QUOT AND HILBERT SPACES

Contents

1. Introduction

The purpose of this chapter is to write about Quot and Hilbert functors and to prove that these are algebraic spaces provided certain technical conditions are satisfied. In this chapter we will discuss this in the setting of algebraic space. A reference is Grothendieck's lectures, see [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d]. Another reference is the paper [OS03]; this paper discusses the more general case of Quot and Hilbert spaces associated to a morphism of algebraic stacks which we will discuss in another chapter, see (insert future reference here).

In the case of Hilbert spaces there is a more general notion of "Hilbert stacks" which we will discuss in a separate chapter, see (insert future reference here).

We have intentionally placed this chapter, as well as the chapters "Examples of Stacks", "Sheaves on Algebraic Stacks", "Criteria for Representability", and "Artin's Axioms" before the general development of the theory of algebraic stacks. The reason for this is that starting with the next chapter (see Properties of Stacks, Section 2) we will no longer distinguish between a scheme and the algebraic stack it gives rise to. Thus our language will become more flexible and easier for a human to parse, but also less precise. These first few chapters, including the initial chapter "Algebraic Stacks", lay the groundwork that later allow us to ignore some of the very technical distinctions between different ways of thinking about algebraic stacks. But especially in the chapters "Artin's Axioms" and "Criteria of Representability" we need to be very precise about what objects exactly we are working with, as we are trying to show that certain constructions produce algebraic stacks or algebraic spaces.

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Unfortunately, this means that some of the notation, conventions and terminology is awkward and may seem backwards to the more experienced reader. We hope the reader will forgive us!

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. The Hom functor

In this section we study the functor of homomorphisms defined below.

Situation 3.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over B we will denote $\mathcal{F}_T$ and $\mathcal{G}_T$ the base changes of $\mathcal{F}$ and $\mathcal{G}$ to T, in other words, the pullbacks via the projection morphism $X_T = X \times_B T \to X$. We consider the functor

$$Hom(\mathcal{F}, \mathcal{G}) : (Sch/B)^{opp} \longrightarrow Sets, \quad T \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T) \tag{3.1.1}$$

In Situation 3.1 we sometimes think of the functor $Hom(\mathcal{F}, \mathcal{G})$ as a functor

$$Hom(\mathcal{F}, \mathcal{G}) : (Sch/S)^{opp} \longrightarrow Sets$$

endowed with a morphism $Hom(\mathcal{F}, \mathcal{G}) \to B$. Namely, if T is a scheme over S, then an element of $Hom(\mathcal{F}, \mathcal{G})(T)$ consists of a pair (h, u), where h is a morphism $h : T \to B$ and $u : \mathcal{F}_T \to \mathcal{G}_T$ is an $\mathcal{O}_{X_T}$-module map where $X_T = T \times_{h,B} X$ and $\mathcal{F}_T$ and $\mathcal{G}_T$ are the pullbacks to X_T. In particular, when we say that $Hom(\mathcal{F}, \mathcal{G})$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

Lemma 3.2. *In Situation 3.1 the functor $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over B. Set $X_i = X_{T_i} = X \times_S T_i$ and $\mathcal{F}_i = u_{T_i}$ and $\mathcal{G}_i = \mathcal{G}_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 3.2. Thus a family of maps $u_i : \mathcal{F}_i \to \mathcal{G}_i$ such that u_i and u_j restrict to the same map on $X_{T_i \times_T T_j}$ comes from a unique map $u : \mathcal{F}_T \to \mathcal{G}_T$ by descent (Descent on Spaces, Proposition 4.1). $\square$

Remark 3.3. In Situation 3.1 let $B' \to B$ be a morphism of algebraic spaces over S. Set $X' = X \times_B B'$ and denote $\mathcal{F}'$, $\mathcal{G}'$ the pullback of $\mathcal{F}$, $\mathcal{G}$ to X'. Then we obtain a functor $Hom(\mathcal{F}', \mathcal{G}') : (Sch/B')^{opp} \to Sets$ associated to the base change $f' : X' \to B'$. For a scheme T over B' it is clear that we have

$$Hom(\mathcal{F}', \mathcal{G}')(T) = Hom(\mathcal{F}, \mathcal{G})(T)$$

where on the right hand side we think of T as a scheme over B via the composition $T \to B' \to B$. This trivial remark will occasionally be useful to change the base algebraic space.

Lemma 3.4. *In Situation 3.1 let $\{X_i \to X\}_{i\in I}$ be an fppf covering and for each $i,j \in I$ let $\{X_{ijk} \to X_i \times_X X_j\}$ be an fppf covering. Denote $\mathcal{F}_i$, resp. $\mathcal{F}_{ijk}$ the pullback of $\mathcal{F}$ to X_i, resp. X_{ijk}. Similarly define $\mathcal{G}_i$ and $\mathcal{G}_{ijk}$. For every scheme T over B the diagram*

$$Hom(\mathcal{F},\mathcal{G})(T) \longrightarrow \prod_i Hom(\mathcal{F}_i,\mathcal{G}_i)(T) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod_{i,j,k} Hom(\mathcal{F}_{ijk},\mathcal{G}_{ijk})(T)$$

presents the first arrow as the equalizer of the other two.

Proof. Let $u_i : \mathcal{F}_{i,T} \to \mathcal{G}_{i,T}$ be an element in the equalizer of pr_0^* and pr_1^*. Since the base change of an fppf covering is an fppf covering (Topologies on Spaces, Lemma 4.2) we see that $\{X_{i,T} \to X_T\}_{i\in I}$ and $\{X_{ijk,T} \to X_{i,T} \times_{X_T} X_{j,T}\}$ are fppf coverings. Applying Descent on Spaces, Proposition 4.1 we first conclude that u_i and u_j restrict to the same morphism over $X_{i,T} \times_{X_T} X_{j,T}$, whereupon a second application shows that there is a unique morphism $u : \mathcal{F}_T \to \mathcal{G}_T$ restricting to u_i for each i. This finishes the proof. □

Lemma 3.5. *In Situation 3.1. If $\mathcal{F}$ is of finite presentation and f is quasi-compact and quasi-separated, then $Hom(\mathcal{F},\mathcal{G})$ is limit preserving.*

Proof. Let $T = \lim_{i\in I} T_i$ be a directed limit of affine B-schemes. We have to show that

$$Hom(\mathcal{F},\mathcal{G})(T) = \operatorname{colim} Hom(\mathcal{F},\mathcal{G})(T_i)$$

Pick $0 \in I$. We may replace B by T_0, X by X_{T_0}, $\mathcal{F}$ by $\mathcal{F}_{T_0}$, $\mathcal{G}$ by $\mathcal{G}_{T_0}$, and I by $\{i \in I \mid i \geq 0\}$. See Remark 3.3. Thus we may assume $B = \mathrm{Spec}(R)$ is affine.

When B is affine, then X is quasi-compact and quasi-separated. Choose a surjective étale morphism $U \to X$ where U is an affine scheme (Properties of Spaces, Lemma 6.3). Since X is quasi-separated, the scheme $U \times_X U$ is quasi-compact and we may choose a surjective étale morphism $V \to U \times_X U$ where V is an affine scheme. Applying Lemma 3.4 we see that $Hom(\mathcal{F},\mathcal{G})$ is the equalizer of two maps between

$$Hom(\mathcal{F}|_U,\mathcal{G}|_U) \quad \text{and} \quad Hom(\mathcal{F}|_V,\mathcal{G}|_V)$$

This reduces us to the case that X is affine.

In the affine case the statement of the lemma reduces to the following problem: Given a ring map $R \to A$, two A-modules M, N and a directed system of R-algebras $C = \operatorname{colim} C_i$. When is it true that the map

$$\operatorname{colim} \mathrm{Hom}_{A\otimes_R C_i}(M \otimes_R C_i, N \otimes_R C_i) \longrightarrow \mathrm{Hom}_{A\otimes_R C}(M \otimes_R C, N \otimes_R C)$$

is bijective? By Algebra, Lemma 123.3 this holds if $M \otimes_R C$ is of finite presentation over $A \otimes_R C$, i.e., when M is of finite presentation over A. □

Lemma 3.6. *Let S be a scheme. Let B be an algebraic space over S. Let $i : X' \to X$ be a closed immersion of algebraic spaces over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\mathcal{G}'$ be a quasi-coherent $\mathcal{O}_{X'}$-module. Then*

$$Hom(\mathcal{F}, i_*\mathcal{G}') = Hom(i^*\mathcal{F},\mathcal{G}')$$

as functors on (Sch/B).

Proof. Let $g : T \to B$ be a morphism where T is a scheme. Denote $i_T : X'_T \to X_T$ the base change of i. Denote $h : X_T \to X$ and $h' : X'_T \to X'$ the projections. Observe that $(h')^*i^*\mathcal{F} = i_T^*h^*\mathcal{F}$. As a closed immersion is affine (Morphisms of Spaces, Lemma 20.6) we have $h^*i_*\mathcal{G} = i_{T,*}(h')^*\mathcal{G}$ by Cohomology of Spaces, Lemma 10.2. Thus we have

$$\begin{aligned} \mathit{Hom}(\mathcal{F}, i_*\mathcal{G}')(T) &= \mathrm{Hom}_{\mathcal{O}_{X_T}}(h^*\mathcal{F}, h^*i_*\mathcal{G}') \\ &= \mathrm{Hom}_{\mathcal{O}_{X_T}}(h^*\mathcal{F}, i_{T,*}(h')^*\mathcal{G}) \\ &= \mathrm{Hom}_{\mathcal{O}_{X'_T}}(i_T^*h^*\mathcal{F}, (h')^*\mathcal{G}) \\ &= \mathrm{Hom}_{\mathcal{O}_{X'_T}}((h')^*i^*\mathcal{F}, (h')^*\mathcal{G}) \\ &= \mathit{Hom}(i^*\mathcal{F}, \mathcal{G}')(T) \end{aligned}$$

as desired. The middle equality follows from the adjointness of the functors $i_{T,*}$ and i_T^*. □

Lemma 3.7. *Let S be a scheme. Let B be an algebraic space over S. Let K be a pseudo-coherent object of $D(\mathcal{O}_B)$.*

(1) *If for all $g : T \to B$ in (Sch/B) the cohomology sheaf $H^{-1}(Lg^*K)$ is zero, then the functor*

$$(Sch/B)^{opp} \longrightarrow Sets, \quad (g : T \to B) \longmapsto H^0(T, H^0(Lg^*K))$$

is an algebraic space affine and of finite presentation over B.

(2) *If for all $g : T \to B$ in (Sch/B) the cohomology sheaves $H^i(Lg^*K)$ are zero for $i < 0$, then K is perfect with tor amplitude in $[0, b]$ for some $b \geq 0$ and the functor*

$$(Sch/B)^{opp} \longrightarrow Sets, \quad (g : T \to B) \longmapsto H^0(T, Lg^*K)$$

is an algebraic space affine and of finite presentation over B.

Proof. Under the assumptions of (2) we have $H^0(T, Lg^*K) = H^0(T, H^0(Lg^*K))$. Let us prove that the rule $T \mapsto H^0(T, H^0(Lg^*K))$ satisfies the sheaf property for the fppf topology. To do this assume we have an fppf covering $\{h_i : T_i \to T\}$ of a scheme $g : T \to B$ over B. Set $g_i = g \circ h_i$. Note that since h_i is flat, we have $Lh_i^* = h_i^*$ and h_i^* commutes with taking cohomology. Hence

$$H^0(T_i, H^0(Lg_i^*K)) = H^0(T_i, H^0(h_i^*Lg^*K)) = H^0(T, h_i^*H^0(Lg^*K))$$

Similarly for the pullback to $T_i \times_T T_j$. Since Lg^*K is a pseudo-coherent complex on T (Cohomology on Sites, Lemma 34.3) the cohomology sheaf $\mathcal{F} = H^0(Lg^*K)$ is quasi-coherent (Derived Categories of Spaces, Lemma 12.5). Hence by Descent on Spaces, Proposition 4.1 we see that

$$H^0(T, \mathcal{F}) = \mathrm{Ker}(\prod H^0(T_i, h_i^*\mathcal{F}) \to \prod H^0(T_i, h_i^*\mathcal{F}))$$

In this way we see that the rules in (1) and (2) satisfy the sheaf property for fppf coverings. This mean we may apply Bootstrap, Lemma 11.4 it suffices to prove the representability étale locally on B. Moreover, we may check whether the end result is affine and of finite presentation étale locally on B, see Morphisms of Spaces, Lemmas 20.3 and 27.4. Hence we may assume that B is an affine scheme.

Assume $B = \mathrm{Spec}(A)$ is an affine scheme. By the results of Derived Categories of Spaces, Lemmas 12.5, 4.2, and 12.2 we deduce that in the rest of the proof we may think of K as a perfect object of the derived category of complexes of modules on B in the Zariski topology. By Derived Categories of Schemes, Lemmas 9.1, 3.4, and 9.3 we can find a pseudo-coherent complex $M^\bullet$ of A-modules such that K is the corresponding object of $D(\mathcal{O}_B)$. Our assumption on pullbacks implies that $M^\bullet \otimes_A^{\mathbf{L}} \kappa(\mathfrak{p})$ has vanishing H^{-1} for all primes $\mathfrak{p} \subset A$. By More on Algebra, Lemma 56.16 we can write

$$M^\bullet = \tau_{\geq 0}M^\bullet \oplus \tau_{\leq -1}M^\bullet$$

with $\tau_{\geq 0}M^\bullet$ perfect with Tor amplitude in $[0, b]$ for some $b \geq 0$ (here we also have used More on Algebra, Lemmas 56.11 and 51.13). Note that in case (2) we also see that $\tau_{\leq -1}M^\bullet = 0$ in $D(A)$ whence $M^\bullet$ and K are perfect with tor amplitude in $[0, b]$. For any B-scheme $g : T \to B$ we have

$$H^0(T, H^0(Lg^*K)) = H^0(T, H^0(Lg^*\tau_{\geq 0}K))$$

(by the dual of Derived Categories, Lemma 17.1) hence we may replace K by $\tau_{\geq 0}K$ and correspondingly $M^\bullet$ by $\tau_{\geq 0}M^\bullet$. In other words, we may assume $M^\bullet$ has tor amplitude in $[0, b]$.

Assume $M^\bullet$ has tor amplitude in $[0, b]$. We may assume $M^\bullet$ is a bounded above complex of finite free A-modules (by our definition of pseudo-coherent complexes, see More on Algebra, Definition 50.1 and the discussion following the definition). By More on Algebra, Lemma 51.2 we see that $M = \mathrm{Coker}(M^{-1} \to M^0)$ is flat. By Algebra, Lemma 75.2 we see that M is finite locally free. Hence $M^\bullet$ is quasi-isomorphic to

$$M \to M^1 \to M^2 \to \ldots \to M^d \to 0 \ldots$$

Note that this is a K-flat complex (Cohomology, Lemma 27.8), hence derived pullback of K via a morphism $T \to B$ is computed by the complex

$$g^*\widetilde{M} \to g^*\widetilde{M^1} \to \ldots$$

Thus it suffices to show that the functor

$$(g : T \to B) \longmapsto \mathrm{Ker}(\Gamma(T, g^*\widetilde{M}) \to \Gamma(T, g^*(\widetilde{M^1}))$$

is representable by an affine scheme of finite presentation over B.

We may still replace B by the members of an affine open covering in order to prove this last statement. Hence we may assume that M is finite free (recall that M^1 is finite free to begin with). Write $M = A^{\oplus n}$ and $M^1 = A^{\oplus m}$. Let the map $M \to M^1$ be given by the $m \times n$ matrix (a_{ij}) with coefficients in A. Then $\widetilde{M} = \mathcal{O}_B^{\oplus n}$ and $\widetilde{M^1} = \mathcal{O}_B^{\oplus m}$. Thus the functor above is equal to the functor

$$(g : T \to B) \longmapsto \{(f_1, \ldots, f_n) \in \Gamma(T, \mathcal{O}_T) \mid \sum g^\sharp(a_{ij}f_i = 0, j = 1, \ldots, m\}$$

Clearly this is representable by the affine scheme

$$\mathrm{Spec}\left(A[x_1, \ldots, x_n]/(\sum a_{ij}x_i; j = 1, \ldots, m)\right)$$

and the lemma has been proved. $\square$

The functor $Hom(\mathcal{F}, \mathcal{G})$ is representable in a number of situations. All of our results will be based on the following basic case. The proof of this lemma as given below is in some sense the natural generalization to the proof of [DG67, III, Cor 7.7.8].

Lemma 3.8. *In Situation 3.1 assume that*

(1) *B is a Noetherian algebraic space,*
(2) *f is locally of finite type and quasi-separated,*
(3) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and*
(4) *$\mathcal{G}$ is a finite type $\mathcal{O}_X$-module, flat over B, with scheme theoretic support proper over B.*

Then the functor $Hom(\mathcal{F}, \mathcal{G})$ is representable by an algebraic space affine and of finite presentation over B.

Proof. We may replace X by a quasi-compact open neighbourhood of the support of $\mathcal{G}$, hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to \mathcal{F}$ by a perfect complex P of the triple $(X, \mathcal{F}, 0)$, see Derived Categories of Spaces, Definition 13.1 and Theorem 13.7). Then the induced map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(P, \mathcal{G})$$

is an isomorphism because $P \to \mathcal{F}$ induces an isomorphism $H^0(P) \to \mathcal{F}$ and $H^i(P) = 0$ for $i > 0$. Moreover, for any morphism $g : T \to B$ denote $h : X_T = T \times_B X \to X$ the projection and set $P_T = Lh^*P$. Then it is equally true that

$$\mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T)$$

is an isomorphism, as $P_T = Lh^*P \to Lh^*\mathcal{F} \to \mathcal{F}_T$ induces an isomorphism $H^0(P_T) \to \mathcal{F}_T$ (because h^* is right exact and $H^i(P) = 0$ for $i > 0$). Thus it suffices to prove the result for the functor

$$T \longmapsto \mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T).$$

By the Leray spectral sequence (see Cohomology on Sites, Remark 14.4) we have

$$\mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T) = H^0(X_T, R\mathcal{H}om(P_T, \mathcal{G}_T)) = H^0(T, Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T))$$

where $f_T : X_T \to T$ is the base change of f. By Derived Categories of Spaces, Lemma 17.6 we have

$$Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T) = Lg^*Rf_*R\mathcal{H}om(P, \mathcal{G}).$$

By Derived Categories of Spaces, Lemma 19.2 the object $K = Rf_*R\mathcal{H}om(P, \mathcal{G})$ of $D(\mathcal{O}_B)$ is perfect. This means we can apply Lemma 3.7 as long as we can prove that the cohomology sheaf $H^i(Lg^*K)$ is 0 for all $i < 0$ and $g : T \to B$ as above. This is clear from the last displayed formula as the cohomology sheaves of $Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T)$ are zero in negative degrees due to the fact that $R\mathcal{H}om(P_T, \mathcal{G}_T)$ has vanishing cohomology sheaves in negative degrees as P_T is perfect with vanishing cohomology sheaves in positive degrees. $\square$

Here is a cheap consequence of Lemma 3.8.

Proposition 3.9. *In Situation 3.1 assume that*

(1) *f is of finite presentation, and*
(2) *$\mathcal{G}$ is a finitely presented $\mathcal{O}_X$-module, flat over B, with scheme theoretic support proper over B.*

Then the functor $Hom(\mathcal{F}, \mathcal{G})$ is representable by an algebraic space affine over B. If $\mathcal{F}$ is of finite presentation, then $Hom(\mathcal{F}, \mathcal{G})$ is of finite presentation over B.

Proof. By Lemma 3.2 the functor $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for fppf coverings. This mean we may[1] apply Bootstrap, Lemma 11.1 to check the representability étale locally on B. Moreover, we may check whether the end result is affine or of finite presentation étale locally on B, see Morphisms of Spaces, Lemmas 20.3 and 27.4. Hence we may assume that B is an affine scheme.

Assume B is an affine scheme. As f is of finite presentation, it follows X is quasi-compact and quasi-separated. Thus we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of $\mathcal{O}_X$-modules of finite presentation (Limits of Spaces, Lemma 9.1). It is clear that

$$Hom(\mathcal{F}, \mathcal{G}) = \lim Hom(\mathcal{F}_i, \mathcal{G})$$

Hence if we can show that each $Hom(\mathcal{F}_i, \mathcal{G})$ is representable by an affine scheme, then we see that the same thing holds for $Hom(\mathcal{F}, \mathcal{G})$. Use the material in Limits, Section 2 and Limits of Spaces, Section 4. Thus we may assume that $\mathcal{F}$ is of finite presentation.

Say $B = \operatorname{Spec}(R)$. Write $R = \operatorname{colim} R_i$ with each R_i a finite type $\mathbf{Z}$-algebra. Set $B_i = \operatorname{Spec}(R_i)$. By the results of Limits of Spaces, Lemmas 7.1 and 7.2 we can find an i, a morphism of algebraic spaces $X_i \to B_i$, and finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{F}_i$ and $\mathcal{G}_i$ such that the base change of $(X_i, \mathcal{F}_i, \mathcal{G}_i)$ to B recovers $(X, \mathcal{F}, \mathcal{G})$. By Limits of Spaces, Lemma 6.11 we may, after increasing i, assume that $\mathcal{G}_i$ is flat over B_i. By Limits of Spaces, Lemma 12.3 we may similarly assume the scheme theoretic support of $\mathcal{G}_i$ is proper over B_i. At this point we can apply Lemma 3.8 to see that $H_i = Hom(\mathcal{F}_i, \mathcal{G}_i)$ is an algebraic space affine of finite presentation over B_i. Pulling back to B (using Remark 3.3) we see that $H_i \times_{B_i} B = Hom(\mathcal{F}, \mathcal{G})$ and we win. □

4. The Isom functor

In Situation 3.1 we can consider the subfunctor

$$Isom(\mathcal{F}, \mathcal{G}) \subset Hom(\mathcal{F}, \mathcal{G})$$

whose value on a scheme T over B is the set of *invertible* $\mathcal{O}_{X_T}$-homomorphisms $u : \mathcal{F}_T \to \mathcal{G}_T$. In this brief section we quickly point out some properties of this functor.

Lemma 4.1. *In Situation 3.1 the functor $Isom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for the fpqc topology.*

Proof. We have already seen that $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property. Hence it remains to show the following: Given an fpqc covering $\{T_i \to T\}_{i \in I}$ of schemes over B and an $\mathcal{O}_{X_T}$-linear map $u : \mathcal{F}_T \to \mathcal{G}_T$ such that u_{T_i} is an isomorphism for all i, then u is an isomorphism. Since $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 3.2, this follows from Descent on Spaces, Proposition 4.1. □

Proposition 4.2. *In Situation 3.1 assume that*

(1) *f is of finite presentation, and*
(2) *$\mathcal{F}$ and $\mathcal{G}$ are finitely presented $\mathcal{O}_X$-modules, flat over B, with scheme theoretic support proper over B.*

[1]We omit the verification of the set theoretical condition (3) of the referenced lemma.

Then the functor $Isom(\mathcal{F}, \mathcal{G})$ is representable by an algebraic space affine of finite presentation over B.

Proof. We will use the abbreviations $H = Hom(\mathcal{F}, \mathcal{G})$, $I = Hom(\mathcal{F}, \mathcal{F})$, $H' = Hom(\mathcal{G}, \mathcal{F})$, and $I' = Hom(\mathcal{G}, \mathcal{G})$. By Proposition 3.9 the functors H, I, H', I' are algebraic spaces and the morphisms $H \to B$, $I \to B$, $H' \to B$, and $I' \to B$ are affine and of finite presentation. The composition of maps gives a morphism

$$c : H' \times_B H \longrightarrow I \times_B I', \quad (u', u) \longmapsto (u \circ u', u' \circ u)$$

of algebraic spaces over B. Since $I \times_B I' \to B$ is separated, the section $\sigma : B \to I \times_B I'$ corresponding to $(\mathrm{id}_\mathcal{F}, \mathrm{id}_\mathcal{G})$ is a closed immersion (Morphisms of Spaces, Lemma 4.7). Moreover, σ is of finite presentation (Morphisms of Spaces, Lemma 27.9). Hence

$$Isom(\mathcal{F}, \mathcal{G}) = (H' \times_B H) \times_{c, I \times_B I', \sigma} B$$

is an algebraic space affine of finite presentation over B as well. Some details omitted. □

5. The stack of coherent sheaves

In this section we prove that the stack of coherent sheaves on X/B is algebraic under suitable hypotheses. This is a special case of [Lie06, Theorem 2.1.1] which treats the case of the stack of coherent sheaves on an Artin stack over a base.

Situation 5.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that f is of finite presentation. We denote $Coh_{X/B}$ the category whose objects are triples $(T, g, \mathcal{F})$ where

(1) T is a scheme over S,
(2) $g : T \to B$ is a morphism over S, and setting $X_T = T \times_{g,B} X$
(3) $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{X_T}$-module of finite presentation, flat over T, with scheme theoretic support proper over T.

A morphism $(T, g, \mathcal{F}) \to (T', g', \mathcal{F}')$ is given by a pair (h, φ) where

(1) $h : T \to T'$ is a morphism of schemes over B (i.e., $g' \circ h = g$), and
(2) $\varphi : (h')^*\mathcal{F}' \to \mathcal{F}$ is an isomorphism of $\mathcal{O}_{X_T}$-modules where $h' : X_T \to X_{T'}$ is the base change of h.

Thus $Coh_{X/B}$ is a category and the rule

$$p : Coh_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (T, g, \mathcal{F}) \longmapsto T$$

is a functor. For a scheme T over S we denote $Coh_{X/B,T}$ the fibre category of p over T. These fibre categories are groupoids.

Lemma 5.2. *In Situation 5.1 the functor $p : Coh_{X/B} \longrightarrow (Sch/S)_{fppf}$ is fibred in groupoids.*

Proof. We show that p is fibred in groupoids by checking conditions (1) and (2) of Categories, Definition 33.1. Given an object $(T', g', \mathcal{F}')$ of $Coh_{X/B}$ and a morphism $h : T \to T'$ of schemes over S we can set $g = h \circ g'$ and $\mathcal{F} = (h')^*\mathcal{F}'$ where $h' : X_T \to X_{T'}$ is the base change of h. Then it is clear that we obtain a morphism $(T, g, \mathcal{F}) \to (T', g', \mathcal{F}')$ of $Coh_{X/B}$ lying over h. This proves (1). For (2) suppose we are given morphisms

$$(h_1, \varphi_1) : (T_1, g_1, \mathcal{F}_1) \to (T, g, \mathcal{F}) \quad \text{and} \quad (h_2, \varphi_2) : (T_2, g_2, \mathcal{F}_2) \to (T, g, \mathcal{F})$$

of $Coh_{X/B}$ and a morphism $h : T_1 \to T_2$ such that $h_2 \circ h = h_1$. Then we can let φ be the composition

$$(h')^*\mathcal{F}_2 \xrightarrow{(h')^*\varphi_2^{-1}} (h')^*(h_2)^*\mathcal{F} = (h_1)^*\mathcal{F} \xrightarrow{\varphi_1} \mathcal{F}_1$$

to obtain the morphism $(h, \varphi) : (T_1, g_1, \mathcal{F}_1) \to (T_2, g_2, \mathcal{F}_2)$ that witnesses the truth of condition (2). $\square$

Lemma 5.3. *In Situation 5.1. Denote $\mathcal{X} = Coh_{X/B}$. Then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.*

Proof. Consider two objects $x = (T, g, \mathcal{F})$ and $y = (T, h, \mathcal{G})$ of $\mathcal{X}$ over a scheme T. We have to show that $Isom_{\mathcal{X}}(x, y)$ is representable by an algebraic space over T, see Algebraic Stacks, Lemma 10.11. If for $a : T' \to T$ the restrictions $x|_{T'}$ and $y|_{T'}$ are isomorphic in the fibre category $\mathcal{X}_{T'}$, then $g \circ a = h \circ a$. Hence there is a transformation of presheaves

$$Isom_{\mathcal{X}}(x, y) \longrightarrow \text{Equalizer}(g, h)$$

Since the diagonal of B is representable by schemes this equalizer is a scheme. Thus we may replace T by this equalizer and the sheaves $\mathcal{F}$ and $\mathcal{G}$ by their pullbacks. Thus we may assume $g = h$. In this case we have $Isom_{\mathcal{X}}(x, y) = Isom(\mathcal{F}, \mathcal{G})$ and the result follows from Proposition 4.2. $\square$

Lemma 5.4. *In Situation 5.1 the functor $p : Coh_{X/B} \longrightarrow (Sch/S)_{fppf}$ is a stack in groupoids.*

Proof. To prove that $Coh_{X/B}$ is a stack in groupoids, we have to show that the presheaves *Isom* are sheaves and that descent data are effective. The statement on *Isom* follows from Lemma 5.3, see Algebraic Stacks, Lemma 10.11. Let us prove the statement on descent data. Suppose that $\{a_i : T_i \to T\}$ is an fppf covering of schemes over S. Let (ξ_i, φ_{ij}) be a descent datum for $\{T_i \to T\}$ with values in $Coh_{X/B}$. For each i we can write $\xi_i = (T_i, g_i, \mathcal{F}_i)$. Denote $\text{pr}_0 : T_i \times_T T_j \to T_i$ and $\text{pr}_1 : T_i \times_T T_j \to T_j$ the projections. The condition that $\xi_i|_{T_i \times_T T_j} = \xi_j|_{T_i \times_T T_j}$ implies in particular that $g_i \circ \text{pr}_0 = g_j \circ \text{pr}_1$. Thus there exists a unique morphism $g : T \to B$ such that $g_i = g \circ a_i$, see Descent on Spaces, Lemma 6.2. Denote $X_T = T \times_{g,B} X$. Set $X_i = X_{T_i} = T_i \times_{g_i,B} X = T_i \times_{a_i,T} X_T$ and

$$X_{ij} = X_{T_i} \times_{X_T} X_{T_j} = X_i \times_{X_T} X_j$$

with projections pr_i and pr_j to X_i and X_j. Observe that the pullback of $(T_i, g_i, \mathcal{F}_i)$ by $\text{pr}_0 : T_i \times_T T_j \to T_i$ is given by $(T_i \times_T T_j, g_i \circ \text{pr}_0, \text{pr}_i^*\mathcal{F}_i)$. Hence a descent datum for $\{T_i \to T\}$ in $Coh_{X/B}$ is given by the objects $(T_i, g \circ a_i, \mathcal{F}_i)$ and for each pair i, j an isomorphism of $\mathcal{O}_{X_{ij}}$-modules

$$\varphi_{ij} : \text{pr}_i^*\mathcal{F}_i \longrightarrow \text{pr}_j^*\mathcal{F}_j$$

satisfying the cocycle condition over (the pullback of X to) $T_i \times_T T_j \times_T T_k$. Ok, and now we simply use that $\{X_i \to X_T\}$ is an fppf covering so that we can view $(\mathcal{F}_i, \varphi_{ij})$ as a descent datum for this covering. By Descent on Spaces, Proposition 4.1 this descent datum is effective and we obtain a quasi-coherent sheaf $\mathcal{F}$ over X_T restricting to $\mathcal{F}_i$ on X_i. By Morphisms of Spaces, Lemma 29.5 we see that $\mathcal{F}$ is flat over T and Descent on Spaces, Lemma 5.2 guarantees that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$-module. Finally, by Descent on Spaces, Lemma 10.17 we see that the scheme theoretic support of $\mathcal{F}$ is proper over T as we've assume the scheme theoretic support of $\mathcal{F}_i$ is proper over T_i (note that taking scheme theoretic support commutes with flat base change by Morphisms of Spaces, Lemma 28.10). In this way and we obtain our desired object over T. $\square$

Remark 5.5. In Situation 5.1 the rule $(T, g, \mathcal{F}) \mapsto (T, g)$ defines a 1-morphism

$$Coh_{X/B} \longrightarrow \mathcal{S}_B$$

of categories fibred in groupoids (see Lemma 5.4, Algebraic Stacks, Section 7, and Examples of Stacks, Section 9). Let $B' \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{S}_{B'} \to \mathcal{S}_B$ be the associated 1-morphism of stacks fibred in sets. Set $X' = X \times_B B'$. We obtain a stack in groupoids $Coh_{X'/B'} \to (Sch/S)_{fppf}$ associated to the base change $f' : X' \to B'$. In this situation the diagram

$$\begin{tikzcd} Coh_{X'/B'} \arrow[r] \arrow[d] & Coh_{X/B} \arrow[d] \\ \mathcal{S}_{B'} \arrow[r] & \mathcal{S}_B \end{tikzcd}$$

is 2-fibre product square. This trivial remark will occasionally be useful to change the base algebraic space.

Lemma 5.6. *In Situation 5.1 assume that $B \to S$ is locally of finite presentation. Then $p : Coh_{X/B} \to (Sch/S)_{fppf}$ is limit preserving (Artin's Axioms, Definition 13.1).*

Proof. Write $B(T)$ for the discrete category whose objects are the S-morphisms $T \to B$. Let $T = \lim T_i$ be a filtered limit of affine schemes over S. Assigning to an object $(T, h, \mathcal{F})$ of $Coh_{X/B,T}$ the object h of $B(T)$ gives us a commutative diagram of fibre categories

$$\begin{tikzcd} \operatorname{colim} Coh_{X/B,T_i} \arrow[r] \arrow[d] & Coh_{X/B,T} \arrow[d] \\ \operatorname{colim} B(T_i) \arrow[r] & B(T) \end{tikzcd}$$

We have to show the top horizontal arrow is an equivalence. Since we have assume that B is locally of finite presentation over S we see from Limits of Spaces, Remark 3.10 that the bottom horizontal arrow is an equivalence. This means that we may assume $T = \lim T_i$ be a filtered limit of affine schemes over B. Denote $g_i : T_i \to B$ and $g : T \to B$ the corresponding morphisms. Set $X_i = T_i \times_{g_i,B} X$ and $X_T = T \times_{g,B} X$. Observe that $X_T = \operatorname{colim} X_i$ and that the algebraic spaces X_i and X_T are quasi-separated and quasi-compact (as they are of finite presentation over the affines T_i and T). By Limits of Spaces, Lemma 7.2 we see that

$$\operatorname{colim} FP(X_i) = FP(X_T).$$

where $FP(W)$ is short hand for the category of finitely presented $\mathcal{O}_W$-modules. The results of Limits of Spaces, Lemmas 6.11 and 12.3 tell us the same thing is true if we replace $FP(X_i)$ and $FP(X_T)$ by the full subcategory of objects flat over T_i and T with scheme theoretic support proper over T_i and T. This proves the lemma. $\square$

Lemma 5.7. *In Situation 5.1. Let*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

be a pushout in the category of schemes over S where $Z \to Z'$ is a thickening and $Z \to Y$ is affine, see More on Morphisms, Lemma 11.1. Then the functor on fibre categories

$$Coh_{X/B,Y'} \longrightarrow Coh_{X/B,Y} \times_{Coh_{X/B,Z}} Coh_{X/B,Z'}$$

is an equivalence.

Proof. Observe that the corresponding map

$$B(Y') \longrightarrow B(Y) \times_{B(Z)} B(Z')$$

is a bijection, see Pushouts of Spaces, Lemma 2.2. Thus using the commutative diagram

$$\begin{tikzcd} Coh_{X/B,Y'} \arrow[r] \arrow[d] & Coh_{X/B,Y} \times_{Coh_{X/B,Z}} Coh_{X/B,Z'} \arrow[d] \\ B(Y') \arrow[r] & B(Y) \times_{B(Z)} B(Z') \end{tikzcd}$$

we see that we may assume that Y' is a scheme over B'. By Remark 5.5 we may replace B by Y' and X by $X \times_B Y'$. Thus we may assume $B = Y'$. In this case the statement follows from Pushouts of Spaces, Lemma 2.7. $\square$

Lemma 5.8. *Let*

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d] & X' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd}$$

be a cartesian square of algebraic spaces where $T \to T'$ is a first order thickening. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module flat over T'. Set $\mathcal{F} = i^*\mathcal{F}'$. The following are equivalent

1. *$\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{X'}$-module of finite presentation,*
2. *$\mathcal{F}$ is an $\mathcal{O}_{X'}$-module of finite presentation,*
3. *$\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite presentation,*
4. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation,*

Proof. Recall that a finitely presented module is quasi-coherent hence the equivalence of (1) and (2) and (3) and (4). The equivalence of (2) and (4) is a special case of Deformation Theory, Lemma 10.3. $\square$

Lemma 5.9. *In Situation 5.1 assume that S is a locally Noetherian scheme and $B \to S$ is locally of finite presentation. Let k be a finite type field over S and let $x_0 = (\mathrm{Spec}(k), g_0, \mathcal{G}_0)$ be an object of $\mathcal{X} = Coh_{X/B}$ over k. Then the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $Inf_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0})$ (Artin's Axioms, Section 8) are finite dimensional.*

Proof. Observe that by Lemma 5.7 our stack in groupoids $\mathcal{X}$ satisfies property (RS*) defined in Artin's Axioms, Section 18. In particular $\mathcal{X}$ satisfies (RS). Hence all associated predeformation categories are deformation categories (Artin's Axioms, Lemma 6.1) and the statement makes sense.

In this paragraph we show that we can reduce to the case $B = \mathrm{Spec}(k)$. Set $X_0 = \mathrm{Spec}(k) \times_{g_0, B} X$ and denote $\mathcal{X}_0 = Coh_{X_0/k}$. In Remark 5.5 we have seen that $\mathcal{X}_0$ is the 2-fibre product of $\mathcal{X}$ with $\mathrm{Spec}(k)$ over B as categories fibred in groupoids over $(Sch/S)_{fppf}$. Thus by Artin's Axioms, Lemma 8.2 we reduce to proving that B, $\mathrm{Spec}(k)$, and $\mathcal{X}_0$ have finite dimensional tangent spaces and infinitesimal automorphism spaces. The tangent space of B and $\mathrm{Spec}(k)$ are finite dimensional by Artin's Axioms, Lemma 8.1 and of course these have vanishing Inf. Thus it suffices to deal with $\mathcal{X}_0$.

Let $k[\epsilon]$ be the dual numbers over k. Let $\mathrm{Spec}(k[\epsilon]) \to B$ be the composition of $g_0 : \mathrm{Spec}(k) \to B$ and the morphism $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ coming from the inclusion $k \to k[\epsilon]$. Set $X_0 = \mathrm{Spec}(k) \times_B X$ and $X_\epsilon = \mathrm{Spec}(k[\epsilon]) \times_B X$. Observe that X_ϵ is a first order thickening of X_0 flat over the first order thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon])$. Unwinding the definitions and using Lemma 5.8 we see that $T\mathcal{F}_{\mathcal{X}_0,k,x_0}$ is the set of lifts of $\mathcal{G}_0$ to a flat module on X_ϵ. By Deformation Theory, Lemma 11.1 we conclude that

$$T\mathcal{F}_{\mathcal{X}_0,k,x_0} = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{G}_0, \mathcal{G}_0)$$

Here we have used the identification $\epsilon k[\epsilon] \cong k$ of $k[\epsilon]$-modules. Using Deformation Theory, Lemma 11.1 once more we see that

$$\mathrm{Inf}_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Ext}^0_{\mathcal{O}_{X_0}}(\mathcal{G}_0, \mathcal{G}_0)$$

These spaces are finite dimensional over k as $\mathcal{G}_0$ has support proper over $\mathrm{Spec}(k)$. Namely, X_0 is of finite presentation over $\mathrm{Spec}(k)$, hence Noetherian. Since $\mathcal{G}_0$ is of finite presentation it is a coherent $\mathcal{O}_{X_0}$-module. Thus we may apply Derived Categories of Spaces, Lemma 19.3 to conclude the desired finiteness. □

Lemma 5.10. *In Situation 5.1 assume that S is a locally Noetherian scheme and that $f : X \to B$ is separated. Let $\mathcal{X} = Coh_{X/B}$. Then the functor Artin's Axioms, Equation (9.2.1) is an equivalence.*

Proof. Let A be an S-algebra which is a complete local Noetherian ring with maximal ideal $\mathfrak{m}$ whose residue field k is of finite type over S. We have to show that the category of objects over A is equivalent to the category of formal objects over A. Since we know this holds for the category $\mathcal{S}_B$ fibred in sets associated to B by Artin's Axioms, Lemma 9.4, it suffices to prove this for those objects lying over a given morphism $\mathrm{Spec}(A) \to B$.

Set $X_A = \mathrm{Spec}(A) \times_B X$ and $X_n = \mathrm{Spec}(A/\mathfrak{m}^n) \times_B X$. By Grothendieck's existence theorem (More on Morphisms of Spaces, Theorem 31.11) we see that the category of coherent modules $\mathcal{F}$ on X_A with support proper over $\mathrm{Spec}(A)$ is equivalent to the category of systems $(\mathcal{F}_n)$ of coherent modules $\mathcal{F}_n$ on X_n with support proper over $\mathrm{Spec}(A/\mathfrak{m}^n)$. The equivalence sends $\mathcal{F}$ to the system $(\mathcal{F} \otimes_A A/\mathfrak{m}^n)$. See discussion in More on Morphisms of Spaces, Remark 31.12. To finish the proof of the lemma, it suffices to show that $\mathcal{F}$ is flat over A if and only if all $\mathcal{F} \otimes_A A/\mathfrak{m}^n$ are flat over $A/\mathfrak{m}^n$. This follows from More on Morphisms of Spaces, Lemma 20.3. □

Lemma 5.11. *In Situation 5.1 assume that S is a locally Noetherian scheme, $S = B$, and $f : X \to B$ is flat. Let $\mathcal{X} = Coh_{X/B}$. Then we have openness of versality for $\mathcal{X}$ (see Artin's Axioms, Definition 14.1).*

Proof. Let $U \to S$ be of finite type morphism of schemes, x an object of $\mathcal{X}$ over U and $u_0 \in U$ a finite type point such that x is versal at u_0. After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 17.1) and $U = \mathrm{Spec}(A)$ with $U \to S$ mapping into an affine open $\mathrm{Spec}(\Lambda)$ of S. We will use Artin's Axioms, Lemma 21.4 to prove the lemma. Let $\mathcal{F}$ be the coherent module on $X_A = \mathrm{Spec}(A) \times_S X$ flat over A corresponding to the given object x.

According to Deformation Theory, Lemma 11.1 we have an isomorphism of functors

$$T_x(M) = \mathrm{Ext}^1_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$$

and given any surjection $A' \to A$ of Λ-algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \mathrm{Ext}^2_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A I)$$

This uses that for any $A' \to A$ as above the base change $X_{A'} = \mathrm{Spec}(A') \times_B X$ is flat over A'. Apply Derived Categories of Spaces, Lemma 19.3 to the computation of the Ext groups $\mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes_A^{\mathbf{L}} M) \longrightarrow \mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K, together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 21.2. Finally, condition (iv) of Artin's Axioms, Lemma 21.3 holds by Deformation Theory, Lemma 11.3. Thus Artin's Axioms, Lemma 21.4 does indeed apply and the lemma is proved. $\square$

Theorem 5.12 (Algebraicity of stack coherent sheaves)**.** *Let S be a scheme. Let $f : X \to B$ be morphism of algebraic spaces over S. Assume that f is of finite presentation, separated, and flat*[2]*. Then $Coh_{X/B}$ is an algebraic stack over S.*

Proof. Set $\mathcal{X} = Coh_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 5.4 and 5.3). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \to \mathcal{X}$.

Let B' be a scheme and let $B' \to B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \to B'$ the projection. Then $\mathcal{X}' = Coh_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 5.5). By the material in Algebraic Stacks, Section 10 the morphism $\mathcal{X}' \to \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B, see Algebraic Stacks, Section 19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(Sch/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth morphisms $W_i \to \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \to \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \mathrm{Spec}(\Lambda_i)$ which is of finite presentation and flat and whose base change to Λ is X. See Limits of Spaces, Lemmas 7.1 and 6.11. If we show

[2]This assumption is not necessary. See discussion in Section 6.

that $Coh_{X_i/\mathrm{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 5.5 and Algebraic Stacks, Section 19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$-algebra.

Assume $S = B = \mathrm{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), and (4) of Artin's Axioms, Lemma 17.1 to conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 39.12. Hence all local rings of S are G-rings. Thus (4) holds. By Lemma 5.11 we have that $\mathcal{X}$ satisfies openness of versality, hence (3) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], [3], and [4] of Artin's Axioms, Section 12. We omit the verification of [-1] and axioms [0], [1], [2], [3], [4] correspond respectively to Lemmas 5.4, 5.6, 5.7, 5.9, and 5.10. Finally, condition (1) is Lemma 5.3. This finishes the proof of the theorem. □

6. The stack of coherent sheaves in the non-flat case

In Theorem 5.12 the assumption that $f : X \to B$ is flat is not necessary. In this section we explain where this assumption is used in the proof and one way to get around it.

For a different approach to this problem the reader may wish to consult [Art69] and follow the method discussed in the papers [OS03], [Lie06], [Ols05], [HR13], [HR10], [Ryd11]. Some of these papers deal with the more general case of the stack of coherent sheaves on an algebraic stack over an algebraic stack and others deal with similar problems in the case of Hilbert stacks or Quot functors. Our strategy will be to show algebraicity of some cases of Hilbert stacks and Quot functors as a consequence of the algebraicity of the stack of coherent sheaves.

The only step in the proof of Theorem 5.12 which uses flatness is in the application of Lemma 5.11. The lemma is used to construct an obstruction theory as in Artin's Axioms, Section 21. The proof of the lemma relies on Deformation Theory, Lemmas 11.1 and 11.3 from Deformation Theory, Section 11. This is how the assumption that f is flat comes about. Before we go on, note that results (2) and (3) of Deformation Theory, Lemmas 11.1 do hold without the assumption that f is flat as they rely on Deformation Theory, Lemmas 10.7. and 10.4 which do not have any flatness assumptions.

Before we give the details we give some motivation for the construction from derived algebraic geometry, since we think it will clarify what follows. Let A be a finite type algebra over the locally Noetherian base S. Denote $X \otimes^{\mathbf{L}} A$ a "derived base change" of X to A and denote $i : X_A \to X \otimes^{\mathbf{L}} A$ the canonical inclusion morphism. The object $X \otimes^{\mathbf{L}} A$ does not (yet) have a definition in the Stacks project; we may think of it as the algebraic space X_A endowed with a simplicial sheaf of rings $\mathcal{O}_{X \otimes^{\mathbf{L}} A}$ whose homology sheaves are

$$H_i(\mathcal{O}_{X \otimes^{\mathbf{L}} A}) = \mathrm{Tor}_i^{\mathcal{O}_S}(\mathcal{O}_X, A).$$

The morphism $X \otimes^{\mathbf{L}} A \to \mathrm{Spec}(A)$ is flat (the terms of the simplicial sheaf of rings being A-flat), so the usual material for deformations of flat modules applies to it. Thus we see that we get an obstruction theory using the groups

$$\mathrm{Ext}^i_{X \otimes^{\mathbf{L}} A}(i_*\mathcal{F}, i_*\mathcal{F} \otimes_A M)$$

where $i = 0, 1, 2$ for inf auts, inf defs, obstructions. Note that a flat deformation of $i_*\mathcal{F}$ to $X \otimes^{\mathbf{L}} A'$ is automatically of the form $i'_*\mathcal{F}'$ where $\mathcal{F}'$ is a flat deformation of $\mathcal{F}$. By adjunction of the functors Li^* and $i_* = Ri_*$ these ext groups are equal to

$$\mathrm{Ext}^i_{X_A}(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$$

Thus we obtain obstruction groups of exactly the same form as in the proof of Lemma 5.11 with the only change being that one replaces the first occurrence of $\mathcal{F}$ by the complex $Li^*(i_*\mathcal{F})$.

Below we prove the non-flat version of the lemma by a "direct" construction of $E(\mathcal{F}) = Li^*(i_*\mathcal{F})$ and direct proof of its relationship to the deformation theory of $\mathcal{F}$. In fact, it suffices to construct $\tau_{\geq -2}E(\mathcal{F})$, as we are only interested in the ext groups $\mathrm{Ext}^i_{X_A}(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i = 0, 1, 2$. We can even identify the cohomology sheaves

$$H^i(E(\mathcal{F})) = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{F} & \text{if } i = 0 \\ 0 & \text{if } i = -1 \\ \mathrm{Tor}_1^{\mathcal{O}_S}(\mathcal{O}_X, A) \otimes_{\mathcal{O}_X} \mathcal{F} & \text{if } i = -2 \end{cases}$$

This observation will guide our construction of $E(\mathcal{F})$ in the remarks below.

Remark 6.1 (Direct construction)**.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let U be another algebraic space over B. Denote $q : X \times_B U \to U$ the second projection. Consider the distinguished triangle

$$Lq^*L_{U/B} \to L_{X \times_B U/B} \to E \to Lq^*L_{U/B}[1]$$

of Cotangent, Section 27. For any sheaf $\mathcal{F}$ of $\mathcal{O}_{X \times_B U}$-modules we have the Atiyah class

$$\mathcal{F} \to L_{X \times_B U/B} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} \mathcal{F}[1]$$

see Cotangent, Section 18. We can compose this with the map to E and choose a distinguished triangle

$$E(\mathcal{F}) \to \mathcal{F} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} E[1] \to E(\mathcal{F})[1]$$

in $D(\mathcal{O}_{X \times_B U})$. By construction the Atiyah class lifts to a map

$$e_{\mathcal{F}} : E(\mathcal{F}) \longrightarrow Lq^*L_{U/B} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} \mathcal{F}[1]$$

fitting into a morphism of distinguished triangles

$$\begin{tikzcd} \mathcal{F} \otimes^{\mathbf{L}} Lq^*L_{U/B}[1] \arrow[r] & \mathcal{F} \otimes^{\mathbf{L}} L_{X \times_B U/B}[1] \arrow[r] & \mathcal{F} \otimes^{\mathbf{L}} E[1] \\ E(\mathcal{F}) \arrow[r] \arrow[u, "e_{\mathcal{F}}"] & \mathcal{F} \arrow[r] \arrow[u, "Atiyah"] & \mathcal{F} \otimes^{\mathbf{L}} E[1] \arrow[u, "="] \end{tikzcd}$$

Given $S, B, X, f, U, \mathcal{F}$ we fix a choice of $E(\mathcal{F})$ and $e_{\mathcal{F}}$.

Remark 6.2 (Construction of obstruction class)**.** With notation as in Remark 6.1 let $i : U \to U'$ be a first order thickening of U over B. Let $\mathcal{I} \subset \mathcal{O}_{U'}$ be the quasi-coherent sheaf of ideals cutting out B in B'. The fundamental triangle

$$Li^*L_{U'/B} \to L_{U/B} \to L_{U/U'} \to Li^*L_{U'/B}[1]$$

together with the map $L_{U/U'} \to \mathcal{I}[1]$ determine a map $e_{U'} : L_{U/B} \to \mathcal{I}[1]$. Combined with the map $e_\mathcal{F}$ of the previous remark we obtain

$$(\mathrm{id}_\mathcal{F} \otimes Lq^* e_{U'}) \cup e_\mathcal{F} : E(\mathcal{F}) \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I}[2]$$

(we have also composed with the map from the derived tensor product to the usual tensor product). In other words, we obtain an element

$$\xi_{U'} \in \mathrm{Ext}^2_{\mathcal{O}_{X \times_B U}}(E(\mathcal{F}), \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I})$$

Lemma 6.3. *In the situation of Remark 6.2 assume that $\mathcal{F}$ is flat over U. Then the vanishing of the class $\xi_{U'}$ is a necessary and sufficient condition for the existence of a $\mathcal{O}_{X \times_B U'}$-module $\mathcal{F}'$ flat over U' with $i^*\mathcal{F}' \cong \mathcal{F}$.*

Proof (sketch). We will use the criterion of Deformation Theory, Lemma 10.8. We will abbreviate $\mathcal{O} = \mathcal{O}_{X \times_B U}$ and $\mathcal{O}' = \mathcal{O}_{X \times_B U'}$. Consider the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{U'} \to \mathcal{O}_U \to 0.$$

Let $\mathcal{J} \subset \mathcal{O}'$ be the quasi-coherent sheaf of ideals cutting out $X \times_B U$. By the above we obtain an exact sequence

$$\mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \to q^*\mathcal{I} \to \mathcal{J} \to 0$$

where the $\mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U)$ is an abbreviation for

$$\mathrm{Tor}_1^{h^{-1}\mathcal{O}_B}(p^{-1}\mathcal{O}_X, q^{-1}\mathcal{O}_U) \otimes_{(p^{-1}\mathcal{O}_X \otimes_{h^{-1}\mathcal{O}_B} q^{-1}\mathcal{O}_U)} \mathcal{O}.$$

Tensoring with $\mathcal{F}$ we obtain the exact sequence

$$\mathcal{F} \otimes_\mathcal{O} \mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \to \mathcal{F} \otimes_\mathcal{O} q^*\mathcal{I} \to \mathcal{F} \otimes_\mathcal{O} \mathcal{J} \to 0$$

(Note that the roles of the letters $\mathcal{I}$ and $\mathcal{J}$ are reversed relative to the notation in Deformation Theory, Lemma 10.8.) Condition (1) of the lemma is that the last map above is an isomorphism, i.e., that the first map is zero. The vanishing of this map may be checked on stalks at geometric points $\overline{z} = (\overline{x}, \overline{u}) : \mathrm{Spec}(k) \to X \times_B U$. Set $R = \mathcal{O}_{B,\overline{b}}$, $A = \mathcal{O}_{X,\overline{x}}$, $B = \mathcal{O}_{U,\overline{u}}$, and $C = \mathcal{O}_{\overline{z}}$. By Cotangent, Lemma 27.2 and the defining triangle for $E(\mathcal{F})$ we see that

$$H^{-2}(E(\mathcal{F}))_{\overline{z}} = \mathcal{F}_{\overline{z}} \otimes \mathrm{Tor}_1^R(A, B)$$

The map $\xi_{U'}$ therefore induces a map

$$\mathcal{F}_{\overline{z}} \otimes \mathrm{Tor}_1^R(A, B) \longrightarrow \mathcal{F}_{\overline{z}} \otimes_B \mathcal{I}_{\overline{u}}$$

We claim this map is the same as the stalk of the map described above (proof omitted; this is a purely ring theoretic statement). Thus we see that condition (1) of Deformation Theory, Lemma 10.8 is equivalent to the vanishing $H^{-2}(\xi_{U'}) : H^{-2}(E(\mathcal{F})) \to \mathcal{F} \otimes \mathcal{I}$.

To finish the proof we show that, assuming that condition (1) is satisfied, condition (2) is equivalent to the vanising of $\xi_{U'}$. In the rest of the proof we write $\mathcal{F} \otimes \mathcal{I}$ to denote $\mathcal{F} \otimes_\mathcal{O} q^*\mathcal{I} = \mathcal{F} \otimes_\mathcal{O} \mathcal{J}$. A consideration of the spectral sequence

$$\mathrm{Ext}^i(H^{-j}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I}) \Rightarrow \mathrm{Ext}^{i+j}(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I})$$

using that $H^0(E(\mathcal{F})) = \mathcal{F}$ and $H^{-1}(E(\mathcal{F})) = 0$ shows that there is an exact sequence

$$0 \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I}) \to \mathrm{Ext}^2(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I}) \to \mathrm{Hom}(H^{-2}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I})$$

Thus our element $\xi_{U'}$ is an element of $\text{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I})$. The proof is finished by showing this element agrees with the element of Deformation Theory, Lemma 10.8 a verification we omit. $\square$

Lemma 6.4. *In Situation 5.1 assume that S is a locally Noetherian scheme and $S = B$. Let $\mathcal{X} = Coh_{X/B}$. Then we have openness of versality for $\mathcal{X}$ (see Artin's Axioms, Definition 14.1).*

Proof (sketch). Let $U \to S$ be of finite type morphism of schemes, x an object of $\mathcal{X}$ over U and $u_0 \in U$ a finite type point such that x is versal at u_0. After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 17.1) and $U = \text{Spec}(A)$ with $U \to S$ mapping into an affine open $\text{Spec}(\Lambda)$ of S. We will use Artin's Axioms, Lemma 21.4 to prove the lemma. Let $\mathcal{F}$ be the coherent module on $X_A = \text{Spec}(A) \times_S X$ flat over A corresponding to the given object x.

Choose $E(\mathcal{F})$ and $e_\mathcal{F}$ as in Remark 6.1. The description of the cohomology sheaves of $E(\mathcal{F})$ shows that

$$\text{Ext}^1(E(\mathcal{F}), \mathcal{F} \otimes_A M) = \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for any A-module M. Using this and using Deformation Theory, Lemma 10.7 we have an isomorphism of functors

$$T_x(M) = \text{Ext}^1_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

By Lemma 6.3 given any surjection $A' \to A$ of Λ-algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \text{Ext}^2_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A I)$$

Apply Derived Categories of Spaces, Lemma 19.3 to the computation of the Ext groups $\text{Ext}^i_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We omit the verification that $E(\mathcal{F})$ is in D^-_{Coh}; hint: use Cotangent, Lemma 5.4. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes^{\mathbf{L}}_A M) \longrightarrow \text{Ext}^i_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K, together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 21.2. Finally, condition (iv) of Artin's Axioms, Lemma 21.3 holds by a variant of Deformation Theory, Lemma 11.3 whose formulation and proof we omit. Thus Artin's Axioms, Lemma 21.4 applies and the lemma is proved. $\square$

Theorem 6.5 (Algebraicity of stack coherent sheaves; general case)**.** *Let S be a scheme. Let $f : X \to B$ be morphism of algebraic spaces over S. Assume that f is of finite presentation and separated. Then $Coh_{X/B}$ is an algebraic stack over S.*

Proof. Identical to the proof of Theorem 5.12 except that we substitute Lemma 6.4 for Lemma 5.11. $\square$

7. Flattening functors

This section is the analogue of More on Flatness, Section 19. We urge the reader to skip this section on a first reading.

Situation 7.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over B we will denote $u_T : \mathcal{F}_T \to \mathcal{G}_T$ the base change of u to T, in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_B T \to X$. In this situation we can consider the functor

$$(7.1.1) \qquad F_{iso} : (Sch/B)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants F_{inj}, F_{surj}, F_{zero} where we ask that u_T is injective, surjective, or zero.

In Situation 7.1 we sometimes think of the functors F_{iso}, F_{inj}, F_{surj}, and F_{zero} as functors $(Sch/S)^{opp} \to Sets$ endowed with a morphism $F_{iso} \to B$, $F_{inj} \to B$, $F_{surj} \to B$, and $F_{zero} \to B$. Namely, if T is a scheme over S, then an element $h \in F_{iso}(T)$ is just a morphism $h : T \to B$, i.e., an element $h \in B(T)$, such that the base change of u via h is an isomorphism. In particular, when we say that F_{iso} is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

Lemma 7.2. *In Situation 7.1. Each of the functors F_{iso}, F_{inj}, F_{surj}, F_{zero} satisfies the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over B. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 3.2. In particular, for every $x \in |X_T|$ there exists an $i \in I$ and an $x_i \in |X_i|$ mapping to x. Since $\mathcal{O}_{X_T,\overline{x}} \to \mathcal{O}_{X_i,\overline{x_i}}$ is flat, hence faithfully flat (see Morphisms of Spaces, Section 28). we conclude that $(u_i)_{x_i}$ is injective, surjective, bijective, or zero if and only if $(u_T)_x$ is injective, surjective, bijective, or zero. The lemma follows. □

Lemma 7.3. *In Situation 7.1 let $X' \to X$ be a flat morphism of algebraic spaces. Denote $u' : \mathcal{F}' \to \mathcal{G}'$ the pullback of u to X'. Denote F'_{iso}, F'_{inj}, F'_{surj}, F'_{zero} the functors on Sch/B associated to u'.*

1. *If $\mathcal{G}$ is of finite type and the image of $|X'| \to |X|$ contains the support of $\mathcal{G}$, then $F_{surj} = F'_{surj}$ and $F_{zero} = F'_{zero}$.*
2. *If $\mathcal{F}$ is of finite type and the image of $|X'| \to |X|$ contains the support of $\mathcal{F}$, then $F_{inj} = F'_{inj}$ and $F_{zero} = F'_{zero}$.*
3. *If $\mathcal{F}$ and $\mathcal{G}$ are of finite type and the image of $|X'| \to |X|$ contains the supports of $\mathcal{F}$ and $\mathcal{G}$, then $F_{iso} = F'_{iso}$.*

Proof. let $v : \mathcal{H} \to \mathcal{E}$ be a map of quasi-coherent modules on an algebraic space Y and let $\varphi : Y' \to Y$ be a surjective flat morphism of algebraic spaces, then v is an isomorphism, injective, surjective, or zero if and only if $\varphi^* v$ is an isomorphism, injective, surjective, or zero. Namely, for every $y \in |Y|$ there exists a $y' \in |Y'|$ and the map of local rings $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{Y',\overline{y'}}$ is faithfully flat (see Morphisms of Spaces, Section 28). Of course, to check for injectivity or being zero it suffices to look at the points in the support of $\mathcal{H}$, and to check for surjectivity it suffices to look at points in the support of $\mathcal{E}$. Moreover, under the finite type assumptions as in the statement of the lemma, taking the supports commutes with base change, see Morphisms of Spaces, Lemma 15.2. Thus the lemma is clear. □

Recall that we've defined the scheme theoretic support of a finite type quasi-coherent module in Morphisms of Spaces, Definition 15.4.

Lemma 7.4. *In Situation 7.1.*

1. *If $\mathcal{G}$ is of finite type and the scheme theoretic support of $\mathcal{G}$ is quasi-compact over B, then F_{surj} is limit preserving.*
2. *If $\mathcal{F}$ of finite type and the scheme theoretic support of $\mathcal{F}$ is quasi-compact over B, then F_{zero} is limit preserving.*
3. *If $\mathcal{F}$ is of finite type, $\mathcal{G}$ is of finite presentation, and the scheme theoretic supports of $\mathcal{F}$ and $\mathcal{G}$ are quasi-compact over B, then F_{iso} is limit preserving.*

Proof. Proof of (1). Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{G}$ and think of $\mathcal{G}$ as a finite type quasi-coherent module on Z. We may replace X by Z and u by the map $i^*\mathcal{F} \to \mathcal{G}$ (details omitted). Hence we may assume f is quasi-compact and $\mathcal{G}$ of finite type. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \to \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove (1) we have to show that u_i is surjective for some i. Pick $0 \in I$ and replace I by $\{i \mid i \geq 0\}$. Since f is quasi-compact we see X_0 is quasi-compact. Hence we may choose a surjective étale morphism $\varphi_0 : W_0 \to X_0$ where W_0 is an affine scheme. Set $W = W_0 \times_{T_0} T$ and $W_i = W_0 \times_{T_0} T_i$ for $i \geq 0$. These are affine schemes endowed with a surjective étale morphisms $\varphi : W \to X_T$ and $\varphi_i : W_i \to X_i$. Note that $W = \lim W_i$. Hence $\varphi^* u_T$ is surjective and it suffices to prove that $\varphi_i^* u_i$ is surjective for some i. Thus we have reduced the problem to the affine case which is Algebra, Lemma 123.3 part (2).

Proof of (2). Assume $\mathcal{F}$ is of finite type with scheme theoretic support $Z \subset B$ quasi-compact over B. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is zero. Set $X_i = T_i \times_B X$ and denote $u_i : \mathcal{F}_i \to \mathcal{G}_i$ the pullback. Choose $0 \in I$ and replace I by $\{i \mid i \geq 0\}$. Set $Z_0 = Z \times_X X_0$. By Morphisms of Spaces, Lemma 15.2 the support of $\mathcal{F}_i$ is $|Z_0|$. Since $|Z_0|$ is quasi-compact we can find an affine scheme W_0 and an étale morphism $W_0 \to X_0$ such that $|Z_0| \subset \text{Im}(|W_0| \to |X_0|)$. Set $W = W_0 \times_{T_0} T$ and $W_i = W_0 \times_{T_0} T_i$ for $i \geq 0$. These are affine schemes endowed with étale morphisms $\varphi : W \to X_T$ and $\varphi_i : W_i \to X_i$. Note that $W = \lim W_i$ and that the support of $\mathcal{F}_T$ and $\mathcal{F}_i$ is contained in the image of $|W| \to |X_T|$ and $|W_i| \to |X_i|$. Now $\varphi^* u_T$ is injective and it suffices to prove that $\varphi_i^* u_i$ is injective for some i. Thus we have reduced the problem to the affine case which is Algebra, Lemma 123.3 part (1).

Proof of (3). This can be proven in exactly the same manner as in the previous two paragraphs using Algebra, Lemma 123.3 part (3). We can also deduce it from (1) and (2) as follows. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is an isomorphism. By part (1) there exists an $0 \in I$ such that u_{T_0} is surjective. Set $\mathcal{K} = \text{Ker}(u_{T_0})$ and consider the map of quasi-coherent modules $v : \mathcal{K} \to \mathcal{F}_{T_0}$. For $i \geq 0$ the base change v_{T_i} is zero if and only if u_i is an isomorphism. Moreover, v_T is zero. Since $\mathcal{G}_{T_0}$ is of finite presentation, $\mathcal{F}_{T_0}$ is of finite type, and u_{T_0} is surjective we conclude that $\mathcal{K}$ is of finite type (Modules on Sites, Lemma 24.1). It is clear that the support of $\mathcal{K}$ is contained in the support of $\mathcal{F}_{T_0}$ which is quasi-compact over T_0. Hence we can apply part (2) to see that v_{T_i} is zero for some i. $\square$

Lemma 7.5. *Let $S = \mathrm{Spec}(R)$ be an affine scheme. Let X be an algebraic space over S. Let $u : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{G}$ flat over S. Let $T \to S$ be a quasi-compact morphism of schemes such that the base change u_T is zero. Then exists a closed subscheme $Z \subset S$ such that (a) $T \to S$ factors through Z and (b) the base change u_Z is zero. If $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module and the scheme theoretic support of $\mathcal{F}$ is quasi-compact, then we can take $Z \to S$ of finite presentation.*

Proof. Let $U \to X$ be a surjective étale morphism of algebraic spaces where $U = \coprod U_i$ is a disjoint union of affine schemes (see Properties of Spaces, Lemma 6.1). By Lemma 7.3 we see that we may replace X by U. In other words, we may assume that $X = \coprod X_i$ is a disjoint union of affine schemes X_i. Suppose that we can prove the lemma for $u_i = u|_{X_i}$. Then we find a closed subscheme $Z_i \subset S$ such that $T \to S$ factors through Z_i and u_{i,Z_i} is zero. If $Z_i = \mathrm{Spec}(R/I_i) \subset \mathrm{Spec}(R) = S$, then taking $Z = \mathrm{Spec}(R/\sum I_i)$ works. Thus we may assume that $X = \mathrm{Spec}(A)$ is affine.

Choose a finite affine open covering $T = T_1 \cup \ldots \cup T_m$. It is clear that we may replace T by $\coprod_{j=1,\ldots,m} T_j$. Hence we may assume T is affine. Say $T = \mathrm{Spec}(R')$. Let $u : M \to N$ be the homomorphisms of A-modules corresponding to $u : \mathcal{F} \to \mathcal{G}$. Then N is a flat R-module as $\mathcal{G}$ is flat over S. The assumption of the lemma means that the composition

$$M \otimes_R R' \to N \otimes_R R'$$

is zero. Let $z \in M$. By Lazard's theorem (Algebra, Theorem 78.4) and the fact that $\otimes$ commutes with colimits we can find free R-module F_z, an element $\tilde{z} \in F_z$, and a map $F_z \to N$ such that $u(z)$ is the image of $\tilde{z}$ and $\tilde{z}$ maps to zero in $F_z \otimes_R R'$. Choose a basis $\{e_{z,\alpha}\}$ of F_z and write $\tilde{z} = \sum f_{z,\alpha} e_{z,\alpha}$ with $f_{z,\alpha} \in R$. Let $I \subset R$ be the ideal generated by the elements $f_{z,\alpha}$ with z ranging over all elements of M. By construction I maps to zero in R' and the elements $\tilde{z}$ map to zero in F_z/IF_z whence in N/IN. Thus $Z = \mathrm{Spec}(R/I)$ is a solution to the problem in this case.

Assume $\mathcal{F}$ is of finite type with quasi-compact scheme theoretic support. Write $Z = \mathrm{Spec}(R/I)$. Write $I = \bigcup I_\lambda$ as a filtered union of finitely generated ideals. Set $Z_\lambda = \mathrm{Spec}(R/I_\lambda)$, so $Z = \mathrm{colim}\, Z_\lambda$. Since u_Z is zero, we see that u_{Z_λ} is zero for some λ by Lemma 7.4. This finishes the proof of the lemma. $\square$

Lemma 7.6. *Let A be a ring. Let $u : M \to N$ be a map of A-modules. If N is projective as an A-module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \to B$ the following are equivalent*

1. *$u \otimes 1 : M \otimes_A B \to N \otimes_A B$ is zero, and*
2. *$\varphi(I) = 0$.*

Proof. As M is projective we can find a projective A-module C such that $F = N \oplus C$ is a free R-module. By replacing u by $u \oplus 1 : F = M \oplus C \to N \oplus C$ we see that we may assume N is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\mathrm{Im}(u)$ with respect to some (fixed) basis of N. $\square$

It would be interesting to find a simple direct proof of the following lemma using the result of Lemma 7.5. A "classical" proof of this lemma when $f : X \to B$ is a projective morphism and B a Noetherian scheme would be: (a) choose a relatively ample invertible sheaf $\mathcal{O}_X(1)$, (b) set $u_n : f_*\mathcal{F}(n) \to f_*\mathcal{G}(n)$, (c) observe that

$f_*\mathcal{G}(n)$ is a finite locally free sheaf for all $n \gg 0$, and (d) F_{zero} is represented by the vanishing locus of u_n for some $n \gg 0$.

Lemma 7.7. *In Situation 7.1. Assume*

1. *f is locally of finite presentation,*
2. *$\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation flat over B,*
3. *the scheme theoretic support of $\mathcal{G}$ is proper over B.*

Then the functor F_{zero} is an algebraic space and $F_{zero} \to B$ is a closed immersion. If $\mathcal{F}$ is of finite type, then $F_{zero} \to B$ is of finite presentation.

Proof. In order to prove that F_{zero} is an algebraic space, it suffices to show that $F_{zero} \to B$ is representable, see Spaces, Lemma 11.1. Let $B' \to B$ be a morphism where B' is a scheme and let $u' : \mathcal{F}' \to \mathcal{G}'$ be the pullback of u to $X' = X_{B'}$. Then the associated functor F'_{zero} equals $F_{zero} \times_B B'$. This reduces us to the case that B is a scheme.

Assume B is a scheme. We will show that F_{zero} is representable by a closed subscheme of B. By Lemma 7.2 and Descent, Lemmas 33.2 and 35.1 the question is local for the étale topology on B. Let $b \in B$. We first replace B by an affine neighbourhood of b. Denote $Z \subset X$ the scheme theoretic support of $\mathcal{G}$. Denote $Z_b \subset X_b$ the fibre of $Z \subset X \to B$ over b. The space $|Z_b|$ is quasi-compact by the last assumption of the lemma. Choose an affine scheme U and an étale morphism $\varphi : U \to X$ such that $|Z_b| \subset \mathrm{Im}(|U| \to |X|)$. After replacing B by an affine elementary étale neighbourhood of b and replacing U by some affine U' étale over U with $U'_b \to U_b$ surjective, we may assume that $\Gamma(U, \varphi^*\mathcal{G})$ is a projective $\Gamma(B, \mathcal{O}_B)$-module, see More on Flatness, Lemma 11.5. Since $Z \to B$ is proper the image of

$$|Z| \setminus \mathrm{Im}(|U| \to |X|)$$

in $|B|$ is a closed subset not containing b. Hence, after replacing B by an affine open containing b, we may assume that $|Z| \subset \mathrm{Im}(|U| \to |X|)$. (To be sure, after this replacement it is still true that $\Gamma(U, \varphi^*\mathcal{G})$ is a projective $\Gamma(B, \mathcal{O}_B)$-module.) By Lemma 7.3 we see that F_{zero} is the same as the corresponding functor for the map $\varphi^*\mathcal{F} \to \varphi^*\mathcal{G}$. This case follows immediately from Lemma 7.6.

We still have to show that $F_{zero} \to B$ is of finite presentation if $\mathcal{F}$ is of finite type. Let $\mathcal{F}' \subset \mathcal{G}$ be the image of u and denote F'_{zero} the functor corresponding to $\mathcal{F}' \to \mathcal{G}$. Then $F_{zero} = F'_{zero}$ and the scheme theoretic support of $\mathcal{F}'$ is a closed subspace of the scheme theoretic support of $\mathcal{G}$, hence proper over B. Thus Lemma 7.4 implies that $F_{zero} = F'_{zero}$ is limit preserving over B. We conclude by Limits of Spaces, Proposition 3.9. $\square$

The following result is a variant of More on Flatness, Theorem 22.3.

Lemma 7.8. *In Situation 7.1. Assume*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is locally of finite presentation and flat over B,*
3. *the scheme theoretic support of $\mathcal{F}$ is proper over B, and*
4. *u is surjective.*

Then the functor F_{iso} is an algebraic space and $F_{iso} \to B$ is a closed immersion. If $\mathcal{G}$ is of finite presentation, then $F_{iso} \to B$ is of finite presentation.

Proof. Let $\mathcal{K} = \mathrm{Ker}(u)$ and apply Lemma 7.7 to $\mathcal{K} \to \mathcal{F}$. Note that $\mathcal{K}$ is of finite type if $\mathcal{G}$ is of finite presentation, see Modules on Sites, Lemma 24.1. $\square$

We will use the following (easy) result when discussing the Quot functor.

Lemma 7.9. *In Situation 7.1. Assume*

1. *f is locally of finite presentation,*
2. *$\mathcal{G}$ is of finite type,*
3. *the scheme theoretic support of $\mathcal{G}$ is proper over B.*

Then F_{surj} is an algebraic space and $F_{surj} \to B$ is an open immersion.

Proof. Consider $\mathrm{Coker}(u)$. Observe that $\mathrm{Coker}(u_T) = \mathrm{Coker}(u)_T$ for any T/B. Note that formation of the support of a finite type quasi-coherent module commutes with pullback (Morphisms of Spaces, Lemma 15.1). Hence F_{surj} is representable by the open subspace of B corresponding to the open set

$$|B| \setminus |f|(\mathrm{Supp}(\mathrm{Coker}(u)))$$

see Properties of Spaces, Lemma 4.8. This is an open because $|f|$ is closed on $\mathrm{Supp}(\mathcal{G})$ and $\mathrm{Supp}(\mathrm{Coker}(u))$ is a closed subset of $\mathrm{Supp}(\mathcal{G})$. $\square$

8. The functor of quotients

In this section we discuss some generalities regarding the functor $Q_{\mathcal{F}/X/B}$ defined below. The notation $\mathrm{Quot}_{\mathcal{F}/X/B}$ is reserved for a subfunctor of $\mathrm{Q}_{\mathcal{F}/X/B}$. We urge the reader to skip this section on a first reading.

Situation 8.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over B we will denote X_T the base change of X to T and $\mathcal{F}_T$ the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Given such a T we set

$$\mathrm{Q}_{\mathcal{F}/X/B}(T) = \left\{ \begin{matrix} \text{quotients } \mathcal{F}_T \to \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a quasi-coherent} \\ \mathcal{O}_{X_T}\text{-module of finite presentation, flat over } T \end{matrix} \right\}$$

We identify quotients if they have the same kernel. Suppose that $T' \to T$ is a morphism of schemes over B and $\mathcal{F}_T \to \mathcal{Q}$ is an element of $\mathrm{Q}_{\mathcal{F}/X/B}(T)$. Then the pullback $\mathcal{Q}' = (X_{T'} \to X_T)^*\mathcal{Q}$ is a quasi-coherent $\mathcal{O}_{X_{T'}}$-module of finite presentation flat over T' (see Properties of Spaces, Section 28 and Morphisms of Spaces, Lemma 29.3). Thus we obtain a functor

$$\mathrm{Q}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow Sets \tag{8.1.1}$$

This is the functor of quotients of $\mathcal{F}/X/B$.

In Situation 8.1 we sometimes think of $\mathrm{Q}_{\mathcal{F}/X/B}$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $\mathrm{Q}_{\mathcal{F}/X/S} \to B$. Namely, if T is a scheme over S, then we can think of an element of $\mathrm{Q}_{\mathcal{F}/X/B}$ as a pair $(h, \mathcal{Q})$ where h a morphism $h : T \to B$, i.e., an element $h \in B(T)$, and $\mathcal{Q}$ is a T-flat quotient $\mathcal{F}_T \to \mathcal{Q}$ of finite presentation on $X_T = X \times_{B,h} T$. In particular, when we say that $\mathrm{Q}_{\mathcal{F}/X/S}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

Remark 8.2. In Situation 8.1 let $B' \to B$ be a morphism of algebraic spaces over S. Set $X' = X \times_B B'$ and denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to X'. Thus we have the functor $Q_{\mathcal{F}'/X'/B'}$ on the category of schemes over B'. For a scheme T over B' it is clear that we have

$$Q_{\mathcal{F}'/X'/B'}(T) = Q_{\mathcal{F}/X/B}(T)$$

where on the right hand side we think of T as a scheme over B via the composition $T \to B' \to B$. This trivial remark will occasionally be useful to change the base algebraic space.

Remark 8.3. Let S be a scheme, X an algebraic space over S, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Suppose that $\{f_i : X_i \to X\}_{i \in I}$ is an fpqc covering and for each $i, j \in I$ we are given an fpqc covering $\{X_{ijk} \to X_i \times_X X_j\}$. In this situation we have a bijection

$$\left\{\begin{matrix} \text{quotients } \mathcal{F} \to \mathcal{Q} \text{ where} \\ \mathcal{Q} \text{ is a quasi-coherent} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \text{families of quotients } f_i^*\mathcal{F} \to \mathcal{Q}_i \text{ where} \\ \mathcal{Q}_i \text{ is quasi-coherent and } \mathcal{Q}_i \text{ and } \mathcal{Q}_j \\ \text{restrict to the same quotient on } X_{ijk} \end{matrix}\right\}$$

Namely, let $(f_i^*\mathcal{F} \to \mathcal{Q}_i)_{i \in I}$ be an element of the right hand side. Then since $\{X_{ijk} \to X_i \times_X X_j\}$ is an fpqc covering we see that the pullbacks of $\mathcal{Q}_i$ and $\mathcal{Q}_j$ restrict to the same quotient of the pullback of $\mathcal{F}$ to $X_i \times_X X_j$ (by fully faithfulness in Descent on Spaces, Proposition 4.1). Hence we obtain a descent datum for quasi-coherent modules with respect to $\{X_i \to X\}_{i \in I}$. By Descent on Spaces, Proposition 4.1 we find a map of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{Q}$ whose restriction to X_i recovers the given maps $f_i^*\mathcal{F} \to \mathcal{Q}_i$. Since the family of morphisms $\{X_i \to X\}$ is jointly surjective and flat, for every point $x \in |X|$ there exists an i and a point $x_i \in |X_i|$ mapping to x. Note that the induced map on local rings $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X_i,\overline{x_i}}$ is faithfully flat, see Morphisms of Spaces, Section 28. Thus we see that $\mathcal{F} \to \mathcal{Q}$ is surjective.

Lemma 8.4. *In Situation 8.1. The functor $Q_{\mathcal{F}/X/B}$ satisfies the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and $\mathcal{F}_i = \mathcal{F}_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T (Topologies on Spaces, Lemma 3.2) and that $X_{T_i \times_T T_{i'}} = X_i \times_{X_T} X_{i'}$. Suppose that $\mathcal{F}_i \to \mathcal{Q}_i$ is a collection of elements of $\mathrm{Q}_{\mathcal{F}/X/B}(T_i)$ such that $\mathcal{Q}_i$ and $\mathcal{Q}_{i'}$ restrict to the same element of $\mathrm{Q}_{\mathcal{F}/X/B}(T_i \times_T T_{i'})$. By Remark 8.3 we obtain a surjective map of quasi-coherent $\mathcal{O}_{X_T}$-modules $\mathcal{F}_T \to \mathcal{Q}$ whose restriction to X_i recovers the given quotients. By Morphisms of Spaces, Lemma 29.5 we see that $\mathcal{Q}$ is flat over T. Finally, Descent on Spaces, Lemma 5.2 guarantees that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$-module. $\square$

Lemma 8.5. *In Situation 8.1 let $\{X_i \to X\}_{i \in I}$ be an fppf covering and for each $i, j \in I$ let $\{X_{ijk} \to X_i \times_X X_j\}$ be an fppf covering. Denote $\mathcal{F}_i$, resp. $\mathcal{F}_{ijk}$ the pullback of $\mathcal{F}$ to X_i, resp. X_{ijk}. For every scheme T over B the diagram*

$$Q_{\mathcal{F}/X/B}(T) \longrightarrow \prod\nolimits_i Q_{\mathcal{F}_i/X_i/B}(T) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod\nolimits_{i,j,k} Q_{\mathcal{F}_{ijk}/X_{ijk}/B}(T)$$

presents the first arrow as the equalizer of the other two.

Proof. Let $\mathcal{F}_{i,T} \to \mathcal{Q}_i$ be an element in the equalizer of pr_0^* and pr_1^*. By Remark 8.3 we obtain a surjection $\mathcal{F}_T \to \mathcal{Q}$ of quasi-coherent $\mathcal{O}_{X_T}$-modules whose restriction to $X_{i,T}$ recovers $\mathcal{F}_i \to \mathcal{Q}_i$. By Morphisms of Spaces, Lemma 29.5 we see that $\mathcal{Q}$ is flat over T as desired. $\square$

Lemma 8.6. *In Situation 8.1 assume also that (a) f is quasi-compact and quasi-separated and (b) $\mathcal{F}$ is of finite presentation. Then the functor $Q_{\mathcal{F}/X/B}$ is limit preserving in the following sense: If $T = \lim T_i$ is a directed limit of affine schemes over B, then $Q_{\mathcal{F}/X/B}(T) = \text{colim}\, Q_{\mathcal{F}/X/B}(T_i)$.*

Proof. Let $T = \lim T_i$ be as in the statement of the lemma. Choose $i_0 \in I$ and replace I by $\{i \in I \mid i \geq i_0\}$. We may set $B = S = T_{i_0}$ and we may replace X by X_{T_0} and $\mathcal{F}$ by the pullback to X_{T_0}. Then $X_T = \lim X_{T_i}$, see Limits of Spaces, Lemma 4.1. Let $\mathcal{F}_T \to \mathcal{Q}$ be an element of $\mathrm{Q}_{\mathcal{F}/X/B}(T)$. By Limits of Spaces, Lemma 7.2 there exists an i and a map $\mathcal{F}_{T_i} \to \mathcal{Q}_i$ of $\mathcal{O}_{X_{T_i}}$-modules of finite presentation whose pullback to X_T is the given quotient map.

We still have to check that, after possibly increasing i, the map $\mathcal{F}_{T_i} \to \mathcal{Q}_i$ is surjective and $\mathcal{Q}_i$ is flat over T_i. To do this, choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 6.3). We may check surjectivity and flatness over T_i after pulling back to the étale cover $U_{T_i} \to X_{T_i}$ (by definition). This reduces us to the case where $X = \text{Spec}(B_0)$ is an affine scheme of finite presentation over $B = S = T_0 = \text{Spec}(A_0)$. Writing $T_i = \text{Spec}(A_i)$, then $T = \text{Spec}(A)$ with $A = \text{colim}\, A_i$ we have reached the following algebra problem. Let $M_i \to N_i$ be a map of finitely presented $B_0 \otimes_{A_0} A_i$-modules such that $M_i \otimes_{A_i} A \to N_i \otimes_{A_i} A$ is surjective and $N_i \otimes_{A_i} A$ is flat over A. Show that for some $i' \geq i$ $M_i \otimes_{A_i} A_{i'} \to N_i \otimes_{A_i} A_{i'}$ is surjective and $N_i \otimes_{A_i} A_{i'}$ is flat over A. The first follows from Algebra, Lemma 123.3 and the second from Algebra, Lemma 156.1. $\square$

Lemma 8.7. *In Situation 8.1 assume $X \to B$ locally of finite presentation. Let*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

be a pushout in the category of schemes over B where $Z \to Z'$ is a thickening and $Z \to Y$ is affine, see More on Morphisms, Lemma 11.1. Then the natural map

$$Q_{\mathcal{F}/X/B}(Y') \longrightarrow Q_{\mathcal{F}/X/B}(Y) \times_{Q_{\mathcal{F}/X/B}(Z)} Q_{\mathcal{F}/X/B}(Z')$$

is bijective.

Proof. We first argue that it suffices to prove this when all the schemes and algebraic spaces in sight are affine schemes. Let $Y' = \bigcup Y'_i$ be an affine open covering and let Y_i, Z'_i, and Z_i be the corresponding (affine) opens of Y, Z', and Z. Since $Q_{\mathcal{F}/X/B}$ satisfies the sheaf property for the fpqc topology (Lemma 8.4), it suffices to prove the result of the lemma for the diagrams

$$\begin{tikzcd} Z_i \arrow[r] \arrow[d] & Z'_i \arrow[d] \\ Y_i \arrow[r] & Y'_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z_i \cap Z_j \arrow[r] \arrow[d] & Z'_i \cap Z'_j \arrow[d] \\ Y_i \cap Y_j \arrow[r] & Y'_i \cap Y'_j \end{tikzcd}$$

This reduces us to the case where the schemes Y', Y, Z', Z are separated and a second application of this argument to the case where Y', Y, Z', Z are affine.

Assume Y' (and hence also Y, Z', and Z) is affine. By Remark 8.2 we may replace B by Y' and X by $X \times_B Y'$, and $\mathcal{F}$ by the pullback. Thus we may assume $B = Y'$.

Assume $B = Y'$ (and hence also Y, Z', and Z) is affine. Choose an étale covering $\{X_i \to X\}_{i \in I}$ with each X_i affine and similarly choose étale coverings $\{X_{ijk} \to X_i \times_X X_j\}$ with each X_{ijk} affine (Properties of Spaces, Lemma 6.1). By Lemma 8.5 it suffices to prove the lemma for each of the functors associated to X_i and X_{ijk}. Hence we may assume X is affine as well. This reduces the lemma to More on Algebra, Remark 4.15. □

9. The quot functor

In this section we prove the Quot functor is representable by an algebraic space.

Situation 9.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over B we will denote X_T the base change of X to T and $\mathcal{F}_T$ the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Given such a T we set

$$\mathrm{Quot}_{\mathcal{F}/X/B}(T) = \left\{ \begin{matrix} \text{quotients } \mathcal{F}_T \to \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a quasi-coherent} \\ \mathcal{O}_{X_T}\text{-module of finite presentation, flat over } T \\ \text{with scheme theoretic support proper over } T \end{matrix} \right\}$$

This is a subfunctor of $Q_{\mathcal{F}/X/T}$ discussed in Section 8. Thus we obtain a functor

$$\mathrm{Quot}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow Sets \tag{9.1.1}$$

This is the *quot functor* associated to $\mathcal{F}/X/B$.

In Situation 9.1 we may think of $\mathrm{Quot}_{\mathcal{F}/X/B}$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $\mathrm{Quot}_{\mathcal{F}/X/S} \to B$. Namely, if T is a scheme over S, then we can think of an element of $\mathrm{Quot}_{\mathcal{F}/X/B}$ as a pair $(h, \mathcal{Q})$ where h a morphism $h : T \to B$, i.e., an element $h \in B(T)$, and Q is a finitely presented, T-flat quotient $\mathcal{F}_T \to \mathcal{Q}$ on $X_T = X \times_{B,h} T$ with support proper over T. In particular, when we say that $\mathrm{Quot}_{\mathcal{F}/X/S}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

Lemma 9.2. *In Situation 9.1. The functor $\mathrm{Quot}_{\mathcal{F}/X/B}$ satisfies the sheaf property for the fpqc topology.*

Proof. In Lemma 8.4 we have seen that the functor $\mathrm{Q}_{\mathcal{F}/X/S}$ is a sheaf. Recall that for a scheme T over S the subset $\mathrm{Quot}_{\mathcal{F}/X/S}(T) \subset \mathrm{Q}_{\mathcal{F}/X/S}(T)$ picks out those quotients whose support is proper over T. This defines a subsheaf by the result of Descent on Spaces, Lemma 10.17 (combined with Morphisms of Spaces, Lemma 28.10) which shows that taking scheme theoretic support commutes with flat base change). □

Proposition 9.3. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. If f is of finite presentation and separated, then $\mathrm{Quot}_{\mathcal{F}/X/B}$ is an algebraic space. If $\mathcal{F}$ is of finite presentation, then $\mathrm{Quot}_{\mathcal{F}/X/B} \to B$ is locally of finite presentation.*

Proof. Note that $\mathrm{Quot}_{\mathcal{F}/X/B}$ is a sheaf in the fppf topology. Let $\mathit{Quot}_{\mathcal{F}/X/B}$ be the stack in groupoids corresponding to $\mathrm{Quot}_{\mathcal{F}/X/S}$, see Algebraic Stacks, Section 7. By Algebraic Stacks, Proposition 13.3 it suffices to show that $\mathit{Quot}_{\mathcal{F}/X/B}$ is an algebraic stack. Consider the 1-morphism of stacks in groupoids

$$\mathit{Quot}_{\mathcal{F}/X/S} \longrightarrow \mathit{Coh}_{X/B}$$

on $(Sch/S)_{fppf}$ which associates to the quotient $\mathcal{F}_T \to \mathcal{Q}$ the coherent sheaf $\mathcal{Q}$. By Theorem 6.5 we know that $\mathit{Coh}_{X/B}$ is an algebraic stack. By Algebraic Stacks, Lemma 15.4 it suffices to show that this 1-morphism is representable by algebraic spaces.

Let T be a scheme over S and let the object $(h, \mathcal{G})$ of $\mathit{Coh}_{X/B}$ over T correspond to a 1-morphism $\xi : (Sch/T)_{fppf} \to \mathit{Coh}_{X/B}$. The 2-fibre product

$$\mathcal{Z} = (Sch/T)_{fppf} \times_{\xi, \mathit{Coh}_{X/B}} \mathit{Quot}_{\mathcal{F}/X/S}$$

is a stack in setoids, see Stacks, Lemma 6.7. The corresponding sheaf of sets (i.e., functor, see Stacks, Lemmas 6.7 and 6.2) assigns to a scheme T'/T the set of surjections $u : \mathcal{F}_{T'} \to \mathcal{G}_{T'}$ of quasi-coherent modules on $X_{T'}$. Thus we see that $\mathcal{Z}$ is representable by an open subspace (by Lemma 7.9) of the algebraic space $\mathit{Hom}(\mathcal{F}_T, \mathcal{G})$ from Proposition 3.9. $\square$

10. Other chapters

References

[Art69] Michael Artin, *Algebraization of formal moduli: I*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 21–71.

[DG67] Jean Dieudonné and Alexander Grothendieck, *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1961–1967).

[Gro95a] Alexander Grothendieck, *Technique de descente et théorèmes d'existence en géometrie algébrique. I. Généralités. Descente par morphismes fidèlement plats*, Séminaire Bourbaki, Vol. 5, Soc. Math. France, Paris, 1995, pp. 299–327.

[Gro95b] ______, *Technique de descente et théorèmes d'existence en géométrie algébrique. II. Le théorème d'existence en théorie formelle des modules*, Séminaire Bourbaki, Vol. 5, Soc. Math. France, Paris, 1995, pp. 369–390.

[Gro95c] ______, *Technique de descente et théorèmes d'existence en géométrie algébrique. V. Les schémas de Picard: théorèmes d'existence*, Séminaire Bourbaki, Vol. 7, Soc. Math. France, Paris, 1995, pp. 143–161.

[Gro95d] ______, *Technique de descente et théorèmes d'existence en géométrie algébrique. VI. Les schémas de Picard: propriétés générales*, Séminaire Bourbaki, Vol. 7, Soc. Math. France, Paris, 1995, pp. 221–243.

[Gro95e] ______, *Techniques de construction et théorèmes d'existence en géométrie algébrique. III. Préschemas quotients*, Séminaire Bourbaki, Vol. 6, Soc. Math. France, Paris, 1995, pp. 99–118.

[Gro95f] ______, *Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6, Soc. Math. France, Paris, 1995, pp. 249–276.

[HR10] Jack Hall and David Rydh, *The hilbert stack*.

[HR13] ______, *General hilbert stacks and quot schemes*.

[Lie06] Max Lieblich, *Remarks on the stack of coherent algebras*, Int. Math. Res. Not. (2006).

[Ols05] Martin Christian Olsson, *On proper coverings of Artin stacks*, Adv. Math. **198** (2005), no. 1, 93–106.

[OS03] Martin Christian Olsson and Jason Starr, *Quot functors for Deligne-Mumford stacks*, Comm. Algebra **31** (2003), no. 8, 4069–4096, Special issue in honor of Steven L. Kleiman.

[Ryd11] David Rydh, *Representability of Hilbert schemes and Hilbert stacks of points*, Comm. Algebra **39** (2011), no. 7, 2632–2646.