus_{\mathfrak{a}} \bigoplus_{\psi \in \text{Hom}_R(\mathfrak{a}, N)} \mathfrak{a} & \longrightarrow & N \\
 & \searrow & \downarrow & & \\
 & & \bigoplus_{\mathfrak{a}} \bigoplus_{\psi \in \text{Hom}_R(\mathfrak{a}, N)} R & & 
 \end{array}$$

which induces the desired map  $\mathbf{M}(M) \rightarrow \mathbf{M}(N)$ . The middle east-south-east arrow maps the summand  $\mathfrak{a}$  corresponding to  $\varphi$  via  $\text{id}_{\mathfrak{a}}$  to the summand  $\mathfrak{a}$  corresponding to  $\psi = \chi \circ \varphi$ . Similarly for the lower east-south-east arrow. Details omitted.  $\square$

The idea will now be to apply the functor  $\mathbf{M}$  a transfinite number of times. We define for each ordinal  $\alpha$  a functor  $\mathbf{M}_{\alpha}$  on the category of  $R$ -modules, together with a natural injection  $N \rightarrow \mathbf{M}_{\alpha}(N)$ . We do this by transfinite induction. First,  $\mathbf{M}_1 = \mathbf{M}$  is the functor defined above. Now, suppose given an ordinal  $\alpha$ , and suppose  $\mathbf{M}_{\alpha'}$  is defined for  $\alpha' < \alpha$ . If  $\alpha$  has an immediate predecessor  $\tilde{\alpha}$ , we let

$$\mathbf{M}_{\alpha} = \mathbf{M} \circ \mathbf{M}_{\tilde{\alpha}}.$$

If not, i.e., if  $\alpha$  is a limit ordinal, we let

$$\mathbf{M}_{\alpha}(N) = \text{colim}_{\alpha' < \alpha} \mathbf{M}_{\alpha'}(N).$$

It is clear (e.g., inductively) that the  $\mathbf{M}_{\alpha}(N)$  form an inductive system over ordinals, so this is reasonable.

**Theorem 2.8.** *Let  $\kappa$  be the cardinality of the set of ideals in  $R$ , and let  $\alpha$  be an ordinal whose cofinality is greater than  $\kappa$ . Then  $\mathbf{M}_{\alpha}(N)$  is an injective  $R$ -module, and  $N \rightarrow \mathbf{M}_{\alpha}(N)$  is a functorial injective embedding.*

**Proof.** By Baer's criterion Lemma 2.6, it suffices to show that if  $\mathfrak{a} \subset R$  is an ideal, then any map  $f : \mathfrak{a} \rightarrow \mathbf{M}_{\alpha}(N)$  extends to  $R \rightarrow \mathbf{M}_{\alpha}(N)$ . However, we know since  $\alpha$  is a limit ordinal that

$$\mathbf{M}_{\alpha}(N) = \text{colim}_{\beta < \alpha} \mathbf{M}_{\beta}(N),$$

so by Proposition 2.5, we find that

$$\text{Hom}_R(\mathfrak{a}, \mathbf{M}_{\alpha}(N)) = \text{colim}_{\beta < \alpha} \text{Hom}_R(\mathfrak{a}, \mathbf{M}_{\beta}(N)).$$

This means in particular that there is some  $\beta' < \alpha$  such that  $f$  factors through the submodule  $\mathbf{M}_{\beta'}(N)$ , as

$$f : \mathfrak{a} \rightarrow \mathbf{M}_{\beta'}(N) \rightarrow \mathbf{M}_{\alpha}(N).$$

However, by the fundamental property of the functor  $\mathbf{M}$ , see Lemma 2.7 part (3), we know that the map  $\mathfrak{a} \rightarrow \mathbf{M}_{\beta'}(N)$  can be extended to

$$R \rightarrow \mathbf{M}(\mathbf{M}_{\beta'}(N)) = \mathbf{M}_{\beta'+1}(N),$$

and the last object imbeds in  $\mathbf{M}_{\alpha}(N)$  (as  $\beta' + 1 < \alpha$  since  $\alpha$  is a limit ordinal). In particular,  $f$  can be extended to  $\mathbf{M}_{\alpha}(N)$ .  $\square$

### 3. G-modules

We will see later (Differential Graded Algebra, Section 12) that the category of modules over an algebra has functorial injective embeddings. The construction is exactly the same as the construction in More on Algebra, Section 42.

**Lemma 3.1.** *Let  $G$  be a topological group. The category  $\text{Mod}_G$  of discrete  $G$ -modules, see *Étale Cohomology*, Definition 58.1 has functorial injective hulls.*

**Proof.** By the remark above the lemma the category  $\text{Mod}_{\mathbf{Z}[G]}$  has functorial injective embeddings. Consider the forgetful functor  $v : \text{Mod}_G \rightarrow \text{Mod}_{\mathbf{Z}[G]}$ . This functor is fully faithful, transforms injective maps into injective maps and has a right adjoint, namely

$$u : M \mapsto u(M) = \{x \in M \mid \text{stabilizer of } x \text{ is open}\}$$

Since it is true that  $v(M) = 0 \Rightarrow M = 0$  we conclude by Homology, Lemma 25.5.  $\square$

### 4. Abelian sheaves on a space

**Lemma 4.1.** *Let  $X$  be a topological space. The category of abelian sheaves on  $X$  has enough injectives. In fact it has functorial injective embeddings.*

**Proof.** For an abelian group  $A$  we denote  $j : A \rightarrow J(A)$  the functorial injective embedding constructed in More on Algebra, Section 42. Let  $\mathcal{F}$  be an abelian sheaf on  $X$ . By Sheaves, Example 7.5 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod_{x \in U} J(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map  $\mathcal{F} \rightarrow \mathcal{I}$  given by mapping  $s \in \mathcal{F}(U)$  to  $\prod_{x \in U} j(s_x)$  where  $s_x \in \mathcal{F}_x$  denotes the germ of  $s$  at  $x$ . This map is injective, see Sheaves, Lemma 11.1 for example.

It remains to prove the following: Given a rule  $x \mapsto I_x$  which assigns to each point  $x \in X$  an injective abelian group the sheaf  $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$  is injective. Note that

$$\mathcal{I} = \prod_{x \in X} i_{x,*} I_x$$

is the product of the skyscraper sheaves  $i_{x,*} I_x$  (see Sheaves, Section 27 for notation.) We have

$$\text{Mor}_{\text{Ab}}(\mathcal{F}_x, I_x) = \text{Mor}_{\text{Ab}(X)}(\mathcal{F}, i_{x,*} I_x).$$

see Sheaves, Lemma 27.3. Hence it is clear that each  $i_{x,*} I_x$  is injective. Hence the injectivity of  $\mathcal{I}$  follows from Homology, Lemma 23.3.  $\square$

### 5. Sheaves of modules on a ringed space

**Lemma 5.1.** *Let  $(X, \mathcal{O}_X)$  be a ringed space, see Sheaves, Section 25. The category of sheaves of  $\mathcal{O}_X$ -modules on  $X$  has enough injectives. In fact it has functorial injective embeddings.*

**Proof.** For any ring  $R$  and any  $R$ -module  $M$  we denote  $j : M \rightarrow J_R(M)$  the functorial injective embedding constructed in More on Algebra, Section 42. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_X$ -modules on  $X$ . By Sheaves, Examples 7.5 and 15.6 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod_{x \in U} J_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map  $\mathcal{F} \rightarrow \mathcal{I}$  given by mapping  $s \in \mathcal{F}(U)$  to  $\prod_{x \in U} j(s_x)$  where  $s_x \in \mathcal{F}_x$  denotes the germ of  $s$  at  $x$ . This map is injective, see Sheaves, Lemma 11.1 for example.

It remains to prove the following: Given a rule  $x \mapsto I_x$  which assigns to each point  $x \in X$  an injective  $\mathcal{O}_{X,x}$ -module the sheaf  $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$  is injective. Note that

$$\mathcal{I} = \prod_{x \in X} i_{x,*} I_x$$

is the product of the skyscraper sheaves  $i_{x,*} I_x$  (see Sheaves, Section 27 for notation.) We have

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{x,*} I_x).$$

see Sheaves, Lemma 27.3. Hence it is clear that each  $i_{x,*} I_x$  is an injective  $\mathcal{O}_X$ -module (see Homology, Lemma 25.1 or argue directly). Hence the injectivity of  $\mathcal{I}$  follows from Homology, Lemma 23.3.  $\square$

## 6. Abelian presheaves on a category

Let  $\mathcal{C}$  be a category. Recall that this means that  $\mathrm{Ob}(\mathcal{C})$  is a set. On the one hand, consider abelian presheaves on  $\mathcal{C}$ , see Sites, Section 2. On the other hand, consider families of abelian groups indexed by elements of  $\mathrm{Ob}(\mathcal{C})$ ; in other words presheaves on the discrete category with underlying set of objects  $\mathrm{Ob}(\mathcal{C})$ . Let us denote this discrete category simply  $\mathrm{Ob}(\mathcal{C})$ . There is a natural functor

$$i : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathcal{C}$$

and hence there is a natural restriction or forgetful functor

$$v = i^p : \mathrm{PAb}(\mathcal{C}) \longrightarrow \mathrm{PAb}(\mathrm{Ob}(\mathcal{C}))$$

compare Sites, Section 5. We will denote presheaves on  $\mathcal{C}$  by  $B$  and presheaves on  $\mathrm{Ob}(\mathcal{C})$  by  $A$ .

There are also two functors, namely  $i_p$  and  $_p i$  which assign an abelian presheaf on  $\mathcal{C}$  to an abelian presheaf on  $\mathrm{Ob}(\mathcal{C})$ , see Sites, Sections 5 and 18. Here we will use  $u = _p i$  which is defined (in the case at hand) as follows:

$$uA(U) = \prod_{U' \rightarrow U} A(U').$$

So an element is a family  $(a_\phi)_\phi$  with  $\phi$  ranging through all morphisms in  $\mathcal{C}$  with target  $U$ . The restriction map on  $uA$  corresponding to  $g : V \rightarrow U$  maps our element  $(a_\phi)_\phi$  to the element  $(a_{g \circ \psi})_\psi$ .

There is a canonical surjective map  $vuA \rightarrow A$  and a canonical injective map  $B \rightarrow uvB$ . We leave it to the reader to show that

$$\mathrm{Mor}_{\mathrm{PAb}(\mathrm{Ob}(\mathcal{C}))}(B, uA) = \mathrm{Mor}_{\mathrm{PAb}(\mathcal{C})}(vB, A).$$

in this simple case; the general case is in Sites, Section 5. Thus the pair  $(u, v)$  is an example of a pair of adjoint functors, see Categories, Section 24.

At this point we can list the following facts about the situation above.

- (1) The functors  $u$  and  $v$  are exact. This follows from the explicit description of these functors given above.
- (2) In particular the functor  $v$  transforms injective maps into injective maps.
- (3) The category  $\mathrm{PAb}(\mathrm{Ob}(\mathcal{C}))$  has enough injectives.

- (4) In fact there is a functorial injective embedding  $A \mapsto (A \rightarrow J(A))$  as in Homology, Definition 23.5. Namely, we can take  $J(A)$  to be the presheaf  $U \mapsto J(A(U))$ , where  $J(-)$  is the functor constructed in More on Algebra, Section 42 for the ring  $\mathbf{Z}$ .

Putting all of this together gives us the following procedure for embedding objects  $B$  of  $PAb(\mathcal{C})$  into an injective object:  $B \rightarrow uJ(vB)$ . See Homology, Lemma 25.5.

**Proposition 6.1.** *For abelian presheaves on a category there is a functorial injective embedding.*

**Proof.** See discussion above.  $\square$

## 7. Abelian Sheaves on a site

Let  $\mathcal{C}$  be a site. In this section we prove that there are enough injectives for abelian sheaves on  $\mathcal{C}$ .

Denote  $i : Ab(\mathcal{C}) \rightarrow PAb(\mathcal{C})$  the forgetful functor from abelian sheaves to abelian presheaves. Let  $\# : PAb(\mathcal{C}) \rightarrow Ab(\mathcal{C})$  denote the sheafification functor. Recall that  $\#$  is a left adjoint to  $i$ , that  $\#$  is exact, and that  $i\mathcal{F}^\# = \mathcal{F}$  for any abelian sheaf  $\mathcal{F}$ . Finally, let  $\mathcal{G} \rightarrow J(\mathcal{G})$  denote the canonical embedding into an injective presheaf we found in Section 6.

For any sheaf  $\mathcal{F}$  in  $Ab(\mathcal{C})$  and any ordinal  $\beta$  we define a sheaf  $J_\beta(\mathcal{F})$  by transfinite induction. We set  $J_0(\mathcal{F}) = \mathcal{F}$ . We define  $J_1(\mathcal{F}) = J(i\mathcal{F})^\#$ . Sheafification of the canonical map  $i\mathcal{F} \rightarrow J(i\mathcal{F})$  gives a functorial map

$$\mathcal{F} \longrightarrow J_1(\mathcal{F})$$

which is injective as  $\#$  is exact. We set  $J_{\alpha+1}(\mathcal{F}) = J_1(J_\alpha(\mathcal{F}))$ . So that there are canonical injective maps  $J_\alpha(\mathcal{F}) \rightarrow J_{\alpha+1}(\mathcal{F})$ . For a limit ordinal  $\beta$ , we define

$$J_\beta(\mathcal{F}) = \operatorname{colim}_{\alpha < \beta} J_\alpha(\mathcal{F}).$$

Note that this is a directed colimit. Hence for any ordinals  $\alpha < \beta$  we have an injective map  $J_\alpha(\mathcal{F}) \rightarrow J_\beta(\mathcal{F})$ .

**Lemma 7.1.** *With notation as above. Suppose that  $\mathcal{G}_1 \rightarrow \mathcal{G}_2$  is an injective map of abelian sheaves on  $\mathcal{C}$ . Let  $\alpha$  be an ordinal and let  $\mathcal{G}_1 \rightarrow J_\alpha(\mathcal{F})$  be a morphism of sheaves. There exists a morphism  $\mathcal{G}_2 \rightarrow J_{\alpha+1}(\mathcal{F})$  such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{G}_1 & \longrightarrow & \mathcal{G}_2 \\ \downarrow & & \downarrow \\ J_\alpha(\mathcal{F}) & \longrightarrow & J_{\alpha+1}(\mathcal{F}) \end{array}$$

**Proof.** This is because the map  $i\mathcal{G}_1 \rightarrow i\mathcal{G}_2$  is injective and hence  $i\mathcal{G}_1 \rightarrow iJ_\alpha(\mathcal{F})$  extends to  $i\mathcal{G}_2 \rightarrow J(iJ_\alpha(\mathcal{F}))$  which gives the desired map after applying the sheafification functor.  $\square$

This lemma says that somehow the system  $\{J_\alpha(\mathcal{F})\}$  is an injective embedding of  $\mathcal{F}$ . Of course we cannot take the limit over all  $\alpha$  because they form a class and not a set. However, the idea is now that you don't have to check injectivity on all injections  $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ , plus the following lemma.



**Lemma 7.2.** *Suppose that  $\mathcal{G}_i, i \in I$  is set of abelian sheaves on  $\mathcal{C}$ . There exists an ordinal  $\beta$  such that for any sheaf  $\mathcal{F}$ , any  $i \in I$ , and any map  $\varphi : \mathcal{G}_i \rightarrow J_\beta(\mathcal{F})$  there exists an  $\alpha < \beta$  such that  $\varphi$  factors through  $J_\alpha(\mathcal{F})$ .*

**Proof.** This reduces to the case of a single sheaf  $\mathcal{G}$  by taking the direct sum of all the  $\mathcal{G}_i$ .

Consider the sets

$$S = \coprod_{U \in \text{Ob}(\mathcal{C})} \mathcal{G}(U).$$

and

$$T_\beta = \coprod_{U \in \text{Ob}(\mathcal{C})} J_\beta(\mathcal{F})(U)$$

Then  $T_\beta = \text{colim}_{\alpha < \beta} T_\alpha$  with injective transition maps. A morphism  $\mathcal{G} \rightarrow J_\beta(\mathcal{F})$  factors through  $J_\alpha(\mathcal{F})$  if and only if the associated map  $S \rightarrow T_\beta$  factors through  $T_\alpha$ . By Sets, Lemma 7.1 if the cofinality of  $\beta$  is bigger than the cardinality of  $S$ , then the result of the lemma is true. Hence the lemma follows from the fact that there are ordinals with arbitrarily large cofinality, see Sets, Proposition 7.2.  $\square$

Recall that for an object  $X$  of  $\mathcal{C}$  we denote  $\mathbf{Z}_X$  the presheaf of abelian groups  $\Gamma(U, \mathbf{Z}_X) = \oplus_{U \rightarrow X} \mathbf{Z}$ , see Modules on Sites, Section 4. The sheaf associated to this presheaf is denoted  $\mathbf{Z}_X^\#$ , see Modules on Sites, Section 5. It can be characterized by the property

$$(7.2.1) \quad \text{Mor}_{\text{Ab}(\mathcal{C})}(\mathbf{Z}_X^\#, \mathcal{G}) = \mathcal{G}(X)$$

where the element  $\varphi$  of the left hand side is mapped to  $\varphi(1 \cdot \text{id}_X)$  in the right hand side. We can use these sheaves to characterize injective abelian sheaves.

**Lemma 7.3.** *Suppose  $\mathcal{J}$  is a sheaf of abelian groups with the following property: For all  $X \in \text{Ob}(\mathcal{C})$ , for any abelian subsheaf  $\mathcal{S} \subset \mathbf{Z}_X^\#$  and any morphism  $\varphi : \mathcal{S} \rightarrow \mathcal{J}$ , there exists a morphism  $\mathbf{Z}_X^\# \rightarrow \mathcal{J}$  extending  $\varphi$ . Then  $\mathcal{J}$  is an injective sheaf of abelian groups.*

**Proof.** Let  $\mathcal{F} \rightarrow \mathcal{G}$  be an injective map of abelian sheaves. Suppose  $\varphi : \mathcal{F} \rightarrow \mathcal{J}$  is a morphism. Arguing as in the proof of More on Algebra, Lemma 41.1 we see that it suffices to prove that if  $\mathcal{F} \neq \mathcal{G}$ , then we can find an abelian sheaf  $\mathcal{F}'$ ,  $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{G}$  such that (a) the inclusion  $\mathcal{F} \subset \mathcal{F}'$  is strict, and (b)  $\varphi$  can be extended to  $\mathcal{F}'$ . To find  $\mathcal{F}'$ , let  $X$  be an object of  $\mathcal{C}$  such that the inclusion  $\mathcal{F}(X) \subset \mathcal{G}(X)$  is strict. Pick  $s \in \mathcal{G}(X)$ ,  $s \notin \mathcal{F}(X)$ . Let  $\psi : \mathbf{Z}_X^\# \rightarrow \mathcal{G}$  be the morphism corresponding to the section  $s$  via (7.2.1). Set  $\mathcal{S} = \psi^{-1}(\mathcal{F})$ . By assumption the morphism

$$\mathcal{S} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{J}$$

can be extended to a morphism  $\varphi' : \mathbf{Z}_X^\# \rightarrow \mathcal{J}$ . Note that  $\varphi'$  annihilates the kernel of  $\psi$  (as this is true for  $\varphi$ ). Thus  $\varphi'$  gives rise to a morphism  $\varphi'' : \text{Im}(\psi) \rightarrow \mathcal{J}$  which agrees with  $\varphi$  on the intersection  $\mathcal{F} \cap \text{Im}(\psi)$  by construction. Thus  $\varphi$  and  $\varphi''$  glue to give an extension of  $\varphi$  to the strictly bigger subsheaf  $\mathcal{F}' = \mathcal{F} + \text{Im}(\psi)$ .  $\square$

**Theorem 7.4.** *The category of sheaves of abelian groups on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 23.5.*

**Proof.** Let  $\mathcal{G}_i$ ,  $i \in I$  be a set of abelian sheaves such that every subsheaf of every  $\mathbf{Z}_X^\#$  occurs as one of the  $\mathcal{G}_i$ . Apply Lemma 7.2 to this collection to get an ordinal  $\beta$ . We claim that for any sheaf of abelian groups  $\mathcal{F}$  the map  $\mathcal{F} \rightarrow J_\beta(\mathcal{F})$  is an injection of  $\mathcal{F}$  into an injective. Note that by construction the assignment  $\mathcal{F} \mapsto (\mathcal{F} \rightarrow J_\beta(\mathcal{F}))$  is indeed functorial.

The proof of the claim comes from the fact that by Lemma 7.3 it suffices to extend any morphism  $\gamma : \mathcal{G} \rightarrow J_\beta(\mathcal{F})$  from a subsheaf  $\mathcal{G}$  of some  $\mathbf{Z}_X^\#$  to all of  $\mathbf{Z}_X^\#$ . Then by Lemma 7.2 the map  $\gamma$  lifts into  $J_\alpha(\mathcal{F})$  for some  $\alpha < \beta$ . Finally, we apply Lemma 7.1 to get the desired extension of  $\gamma$  to a morphism into  $J_{\alpha+1}(\mathcal{F}) \rightarrow J_\beta(\mathcal{F})$ .  $\square$

## 8. Modules on a ringed site

Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ . By analogy with More on Algebra, Section 42 let us try to prove that there are enough injective  $\mathcal{O}$ -modules. First of all, we pick an injective embedding

$$\bigoplus_{U, \mathcal{I}} j_{U!} \mathcal{O}_U / \mathcal{I} \longrightarrow \mathcal{J}$$

where  $\mathcal{J}$  is an injective abelian sheaf (which exists by the previous section). Here the direct sum is over all objects  $U$  of  $\mathcal{C}$  and over all  $\mathcal{O}$ -submodules  $\mathcal{I} \subset j_{U!} \mathcal{O}_U$ . Please see Modules on Sites, Section 19 to read about the functors restriction and extension by 0 for the localization functor  $j_U : \mathcal{C}/U \rightarrow \mathcal{C}$ .

For any sheaf of  $\mathcal{O}$ -modules  $\mathcal{F}$  denote

$$\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{J})$$

with its natural  $\mathcal{O}$ -module structure. Insert here future reference to internal hom. We will also need a canonical flat resolution of a sheaf of  $\mathcal{O}$ -modules. This we can do as follows: For any  $\mathcal{O}$ -module  $\mathcal{F}$  we denote

$$F(\mathcal{F}) = \bigoplus_{U \in \text{Ob}(\mathcal{C}), s \in \mathcal{F}(U)} j_{U!} \mathcal{O}_U.$$

This is a flat sheaf of  $\mathcal{O}$ -modules which comes equipped with a canonical surjection  $F(\mathcal{F}) \rightarrow \mathcal{F}$ , see Modules on Sites, Lemma 28.6. Moreover the construction  $\mathcal{F} \mapsto F(\mathcal{F})$  is functorial in  $\mathcal{F}$ .

**Lemma 8.1.** *The functor  $\mathcal{F} \mapsto \mathcal{F}^\vee$  is exact.*

**Proof.** This because  $\mathcal{J}$  is an injective abelian sheaf.  $\square$

There is a canonical map  $ev : \mathcal{F} \rightarrow (\mathcal{F}^\vee)^\vee$  given by evaluation: given  $x \in \mathcal{F}(U)$  we let  $ev(x) \in (\mathcal{F}^\vee)^\vee = \mathcal{H}om(\mathcal{F}^\vee, \mathcal{J})$  be the map  $\varphi \mapsto \varphi(x)$ .

**Lemma 8.2.** *For any  $\mathcal{O}$ -module  $\mathcal{F}$  the evaluation map  $ev : \mathcal{F} \rightarrow (\mathcal{F}^\vee)^\vee$  is injective.*

**Proof.** You can check this using the definition of  $\mathcal{J}$ . Namely, if  $s \in \mathcal{F}(U)$  is not zero, then let  $j_{U!} \mathcal{O}_U \rightarrow \mathcal{F}$  be the map of  $\mathcal{O}$ -modules it corresponds to via adjunction. Let  $\mathcal{I}$  be the kernel of this map. There exists a nonzero map  $\mathcal{F} \supset j_{U!} \mathcal{O}_U / \mathcal{I} \rightarrow \mathcal{J}$  which does not annihilate  $s$ . As  $\mathcal{J}$  is an injective  $\mathcal{O}$ -module, this extends to a map  $\varphi : \mathcal{F} \rightarrow \mathcal{J}$ . Then  $ev(s)(\varphi) = \varphi(s) \neq 0$  which is what we had to prove.  $\square$

The canonical surjection  $F(\mathcal{F}) \rightarrow \mathcal{F}$  of  $\mathcal{O}$ -modules turns into a canonical injection, see above, of  $\mathcal{O}$ -modules

$$(\mathcal{F}^\vee)^\vee \longrightarrow (F(\mathcal{F}^\vee))^\vee.$$

Set  $J(\mathcal{F}) = (F(\mathcal{F}^\vee))^\vee$ . The composition of  $ev$  with this the displayed map gives  $\mathcal{F} \rightarrow J(\mathcal{F})$  functorially in  $\mathcal{F}$ .

**Lemma 8.3.** *Let  $\mathcal{O}$  be a sheaf of rings. For every  $\mathcal{O}$ -module  $\mathcal{F}$  the  $\mathcal{O}$ -module  $J(\mathcal{F})$  is injective.*

**Proof.** We have to show that the functor  $\text{Hom}_{\mathcal{O}}(\mathcal{G}, J(\mathcal{F}))$  is exact. Note that

$$\begin{aligned} \text{Hom}_{\mathcal{O}}(\mathcal{G}, J(\mathcal{F})) &= \text{Hom}_{\mathcal{O}}(\mathcal{G}, (F(\mathcal{F}^\vee))^\vee) \\ &= \text{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{H}om(F(\mathcal{F}^\vee), \mathcal{I})) \\ &= \text{Hom}(\mathcal{G} \otimes_{\mathcal{O}} F(\mathcal{F}^\vee), \mathcal{I}) \end{aligned}$$

Thus what we want follows from the fact that  $F(\mathcal{F}^\vee)$  is flat and  $\mathcal{I}$  is injective.  $\square$

**Theorem 8.4.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ . The category of sheaves of  $\mathcal{O}$ -modules on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 23.5.*

**Proof.** From the discussion in this section.  $\square$

**Proposition 8.5.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings on  $\mathcal{C}$ . The category  $P\text{Mod}(\mathcal{O})$  of presheaves of  $\mathcal{O}$ -modules has functorial injective embeddings.*

**Proof.** We could prove this along the lines of the discussion in Section 6. But instead we argue using the theorem above. Endow  $\mathcal{C}$  with the structure of a site by letting the set of coverings of an object  $U$  consist of all singletons  $\{f : V \rightarrow U\}$  where  $f$  is an isomorphism. We omit the verification that this defines a site. A sheaf for this topology is the same as a presheaf (proof omitted). Hence the theorem applies.  $\square$

## 9. Embedding abelian categories

In this section we show that an abelian category embeds in the category of abelian sheaves on a site having enough points. The site will be the one described in the following lemma.

**Lemma 9.1.** *Let  $\mathcal{A}$  be an abelian category. Let*

$$\text{Cov} = \{\{f : V \rightarrow U\} \mid f \text{ is surjective}\}.$$

*Then  $(\mathcal{A}, \text{Cov})$  is a site, see Sites, Definition 6.2.*

**Proof.** Note that  $\text{Ob}(\mathcal{A})$  is a set by our conventions about categories. An isomorphism is a surjective morphism. The composition of surjective morphisms is surjective. And the base change of a surjective morphism in  $\mathcal{A}$  is surjective, see Homology, Lemma 5.14.  $\square$

Let  $\mathcal{A}$  be a pre-additive category. In this case the Yoneda embedding  $\mathcal{A} \rightarrow P\text{Sh}(\mathcal{A})$ ,  $X \mapsto h_X$  factors through a functor  $\mathcal{A} \rightarrow P\text{Ab}(\mathcal{A})$ .

**Lemma 9.2.** *Let  $\mathcal{A}$  be an abelian category. Let  $\mathcal{C} = (\mathcal{A}, \text{Cov})$  be the site defined in Lemma 9.1. Then  $X \mapsto h_X$  defines a fully faithful, exact functor*

$$\mathcal{A} \longrightarrow \text{Ab}(\mathcal{C}).$$

*Moreover, the site  $\mathcal{C}$  has enough points.*

**Proof.** Suppose that  $f : V \rightarrow U$  is a surjective morphism of  $\mathcal{A}$ . Let  $K = \text{Ker}(f)$ . Recall that  $V \times_U V = \text{Ker}((f, -f) : V \oplus V \rightarrow U)$ , see Homology, Example 5.6. In particular there exists an injection  $K \oplus K \rightarrow V \times_U V$ . Let  $p, q : V \times_U V \rightarrow V$  be the two projection morphisms. Note that  $p - q : V \times_U V \rightarrow V$  is a morphism such that  $f \circ (p - q) = 0$ . Hence  $p - q$  factors through  $K \rightarrow V$ . Let us denote this morphism by  $c : V \times_U V \rightarrow K$ . And since the composition  $K \oplus K \rightarrow V \times_U V \rightarrow K$  is surjective, we conclude that  $c$  is surjective. It follows that

$$V \times_U V \xrightarrow{p-q} V \rightarrow U \rightarrow 0$$

is an exact sequence of  $\mathcal{A}$ . Hence for an object  $X$  of  $\mathcal{A}$  the sequence

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(U, X) \rightarrow \text{Hom}_{\mathcal{A}}(V, X) \rightarrow \text{Hom}_{\mathcal{A}}(V \times_U V, X)$$

is an exact sequence of abelian groups, see Homology, Lemma 5.8. This means that  $h_X$  satisfies the sheaf condition on  $\mathcal{C}$ .

The functor is fully faithful by Categories, Lemma 3.5. The functor is a left exact functor between abelian categories by Homology, Lemma 5.8. To show that it is right exact, let  $X \rightarrow Y$  be a surjective morphism of  $\mathcal{A}$ . Let  $U$  be an object of  $\mathcal{A}$ , and let  $s \in h_Y(U) = \text{Mor}_{\mathcal{A}}(U, Y)$  be a section of  $h_Y$  over  $U$ . By Homology, Lemma 5.14 the projection  $U \times_Y X \rightarrow U$  is surjective. Hence  $\{V = U \times_Y X \rightarrow U\}$  is a covering of  $U$  such that  $s|_V$  lifts to a section of  $h_X$ . This proves that  $h_X \rightarrow h_Y$  is a surjection of abelian sheaves, see Sites, Lemma 12.2.

The site  $\mathcal{C}$  has enough points by Sites, Proposition 38.3. □

**Remark 9.3.** The Freyd-Mitchell embedding theorem says there exists a fully faithful exact functor from any abelian category  $\mathcal{A}$  to the category of modules over a ring. Lemma 9.2 is not quite as strong. But the result is suitable for the stacks project as we have to understand sheaves of abelian groups on sites in detail anyway. Moreover, “diagram chasing” works in the category of abelian sheaves on  $\mathcal{C}$ , for example by working with sections over objects, or by working on the level of stalks using that  $\mathcal{C}$  has enough points. To see how to deduce the Freyd-Mitchell embedding theorem from Lemma 9.2 see Remark 9.5.

**Remark 9.4.** If  $\mathcal{A}$  is a “big” abelian category, i.e., if  $\mathcal{A}$  has a class of objects, then Lemma 9.2 does not work. In this case, given any set of objects  $E \subset \text{Ob}(\mathcal{A})$  there exists an abelian full subcategory  $\mathcal{A}' \subset \mathcal{A}$  such that  $\text{Ob}(\mathcal{A}')$  is a set and  $E \subset \text{Ob}(\mathcal{A}')$ . Then one can apply Lemma 9.2 to  $\mathcal{A}'$ . One can use this to prove that results depending on a diagram chase hold in  $\mathcal{A}$ .

**Remark 9.5.** Let  $\mathcal{C}$  be a site. Note that  $\text{Ab}(\mathcal{C})$  has enough injectives, see Theorem 7.4. (In the case that  $\mathcal{C}$  has enough points this is straightforward because  $p_*I$  is an injective sheaf if  $I$  is an injective  $\mathbf{Z}$ -module and  $p$  is a point.) Also,  $\text{Ab}(\mathcal{C})$  has a cogenerator (details omitted). Hence Lemma 9.2 proves that we have a fully faithful, exact embedding  $\mathcal{A} \rightarrow \mathcal{B}$  where  $\mathcal{B}$  has a cogenerator and enough injectives. We can apply this to  $\mathcal{A}^{\text{opp}}$  and we get a fully faithful exact functor  $i : \mathcal{A} \rightarrow \mathcal{D} = \mathcal{B}^{\text{opp}}$  where  $\mathcal{D}$  has enough projectives and a generator. Hence  $\mathcal{D}$  has a projective generator  $P$ . Set  $R = \text{Mor}_{\mathcal{D}}(P, P)$ . Then

$$\mathcal{A} \longrightarrow \text{Mod}_R, \quad X \longmapsto \text{Hom}_{\mathcal{D}}(P, X).$$

One can check this is a fully faithful, exact functor. In other words, one retrieves the Freyd-Mitchell theorem mentioned in Remark 9.3 above.

**Remark 9.6.** The arguments proving Lemmas 9.1 and 9.2 work also for *exact categories*, see [Büh10, Appendix A] and [BBD82, 1.1.4]. We quickly review this here and we add more details if we ever need it in the stacks project.

Let  $\mathcal{A}$  be an additive category. A *kernel-cokernel* pair is a pair  $(i, p)$  of morphisms of  $\mathcal{A}$  with  $i : A \rightarrow B$ ,  $p : B \rightarrow C$  such that  $i$  is the kernel of  $p$  and  $p$  is the cokernel of  $i$ . Given a set  $\mathcal{E}$  of kernel-cokernel pairs we say  $i : A \rightarrow B$  is an *admissible monomorphism* if  $(i, p) \in \mathcal{E}$  for some morphism  $p$ . Similarly we say a morphism  $p : B \rightarrow C$  is an *admissible epimorphism* if  $(i, p) \in \mathcal{E}$  for some morphism  $i$ . The pair  $(\mathcal{A}, \mathcal{E})$  is said to be an *exact category* if the following axioms hold

- (1)  $\mathcal{E}$  is closed under isomorphisms of kernel-cokernel pairs,
- (2) for any object  $A$  the morphism  $1_A$  is both an admissible epimorphism and an admissible monomorphism,
- (3) admissible monomorphisms are stable under composition,
- (4) admissible epimorphisms are stable under composition,
- (5) the push-out of an admissible monomorphism  $i : A \rightarrow B$  via any morphism  $A \rightarrow A'$  exist and the induced morphism  $i' : A' \rightarrow B'$  is an admissible monomorphism, and
- (6) the base change of an admissible epimorphism  $p : B \rightarrow C$  via any morphism  $C' \rightarrow C$  exist and the induced morphism  $p' : B' \rightarrow C'$  is an admissible epimorphism.

Given such a structure let  $\mathcal{C} = (\mathcal{A}, \text{Cov})$  where coverings (i.e., elements of  $\text{Cov}$ ) are given by admissible epimorphisms. The axioms listed above immediately imply that this is a site. Consider the functor

$$F : \mathcal{A} \longrightarrow \text{Ab}(\mathcal{C}), \quad X \longmapsto h_X$$

exactly as in Lemma 9.2. It turns out that this functor is fully faithful, exact, and reflects exactness. Moreover, any extension of objects in the essential image of  $F$  is in the essential image of  $F$ .

## 10. Grothendieck's AB conditions

This and the next few sections are mostly interesting for “big” abelian categories, i.e., those categories listed in Categories, Remark 2.2. A good case to keep in mind is the category of sheaves of modules on a ringed site.

Grothendieck proved the existence of injectives in great generality in the paper [Gro57]. He used the following conditions to single out abelian categories with special properties.

**Definition 10.1.** Let  $\mathcal{A}$  be an abelian category. We name some conditions

- AB3  $\mathcal{A}$  has direct sums,
- AB4  $\mathcal{A}$  has AB3 and direct sums are exact,
- AB5  $\mathcal{A}$  has AB3 and filtered colimits are exact.

Here are the dual notions

- AB3\*  $\mathcal{A}$  has products,
- AB4\*  $\mathcal{A}$  has AB3\* and products are exact,
- AB5\*  $\mathcal{A}$  has AB3\* and filtered limits are exact.

We say an object  $U$  of  $\mathcal{A}$  is a *generator* if for every  $N \subset M$ ,  $N \neq M$  in  $\mathcal{A}$  there exists a morphism  $U \rightarrow M$  which does not factor through  $N$ . We say  $\mathcal{A}$  is a *Grothendieck abelian category* if it has AB5 and a generator.

Discussion: A direct sum in an abelian category is a coproduct. If an abelian category has direct sums (i.e., AB3), then it has colimits, see Categories, Lemma 14.11. Similarly if  $\mathcal{A}$  has AB3\* then it has limits, see Categories, Lemma 14.10. Exactness of direct sums means the following: given an index set  $I$  and short exact sequences

$$0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0, \quad i \in I$$

in  $\mathcal{A}$  then the sequence

$$0 \rightarrow \bigoplus_{i \in I} A_i \rightarrow \bigoplus_{i \in I} B_i \rightarrow \bigoplus_{i \in I} C_i \rightarrow 0$$

is exact as well. Without assuming AB4 it is only true in general that the sequence is exact on the right (i.e., taking direct sums is a right exact functor if direct sums exist). Similarly, exactness of filtered colimits means the following: given a directed partially ordered set  $I$  and a system of short exact sequences

$$0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$$

over  $I$  in  $\mathcal{A}$  then the sequence

$$0 \rightarrow \operatorname{colim}_{i \in I} A_i \rightarrow \operatorname{colim}_{i \in I} B_i \rightarrow \operatorname{colim}_{i \in I} C_i \rightarrow 0$$

is exact as well. Without assuming AB5 it is only true in general that the sequence is exact on the right (i.e., taking colimits is a right exact functor if colimits exist). A similar explanation holds for AB4\* and AB5\*.

## 11. Injectives in Grothendieck categories

The existence of a generator implies that given an object  $M$  of a Grothendieck abelian category  $\mathcal{A}$  there is a set of subobjects. (This may not be true for a general “big” abelian category.)

**Definition 11.1.** Let  $\mathcal{A}$  be a Grothendieck abelian category. Let  $M$  be an object of  $\mathcal{A}$ . The *size*  $|M|$  of  $M$  is the cardinality of the set of subobjects of  $M$ .

**Lemma 11.2.** Let  $\mathcal{A}$  be a Grothendieck abelian category. If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence of  $\mathcal{A}$ , then  $|M'|, |M''| \leq |M|$ .

**Proof.** Immediate from the definitions.  $\square$

**Lemma 11.3.** Let  $\mathcal{A}$  be a Grothendieck abelian category with generator  $U$ .

- (1) If  $|M| \leq \kappa$ , then  $M$  is the quotient of a direct sum of at most  $\kappa$  copies of  $U$ .
- (2) For every cardinal  $\kappa$  there exists a set of isomorphism classes of objects  $M$  with  $|M| \leq \kappa$ .

**Proof.** For (1) choose for every proper subobject  $M' \subset M$  a morphism  $\varphi_{M'} : U \rightarrow M$  whose image is not contained in  $M'$ . Then  $\bigoplus_{M' \subset M} \varphi_{M'} : \bigoplus_{M' \subset M} U \rightarrow M$  is surjective. It is clear that (1) implies (2).  $\square$

**Proposition 11.4.** Let  $\mathcal{A}$  be a Grothendieck abelian category. Let  $M$  be an object of  $\mathcal{A}$ . Let  $\kappa = |M|$ . If  $\alpha$  is an ordinal whose cofinality is bigger than  $\kappa$ , then  $M$  is  $\alpha$ -small with respect to injections.

**Proof.** Please compare with Proposition 2.5. We need only show that the map (2.0.1) is a surjection. Let  $f : M \rightarrow \operatorname{colim} B_\beta$  be a map. Consider the subobjects  $\{f^{-1}(B_\beta)\}$  of  $M$ , where  $B_\beta$  is considered as a subobject of the colimit  $B = \bigcup_\beta B_\beta$ . If one of these, say  $f^{-1}(B_\beta)$ , fills  $M$ , then the map factors through  $B_\beta$ .

So suppose to the contrary that all of the  $f^{-1}(B_\beta)$  were proper subobjects of  $M$ . However, because  $\mathcal{A}$  has AB5 we have

$$\operatorname{colim} f^{-1}(B_\beta) = f^{-1}(\operatorname{colim} B_\beta) = M.$$

Now there are at most  $\kappa$  different subobjects of  $M$  that occur among the  $f^{-1}(B_\alpha)$ , by hypothesis. Thus we can find a subset  $S \subset \alpha$  of cardinality at most  $\kappa$  such that as  $\beta'$  ranges over  $S$ , the  $f^{-1}(B_{\beta'})$  range over *all* the  $f^{-1}(B_\alpha)$ .

However,  $S$  has an upper bound  $\tilde{\alpha} < \alpha$  as  $\alpha$  has cofinality bigger than  $\kappa$ . In particular, all the  $f^{-1}(B_{\beta'})$ ,  $\beta' \in S$  are contained in  $f^{-1}(B_{\tilde{\alpha}})$ . It follows that  $f^{-1}(B_{\tilde{\alpha}}) = M$ . In particular, the map  $f$  factors through  $B_{\tilde{\alpha}}$ .  $\square$

**Lemma 11.5.** *Let  $\mathcal{A}$  be a Grothendieck abelian category with generator  $U$ . An object  $I$  of  $\mathcal{A}$  is injective if and only if in every commutative diagram*

$$\begin{array}{ccc} M & \longrightarrow & I \\ \downarrow & \nearrow & \\ U & & \end{array}$$

for  $M \subset U$  a subobject, the dotted arrow exists.

**Proof.** Please see Lemma 2.6 for the case of modules. Choose an injection  $A \subset B$  and a morphism  $\varphi : A \rightarrow I$ . Consider the set  $S$  of pairs  $(A', \varphi')$  consisting of subobjects  $A \subset A' \subset B$  and a morphism  $\varphi' : A' \rightarrow I$  extending  $\varphi$ . Define a partial ordering on this set in the obvious manner. Choose a totally ordered subset  $T \subset S$ . Then

$$A' = \operatorname{colim}_{t \in T} A_t \xrightarrow{\operatorname{colim}_{t \in T} \varphi_t} I$$

is an upper bound. Hence by Zorn's lemma the set  $S$  has a maximal element  $(A', \varphi')$ . We claim that  $A' = B$ . If not, then choose a morphism  $\psi : U \rightarrow B$  which does not factor through  $A'$ . Set  $N = A' \cap \psi(U)$ . Set  $M = \psi^{-1}(N)$ . Then the map

$$M \rightarrow N \rightarrow A' \xrightarrow{\varphi'} I$$

can be extended to a morphism  $\chi : U \rightarrow I$ . Since  $\chi|_{\operatorname{Ker}(\psi)} = 0$  we see that  $\chi$  factors as

$$U \rightarrow \operatorname{Im}(\psi) \xrightarrow{\varphi''} I$$

Since  $\varphi'$  and  $\varphi''$  agree on  $N = A' \cap \operatorname{Im}(\psi)$  we see that combined they define a morphism  $A' + \operatorname{Im}(\psi) \rightarrow I$  contradicting the assumed maximality of  $A'$ .  $\square$

**Theorem 11.6.** *Let  $\mathcal{A}$  be a Grothendieck abelian category. Then  $\mathcal{A}$  has functorial injective embeddings.*

**Proof.** Please compare with the proof of Theorem 2.8. Choose a generator  $U$  of  $\mathcal{A}$ . For an object  $M$  we define  $\mathbf{M}(M)$  by the following pushout diagram

$$\begin{array}{ccc} \bigoplus_{N \subset U} \bigoplus_{\varphi \in \text{Hom}(N, M)} N & \longrightarrow & M \\ \downarrow & & \downarrow \\ \bigoplus_{N \subset U} \bigoplus_{\varphi \in \text{Hom}(U, M)} U & \longrightarrow & \mathbf{M}(M). \end{array}$$

Note that  $M \rightarrow \mathbf{M}(M)$  is a functor and that there exist functorial injective maps  $M \rightarrow \mathbf{M}(M)$ . By transfinite induction we define functors  $\mathbf{M}_\alpha(M)$  for every ordinal  $\alpha$ . Namely, set  $\mathbf{M}_0(M) = M$ . Given  $\mathbf{M}_\alpha(M)$  set  $\mathbf{M}_{\alpha+1}(M) = \mathbf{M}(\mathbf{M}_\alpha(M))$ . For a limit ordinal  $\beta$  set

$$\mathbf{M}_\beta(M) = \text{colim}_{\alpha < \beta} \mathbf{M}_\alpha(M).$$

Finally, choose an ordinal  $\alpha$  whose cofinality is greater than  $|U|$ , see Sets, Proposition 7.2. We claim that  $M \rightarrow \mathbf{M}_\alpha(M)$  is the desired functorial injective embedding. Namely, if  $N \subset U$  is a subobject and  $\varphi : N \rightarrow \mathbf{M}_\alpha(M)$  is a morphism, then we see that  $\varphi$  factors through  $\mathbf{M}_{\alpha'}(M)$  for some  $\alpha' < \alpha$  by Proposition 11.4. By construction of  $\mathbf{M}(-)$  we see that  $\varphi$  extends to a morphism from  $U$  into  $\mathbf{M}_{\alpha'+1}(M)$  and hence into  $\mathbf{M}_\alpha(M)$ . By Lemma 11.5 we conclude that  $\mathbf{M}_\alpha(M)$  is injective.  $\square$

## 12. K-injectives in Grothendieck categories

The material in this section is taken from the paper [Ser03] authored by Serpé. This paper generalizes some of the results of [Spa88] by Spaltenstein to general Grothendieck abelian categories. Our Lemma 12.3 is only implicit in the paper by Serpé. Our approach is to mimic Grothendieck's proof of Theorem 11.6.

**Lemma 12.1.** *Let  $\mathcal{A}$  be a Grothendieck abelian category with generator  $U$ . Let  $c$  be the function on cardinals defined by  $c(\kappa) = |\bigoplus_{\alpha \in \kappa} U|$ . If  $\pi : M \rightarrow N$  is a surjection then there exists a subobject  $M' \subset M$  which surjects onto  $N$  with  $|N'| \leq c(|N|)$ .*

**Proof.** For every proper subobject  $N' \subset N$  choose a morphism  $\varphi_{N'} : U \rightarrow M$  such that  $U \rightarrow M \rightarrow N$  does not factor through  $N'$ . Set

$$N' = \text{Im} \left( \bigoplus_{N' \subset N} \varphi_{N'} : \bigoplus_{N' \subset N} U \longrightarrow M \right)$$

Then  $N'$  works.  $\square$

**Lemma 12.2.** *Let  $\mathcal{A}$  be a Grothendieck abelian category. There exists a cardinal  $\kappa$  such that given any acyclic complex  $M^\bullet$  we have*

- (1) *if  $M^\bullet$  is nonzero, there is a nonzero subcomplex  $N^\bullet$  which is bounded above, acyclic, and  $|N^n| \leq \kappa$ ,*
- (2) *there exists a surjection of complexes*

$$\bigoplus_{i \in I} M_i^\bullet \longrightarrow M^\bullet$$

*where  $M_i^\bullet$  is bounded above, acyclic, and  $|M_i^n| \leq \kappa$ .*

**Proof.** Choose a generator  $U$  of  $\mathcal{A}$ . Denote  $c$  the function of Lemma 12.1. Set  $\kappa = \sup\{c^n(|U|), n = 1, 2, 3, \dots\}$ . Let  $n \in \mathbf{Z}$  and let  $\psi : U \rightarrow M^n$  be a morphism. In order to prove (1) and (2) it suffices to prove there exists a subcomplex  $N^\bullet \subset M^\bullet$  which is bounded above, acyclic, and  $|N^m| \leq \kappa$ , such that  $\psi$  factors through  $N^n$ .



To do this set  $N^n = \text{Im}(\psi)$ ,  $N^{n+1} = \text{Im}(U \rightarrow M^n \rightarrow M^{n+1})$ , and  $N^m = 0$  for  $m \geq n+2$ . Suppose we have constructed  $N^m \subset M^m$  for all  $m \geq k$  such that

- (1)  $d(N^m) \subset N^{m+1}$ ,  $m \geq k$ ,
- (2)  $\text{Im}(N^{m-1} \rightarrow N^m) = \text{Ker}(N^m \rightarrow N^{m+1})$  for all  $m \geq k+1$ , and
- (3)  $|N^m| \leq c^{\max\{n-m, 0\}}(|U|)$ .

for some  $k \leq n$ . Because  $M^\bullet$  is acyclic, we see that the subobject  $d^{-1}(\text{Ker}(N^k \rightarrow N^{k+1})) \subset M^{k-1}$  surjects onto  $\text{Ker}(N^k \rightarrow N^{k+1})$ . Thus we can choose  $N^{k-1} \subset M^{k-1}$  surjecting onto  $\text{Ker}(N^k \rightarrow N^{k+1})$  with  $|N^{k-1}| \leq c^{n-k+1}(|U|)$  by Lemma 12.1. The proof is finished by induction on  $k$ .  $\square$

**Lemma 12.3.** *Let  $\mathcal{A}$  be a Grothendieck abelian category. Let  $\kappa$  be a cardinal as in Lemma 12.2. Suppose that  $I^\bullet$  is a complex such that*

- (1) *each  $I^j$  is injective, and*
- (2) *for every bounded above acyclic complex  $M^\bullet$  such that  $|M^n| \leq \kappa$  we have  $\text{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$ .*

*Then  $I^\bullet$  is an  $K$ -injective complex.*

**Proof.** Let  $M^\bullet$  be an acyclic complex. We are going to construct by induction on the ordinal  $\alpha$  an acyclic subcomplex  $K_\alpha^\bullet \subset M^\bullet$  as follows. For  $\alpha = 0$  we set  $N_0^\bullet = 0$ . For  $\alpha > 0$  we proceed as follows:

- (1) If  $\alpha = \beta + 1$  and  $K_\beta^\bullet = M^\bullet$  then we choose  $K_\alpha^\bullet = K_\beta^\bullet$ .
- (2) If  $\alpha = \beta + 1$  and  $K_\beta^\bullet \neq M^\bullet$  then  $M^\bullet/K_\beta^\bullet$  is a nonzero acyclic complex. We choose a subcomplex  $N_\alpha^\bullet \subset M^\bullet/K_\beta^\bullet$  as in Lemma 12.2. Finally, we let  $K_\alpha^\bullet \subset M^\bullet$  be the inverse image of  $N_\alpha^\bullet$ .
- (3) If  $\alpha$  is a limit ordinal we set  $N_\beta^\bullet = \text{colim } N_\alpha^\bullet$ .

It is clear that  $M^\bullet = K_\alpha^\bullet$  for a suitably large ordinal  $\alpha$ . We will prove that

$$\text{Hom}_{K(\mathcal{A})}(K_\alpha^\bullet, I^\bullet)$$

is zero by transfinite induction on  $\alpha$ . It holds for  $\alpha = 0$  since  $K_0^\bullet$  is zero. Suppose it holds for  $\beta$  and  $\alpha = \beta + 1$ . In case (1) of the list above the result is clear. In case (2) there is a short exact sequence of complexes

$$0 \rightarrow K_\beta^\bullet \rightarrow K_\alpha^\bullet \rightarrow N_\alpha^\bullet \rightarrow 0$$

Since each component of  $I^\bullet$  is injective we see that we obtain an exact sequence

$$\text{Hom}_{K(\mathcal{A})}(K_\beta^\bullet, I^\bullet) \rightarrow \text{Hom}_{K(\mathcal{A})}(K_\alpha^\bullet, I^\bullet) \rightarrow \text{Hom}_{K(\mathcal{A})}(N_\alpha^\bullet, I^\bullet)$$

By induction the term on the left is zero and by assumption on  $I^\bullet$  the term on the right is zero. Thus the middle group is zero too. Finally, suppose that  $\alpha$  is a limit ordinal. Then we see that

$$\text{Hom}^\bullet(K_\alpha^\bullet, I^\bullet) = \lim_{\beta < \alpha} \text{Hom}^\bullet(K_\beta^\bullet, I^\bullet)$$

with notation as in More on Algebra, Section 54. These complexes compute morphisms in  $K(\mathcal{A})$  by More on Algebra, Equation (54.0.1). Note that the transition maps in the system are surjective because  $I^j$  is surjective for each  $j$ . Moreover, for a limit ordinal  $\alpha$  we have equality of limit and value (see displayed formula above). Thus we may apply Homology, Lemma 27.8 to conclude.  $\square$

**Lemma 12.4.** *Let  $\mathcal{A}$  be a Grothendieck abelian category. Let  $(K_i^\bullet)_{i \in I}$  be a set of acyclic complexes. There exists a functor  $M^\bullet \mapsto \mathbf{M}^\bullet(M^\bullet)$  and a natural transformation  $j_{M^\bullet} : M^\bullet \rightarrow \mathbf{M}^\bullet(M^\bullet)$  such*

- (1)  $j_{M^\bullet}$  is a (termwise) injective quasi-isomorphism, and
- (2) for every  $i \in I$  and  $w : K_i^\bullet \rightarrow M^\bullet$  the morphism  $j_{M^\bullet} \circ w$  is homotopic to zero.

**Proof.** For every  $i \in I$  choose a (termwise) injective map of complexes  $K_i^\bullet \rightarrow L_i^\bullet$  which is homotopic to zero with  $L_i^\bullet$  quasi-isomorphic to zero. For example, take  $L_i^\bullet$  to be the cone on the identity of  $K_i^\bullet$ . We define  $\mathbf{M}^\bullet(M^\bullet)$  by the following pushout diagram

$$\begin{array}{ccc} \bigoplus_{i \in I} \bigoplus_{w: K_i^\bullet \rightarrow M^\bullet} K_i^\bullet & \longrightarrow & M^\bullet \\ \downarrow & & \downarrow \\ \bigoplus_{i \in I} \bigoplus_{w: K_i^\bullet \rightarrow M^\bullet} L_i^\bullet & \longrightarrow & \mathbf{M}^\bullet(M^\bullet). \end{array}$$

Then  $M^\bullet \rightarrow \mathbf{M}^\bullet(M^\bullet)$  is a functor. The right vertical arrow defines the functorial injective map  $j_{M^\bullet}$ . The cokernel of  $j_{M^\bullet}$  is isomorphic to the direct sum of the cokernels of the maps  $K_i^\bullet \rightarrow L_i^\bullet$  hence acyclic. Thus  $j_{M^\bullet}$  is a quasi-isomorphism. Part (2) holds by construction.  $\square$

**Lemma 12.5.** *Let  $\mathcal{A}$  be a Grothendieck abelian category. There exists a functor  $M^\bullet \mapsto \mathbf{N}^\bullet(M^\bullet)$  and a natural transformation  $j_{M^\bullet} : M^\bullet \rightarrow \mathbf{N}^\bullet(M^\bullet)$  such*

- (1)  $j_{M^\bullet}$  is a (termwise) injective quasi-isomorphism, and
- (2) for every  $n \in \mathbf{Z}$  the map  $M^n \rightarrow \mathbf{N}^n(M^\bullet)$  factors through a subobject  $I^n \subset \mathbf{N}^n(M^\bullet)$  where  $I^n$  is an injective object of  $\mathcal{A}$ .

**Proof.** Choose a functorial injective embeddings  $i_M : M \rightarrow I(M)$ , see Theorem 11.6. For every complex  $M^\bullet$  denote  $J^\bullet(M^\bullet)$  the complex with terms  $J^n(M^\bullet) = I(M^n) \oplus I(M^{n+1})$  and differential

$$d_{J^\bullet(M^\bullet)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

There exists a canonical injective map of complexes  $u_{M^\bullet} : M^\bullet \rightarrow J^\bullet(M^\bullet)$  by mapping  $M^n$  to  $I(M^n) \oplus I(M^{n+1})$  via the maps  $i_{M^n} : M^n \rightarrow I(M^n)$  and  $i_{M^{n+1}} \circ d : M^n \rightarrow M^{n+1} \rightarrow I(M^{n+1})$ . Hence a short exact sequence of complexes

$$0 \rightarrow M^\bullet \xrightarrow{u_{M^\bullet}} J^\bullet(M^\bullet) \xrightarrow{v_{M^\bullet}} Q^\bullet(M^\bullet) \rightarrow 0$$

functorial in  $M^\bullet$ . Set

$$\mathbf{N}^\bullet(M^\bullet) = C(v_{M^\bullet})^\bullet[-1].$$

Note that

$$\mathbf{N}^n(M^\bullet) = Q^{n-1}(M^\bullet) \oplus J^n(M^\bullet)$$

with differential

$$\begin{pmatrix} -d_{Q^\bullet(M^\bullet)}^{n-1} & -v_{M^\bullet}^n \\ 0 & d_{J^\bullet(M^\bullet)}^n \end{pmatrix}$$

Hence we see that there is a map of complexes  $j_{M^\bullet} : M^\bullet \rightarrow \mathbf{N}^\bullet(M^\bullet)$  induced by  $u$ . It is injective and factors through an injective subobject by construction. The map  $j_{M^\bullet}$  is a quasi-isomorphism as one can prove by looking at the long exact sequence of cohomology associated to the short exact sequences of complexes above.  $\square$

**Theorem 12.6.** *Let  $\mathcal{A}$  be a Grothendieck abelian category. For every complex  $M^\bullet$  there exists a quasi-isomorphism  $M^\bullet \rightarrow I^\bullet$  where  $I^\bullet$  is a  $K$ -injective complex. In fact, we may also assume that  $I^n$  is an injective object of  $\mathcal{A}$  for all  $n$ . Moreover, there exists a functorial injective quasi-isomorphism into such a  $K$ -injective complex.*

**Proof.** Please compare with the proof of Theorem 2.8 and Theorem 11.6. Choose a cardinal  $\kappa$  as in Lemmas 12.2 and 12.3. Choose a set  $(K_i^\bullet)_{i \in I}$  of bounded above, acyclic complexes such that every bounded above acyclic complex  $K^\bullet$  such that  $|K^n| \leq \kappa$  is isomorphic to  $K_i^\bullet$  for some  $i \in I$ . This is possible by Lemma 11.3. Denote  $\mathbf{M}^\bullet(-)$  the functor constructed in Lemma 12.4. Denote  $\mathbf{N}^\bullet(-)$  the functor constructed in Lemma 12.5. Both of these functors come with injective transformations  $\text{id} \rightarrow \mathbf{M}$  and  $\text{id} \rightarrow \mathbf{N}$ .

By transfinite induction we define a sequence of functors  $\mathbf{T}_\alpha(-)$  and corresponding transformations  $\text{id} \rightarrow \mathbf{T}_\alpha$ . Namely we set  $\mathbf{T}_0(M^\bullet) = M^\bullet$ . If  $\mathbf{T}_\alpha$  is given then we set

$$\mathbf{T}_{\alpha+1}(M^\bullet) = \mathbf{N}^\bullet(\mathbf{M}^\bullet(\mathbf{T}_\alpha(M^\bullet)))$$

If  $\beta$  is a limit ordinal we set

$$\mathbf{T}_\beta(M^\bullet) = \text{colim}_{\alpha < \beta} \mathbf{T}_\alpha(M^\bullet)$$

The transition maps of the system are injective quasi-isomorphisms. By AB5 we see that the colimit is still quasi-isomorphic to  $M^\bullet$ . We claim that  $M^\bullet \rightarrow \mathbf{T}_\alpha(M^\bullet)$  does the job if the cofinality of  $\alpha$  is larger than  $\max(\kappa, |U|)$  where  $U$  is a generator of  $\mathcal{A}$ . Namely, it suffices to check conditions (1) and (2) of Lemma 12.3.

For (1) we use the criterion of Lemma 11.5. Suppose that  $M \subset U$  and  $\varphi : M \rightarrow \mathbf{T}_\alpha^n(M^\bullet)$  is a morphism for some  $n \in \mathbf{Z}$ . By Proposition 11.4 we see that  $\varphi$  factor through  $\mathbf{T}_{\alpha'}^n(M^\bullet)$  for some  $\alpha' < \alpha$ . In particular, by the construction of the functor  $\mathbf{N}^\bullet(-)$  we see that  $\varphi$  factors through an injective object of  $\mathcal{A}$  which shows that  $\varphi$  lifts to a morphism on  $U$ .

For (2) let  $w : K^\bullet \rightarrow \mathbf{T}_\alpha(M^\bullet)$  be a morphism of complexes where  $K^\bullet$  is a bounded above acyclic complex such that  $|K^n| \leq \kappa$ . Then  $K^\bullet \cong K_i^\bullet$  for some  $i \in I$ . Moreover, by Proposition 11.4 once again we see that  $w$  factor through  $\mathbf{T}_{\alpha'}^n(M^\bullet)$  for some  $\alpha' < \alpha$ . In particular, by the construction of the functor  $\mathbf{M}^\bullet(-)$  we see that  $w$  is homotopic to zero. This finishes the proof.  $\square$

### 13. Additional remarks on Grothendieck abelian categories

In this section we put some results on Grothendieck abelian categories which are folklore.

**Lemma 13.1.** *Let  $\mathcal{A}$  be a Grothendieck abelian category. Let  $F : \mathcal{A}^{opp} \rightarrow \text{Sets}$  be a functor. Then  $F$  is representable if and only if  $F$  commutes with colimits, i.e.,*

$$F(\text{colim}_i N_i) = \lim F(N_i)$$

for any diagram  $\mathcal{I} \rightarrow \mathcal{A}$ ,  $i \in \mathcal{I}$ .

**Proof.** If  $F$  is representable, then it commutes with colimits by definition of colimits.

Assume that  $F$  commutes with colimits. Then  $F(M \oplus N) = F(M) \coprod F(N)$  and we can use this to define a group structure on  $F(M)$ . Hence we get  $F : \mathcal{A} \rightarrow \text{Ab}$

which is additive and right exact, i.e., transforms a short exact sequence  $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$  into an exact sequence  $F(K) \leftarrow F(L) \leftarrow F(M) \leftarrow 0$  (compare with Homology, Section 7).

Let  $U$  be a generator for  $\mathcal{A}$ . Set  $A = \bigoplus_{s \in F(U)} U$ . Let  $s_{univ} = (s)_{s \in F(U)} \in F(A) = \prod_{s \in F(U)} F(U)$ . Let  $A' \subset A$  be the largest subobject such that  $s_{univ}$  restricts to zero on  $A'$ . This exists because  $\mathcal{A}$  is a grothendieck category and because  $F$  commutes with colimits. Because  $F$  commutes with colimits there exists a unique element  $\bar{s}_{univ} \in F(A/A')$  which maps to  $s_{univ}$  in  $F(A)$ . We claim that  $A/A'$  represents  $F$ , in other words, the Yoneda map

$$\bar{s}_{univ} : h_{A/A'} \longrightarrow F$$

is an isomorphism. Let  $M \in \text{Ob}(\mathcal{A})$  and  $s \in F(M)$ . Consider the surjection

$$c_M : A_M = \bigoplus_{\varphi \in \text{Hom}_{\mathcal{A}}(U, M)} U \longrightarrow M.$$

This gives  $F(c_M)(s) = (s_\varphi) \in \prod_{\varphi} F(U)$ . Consider the map

$$\psi : A_M = \bigoplus_{\varphi \in \text{Hom}_{\mathcal{A}}(U, M)} U \longrightarrow \bigoplus_{s \in F(U)} U = A$$

which maps the summand corresponding to  $\varphi$  to the summand corresponding to  $s_\varphi$  by the identity map on  $U$ . Then  $s_{univ}$  maps to  $(s_\varphi)_\varphi$  by construction. In other words the right square in the diagram

$$\begin{array}{ccccc} A' & \longrightarrow & A & \xrightarrow{s_{univ}} & F \\ \uparrow ? & & \uparrow \psi & & \uparrow s \\ K & \longrightarrow & A_M & \longrightarrow & M \end{array}$$

commutes. Let  $K = \text{Ker}(A_M \rightarrow M)$ . Since  $s$  restricts to zero on  $K$  we see that  $\psi(K) \subset A'$  by definition of  $A'$ . Hence there is an induced morphism  $M \rightarrow A/A'$ . This construction gives an inverse to the map  $h_{A/A'}(M) \rightarrow F(M)$  (details omitted).  $\square$

**Lemma 13.2.** *A Grothendieck abelian category has  $\text{Ab}3^*$ .*

**Proof.** Let  $M_i$ ,  $i \in I$  be a family of objects of  $\mathcal{A}$  indexed by a set  $I$ . The functor  $F = \prod_{i \in I} h_{M_i}$  commutes with colimits. Hence Lemma 13.1 applies.  $\square$

**Remark 13.3.** In the chapter on derived categories we consistently work with “small” abelian categories (as is the convention in the Stacks project). For a “big” abelian category  $\mathcal{A}$  it isn’t clear that the derived category  $D(\mathcal{A})$  exists because it isn’t clear that morphisms in the derived category are sets. In general this isn’t true, see Examples, Lemma 52.1. However, if  $\mathcal{A}$  is a Grothendieck abelian category, and given  $K^\bullet, L^\bullet$  in  $K(\mathcal{A})$ , then by Theorem 12.6 there exists a quasi-isomorphism  $L^\bullet \rightarrow I^\bullet$  to a K-injective complex  $I^\bullet$  and Derived Categories, Lemma 29.2 shows that

$$\text{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \text{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$$

which is a set. Some examples of Grothendieck abelian categories are the category of modules over a ring, or more generally the category of sheaves of modules on a ringed site.

**Lemma 13.4.** *Let  $\mathcal{A}$  be a Grothendieck abelian category. Then*

- (1)  $D(\mathcal{A})$  has both direct sums and products,
- (2) direct sums are obtained by taking termwise direct sums of any complexes,
- (3) products are obtained by taking termwise products of K-injective complexes.

**Proof.** Let  $K_i^\bullet$ ,  $i \in I$  be a family of objects of  $D(\mathcal{A})$  indexed by a set  $I$ . We claim that the termwise direct sum  $\bigoplus_{i \in I} K_i^\bullet$  is a direct sum in  $D(\mathcal{A})$ . Namely, let  $I^\bullet$  be a K-injective complex. Then we have

$$\begin{aligned} \operatorname{Hom}_{D(\mathcal{A})}(\bigoplus_{i \in I} K_i^\bullet, I^\bullet) &= \operatorname{Hom}_{K(\mathcal{A})}(\bigoplus_{i \in I} K_i^\bullet, I^\bullet) \\ &= \prod_{i \in I} \operatorname{Hom}_{K(\mathcal{A})}(K_i^\bullet, I^\bullet) \\ &= \prod_{i \in I} \operatorname{Hom}_{D(\mathcal{A})}(K_i^\bullet, I^\bullet) \end{aligned}$$

as desired. This is sufficient since any complex can be represented by a K-injective complex by Theorem 12.6. To construct the product, choose a K-injective resolution  $K_i^\bullet \rightarrow I_i^\bullet$  for each  $i$ . Then we claim that  $\prod_{i \in I} I_i^\bullet$  is a product in  $D(\mathcal{A})$ . Namely, let  $K^\bullet$  be an complex. Note that a product of K-injective complexes is K-injective (follows immediately from the definition). Thus we have

$$\begin{aligned} \operatorname{Hom}_{D(\mathcal{A})}(K^\bullet, \prod_{i \in I} I_i^\bullet) &= \operatorname{Hom}_{K(\mathcal{A})}(K^\bullet, \prod_{i \in I} I_i^\bullet) \\ &= \prod_{i \in I} \operatorname{Hom}_{K(\mathcal{A})}(K^\bullet, I_i^\bullet) \\ &= \prod_{i \in I} \operatorname{Hom}_{D(\mathcal{A})}(K^\bullet, I_i^\bullet) \end{aligned}$$

which proves the result.  $\square$

**Remark 13.5.** Let  $R$  be a ring. Suppose that  $M_n$ ,  $n \in \mathbf{Z}$  are  $R$ -modules. Denote  $E_n = M_n[-n] \in D(R)$ . We claim that  $E = \bigoplus M_n[-n]$  is *both* the direct sum and the product of the objects  $E_n$  in  $D(R)$ . To see that it is the direct sum, take a look at the proof of Lemma 13.4. To see that it is the direct product, take injective resolutions  $M_n \rightarrow I_n^\bullet$ . By the proof of Lemma 13.4 we have

$$\prod E_n = \prod I_n^\bullet[-n]$$

in  $D(R)$ . Since products in  $\operatorname{Mod}_R$  are exact, we see that  $\prod I_n^\bullet$  is quasi-isomorphic to  $E$ . This works more generally in  $D(\mathcal{A})$  where  $\mathcal{A}$  is a Grothendieck abelian category with  $\operatorname{Ab}4^*$ .

**Lemma 13.6.** *Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor of abelian categories. Assume*

- (1)  $\mathcal{A}$  is a Grothendieck abelian category,
- (2)  $\mathcal{B}$  has exact countable products, and
- (3)  $F$  commutes with countable products.

*Then  $RF : D(\mathcal{A}) \rightarrow D(\mathcal{B})$  commutes with derived limits.*

**Proof.** Observe that  $RF$  exists as  $\mathcal{A}$  has enough K-injectives (Theorem 12.6 and Derived Categories, Lemma 29.5). The statement means that if  $K = R \lim K_n$ , then  $RF(K) = R \lim RF(K_n)$ . See Derived Categories, Definition 32.1 for notation. Since  $RF$  is an exact functor of triangulated categories it suffices to see that  $RF$  commutes with countable products of objects of  $D(\mathcal{A})$ . In the proof of Lemma 13.4 we have seen that products in  $D(\mathcal{A})$  are computed by taking products of K-injective complexes and moreover that a product of K-injective complexes is K-injective. Moreover, in Derived Categories, Lemma 32.2 we have seen that products in  $D(\mathcal{B})$

are computed by taking termwise products. Since  $RF$  is computed by applying  $F$  to a  $K$ -injective representative and since we've assumed  $F$  commutes with countable products, the lemma follows.  $\square$

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