VARIETIES

Contents

1. Introduction

In this chapter we start studying varieties and more generally schemes over a field. A fundamental reference is [DG67].

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2. Notation

Throughout this chapter we use the letter k to denote the ground field.

3. Varieties

In the stacks project we will use the following as our definition of a variety.

Definition 3.1. Let k be a field. A *variety* is a scheme X over k such that X is integral and the structure morphism $X \to \mathrm{Spec}(k)$ is separated and of finite type.

This definition has the following drawback. Suppose that $k \subset k'$ is an extension of fields. Suppose that X is a variety over k. Then the base change $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is not necessarily a variety over k'. This phenomenon (in greater generality) will be discussed in detail in the following sections. The product of two varieties need not be a variety (this is really the same phenomenon). Here is an example.

Example 3.2. Let $k = \mathbf{Q}$. Let $X = \mathrm{Spec}(\mathbf{Q}(i))$ and $Y = \mathrm{Spec}(\mathbf{Q}(i))$. Then the product $X \times_{\mathrm{Spec}(k)} Y$ of the varieties X and Y is not a variety, since it is reducible. (It is isomorphic to the disjoint union of two copies of X.)

If the ground field is algebraically closed however, then the product of varieties is a variety. This follows from the results in the algebra chapter, but there we treat much more general situations. There is also a simple direct proof of it which we present here.

Lemma 3.3. *Let k be an algebraically closed field. Let X, Y be varieties over k. Then $X \times_{\mathrm{Spec}(k)} Y$ is a variety over k.*

Proof. The morphism $X \times_{\mathrm{Spec}(k)} Y \to \mathrm{Spec}(k)$ is of finite type and separated because it is the composition of the morphisms $X \times_{\mathrm{Spec}(k)} Y \to Y \to \mathrm{Spec}(k)$ which are separated and of finite type, see Morphisms, Lemmas 16.4 and 16.3 and Schemes, Lemma 21.13. To finish the proof it suffices to show that $X \times_{\mathrm{Spec}(k)} Y$ is integral. Let $X = \bigcup_{i=1,\ldots,n} U_i$, $Y = \bigcup_{j=1,\ldots,m} V_j$ be finite affine open coverings. If we can show that each $U_i \times_{\mathrm{Spec}(k)} V_j$ is integral, then we are done by Properties, Lemmas 3.2, 3.3, and 3.4. This reduces us to the affine case.

The affine case translates into the following algebra statement: Suppose that A, B are integral domains and finitely generated k-algebras. Then $A \otimes_k B$ is an integral domain. To get a contradiction suppose that

$$(\sum\nolimits_{i=1,\ldots,n} a_i \otimes b_i)(\sum\nolimits_{j=1,\ldots,m} c_j \otimes d_j) = 0$$

in $A \otimes_k B$ with both factors nonzero in $A \otimes_k B$. We may assume that $b_1, \ldots, b_n$ are k-linearly independent in B, and that $d_1, \ldots, d_m$ are k-linearly independent in B. Of course we may also assume that a_1 and c_1 are nonzero in A. Hence $D(a_1c_1) \subset \mathrm{Spec}(A)$ is nonempty. By the Hilbert Nullstellensatz (Algebra, Theorem 33.1) we can find a maximal ideal $\mathfrak{m} \subset A$ contained in $D(a_1c_1)$ and $A/\mathfrak{m} = k$ as k is algebraically closed. Denote $\overline{a}_i, \overline{c}_j$ the residue classes of a_i, c_j in $A/\mathfrak{m} = k$. Then equation above becomes

$$(\sum\nolimits_{i=1,\ldots,n} \overline{a}_i b_i)(\sum\nolimits_{j=1,\ldots,m} \overline{c}_j d_j) = 0$$

which is a contradiction with $\mathfrak{m} \in D(a_1c_1)$, the linear independence of $b_1, \ldots, b_n$ and $d_1, \ldots, d_m$, and the fact that B is a domain. $\square$

4. Geometrically reduced schemes

If X is a reduced scheme over a field, then it can happen that X becomes nonreduced after extending the ground field. This does not happen for geometrically reduced schemes.

Definition 4.1. Let k be a field. Let X be a scheme over k. Let $x \in X$ be a point.

(1) Let $x \in X$ be a point. We say X is *geometrically reduced at* x if for any field extension $k \subset k'$ and any point $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is reduced.
(2) We say X is *geometrically reduced* over k if X is geometrically reduced at every point of X.

This may seem a little mysterious at first, but it is really the same thing as the notion discussed in the algebra chapter. Here are some basic results explaining the connection.

Lemma 4.2. *Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically reduced at x, and*
(2) *the ring $\mathcal{O}_{X,x}$ is geometrically reduced over k (see Algebra, Definition 42.1).*

Proof. Assume (1). This in particular implies that $\mathcal{O}_{X,x}$ is reduced. Let $k \subset k'$ be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 45.2 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 17.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a reduced ring. Hence we deduce (2) by Algebra, Lemma 43.3.

Assume (2). Let $k \subset k'$ be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 11.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is reduced by assumption and (1) is proved. $\square$

The notion isn't interesting in characteristic zero.

Lemma 4.3. *Let X be a scheme over a perfect field k (e.g. k has characteristic zero). Let $x \in X$. If $\mathcal{O}_{X,x}$ is reduced, then X is geometrically reduced at x. If X is reduced, then X is geometrically reduced over k.*

Proof. The first statement follows from Lemma 4.2 and Algebra, Lemma 42.6 and the definition of a perfect field (Algebra, Definition 44.1). The second statement follows from the first. $\square$

Lemma 4.4. *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically reduced,*
(2) *$X_{k'}$ is reduced for every field extension $k \subset k'$,*
(3) *$X_{k'}$ is reduced for every finite purely inseparable field extension $k \subset k'$,*
(4) *$X_{k^{1/p}}$ is reduced,*
(5) *$X_{k^{perf}}$ is reduced,*
(6) *$X_{\bar{k}}$ is reduced,*
(7) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically reduced (see Algebra, Definition 42.1).*

Proof. Assume (1). Then for every field extension $k \subset k'$ and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is reduced. In other words $X_{k'}$ is reduced. Hence (2).

Assume (2). Let $U \subset X$ be an affine open. Then for every field extension $k \subset k'$ the scheme $X_{k'}$ is reduced, hence $U_{k'} = \mathrm{Spec}(\mathcal{O}(U) \otimes_k k')$ is reduced, hence $\mathcal{O}(U) \otimes_k k'$ is reduced (see Properties, Section 3). In other words $\mathcal{O}(U)$ is geometrically reduced, so (7) holds.

Assume (7). For any field extension $k \subset k'$ the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 17). Hence $X_{k'}$ is reduced. So (1) holds.

This proves that (1), (2), and (7) are equivalent. These are equivalent to (3), (4), (5), and (6) because we can apply Algebra, Lemma 43.3 to $\mathcal{O}_X(U)$ for $U \subset X$ affine open. $\square$

Lemma 4.5. *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

1. *X is geometrically reduced at x,*
2. *$\mathcal{O}_{X_{k'}, x'}$ is reduced for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ the unique point lying over x,*
3. *$\mathcal{O}_{X_{k^{1/p}}, x'}$ is reduced for $x' \in X_{k'}$ the unique point lying over x, and*
4. *$\mathcal{O}_{X_{k^{perf}}, x'}$ is reduced for $x' \in X_{k^{perf}}$ the unique point lying over x.*

Proof. Note that if $k \subset k'$ is purely inseparable, then $X_{k'} \to X$ induces a homeomorphism on underlying topological spaces, see Algebra, Lemma 45.2. Whence the uniqueness of x' lying over x mentioned in the statement. Moreover, in this case $\mathcal{O}_{X_{k'}, x'} = \mathcal{O}_{X,x} \otimes_k k'$. Hence the lemma follows from Lemma 4.2 above and Algebra, Lemma 43.3. $\square$

Lemma 4.6. *Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

1. *X is geometrically reduced at x,*
2. *$X_{k'}$ is geometrically reduced at x'.*

In particular, X is geometrically reduced over k if and only if $X_{k'}$ is geometrically reduced over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k''$ be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common field extension $k \subset k'''$ (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''}, x''} \longrightarrow \mathcal{O}_{X_{k'''}, x''''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is reduced. Thus by Algebra, Lemma 152.2 we conclude that $\mathcal{O}_{X_{k''}, x''}$ is reduced. Thus by Lemma 4.5 we conclude that X is geometrically reduced at x. $\square$

Lemma 4.7. *Let k be a field. Let X, Y be schemes over k.*

1. *If X is geometrically reduced at x, and Y reduced, then $X \times_k Y$ is reduced at every point lying over x.*

(2) *If X geometrically reduced over k and Y reduced. Then $X \times_k Y$ is reduced.*

Proof. Combine, Lemmas 4.2 and 4.4 and Algebra, Lemma 42.5. □

Lemma 4.8. *Let k be a field. Let X be a scheme over k.*

(1) *If $x' \rightsquigarrow x$ is a specialization and X is geometrically reduced at x, then X is geometrically reduced at x'.*
(2) *If $x \in X$ such that (a) $\mathcal{O}_{X,x}$ is reduced, and (b) for each specialization $x' \rightsquigarrow x$ where x' is a generic point of an irreducible component of X the scheme X is geometrically reduced at x', then X is geometrically reduced at x.*
(3) *If X is reduced and geometrically reduced at all generic points of irreducible components of X, then X is geometrically reduced.*

Proof. Part (1) follows from Lemma 4.2 and the fact that if A is a geometrically reduced k-algebra, then $S^{-1}A$ is a geometrically reduced k-algebra for any multiplicative subset S of A, see Algebra, Lemma 42.3.

Let $A = \mathcal{O}_{X,x}$. The assumptions (a) and (b) of (2) imply that A is reduced, and that $A_{\mathfrak q}$ is geometrically reduced over k for every minimal prime $\mathfrak q$ of A. Hence A is geometrically reduced over k, see Algebra, Lemma 42.7. Thus X is geometrically reduced at x, see Lemma 4.2.

Part (3) follows trivially from part (2). □

Lemma 4.9. *Let k be a field. Let X be a scheme over k. Let $x \in X$. Assume X locally Noetherian and geometrically reduced at x. Then there exists an open neighbourhood $U \subset X$ of x which is geometrically reduced over k.*

Proof. Let R be a Noetherian k-algebra. Let $\mathfrak p \subset R$ be a prime. Let $I = \mathrm{Ker}(R \to R_{\mathfrak p}$. Since $IR_{\mathfrak p} = 0$ and I is finitely generated there exists an $f \in R$, $f \notin \mathfrak p$ such that $fI = 0$. Hence $R_f \subset R_{\mathfrak p}$.

Assume X locally Noetherian and geometrically reduced at x. If we apply the above to $R = \mathcal{O}_X(U)$ for some affine open neighbourhood of x, and $\mathfrak p \subset R$ the prime corresponding to x, then we see that after shrinking U we may assume $R \subset R_{\mathfrak p}$. By Lemma 4.2 the assumption means that $R_{\mathfrak p}$ is geometrically reduced over k. By Algebra, Lemma 42.2 this implies that R is geometrically reduced over k, which in turn implies that U is geometrically reduced. □

Example 4.10. Let $k = \mathbf{F}_p(s,t)$, i.e., a purely transcendental extension of the prime field. Consider the variety $X = \mathrm{Spec}(k[x,y]/(1 + sx^p + ty^p))$. Let $k \subset k'$ be any extension such that both s and t have a pth root in k'. Then the base change $X_{k'}$ is not reduced. Namely, the ring $k'[x,y]/(1 + sx^p + ty^p)$ contains the element $1 + s^{1/p}x + t^{1/p}y$ whose pth power is zero but which is not zero (since the ideal $(1 + sx^p + ty^p)$ certainly does not contain any nonzero element of degree $< p$).

Lemma 4.11. *Let k be a field. Let $X \to \mathrm{Spec}(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite purely inseparable extension $k \subset k'$ such that $(X_{k'})_{red}$ is geometrically reduced over k'.*

Proof. To prove this lemma we may replace X by its reduction X_{red}. Hence we may assume that X is reduced and locally of finite type over k. Let $x_1, \ldots, x_n \in X$ be the generic points of the irreducible components of X. Note that for every purely inseparable algebraic extension $k \subset k'$ the morphism $(X_{k'})_{red} \to X$ is a homeomorphism, see Algebra, Lemma 45.2. Hence the points $x'_1, \ldots, x'_n$ lying over $x_1, \ldots, x_n$ are the generic points of the irreducible components of $(X_{k'})_{red}$. As X is reduced the local rings $K_i = \mathcal{O}_{X,x_i}$ are fields, see Algebra, Lemma 24.1. As X is locally of finite type over k the field extensions $k \subset K_i$ are finitely generated field extensions. Finally, the local rings $\mathcal{O}_{(X_{k'})_{red},x'_i}$ are the fields $(K_i \otimes_k k')_{red}$. By Algebra, Lemma 44.3 we can find a finite purely inseparable extension $k \subset k'$ such that $(K_i \otimes_k k')_{red}$ are separable field extensions of k'. In particular each $(K_i \otimes_k k')_{red}$ is geometrically reduced over k' by Algebra, Lemma 43.1. At this point Lemma 4.8 part (3) implies that $(X_{k'})_{red}$ is geometrically reduced. □

5. Geometrically connected schemes

If X is a connected scheme over a field, then it can happen that X becomes disconnected after extending the ground field. This does not happen for geometrically connected schemes.

Definition 5.1. Let X be a scheme over the field k. We say X is *geometrically connected* over k if the scheme $X_{k'}$ is connected for every field extension k' of k.

By convention a connected topological space is nonempty; hence a fortiori geometrically connected schemes are nonempty. Here is an example of a variety which is not geometrically connected.

Example 5.2. Let $k = \mathbf{Q}$. The scheme $X = \text{Spec}(\mathbf{Q}(i))$ is a variety over $\text{Spec}(\mathbf{Q})$. But the base change $X_{\mathbf{C}}$ is the spectrum of $\mathbf{C} \otimes_{\mathbf{Q}} \mathbf{Q}(i) \cong \mathbf{C} \times \mathbf{C}$ which is the disjoint union of two copies of $\text{Spec}(\mathbf{C})$. So in fact, this is an example of a non-geometrically connected variety.

Lemma 5.3. *Let X be a scheme over the field k. Let $k \subset k'$ be a field extension. Then X is geometrically connected over k if and only if $X_{k'}$ is geometrically connected over k'.*

Proof. If X is geometrically connected over k, then it is clear that $X_{k'}$ is geometrically connected over k'. For the converse, note that for any field extension $k \subset k''$ there exists a common field extension $k' \subset k'''$ and $k'' \subset k'''$. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over k' then X is geometrically connected over k. □

Lemma 5.4. *Let k be a field. Let X, Y be schemes over k. Assume X is geometrically connected over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between connected components.

Proof. The scheme theoretic fibres of p are connected, since they are base changes of the geometrically connected scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 18.5. By Morphisms, Lemma 24.4 the map p is open. Thus we may apply Topology, Lemma 6.5 to conclude. □

Lemma 5.5. *Let k be a field. Let A be a k-algebra. Then $X = \mathrm{Spec}(A)$ is geometrically connected over k if and only if A is geometrically connected over k (see Algebra, Definition 46.3).*

Proof. Immediate from the definitions. $\square$

Lemma 5.6. *Let $k \subset k'$ be an extension of fields. Let X be a scheme over k. Assume k separably algebraically closed. Then the morphism $X_{k'} \to X$ induces a bijection of connected components. In particular, X is geometrically connected over k if and only if X is connected.*

Proof. Since k is separably algebraically closed we see that k' is geometrically connected over k, see Algebra, Lemma 46.4. Hence $Z = \mathrm{Spec}(k')$ is geometrically connected over k by Lemma 5.5 above. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 5.4. $\square$

Lemma 5.7. *Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a separable algebraic closure of k. Then X is geometrically connected if and only if the base change $X_{\overline{k}}$ is connected.*

Proof. Assume $X_{\overline{k}}$ is connected. Let $k \subset k'$ be a field extension. There exists a field extension $\overline{k} \subset \overline{k}'$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 5.6 we see that $X_{\overline{k}'}$ is connected. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. $\square$

Lemma 5.8. *Let k be a field. Let X be a scheme over k. Let A be a k-algebra. Let $V \subset X_A$ be a quasi-compact open. Then there exists a finitely generated k-subalgebra $A' \subset A$ and a quasi-compact open $V' \subset X_{A'}$ such that $V = V'_A$.*

Proof. We remark that if X is also quasi-separated this follows from Limits, Lemma 3.8. Let $U_1, \ldots, U_n$ be finitely many affine opens of X such that $V \subset \bigcup U_{i,A}$. Say $U_i = \mathrm{Spec}(R_i)$. Since V is quasi-compact we can find finitely many $f_{ij} \in R_i \otimes_k A$, $j = 1, \ldots, n_i$ such that $V = \bigcup_i \bigcup_{j=1,\ldots,n_i} D(f_{ij})$ where $D(f_{ij}) \subset U_{i,A}$ is the corresponding standard open. (We do not claim that $V \cap U_{i,A}$ is the union of the $D(f_{ij})$, $j = 1, \ldots, n_i$.) It is clear that we can find a finitely generated k-subalgebra $A' \subset A$ such that f_{ij} is the image of some $f'_{ij} \in R_i \otimes_k A'$. Set $V' = \bigcup D(f'_{ij})$ which is a quasi-compact open of $X_{A'}$. Denote $\pi : X_A \to X_{A'}$ the canonical morphism. We have $\pi(V) \subset V'$ as $\pi(D(f_{ij})) \subset D(f'_{ij})$. If $x \in X_A$ with $\pi(x) \in V'$, then $\pi(x) \in D(f'_{ij})$ for some i, j and we see that $x \in D(f_{ij})$ as f'_{ij} maps to f_{ij}. Thus we see that $V = \pi^{-1}(V')$ as desired. $\square$

Let k be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. For example $\overline{k}$ could be the separable algebraic closure of k. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we get a corresponding automorphism $\mathrm{Spec}(\sigma) : \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$. Note that $\mathrm{Spec}(\sigma) \circ \mathrm{Spec}(\tau) = \mathrm{Spec}(\tau \circ \sigma)$. Hence we get an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$$

of the opposite group on the scheme $\mathrm{Spec}(\overline{k})$. Let X be a scheme over k. Since $X_{\overline{k}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} X$ by definition we see that the action above induces a canonical action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times X_{\overline{k}} \longrightarrow X_{\overline{k}}. \tag{5.8.1}$$

Lemma 5.9. *Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a (possibly infinite) Galois extension of k. Let $V \subset X_{\overline{k}}$ be a quasi-compact open. Then*

1. *there exists a finite subextension $k \subset k' \subset \overline{k}$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\overline{k}}$,*
2. *there exists an open subgroup $H \subset Gal(\overline{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.*

Proof. By Lemma 5.8 there exists a finite subextension $k \subset k' \subset \overline{k}$ and an open $V' \subset X_{k'}$ which pulls back to V. This proves (1). Since $\text{Gal}(\overline{k}/k')$ is open in $\text{Gal}(\overline{k}/k)$ part (2) is clear as well. $\square$

Lemma 5.10. *Let k be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. Let X be a scheme over k. Let $\overline{T} \subset X_{\overline{k}}$ have the following properties*

1. *$\overline{T}$ is a closed subset of $X_{\overline{k}}$,*
2. *for every $\sigma \in Gal(\overline{k}/k)$ we have $\sigma(\overline{T}) = \overline{T}$.*

Then there exists a closed subset $T \subset X$ whose inverse image in $X_{k'}$ is $\overline{T}$.

Proof. This lemma immediately reduces to the case where $X = \text{Spec}(A)$ is affine. In this case, let $\overline{I} \subset A \otimes_k \overline{k}$ be the radical ideal corresponding to $\overline{T}$. Assumption (2) implies that $\sigma(\overline{I}) = \overline{I}$ for all $\sigma \in \text{Gal}(\overline{k}/k)$. Pick $x \in \overline{I}$. There exists a finite Galois extension $k \subset k'$ contained in $\overline{k}$ such that $x \in A \otimes_k k'$. Set $G = \text{Gal}(k'/k)$. Set

$$P(T) = \prod\nolimits_{\sigma \in G} (T - \sigma(x)) \in (A \otimes_k k')[T]$$

It is clear that $P(T)$ is monic and is actually an element of $(A \otimes_k k')^G[T] = A[T]$ (by basic Galois theory). Moreover, if we write $P(T) = T^d + a_1 T^{d-1} + \ldots + a_0$ the we see that $a_i \in I := A \cap \overline{I}$. By Algebra, Lemma 37.5 we see that x is contained in the radical of $I(A \otimes_k \overline{k})$. Hence $\overline{I}$ is the radical of $I(A \otimes_k \overline{k})$ and setting $T = V(I)$ is a solution. $\square$

Lemma 5.11. *Let k be a field. Let X be a scheme over k. The following are equivalent*

1. *X is geometrically connected,*
2. *for every finite separable field extension $k \subset k'$ the scheme $X_{k'}$ is connected.*

Proof. It follows immediately from the definition that (1) implies (2). Assume that X is not geometrically connected. Let $k \subset \overline{k}$ be a separable algebraic closure of k. By Lemma 5.7 it follows that $X_{\overline{k}}$ is disconnected. Say $X_{\overline{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open, closed, and nonempty.

Suppose that $W \subset X$ is any quasi-compact open. Then $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are open and closed in $W_{\overline{k}}$. In particular $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are quasi-compact, and by Lemma 5.9 both $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are defined over a finite subextension and invariant under an open subgroup of $\text{Gal}(\overline{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open such that both $W_{0,\overline{k}} \cap \overline{U}$ and $W_{0,\overline{k}} \cap \overline{V}$ are nonempty. Choose a finite subextension $k \subset k' \subset \overline{k}$ and a decomposition $W_{0,k'} = U'_0 \amalg V'_0$ into open and closed subsets such that $W_{0,\overline{k}} \cap \overline{U} = (U'_0)_{\overline{k}}$ and $W_{0,\overline{k}} \cap \overline{V} = (V'_0)_{\overline{k}}$. Let $H = \text{Gal}(\overline{k}/k') \subset \text{Gal}(\overline{k}/k)$. In particular $\sigma(W_{0,\overline{k}} \cap \overline{U}) = W_{0,\overline{k}} \cap \overline{U}$ and similarly for $\overline{V}$.

Having chosen W_0, k' as above, for every quasi-compact open $W \subset X$ we set

$$U_W = \bigcap\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{U}), \quad V_W = \bigcup\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{V}).$$

Now, since $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are fixed by an open subgroup of $\mathrm{Gal}(\overline{k}/k)$ we see that the union and intersection above are finite. Hence U_W and V_W are both open and closed. Also, by construction $W_{\overline{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W_{\overline{k}} \cap U_{W'} = U_W$ and $W_{\overline{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\overline{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $W_0 \cap \overline{U}$ by construction. Finally, $U, V \subset X_{\bar{k}}$ are closed and H-invariant by construction. Hence by Lemma 5.10 we have $U = (U')_{\bar{k}}$, and $V = (V')_{\bar{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired. □

Lemma 5.12. *Let k be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. Let $f : T \to X$ be a morphism of schemes over k. Assume $T_{\overline{k}}$ connected and $X_{\overline{k}}$ disconnected. Then X is disconnected.*

Proof. Write $X_{\overline{k}} = \overline{U} \coprod \overline{V}$ with $\overline{U}$ and $\overline{V}$ open and closed. Denote $\overline{f} : T_{\overline{k}} \to X_{\overline{k}}$ the base change of f. Since $T_{\overline{k}}$ is connected we see that $T_{\overline{k}}$ is contained in either $\overline{f}^{-1}(\overline{U})$ or $\overline{f}^{-1}(\overline{V})$. Say $T_{\overline{k}} \subset \overline{f}^{-1}(\overline{U})$.

Fix a quasi-compact open $W \subset X$. There exists a finite Galois subextension $k \subset k' \subset \overline{k}$ such that $\overline{U} \cap W_{\overline{k}}$ and $\overline{V} \cap W_{\overline{k}}$ come from quasi-compact opens $U', V' \subset W_{k'}$. Then also $W_{k'} = U' \coprod V'$. Consider

$$U'' = \bigcap\nolimits_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(U'), \quad V'' = \bigcup\nolimits_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(V').$$

These are Galois invariant, open and closed, and $W_{k'} = U'' \coprod V''$. By Lemma 5.10 we get open and closed subsets $U_W, V_W \subset W$ such that $U'' = (U_W)_{k'}$, $V'' = (V_W)_{k'}$ and $W = U_W \coprod V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W \cap U_{W'} = U_W$ and $W \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X = U \coprod V$. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $f(T)$ by construction. □

Lemma 5.13. *Let k be a field. Let $T \to X$ be a morphism of schemes over k. Assume T is geometrically connected and X connected. Then X is geometrically connected.*

Proof. This is a reformulation of Lemma 5.12. □

Lemma 5.14. *Let k be a field. Let X be a scheme over k. Assume X is connected and has a point x such that k is algebraically closed in $\kappa(x)$. Then X is geometrically connected. In particular, if X has a k-rational point and X is connected, then X is geometrically connected.*

Proof. Set $T = \mathrm{Spec}(\kappa(x))$. Let $k \subset \overline{k}$ be a separable algebraic closure of k. The assumption on $k \subset \kappa(x)$ implies that $T_{\overline{k}}$ is irreducible, see Algebra, Lemma 45.10. Hence by Lemma 5.13 we see that $X_{\overline{k}}$ is connected. By Lemma 5.7 we conclude that X is geometrically connected. $\square$

Lemma 5.15. *Let $k \subset K$ be an extension of fields. Let X be a scheme over k. For every connected component T of X the inverse image $T_K \subset X_K$ is a union of connected components of X_K.*

Proof. This is a purely topological statement. Denote $p : X_K \to X$ the projection morphism. Let $T \subset X$ be a connected component of X. Let $t \in T_K = p^{-1}(T)$. Let $C \subset X_K$ be a connected component containing t. Then $p(C)$ is a connected subset of X which meets T, hence $p(C) \subset T$. Hence $C \subset T_K$. $\square$

Lemma 5.16. *Let $k \subset K$ be a finite extension of fields and let X be a scheme over k. Denote by $p : X_K \to X$ the projection morphism. For every connected component T of X_K the image $p(T)$ is a connected component of X.*

Proof. The image $p(T)$ is contained in some connected component X' of X. Consider X' as a closed subscheme of X in any way. Then T is also a connected component of $X'_K = p^{-1}(X')$ and we may therefore assume that X is connected. The morphism p is open (Morphisms, Lemma 24.4), closed (Morphisms, Lemma 44.7) and the fibers of p are finite sets (Morphisms, Lemma 44.9). Thus we may apply Topology, Lemma 6.6 to conclude. $\square$

Remark 5.17. Let $k \subset K$ be an extension of fields. Let X be a scheme over k. Denote $p : X_K \to X$ the projection morphism. Let $\overline{T} \subset X_K$ be a connected component. Is it true that $p(\overline{T})$ is a connected component of X? When $k \subset K$ is finite Lemma 5.16 tells us the answer is "yes". In general we do not know the answer. If you do, or if you have a reference, please email stacks.project@gmail.com.

Let X be a scheme. We denote $\pi_0(X)$ the set of connected components of X.

Lemma 5.18. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action*

$$Gal(\overline{k}/k)^{opp} \times \pi_0(X_{\overline{k}}) \longrightarrow \pi_0(X_{\overline{k}})$$

with the following properties:

(1) *An element $\overline{T} \in \pi_0(X_{\overline{k}})$ is fixed by the action if and only if there exists a connected component $T \subset X$, which is geometrically connected over k, such that $T_{\overline{k}} = \overline{T}$.*
(2) *For any field extension $k \subset k'$ with separable algebraic closure $\overline{k}'$ the diagram*

$$\begin{tikzcd} Gal(\overline{k}'/k') \times \pi_0(X_{\overline{k}'}) \arrow[r] \arrow[d] & \pi_0(X_{\overline{k}'}) \arrow[d] \\ Gal(\overline{k}/k) \times \pi_0(X_{\overline{k}}) \arrow[r] & \pi_0(X_{\overline{k}}) \end{tikzcd}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 5.6).

Proof. The action (5.8.1) of $\text{Gal}(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its connected components. Connected components are always closed (Topology, Lemma 6.3). Hence if $\overline{T}$ is as in (1), then by Lemma 5.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically connected over k, see Lemma 5.7. To see that T is a connected component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is a connected component of X. In this case $T'_{k'}$ strictly contains $\overline{T}$ and hence is disconnected. By Lemma 5.12 this means that T' is disconnected! Contradiction.

We omit the proof of the functoriality in (2). □

Lemma 5.19. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. Assume*

1. *X is quasi-compact, and*
2. *the connected components of $X_{\overline{k}}$ are open.*

Then

- (a) *$\pi_0(X_{\overline{k}})$ is finite, and*
- (b) *the action of $Gal(\overline{k}/k)$ on $\pi_0(X_{\overline{k}})$ is continuous.*

Moreover, assumptions (1) and (2) are satisfied when X is of finite type over k.

Proof. Since the connected components are open, cover $X_{\overline{k}}$ (Topology, Lemma 6.3) and $X_{\overline{k}}$ is quasi-compact, we conclude that there are only finitely many of them. Thus (a) holds. By Lemma 5.8 these connected components are each defined over a finite subextension of $k \subset \overline{k}$ and we get (b). If X is of finite type over k, then $X_{\overline{k}}$ is of finite type over $\overline{k}$ (Morphisms, Lemma 16.4). Hence $X_{\overline{k}}$ is a Noetherian scheme (Morphisms, Lemma 16.6) and has an underlying Noetherian topological space (Properties, Lemma 5.5). Thus $X_{\overline{k}}$ has finitely many irreducible components (Topology, Lemma 8.2) and a fortiori finitely many connected components (which are therefore open). □

6. Geometrically irreducible schemes

If X is an irreducible scheme over a field, then it can happen that X becomes reducible after extending the ground field. This does not happen for geometrically irreducible schemes.

Definition 6.1. Let X be a scheme over the field k. We say X is *geometrically irreducible* over k if the scheme $X_{k'}$ is irreducible[1] for any field extension k' of k.

Lemma 6.2. *Let X be a scheme over the field k. Let $k \subset k'$ be a field extension. Then X is geometrically irreducible over k if and only if $X_{k'}$ is geometrically irreducible over k'.*

Proof. If X is geometrically irreducible over k, then it is clear that $X_{k'}$ is geometrically irreducible over k'. For the converse, note that for any field extension $k \subset k''$ there exists a common field extension $k' \subset k'''$ and $k'' \subset k'''$. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the irreducibility of $X_{k'''}$ implies the irreducibility of $X_{k''}$. Thus if $X_{k'}$ is geometrically irreducible over k' then X is geometrically irreducible over k. □

[1] An irreducible space is nonempty.

Lemma 6.3. *Let X be a scheme over a separably closed field k. If X is irreducible, then X_K is irreducible for any field extension $k \subset K$. I.e., X is geometrically irreducible over k.*

Proof. Use Properties, Lemma 3.3 and Algebra, Lemma 45.4. □

Lemma 6.4. *Let k be a field. Let X, Y be schemes over k. Assume X is geometrically irreducible over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between irreducible components.

Proof. First, note that the scheme theoretic fibres of p are irreducible, since they are base changes of the geometrically irreducible scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 18.5. By Morphisms, Lemma 24.4 the map p is open. Thus we may apply Topology, Lemma 7.8 to conclude. □

Lemma 6.5. *Let k be a field. Let X be a scheme over k. The following are equivalent*

1. *X is geometrically irreducible over k,*
2. *for every nonempty affine open U the k-algebra $\mathcal{O}_X(U)$ is geometrically irreducible over k (see Algebra, Definition 45.6),*
3. *X is irreducible and there exists an affine open covering $X = \bigcup U_i$ such that each k-algebra $\mathcal{O}_X(U_i)$ is geometrically irreducible, and*
4. *there exists an open covering $X = \bigcup_{i \in I} X_i$ with $I \neq \emptyset$ such that X_i is geometrically irreducible for each i and such that $X_i \cap X_j \neq \emptyset$ for all $i, j \in I$.*

Moreover, if X is geometrically irreducible so is every nonempty open subscheme of X.

Proof. An affine scheme $\text{Spec}(A)$ over k is geometrically irreducible if and only if A is geometrically irreducible over k; this is immediate from the definitions. Recall that if a scheme is irreducible so is every nonempty open subscheme of X, any two nonempty open subsets have a nonempty intersection. Also, if every affine open is irreducible then the scheme is irreducible, see Properties, Lemma 3.3. Hence the final statement of the lemma is clear, as well as the implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (4). If (4) holds, then for any field extension k'/k the scheme $X_{k'}$ has a covering by irreducible opens which pairwise intersect. Hence $X_{k'}$ is irreducible. Hence (4) implies (1). □

Lemma 6.6. *Let X be a geometrically irreducible scheme over the field k. Let $\xi \in X$ be its generic point. Then $\kappa(\xi)$ is a geometrically irreducible over k.*

Proof. Combining Lemma 6.5 and Algebra, Lemma 45.8 we see that $\mathcal{O}_{X,\xi}$ is geometrically irreducible over k. Since $\mathcal{O}_{X,\xi} \to \kappa(\xi)$ is a surjection with locally nilpotent kernel (see Algebra, Lemma 24.1) it follows that $\kappa(\xi)$ is geometrically irreducible, see Algebra, Lemma 45.2. □

Lemma 6.7. *Let $k \subset k'$ be an extension of fields. Let X be a scheme over k. Set $X' = X_{k'}$. Assume k separably algebraically closed. Then the morphism $X' \to X$ induces a bijection of irreducible components.*

Proof. Since k is separably algebraically closed we see that k' is geometrically irreducible over k, see Algebra, Lemma 45.7. Hence $Z = \mathrm{Spec}(k')$ is geometrically irreducible over k. by Lemma 6.5 above. Since $X' = Z \times_k X$ the result is a special case of Lemma 6.4. $\square$

Lemma 6.8. *Let k be a field. Let X be a scheme over k. The following are equivalent:*

1. *X is geometrically irreducible over k,*
2. *for every finite separable field extension $k \subset k'$ the scheme $X_{k'}$ is irreducible, and*
3. *$X_{\overline{k}}$ is irreducible, where $k \subset \overline{k}$ is a separable algebraic closure of k.*

Proof. Assume $X_{\overline{k}}$ is irreducible, i.e., assume (3). Let $k \subset k'$ be a field extension. There exists a field extension $\overline{k} \subset \overline{k}'$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 6.7 we see that $X_{\overline{k}'}$ is irreducible. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is irreducible. Hence (1) holds.

Let $k \subset \overline{k}$ be a separable algebraic closure of k. Assume not (3), i.e., assume $X_{\overline{k}}$ is reducible. Our goal is to show that also $X_{k'}$ is reducible for some finite subextension $k \subset k' \subset \overline{k}$. Let $X = \bigcup_{i \in I} U_i$ be an affine open covering with U_i not empty. If for some i the scheme U_i is reducible, or if for some pair $i \neq j$ the intersection $U_i \cap U_j$ is empty, then X is reducible (Properties, Lemma 3.3) and we are done. In particular we may assume that $U_{i,\overline{k}} \cap U_{j,\overline{k}}$ for all $i, j \in I$ is nonempty and we conclude that $U_{i,\overline{k}}$ has to be reducible for some i. According to Algebra, Lemma 45.5 this means that $U_{i,k'}$ is reducible for some finite separable field extension $k \subset k'$. Hence also $X_{k'}$ is reducible. Thus we see that (2) implies (3).

The implication (1) $\Rightarrow$ (2) is immediate. This proves the lemma. $\square$

Lemma 6.9. *Let $k \subset K$ be an extension of fields. Let X be a scheme over k. For every irreducible component T of X the inverse image $T_K \subset X_K$ is a union of irreducible components of X_K.*

Proof. Let $T \subset X$ be an irreducible component of X. The morphism $T_K \to T$ is flat, so generalizations lift along $T_K \to T$. Hence every $\xi \in T_K$ which is a generic point of an irreducible component of T_K maps to the generic point η of T. If $\xi' \rightsquigarrow \xi$ is a specialization in X_K then ξ' maps to η since there are no points specializing to η in X. Hence $\xi' \in T_K$ and we conclude that $\xi = \xi'$. In other words ξ is the generic point of an irreducible component of X_K. This means that the irreducible components of T_K are all irreducible components of X_K. $\square$

For a scheme X we denote $\mathrm{IrredComp}(X)$ the set of irreducible components of X.

Lemma 6.10. *Let $k \subset K$ be an extension of fields. Let X be a scheme over k. For every irreducible component $\overline{T} \subset X_K$ the image of $\overline{T}$ in X is an irreducible component in X. This defines a canonical map*

$$IrredComp(X_K) \longrightarrow IrredComp(X)$$

which is surjective.

Proof. Consider the diagram

$$\begin{tikzcd} X_K & X_{\overline{K}} \arrow[l] \arrow[d] \\ X \arrow[u, leftarrow] & X_{\overline{k}} \arrow[l] \end{tikzcd}$$

where $\overline{K}$ is the separable algebraic closure of K, and where $\overline{k}$ is the separable algebraic closure of k. By Lemma 6.7 the morphism $X_{\overline{K}} \to X_{\overline{k}}$ induces a bijection between irreducible components. Hence it suffices to show the lemma for the morphisms $X_{\overline{k}} \to X$ and $X_{\overline{K}} \to X_K$. In other words we may assume that $K = \overline{k}$.

The morphism $p : X_{\overline{k}} \to X$ is integral, flat and surjective. Flatness implies that generalizations lift along p, see Morphisms, Lemma 26.8. Hence generic points of irreducible components of $X_{\overline{k}}$ map to generic points of irreducible components of X. Integrality implies that p is universally closed, see Morphisms, Lemma 44.7. Hence we conclude that the image $p(\overline{T})$ of an irreducible component is a closed irreducible subset which contains a generic point of an irreducible component of X, hence $p(\overline{T})$ is an irreducible component of X. This proves the first assertion. If $T \subset X$ is an irreducible component, then $p^{-1}(T) = T_K$ is a nonempty union of irreducible components, see Lemma 6.9. Each of these necessarily maps onto T by the first part. Hence the map is surjective. $\square$

Lemma 6.11. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action*

$$Gal(\overline{k}/k)^{opp} \times IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X_{\overline{k}})$$

with the following properties:

(1) *An element $\overline{T} \in IrredComp(X_{\overline{k}})$ is fixed by the action if and only if there exists an irreducible component $T \subset X$, which is geometrically irreducible over k, such that $T_{\overline{k}} = \overline{T}$.*

(2) *For any field extension $k \subset k'$ with separable algebraic closure $\overline{k}'$ the diagram*

$$\begin{tikzcd} Gal(\overline{k}'/k') \times IrredComp(X_{\overline{k}'}) \arrow[r] \arrow[d] & IrredComp(X_{\overline{k}'}) \arrow[d] \\ Gal(\overline{k}/k) \times IrredComp(X_{\overline{k}}) \arrow[r] & IrredComp(X_{\overline{k}}) \end{tikzcd}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 6.7).

Proof. The action (5.8.1) of $\mathrm{Gal}(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its irreducible components. Irreducible components are always closed (Topology, Lemma 6.3). Hence if $\overline{T}$ is as in (1), then by Lemma 5.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically irreducible over k, see Lemma 6.8. To see that T is an irreducible component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is an irreducible component of X. Let $\overline{\eta}$ be the generic point of $\overline{T}$. It maps to the generic point η of T. Then the generic point $\xi \in T'$ specializes to η. As $X_{\overline{k}} \to X$ is flat there exists a point $\overline{\xi} \in X_{\overline{k}}$ which maps to ξ and specializes to

$\overline{\eta}$. It follows that the closure of the singleton $\{\overline{\xi}\}$ is an irreducible closed subset of $X_{\overline{\xi}}$ which strictly contains $\overline{T}$. This is the desired contradiction.

We omit the proof of the functoriality in (2). $\square$

Lemma 6.12. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. The fibres of the map*

$$IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X)$$

of Lemma 6.10 are exactly the orbits of $Gal(\overline{k}/k)$ under the action of Lemma 6.11.

Proof. Let $T \subset X$ be an irreducible component of X. Let $\eta \in T$ be its generic point. By Lemmas 6.9 and 6.10 the generic points of irreducible components of $\overline{T}$ which map into T map to η. By Algebra, Lemma 45.12 the Galois group acts transitively on all of the points of $X_{\overline{k}}$ mapping to η. Hence the lemma follows. $\square$

Lemma 6.13. *Let k be a field. Assume $X \to \mathrm{Spec}(k)$ locally of finite type. In this case*

(1) *the action*
$$Gal(\overline{k}/k)^{opp} \times IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X_{\overline{k}})$$
is continuous if we give $IrredComp(X_{\overline{k}})$ the discrete topology,
(2) *every irreducible component of $X_{\overline{k}}$ can be defined over a finite extension of k, and*
(3) *given any irreducible component $T \subset X$ the scheme $T_{\overline{k}}$ is a finite union of irreducible components of $X_{\overline{k}}$ which are all in the same $Gal(\overline{k}/k)$-orbit.*

Proof. Let $\overline{T}$ be an irreducible component of $X_{\overline{k}}$. We may choose an affine open $U \subset X$ such that $\overline{T} \cap U_{\overline{k}}$ is not empty. Write $U = \mathrm{Spec}(A)$, so A is a finite type k-algebra, see Morphisms, Lemma 16.2. Hence $A_{\overline{k}}$ is a finite type $\overline{k}$-algebra, and in particular Noetherian. Let $\mathfrak{p} = (f_1, \ldots, f_n)$ be the prime ideal corresponding to $\overline{T} \cap U_{\overline{k}}$. Since $A_{\overline{k}} = A \otimes_k \overline{k}$ we see that there exists a finite subextension $k \subset k' \subset \overline{k}$ such that each $f_i \in A_{k'}$. It is clear that $\mathrm{Gal}(\overline{k}/k')$ fixes $\overline{T}$, which proves (1).

Part (2) follows by applying Lemma 6.11 (1) to the situation over k' which implies the irreducible component $\overline{T}$ is of the form $T'_{\overline{k}}$ for some irreducible $T' \subset X_{k'}$.

To prove (3), let $T \subset X$ be an irreducible component. Choose an irreducible component $\overline{T} \subset X_{\overline{k}}$ which maps to T, see Lemma 6.10. By the above the orbit of $\overline{T}$ is finite, say it is $\overline{T}_1, \ldots, \overline{T}_n$. Then $\overline{T}_1 \cup \ldots \cup \overline{T}_n$ is a $\mathrm{Gal}(\overline{k}/k)$-invariant closed subset of $X_{\overline{k}}$ hence of the form $W_{\overline{k}}$ for some $W \subset X$ closed by Lemma 5.10. Clearly $W = T$ and we win. $\square$

Lemma 6.14. *Let k be a field. Let $X \to \mathrm{Spec}(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite separable extension $k \subset k'$ such that every irreducible component of $X_{k'}$ is geometrically irreducible over k'.*

Proof. Let $\overline{k}$ be a separable algebraic closure of k. The assumption that X has finitely many irreducible components combined with Lemma 6.13 (3) shows that $X_{\overline{k}}$ has finitely many irreducible components $\overline{T}_1, \ldots, \overline{T}_n$. By Lemma 6.13 (2) there exists a finite extension $k \subset k' \subset \overline{k}$ and irreducible components $T_i \subset X_{k'}$ such that $\overline{T}_i = T_{i,\overline{k}}$ and we win. $\square$

Lemma 6.15. *Let X be a scheme over the field k. Assume X has finitely many irreducible components which are all geometrically irreducible. Then X has finitely many connected components each of which is geometrically connected.*

Proof. This is clear because a connected component is a union of irreducible components. Details omitted. $\square$

7. Geometrically integral schemes

If X is an irreducible scheme over a field, then it can happen that X becomes reducible after extending the ground field. This does not happen for geometrically irreducible schemes.

Definition 7.1. Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is *geometrically pointwise integral at x* if for every field extension $k \subset k'$ and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is integral.
(2) We say X is *geometrically pointwise integral* if X is geometrically pointwise integral at every point.
(3) We say X is *geometrically integral* over k if the scheme $X_{k'}$ is integral for every field extension k' of k.

The distinction between notions (2) and (3) is necessary. For example if $k = \mathbf{R}$ and $X = \mathrm{Spec}(\mathbf{C}[x])$, then X is geometrically pointwise integral over $\mathbf{R}$ but of course not geometrically integral.

Lemma 7.2. *Let k be a field. Let X be a scheme over k. Then X is geometrically integral over k if and only if X is both geometrically reduced and geometrically irreducible over k.*

Proof. See Properties, Lemma 3.4. $\square$

8. Geometrically normal schemes

In Properties, Definition 7.1 we have defined the notion of a normal scheme. This notion is defined even for non-Noetherian schemes. Hence, contrary to our discussion of "geometrically regular" schemes we consider all field extensions of the ground field.

Definition 8.1. Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is *geometrically normal at x* if for every field extension $k \subset k'$ and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is normal.
(2) We say X is *geometrically normal* over k if X is geometrically normal at every $x \in X$.

Lemma 8.2. *Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically normal at x,*
(2) *for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over over x the local ring $\mathcal{O}_{X_{k'},x'}$ is normal, and*
(3) *the ring $\mathcal{O}_{X,x}$ is geometrically normal over k (see Algebra, Definition 153.2).*

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k'$ be a finite purely inseparable field extension (for example $k = k'$). Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 45.2 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 17.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a normal ring. Hence we deduce (3) by Algebra, Lemma 153.1.

Assume (3). Let $k \subset k'$ be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 11.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is normal by assumption and (1) is proved. $\square$

Lemma 8.3. *Let k be a field. Let X be a scheme over k. The following are equivalent*

1. *X is geometrically normal,*
2. *$X_{k'}$ is a normal scheme for every field extension $k \subset k'$,*
3. *$X_{k'}$ is a normal scheme for every finitely generated field extension $k \subset k'$,*
4. *$X_{k'}$ is a normal scheme for every finite purely inseparable field extension $k \subset k'$, and*
5. *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically normal (see Algebra, Definition 153.2).*

Proof. Assume (1). Then for every field extension $k \subset k'$ and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is normal. By definition this means that $X_{k'}$ is normal. Hence (2).

It is clear that (2) implies (3) implies (4).

Assume (4) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a normal scheme for any finite purely inseparable extension $k \subset k'$ (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a normal ring for all finite purely inseparable extensions $k \subset k'$. Hence $\mathcal{O}(U)$ is a geometrically normal k-algebra by definition.

Assume (5). For any field extension $k \subset k'$ the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 17). Hence $X_{k'}$ is normal. So (1) holds. $\square$

Lemma 8.4. *Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

1. *X is geometrically normal at x,*
2. *$X_{k'}$ is geometrically normal at x'.*

In particular, X is geometrically normal over k if and only if $X_{k'}$ is geometrically normal over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k''$ be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common field extension $k \subset k'''$ (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x''''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is normal. Thus by Algebra, Lemma 152.3 we conclude that $\mathcal{O}_{X_{k''},x''}$ is normal. By Lemma 8.2 we see that X is geometrically normal at x. □

Lemma 8.5. *Let k be a field. Let X be a geometrically normal scheme over k and let Y be a normal scheme over k. Then $X \times_k Y$ is a normal scheme.*

Proof. This reduces to Algebra, Lemma 153.4 by Lemma 8.3. □

9. Change of fields and locally Noetherian schemes

Let X a locally Noetherian scheme over a field k. It is not always that case that $X_{k'}$ is locally Noetherian too. For example if $X = \mathrm{Spec}(\overline{\mathbf{Q}})$ and $k = \mathbf{Q}$, then $X_{\overline{\mathbf{Q}}}$ is the spectrum of $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ which is not Noetherian. (Hint: It has too many idempotents). But if we only base change using finitely generated field extensions then the Noetherian property is preserved. (Or if X is locally of finite type over k, since this property is preserved under base change.)

Lemma 9.1. *Let k be a field. Let X be a scheme over k. Let $k \subset k'$ be a finitely generated field extension. Then X is locally Noetherian if and only if $X_{k'}$ is locally Noetherian.*

Proof. Using Properties, Lemma 5.2 we reduce to the case where X is affine, say $X = \mathrm{Spec}(A)$. In this case we have to prove that A is Noetherian if and only if $A_{k'}$ is Noetherian. Since $A \to A_{k'} = k' \otimes_k A$ is faithfully flat, we see that if $A_{k'}$ is Noetherian, then so is A, by Algebra, Lemma 152.1. Conversely, if A is Noetherian then $A_{k'}$ is Noetherian by Algebra, Lemma 30.7. □

10. Geometrically regular schemes

A geometrically regular scheme over a field k is a locally Noetherian scheme over k which remains regular upon suitable changes of base field. A finite type scheme over k is geometrically regular if and only if it is smooth over k (see Lemma 10.6). The notion of geometric regularity is most interesting in situations where smoothness cannot be used such as formal fibres (insert future reference here).

In the following definition we restrict ourselves to locally Noetherian schemes, since the property of being a regular local ring is only defined for Noetherian local rings. By Lemma 8.3 above, if we restrict ourselves to finitely generated field extensions then this property is preserved under change of base field. This comment will be used without further reference in this section. In particular the following definition makes sense.

Definition 10.1. Let k be a field. Let X be a locally Noetherian scheme over k.

(1) Let $x \in X$. We say X is *geometrically regular at x* over k if for every finitely generated field extension $k \subset k'$ and any $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular.
(2) We say X is *geometrically regular over k* if X is geometrically regular at all of its points.

A similar definition works to define geometrically Cohen-Macaulay, (R_k), and (S_k) schemes over a field. We will add a section for these separately as needed.

Lemma 10.2. *Let k be a field. Let X be a locally Noetherian scheme over k. Let $x \in X$. The following are equivalent*

1. *X is geometrically regular at x,*
2. *for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular, and*
3. *the ring $\mathcal{O}_{X,x}$ is geometrically regular over k (see Algebra, Definition 154.2).*

Proof. It is clear that (1) implies (2). Assume (2). This in particular implies that $\mathcal{O}_{X,x}$ is a regular local ring. Let $k \subset k'$ be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 45.2 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 17.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a regular ring. Hence we deduce (3) from the definition of a geometrically regular ring.

Assume (3). Let $k \subset k'$ be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 11.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is regular by assumption and (1) is proved. $\square$

Lemma 10.3. *Let k be a field. Let X be a locally Noetherian scheme over k. The following are equivalent*

1. *X is geometrically regular,*
2. *$X_{k'}$ is a regular scheme for every finitely generated field extension $k \subset k'$,*
3. *$X_{k'}$ is a regular scheme for every finite purely inseparable field extension $k \subset k'$,*
4. *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically regular (see Algebra, Definition 154.2), and*
5. *there exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is geometrically regular over k.*

Proof. Assume (1). Then for every finitely generated field extension $k \subset k'$ and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is regular. By Properties, Lemma 9.2 this means that $X_{k'}$ is regular. Hence (2).

It is clear that (2) implies (3).

Assume (3) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a regular scheme for any finite purely inseparable extension $k \subset k'$ (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a regular ring for all finite purely inseparable extensions $k \subset k'$. Hence $\mathcal{O}(U)$ is a geometrically regular k-algebra and we see that (4) holds.

It is clear that (4) implies (5). Let $X = \bigcup U_i$ be an affine open covering as in (5). For any field extension $k \subset k'$ the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U_i) \otimes_k k'$ (see Schemes, Section 17). Hence $X_{k'}$ is regular. So (1) holds. $\square$

Lemma 10.4. *Let k be a field. Let X be a scheme over k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

1. *X is geometrically regular at x,*
2. *$X_{k'}$ is geometrically regular at x'.*

In particular, X is geometrically regular over k if and only if $X_{k'}$ is geometrically regular over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k''$ be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common, finitely generated, field extension $k \subset k'''$ (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x''''}.$$

This is a flat local ring homomorphism of Noetherian local rings and hence faithfully flat. By (2) we see that the local ring on the right is regular. Thus by Algebra, Lemma 106.9 we conclude that $\mathcal{O}_{X_{k''},x''}$ is regular. By Lemma 10.2 we see that X is geometrically regular at x. □

The following lemma is a geometric variant of Algebra, Lemma 154.3.

Lemma 10.5. *Let k be a field. Let $f : X \to Y$ be a morphism of locally Noetherian schemes over k. Let $x \in X$ be a point and set $y = f(x)$. If X is geometrically regular at x and f is flat at x then Y is geometrically regular at y. In particular, if X is geometrically regular over k and f is flat and surjective, then Y is geometrically regular over k.*

Proof. Let k' be finite purely inseparable extension of k. Let $f' : X_{k'} \to Y_{k'}$ be the base change of f. Let $x' \in X_{k'}$ be the unique point lying over x. If we show that $Y_{k'}$ is regular at $y' = f'(x')$, then Y is geometrically regular over k at y', see Lemma 10.3. By Morphisms, Lemma 26.6 the morphism $X_{k'} \to Y_{k'}$ is flat at x'. Hence the ring map

$$\mathcal{O}_{Y_{k'},y'} \longrightarrow \mathcal{O}_{X_{k'},x'}$$

is a flat local homomorphism of local Noetherian rings with right hand side regular by assumption. Hence the left hand side is a regular local ring by Algebra, Lemma 106.9. □

Lemma 10.6. *Let k be a field. Let X be a scheme of finite type over k. Let $x \in X$. Then X is geometrically regular at x if and only if $X \to \mathrm{Spec}(k)$ is smooth at x (Morphisms, Definition 35.1).*

Proof. The question is local around x, hence we may assume that $X = \mathrm{Spec}(A)$ for some finite type k-algebra. Let x correspond to the prime $\mathfrak{p}$.

If A is smooth over k at $\mathfrak{p}$, then we may localize A and assume that A is smooth over k. In this case $k' \otimes_k A$ is smooth over k' for all extension fields k'/k, and each of these Noetherian rings is regular by Algebra, Lemma 135.3.

Assume X is geometrically regular at x. Consider the residue field $K := \kappa(x) = \kappa(\mathfrak{p})$ of x. It is a finitely generated extension of k. By Algebra, Lemma 44.3 there exists a finite purely inseparable extension $k \subset k'$ such that the compositum $k'K$ is a separable field extension of k'. Let $\mathfrak{p}' \subset A' = k' \otimes_k A$ be a prime ideal lying over $\mathfrak{p}$. It is the unique prime lying over $\mathfrak{p}$, see Algebra, Lemma 45.2. Hence the residue field $K' := \kappa(\mathfrak{p}')$ is the compositum $k'K$. By assumption the local ring $(A')_{\mathfrak{p}'}$ is regular. Hence by Algebra, Lemma 135.5 we see that $k' \to A'$ is smooth at $\mathfrak{p}'$. This in turn implies that $k \to A$ is smooth at $\mathfrak{p}$ by Algebra, Lemma 132.18. The lemma is proved. □

Example 10.7. Let $k = \mathbf{F}_p(t)$. It is quite easy to give an example of a regular variety V over k which is not geometrically reduced. For example we can take $\text{Spec}(k[x]/(x^p - t))$. In fact, there exists an example of a regular variety V which is geometrically reduced, but not even geometrically normal. Namely, take for $p > 2$ the scheme $V = \text{Spec}(k[x,y]/(y^2 - x^p + t))$. This is a variety as the polynomial $y^2 - x^p + t \in k[x,y]$ is irreducible. The morphism $V \to \text{Spec}(k)$ is smooth at all points except at the point $v_0 \in V$ corresponding to the maximal ideal $(y, x^p - t)$ (because $2y$ is invertible). In particular we see that V is (geometrically) regular at all points, except possibly v_0. The local ring

$$\mathcal{O}_{V,v_0} = \left(k[x,y]/(y^2 - x^p + t)\right)_{(y, x^p - t)}$$

is a domain of dimension 1. Its maximal ideal is generated by 1 element, namely y. Hence it is a discrete valuation ring and regular. Let $k' = k[t^{1/p}]$. Denote $t' = t^{1/p} \in k'$, $V' = V_{k'}$, $v'_0 \in V'$ the unique point lying over v_0. Over k' we can write $x^p - t = (x - t')^p$, but the polynomial $y^2 - (x - t')^p$ is still irreducible and V' is still a variety. But the element

$$\frac{y}{x - t'} \in f.f.(\mathcal{O}_{V',v'_0})$$

is integral over $\mathcal{O}_{V',v'_0}$ (just compute its square) and not contained in it, so V' is not normal at v'_0. This concludes the example.

11. Change of fields and the Cohen-Macaulay property

The following lemma says that it does not make sense to define geometrically Cohen-Macaulay schemes, since these would be the same as Cohen-Macaulay schemes.

Lemma 11.1. *Let X be a locally Noetherian scheme over the field k. Let $k \subset k'$ be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. Then we have*

$$\mathcal{O}_{X,x} \text{ is Cohen-Macaulay} \Leftrightarrow \mathcal{O}_{X_{k'},x'} \text{ is Cohen-Macaulay}$$

If X is locally of finite type over k, the same holds for any field extension $k \subset k'$.

Proof. The first case of the lemma follows from Algebra, Lemma 155.2. The second case of the lemma is equivalent to Algebra, Lemma 126.6. $\square$

12. Change of fields and the Jacobson property

A scheme locally of finite type over a field has plenty of closed points, namely it is Jacobson. Moreover, the residue fields are finite extensions of the ground field.

Lemma 12.1. *Let X be a scheme which is locally of finite type over k. Then*

1. *for any closed point $x \in X$ the extension $k \subset \kappa(x)$ is algebraic, and*
2. *X is a Jacobson scheme (Properties, Definition 6.1).*

Proof. A scheme is Jacobson if and only if it has an affine open covering by Jacobson schemes, see Properties, Lemma 6.3. The property on residue fields at closed points is also local on X. Hence we may assume that X is affine. In this case the result is a consequence of the Hilbert Nullstellensatz, see Algebra, Theorem 33.1. It also follows from a combination of Morphisms, Lemmas 17.8, 17.9, and 17.10. $\square$

It turns out that if X is not locally of finite type, then we can achieve the same result after making a suitably large base field extension.

Lemma 12.2. *Let X be a scheme over a field k. For any field extension $k \subset K$ whose cardinality is large enough we have*

1. *for any closed point $x \in X_K$ the extension $K \subset \kappa(x)$ is algebraic, and*
2. *X_K is a Jacobson scheme (Properties, Definition 6.1).*

Proof. Choose an affine open covering $X = \bigcup U_i$. By Algebra, Lemma 34.12 and Properties, Lemma 6.2 there exist cardinals κ_i such that $U_{i,K}$ has the desired properties over K if $\#(K) \geq \kappa_i$. Set $\kappa = \max\{\kappa_i\}$. Then if the cardinality of K is larger than κ we see that each $U_{i,K}$ satisfies the conclusions of the lemma. Hence X_K is Jacobson by Properties, Lemma 6.3. The statement on residue fields at closed points of X_K follows from the corresponding statements for residue fields of closed points of the $U_{i,K}$. □

13. Algebraic schemes

The following definition is taken from [DG67, I Definition 6.4.1].

Definition 13.1. Let k be a field. An *algebraic k-scheme* is a scheme X over k such that the structure morphism $X \to \mathrm{Spec}(k)$ is of finite type. A *locally algebraic k-scheme* is a scheme X over k such that the structure morphism $X \to \mathrm{Spec}(k)$ is locally of finite type.

Note that every (locally) algebraic k-scheme is (locally) Noetherian, see Morphisms, Lemma 16.6. The category of algebraic k-schemes has all products and fibre products (unlike the category of varieties over k). Similarly for the category of locally algebraic k-schemes.

Lemma 13.2. *Let k be a field. Let X be a locally algebraic k-scheme of dimension 0. Then X is a disjoint union of spectra of local Artinian k-algebras A with $\dim_k(A) < \infty$. If X is an algebraic k-scheme of dimension 0, then in addition X is affine and the morphism $X \to \mathrm{Spec}(k)$ is finite.*

Proof. Let X be a locally algebraic k-scheme of dimension 0. Let $U = \mathrm{Spec}(A) \subset X$ be an affine open subscheme. Since $\dim(X) = 0$ we see that $\dim(A) = 0$. By Noether normalization, see Algebra, Lemma 111.4 we see that there exists a finite injection $k \to A$, i.e., $\dim_k(A) < \infty$. Hence A is Artinian, see Algebra, Lemma 51.2. This implies that $A = A_1 \times \ldots \times A_r$ is a product of finitely many Artinian local rings, see Algebra, Lemma 51.6. Of course $\dim_k(A_i) < \infty$ for each i as the sum of these dimensions equals $\dim_k(A)$.

The arguments above show that X has an open covering whose members are finite discrete topological spaces. Hence X is a discrete topological space. It follows that X is isomorphic to the disjoint union of its connected components each of which is a singleton. Since a singleton scheme is affine we conclude (by the results of the paragraph above) that each of these singletons is the spectrum of a local Artinian k-algebra A with $\dim_k(A) < \infty$.

Finally, if X is an algebraic k-scheme of dimension 0, then X is quasi-compact hence is a finite disjoint union $X = \mathrm{Spec}(A_1) \amalg \ldots \amalg \mathrm{Spec}(A_r)$ hence affine (see Schemes, Lemma 6.8) and we have seen the finiteness of $X \to \mathrm{Spec}(k)$ in the first paragraph of the proof. □

Lemma 13.3. *Let k be a field. Let X be a locally algebraic k-scheme.*

(1) *The dimension of k is the supremum of the numbers $trdeg_k(\kappa(\eta)$ where η runs over the generic points of the irreducible components of X.*
(2) *If X is irreducible, then all maximal chains of irreducible closed subsets have length equal to the dimension of X.*

Proof. It is clear that the dimension of X is the supremum of the dimensions of all affine opens. Simiarly, any maximal chain in X gives rise to a maximal chain in an affine open. Hence it suffices to prove the lemma for an affine open. Part (2) follows from Algebra, Lemma 110.4. Part (1) follows from Algebra, Lemma 112.3. $\square$

14. Closures of products

Some results on the relation between closure and products.

Lemma 14.1. *Let k be a field. Let X, Y be schemes over k, and let $A \subset X$, $B \subset Y$ be subsets. Set*

$$AB = \{z \in X \times_k Y \mid pr_X(\gamma) \in A,\ pr_Y(\gamma) \in B\} \subset X \times_k Y$$

Then set theoretically we have

$$\overline{A} \times_k \overline{B} = \overline{AB}$$

Proof. The inclusion $\overline{AB} \subset \overline{A} \times_k \overline{B}$ is immediate. We may replace X and Y by the reduced closed subschemes $\overline{A}$ and $\overline{B}$. Let $W \subset X \times_k Y$ be a nonempty open subset. By Morphisms, Lemma 24.4 the subset $U = \text{pr}_X(W)$ is nonempty open in X. Hence $A \cap U$ is nonempty. Pick $a \in A \cap U$. Denote $Y_{\kappa(a)} = \{a\} \times_k Y$ the fibre of $\text{pr}_X : X \times_k Y \to X$ over a. By Morphisms, Lemma 24.4 again the morphism $Y_a \to Y$ is open as $\text{Spec}(\kappa(a)) \to \text{Spec}(k)$ is universally open. Hence the nonempty open subset $W_a = W \times_{X \times_k Y} Y_a$ maps to a nonempty open subset of Y. We conclude there exists a $b \in B$ in the image. Hence $AB \cap W \neq \emptyset$ as desired. $\square$

Lemma 14.2. *Let k be a field. Let $f : A \to X$, $g : B \to Y$ be morphisms of schemes over k. Then set theoretically we have*

$$\overline{f(A)} \times_k \overline{g(B)} = \overline{(f \times g)(A \times_k B)}$$

Proof. This follows from Lemma 14.1 as the image of $f \times g$ is $f(A)g(B)$ in the notation of that lemma. $\square$

Lemma 14.3. *Let k be a field. Let $f : A \to X$, $g : B \to Y$ be quasi-compact morphisms of schemes over k. Let $Z \subset X$ be the scheme theoretic image of f, see Morphisms, Definition 6.2. Similarly, let $Z' \subset Y$ be the scheme theoretic image of g. Then $Z \times_k Z'$ is the scheme theoretic image of $f \times g$.*

Proof. Recall that Z is the smallest closed subscheme of X through which f factors. Similarly for Z'. Let $W \subset X \times_k Y$ be the scheme theoretic image of $f \times g$. As $f \times g$ factors through $Z \times_k Z'$ we see that $W \subset Z \times_k Z'$.

To prove the other inclusion let $U \subset X$ and $V \subset Y$ be affine opens. By Morphisms, Lemma 6.3 the scheme $Z \cap U$ is the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$, and similarly for $Z' \cap V$ and $W \cap U \times_k V$. Hence we may assume X and Y affine. As f and g are quasi-compact this implies that $A = \bigcup U_i$ is a finite union of affines and $B = \bigcup V_j$ is a finite union of affines. Then we may replace A by $\coprod U_i$ and B by $\coprod V_j$, i.e., we may assume that A and B are affine as well. In this case Z

is cut out by $\mathrm{Ker}(\Gamma(X, \mathcal{O}_X) \to \Gamma(A, \mathcal{O}_A))$ and similarly for Z' and W. Hence the result follows from the equality

$$\Gamma(A \times_k B, \mathcal{O}_{A \times_k B}) = \Gamma(A, \mathcal{O}_A) \otimes_k \Gamma(B, \mathcal{O}_B)$$

which holds as A and B are affine. Details omitted. □

15. Schemes smooth over fields

Here are two lemmas characterizing smooth schemes over fields.

Lemma 15.1. *Let k be a field. Let X be a scheme over k. Assume*

1. *X is locally of finite type over k,*
2. *$\Omega_{X/k}$ is locally free, and*
3. *k has characteristic zero.*

Then the structure morphism $X \to \mathrm{Spec}(k)$ is smooth.

Proof. This follows from Algebra, Lemma 135.7. □

In positive characteristic there exist nonreduced schemes of finite type whose sheaf of differentials is free, for example $\mathrm{Spec}(\mathbf{F}_p[t]/(t^p))$ over $\mathrm{Spec}(\mathbf{F}_p)$. If the ground field k is nonperfect of characteristic p, there exist reduced schemes X/k with free $\Omega_{X/k}$ which are nonsmooth, for example $\mathrm{Spec}(k[t]/(t^p - a)$ where $a \in k$ is not a pth power.

Lemma 15.2. *Let k be a field. Let X be a scheme over k. Assume*

1. *X is locally of finite type over k,*
2. *$\Omega_{X/k}$ is locally free,*
3. *X is reduced, and*
4. *k is perfect.*

Then the structure morphism $X \to \mathrm{Spec}(k)$ is smooth.

Proof. Let $x \in X$ be a point. As X is locally Noetherian (see Morphisms, Lemma 16.6) there are finitely many irreducible components $X_1, \ldots, X_n$ passing through x (see Properties, Lemma 5.5 and Topology, Lemma 8.2). Let $\eta_i \in X_i$ be the generic point. As X is reduced we have $\mathcal{O}_{X,\eta_i} = \kappa(\eta_i)$, see Algebra, Lemma 24.1. Moreover, $\kappa(\eta_i)$ is a finitely generated field extension of the perfect field k hence separably generated over k (see Algebra, Section 41). It follows that $\Omega_{X/k,\eta_i} = \Omega_{\kappa(\eta_i)/k}$ is free of rank the transcendence degree of $\kappa(\eta_i)$ over k. By Morphisms, Lemma 29.1 we conclude that $\dim_{\eta_i}(X_i) = \mathrm{rank}_{\eta_i}(\Omega_{X/k})$. Since $x \in X_1 \cap \ldots \cap X_n$ we see that

$$\mathrm{rank}_x(\Omega_{X/k}) = \mathrm{rank}_{\eta_i}(\Omega_{X/k}) = \dim(X_i).$$

Therefore $\dim_x(X) = \mathrm{rank}_x(\Omega_{X/k})$, see Algebra, Lemma 110.5. It follows that $X \to \mathrm{Spec}(k)$ is smooth at x for example by Algebra, Lemma 135.3. □

Lemma 15.3. *Let $X \to \mathrm{Spec}(k)$ be a smooth morphism where k is a field. Then X is a regular scheme.*

Proof. (See also Lemma 10.6.) By Algebra, Lemma 135.3 every local ring $\mathcal{O}_{X,x}$ is regular. And because X is locally of finite type over k it is locally Noetherian. Hence X is regular by Properties, Lemma 9.2. □

Lemma 15.4. *Let $X \to \mathrm{Spec}(k)$ be a smooth morphism where k is a field. Then X is geometrically regular, geometrically normal, and geometrically reduced over k.*

Proof. (See also Lemma 10.6.) Let k' be a finite purely inseparable extension of k. It suffices to prove that $X_{k'}$ is regular, normal, reduced, see Lemmas 10.3, 8.3, and 4.5. By Morphisms, Lemma 35.5 the morphism $X_{k'} \to \mathrm{Spec}(k')$ is smooth too. Hence it suffices to show that a scheme X smooth over a field is regular, normal, and reduced. We see that X is regular by Lemma 15.3. Hence Properties, Lemma 9.4 guarantees that X is normal. $\square$

Lemma 15.5. *Let k be a field. Let $d \geq 0$. Let $W \subset \mathbf{A}^d_k$ be nonempty open. Then there exists a closed point $w \in W$ such that $k \subset \kappa(w)$ is finite separable.*

Proof. After possible shrinking W we may assume that $W = \mathbf{A}^d_k \setminus V(f)$ for some $f \in k[x_1, \ldots, x_n]$. If the lemma is wrong then $f(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in (k^{sep})^n$. This is absurd as k^{sep} is an infinite field. $\square$

Lemma 15.6. *Let k be a field. If X is smooth over* $\mathrm{Spec}(k)$ *then the set*

$$\{x \in X \text{ closed such that } k \subset \kappa(x) \text{ is finite separable}\}$$

is dense in X.

Proof. It suffices to show that given a nonempty smooth X over k there exists at least one closed point whose residue field is finite separable over k. To see this, choose a diagram

$$X \longleftarrow U \xrightarrow{\pi} \mathbf{A}^d_k$$

with π étale, see Morphisms, Lemma 37.20. The morphism $\pi : U \to \mathbf{A}^d_k$ is open, see Morphisms, Lemma 37.13. By Lemma 15.5 we may choose a closed point $w \in \pi(V)$ whose residue field is finite separable over k. Pick any $x \in V$ with $\pi(x) = w$. By Morphisms, Lemma 37.7 the field extension $\kappa(w) \subset \kappa(x)$ is finite separable. Hence $k \subset \kappa(x)$ is finite separable. The point x is a closed point of X by Morphisms, Lemma 21.2. $\square$

Lemma 15.7. *Let X be a scheme over a field k. If X is locally of finite type and geometrically reduced over k then X contains a dense open which is smooth over k.*

Proof. The problem is local on X, hence we may assume X is quasi-compact. Let $X = X_1 \cup \ldots \cup X_n$ be the irreducible components of X. Then $Z = \bigcup_{i \neq j} X_i \cap X_j$ is nowhere dense in X. Hence we may replace X by $X \setminus Z$. As $X \setminus Z$ is a disjoint union of irreducible schemes, this reduces us to the case where X is irreducible. As X is irreducible and reduced, it is integral, see Properties, Lemma 3.4. Let $\eta \in X$ be its generic point. Then the function field $K = k(X) = \kappa(\eta)$ is geometrically reduced over k, hence separable over k, see Algebra, Lemma 43.1. Let $U = \mathrm{Spec}(A) \subset X$ be any nonempty affine open so that $K = f.f.(A) = A_{(0)}$. Apply Algebra, Lemma 135.5 to conclude that A is smooth at (0) over k. By definition this means that some principal localization of A is smooth over k and we win. $\square$

Lemma 15.8. *Let k be a field. Let $f : X \to Y$ be a morphism of schemes locally of finite type over k. Let $x \in X$ be a point and set $y = f(x)$. If $X \to$* $\mathrm{Spec}(k)$ *is smooth at x and f is flat at x then $Y \to$* $\mathrm{Spec}(k)$ *is smooth at y. In particular, if X is smooth over k and f is flat and surjective, then Y is smooth over k.*

Proof. It suffices to show that Y is geometrically regular at y, see Lemma 10.6. This follows from Lemma 10.5 (and Lemma 10.6 applied to (X, x)). $\square$

16. Types of varieties

Short section discussion some elementary global properties of varieties.

Definition 16.1. Let k be a field. Let X be a variety over k.

(1) We say X is an *affine variety* if X is an affine scheme. This is equivalent to requiring X it be isomorphic to a closed subscheme of $\mathbf{A}^n_k$ for some n.
(2) We say X is a *projective variety* if the structure morphism $X \to \mathrm{Spec}(k)$ is projective. By Morphisms, Lemma 43.4 this is true if and only if X is isomorphic to a closed subscheme of $\mathbf{P}^n_k$ for some n.
(3) We say X is a *quasi-projective variety* if the structure morphism $X \to \mathrm{Spec}(k)$ is quasi-projective. By Morphisms, Lemma 41.4 this is true if and only if X is isomorphic to a locally closed subscheme of $\mathbf{P}^n_k$ for some n.
(4) A *proper variety* is a variety such that the morphism $X \to \mathrm{Spec}(k)$ is proper.

Note that a projective variety is a proper variety, see Morphisms, Lemma 43.5. Also, an affine variety is quasi-projective as $\mathbf{A}^n_k$ is isomorphic to an open subscheme of $\mathbf{P}^n_k$, see Constructions, Lemma 13.3.

Lemma 16.2. *Let X be a proper variety over k. Then $\Gamma(X, \mathcal{O}_X)$ is a field which is a finite extension of the field k.*

Proof. By Cohomology of Schemes, Proposition 17.2 we see that $\Gamma(X, \mathcal{O}_X)$ is a finite dimensional k-vector space. It is also a k-algebra without zero-divisors. Hence it is a field, see Algebra, Lemma 35.17. $\square$

17. Groups of invertible functions

It is often (but not always) the case that $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group if X is a variety over k. We show this by a series of lemmas. Everything rests on the following special case.

Lemma 17.1. *Let k be an algebraically closed field. Let $\overline{X}$ be a proper variety over k. Let $X \subset \overline{X}$ be an open subscheme. Assume X is normal. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. We will use without further mention that for any affine open U of $\overline{X}$ the ring $\mathcal{O}(U)$ is a finitely generated k-algebra, which is Noetherian, a domain and normal, see Algebra, Lemma 30.1, Properties, Definition 3.1, Properties, Lemmas 5.2 and 7.2, Morphisms, Lemma 16.2.

Let $\xi_1, \ldots, \xi_r$ be the generic points of the complement of X in $\overline{X}$. There are finitely many since $\overline{X}$ has a Noetherian underlying topological space (see Morphisms, Lemma 16.6, Properties, Lemma 5.5, and Topology, Lemma 8.2). For each i the local ring $\mathcal{O}_i = \mathcal{O}_{X, \xi_i}$ is a normal Noetherian local domain (as a localization of a Noetherian normal domain). Let $J \subset \{1, \ldots, r\}$ be the set of indices i such that $\dim(\mathcal{O}_i) = 1$. For $j \in J$ the local ring $\mathcal{O}_j$ is a discrete valuation ring, see Algebra, Lemma 115.6. Hence we obtain a valuation

$$v_j : k(\overline{X})^* \longrightarrow \mathbf{Z}$$

with the property that $v_j(f) \geq 0 \Leftrightarrow f \in \mathcal{O}_j$.

Think of $\mathcal{O}(X)$ as a sub k-algebra of $k(X) = k(\overline{X})$. We claim that the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod\nolimits_{j \in J} \mathbf{Z}, \quad f \longmapsto \prod v_j(f)$$

is k^*. It is clear that this claim proves the lemma. Namely, suppose that $f \in \mathcal{O}(X)$ is an element of the kernel. Let $U = \text{Spec}(B) \subset \overline{X}$ be any affine open. Then B is a Noetherian normal domain. For every height one prime $\mathfrak{q} \subset B$ with corresponding point $\xi \in X$ we see that either $\xi = \xi_j$ for some $j \in J$ or that $\xi \in X$. The reason is that $\text{codim}(\overline{\{\xi\}}, \overline{X}) = 1$ by Properties, Lemma 11.4 and hence if $\xi \in \overline{X} \setminus X$ it must be a generic point of $\overline{X} \setminus X$, hence equal to some ξ_j, $j \in J$. We conclude that $f \in \mathcal{O}_{X,\xi} = B_{\mathfrak{q}}$ in either case as f is in the kernel of the map. Thus $f \in \bigcap_{\text{ht}(\mathfrak{q})=1} B_{\mathfrak{q}} = B$, see Algebra, Lemma 146.6. In other words, we see that $f \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$. But since k is algebraically closed we conclude that $f \in k$ by Lemma 16.2. $\square$

Next, we generalize the case above by some elementary arguments, still keeping the field algebraically closed.

Lemma 17.2. *Let k be an algebraically closed field. Let X be an integral scheme locally of finite type over k. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. As X is integral the restriction mapping $\mathcal{O}(X) \to \mathcal{O}(U)$ is injective for any nonempty open subscheme $U \subset X$. Hence we may assume that X is affine. Choose a closed immersion $X \to \mathbf{A}^n_k$ and denote $\overline{X}$ the closure of X in $\mathbf{P}^n_k$ via the usual immersion $\mathbf{A}^n_k \to \mathbf{P}^n_k$. Thus we may assume that X is an affine open of a projective variety $\overline{X}$.

Let $\nu : \overline{X}^\nu \to \overline{X}$ be the normalization morphism, see Morphisms, Definition 48.12. We know that ν is finite, dominant, and that $\overline{X}^\nu$ is a normal irreducible scheme, see Morphisms, Lemmas 48.15, 48.17, and 19.2. It follows that $\overline{X}^\nu$ is a proper variety, because $\overline{X} \to \text{Spec}(k)$ is proper as a composition of a finite and a proper morphism (see results in Morphisms, Sections 42 and 44). It also follows that ν is a surjective morphism, because the image of ν is closed and contains the generic point of $\overline{X}$. Hence setting $X^\nu = \nu^{-1}(X)$ we see that it suffices to prove the result for X^ν. In other words, we may assume that X is a nonempty open of a normal proper variety $\overline{X}$. This case is handled by Lemma 17.1. $\square$

The preceding lemma implies the following slight generalization.

Lemma 17.3. *Let k be an algebraically closed field. Let X be a connected reduced scheme which is locally of finite type over k with finitely many irreducible components. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. Let $X = \bigcup X_i$ be the irreducible components. By Lemma 17.2 we see that $\mathcal{O}(X_i)^*/k^*$ is a finitely generated abelian group. Let $f \in \mathcal{O}(X)^*$ be in the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod \mathcal{O}(X_i)^*/k^*.$$

Then for each i there exists an element $\lambda_i \in k$ such that $f|_{X_i} = \lambda_i$. By restricting to $X_i \cap X_j$ we conclude that $\lambda_i = \lambda_j$ if $X_i \cap X_j \neq \emptyset$. Since X is connected we conclude that all λ_i agree and hence that $f \in k^*$. This proves that

$$\mathcal{O}(X)^*/k^* \subset \prod \mathcal{O}(X_i)^*/k^*$$

and the lemma follows as on the right we have a product of finitely many finitely generated abelian groups. $\square$

Lemma 17.4. *Let k be a field. Let X be a scheme over k which is connected and reduced. Then the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is a field.*

Proof. Let $k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k. Then $X \to \mathrm{Spec}(k)$ factors through $\mathrm{Spec}(k')$, see Schemes, Lemma 6.4. As X is reduced we see that k' has no nonzero nilpotent elements. As $k \to k'$ is integral we see that every prime ideal of k' is both a maximal ideal and a minimal prime, and $\mathrm{Spec}(k')$ is totally disconnected, see Algebra, Lemmas 35.18 and 25.5. As X is connected the morphism $X \to \mathrm{Spec}(k')$ is constant, say with image the point corresponding to $\mathfrak{p} \subset k'$. Then any $f \in k'$, $f \notin \mathfrak{p}$ maps to an invertible element of $\mathcal{O}_X$. By definition of k' this then forces f to be a unit of k'. Hence we see that k' is local with maximal ideal $\mathfrak{p}$, see Algebra, Lemma 17.2. Since we've already seen that k' is reduced this implies that k' is a field, see Algebra, Lemma 24.1. $\square$

Proposition 17.5. *Let k be a field. Let X be a scheme over k. Assume that X is locally of finite type over k, connected, reduced, and has finitely many irreducible components. Then $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group if in addition to the conditions above at least one of the following conditions is satisfied:*

1. *the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is k,*
2. *X has a k-rational point, or*
3. *X is geometrically integral.*

Proof. Let $\overline{k}$ be an algebraic closure of k. Let Y be a connected component of $(X_{\overline{k}})_{red}$. Note that the canonical morphism $p : Y \to X$ is open (by Morphisms, Lemma 24.4) and closed (by Morphisms, Lemma 44.7). Hence $p(Y) = X$ as X was assumed connected. In particular, as X is reduced this implies $\mathcal{O}(X) \subset \mathcal{O}(Y)$. By Lemma 6.13 we see that Y has finitely many irreducible components. Thus Lemma 17.3 applies to Y. This implies that if $\mathcal{O}(X)^*/k^*$ is not a finitely generated abelian group, then there exist elements $f \in \mathcal{O}(X)$, $f \notin k$ which map to an element of $\overline{k}$ via the map $\mathcal{O}(X) \to \mathcal{O}(Y)$. In this case f is algebraic over k, hence integral over k. Thus, if condition (1) holds, then this cannot happen. To finish the proof we show that conditions (2) and (3) imply (1).

Let $k \subset k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k in $\Gamma(X, \mathcal{O}_X)$. By Lemma 17.4 we see that k' is a field. If $e : \mathrm{Spec}(k) \to X$ is a k-rational point, then $e^\sharp : \Gamma(X, \mathcal{O}_X) \to k$ is a section to the inclusion map $k \to \Gamma(X, \mathcal{O}_X)$. In particular the restriction of $e^\sharp$ to k' is a field map $k' \to k$ over k, which clearly shows that (2) implies (1).

If the integral closure k' of k in $\Gamma(X, \mathcal{O}_X)$ is not trivial, then we see that X is either not geometrically connected (if $k \subset k'$ is not purely inseparable) or that X is not geometrically reduced (if $k \subset k'$ is nontrivial purely inseparable). Details omitted. Hence (3) implies (1). $\square$

Lemma 17.6. *Let k be a field. Let X be a variety over k. The group $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group provided at least one of the following conditions holds:*

1. *k is integrally closed in $\Gamma(X, \mathcal{O}_X)$,*
2. *k is algebraically closed in $k(X)$,*
3. *X is geometrically integral over k, or*

(4) *k is the "intersection" of the field extensions $k \subset \kappa(x)$ where x runs over the closed points of x.*

Proof. We see that (1) is enough by Proposition 17.5. We omit the verification that each of (2), (3), (4) implies (1). $\square$

18. Uniqueness of base field

The phrase "let X be a scheme over k" means that X is a scheme which comes equipped with a morphism $X \to \mathrm{Spec}(k)$. Now we can ask whether the field k is uniquely determined by the scheme X. Of course this is not the case, since for example $\mathbf{A}^1_{\mathbf{C}}$ which we ordinarily consider as a scheme over the field $\mathbf{C}$ of complex numbers, could also be considered as a scheme over $\mathbf{Q}$. But what if we ask that the morphism $X \to \mathrm{Spec}(k)$ does not factor as $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ for any nontrivial field extension $k \subset k'$? In other words we ask that k is somehow maximal such that X lives over k.

An example to show that this still does not guarantee uniqueness of k is the scheme

$$X = \mathrm{Spec}\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(y)}, P \in \mathbf{Q}[y], P \neq 0\right]\right)$$

At first sight this seems to be a scheme over $\mathbf{Q}(x)$, but on a second look it is clear that it is also a scheme over $\mathbf{Q}(y)$. Moreover, the fields $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are subfields of $R = \Gamma(X, \mathcal{O}_X)$ which are maximal among the subfields of R (details omitted). In particular, both $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are maximal in the sense above. Note that both morphisms $X \to \mathrm{Spec}(\mathbf{Q}(x))$ and $X \to \mathrm{Spec}(\mathbf{Q}(y))$ are "essentially of finite type" (i.e., the corresponding ring map is essentially of finite type). Hence X is a Noetherian scheme of finite dimension, i.e., it is not completely pathological.

Another issue that can prevent uniqueness is that the scheme X may be nonreduced. In that case there can be many different morphisms from X to the spectrum of a given field. As an explicit example consider the dual numbers $D = \mathbf{C}[y]/(y^2) = \mathbf{C} \oplus \epsilon\mathbf{C}$. Given any derivation $\theta : \mathbf{C} \to \mathbf{C}$ over $\mathbf{Q}$ we get a ring map

$$\mathbf{C} \longrightarrow D, \quad c \longmapsto c + \epsilon\theta(c).$$

The subfield of $\mathbf{C}$ on which all of these maps are the same is the algebraic closure of $\mathbf{Q}$. This means that taking the intersection of all the fields that X can live over may end up being a very small field if X is nonreduced.

One observation in this regard is the following: given a field k and two subfields k_1, k_2 of k such that k is finite over k_1 and over k_2, then in general it is *not* the case that k is finite over $k_1 \cap k_2$. An example is the field $k = \mathbf{Q}(t)$ and its subfields $k_1 = \mathbf{Q}(t^2)$ and $\mathbf{Q}((t+1)^2)$. Namely we have $k_1 \cap k_2 = \mathbf{Q}$ in this case. So in the following we have to be careful when taking intersections of fields.

Having said all of this we now show that if X is locally of finite type over a field, then some uniqueness holds. Here is the precise result.

Proposition 18.1. *Let X be a scheme. Let $a : X \to \mathrm{Spec}(k_1)$ and $b : X \to \mathrm{Spec}(k_2)$ be morphisms from X to spectra of fields. Assume a, b are locally of finite type, and X is reduced, and connected. Then we have $k_1' = k_2'$, where $k_i' \subset \Gamma(X, \mathcal{O}_X)$ is the integral closure of k_i in $\Gamma(X, \mathcal{O}_X)$.*

Proof. First, assume the lemma holds in case X is quasi-compact (we will do the quasi-compact case below). As X is locally of finite type over a field, it is locally Noetherian, see Morphisms, Lemma 16.6. In particular this means that it is locally connected, connected components of open subsets are open, and intersections of quasi-compact opens are quasi-compact, see Properties, Lemma 5.5, Topology, Lemma 6.10, Topology, Section 8, and Topology, Lemma 15.1. Pick an open covering $X = \bigcup_{i \in I} U_i$ such that each U_i is quasi-compact and connected. For each i let $K_i \subset \mathcal{O}_X(U_i)$ be the integral closure of k_1 and of k_2. For each pair $i, j \in I$ we decompose

$$U_i \cap U_j = \coprod U_{i,j,l}$$

into its finitely many connected components. Write $K_{i,j,l} \subset \mathcal{O}(U_{i,j,l})$ for the integral closure of k_1 and of k_2. By Lemma 17.4 the rings K_i and $K_{i,j,l}$ are fields. Now we claim that k'_1 and k'_2 both equal the kernel of the map

$$\prod K_i \longrightarrow \prod K_{i,j,l}, \quad (x_i)_i \longmapsto x_i|_{U_{i,j,l}} - x_j|_{U_{i,j,l}}$$

which proves what we want. Namely, it is clear that k'_1 is contained in this kernel. On the other hand, suppose that $(x_i)_i$ is in the kernel. By the sheaf condition $(x_i)_i$ corresponds to $f \in \mathcal{O}(X)$. Pick some $i_0 \in I$ and let $P(T) \in k_1[T]$ be a monic polynomial with $P(x_{i_0}) = 0$. Then we claim that $P(f) = 0$ which proves that $f \in k_1$. To prove this we have to show that $P(x_i) = 0$ for all i. Pick $i \in I$. As X is connected there exists a sequence $i_0, i_1, \ldots, i_n = i \in I$ such that $U_{i_t} \cap U_{i_{t+1}} \neq \emptyset$. Now this means that for each t there exists an l_t such that x_{i_t} and $x_{i_{t+1}}$ map to the same element of the field $K_{i,j,l}$. Hence if $P(x_{i_t}) = 0$, then $P(x_{i_{t+1}}) = 0$. By induction, starting with $P(x_{i_0}) = 0$ we deduce that $P(x_i) = 0$ as desired.

To finish the proof of the lemma we prove the lemma under the additional hypothesis that X is quasi-compact. By Lemma 17.4 after replacing k_i by k'_i we may assume that k_i is integrally closed in $\Gamma(X, \mathcal{O}_X)$. This implies that $\mathcal{O}(X)^*/k_i^*$ is a finitely generated abelian group, see Proposition 17.5. Let $k_{12} = k_1 \cap k_2$ as a subring of $\mathcal{O}(X)$. Note that k_{12} is a field. Since

$$k_1^*/k_{12}^* \longrightarrow \mathcal{O}(X)^*/k_2^*$$

we see that k_1^*/k_{12}^* is a finitely generated abelian group as well. Hence there exist $\alpha_1, \ldots, \alpha_n \in k_1^*$ such that every element $\lambda \in k_1$ has the form

$$\lambda = c\alpha_1^{e_1} \ldots \alpha_n^{e_n}$$

for some $e_i \in \mathbf{Z}$ and $c \in k_{12}$. In particular, the ring map

$$k_{12}[x_1, \ldots, x_n, \frac{1}{x_1 \ldots x_n}] \longrightarrow k_1, \quad x_i \longmapsto \alpha_i$$

is surjective. By the Hilbert Nullstellensatz, Algebra, Theorem 33.1 we conclude that k_1 is a finite extension of k_{12}. In the same way we conclude that k_2 is a finite extension of k_{12}. In particular both k_1 and k_2 are contained in the integral closure k'_{12} of k_{12} in $\Gamma(X, \mathcal{O}_X)$. But since k'_{12} is a field by Lemma 17.4 and since we chose k_i to be integrally closed in $\Gamma(X, \mathcal{O}_X)$ we conclude that $k_1 = k_{12} = k_2$ as desired. $\square$

19. Coherent sheaves on projective space

In this section we prove some results on the cohomology of coherent sheaves on $\mathbf{P}^n$ over a field which can be found in [Mum66]. These will be useful later when discussing Quot and Hilbert schemes.

19.1. Preliminaries. Let k be a field, $n \geq 1$, $d \geq 1$, and let $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(d))$ be a nonzero section. In this section we will write $\mathcal{O}(d)$ for the dth twist of the structure sheaf on projective space (Constructions, Definitions 10.1 and 13.2). Since $\mathbf{P}^n_k$ is a variety this section is regular, hence s is a regular section of $\mathcal{O}(d)$ and defines an effective Cartier divisor $H = Z(s) \subset \mathbf{P}^n_k$, see Divisors, Section 9. Such a divisor H is called a *hypersurface* and if $d = 1$ it is called a *hyperplane*.

Lemma 19.2. *Let k be a field. Let $n \geq 1$. Let $i : H \to \mathbf{P}^n_k$ be a hyperplane. Then there exists an isomorphism*

$$\varphi : \mathbf{P}^{n-1}_k \longrightarrow H$$

such that $i^*\mathcal{O}(1)$ pulls back to $\mathcal{O}(1)$.

Proof. We have $\mathbf{P}^n_k = \text{Proj}(k[T_0, \ldots, T_n])$. The section s corresponds to a homogeneous form in $T_0, \ldots, T_n$ of degree 1, see Cohomology of Schemes, Section 8. Say $s = \sum a_i T_i$. Constructions, Lemma 13.6 gives that $H = \text{Proj}(k[T_0, \ldots, T_n]/I)$ for the graded ideal I defined by setting I_d equal to the kernel of the map $\Gamma(\mathbf{P}^n_k, \mathcal{O}(d)) \to \Gamma(H, i^*\mathcal{O}(d))$. By our construction of $Z(s)$ in Divisors, Definition 9.18 we see that on $D_+(T_j)$ the ideal of H is generated by $\sum a_i T_i/T_j$ in the polynomial ring $k[T_0/T_j, \ldots, T_n/T_j]$. Thus it is clear that I is the ideal generated by $\sum a_i T_i$. Note that

$$k[T_0, \ldots, T_n]/I = k[T_0, \ldots, T_n]/(\sum a_i T_i) \cong k[S_0, \ldots, S_{n-1}]$$

as graded rings. For example, if $a_n \neq 0$, then mapping S_i equal to the class of T_i works. We obtain the desired isomorphism by functoriality of Proj. Equality of twists of structure sheaves follows for example from Constructions, Lemma 11.5. $\square$

Lemma 19.3. *Let k be an infinite field. Let $n \geq 1$. Let $\mathcal{F}$ be a coherent module on $\mathbf{P}^n_k$. Then there exist a nonzero section $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ and a short exact sequence*

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to i_*\mathcal{G} \to 0$$

where $i : H \to \mathbf{P}^n_k$ is the hyperplane H associated to s and $\mathcal{G} = i^*\mathcal{F}$.

Proof. The map $\mathcal{F}(-1) \to \mathcal{F}$ comes from Constructions, Equation (10.1.2) with $n = 1$, $m = -1$ and the section s of $\mathcal{O}(1)$. Let's work out what this map looks like if we restrict it to $D_+(T_0)$. Write $D_+(T_0) = \text{Spec}(k[x_1, \ldots, x_n])$ with $x_i = T_i/T_0$. Identify $\mathcal{O}(1)|_{D_+(T_0)}$ with $\mathcal{O}$ using the section T_0. Hence if $s = \sum a_i T_i$ then $s|_{D_+(T_0)} = a_0 + \sum a_i x_i$ with the identification chosen above. Furthermore, suppose $\mathcal{F}|_{D_+(T_0)}$ corresponds to the finite $k[x_1, \ldots, x_n]$-module M. Via the identification $\mathcal{F}(-1) = \mathcal{F} \otimes \mathcal{O}(-1)$ and our chosen trivialization of $\mathcal{O}(1)$ we see that $\mathcal{F}(-1)$ corresponds to M as well. Thus restricting $\mathcal{F}(-1) \to \mathcal{F}$ to $D_+(T_0)$ gives the map

$$M \xrightarrow{a_0 + \sum a_i x_i} M$$

To see that the arrow is injective, it suffices to pick $a_0 + \sum a_i x_i$ outside any of the associated primes of M, see Algebra, Lemma 62.9. By Algebra, Lemma 62.5 the set $\text{Ass}(M)$ of associated primes of M is finite. Note that for $\mathfrak{p} \in \text{Ass}(M)$ the intersection $\mathfrak{p} \cap \{a_0 + \sum a_i x_i\}$ is a proper k-subvector space. We conclude that there is a finite family of proper sub vector spaces $V_1, \ldots, V_m \subset \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ such that if we take s outside of $\bigcup V_i$, then multiplication by s is injective over $D_+(T_0)$. Similarly for the restriction to $D_+(T_j)$ for $j = 1, \ldots, n$. Since k is infinite, a finite union of proper sub vector spaces is never equal to the whole space, hence we may choose s such that the map is injective. The cokernel of $\mathcal{F}(-1) \to \mathcal{F}$ is annihilated by $\mathrm{Im}(s : \mathcal{O}(-1) \to \mathcal{O})$ which is the ideal sheaf of H by Divisors, Definition 9.18. Hence we obtain $\mathcal{G}$ on H using Cohomology of Schemes, Lemma 9.8. □

Remark 19.4. Let k be an infinite field. Let $n \geq 1$. Given a finite number of coherent modules $\mathcal{F}_i$ on $\mathbf{P}^n_k$ we can choose a single $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ such that the statement of Lemma 19.3 works for each of them. To prove this, just apply the lemma to $\bigoplus \mathcal{F}_i$.

19.5. Regularity.

Definition 19.6. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. We say $\mathcal{F}$ is *m-regular* if

$$H^i(\mathbf{P}^n_k, \mathcal{F}(m - i)) = 0$$

for $i = 1, \ldots, n$.

Note that $\mathcal{F} = \mathcal{O}(d)$ is m-regular if and only if $d \geq m$. This follows from the computation of cohomology groups in Cohomology of Schemes, Equation (8.1.1). Namely, we see that $H^n(\mathbf{P}^n_k, \mathcal{O}(d)) = 0$ if and only if $d \geq -n$.

Lemma 19.7. *Let $k \subset k'$ be an extension of fields. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $\mathbf{P}^n_{k'}$. Then $\mathcal{F}$ is m-regular if and only if $\mathcal{F}'$ is m-regular.*

Proof. This is true because

$$H^i(\mathbf{P}^n_{k'}, \mathcal{F}') = H^i(\mathbf{P}^n_k, \mathcal{F}) \otimes_k k'$$

by flat base change, see Cohomology of Schemes, Lemma 5.2. □

Lemma 19.8. *In the situation of Lemma 19.3, if $\mathcal{F}$ is m-regular, then $\mathcal{G}$ is m-regular on $H \cong \mathbf{P}^{n-1}_k$.*

Proof. Recall that $H^i(\mathbf{P}^n_k, i_*\mathcal{G}) = H^i(H, \mathcal{G})$ by Cohomology of Schemes, Lemma 2.4. Hence we see that for $i \geq 1$ we get

$$H^i(\mathbf{P}^n_k, \mathcal{F}(m - i)) \to H^i(H, \mathcal{G}(m - i)) \to H^{i+1}(\mathbf{P}^n_k, \mathcal{F}(m - 1 - i))$$

as part of the long exact sequence associated to the short exact sequence $0 \to \mathcal{F}(m - 1 - i) \to \mathcal{F}(m - i) \to i_*\mathcal{G}(m - i) \to 0$ we obtain from the exact sequence of Lemma 19.3 by tensoring with the invertible sheaf $\mathcal{O}(m-i)$. The lemma follows. □

Lemma 19.9. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}$ is $(m + 1)$-regular.*

Proof. We prove this by induction on n. If $n = 0$ every sheaf is m-regular for all m and there is nothing to prove. By Lemma 19.7 we may replace k by an infinite overfield and assume k is infinite. Thus we may apply Lemma 19.3. By Lemma 19.8 we know that $\mathcal{G}$ is m-regular. By induction on n we see that $\mathcal{G}$ is $(m + 1)$-regular. Considering the long exact cohomology sequence associated to the sequence

$$0 \to \mathcal{F}(m - i) \to \mathcal{F}(m + 1 - i) \to i_*\mathcal{G}(m + 1 - i) \to 0$$

the reader easily deduces for $i \geq 1$ the vanishing of $H^i(\mathbf{P}^n_k, \mathcal{F}(m+1-i))$ from the (known) vanishing of $H^i(\mathbf{P}^n_k, \mathcal{F}(m-i))$ and $H^i(\mathbf{P}^n_k, \mathcal{G}(m+1-i))$. $\square$

Lemma 19.10. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then the multiplication map*

$$H^0(\mathbf{P}^n_k, \mathcal{F}(m)) \otimes_k H^0(\mathbf{P}^n_k, \mathcal{O}(1)) \longrightarrow H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$$

is surjective.

Proof. Let $k \subset k'$ be an extension of fields. Let $\mathcal{F}'$ be as in Lemma 19.7. By Cohomology of Schemes, Lemma 5.2 the base change of the linear map of the lemma to k' is the same linear map for the sheaf $\mathcal{F}'$. Since $k \to k'$ is faithfully flat it suffices to prove the lemma over k', i.e., we may assume k is infinite.

Assume k is infinite. We prove the lemma by induction on n. The case $n = 0$ is trivial as $\mathcal{O}(1) \cong \mathcal{O}$ is generated by T_0. For $n > 0$ apply Lemma 19.3 and tensor the sequence by $\mathcal{O}(m+1)$ to get

$$0 \to \mathcal{F}(m) \xrightarrow{s} \mathcal{F}(m+1) \to i_*\mathcal{G}(m+1) \to 0$$

Let $t \in H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$. By induction the image $\overline{t} \in H^0(H, \mathcal{G}(m+1))$ is the image of $\sum g_i \otimes \overline{s}_i$ with $\overline{s}_i \in \Gamma(H, \mathcal{O}(1))$ and $g_i \in H^0(H, \mathcal{G}(m))$. Since $\mathcal{F}$ is m-regular we have $H^1(\mathbf{P}^n_k, \mathcal{F}(m-1)) = 0$, hence long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{F}(m-1) \xrightarrow{s} \mathcal{F}(m) \to i_*\mathcal{G}(m) \to 0$$

shows we can lift g_i to $f_i \in H^0(\mathbf{P}^n_k, \mathcal{F}(m))$. We can also lift $\overline{s}_i$ to $s_i \in H^0(\mathbf{P}^n_k, \mathcal{O}(1))$ (see proof of Lemma 19.2 for example). After substracting the image of $\sum f_i \otimes s_i$ from t we see that we may assume $\overline{t} = 0$. But this exactly means that t is the image of $f \otimes s$ for some $f \in H^0(\mathbf{P}^n_k, \mathcal{F}(m))$ as desired. $\square$

Lemma 19.11. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}(m)$ is globally generated.*

Proof. For all $d \gg 0$ the sheaf $\mathcal{F}(d)$ is globally generated. This follows for example from the first part of Cohomology of Schemes, Lemma 15.1. Pick $d \geq m$ such that $\mathcal{F}(d)$ is globally generated. Choose a basis $f_1, \ldots, f_r \in H^0(\mathbf{P}^n_k, \mathcal{F})$. By Lemma 19.10 every element $f \in H^0(\mathbf{P}^n_k, \mathcal{F}(d))$ can be written as $f = \sum P_i f_i$ for some $P_i \in k[T_0, \ldots, T_n]$ homogeneous of degree $d - m$. Since the sections f generate $\mathcal{F}(d)$ it follows that the sections f_i generate $\mathcal{F}(m)$. $\square$

19.12. Hilbert polynomials. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. In this situation the *Euler characteristic of $\mathcal{F}$* is the integer

$$\chi(X, \mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Note that only a finite number of the vector spaces $H^i(X, \mathcal{F})$ are nonzero (Cohomology of Schemes, Lemma 4.4) and that each of these spaces is finite dimensional (Cohomology of Schemes, Lemma 17.4). Thus $\chi(X, \mathcal{F}) \in \mathbf{Z}$ is well defined. Observe that this definition depends on the field k and not just on the pair $(X, \mathcal{F})$.

Lemma 19.13. *Let k be a field. Let X be a proper scheme over k. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent modules on X. Then*

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

Proof. Consider the long exact sequence of cohomology

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1) \to \ldots$$

associated to the short exact sequence of the lemma. The rank-nullity theorem in linear algebra shows that

$$0 = \dim H^0(X, \mathcal{F}_1) - \dim H^0(X, \mathcal{F}_2) + \dim H^0(X, \mathcal{F}_3) - \dim H^1(X, \mathcal{F}_1) + \ldots$$

This immediately implies the lemma. $\square$

Lemma 19.14. *Let $k \subset k'$ be an extension of fields. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $X_{k'}$. Then $\chi(X, \mathcal{F}) = \chi(X', \mathcal{F}')$.*

Proof. This is true because

$$H^i(X_{k'}, \mathcal{F}') = H^i(X, \mathcal{F}) \otimes_k k'$$

by flat base change, see Cohomology of Schemes, Lemma 5.2. $\square$

Lemma 19.15. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function*

$$d \longmapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$$

is a polynomial.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \mathrm{Spec}(k)$ and $\mathcal{F}(d) = \mathcal{F}$. Hence in this case the function is constant, i.e., a polynomial of degree 0. Assume $n > 0$. By Lemma 19.14 we may assume k is infinite. Apply Lemma 19.3. Applying Lemma 19.13 to the twisted sequences $0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_*\mathcal{G}(d) \to 0$ we obtain

$$\chi(\mathbf{P}^n_k, \mathcal{F}(d)) - \chi(\mathbf{P}^n_k, \mathcal{F}(d-1)) = \chi(H, \mathcal{G}(d))$$

(this also uses the identification of the cohomology of $i_*\mathcal{G}$ with the cohomology of $\mathcal{G}$, see Cohomology of Schemes, Lemma 2.4). Since $H \cong \mathbf{P}^{n-1}_k$ (Lemma 19.2) by induction the right hand side is a polynomial. The lemma is finished by noting that any function $f : \mathbf{Z} \to \mathbf{Z}$ with the property that the map $d \mapsto f(d) - f(d-1)$ is a polynomial, is itself a polynomial. We omit the proof of this fact (hint: compare with Algebra, Lemma 57.5). $\square$

Definition 19.16. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function $d \mapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$ is called the *Hilbert polynomial* of $\mathcal{F}$.

The Hilbert polynomial has coefficients in $\mathbf{Q}$ and not in general in $\mathbf{Z}$. For example the Hilbert polynomial of $\mathcal{O}_{\mathbf{P}^n_k}$ is

$$d \longmapsto \binom{d+n}{n} = \frac{d^n}{n!} + \ldots$$

This follows from the following lemma and the fact that

$$H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}) = k[T_0, \ldots, T_n]_d$$

(degree d part) whose dimension over k is $\binom{d+n}{n}$.

Lemma 19.17. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$ with Hilbert polynomial $P \in \mathbf{Q}[t]$. Then*

$$P(d) = \dim_k H^0(\mathbf{P}^n_k, \mathcal{F}(d))$$

for all $d \gg 0$.

Proof. This follows from the vanishing of cohomology of high enough twists of $\mathcal{F}$. See Cohomology of Schemes, Lemma 15.1. □

19.18. Boundedness of quotients. In this subsection we bound the regularity of quotients of a given coherent sheaf on $\mathbf{P}^n$ in terms of the Hilbert polynomial.

Lemma 19.19. *Let k be a field. Let $n \geq 0$. Let $r \geq 1$. Let $P \in \mathbf{Q}[t]$. There exists an integer m depending on n, r, and P with the following property: if*

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus r} \to \mathcal{F} \to 0$$

is a short exact sequence of coherent sheaves on $\mathbf{P}^n_k$ and $\mathcal{F}$ has Hilbert polynomial P, then $\mathcal{K}$ is m-regular.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \mathrm{Spec}(k)$ and any coherent module is 0-regular and any surjective map is surjective on global sections. Assume $n > 0$. Consider an exact sequence as in the lemma. Let $P' \in \mathbf{Q}[t]$ be the polynomial $P'(t) = P(t) - P(t-1)$. Let m' be the integer which works for $n-1$, r, and P'. By Lemmas 19.7 and 19.14 we may replace k by a field extension, hence we may assume k is infinite. Apply Lemma 19.3 to the coherent sheaf $\mathcal{F}$. The Hilbert polynomial of $\mathcal{F}' = i^*\mathcal{F}$ is P' (see proof of Lemma 19.15). Since i^* is right exact we see that $\mathcal{F}'$ is a quotient of $\mathcal{O}_H^{\oplus r} = i^*\mathcal{O}^{\oplus r}$. Thus the induction hypothesis applies to $\mathcal{F}'$ on $H \cong \mathbf{P}^{n-1}_k$ (Lemma 19.2). Note that the map $\mathcal{K}(-1) \to \mathcal{K}$ is injective as $\mathcal{K} \subset \mathcal{O}^{\oplus r}$ and has cokernel $i_*\mathcal{H}$ where $\mathcal{H} = i^*\mathcal{K}$. By the snake lemma (Homology, Lemma 5.17) we obtain a commutative diagram with exact columns and rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{K}(-1) \arrow[r] \arrow[d] & \mathcal{O}^{\oplus r}(-1) \arrow[r] \arrow[d] & \mathcal{F}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{O}^{\oplus r} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & i_*\mathcal{H} \arrow[r] \arrow[d] & i_*\mathcal{O}_H^{\oplus r} \arrow[r] \arrow[d] & i_*\mathcal{F}' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Thus the induction hypothesis applies to the exact sequence $0 \to \mathcal{H} \to \mathcal{O}_H^{\oplus r} \to \mathcal{F}' \to 0$ on $H \cong \mathbf{P}^{n-1}_k$ (Lemma 19.2) and $\mathcal{H}$ is m'-regular. Recall that this implies that $\mathcal{H}$ is d-regular for all $d \geq m'$ (Lemma 19.9).

Let $i \geq 2$ and $d \geq m'$. It follows from the long exact cohomology sequence associated to the left column of the diagram above and the vanishing of $H^{i-1}(H, \mathcal{H}(d))$

that the map

$$H^i(\mathbf{P}^n_k, \mathcal{K}(d-1)) \longrightarrow H^i(\mathbf{P}^n_k, \mathcal{K}(d))$$

is injective. As these groups are zero for $d \gg 0$ (Cohomology of Schemes, Lemma 15.1) we conclude $H^i(\mathbf{P}^n_k, \mathcal{K}(d))$ are zero for all $d \geq m'$ and $i \geq 2$.

We still have to control H^1. First we observe that all the maps

$$H^1(\mathbf{P}^n_k, \mathcal{K}(m'-1)) \to H^1(\mathbf{P}^n_k, \mathcal{K}(m')) \to H^1(\mathbf{P}^n_k, \mathcal{K}(m'+1)) \to \ldots$$

are surjective by the vanishing of $H^1(H, \mathcal{H}(d))$ for $d \geq m'$. Suppose $d > m'$ is such that

$$H^1(\mathbf{P}^n_k, \mathcal{K}(d-1)) \longrightarrow H^1(\mathbf{P}^n_k, \mathcal{K}(d))$$

is injective. Then $H^0(\mathbf{P}^n_k, \mathcal{K}(d)) \to H^0(H, \mathcal{H}(d))$ is surjective. Consider the commutative diagram

$$\begin{tikzcd}
H^0(\mathbf{P}^n_k, \mathcal{K}(d)) \otimes_k H^0(\mathbf{P}^n_k, \mathcal{O}(1)) \arrow[r] \arrow[d] & H^0(\mathbf{P}^n_k, \mathcal{K}(d+1)) \arrow[d] \\
H^0(H, \mathcal{H}(d)) \otimes_k H^0(H, \mathcal{O}_H(1)) \arrow[r] & H^0(H, \mathcal{H}(d+1))
\end{tikzcd}$$

By Lemma 19.10 we see that the bottom horizontal arrow is surjective. Hence the right vertical arrow is surjective. We conclude that

$$H^1(\mathbf{P}^n_k, \mathcal{K}(d)) \longrightarrow H^1(\mathbf{P}^n_k, \mathcal{K}(d+1))$$

is injective. By induction we see that

$$H^1(\mathbf{P}^n_k, \mathcal{K}(d-1)) \to H^1(\mathbf{P}^n_k, \mathcal{K}(d)) \to H^1(\mathbf{P}^n_k, \mathcal{K}(d+1)) \to \ldots$$

are all injective and we conclude that $H^1(\mathbf{P}^n_k, \mathcal{K}(d-1)) = 0$ because of the eventual vanishing of these groups. Thus the dimensions of the groups $H^1(\mathbf{P}^n_k, \mathcal{K}(d))$ for $d \geq m'$ are strictly decreasing until they become zero. It follows that the regularity of $\mathcal{K}$ is bounded by $m' + \dim_k H^1(\mathbf{P}^n_k, \mathcal{K}(m'))$. On the other hand, by the vanishing of the higher cohomology groups we have

$$\dim_k H^1(\mathbf{P}^n_k, \mathcal{K}(m')) = -\chi(\mathbf{P}^n_k, \mathcal{K}(m')) + \dim_k H^0(\mathbf{P}^n_k, \mathcal{K}(m'))$$

Note that the H^0 has dimension bounded by the dimension of $H^0(\mathbf{P}^n_k, \mathcal{O}^{\oplus r}(m'))$ which is at most $r\binom{n+m'}{n}$ if $m' > 0$ and zero if not. Finally, the term $\chi(\mathbf{P}^n_k, \mathcal{K}(m'))$ is equal to $r\binom{n+m'}{n} - P(m')$. This gives a bound of the desired type finishing the proof of the lemma. $\square$

20. Glueing dimension one rings

This section contains some algebraic preliminaries to proving that a finite set of codimension 1 points of a separated scheme is contained in an affine open.

Situation 20.1. Here we are given a commutative diagram of rings

$$\begin{tikzcd}
A \arrow[r] & K \\
R \arrow[r] \arrow[u] & B \arrow[u]
\end{tikzcd}$$

where K is a field and A, B are subrings of K with fraction field K. Finally, $R = A \times_K B = A \cap B$.

Lemma 20.2. *In Situation 20.1 assume that B is a valuation ring. Then for every unit u of A either $u \in R$ or $u^{-1} \in R$.*

Proof. Namely, if the image c of u in K is in B, then $u \in R$. Otherwise, $c^{-1} \in B$ (Algebra, Lemma 48.3) and $u^{-1} \in R$. □

The following lemma explains the meaning of the condition "$A \otimes B \to K$ is surjective" which comes up quite a bit in the following.

Lemma 20.3. *In Situation 20.1 assume A is a Noetherian ring of dimension 1. The following are equivalent*

(1) *$A \otimes B \to K$ is not surjective,*
(2) *there exists a discrete valuation ring $\mathcal{O} \subset K$ containing both A and B.*

Proof. It is clear that (2) implies (1). On the other hand, if $A \otimes B \to K$ is not surjective, then the image $C \subset K$ is not a field hence C has a nonzero maximal ideal $\mathfrak{m}$. Choose a valuation ring $\mathcal{O} \subset K$ dominating $C_{\mathfrak{m}}$. By Algebra, Lemma 115.11 applied to $A \subset \mathcal{O}$ the ring $\mathcal{O}$ is Noetherian. Hence $\mathcal{O}$ is a valuation ring by Algebra, Lemma 48.18. □

Lemma 20.4. *In Situation 20.1 assume*

(1) *A is a Noetherian semi-local domain of dimension 1,*
(2) *B is a discrete valuation ring,*

Then we have the following two possibilities

(a) *If A^* is not contained in R, then* $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *and* $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ *are open immersions and $K = A \otimes_R B$.*
(b) *If A^* is contained in R, then B dominates one of the local rings of A at a maximal ideal and $A \otimes B \to K$ is not surjective.*

Proof. Assumption (a) implies there is a unit of A whose image in K lies in the maximal ideal of B. Then u is a nonzerodivisor of R and for every $a \in A$ there exists an n such that $u^n a \in R$. It follows that $A = R_u$.

Let $\mathfrak{m}_A$ be the radical of A. Let $x \in \mathfrak{m}_A$ be a nonzero element. Since $\dim(A) = 1$ we see that $K = A_x$. After replacing x by $x^n u^m$ for some $n \geq 1$ and $m \in \mathbf{Z}$ we may assume x maps to a unit of B. We see that for every $b \in B$ we have that $x^n b$ in the image of R for some n. Thus $B = R_x$.

Let $z \in R$. If $z \notin \mathfrak{m}_A$ and z does not map to an element of $\mathfrak{m}_B$, then z is invertible. Thus $x + u$ is invertible in R. Hence $\mathrm{Spec}(R) = D(x) \cup D(u)$. We have seen above that $D(u) = \mathrm{Spec}(A)$ and $D(x) = \mathrm{Spec}(B)$.

Case (b). If $x \in \mathfrak{m}_A$, then $1 + x$ is a unit and hence $1 + x \in R$, i.e, $x \in R$. Thus we see that $\mathfrak{m}_A \subset R \subset A$. In fact, in this case A is integral over R. Namely, write $A/\mathfrak{m}_A = \kappa_1 \times \ldots \times \kappa_n$ as a product of fields. Say $x = (c_1, \ldots, c_r, 0, \ldots, 0)$ is an element with $c_i \neq 0$. Then

$$x^2 - x(c_1, \ldots, c_r, 1, \ldots, 1) = 0$$

Since R contains all units we see that $A/\mathfrak{m}_A$ is integral over the image of R in it, and hence A is integral over R. It follows that $R \subset A \subset B$ as B is integrally closed. Moreover, if $x \in \mathfrak{m}_A$ is nonzero, then $K = A_x = \bigcup x^{-n}A = \bigcup x^{-n}R$. Hence $x^{-1} \notin B$, i.e., $x \in \mathfrak{m}_B$. We conclude $\mathfrak{m}_A \subset \mathfrak{m}_B$. Thus $A \cap \mathfrak{m}_B$ is a maximal ideal of A thereby finishing the proof. □

Lemma 20.5. *Let B be a semi-local Noetherian domain of dimension 1. Let B' be the integral closure of B in its fraction field. Then B' is a semi-local Dedekind domain. Let x be a nonzero element of the radical of B'. Then for every $y \in B'$ there exists an n such that $x^n y \in B$.*

Proof. Let $\mathfrak{m}_B$ be the radical of B. The structure of B' results from Algebra, Lemma 116.14. Given $x, y \in B'$ as in the statement of the lemma consider the subring $B \subset A \subset B'$ generated by x and y. Then A is finite over B (Algebra, Lemma 35.5). Since the fraction fields of B and A are the same we see that the finite module A/B is supported on the set of closed points of B. Thus $\mathfrak{m}_B^n A \subset B$ for a suitable n. Moreover, $\text{Spec}(B') \to \text{Spec}(A)$ is surjective (Algebra, Lemma 35.15), hence A is semi-local as well. It also follows that x is in the radical $\mathfrak{m}_A$ of A. Note that $\mathfrak{m}_A = \sqrt{\mathfrak{m}_B A}$. Thus $x^m y \in \mathfrak{m}_B A$ for some m. Then $x^{nm} y \in B$. $\square$

Lemma 20.6. *In Situation 20.1 assume*

1. *A is a Noetherian semi-local domain of dimension 1,*
2. *B is a Noetherian semi-local domain of dimension 1,*
3. *$A \otimes B \to K$ is surjective.*

Then $\text{Spec}(A) \to \text{Spec}(R)$ and $\text{Spec}(B) \to \text{Spec}(R)$ are open immersions and $K = A \otimes_R B$.

Proof. Special case: B is integrally closed in K. This means that B is a Dedekind domain (Algebra, Lemma 116.13) whence all of its localizations at maximal ideals are discrete valuation rings. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals of B. We set

$$R_1 = A \times_K B_{\mathfrak{m}_1}$$

Observing that $A \otimes_{R_1} B_{\mathfrak{m}_1} \to K$ is surjective we conclude from Lemma 20.4 that A and $B_{\mathfrak{m}_1}$ define open subschemes covering $\text{Spec}(R_1)$ and that $K = A \otimes_{R_1} B_{\mathfrak{m}_1}$. In particular R_1 is a semi-local Noetherian ring of dimension 1. By induction we define

$$R_{i+1} = R_i \times_K B_{\mathfrak{m}_{i+1}}$$

for $i = 1, \ldots, r-1$. Observe that $R = R_n$ because $B = B_{\mathfrak{m}_1} \cap \ldots \cap B_{\mathfrak{m}_r}$ (see Algebra, Lemma 146.6). It follows from the inductive procedure that $R \to A$ defines an open immersion $\text{Spec}(A) \to \text{Spec}(R)$. On the other hand, the maximal ideals $\mathfrak{n}_i$ of R not in this open correspond to the maximal ideals $\mathfrak{m}_i$ of B and in fact the ring map $R \to B$ defines an isomorphisms $R_{\mathfrak{n}_i} \to B_{\mathfrak{m}_i}$ (details omitted; hint: in each step we added exactly one maximal ideal to $\text{Spec}(R_i)$). It follows that $\text{Spec}(B) \to \text{Spec}(R)$ is an open immersion as desired.

General case. Let $B' \subset K$ be the integral closure of B. See Lemma 20.5. Then the special case applies to $R' = A \times_K B'$. Pick $x \in R'$ which is not contained in the maximal ideals of A and is contained in the maximal ideals of B' (see Algebra, Lemma 14.3). By Lemma 20.5 there exists an integer n such that $x^n \in R = A \times_K B$. Replace x by x^n so $x \in R$. For every $y \in R'$ there exists an integer n such that $x^n y \in R$. On the other hand, it is clear that $R'_x = A$. Thus $R_x = A$. Exchanging the roles of A and B we also find an $y \in R$ such that $B = R_y$. Note that inverting both x and y leaves no primes except (0). Thus $K = R_{xy} = R_x \otimes_R R_y$. This finishes the proof. $\square$

Lemma 20.7. *Let K be a field. Let $A_1, \ldots, A_r \subset K$ be Noetherian semi-local rings of dimension 1 with fraction field K. If $A_i \otimes A_j \to K$ is surjective for all $i \neq j$,* *then there exists a Noetherian semi-local domain $A \subset K$ of dimension 1 containing $A_1, \ldots, A_r$ such that*

1. *$A \to A_i$ induces an open immersion $j_i : \mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$,*
2. *$\mathrm{Spec}(A)$ is the union of the opens $j_i(\mathrm{Spec}(A_i))$,*
3. *each closed point of $\mathrm{Spec}(A)$ lies in exactly one of these opens.*

Proof. Namely, we can take $A = A_1 \cap \ldots \cap A_r$. First we note that (3), once (1) and (2) have been proven, follows from the assumption that $A_i \otimes A_j \to K$ is surjective since if $\mathfrak{m} \in j_i(\mathrm{Spec}(A_i)) \cap j_j(\mathrm{Spec}(A_j))$, then $A_i \otimes A_j \to K$ ends up in $A_{\mathfrak{m}}$. To prove (1) and (2) we argue by induction on r. If $r > 1$ by induction we have the results (1) and (2) for $B = A_2 \cap \ldots \cap A_r$. Then we apply Lemma 20.6 to see they hold for $A = A_1 \cap B$. $\square$

Lemma 20.8. *Let A be a domain with fraction field K. Let $B_1, \ldots, B_r \subset K$ be Noetherian 1-dimensional semi-local rings whose fraction fields are K. If $A \otimes B_i \to K$ are surjective for $i = 1, \ldots, r$, then there exists an $x \in A$ such that x^{-1} is in the radical of B_i for $i = 1, \ldots, r$.*

Proof. Let B_i' be the integral closure of B_i in K. Suppose we find a nonzero $x \in A$ such that x^{-1} is in the radical of B_i' for $i = 1, \ldots, r$. Then by Lemma 20.5, after replacing x by a power we get $x^{-1} \in B_i$. Since $\mathrm{Spec}(B_i') \to \mathrm{Spec}(B_i)$ is surjective we see that x^{-1} is then also in the radical of B_i. Thus we may assume that each B_i is a semi-local Dedekind domain.

If B_i is not local, then remove B_i from the list and add back the finite collection of local rings $(B_i)_{\mathfrak{m}}$. Thus we may assume that B_i is a discrete valuation ring for $i = 1, \ldots, r$.

Let $v_i : K \to \mathbf{Z}$, $i = 1, \ldots, r$ be the corresponding discrete valuations (see Algebra, Lemma 116.13). We are looking for a nonzero $x \in A$ with $v_i(x) < 0$ for $i = 1, \ldots, r$. We will prove this by induction on r.

If $r = 1$ and the result is wrong, then $A \subset B$ and the map $A \otimes B \to K$ is not surjective, contradiction.

If $r > 1$, then by induction we can find a nonzero $x \in A$ such that $v_i(x) < 0$ for $i = 1, \ldots, r-1$. If $v_r(x) < 0$ then we are done, so we may assume $v_r(x) \geq 0$. By the base case we can find $y \in A$ nonzero such that $v_r(y) < 0$. After replacing x by a power we may assume that $v_i(x) < v_i(y)$ for $i = 1, \ldots, r-1$. Then $x + y$ is the element we are looking for. $\square$

Lemma 20.9. *Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_{\mathfrak{p}}$ where the product is over the minimal primes of A. Let $a_1, a_2 \in \mathfrak{m}_A$ map to the same element of L. Then $a_1^n = a_2^n$ for some $n > 0$.*

Proof. Write $a_1 = a_2 + x$. Then x maps to zero in L. Hence x is a nilpotent element of A because $\bigcap \mathfrak{p}$ is the radical of (0) and the annihilator I of x contains a power of the maximal ideal because $\mathfrak{p} \not\in V(I)$ for all minimal primes. Say $x^k = 0$ and $\mathfrak{m}^n \subset I$. Then

$$a_1^{k+n} = a_2^{k+n} + \binom{n+k}{1} a_2^{n+k-1} x + \binom{n+k}{2} a_2^{n+k-2} x^2 + \ldots + \binom{n+k}{k-1} a_2^{n+1} x^{k-1} = a_2^{n+k}$$

because $a_2 \in \mathfrak{m}_A$. $\square$

Lemma 20.10. *Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_{\mathfrak p}$ and $I = \bigcap \mathfrak p$ where the product and intersection are over the minimal primes of A. Let $f \in L$ be an element of the form $f = i + a$ where $a \in \mathfrak m_A$ and $i \in IL$. Then some power of f is in the image of $A \to L$.*

Proof. Since A is Noetherian we have $I^t = 0$ for some $t > 0$. Suppose that we know that $f = a + i$ with $i \in I^kL$. Then $f^n = a^n + na^{n-1}i \bmod I^{k+1}L$. Hence it suffices to show that $na^{n-1}i$ is in the image of $I^k \to I^kL$ for some $n \gg 0$. To see this, pick a $g \in A$ such that $\mathfrak m_A = \sqrt{(g)}$ (Algebra, Lemma 59.7). Then $L = A_g$ for example by Algebra, Proposition 59.6. On the other hand, there is an n such that $a^n \in (g)$. Hence we can clear denominators for elements of L by multiplying by a high power of a. $\square$

Lemma 20.11. *Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_{\mathfrak p}$ where the product is over the minimal primes of A. Let $K \to L$ be an integral ring map. Then there exist $a \in \mathfrak m_A$ and $x \in K$ which map to the same element of L such that $\mathfrak m_A = \sqrt{(a)}$.*

Proof. By Lemma 20.10 we may replace A by $A/(\bigcap \mathfrak p)$ and assume that A is a reduced ring (some details omitted). We may also replace K by the image of $K \to L$. Then K is a reduced ring. The map $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is surjective and closed (details omitted). Hence $\mathrm{Spec}(K)$ is a finite discrete space. It follows that K is a finite product of fields.

Let $\mathfrak p_j$, $j = 1, \ldots, m$ be the minimal primes of A. Set $L_j = f.f.(A_j)$ so that $L = \prod_{j=1,\ldots,m} L_j$. Let A_j be the normalization of $A/\mathfrak p_j$. Then A_j is a semi-local Dedekind domain with at least one maximal ideal, see Algebra, Lemma 116.14. Let n be the sum of the numbers of maximal ideals in $A_1, \ldots, A_m$. For such a maximal ideal $\mathfrak m \subset A_j$ we consider the function

$$v_{\mathfrak m} : L \to L_j \to \mathbf{Z} \cup \{\infty\}$$

where the second arrow is the discrete valuation corresponding to the discrete valuation ring $(A_j)_{\mathfrak m}$ extended by mapping 0 to ∞. In this way we obtain n functions $v_1, \ldots, v_n : L \to \mathbf{Z} \cup \{\infty\}$. We will find an element $x \in K$ such that $v_i(x) < 0$ for all $i = 1, \ldots, n$.

First we claim that for each i there exists an element $x \in K$ with $v_i(x) < 0$. Namely, suppose that v_i corresponds to $\mathfrak m \subset A_j$. If $v_i(x) \geq 0$ for all $x \in K$, then K maps into $(A_j)_{\mathfrak m}$ inside of $L_j = f.f.(A_j)$. The image of K in L_j is a field over L_j is algebraic by Algebra, Lemma 35.16. Combined we get a contradiction with Algebra, Lemma 48.7.

Suppose we have found an element $x \in K$ such that $v_1(x) < 0, \ldots, v_r(x) < 0$ for some $r < n$. If $v_{r+1}(x) < 0$, then x works for $r + 1$. If not, then choose some $y \in K$ with $v_{r+1}(y) < 0$ as is possible by the result of the previous paragraph. After replacing x by x^n for some $n > 0$, we may assume $v_i(x) < v_i(y)$ for $i = 1, \ldots, r$. Then $v_j(x+y) = v_j(x) < 0$ for $j = 1, \ldots, r$ by properties of valuations and similarly $v_{r+1}(x + y) = v_{r+1}(y) < 0$. Arguing by induction, we find $x \in K$ with $v_i(x) < 0$ for $i = 1, \ldots, n$.

In particular, the element $x \in K$ has nonzero projection in each factor of K (recall that K is a finite product of fields and if some component of x was zero, then one of the values $v_i(x)$ would be ∞). Hence x is invertible and $x^{-1} \in K$ is an element with $\infty > v_i(x^{-1}) > 0$ for all i. It follows from Lemma 20.5 that for some $e < 0$ the element $x^e \in K$ maps to an element of $\mathfrak{m}_A/\mathfrak{p}_j \subset A/\mathfrak{p}_j$ for all $j = 1, \ldots, m$. Observe that the cokernel of the map $\mathfrak{m}_A \to \prod \mathfrak{m}_A/\mathfrak{p}_j$ is annihilated by a power of $\mathfrak{m}_A$. Hence after replacing e by a more negative e, we find an element $a \in \mathfrak{m}_A$ whose image in $\mathfrak{m}_A/\mathfrak{p}_j$ is equal to the image of x^e. The pair (a, x^e) satisfies the conclusions of the lemma. $\square$

Lemma 20.12. *Let A be a ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be a finite set of a primes of A. Let $S = A \setminus \bigcup \mathfrak{p}_i$. Then S is a multiplicative system and $S^{-1}A$ is a semi-local ring whose maximal ideals correspond to the maximal elements of the set $\{\mathfrak{p}_i\}$.*

Proof. If $a, b \in A$ and $a, b \in S$, then $a, b \notin \mathfrak{p}_i$ hence $ab \notin \mathfrak{p}_i$, hence $ab \in S$. Also $1 \in S$. Thus S is a multiplicative subset of A. By the description of $\mathrm{Spec}(S^{-1}A)$ in Algebra, Lemma 16.5 and by Algebra, Lemma 14.2 we see that the primes of $S^{-1}A$ correspond to the primes of A contained in one of the $\mathfrak{p}_i$. Hence the maximal ideals of $S^{-1}A$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. $\square$

21. One dimensional Noetherian schemes

Some material leading up to a discussion of algebraic curves.

Lemma 21.1. *Let X be a scheme all of whose local rings are Noetherian of dimension ≤ 1. Let $U \subset X$ be a retrocompact open. Denote $j : U \to X$ the inclusion morphism. Then $R^p j_* \mathcal{F} = 0$, $p > 0$ for every quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$.*

Proof. We may check the vanishing of $R^p j_* \mathcal{F}$ at stalks. Formation of $R^q j_*$ commutes with flat base change, see Cohomology of Schemes, Lemma 5.2. Thus we may assume that X is the spectrum of a Noetherian local ring of dimension ≤ 1. In this case X has a closed points x and finitely many other points $x_1, \ldots, x_n$ which specialize to x but not each other (see Algebra, Lemma 30.6). If $x \in U$, then $U = X$ and the result is clear. If not, then $U = \{x_1, \ldots, x_r\}$ for some r after possibly renumbering the points. Then U is affine (Schemes, Lemma 11.7). Thus the result by Cohomology of Schemes, Lemma 2.3. $\square$

Lemma 21.2. *Let X be an affine scheme all of whose local rings are Noetherian of dimension ≤ 1. Then any quasi-compact open $U \subset X$ is affine.*

Proof. Denote $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. By Lemma 21.1 the higher direct images $R^p j_* \mathcal{F}$ are zero. The $\mathcal{O}_X$-module $j_* \mathcal{F}$ is quasi-coherent (Schemes, Lemma 24.1). Hence it has vanishing higher cohomology groups by Cohomology of Schemes, Lemma 2.2. By the Leray spectral sequence Cohomology, Lemma 14.6 we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$. Thus U is affine, for example by Cohomology of Schemes, Lemma 3.1. $\square$

Lemma 21.3. *Let X be a scheme. Let $U \subset X$ be an open. Assume*

1. *U is a retrocompact open of X,*
2. *$X \setminus U$ is discrete, and*
3. *for $x \in X \setminus U$ the local ring $\mathcal{O}_{X,x}$ is Noetherian of dimension ≤ 1.*

Then (1) there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a section s such that $U = X_s$ and (2) the map $Pic(X) \to Pic(U)$ is surjective.

Proof. Let $X \setminus U = \{x_i; i \in I\}$. Choose affine opens $U_i \subset X$ with $x_i \in X$ and $x_j \notin U_i$ for $j \neq i$. This is possible by condition (2). Say $U_i = \text{Spec}(A_i)$. Let $\mathfrak{m}_i \subset A_i$ be the maximal ideal corresponding to x_i. By our assumption on the local rings there are only a finite number of prime ideals $\mathfrak{q} \subset \mathfrak{m}_i$, $\mathfrak{q} \neq \mathfrak{m}_i$ (see Algebra, Lemma 30.6). Thus by prime avoidance (Algebra, Lemma 14.2) we can find $f_i \in \mathfrak{m}_i$ not contained in any of those primes. Then $V(f_i) = \{\mathfrak{m}_i\} \amalg Z_i$ for some closed subset $Z_i \subset U_i$ because Z_i is a retrocompact open subset of $V(f_i)$ closed under specialization, see Algebra, Lemma 40.7. After shrinking U_i we may assume $V(f_i) = \{x_i\}$. Then

$$\mathcal{U} : X = U \cup \bigcup U_i$$

is an open covering of X. Consider the 2-cocycle with values in $\mathcal{O}_X^*$ given by f_i on $U \cap U_i$ and by f_i/f_j on $U_i \cap U_j$. This defines a line bundle $\mathcal{L}$ such that the section s defined by 1 on U and f_i on U_i is as in the statement of the lemma.

Let $\mathcal{N}$ be an invertible $\mathcal{O}_U$-module. Let N_i be the invertible $(A_i)_{f_i}$ module such that $\mathcal{N}|_{U \cap U_i}$ is equal to $\tilde{N}_i$. Observe that $(A_{\mathfrak{m}_i})_{f_i}$ is an Artinian ring (as a dimension zero Noetherian ring, see Algebra, Lemma 59.4). Thus it is a product of local rings (Algebra, Lemma 51.6) and hence has trivial Picard group. Thus, after shrinking U_i (i.e., after replacing A_i by $(A_i)_g$ for some $g \in A_i$, $g \notin \mathfrak{m}_i$) we can assume that $N_i = (A_i)_{f_i}$, i.e., that $\mathcal{N}|_{U \cap U_i}$ is trivial. In this case it is clear how to extend $\mathcal{N}$ to an invertible sheaf over X (by extending it by a trivial invertible module over each U_i). $\square$

Lemma 21.4. *Let X be an integral separated scheme. Let $U \subset X$ be a nonempty affine open such that $X \setminus U$ is a finite set of points $x_1, \ldots, x_r$ with $\mathcal{O}_{X,x_i}$ Noetherian of dimension 1. Then there exists a globally generated invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a section s such that $U = X_s$.*

Proof. Say $U = \text{Spec}(A)$ and let K be the fraction field of X. Write $B_i = \mathcal{O}_{X,x_i}$ and $\mathfrak{m}_i = \mathfrak{m}_{x_i}$. Since $x_i \notin U$ we see that the open $U \times_X \text{Spec}(B_i)$ of $\text{Spec}(B_i)$ has only one point, i.e., $U \times_X \text{Spec}(B_i) = \text{Spec}(K)$. Since X is separated, we find that $\text{Spec}(K)$ is a closed subscheme of $U \times \text{Spec}(B_i)$, i.e., the map $A \otimes B_i \to K$ is a surjection. By Lemma 20.8 we can find a nonzero $f \in A$ such that $f^{-1} \in \mathfrak{m}_i$ for $i = 1, \ldots, r$. Pick opens $x_i \in U_i \subset X$ such that $f^{-1} \in \mathcal{O}(U_i)$. Then

$$\mathcal{U} : X = U \cup \bigcup U_i$$

is an open covering of X. Consider the 2-cocycle with values in $\mathcal{O}_X^*$ given by f on $U \cap U_i$ and by 1 on $U_i \cap U_j$. This defines a line bundle $\mathcal{L}$ with two sections:

(1) a section s defined by 1 on U and f^{-1} on U_i is as in the statement of the lemma, and
(2) a section t defined by f on U and 1 on U_i.

Note that $X_t \supset U_1 \cup \ldots \cup U_r$. Hence s, t generate $\mathcal{L}$ and the lemma is proved. $\square$

Lemma 21.5. *Let X be a quasi-compact scheme. If for every $x \in X$ there exists a pair $(\mathcal{L}, s)$ consisting of a globally generated invertible sheaf $\mathcal{L}$ and a global section s such that $x \in X_s$ and X_s is affine, then X has an ample invertible sheaf.*

Proof. Since X is quasi-compact we can find a finite collection $(\mathcal{L}_i, s_i)$, $i = 1, \ldots, n$ of pairs such that X_{s_i} is affine and $X = \bigcup X_{s_i}$. Again because X is quasi-compact we can find, for each i, a finite collection of sections $t_{i,j}$, $j = 1, \ldots, m_i$ such that $X = \bigcup X_{t_{i,j}}$. Set $t_{i,0} = s_i$. Consider the invertible sheaf

$$\mathcal{L} = \mathcal{L}_1 \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}_n$$

and the global sections

$$\tau_J = t_{1,j_1} \otimes \ldots \otimes t_{n,j_n}$$

By Properties, Lemma 24.4 the open X_{τ_J} is affine as soon as $j_i = 0$ for some i. It is a simple matter to see that these opens cover X. Hence $\mathcal{L}$ is ample by definition. $\square$

Lemma 21.6. *Let X be a Noetherian integral separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Choose an affine open covering $X = U_1 \cup \ldots \cup U_n$. Since X is Noetherian, each of the sets $X \setminus U_i$ is finite. Thus by Lemma 21.4 we can find a pair $(\mathcal{L}_i, s_i)$ consisting of a globally generated invertible sheaf $\mathcal{L}_i$ and a global section s_i such that $U_i = X_{s_i}$. We conclude that X has an ample invertible sheaf by Lemma 21.5. $\square$

Lemma 21.7. *Let X be a scheme. Let $Z_1, \ldots, Z_n \subset X$ be closed subschemes. Let $\mathcal{L}_i$ be an invertible sheaf on Z_i. Assume that*

1. *X is reduced,*
2. *$X = \bigcup Z_i$ set theoretically, and*
3. *$Z_i \cap Z_j$ is a discrete topological space for $i \neq j$.*

Then there exists an invertible sheaf $\mathcal{L}$ on X whose restriction to Z_i is $\mathcal{L}_i$. Moreover, if we are given sections $s_i \in \Gamma(Z_i, \mathcal{L}_i)$ which are nonvanishing at the points of $Z_i \cap Z_j$, then we can choose $\mathcal{L}$ such that there exists a $s \in \Gamma(X, \mathcal{L})$ with $s|_{Z_i} = s_i$ for all i.

Proof. Set $T = \bigcup_{i \neq j} Z_i \cap Z_j$. As X is reduced we have

$$X \setminus T = \bigcup (Z_i \setminus T)$$

as schemes. Assumption (3) implies T is a discrete subset of X. Thus for each $t \in T$ we can find an open $U_t \subset X$ with $t \in U_t$ but $t' \notin U_t$ for $t' \in T$, $t' \neq t$. By shrinking U_t if necessary, we may assume that there exist isomorphisms $\varphi_{t,i} : \mathcal{L}_i|_{U_t \cap Z_i} \to \mathcal{O}_{U_t \cap Z_i}$. Furthermore, for each i choose an open covering

$$Z_i \setminus T = \bigcup\nolimits_j U_{ij}$$

such that there exist isomorphisms $\varphi_{i,j} : \mathcal{L}_i|_{U_{ij}} \cong \mathcal{O}_{U_{ij}}$. Observe that

$$\mathcal{U} : X = \bigcup U_t \cup \bigcup U_{ij}$$

is an open covering of X. We claim that we can use the isomorphisms $\varphi_{t,i}$ and $\varphi_{i,j}$ to define a 2-cocycle with values in $\mathcal{O}_X^*$ for this covering that defines $\mathcal{L}$ as in the statement of the lemma.

Namely, if $i \neq i'$, then $U_{i,j} \cap U_{i',j'} = \emptyset$ and there is nothing to do. For $U_{i,j} \cap U_{i,j'}$ we have $\mathcal{O}_X(U_{i,j} \cap U_{i,j'}) = \mathcal{O}_{Z_i}(U_{i,j} \cap U_{i,j'})$ by the first remark of the proof. Thus the transition function for $\mathcal{L}_i$ (more precisely $\varphi_{i,j} \circ \varphi_{i,j'}^{-1}$) defines the value of our cocycle on this intersection. For $U_t \cap U_{i,j}$ we can do the same thing as before. Finally, for $t \neq t'$ we have

$$U_t \cap U_{t'} = \coprod (U_t \cap U_{t'}) \cap Z_i$$

and moreover the intersections $U_t \cap U_{t'} \cap Z_i$ is contained in $Z_t i \setminus T$. Hence by the same reasoning as before we see that

$$\mathcal{O}_X(U_t \cap U_{t'}) = \prod \mathcal{O}_{Z_i}(U_t \cap U_{t'} \cap Z_i)$$

and we can use the transition functions fpr $\mathcal{L}_i$ (more precisely $\varphi_{t,i} \circ \varphi_{t',i}^{-1}$) to define the value of our cocycle on $U_t \cap U_{t'}$. This finishes the proof of existence of $\mathcal{L}$.

Given sections s_i as in the last assertion of the lemma, in the argument above, we choose U_t such that $s_i|_{U_t \cap Z_i}$ is nonvanishing and we choose $\varphi_{t,i}$ such that $\varphi_{t,i}(s_i|_{U_t \cap Z_i}) = 1$. Then using 1 over U_t and $\varphi_{i,j}(s_i|_{U_{i,j}})$ over $U_{i,j}$ will define a section of $\mathcal{L}$ which restricts to s_i over Z_i. □

Remark 21.8. Let A be a reduced ring. Let I, J be ideals of A such that $V(I) \cup V(J) = \mathrm{Spec}(A)$. Set $B = A/J$. Then $I \to IB$ is an isomorphism of A-modules. Namely, we have $IB = I + J/J = I/(I \cap J)$ and $I \cap J$ is zero because A is reduced and $\mathrm{Spec}(A) = V(I) \cup V(J) = V(I \cap J)$. Thus for any projective A-module P we also have $IP = I(P/JP)$.

Lemma 21.9. *Let X be a Noetherian reduced separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Let Z_i, $i = 1, \ldots, n$ be the irreducible components of X. We view these as reduced closed subschemes of X. By Lemma 21.6 there exist ample invertible sheaves $\mathcal{L}_i$ on Z_i. Set $T = \bigcup_{i \neq j} Z_i \cap Z_j$. As X is Noetherian of dimension 1, the set T is finite and consists of closed points of X. For each i we may, possibly after replacing $\mathcal{L}_i$ by a power, choose $s_i \in \Gamma(Z_i, \mathcal{L}_i)$ such that $(Z_i)_{s_i}$ is affine and contains $T \cap Z_i$, see Properties, Lemma 27.6.

By Lemma 21.7 we can find an invertible sheaf $\mathcal{L}$ on X and $s \in \Gamma(X, \mathcal{L})$ such that $(\mathcal{L}, s)|_{Z_i} = (\mathcal{L}_i, s_i)$. Observe that X_s contains T and is set theoretically equal to the affine closed subschemes $(Z_i)_{s_i}$. Thus it is affine by Limits, Lemma 10.3. To finish the proof, it suffices to find for every $x \in X$, $x \notin T$ an integer $m > 0$ and a section $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that X_t is affine and $x \in X_t$. Since $x \notin T$ we see that $x \in Z_i$ for some unique i, say $i = 1$. Let $Z \subset X$ be the reduced closed subscheme whose underlying topological space is $Z_2 \cup \ldots \cup Z_n$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Denote that $\mathcal{I}_1 \subset \mathcal{O}_{Z_1}$ the inverse image of this ideal sheaf under the inclusion morphism $Z_1 \to X$. Observe that

$$\Gamma(X, \mathcal{I}\mathcal{L}^{\otimes m}) = \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$$

see Remark 21.8. Thus it suffices to find $m > 0$ and $t \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$ with $x \in (Z_1)_t$ affine. Since $\mathcal{L}_1$ is ample and since x is not in $Z_1 \cap T = V(\mathcal{I}_1)$ we can find a section $t_1 \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m_1})$ with $x \in (Z_1)_{t_1}$, see Properties, Proposition 24.14. Since $\mathcal{L}_1$ is ample we can find a section $t_2 \in \Gamma(Z_1, \mathcal{L}_1^{\otimes m_2})$ with $x \in (Z_1)_{t_2}$ and $(Z_1)_{t_2}$ affine, see Properties, Definition 24.1. Set $m = m_1 + m_2$ and $t = t_1 t_2$. Then $t \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$ with $x \in (Z_1)_t$ by construction and $(Z_1)_t$ is affine by Properties, Lemma 24.4. □

Lemma 21.10. *Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism on underlying topological spaces. If the underlying topological space of X is Noetherian and $\dim(X) \leq 1$, then $Pic(X) \to Pic(Z)$ is surjective.*

Proof. Consider the short exact sequence

$$0 \to (1 + \mathcal{I})^* \to \mathcal{O}_X^* \to \mathcal{O}_Z^* \to 0$$

of sheaves of abelian groups on X. Since $\dim(X) \leq 1$ we see that $H^2(X, \mathcal{F}) = 0$ for any abelian sheaf $\mathcal{F}$, see Cohomology, Proposition 21.6. Hence the map $H^1(X, \mathcal{O}_X^*) \to H^1(Z, \mathcal{O}_Z^*)$ is surjective. This proves the lemma by Cohomology, Lemma 6.1. $\square$

Proposition 21.11. *Let X be a Noetherian separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Let $Z \subset X$ be the reduction of X. By Lemma 21.9 the scheme Z has an ample invertible sheaf. Thus by Lemma 21.10 there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X whose restriction to Z is ample. Then $\mathcal{L}$ is ample by an application of Cohomology of Schemes, Lemma 14.5. $\square$

Remark 21.12. In fact, if X is a scheme whose reduction is a Noetherian separated scheme of dimension 1, then X has an ample invertible sheaf. The argument to prove this is the same as the proof of Proposition 21.11 except one uses Limits, Lemma 10.4 instead of Cohomology of Schemes, Lemma 14.5.

22. Finding affine opens

We continue the discussion started in Properties, Section 27. It turns out that we can find affines containing a finite given set of codimension 1 points on a separated scheme. See Proposition 22.7.

We will improve on the following lemma in Descent, Lemma 21.4.

Lemma 22.1. *Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If*

1. *f is separated,*
2. *there is an open covering $X = \bigcup U_i$ such that $f|_{U_i} : U_i \to X$ is an open immersion, and*
3. *if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,*

then f is an open immersion.

Proof. Suppose that $y = f(x) = f(x')$. Pick a specialization $y_0 \rightsquigarrow y$ where y_0 is a generic point of an irreducible component of Y. Since f is locally on the source an isomorphism we can pick specializations $x_0 \rightsquigarrow x$ and $x'_0 \rightsquigarrow x'$ mapping to $y_0 \rightsquigarrow y$. Note that $x_0, x'_0 \in X^0$. Hence $x_0 = x'_0$ by assumption (3). As f is separated we conclude that $x = x'$. Thus f is an open immersion. $\square$

Lemma 22.2. *Let $X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. If*

1. *$\mathcal{O}_{X,x} = \mathcal{O}_{S,s}$,*
2. *X is reduced,*
3. *$X \to S$ is of finite type, and*
4. *S has finitely many irreducible components,*

then there exists an open neighbourhood U of x such that $f|_U$ is an open immersion.

Proof. We may remove the (finitely many) irreducible components of S which do not contain s. We may replace S by an affine open neighbourhood of s. We may replace X by an affine open neighbourhood of x. Say $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$. Let $\mathfrak{q} \subset B$, resp. $\mathfrak{p} \subset A$ be the prime ideal corresponding to x, resp. s. As A is a reduced and all of the minimal primes of A are contained in $\mathfrak{p}$ we see that $A \subset A_\mathfrak{p}$. As $X \to S$ is of finite type, B is of finite type over A. Let $b_1, \ldots, b_n \in B$ be elements which generate B over A Since $A_\mathfrak{p} = B_\mathfrak{q}$ we can find $f \in A$, $f \notin \mathfrak{p}$ and $a_i \in A$ such that b_i and a_i/f have the same image in $B_\mathfrak{q}$. Thus we can find $g \in B$, $g \notin \mathfrak{q}$ such that $g(fb_i - a_i) = 0$ in B. It follows that the image of $A_f \to B_{fg}$ contains the images of $b_1, \ldots, b_n$, in particular also the image of g. Choose $n \geq 0$ and $f' \in A$ such that f'/f^n maps to the image of g in B_{fg}. Since $A_\mathfrak{p} = B_\mathfrak{q}$ we see that $f' \notin \mathfrak{p}$. We conclude that $A_{ff'} \to B_{fg}$ is surjective. Finally, as $A_{ff'} \subset A_\mathfrak{p} = B_\mathfrak{q}$ (see above) the map $A_{ff'} \to B_{fg}$ is injective, hence an isomorphism. $\square$

Lemma 22.3. *Let $f : T \to X$ be a morphism of schemes. Let X^0, resp. T^0 denote the sets of generic points of irreducible components. Let $t_1, \ldots, t_m \in T$ be a finite set of points with images $x_j = f(t_j)$. If*

1. *T is affine,*
2. *X is quasi-separated,*
3. *X^0 is finite*
4. *$f(T^0) \subset X^0$ and $f : T^0 \to X^0$ is injective, and*
5. *$\mathcal{O}_{X,x_j} = \mathcal{O}_{T,t_j}$,*

then there exists an affine open of X containing $x_1, \ldots, x_r$.

Proof. Using Limits, Proposition 10.2 there is an immediate reduction to the case where X and T are reduced. Details omitted.

Assume X and T are reduced. We may write $T = \lim_{i \in I} T_i$ as a directed limit of schemes of finite presentation over X with affine transition morphisms, see Limits, Lemma 6.1. Pick $i \in I$ such that T_i is affine, see Limits, Lemma 3.10. Say $T_i = \text{Spec}(R_i)$ and $T = \text{Spec}(R)$. Let $R' \subset R$ be the image of $R_i \to R$. Then $T' = \text{Spec}(R')$ is affine, reduced, of finite type over X, and $T \to T'$ dominant. For $j = 1, \ldots, r$ let $t'_j \in T'$ be the image of t_j. Consider the local ring maps

$$\mathcal{O}_{X,x_j} \to \mathcal{O}_{T',t'_j} \to \mathcal{O}_{T,t_j}$$

Denote $(T')^0$ the set of generic points of irreducible components of T'. Let $\xi \rightsquigarrow t'_j$ be a specialization with $\xi \in (T')^0$. As $T \to T'$ is dominant we can choose $\eta \in T^0$ mapping to ξ (warning: a priori we do not know that η specializes to t_j). Assumption (3) applied to η tells us that the image θ of ξ in X corresponds to a minimal prime of $\mathcal{O}_{X,x_j}$. Lifting ξ via the isomorphism of (5) we obtain a specialization $\eta' \rightsquigarrow t_j$ with $\eta' \in X^0$ mapping to $\theta \rightsquigarrow x_j$. The injectivity of (4) shows that $\eta = \eta'$. Thus every minimal prime of $\mathcal{O}_{T',t'_j}$ lies below a minimal prime of $\mathcal{O}_{T,t_j}$. We conclude that $\mathcal{O}_{T',t'_j} \to \mathcal{O}_{T,t_j}$ is injective, hence both maps above are isomorphisms.

By Lemma 22.2 there exists an open $U \subset T'$ containing all the points t'_j such that $U \to X$ is a local isomorphism as in Lemma 22.1. By that lemma we see that $U \to X$ is an open immersion. Finally, by Properties, Lemma 27.5 we can find an open $W \subset U \subset T'$ containing all the t'_j. The image of W in X is the desired affine open. $\square$

Lemma 22.4. *Let X be an integral separated scheme. Let $x_1, \ldots, x_r \in X$ be a finite set of points such that $\mathcal{O}_{X,x_i}$ is Noetherian of dimension ≤ 1. Then there exists an affine open subscheme of X containing all of $x_1, \ldots, x_r$.*

Proof. Let K be the field of rational functions of X. Set $A_i = \mathcal{O}_{X,x_i}$. Then $A_i \subset K$ and K is the fraction field of A_i. Since X is separated, and $x_i \neq x_j$ there cannot be a valuation ring $\mathcal{O} \subset K$ dominating both A_i and A_j. Namely, considering the diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathcal{O}) \arrow[r] \arrow[d] & \mathrm{Spec}(A_1) \arrow[d] \\ \mathrm{Spec}(A_2) \arrow[r] & X \end{tikzcd}$$

and applying the valuative criterion of separatedness (Schemes, Lemma 22.1) we would get $x_i = x_j$. Thus we see by Lemma 20.3 that $A_i \otimes A_j \to K$ is surjective for all $i \neq j$. By Lemma 20.7 we see that $A = A_1 \cap \ldots \cap A_r$ is a Noetherian semi-local rings with exactly r maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ such that $A_i = A_{\mathfrak{m}_i}$. Moreover,

$$\mathrm{Spec}(A) = \mathrm{Spec}(A_1) \cup \ldots \cup \mathrm{Spec}(A_r)$$

is an open covering and the intersection of any two pieces of this covering is $\mathrm{Spec}(K)$. Thus the given morphisms $\mathrm{Spec}(A_i) \to X$ glue to a morphism of schemes

$$\mathrm{Spec}(A) \longrightarrow X$$

mapping $\mathfrak{m}_i$ to x_i and inducing isomorphisms of local rings. Thus the result follows from Lemma 22.3. $\square$

Lemma 22.5. *Let A be a ring, $I \subset A$ an ideal, $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ primes of A, and $\overline{f} \in A/I$ an element. If $I \not\subset \mathfrak{p}_i$ for all i, then there exists an $f \in A$, $f \notin \mathfrak{p}_i$ which maps to $\overline{f}$ in A/I.*

Proof. We may assume there are no inclusion relations among the $\mathfrak{p}_i$ (by removing the smaller primes). First pick any $f \in A$ lifting $\overline{f}$. Let S be the set $s \in \{1, \ldots, r\}$ such that $f \in \mathfrak{p}_s$. If S is empty we are done. If not, consider the ideal $J = I \prod_{i \notin S} \mathfrak{p}_i$. Note that J is not contained in $\mathfrak{p}_s$ for $s \in S$ because there are no inclusions among the $\mathfrak{p}_i$ and because I is not contained in any $\mathfrak{p}_i$. Hence we can choose $g \in J$, $g \notin \mathfrak{p}_s$ for $s \in S$ by Algebra, Lemma 14.2. Then $f + g$ is a solution to the problem posed by the lemma. $\square$

Lemma 22.6. *Let X be a scheme. Let $T \subset X$ be finite set of points. Assume*

(1) *X has finitely many irreducible components $Z_1, \ldots, Z_t$, and*
(2) *$Z_i \cap T$ is contained in an affine open of the reduced induced subscheme corresponding to Z_i.*

Then there exists an affine open subscheme of X containing T.

Proof. Using Limits, Proposition 10.2 there is an immediate reduction to the case where X is reduced. Details omitted. In the rest of the proof we endow every closed subset of X with the induced reduced closed subscheme structure.

We argue by induction that we can find an affine open $U \subset Z_1 \cup \ldots \cup Z_r$ containing $T \cap (Z_1 \cup \ldots \cup Z_r)$. For $r = 1$ this holds by assumption. Say $r > 1$ and let

$U \subset Z_1 \cup \ldots \cup Z_{r-1}$ be an affine open containing $T \cap (Z_1 \cup \ldots \cup Z_{r-1})$. Let $V \subset X_r$ be an affine open containing $T \cap Z_r$ (exists by assumption). Then $U \cap V$ contains $T \cap (Z_1 \cup \ldots \cup Z_{r-1}) \cap Z_r$. Hence

$$\Delta = (U \cap Z_r) \setminus (U \cap V)$$

does not contain any element of T. Note that Δ is a closed subset of U. By prime avoidance (Algebra, Lemma 14.2), we can find a standard open U' of U containing $T \cap U$ and avoiding Δ, i.e., $U' \cap Z_r \subset U \cap V$. After replacing U by U' we may assume that $U \cap V$ is closed in U.

Using that by the same arguments as above also the set $\Delta' = (U \cap (Z_1 \cup \ldots \cup Z_{r-1})) \setminus (U \cap V)$ does not contain any element of T we find a $h \in \mathcal{O}(V)$ such that $D(h) \subset V$ contains $T \cap V$ and such that $U \cap D(h) \subset U \cap V$. Using that $U \cap V$ is closed in U we can use Lemma 22.5 to find an element $g \in \mathcal{O}(U)$ whose restriction to $U \cap V$ equals the restriction of h to $U \cap V$ and such that $T \cap U \subset D(g)$. Then we can replace U by $D(g)$ and V by $D(h)$ to reach the situation where $U \cap V$ is closed in both U and V. In this case the scheme $U \cup V$ is affine by Limits, Lemma 10.3. This proves the induction step and thereby the lemma. □

Here is a conclusion we can draw from the material above.

Proposition 22.7. *Let X be a separated scheme such that every quasi-compact open has a finite number of irreducible components. Let $x_1, \ldots, x_r \in X$ be points such that $\mathcal{O}_{X,x_i}$ is Noetherian of dimension ≤ 1. Then there exists an affine open subscheme of X containing all of $x_1, \ldots, x_r$.*

Proof. We can replace X by a quasi-compact open containing $x_1, \ldots, x_r$ hence we may assume that X has finitely many irreducible components. By Lemma 22.6 we reduce to the case where X is integral. This case is Lemma 22.4. □

23. Curves

In the stacks project we will use the following as our definition of a curve.

Definition 23.1. Let k be a field. A *curve* is a variety of dimension 1 over k.

Two standard examples of curves over k are the affine line $\mathbf{A}^1_k$ and the projective line $\mathbf{P}^1_k$. The scheme $X = \text{Spec}(k[x,y]/(f))$ is a curve if and only if $f \in k[x,y]$ is irreducible.

Our definition of a curve has the same problems as our definition of a variety, see the discussion following Definition 3.1. Moreover, it means that every curve comes with a specified field of definition. For example $X = \text{Spec}(\mathbf{C}[x])$ is a curve over $\mathbf{C}$ but we can also view it as a curve over $\mathbf{R}$. The scheme $\text{Spec}(\mathbf{Z})$ isn't a curve, even though the schemes $\text{Spec}(\mathbf{Z})$ and $\mathbf{A}^1_{\mathbf{F}_p}$ behave similarly in many respects.

Lemma 23.2. *Let X be an irreducible scheme of dimension > 0 over a field k. Let $x \in X$ be a closed point. The open subscheme $X \setminus \{x\}$ is not proper over k.*

Proof. Namely, choose a specialization $x' \leadsto x$ with $x' \neq x$ (for example take x' to be the generic point). By Schemes, Lemma 20.4 there exists a morphism $\text{Spec}(A) \to X$ where A is a valuation ring such that the generic point of A maps to x' and the closed point of $\text{Spec}(A)$ maps to x. Clearly the morphism $\text{Spec}(f.f.(A)) \to X \setminus \{x\}$ does not extend to a morphism $\text{Spec}(A) \to X \setminus \{x\}$. Hence the valuative criterion

(Schemes, Proposition 20.6) shows that $X \to \mathrm{Spec}(k)$ is not universally closed, hence not proper. $\square$

Lemma 23.3. *Let X be a separated finite type scheme over a field k. If $\dim(X) \leq 1$ then X is H-quasi-projective over k.*

Proof. By Proposition 21.11 the scheme X has an ample invertible sheaf $\mathcal{L}$. By Morphisms, Lemma 40.3 we see that X is isomorphic to a locally closed subscheme of $\mathbf{P}^n_k$ over $\mathrm{Spec}(k)$. This is the definiton of being H-quasi-projective over k, see Morphisms, Definition 41.1. $\square$

Lemma 23.4. *Let X be a proper scheme over a field k. If $\dim(X) \leq 1$ then X is H-projective over k.*

Proof. By Lemma 23.3 we see that X is a locally closed subscheme of $\mathbf{P}^n_k$ for some field k. Since X is proper over k it follows that X is a closed subscheme of $\mathbf{P}^n_k$ (Morphisms, Lemma 42.7). $\square$

Observe that if an affine scheme X over k is proper over k then X is finite over k (Morphisms, Lemma 44.7) and hence has dimension 0 (Algebra, Lemma 51.2 and Proposition 59.6). Hence a scheme of dimension > 0 over k cannot be both affine and proper over k. Thus the possibilities in the following lemma are mutually exclusive.

Lemma 23.5. *Let X be a curve over k. Then either X is an affine scheme or X is H-projective over k.*

Proof. By Lemma 23.3 we may assume X is a locally closed subscheme of $\mathbf{P}^n_k$ for some n. Let $\overline{X} \subset \mathbf{P}^n_k$ be the scheme theoretic image of $X \to \mathbf{P}^n_k$, see Morphisms, Definition 6.2 and the description in Morphisms, Lemma 7.7. Since X is irreducible, we see that $\overline{X}$ is irreducible. Then $\dim(X) = 1 \Rightarrow \dim(\overline{X}) = 1$ for example by looking at the generic point, see Lemma 13.3. As $\overline{X}$ is Noetherian, it then follows that $\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ is a finite set of closed points. By Lemma 21.4 we can find a globally generated invertible sheaf $\mathcal{L}$ on $\overline{X}$ and a section $s \in \Gamma(\overline{X}, \mathcal{L})$ such that $X = \overline{X}_s$.

Choose a basis $s = s_0, s_1, \ldots, s_m$ of the finite dimensional k-vector space $\Gamma(\overline{X}, \mathcal{L})$ (Cohomology of Schemes, Lemma 17.4). We obtain a corresponding morphism

$$f : \overline{X} \longrightarrow \mathbf{P}^m_k$$

such that the inverse image of $D_+(T_0)$ is X, see Constructions, Lemma 13.1. In particular, f is non-constant, i.e., $\mathrm{Im}(f)$ has more than one point. A topological argument shows that f maps the generic point η of $\overline{X}$ to a nonclosed point of $\mathbf{P}^n_k$. Hence if $y \in \mathbf{P}^n_k$ is a closed point, then $f^{-1}(\{y\})$ is a closed set of $\overline{X}$ not containing η, hence finite. By Cohomology of Schemes, Lemma 19.2[2] we conclude that f is finite. Hence $X = f^{-1}(D_+(T_0))$ is affine. $\square$

[2] One can avoid using this lemma which relies on the theorem of formal functions. Namely, $\overline{X}$ is projective hence it suffices to show a proper morphism $f : X \to Y$ with finite fibres between quasi-projective schemes over k is finite. To do this, one chooses an affine open of X containing the fibre of f over a point y using that any finite set of points of a quasi-projective scheme over k is contained in an affine. Shrinking Y to a small affine neighbourhood of y one reduces to the case of a proper morphism between affines. Such a morphism is finite by Morphisms, Lemma 44.7.

The following lemma combined with Lemma 23.2 tells us that given a separated scheme of finite type over k, then $X \setminus Z$ is affine, whenever the closed subset Z meets every irreducible component of X.

Lemma 23.6. *Let X be a separated scheme of finite type over k. If $\dim(X) \leq 1$ and no irreducible component of X is proper of dimension 1, then X is affine.*

Proof. Let $X = \bigcup X_i$ be the decomposition of X into irreducible components. We think of X_i as an integral scheme (using the reduced induced scheme structure, see Schemes, Definition 12.5). In particular X_i is a singleton (hence affine) or a curve hence affine by Lemma 23.5. Then $\coprod X_i \to X$ is finite surjective and $\coprod X_i$ is affine. Thus we see that X is affine by Cohomology of Schemes, Lemma 13.3. $\square$

24. Generically finite morphisms

In this section we revisit the notion of a generically finite morphism of schemes as studied in Morphisms, Section 47.

Lemma 24.1. *Let $f : X \to Y$ be locally of finite type. Let $y \in Y$ be a point such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Assume in addition one of the following conditions is satisfied*

1. *for every generic point η of an irreducible component of X the field extension $\kappa(\eta) \supset \kappa(f(\eta))$ is finite (or algebraic),*
2. *for every generic point η of an irreducible component of X such that $f(\eta) \leadsto y$ the field extension $\kappa(\eta) \supset \kappa(f(\eta))$ is finite (or algebraic),*
3. *f is quasi-finite at every generic point of an irreducible component of X,*
4. *Y is locally Noetherian and f is quasi-finite at a dense set of points of X,*
5. *add more here.*

Then f is quasi-finite at every point of X lying over y.

Proof. Condition (4) implies X is locally Noetherian (Morphisms, Lemma 16.6). The set of points at which morphism is quasi-finite is open (Morphisms, Lemma 49.2). A dense open of a locally Noetherian scheme contains all generic point of irreducible components, hence (4) implies (3). Condition (3) implies condition (1) by Morphisms, Lemma 21.5. Condition (1) implies condition (2). Thus it suffices to prove the lemma in case (2) holds.

Assume (2) holds. Recall that $\mathrm{Spec}(\mathcal{O}_{Y,y})$ is the set of points of Y specializing to y, see Schemes, Lemma 13.2. Combined with Morphisms, Lemma 21.13 this shows we may replace Y by $\mathrm{Spec}(\mathcal{O}_{Y,y})$. Thus we may assume $Y = \mathrm{Spec}(B)$ where B is a Noetherian local ring of dimension ≤ 1 and y is the closed point.

Let $X = \bigcup X_i$ be the irreducible components of X viewed as reduced closed subschemes. If we can show each fibre $X_{i,y}$ is a discrete space, then $X_y = \bigcup X_{i,y}$ is discrete as well and we conclude that $X \to Y$ is quasi-finite at all points of X_y by Morphisms, Lemma 21.6. Thus we may assume X is an integral scheme.

If $X \to Y$ maps the generic point η of X to y, then X is the spectrum of a finite extension of $\kappa(y)$ and the result is true. Assume that X maps η to a point corresponding to a minimal prime $\mathfrak{p}$ of B different from $\mathfrak{m}_B$. We obtain a factorization $X \to \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(B)$. Let $x \in X$ be a point lying over y. By the dimension formula (Morphisms, Lemma 31.1) we have

$$\dim(\mathcal{O}_{X,x}) \leq \dim(B/\mathfrak{q}) + \mathrm{trdeg}_{\kappa(\mathfrak{q})}(R(X)) - \mathrm{trdeg}_{\kappa(y)}\kappa(x)$$

We know that $\dim(B/\mathfrak{q}) = 1$, that the generic point of X is not equal to x and specializes to x and that $R(X)$ is algebraic over $\kappa(\mathfrak{q})$. Thus we get

$$1 \leq 1 - \text{trdeg}_{\kappa(y)}\kappa(x)$$

Hence every point x of X_y is closed in X_y by Morphisms, Lemma 21.2 and hence $X \to Y$ is quasi-finite at every point x of X_y by Morphisms, Lemma 21.6 (which also implies that X_y is a discrete topological space). □

Lemma 24.2. *Let $f : X \to Y$ be a proper morphism. Let $y \in Y$ be a point such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Assume in addition one of the following conditions is satisfied*

1. *for every generic point η of an irreducible component of X the field extension $\kappa(\eta) \supset \kappa(f(\eta))$ is finite (or algebraic),*
2. *for every generic point η of an irreducible component of X such that $f(\eta) \rightsquigarrow y$ the field extension $\kappa(\eta) \supset \kappa(f(\eta))$ is finite (or algebraic),*
3. *f is quasi-finite at every generic point of X,*
4. *Y is locally Noetherian and f is quasi-finite at a dense set of points of X,*
5. *add more here.*

Then there exists an open neighbourhood $V \subset Y$ of y such that $f^{-1}(V) \to V$ is finite.

Proof. By Lemma 24.1 the morphism f is quasi-finite at every point of the fibre X_y. Hence X_y is a discrete topological space (Morphisms, Lemma 21.6). As f is proper the fibre X_y is quasi-compact, i.e., finite. Thus we can apply Cohomology of Schemes, Lemma 19.2 to conclude. □

25. Other chapters

References

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