

EXAMPLES OF STACKS

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1. Introduction

This is a discussion of examples of stacks in algebraic geometry. Some of them are algebraic stacks, some are not. We will discuss which are algebraic stacks in a later chapter. This means that in this chapter we mainly worry about the descent conditions. See [Vis04] for example.

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 1 for an explanation.

2. Notation

In this chapter we fix a suitable big fppf site Sch_{fppf} as in Topologies, Definition 7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf} . We will always work relative to a base S contained in Sch_{fppf} . And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 7.8. The absolute case can be recovered by taking $S = \text{Spec}(\mathbf{Z})$.

3. Examples of stacks

We first give some important examples of stacks over $(Sch/S)_{fppf}$.

4. Quasi-coherent sheaves

We define a category $QCoh$ as follows:

- (1) An object of $QCoh$ is a pair (X, \mathcal{F}) , where X/S is an object of $(Sch/S)_{fppf}$, and \mathcal{F} is a quasi-coherent \mathcal{O}_X -module, and
- (2) a morphism $(f, \varphi) : (Y, \mathcal{G}) \rightarrow (X, \mathcal{F})$ is a pair consisting of a morphism $f : Y \rightarrow X$ of schemes over S and an f -map (see Sheaves, Section 26) $\varphi : \mathcal{F} \rightarrow \mathcal{G}$.
- (3) The composition of morphisms

$$(Z, \mathcal{H}) \xrightarrow{(g, \psi)} (Y, \mathcal{G}) \xrightarrow{(f, \phi)} (X, \mathcal{F})$$

is $(f \circ g, \psi \circ \phi)$ where $\psi \circ \phi$ is the composition of f -maps.

Thus $QCoh$ is a category and

$$p : QCoh \rightarrow (Sch/S)_{fppf}, \quad (X, \mathcal{F}) \mapsto X$$

is a functor. Note that the fibre category of $QCoh$ over a scheme X is just the category $QCoh(\mathcal{O}_X)$ of quasi-coherent \mathcal{O}_X -modules. We remark for later use that given $(X, \mathcal{F}), (Y, \mathcal{G}) \in \text{Ob}(QCoh)$ we have

$$(4.0.1) \quad \text{Mor}_{QCoh}((Y, \mathcal{G}), (X, \mathcal{F})) = \coprod_{f \in \text{Mor}_S(Y, X)} \text{Mor}_{QCoh(\mathcal{O}_Y)}(f^* \mathcal{F}, \mathcal{G})$$

See the discussion on f -maps of modules in Sheaves, Section 26.

The category $QCoh$ is not a stack over $(Sch/S)_{fppf}$ because its collection of objects is a proper class. On the other hand we will see that it does satisfy all the axioms of a stack. We will get around the set theoretical issue in Section 5.

Lemma 4.1. *A morphism $(f, \varphi) : (Y, \mathcal{G}) \rightarrow (X, \mathcal{F})$ of $QCoh$ is strongly cartesian if and only if the map φ induces an isomorphism $f^* \mathcal{F} \rightarrow \mathcal{G}$.*

Proof. Let $(X, \mathcal{F}) \in \text{Ob}(QCoh)$. Let $f : Y \rightarrow X$ be a morphism of $(Sch/S)_{fppf}$. Note that there is a canonical f -map $c : \mathcal{F} \rightarrow f^* \mathcal{F}$ and hence we get a morphism $(f, c) : (Y, f^* \mathcal{F}) \rightarrow (X, \mathcal{F})$. We claim that (f, c) is strongly cartesian. Namely, for any object (Z, \mathcal{H}) of $QCoh$ we have

$$\begin{aligned} \text{Mor}_{QCoh}((Z, \mathcal{H}), (Y, f^* \mathcal{F})) &= \coprod_{g \in \text{Mor}_S(Z, Y)} \text{Mor}_{QCoh(\mathcal{O}_Z)}(g^* f^* \mathcal{F}, \mathcal{H}) \\ &= \coprod_{g \in \text{Mor}_S(Z, Y)} \text{Mor}_{QCoh(\mathcal{O}_Z)}((f \circ g)^* \mathcal{F}, \mathcal{H}) \\ &= \text{Mor}_{QCoh}((Z, \mathcal{H}), (X, \mathcal{F})) \times_{\text{Mor}_S(Z, X)} \text{Mor}_S(Z, Y) \end{aligned}$$

where we have used Equation (4.0.1) twice. This proves that the condition of Categories, Definition 31.1 holds for (f, c) , and hence our claim is true. Now by

Categories, Lemma 31.2 we see that isomorphisms are strongly cartesian and compositions of strongly cartesian morphisms are strongly cartesian which proves the “if” part of the lemma. For the converse, note that given (X, \mathcal{F}) and $f : Y \rightarrow X$, if there exists a strongly cartesian morphism lifting f with target (X, \mathcal{F}) then it has to be isomorphic to (f, c) (see discussion following Categories, Definition 31.1). Hence the “only if” part of the lemma holds. \square

Lemma 4.2. *The functor $p : QCoh \rightarrow (Sch/S)_{fppf}$ satisfies conditions (1), (2) and (3) of Stacks, Definition 4.1.*

Proof. It is clear from Lemma 4.1 that $QCoh$ is a fibred category over $(Sch/S)_{fppf}$. Given covering $\mathcal{U} = \{X_i \rightarrow X\}_{i \in I}$ of $(Sch/S)_{fppf}$ the functor

$$QCoh(\mathcal{O}_T) \longrightarrow DD(\mathcal{U})$$

is fully faithful and essentially surjective, see Descent, Proposition 5.2. Hence Stacks, Lemma 4.2 applies to show that $QCoh$ satisfies all the axioms of a stack. \square

5. The stack of finitely generated quasi-coherent sheaves

It turns out that we can get a stack of quasi-coherent sheaves if we only consider finite type quasi-coherent modules. Let us denote

$$p_{fg} : QCoh_{fg} \rightarrow (Sch/S)_{fppf}$$

the full subcategory of $QCoh$ over $(Sch/S)_{fppf}$ consisting of pairs (T, \mathcal{F}) such that \mathcal{F} is a quasi-coherent \mathcal{O}_T -module of finite type.

Lemma 5.1. *The functor $p_{fg} : QCoh_{fg} \rightarrow (Sch/S)_{fppf}$ satisfies conditions (1), (2) and (3) of Stacks, Definition 4.1.*

Proof. We will verify assumptions (1), (2), (3) of Stacks, Lemma 4.3 to prove this. By Lemma 4.1 a morphism $(Y, \mathcal{G}) \rightarrow (X, \mathcal{F})$ is strongly cartesian if and only if it induces an isomorphism $f^*\mathcal{F} \rightarrow \mathcal{G}$. By Modules, Lemma 9.2 the pullback of a finite type \mathcal{O}_X -module is of finite type. Hence assumption (1) of Stacks, Lemma 4.3 holds. Assumption (2) holds trivially. Finally, to prove assumption (3) we have to show: If \mathcal{F} is a quasi-coherent \mathcal{O}_X -module and $\{f_i : X_i \rightarrow X\}$ is an fppf covering such that each $f_i^*\mathcal{F}$ is of finite type, then \mathcal{F} is of finite type. Considering the restriction of \mathcal{F} to an affine open of X this reduces to the following algebra statement: Suppose that $R \rightarrow S$ is a finitely presented, faithfully flat ring map and M an R -module. If $M \otimes_R S$ is a finitely generated S -module, then M is a finitely generated R -module. A stronger form of the algebra fact can be found in Algebra, Lemma 80.2. \square

Lemma 5.2. *Let (X, \mathcal{O}_X) be a ringed space.*

- (1) *The category of finite type \mathcal{O}_X -modules has a set of isomorphism classes.*
- (2) *The category of finite type quasi-coherent \mathcal{O}_X -modules has a set of isomorphism classes.*

Proof. Part (2) follows from part (1) as the category in (2) is a full subcategory of the category in (1). Consider any open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$. Denote $j_i : U_i \rightarrow X$ the inclusion maps. Consider any map $r : I \rightarrow \mathbb{N}$. If \mathcal{F} is an \mathcal{O}_X -module whose restriction to U_i is generated by at most $r(i)$ sections from $\mathcal{F}(U_i)$, then \mathcal{F} is a quotient of the sheaf

$$\mathcal{H}_{\mathcal{U}, r} = \bigoplus_{i \in I} j_{i,*} \mathcal{O}_{U_i}^{\oplus r(i)}$$

By definition, if \mathcal{F} is of finite type, then there exists some open covering with \mathcal{U} whose index set is $I = X$ such that this condition is true. Hence it suffices to show that there is a set of possible choices for \mathcal{U} (obvious), a set of possible choices for $r : I \rightarrow \mathbf{N}$ (obvious), and a set of possible quotient modules of $\mathcal{H}_{\mathcal{U},r}$ for each \mathcal{U} and r . In other words, it suffices to show that given an \mathcal{O}_X -module \mathcal{H} there is at most a set of isomorphism classes of quotients. This last assertion becomes obvious by thinking of the kernels of a quotient map $\mathcal{H} \rightarrow \mathcal{F}$ as being parametrized by a subset of the power set of $\prod_{U \subset X \text{ open}} \mathcal{H}(U)$. \square

Lemma 5.3. *There exists a subcategory $QCoh_{fg,small} \subset QCoh_{fg}$ with the following properties:*

- (1) *the inclusion functor $QCoh_{fg,small} \rightarrow QCoh_{fg}$ is fully faithful and essentially surjective, and*
- (2) *the functor $p_{fg,small} : QCoh_{fg,small} \rightarrow (Sch/S)_{fppf}$ turns $QCoh_{fg,small}$ into a stack over $(Sch/S)_{fppf}$.*

Proof. We have seen in Lemmas 5.1 and 5.2 that $p_{fg} : QCoh_{fg} \rightarrow (Sch/S)_{fppf}$ satisfies (1), (2) and (3) of Stacks, Definition 4.1 as well as the additional condition (4) of Stacks, Remark 4.9. Hence we obtain $QCoh_{fg,small}$ from the discussion in that remark. \square

We will often perform the replacement

$$QCoh_{fg} \rightsquigarrow QCoh_{fg,small}$$

without further remarking on it, and by abuse of notation we will simply denote $QCoh_{fg}$ this replacement.

Remark 5.4. Note that the whole discussion in this section works if we want to consider those quasi-coherent sheaves which are locally generated by at most κ sections, for some infinite cardinal κ , e.g., $\kappa = \aleph_0$.

6. Algebraic spaces

We define a category $Spaces$ as follows:

- (1) An object of $Spaces$ is a morphism $X \rightarrow U$ of algebraic spaces over S , where U is representable by an object of $(Sch/S)_{fppf}$, and
- (2) a morphism $(f, g) : (X \rightarrow U) \rightarrow (Y \rightarrow V)$ is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow & & \downarrow \\ U & \xrightarrow{\quad g \quad} & V \end{array}$$

of morphisms of algebraic spaces over S .

Thus $Spaces$ is a category and

$$p : Spaces \rightarrow (Sch/S)_{fppf}, \quad (X \rightarrow U) \mapsto U$$

is a functor. Note that the fibre category of $Spaces$ over a scheme U is just the category $Spaces/U$ of algebraic spaces over U (see Topologies on Spaces, Section 2). Hence we sometimes think of an object of $Spaces$ as a pair X/U consisting of

a scheme U and an algebraic space X over U . We remark for later use that given $(X/U), (Y/V) \in \text{Ob}(\text{Spaces})$ we have

$$(6.0.1) \quad \text{Mor}_{\text{Spaces}}(X/U, Y/V) = \coprod_{g \in \text{Mor}_S(U, V)} \text{Mor}_{\text{Spaces}/U}(X, U \times_{g, V} Y)$$

The category Spaces is almost, but not quite a stack over $(\text{Sch}/S)_{\text{fppf}}$. The problem is a set theoretical issue as we will explain below.

Lemma 6.1. *A morphism $(f, g) : X/U \rightarrow Y/V$ of Spaces is strongly cartesian if and only if the map f induces an isomorphism $X \rightarrow U \times_{g, V} Y$.*

Proof. Let $Y/V \in \text{Ob}(\text{Spaces})$. Let $g : U \rightarrow V$ be a morphism of $(\text{Sch}/S)_{\text{fppf}}$. Note that the projection $p : U \times_{g, V} Y \rightarrow Y$ gives rise a morphism $(p, g) : U \times_{g, V} Y/U \rightarrow Y/V$ of Spaces . We claim that (p, g) is strongly cartesian. Namely, for any object Z/W of Spaces we have

$$\begin{aligned} \text{Mor}_{\text{Spaces}}(Z/W, U \times_{g, V} Y/U) &= \coprod_{h \in \text{Mor}_S(W, U)} \text{Mor}_{\text{Spaces}/W}(Z, W \times_{h, U} U \times_{g, V} Y) \\ &= \coprod_{h \in \text{Mor}_S(W, U)} \text{Mor}_{\text{Spaces}/W}(Z, W \times_{g \circ h, V} Y) \\ &= \text{Mor}_{\text{Spaces}}(Z/W, Y/V) \times_{\text{Mor}_S(W, V)} \text{Mor}_S(W, U) \end{aligned}$$

where we have used Equation (6.0.1) twice. This proves that the condition of Categories, Definition 31.1 holds for (p, g) , and hence our claim is true. Now by Categories, Lemma 31.2 we see that isomorphisms are strongly cartesian and compositions of strongly cartesian morphisms are strongly cartesian which proves the “if” part of the lemma. For the converse, note that given Y/V and $g : U \rightarrow V$, if there exists a strongly cartesian morphism lifting g with target Y/V then it has to be isomorphic to (p, g) (see discussion following Categories, Definition 31.1). Hence the “only if” part of the lemma holds. \square

Lemma 6.2. *The functor $p : \text{Spaces} \rightarrow (\text{Sch}/S)_{\text{fppf}}$ satisfies conditions (1) and (2) of Stacks, Definition 4.1.*

Proof. It follows from Lemma 6.1 that Spaces is a fibred category over $(\text{Sch}/S)_{\text{fppf}}$ which proves (1). Suppose that $\{U_i \rightarrow U\}_{i \in I}$ is a covering of $(\text{Sch}/S)_{\text{fppf}}$. Suppose that X, Y are algebraic spaces over U . Finally, suppose that $\varphi_i : X_{U_i} \rightarrow Y_{U_i}$ are morphisms of Spaces/U_i such that φ_i and φ_j restrict to the same morphisms $X_{U_i \times_U U_j} \rightarrow Y_{U_i \times_U U_j}$ of algebraic spaces over $U_i \times_U U_j$. To prove (2) we have to show that there exists a unique morphism $\varphi : X \rightarrow Y$ over U whose base change to U_i is equal to φ_i . As a morphism from X to Y is the same thing as a map of sheaves this follows directly from Sites, Lemma 25.1. \square

Remark 6.3. Ignoring set theoretical difficulties¹ Spaces also satisfies descent for objects and hence is a stack. Namely, we have to show that given

- (1) an fppf covering $\{U_i \rightarrow U\}_{i \in I}$,
- (2) for each $i \in I$ an algebraic space X_i/U_i , and
- (3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \rightarrow U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$,

¹The difficulty is not that Spaces is a proper class, since by our definition of an algebraic space over S there is only a set worth of isomorphism classes of algebraic spaces over S . It is rather that arbitrary disjoint unions of algebraic spaces may end up being too large, hence lie outside of our chosen “partial universe” of sets.

there exists an algebraic space X/U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij} . First, note that by Sites, Lemma 25.2 there exists a sheaf X on $(Sch/U)_{fppf}$ recovering the X_i and the φ_{ij} . Then by Bootstrap, Lemma 11.1 we see that X is an algebraic space (if we ignore the set theoretic condition of that lemma). We will use this argument in the next section to show that if we consider only algebraic spaces of finite type, then we obtain a stack.

7. The stack of finite type algebraic spaces

It turns out that we can get a stack of spaces if we only consider spaces of finite type. Let us denote

$$p_{ft} : Spaces_{ft} \rightarrow (Sch/S)_{fppf}$$

the full subcategory of $Spaces$ over $(Sch/S)_{fppf}$ consisting of pairs X/U such that $X \rightarrow U$ is a morphism of finite type.

Lemma 7.1. *The functor $p_{ft} : Spaces_{ft} \rightarrow (Sch/S)_{fppf}$ satisfies the conditions (1), (2) and (3) of Stacks, Definition 4.1.*

Proof. We are going to write this out in ridiculous detail (which may make it hard to see what is going on).

We have seen in Lemma 6.1 that a morphism $(f, g) : X/U \rightarrow Y/V$ of $Spaces$ is strongly cartesian if the induced morphism $f : X \rightarrow U \times_V Y$ is an isomorphism. Note that if $Y \rightarrow V$ is of finite type then also $U \times_V Y \rightarrow U$ is of finite type, see Morphisms of Spaces, Lemma 23.3. So if $(f, g) : X/U \rightarrow Y/V$ of $Spaces$ is strongly cartesian in $Spaces$ and Y/V is an object of $Spaces_{ft}$ then automatically also X/U is an object of $Spaces_{ft}$, and of course (f, g) is also strongly cartesian in $Spaces_{ft}$. In this way we conclude that $Spaces_{ft}$ is a fibred category over $(Sch/S)_{fppf}$. This proves (1).

The argument above also shows that the inclusion functor $Spaces_{ft} \rightarrow Spaces$ transforms strongly cartesian morphisms into strongly cartesian morphisms. In other words $Spaces_{ft} \rightarrow Spaces$ is a 1-morphism of fibred categories over $(Sch/S)_{fppf}$.

Let $U \in \text{Ob}((Sch/S)_{fppf})$. Let X, Y be algebraic spaces of finite type over U . By Stacks, Lemma 2.3 we obtain a map of presheaves

$$Mor_{Spaces_{ft}}(X, Y) \longrightarrow Mor_{Spaces}(X, Y)$$

which is an isomorphism as $Spaces_{ft}$ is a full subcategory of $Spaces$. Hence the left hand side is a sheaf, because in Lemma 6.2 we showed the right hand side is a sheaf. This proves (2).

To prove condition (3) of Stacks, Definition 4.1 we have to show the following: Given

- (1) a covering $\{U_i \rightarrow U\}_{i \in I}$ of $(Sch/S)_{fppf}$,
- (2) for each $i \in I$ an algebraic space X_i of finite type over U_i , and
- (3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \rightarrow U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$,

there exists an algebraic space X of finite type over U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij} . By Sites, Lemma 25.2 there exists a sheaf X on $(Sch/U)_{fppf}$ recovering the X_i and the φ_{ij} . Then by Bootstrap, Lemma 11.4 we see that X is an algebraic space. By Descent on Spaces, Lemma 10.8 we see that $X \rightarrow U$ is of finite type which concludes the proof. \square

Lemma 7.2. *There exists a subcategory $\text{Spaces}_{ft,small} \subset \text{Spaces}_{ft}$ with the following properties:*

- (1) *the inclusion functor $\text{Spaces}_{ft,small} \rightarrow \text{Spaces}_{ft}$ is fully faithful and essentially surjective, and*
- (2) *the functor $p_{ft,small} : \text{Spaces}_{ft,small} \rightarrow (Sch/S)_{fppf}$ turns $\text{Spaces}_{ft,small}$ into a stack over $(Sch/S)_{fppf}$.*

Proof. We have seen in Lemmas 7.1 that $p_{fg} : QCoh_{fg} \rightarrow (Sch/S)_{fppf}$ satisfies (1), (2) and (3) of Stacks, Definition 4.1. The additional condition (4) of Stacks, Remark 4.9 holds because every algebraic space X over S is of the form U/R for $U, R \in \text{Ob}((Sch/S)_{fppf})$, see Spaces, Lemma 9.1. Thus there is only a set worth of isomorphism classes of objects. Hence we obtain $\text{Spaces}_{ft,small}$ from the discussion in that remark. \square

We will often perform the replacement

$$\text{Spaces}_{ft} \rightsquigarrow \text{Spaces}_{ft,small}$$

without further remarking on it, and by abuse of notation we will simply denote Spaces_{ft} this replacement.

Remark 7.3. Note that the whole discussion in this section works if we want to consider those algebraic spaces X/U which are locally of finite type such that the inverse image in X of an affine open of U can be covered by countably many affines. If needed we can also introduce the notion of a morphism of κ -type (meaning some bound on the number of generators of ring extensions and some bound on the cardinality of the affines over a given affine in the base) where κ is a cardinal, and then we can produce a stack

$$\text{Spaces}_{\kappa} \longrightarrow (Sch/S)_{fppf}$$

in exactly the same manner as above (provided we make sure that Sch is large enough depending on κ).

8. Examples of stacks in groupoids

The examples above are examples of stacks which are not stacks in groupoids. In the rest of this chapter we give algebraic geometric examples of stacks in groupoids.

9. The stack associated to a sheaf

Let $F : (Sch/S)_{fppf}^{opp} \rightarrow \text{Sets}$ be a presheaf. We obtain a category fibred in sets

$$p_F : \mathcal{S}_F \rightarrow (Sch/S)_{fppf},$$

see Categories, Example 36.5. This is a stack in sets if and only if F is a sheaf, see Stacks, Lemma 6.3.

10. The stack in groupoids of finitely generated quasi-coherent sheaves

Let $p : QCoh_{fg} \rightarrow (Sch/S)_{fppf}$ be the stack introduced in Section 5 (using the abuse of notation introduced there). We can turn this into a stack in groupoids $p' : QCoh'_{fg} \rightarrow (Sch/S)_{fppf}$ by the procedure of Categories, Lemma 33.3, see Stacks, Lemma 5.3. In this particular case this simply means $QCoh'_{fg}$ has the same objects as $QCoh_{fg}$ but the morphisms are pairs $(f, g) : (U, \mathcal{F}) \rightarrow (U', \mathcal{F}')$ where g is an isomorphism $g : f^* \mathcal{F}' \rightarrow \mathcal{F}$.

11. The stack in groupoids of finite type algebraic spaces

Let $p : \mathcal{S}paces_{ft} \rightarrow (Sch/S)_{fppf}$ be the stack introduced in Section 7 (using the abuse of notation introduced there). We can turn this into a stack in groupoids $p' : \mathcal{S}paces'_{ft} \rightarrow (Sch/S)_{fppf}$ by the procedure of Categories, Lemma 33.3, see Stacks, Lemma 5.3. In this particular case this simply means $\mathcal{S}paces'_{ft}$ has the same objects as $\mathcal{S}paces_{ft}$, i.e., finite type morphisms $X \rightarrow U$ where X is an algebraic space over S and U is a scheme over S . But the morphisms $(f, g) : X/U \rightarrow Y/V$ are now commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow & & \downarrow \\ U & \xrightarrow{\quad g \quad} & V \end{array}$$

which are cartesian.

12. Quotient stacks

Let (U, R, s, t, c) be a groupoid in algebraic spaces over S . In this case the quotient stack

$$[U/R] \longrightarrow (Sch/S)_{fppf}$$

is a stack in groupoids by construction, see Groupoids in Spaces, Definition 19.1. It is even the case that the *Isom*-sheaves are representable by algebraic spaces, see Bootstrap, Lemma 11.5. These quotient stacks are of fundamental importance to the theory of algebraic stacks.

A special case of the construction above is the quotient stack

$$[X/G] \longrightarrow (Sch/S)_{fppf}$$

associated to a datum $(B, G/B, m, X/B, a)$. Here

- (1) B is an algebraic space over S ,
- (2) (G, m) is a group algebraic space over B ,
- (3) X is an algebraic space over B , and
- (4) $a : G \times_B X \rightarrow X$ is an action of G on X over B .

Namely, by Groupoids in Spaces, Definition 19.1 the stack in groupoids $[X/G]$ is the quotient stack $[X/G \times_B X]$ given above. It behooves us to spell out what the category $[X/G]$ really looks like. We will do this in Section 14.

13. Classifying torsors

We want to carefully explain a number of variants of what it could mean to study the stack of torsors for a group algebraic space G or a sheaf of groups \mathcal{G} .

13.1. Torsors for a sheaf of groups. Let \mathcal{G} be a sheaf of groups on $(Sch/S)_{fppf}$. For $U \in \text{Ob}((Sch/S)_{fppf})$ we denote $\mathcal{G}|_U$ the restriction of \mathcal{G} to $(Sch/U)_{fppf}$. We define a category $\mathcal{G}\text{-Torsors}$ as follows:

- (1) An object of $\mathcal{G}\text{-Torsors}$ is a pair (U, \mathcal{F}) where U is an object of $(Sch/S)_{fppf}$ and \mathcal{F} is a $\mathcal{G}|_U$ -torsor, see Cohomology on Sites, Definition 5.1.
- (2) A morphism $(U, \mathcal{F}) \rightarrow (V, \mathcal{H})$ is given by a pair (f, α) , where $f : U \rightarrow V$ is a morphism of schemes over S , and $\alpha : f^{-1}\mathcal{H} \rightarrow \mathcal{F}$ is an isomorphism of $\mathcal{G}|_U$ -torsors.

Thus $\mathcal{G}\text{-Torsors}$ is a category and

$$p : \mathcal{G}\text{-Torsors} \longrightarrow (Sch/S)_{fppf}, \quad (U, \mathcal{F}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{G}\text{-Torsors}$ over U is the category of $\mathcal{G}|_U$ -torsors which is a groupoid.

Lemma 13.2. *Up to a replacement as in Stacks, Remark 4.9 the functor*

$$p : \mathcal{G}\text{-Torsors} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects. Let $\{U_i \rightarrow U\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Suppose that for each i we are given a $\mathcal{G}|_{U_i}$ -torsor \mathcal{F}_i , and for each $i, j \in I$ an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \times_U U_j} \rightarrow \mathcal{F}_j|_{U_i \times_U U_j}$ of $\mathcal{G}|_{U_i \times_U U_j}$ -torsors satisfying a suitable cocycle condition on $U_i \times_U U_j \times_U U_k$. Then by Sites, Section 25 we obtain a sheaf \mathcal{F} on $(Sch/U)_{fppf}$ whose restriction to each U_i recovers \mathcal{F}_i as well as recovering the descent data. By the equivalence of categories in Sites, Lemma 25.3 the action maps $\mathcal{G}|_{U_i} \times \mathcal{F}_i \rightarrow \mathcal{F}_i$ glue to give a map $a : \mathcal{G}|_U \times \mathcal{F} \rightarrow \mathcal{F}$. Now we have to show that a is an action and that \mathcal{F} becomes a $\mathcal{G}|_U$ -torsor. Both properties may be checked locally, and hence follow from the corresponding properties of the actions $\mathcal{G}|_{U_i} \times \mathcal{F}_i \rightarrow \mathcal{F}_i$. This proves that descent for objects holds in $\mathcal{G}\text{-Torsors}$. Some details omitted. \square

13.3. Variant on torsors for a sheaf. The construction of Subsection 13.1 can be generalized slightly. Namely, let $\mathcal{G} \rightarrow \mathcal{B}$ be a map of sheaves on $(Sch/S)_{fppf}$ and let

$$m : \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \longrightarrow \mathcal{G}$$

be a group law on \mathcal{G}/\mathcal{B} . In other words, the pair (\mathcal{G}, m) is a group object of the topos $Sh((Sch/S)_{fppf})/\mathcal{B}$. See Sites, Section 29 for information regarding localizations of topoi. In this setting we can define a category $\mathcal{G}/\mathcal{B}\text{-Torsors}$ as follows (where we use the Yoneda embedding to think of schemes as sheaves):

- (1) An object of $\mathcal{G}/\mathcal{B}\text{-Torsors}$ is a triple (U, b, \mathcal{F}) where
 - (a) U is an object of $(Sch/S)_{fppf}$,
 - (b) $b : U \rightarrow \mathcal{B}$ is a section of \mathcal{B} over U , and
 - (c) \mathcal{F} is a $U \times_{b, \mathcal{B}} \mathcal{G}$ -torsor over U .
- (2) A morphism $(U, b, \mathcal{F}) \rightarrow (U', b', \mathcal{F}')$ is given by a pair (f, g) , where $f : U \rightarrow U'$ is a morphism of schemes over S such that $b = b' \circ f$, and $g : f^{-1}\mathcal{F}' \rightarrow \mathcal{F}$ is an isomorphism of $U \times_{b, \mathcal{B}} \mathcal{G}$ -torsors.

Thus $\mathcal{G}/\mathcal{B}\text{-Torsors}$ is a category and

$$p : \mathcal{G}/\mathcal{B}\text{-Torsors} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathcal{F}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{G}/\mathcal{B}\text{-Torsors}$ over U is the disjoint union over $b : U \rightarrow \mathcal{B}$ of the categories of $U \times_{b, \mathcal{B}} \mathcal{G}$ -torsors, hence is a groupoid.

In the special case $\mathcal{B} = S$ we recover the category $\mathcal{G}\text{-Torsors}$ introduced in Subsection 13.1.

Lemma 13.4. *Up to a replacement as in Stacks, Remark 4.9 the functor*

$$p : \mathcal{G}/\mathcal{B}\text{-Torsors} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. This proof is a repeat of the proof of Lemma 13.2. The reader is encouraged to read that proof first since the notation is less cumbersome. The most difficult part of the proof is to show that we have descent for objects. Let $\{U_i \rightarrow U\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Suppose that for each i we are given a pair (b_i, \mathcal{F}_i) consisting of a morphism $b_i : U_i \rightarrow \mathcal{B}$ and a $U_i \times_{b_i, \mathcal{B}} \mathcal{G}$ -torsor \mathcal{F}_i , and for each $i, j \in I$ we have $b_i|_{U_i \times_U U_j} = b_j|_{U_i \times_U U_j}$ and we are given an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \times_U U_j} \rightarrow \mathcal{F}_j|_{U_i \times_U U_j}$ of $(U_i \times_U U_j) \times_{\mathcal{B}} \mathcal{G}$ -torsors satisfying a suitable cocycle condition on $U_i \times_U U_j \times_U U_k$. Then by Sites, Section 25 we obtain a sheaf \mathcal{F} on $(Sch/U)_{fppf}$ whose restriction to each U_i recovers \mathcal{F}_i as well as recovering the descent data. By the sheaf axiom for \mathcal{B} the morphisms b_i come from a unique morphism $b : U \rightarrow \mathcal{B}$. By the equivalence of categories in Sites, Lemma 25.3 the action maps $(U_i \times_{b_i, \mathcal{B}} \mathcal{G}) \times_{U_i} \mathcal{F}_i \rightarrow \mathcal{F}_i$ glue to give a map $(U \times_{b, \mathcal{B}} \mathcal{G}) \times \mathcal{F} \rightarrow \mathcal{F}$. Now we have to show that this is an action and that \mathcal{F} becomes a $U \times_{b, \mathcal{B}} \mathcal{G}$ -torsor. Both properties may be checked locally, and hence follow from the corresponding properties of the actions on the \mathcal{F}_i . This proves that descent for objects holds in \mathcal{G}/\mathcal{B} -Torsors. Some details omitted. \square

13.5. Principal homogeneous spaces. Let B be an algebraic space over S . Let G be a group algebraic space over B . We define a category G -Principal as follows:

- (1) An object of G -Principal is a triple (U, b, X) where
 - (a) U is an object of $(Sch/S)_{fppf}$,
 - (b) $b : U \rightarrow B$ is a morphism over S , and
 - (c) X is a principal homogeneous G_U -space over U where $G_U = U \times_{b, B} G$. See Groupoids in Spaces, Definition 9.3.
- (2) A morphism $(U, b, X) \rightarrow (U', b', X')$ is given by a pair (f, g) , where $f : U \rightarrow U'$ is a morphism of schemes over B , and $g : X \rightarrow U \times_{f, U'} X'$ is an isomorphism of principal homogeneous G_U -spaces.

Thus G -Principal is a category and

$$p : G\text{-Principal} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, X) \longmapsto U$$

is a functor. Note that the fibre category of G -Principal over U is the disjoint union over $b : U \rightarrow B$ of the categories of principal homogeneous $U \times_{b, B} G$ -spaces, hence is a groupoid.

In the special case $S = B$ the objects are simply pairs (U, X) where U is a scheme over S , and X is a principal homogeneous G_U -space over U . Moreover, morphisms are simply cartesian diagrams

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & U' \end{array}$$

where g is G -equivariant.

Remark 13.6. We conjecture that up to a replacement as in Stacks, Remark 4.9 the functor

$$p : G\text{-Principal} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$. This would follow if one could show that given

- (1) a covering $\{U_i \rightarrow U\}_{i \in I}$ of $(Sch/S)_{fppf}$,

- (2) an group algebraic space H over U ,
- (3) for every i a principal homogeneous H_{U_i} -space X_i over U_i , and
- (4) H -equivariant isomorphisms $\varphi_{ij} : X_{i,U_i \times_U U_j} \rightarrow X_{j,U_i \times_U U_j}$ satisfying the cocycle condition,

there exists a principal homogeneous H -space X over U which recovers (X_i, φ_{ij}) . The technique of the proof of Bootstrap, Lemma 11.8 reduces this to a set theoretical question, so the reader who ignores set theoretical questions will “know” that the result is true. In <http://math.columbia.edu/~dejong/wordpress/?p=591> there is a suggestion as to how to approach this problem.

13.7. Variant on principal homogeneous spaces. Let S be a scheme. Let $B = S$. Let G be a group scheme over $B = S$. In this setting we can define a full subcategory $G\text{-Principal-Schemes} \subset G\text{-Principal}$ whose objects are pairs (U, X) where U is an object of $(Sch/S)_{fppf}$ and $X \rightarrow U$ is a principal homogeneous G -space over U which is representable, i.e., a scheme.

It is in general not the case that $G\text{-Principal-Schemes}$ is a stack in groupoids over $(Sch/S)_{fppf}$. The reason is that in general there really do exist principal homogeneous spaces which are not schemes, hence descent for objects will not be satisfied in general.

13.8. Torsors in fppf topology. Let B be an algebraic space over S . Let G be a group algebraic space over B . We define a category $G\text{-Torsors}$ as follows:

- (1) An object of $G\text{-Torsors}$ is a triple (U, b, X) where
 - (a) U is an object of $(Sch/S)_{fppf}$,
 - (b) $b : U \rightarrow B$ is a morphism, and
 - (c) X is an fppf G_U -torsor over U where $G_U = U \times_{b,B} G$.
 See Groupoids in Spaces, Definition 9.3.
- (2) A morphism $(U, b, X) \rightarrow (U', b', X')$ is given by a pair (f, g) , where $f : U \rightarrow U'$ is a morphism of schemes over B , and $g : X \rightarrow U \times_{f,U'} X'$ is an isomorphism of G_U -torsors.

Thus $G\text{-Torsors}$ is a category and

$$p : G\text{-Torsors} \longrightarrow (Sch/S)_{fppf}, \quad (U, a, X) \longmapsto U$$

is a functor. Note that the fibre category of $G\text{-Torsors}$ over U is the disjoint union over $b : U \rightarrow B$ of the categories of fppf $U \times_{b,B} G$ -torsors, hence is a groupoid.

In the special case $S = B$ the objects are simply pairs (U, X) where U is a scheme over S , and X is an fppf G_U -torsor over U . Moreover, morphisms are simply cartesian diagrams

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & U' \end{array}$$

where g is G -equivariant.

Lemma 13.9. *Up to a replacement as in Stacks, Remark 4.9 the functor*

$$p : G\text{-Torsors} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects, which is Bootstrap, Lemma 11.8. We omit the proof of axioms (1) and (2) of Stacks, Definition 5.1. \square

Lemma 13.10. *Let B be an algebraic space over S . Let G be a group algebraic space over B . Denote \mathcal{G} , resp. \mathcal{B} the algebraic space G , resp. B seen as a sheaf on $(Sch/S)_{fppf}$. The functor*

$$G\text{-Torsors} \longrightarrow \mathcal{G}/\mathcal{B}\text{-Torsors}$$

which associates to a triple (U, b, X) the triple (U, b, \mathcal{X}) where \mathcal{X} is X viewed as a sheaf is an equivalence of stacks in groupoids over $(Sch/S)_{fppf}$.

Proof. We will use the result of Stacks, Lemma 4.8 to prove this. The functor is fully faithful since the category of algebraic spaces over S is a full subcategory of the category of sheaves on $(Sch/S)_{fppf}$. Moreover, all objects (on both sides) are locally trivial torsors so condition (2) of the lemma referenced above holds. Hence the functor is an equivalence. \square

13.11. Variant on torsors in fppf topology. Let S be a scheme. Let $B = S$. Let G be a group scheme over $B = S$. In this setting we can define a full subcategory $G\text{-Torsors-Schemes} \subset G\text{-Torsors}$ whose objects are pairs (U, X) where U is an object of $(Sch/S)_{fppf}$ and $X \rightarrow U$ is an fppf G -torsor over U which is representable, i.e., a scheme.

It is in general not the case that $G\text{-Torsors-Schemes}$ is a stack in groupoids over $(Sch/S)_{fppf}$. The reason is that in general there really do exist fppf G -torsors which are not schemes, hence descent for objects will not be satisfied in general.

14. Quotients by group actions

At this point we have introduced enough notation that we can work out in more detail what the stacks $[X/G]$ of Section 12 look like.

Situation 14.1. Here

- (1) S is a scheme contained in Sch_{fppf} ,
- (2) B is an algebraic space over S ,
- (3) (G, m) is a group algebraic space over B ,
- (4) $\pi : X \rightarrow B$ is an algebraic space over B , and
- (5) $a : G \times_B X \rightarrow X$ is an action of G on X over B .

In this situation we construct a category $[[X/G]]^2$ as follows:

- (1) An object of $[[X/G]]$ consists of a quadruple $(U, b, P, \varphi : P \rightarrow X)$ where
 - (a) U is an object of $(Sch/S)_{fppf}$,
 - (b) $b : U \rightarrow B$ is a morphism over S ,
 - (c) P is an fppf G_U -torsor over U where $G_U = U \times_{b,B} G$, and

²The notation $[[X/G]]$ with double brackets serves to distinguish this category from the stack $[X/G]$ introduced earlier. In Proposition 14.3 we show that the two are canonically equivalent. Afterwards we will use the notation $[X/G]$ to indicate either.

- (d) $\varphi : P \rightarrow X$ is a G -equivariant morphism fitting into the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ U & \xrightarrow{b} & B \end{array}$$

- (2) A morphism of $[[X/G]]$ is a pair $(f, g) : (U, b, P, \varphi) \rightarrow (U', b', P', \varphi')$ where $f : U \rightarrow U'$ is a morphism of schemes over B and $g : P \rightarrow P'$ is a G -equivariant morphism over f which induces an isomorphism $P \cong U \times_{f, U'} P'$, and has the property that $\varphi = \varphi' \circ g$. In other words (f, g) fits into the following commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{g} & P' & & \\ \downarrow & & \downarrow & \searrow \varphi' & \\ U & \xrightarrow{f} & U' & & X \\ & \searrow b & \searrow b' & \searrow & \downarrow \\ & & & & B \end{array}$$

Thus $[[X/G]]$ is a category and

$$p : [[X/G]] \longrightarrow (Sch/S)_{fppf}, \quad (U, b, P, \varphi) \longmapsto U$$

is a functor. Note that the fibre category of $[[X/G]]$ over U is the disjoint union over $b \in \text{Mor}_S(U, B)$ of $U \times_{b, B} G$ -torsors P endowed with a G -equivariant morphism to X . Hence the fibre categories of $[[X/G]]$ are groupoids.

Note that the functor

$$[[X/G]] \longrightarrow G\text{-Torsors}, \quad (U, b, P, \varphi) \longmapsto (U, b, P)$$

is a 1-morphism of categories over $(Sch/S)_{fppf}$.

Lemma 14.2. *Up to a replacement as in Stacks, Remark 4.9 the functor*

$$p : [[X/G]] \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects. Suppose that $\{U_i \rightarrow U\}_{i \in I}$ is a covering in $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, b_i, P_i, \varphi_i)$ be objects of $[[X/G]]$ over U_i , and let $\varphi_{ij} : \text{pr}_0^* \xi_i \rightarrow \text{pr}_1^* \xi_j$ be a descent datum. This in particular implies that we get a descent datum on the triples (U_i, b_i, P_i) for the stack in groupoids $G\text{-Torsors}$ by applying the functor $[[X/G]] \rightarrow G\text{-Torsors}$ above. We have seen that $G\text{-Torsors}$ is a stack in groupoids (Lemma 13.9). Hence we may assume that $b_i = b|_{U_i}$ for some morphism $b : U \rightarrow B$, and that $P_i = U_i \times_U P$ for some fppf $G_U = U \times_{b, B} G$ -torsor P over U . The morphisms φ_i are compatible with the canonical descent datum on the restrictions $U_i \times_U P$ and hence define a morphism $\varphi : P \rightarrow X$. (For example you can use Sites, Lemma 25.3 or you can use Descent on Spaces, Lemma 6.2 to get φ .) This proves descent for objects. We omit the proof of axioms (1) and (2) of Stacks, Definition 5.1. \square

Proposition 14.3. *In Situation 14.1 there exists a canonical equivalence*

$$[X/G] \longrightarrow [[X/G]]$$

of stacks in groupoids over $(Sch/S)_{fppf}$.

Proof. We write this out in detail, to make sure that all the definitions work out in exactly the correct manner. Recall that $[X/G]$ is the quotient stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$, see Groupoids in Spaces, Definition 19.1. This means that $[X/G]$ is the stackification of the category fibred in groupoids $[X/_p G]$ associated to the functor

$$(Sch/S)_{fppf} \longrightarrow \text{Groupoids}, \quad U \longmapsto (X(U), G(U) \times_{B(U)} X(U), s, t, c)$$

where $s(g, x) = x$, $t(g, x) = a(g, x)$, and $c((g, x), (g', x')) = (m(g, g'), x')$. By the construction of Categories, Example 35.1 an object of $[X/_p G]$ is a pair (U, x) with $x \in X(U)$ and a morphism $(f, g) : (U, x) \rightarrow (U', x')$ of $[X/_p G]$ is given by a morphism of schemes $f : U \rightarrow U'$ and an element $g \in G(U)$ such that $a(g, x) = x' \circ f$. Hence we can define a 1-morphism of stacks in groupoids

$$F_p : [X/_p G] \longrightarrow [[X/G]]$$

by the following rules: On objects we set

$$F_p(U, x) = (U, \pi \circ x, G \times_{B, \pi \circ x} U, a \circ (\text{id}_G \times x))$$

This makes sense because the diagram

$$\begin{array}{ccccc} G \times_{B, \pi \circ x} U & \xrightarrow{\text{id}_G \times x} & G \times_{B, \pi} X & \xrightarrow{a} & X \\ \downarrow & & & & \downarrow \pi \\ U & \xrightarrow{\pi \circ x} & & & B \end{array}$$

commutes, and the two horizontal arrows are G -equivariant if we think of the fibre products as trivial G -torsors over U , resp. X . On morphisms $(f, g) : (U, x) \rightarrow (U', x')$ we set $F_p(f, g) = (f, R_g)$ where R_g denotes right translation by g . More precisely, the morphism of $F_p(f, g) : F_p(U, x) \rightarrow F_p(U', x')$ is given by the cartesian diagram

$$\begin{array}{ccc} G \times_{B, \pi \circ x} U & \xrightarrow{R_g} & G \times_{B, \pi \circ x'} U' \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & U' \end{array}$$

where R_g on T -valued points is given by

$$R_g(g', u) = (m(g', g), f(u))$$

To see that this works we have to verify that

$$a \circ (\text{id}_G \times x) = a \circ (\text{id}_G \times x') \circ R_g$$

which is true because the right hand side applied to the T -valued point (g', u) gives

$$\begin{aligned} a((\text{id}_G \times x')(m(g', g), f(u))) &= a(m(g', g), x'(f(u))) \\ &= a(g', a(g, x'(f(u)))) \\ &= a(g', x(u)) \end{aligned}$$

because $a(g, x) = x' \circ f$ as desired.

By the universal property of stackification from Stacks, Lemma 9.2 we obtain a canonical extension $F : [X/G] \rightarrow [[X/G]]$ of the 1-morphism F_p above. We first prove that F is fully faithful. To do this, since both source and target are stacks in groupoids, it suffices to prove that the *Isom*-sheaves are identified under F . Pick a scheme U and objects ξ, ξ' of $[X/G]$ over U . We want to show that

$$F : \text{Isom}_{[X/G]}(\xi, \xi') \longrightarrow \text{Isom}_{[[X/G]]}(F(\xi), F(\xi'))$$

is an isomorphism of sheaves. To do this it suffices to work locally on U , and hence we may assume that ξ, ξ' come from objects $(U, x), (U, x')$ of $[X/pG]$ over U ; this follows directly from the construction of the stackification, and it is also worked out in detail in Groupoids in Spaces, Section 23. Either by directly using the description of morphisms in $[X/pG]$ above, or using Groupoids in Spaces, Lemma 21.1 we see that in this case

$$\text{Isom}_{[X/G]}(\xi, \xi') = U \times_{(x, x'), X \times_S X, (s, t)} (G \times_B X)$$

A T -valued point of this fibre product corresponds to a pair (u, g) with $u \in U(T)$, and $g \in G(T)$ such that $a(g, x \circ u) = x' \circ u$. (Note that this implies $\pi \circ x \circ u = \pi \circ x' \circ u$.) On the other hand, a T -valued point of $\text{Isom}_{[[X/G]]}(F(\xi), F(\xi'))$ by definition corresponds to a morphism $u : T \rightarrow U$ such that $\pi \circ x \circ u = \pi \circ x' \circ u : T \rightarrow B$ and an isomorphism

$$R : G \times_{B, \pi \circ x \circ u} T \longrightarrow G \times_{B, \pi \circ x' \circ u} T$$

of trivial G_T -torsors compatible with the given maps to X . Since the torsors are trivial we see that $R = R_g$ (right multiplication) by some $g \in G(T)$. Compatibility with the maps $a \circ (1_G, x \circ u), a \circ (1_G, x' \circ u) : G \times_B T \rightarrow X$ is equivalent to the condition that $a(g, x \circ u) = x' \circ u$. Hence we obtain the desired equality of *Isom*-sheaves.

Now that we know that F is fully faithful we see that Stacks, Lemma 4.8 applies. Thus to show that F is an equivalence it suffices to show that objects of $[[X/G]]$ are fppf locally in the essential image of F . This is clear as fppf torsors are locally trivial, and hence we win. \square

Remark 14.4. Let S be a scheme. Let G be an abstract group. Let X be an algebraic space over S . Let $G \rightarrow \text{Aut}_S(X)$ be a group homomorphism. In this setting we can define $[[X/G]]$ similarly to the above as follows:

- (1) An object of $[[X/G]]$ consists of a triple $(U, P, \varphi : P \rightarrow X)$ where
 - (a) U is an object of $(\text{Sch}/S)_{\text{fppf}}$,
 - (b) P is a sheaf on $(\text{Sch}/U)_{\text{fppf}}$ which comes with an action of G that turns it into a torsor under the constant sheaf with value G , and
 - (c) $\varphi : P \rightarrow X$ is a G -equivariant map of sheaves.
- (2) A morphism $(f, g) : (U, P, \varphi) \rightarrow (U', P', \varphi')$ is given by a morphism of schemes $f : T \rightarrow T'$ and a G -equivariant isomorphism $g : P \rightarrow f^{-1}P'$ such that $\varphi = \varphi' \circ g$.

In exactly the same manner as above we obtain a functor

$$[[X/G]] \longrightarrow (\text{Sch}/S)_{\text{fppf}}$$

which turns $[[X/G]]$ into a stack in groupoids over $(\text{Sch}/S)_{\text{fppf}}$. The constant sheaf \underline{G} is (provided the cardinality of G is not too large) representable by G_S on $(\text{Sch}/S)_{\text{fppf}}$ and this version of $[[X/G]]$ is equivalent to the stack $[[X/G_S]]$ introduced above.

15. The Picard stack

Let S be a scheme. Let $\pi : X \rightarrow B$ be a morphism of algebraic spaces over S . We define a category $Pic_{X/B}$ as follows:

- (1) An object is a triple (U, b, \mathcal{L}) , where
 - (a) U is an object of $(Sch/S)_{fppf}$,
 - (b) $b : U \rightarrow B$ is a morphism over S , and
 - (c) \mathcal{L} is an invertible sheaf on the base change $X_U = U \times_{b,B} X$.
- (2) A morphism $(f, g) : (U, b, \mathcal{L}) \rightarrow (U', b', \mathcal{L}')$ is given by a morphism of schemes $f : U \rightarrow U'$ over B and an isomorphism $g : f^* \mathcal{L}' \rightarrow \mathcal{L}$.

The composition of $(f, g) : (U, b, \mathcal{L}) \rightarrow (U', b', \mathcal{L}')$ with $(f', g') : (U', b', \mathcal{L}') \rightarrow (U'', b'', \mathcal{L}'')$ is given by $(f \circ f', g \circ f'^*(g'))$. Thus we get a category $Pic_{X/B}$ and

$$p : Pic_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathcal{L}) \longmapsto U$$

is a functor. Note that the fibre category of $Pic_{X/B}$ over U is the disjoint union over $b \in \text{Mor}_S(U, B)$ of the categories of invertible sheaves on $X_U = U \times_{b,B} X$. Hence the fibre categories are groupoids.

Lemma 15.1. *Up to a replacement as in Stacks, Remark 4.9 the functor*

$$Pic_{X/B} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. As usual, the hardest part is to show descent for objects. To see this let $\{U_i \rightarrow U\}$ be a covering of $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, b_i, \mathcal{L}_i)$ be an object of $Pic_{X/B}$ lying over U , and let $\varphi_{ij} : \text{pr}_0^* \xi_i \rightarrow \text{pr}_1^* \xi_j$ be a descent datum. This implies in particular that the morphisms b_i are the restrictions of a morphism $b : U \rightarrow B$. Write $X_U = U \times_{b,B} X$ and $X_i = U_i \times_{b_i,B} X = U_i \times_U U \times_{b,B} X = U_i \times_U X_U$. Observe that \mathcal{L}_i is an invertible \mathcal{O}_{X_i} -module. Note that $\{X_i \rightarrow X_U\}$ forms an fppf covering as well. Moreover, the descent datum φ_{ij} translates into a descent datum on the invertible sheaves \mathcal{L}_i relative to the fppf covering $\{X_i \rightarrow X_U\}$. Hence by Descent on Spaces, Proposition 4.1 we obtain a unique invertible sheaf \mathcal{L} on X_U which recovers \mathcal{L}_i and the descent data over X_i . The triple (U, b, \mathcal{L}) is therefore the object of $Pic_{X/B}$ over U we were looking for. Details omitted. \square

16. Examples of inertia stacks

Here are some examples of inertia stacks.

Example 16.1. Let S be a scheme. Let G be a commutative group. Let $X \rightarrow S$ be a scheme over S . Let $a : G \times X \rightarrow X$ be an action of G on X . For $g \in G$ we denote $g : X \rightarrow X$ the corresponding automorphism. In this case the inertia stack of $[X/G]$ (see Remark 14.4) is given by

$$I_{[X/G]} = \coprod_{g \in G} [X^g/G],$$

where, given an element g of G , the symbol X^g denotes the scheme $X^g = \{x \in X \mid g(x) = x\}$. In a formula X^g is really the fibre product

$$X^g = X \times_{(1,1), X \times_S X, (g,1)} X.$$

Indeed, for any S -scheme T , a T -point on the inertia stack of $[X/G]$ consists of a triple $(P/T, \phi, \alpha)$ consisting of a G -torsor $P \rightarrow T$ together with a G -equivariant isomorphism $\phi : P \rightarrow X$, together with an automorphism α of $P \rightarrow T$ over T such

that $\phi \circ \alpha = \phi$. Since G is a sheaf of *commutative* groups, α is, locally in the fppf topology over T , given by multiplication by some element g of G . The condition that $\phi \circ \alpha = \phi$ means that ϕ factors through the inclusion of X^g in X , i.e., ϕ is obtained by composing that inclusion with a morphism $P \rightarrow X^g$. The above discussion allows us to define a morphism of fibred categories $I_{[X/G]} \rightarrow \coprod_{g \in G} [X^g/G]$ given on T -points by the discussion above. We omit showing that this is an equivalence.

Example 16.2. Let $X \rightarrow S$ be a morphism of schemes. Assume that for any $T \rightarrow S$ the base change $f_T : X_T \rightarrow T$ has the property that the map $\mathcal{O}_T \rightarrow f_{T,*} \mathcal{O}_{X_T}$ is an isomorphism. (This implies that f is *cohomologically flat in dimension 0* (insert future reference here) but is stronger.) Consider the Picard stack $\text{Pic}_{X/S}$, see Section 15. The points of its inertia stack over an S -scheme T consist of pairs (\mathcal{L}, α) where \mathcal{L} is a line bundle on X_T and α is an automorphism of that line bundle. I.e., we can think of α as an element of $H^0(X_T, \mathcal{O}_{X_T}^\times) = H^0(T, \mathcal{O}_T^\times)$ by our condition. Note that $H^0(T, \mathcal{O}_T^\times) = \mathbf{G}_{m,S}(T)$, see Groupoids, Example 5.1. Hence the inertia stack of $\text{Pic}_{X/S}$ is

$$I_{\text{Pic}_{X/S}} = \mathbf{G}_{m,S} \times_S \text{Pic}_{X/S}.$$

as a stack over $(\text{Sch}/S)_{\text{fppf}}$.

17. Finite Hilbert stacks

We formulate this in somewhat greater generality than is perhaps strictly needed. Fix a 1-morphism

$$F : \mathcal{X} \longrightarrow \mathcal{Y}$$

of stacks in groupoids over $(\text{Sch}/S)_{\text{fppf}}$. For each integer $d \geq 1$ consider a category $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ defined as follows:

- (1) An object (U, Z, y, x, α) where U, Z are objects of in $(\text{Sch}/S)_{\text{fppf}}$ and Z is a finite locally free of degree d over U , where $y \in \text{Ob}(\mathcal{Y}_U)$, $x \in \text{Ob}(\mathcal{X}_Z)$ and $\alpha : y|_Z \rightarrow F(x)$ is an isomorphism³.
- (2) A morphism $(U, Z, y, x, \alpha) \rightarrow (U', Z', y', x', \alpha')$ is given by a morphism of schemes $f : U \rightarrow U'$, a morphism of schemes $g : Z \rightarrow Z'$ which induces an isomorphism $Z \rightarrow Z' \times_U U'$, and isomorphisms $b : y \rightarrow f^* y'$, $a : x \rightarrow g^* x'$

³ This means the data gives rise, via the 2-Yoneda lemma (Categories, Lemma 39.1), to a 2-commutative diagram

$$\begin{array}{ccc} (\text{Sch}/Z)_{\text{fppf}} & \xrightarrow{x} & \mathcal{X} \\ \downarrow & & \downarrow F \\ (\text{Sch}/U)_{\text{fppf}} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

of stacks in groupoids over $(\text{Sch}/S)_{\text{fppf}}$. Alternatively, we may picture α as a 2-morphism

$$\begin{array}{ccc} & \xrightarrow{y \circ (Z \rightarrow U)} & \\ (\text{Sch}/Z)_{\text{fppf}} & \Downarrow \alpha & \mathcal{Y} \\ & \xrightarrow{F \circ x} & \end{array}$$

inducing a commutative diagram

$$\begin{array}{ccc} y|_Z & \xrightarrow{\alpha} & F(x) \\ b|_Z \downarrow & & \downarrow F(a) \\ f^*y'|_Z & \xrightarrow{\alpha'} & F(g^*x') \end{array}$$

It is clear from the definitions that there is a canonical forgetful functor

$$p : \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow (Sch/S)_{fppf}$$

which assigns to the quintuple (U, Z, y, x, α) the scheme U and to the morphism $(f, g, b, a) : (U, Z, y, x, \alpha) \rightarrow (U', Z', y', x', \alpha')$ the morphism $f : U \rightarrow U'$.

Lemma 17.1. *The category $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ endowed with the functor p above defines a stack in groupoids over $(Sch/S)_{fppf}$.*

Proof. As usual, the hardest part is to show descent for objects. To see this let $\{U_i \rightarrow U\}$ be a covering of $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, Z_i, y_i, x_i, \alpha_i)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ lying over U_i , and let $\varphi_{ij} : \text{pr}_0^* \xi_i \rightarrow \text{pr}_1^* \xi_j$ be a descent datum. First, observe that φ_{ij} induces a descent datum $(Z_i/U_i, \varphi_{ij})$ which is effective by Descent, Lemma 33.1 This produces a scheme Z/U which is finite locally free of degree d by Descent, Lemma 19.28. From now on we identify Z_i with $Z \times_U U_i$. Next, the objects y_i in the fibre categories \mathcal{Y}_{U_i} descend to an object y in \mathcal{Y}_U because \mathcal{Y} is a stack in groupoids. Similarly the objects x_i in the fibre categories \mathcal{X}_{Z_i} descend to an object x in \mathcal{X}_Z because \mathcal{X} is a stack in groupoids. Finally, the given isomorphisms

$$\alpha_i : (y|_Z)_{Z_i} = y_i|_{Z_i} \longrightarrow F(x_i) = F(x|_{Z_i})$$

glue to a morphism $\alpha : y|_Z \rightarrow F(x)$ as the \mathcal{Y} is a stack and hence $\text{Isom}_{\mathcal{Y}}(y|_Z, F(x))$ is a sheaf. Details omitted. \square

Definition 17.2. We will denote $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ the *degree d finite Hilbert stack of \mathcal{X} over \mathcal{Y}* constructed above. If $\mathcal{Y} = S$ we write $\mathcal{H}_d(\mathcal{X}) = \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. If $\mathcal{X} = \mathcal{Y} = S$ we denote it \mathcal{H}_d .

Note that given $F : \mathcal{X} \rightarrow \mathcal{Y}$ as above we have the following natural 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$:

$$(17.2.1) \quad \begin{array}{ccccc} \mathcal{H}_d(\mathcal{X}) & \longleftarrow & \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) & \longrightarrow & \mathcal{Y} \\ & & \downarrow & & \\ & & \mathcal{H}_d & & \end{array}$$

Each of the arrows is given by a "forgetful functor".

Lemma 17.3. *The 1-morphism $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \rightarrow \mathcal{H}_d(\mathcal{X})$ is faithful.*

Proof. To check that $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \rightarrow \mathcal{H}_d(\mathcal{X})$ is faithful it suffices to prove that it is faithful on fibre categories. Suppose that $\xi = (U, Z, y, x, \alpha)$ and $\xi' = (U, Z', y', x', \alpha')$ are two objects of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over the scheme U . Let $(g, b, a), (g', b', a') : \xi \rightarrow \xi'$ be two morphisms in the fibre category of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U . The image of these

morphisms in $\mathcal{H}_d(\mathcal{X})$ agree if and only if $g = g'$ and $a = a'$. Then the commutative diagram

$$\begin{array}{ccc} y|_Z & \xrightarrow{\alpha} & F(x) \\ b|_Z, b'|_Z \downarrow & & \downarrow F(a)=F(a') \\ y'|_Z & \xrightarrow{\alpha'} & F(g^*x') = F((g')^*x') \end{array}$$

implies that $b|_Z = b'|_Z$. Since $Z \rightarrow U$ is finite locally free of degree d we see $\{Z \rightarrow U\}$ is an fppf covering, hence $b = b'$. \square

18. Other chapters

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