

# MODULES ON SITES

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## 1. Introduction

In this document we work out basic notions of sheaves of modules on ringed topoi or ringed sites. We first work out some basic facts on abelian sheaves. After this we introduce ringed sites and ringed topoi. We work through some of the very basic notions on (pre)sheaves of  $\mathcal{O}$ -modules, analogous to the material on (pre)sheaves of  $\mathcal{O}$ -modules in the chapter on sheaves on spaces. Having done this, we duplicate much of the discussion in the chapter on sheaves of modules (see Modules, Section 1). Basic references are [Ser55], [DG67] and [AGV71].

## 2. Abelian presheaves

Let  $\mathcal{C}$  be a category. Abelian presheaves were introduced in Sites, Sections 2 and 7 and discussed a bit more in Sites, Section 43. We will follow the convention of this last reference, in that we think of an abelian presheaf as a presheaf of sets endowed with addition rules on all sets of sections compatible with the restriction mappings. Recall that the category of abelian presheaves on  $\mathcal{C}$  is denoted  $PAb(\mathcal{C})$ .

The category  $PAb(\mathcal{C})$  is abelian as defined in Homology, Definition 5.1. Given a map of presheaves  $\varphi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  the kernel of  $\varphi$  is the abelian presheaf  $U \mapsto \text{Ker}(\mathcal{G}_1(U) \rightarrow \mathcal{G}_2(U))$  and the cokernel of  $\varphi$  is the presheaf  $U \mapsto \text{Coker}(\mathcal{G}_1(U) \rightarrow \mathcal{G}_2(U))$ . Since the category of abelian groups is abelian it follows that  $\text{Coim} = \text{Im}$  because this holds over each  $U$ . A sequence of abelian presheaves

$$\mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_3$$

is exact if and only if  $\mathcal{G}_1(U) \rightarrow \mathcal{G}_2(U) \rightarrow \mathcal{G}_3(U)$  is an exact sequence of abelian groups for all  $U \in \text{Ob}(\mathcal{C})$ . We leave the verifications to the reader.

**Lemma 2.1.** *Let  $\mathcal{C}$  be a category.*

- (1) *All limits and colimits exist in  $PAb(\mathcal{C})$ .*
- (2) *All limits and colimits commute with taking sections over objects of  $\mathcal{C}$ .*

**Proof.** Let  $\mathcal{I} \rightarrow PAb(\mathcal{C})$ ,  $i \mapsto \mathcal{F}_i$  be a diagram. We can simply define abelian presheaves  $L$  and  $C$  by the rules

$$L : U \longmapsto \lim_i \mathcal{F}_i(U)$$

and

$$C : U \longmapsto \text{colim}_i \mathcal{F}_i(U).$$

It is clear that there are maps of abelian presheaves  $L \rightarrow \mathcal{F}_i$  and  $\mathcal{F}_i \rightarrow C$ , by using the corresponding maps on groups of sections over each  $U$ . It is straightforward to check that  $L$  and  $C$  endowed with these maps are the limit and colimit of the diagram in  $PAb(\mathcal{C})$ . This proves (1) and (2). Details omitted.  $\square$

### 3. Abelian sheaves

Let  $\mathcal{C}$  be a site. The category of abelian sheaves on  $\mathcal{C}$  is denoted  $Ab(\mathcal{C})$ . It is the full subcategory of  $PAb(\mathcal{C})$  consisting of those abelian presheaves whose underlying presheaves of sets are sheaves. Properties  $(\alpha) - (\zeta)$  of Sites, Section 43 hold, see Sites, Proposition 43.3. In particular the inclusion functor  $Ab(\mathcal{C}) \rightarrow PAb(\mathcal{C})$  has a left adjoint, namely the sheafification functor  $\mathcal{G} \mapsto \mathcal{G}^\#$ .

We suggest the reader prove the lemma on a piece of scratch paper rather than reading the proof.

**Lemma 3.1.** *Let  $\mathcal{C}$  be a site. Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of abelian sheaves on  $\mathcal{C}$ .*

- (1) *The category  $Ab(\mathcal{C})$  is an abelian category.*
- (2) *The kernel  $Ker(\varphi)$  of  $\varphi$  is the same as the kernel of  $\varphi$  as a morphism of presheaves.*
- (3) *The morphism  $\varphi$  is injective (Homology, Definition 5.3) if and only if  $\varphi$  is injective as a map of presheaves (Sites, Definition 3.1), if and only if  $\varphi$  is injective as a map of sheaves (Sites, Definition 12.1).*
- (4) *The cokernel  $Coker(\varphi)$  of  $\varphi$  is the sheafification of the cokernel of  $\varphi$  as a morphism of presheaves.*
- (5) *The morphism  $\varphi$  is surjective (Homology, Definition 5.3) if and only if  $\varphi$  is surjective as a map of sheaves (Sites, Definition 12.1).*
- (6) *A complex of abelian sheaves*

$$\mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H}$$

*is exact at  $\mathcal{G}$  if and only if for all  $U \in \text{Ob}(\mathcal{C})$  and all  $s \in \mathcal{G}(U)$  mapping to zero in  $\mathcal{H}(U)$  there exists a covering  $\{U_i \rightarrow U\}_{i \in I}$  in  $\mathcal{C}$  such that each  $s|_{U_i}$  is in the image of  $\mathcal{F}(U_i) \rightarrow \mathcal{G}(U_i)$ .*

**Proof.** We claim that Homology, Lemma 7.3 applies to the categories  $\mathcal{A} = Ab(\mathcal{C})$  and  $\mathcal{B} = PAb(\mathcal{C})$ , and the functors  $a : \mathcal{A} \rightarrow \mathcal{B}$  (inclusion), and  $b : \mathcal{B} \rightarrow \mathcal{A}$  (sheafification). Let us check the assumptions of Homology, Lemma 7.3. Assumption (1) is that  $\mathcal{A}, \mathcal{B}$  are additive categories,  $a, b$  are additive functors, and  $a$  is right adjoint to  $b$ . The first two statements are clear and adjointness is Sites, Section 43  $(\epsilon)$ . Assumption (2) says that  $PAb(\mathcal{C})$  is abelian which we saw in Section 2 and that sheafification is left exact, which is Sites, Section 43  $(\zeta)$ . The final assumption is that  $ba \cong \text{id}_{\mathcal{A}}$  which is Sites, Section 43  $(\delta)$ . Hence Homology, Lemma 7.3 applies and we conclude that  $Ab(\mathcal{C})$  is abelian.

In the proof of Homology, Lemma 7.3 it is shown that  $Ker(\varphi)$  and  $Coker(\varphi)$  are equal to the sheafification of the kernel and cokernel of  $\varphi$  as a morphism of abelian presheaves. This proves (4). Since the kernel is a equalizer (i.e., a limit) and since sheafification commutes with finite limits, we conclude that (2) holds.

Statement (2) implies (3). Statement (4) implies (5) by our description of sheafification. The characterization of exactness in (6) follows from (2) and (5), and the fact that the sequence is exact if and only if  $\text{Im}(\mathcal{F} \rightarrow \mathcal{G}) = \text{Ker}(\mathcal{G} \rightarrow \mathcal{H})$ .  $\square$

Another way to say part (6) of the lemma is that a sequence of abelian sheaves

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$$

is exact if and only if the sheafification of  $U \mapsto \mathcal{F}_2(U)/\mathcal{F}_1(U)$  is equal to the kernel of  $\mathcal{F}_2 \rightarrow \mathcal{F}_3$ .

**Lemma 3.2.** *Let  $\mathcal{C}$  be a site.*

- (1) *All limits and colimits exist in  $Ab(\mathcal{C})$ .*
- (2) *Limits are the same as the corresponding limits of abelian presheaves over  $\mathcal{C}$  (i.e., commute with taking sections over objects of  $\mathcal{C}$ ).*
- (3) *Finite direct sums are the same as the corresponding finite direct sums in the category of abelian pre-sheaves over  $\mathcal{C}$ .*
- (4) *A colimit is the sheafification of the corresponding colimit in the category of abelian presheaves.*
- (5) *Filtered colimits are exact.*

**Proof.** By Lemma 2.1 limits and colimits of abelian presheaves exist, and are described by taking limits and colimits on the level of sections over objects.

Let  $\mathcal{I} \rightarrow Ab(\mathcal{C})$ ,  $i \mapsto \mathcal{F}_i$  be a diagram. Let  $\lim_i \mathcal{F}_i$  be the limit of the diagram as an abelian presheaf. By Sites, Lemma 10.1 this is an abelian sheaf. Then it is quite easy to see that  $\lim_i \mathcal{F}_i$  is the limit of the diagram in  $Ab(\mathcal{C})$ . This proves limits exist and (2) holds.

By Categories, Lemma 24.4, and because sheafification is left adjoint to the inclusion functor we see that  $\text{colim}_i \mathcal{F}$  exists and is the sheafification of the colimit in  $PAb(\mathcal{C})$ . This proves colimits exist and (4) holds.

Finite direct sums are the same thing as finite products in any abelian category. Hence (3) follows from (2).

Proof of (5). The statement means that given a system  $0 \rightarrow \mathcal{F}_i \rightarrow \mathcal{G}_i \rightarrow \mathcal{H}_i \rightarrow 0$  of exact sequences of abelian sheaves over a directed partially ordered set  $I$  the sequence  $0 \rightarrow \text{colim}_i \mathcal{F}_i \rightarrow \text{colim}_i \mathcal{G}_i \rightarrow \text{colim}_i \mathcal{H}_i \rightarrow 0$  is exact as well. A formal argument using Homology, Lemma 5.8 and the definition of colimits shows that the sequence  $\text{colim}_i \mathcal{F}_i \rightarrow \text{colim}_i \mathcal{G}_i \rightarrow \text{colim}_i \mathcal{H}_i \rightarrow 0$  is exact. Note that  $\text{colim}_i \mathcal{F}_i \rightarrow \text{colim}_i \mathcal{G}_i$  is the sheafification of the map of presheaf colimits which is injective as each of the maps  $\mathcal{F}_i \rightarrow \mathcal{G}_i$  is injective. Since sheafification is exact we conclude.  $\square$

#### 4. Free abelian presheaves

In order to prepare notation for the following definition, let us agree to denote the free abelian group on a set  $S$  as<sup>1</sup>  $\mathbf{Z}[S] = \bigoplus_{s \in S} \mathbf{Z}$ . It is characterized by the property

$$\text{Mor}_{Ab}(\mathbf{Z}[S], A) = \text{Mor}_{Sets}(S, A)$$

In other words the construction  $S \mapsto \mathbf{Z}[S]$  is a left adjoint to the forgetful functor  $Ab \rightarrow Sets$ .

**Definition 4.1.** Let  $\mathcal{C}$  be a category. Let  $\mathcal{G}$  be a presheaf of sets. The *free abelian presheaf*  $\mathbf{Z}_{\mathcal{G}}$  on  $\mathcal{G}$  is the abelian presheaf defined by the rule

$$U \mapsto \mathbf{Z}[\mathcal{G}(U)].$$

In the special case  $\mathcal{G} = h_X$  of a representable presheaf associated to an object  $X$  of  $\mathcal{C}$  we use the notation  $\mathbf{Z}_X = \mathbf{Z}_{h_X}$ . In other words

$$\mathbf{Z}_X(U) = \mathbf{Z}[\text{Mor}_{\mathcal{C}}(U, X)].$$

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<sup>1</sup>In other chapters the notation  $\mathbf{Z}[S]$  sometimes indicates the polynomial ring over  $\mathbf{Z}$  on  $S$ .

This construction is clearly functorial in the presheaf  $\mathcal{G}$ . In fact it is adjoint to the forgetful functor  $PAb(\mathcal{C}) \rightarrow PSh(\mathcal{C})$ . Here is the precise statement.

**Lemma 4.2.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{G}, \mathcal{F}$  be a presheaves of sets. Let  $\mathcal{A}$  be an abelian presheaf. Let  $U$  be an object of  $\mathcal{C}$ . Then we have*

$$\begin{aligned} \text{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) &= \mathcal{F}(U), \\ \text{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{G}}, \mathcal{A}) &= \text{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\ \text{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_U, \mathcal{A}) &= \mathcal{A}(U). \end{aligned}$$

*All of these equalities are functorial.*

**Proof.** Omitted. □

**Lemma 4.3.** *Let  $\mathcal{C}$  be a category. Let  $I$  be a set. For each  $i \in I$  let  $\mathcal{G}_i$  be a presheaf of sets. Then*

$$\mathbf{Z}_{\coprod_i \mathcal{G}_i} = \bigoplus_{i \in I} \mathbf{Z}_{\mathcal{G}_i}$$

*in  $PAb(\mathcal{C})$ .*

**Proof.** Omitted. □

## 5. Free abelian sheaves

Here is the notion of a free abelian sheaf on a sheaf of sets.

**Definition 5.1.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{G}$  be a presheaf of sets. The *free abelian sheaf*  $\mathbf{Z}_{\mathcal{G}}^{\#}$  on  $\mathcal{G}$  is the abelian sheaf  $\mathbf{Z}_{\mathcal{G}}^{\#}$  which is the sheafification of the abelian presheaf on  $\mathcal{G}$ . In the special case  $\mathcal{G} = h_X$  of a representable presheaf associated to an object  $X$  of  $\mathcal{C}$  we use the notation  $\mathbf{Z}_X^{\#}$ .

This construction is clearly functorial in the presheaf  $\mathcal{G}$ . In fact it provides an adjoint to the forgetful functor  $Ab(\mathcal{C}) \rightarrow Sh(\mathcal{C})$ . Here is the precise statement.

**Lemma 5.2.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{G}, \mathcal{F}$  be a sheaves of sets. Let  $\mathcal{A}$  be an abelian sheaf. Let  $U$  be an object of  $\mathcal{C}$ . Then we have*

$$\begin{aligned} \text{Mor}_{Sh(\mathcal{C})}(h_U^{\#}, \mathcal{F}) &= \mathcal{F}(U), \\ \text{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}_{\mathcal{G}}^{\#}, \mathcal{A}) &= \text{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\ \text{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}_U^{\#}, \mathcal{A}) &= \mathcal{A}(U). \end{aligned}$$

*All of these equalities are functorial.*

**Proof.** Omitted. □

**Lemma 5.3.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{G}$  be a presheaf of sets. Then  $\mathbf{Z}_{\mathcal{G}}^{\#} = (\mathbf{Z}_{\mathcal{G}^{\#}})^{\#}$ .*

**Proof.** Omitted. □

## 6. Ringed sites

In this chapter we mainly work with sheaves of modules on a ringed site. Hence we need to define this notion.

**Definition 6.1.** Ringed sites.

- (1) A *ringed site* is a pair  $(\mathcal{C}, \mathcal{O})$  where  $\mathcal{C}$  is a site and  $\mathcal{O}$  is a sheaf of rings on  $\mathcal{C}$ . The sheaf  $\mathcal{O}$  is called the *structure sheaf* of the ringed site.

- (2) Let  $(\mathcal{C}, \mathcal{O})$ ,  $(\mathcal{C}', \mathcal{O}')$  be ringed sites. A *morphism of ringed sites*

$$(f, f^\#) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{C}', \mathcal{O}')$$

is given by a morphism of sites  $f : \mathcal{C} \rightarrow \mathcal{C}'$  (see Sites, Definition 15.1) together with a map of sheaves of rings  $f^\# : f^{-1}\mathcal{O}' \rightarrow \mathcal{O}$ , which by adjunction is the same thing as a map of sheaves of rings  $f^\# : \mathcal{O}' \rightarrow f_*\mathcal{O}$ .

- (3) Let  $(f, f^\#) : (\mathcal{C}_1, \mathcal{O}_1) \rightarrow (\mathcal{C}_2, \mathcal{O}_2)$  and  $(g, g^\#) : (\mathcal{C}_2, \mathcal{O}_2) \rightarrow (\mathcal{C}_3, \mathcal{O}_3)$  be morphisms of ringed sites. Then we define the *composition of morphisms of ringed sites* by the rule

$$(g, g^\#) \circ (f, f^\#) = (g \circ f, f^\# \circ g^\#).$$

Here we use composition of morphisms of sites defined in Sites, Definition 15.4 and  $f^\# \circ g^\#$  indicates the morphism of sheaves of rings

$$\mathcal{O}_3 \xrightarrow{g^\#} g_*\mathcal{O}_2 \xrightarrow{g_*f^\#} g_*f_*\mathcal{O}_1 = (g \circ f)_*\mathcal{O}_1$$

## 7. Ringed topoi

A ringed topos is just a ringed site, except that the notion of a morphism of ringed topoi is different from the notion of a morphism of ringed sites.

**Definition 7.1.** Ringed topoi.

- (1) A *ringed topos* is a pair  $(Sh(\mathcal{C}), \mathcal{O})$  where  $\mathcal{C}$  is a site and  $\mathcal{O}$  is a sheaf of rings on  $\mathcal{C}$ . The sheaf  $\mathcal{O}$  is called the *structure sheaf* of the ringed site.
- (2) Let  $(Sh(\mathcal{C}), \mathcal{O})$ ,  $(Sh(\mathcal{C}'), \mathcal{O}')$  be ringed topoi. A *morphism of ringed topoi*

$$(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$$

is given by a morphism of topoi  $f : \mathcal{C} \rightarrow \mathcal{C}'$  (see Sites, Definition 16.1) together with a map of sheaves of rings  $f^\# : f^{-1}\mathcal{O}' \rightarrow \mathcal{O}$ , which by adjunction is the same thing as a map of sheaves of rings  $f^\# : \mathcal{O}' \rightarrow f_*\mathcal{O}$ .

- (3) Let  $(f, f^\#) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \rightarrow (Sh(\mathcal{C}_2), \mathcal{O}_2)$  and  $(g, g^\#) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \rightarrow (Sh(\mathcal{C}_3), \mathcal{O}_3)$  be morphisms of ringed topoi. Then we define the *composition of morphisms of ringed topoi* by the rule

$$(g, g^\#) \circ (f, f^\#) = (g \circ f, f^\# \circ g^\#).$$

Here we use composition of morphisms of topoi defined in Sites, Definition 16.1 and  $f^\# \circ g^\#$  indicates the morphism of sheaves of rings

$$\mathcal{O}_3 \xrightarrow{g^\#} g_*\mathcal{O}_2 \xrightarrow{g_*f^\#} g_*f_*\mathcal{O}_1 = (g \circ f)_*\mathcal{O}_1$$

Every morphism of ringed topoi is the composition of an equivalence of ringed topoi with a morphism of ringed topoi associated to a morphism of ringed sites. Here is the precise statement.

**Lemma 7.2.** Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. There exists a factorization

$$\begin{array}{ccc} (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) & \xrightarrow{\quad (f, f^\#) \quad} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \\ (g, g^\#) \downarrow & & \downarrow (e, e^\#) \\ (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{\quad (h, h^\#) \quad} & (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \end{array}$$

where

- (1)  $g : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{C}')$  is an equivalence of topoi induced by a special cocontinuous functor  $\mathcal{C} \rightarrow \mathcal{C}'$  (see Sites, Definition 28.2),
- (2)  $e : Sh(\mathcal{D}) \rightarrow Sh(\mathcal{D}')$  is an equivalence of topoi induced by a special cocontinuous functor  $\mathcal{D} \rightarrow \mathcal{D}'$  (see Sites, Definition 28.2),
- (3)  $\mathcal{O}_{\mathcal{C}'} = g_*\mathcal{O}_{\mathcal{C}}$  and  $g^\#$  is the obvious map,
- (4)  $\mathcal{O}_{\mathcal{D}'} = e_*\mathcal{O}_{\mathcal{D}}$  and  $e^\#$  is the obvious map,
- (5) the sites  $\mathcal{C}'$  and  $\mathcal{D}'$  have final objects and fibre products (i.e., all finite limits),
- (6)  $h$  is a morphism of sites induced by a continuous functor  $u : \mathcal{D}' \rightarrow \mathcal{C}'$  which commutes with all finite limits (i.e., it satisfies the assumptions of Sites, Proposition 15.6), and
- (7) given any set of sheaves  $\mathcal{F}_i$  (resp.  $\mathcal{G}_j$ ) on  $\mathcal{C}$  (resp.  $\mathcal{D}$ ) we may assume each of these is a representable sheaf on  $\mathcal{C}'$  (resp.  $\mathcal{D}'$ ).

Moreover, if  $(f, f^\#)$  is an equivalence of ringed topoi, then we can choose the diagram such that  $\mathcal{C}' = \mathcal{D}'$ ,  $\mathcal{O}_{\mathcal{C}'} = \mathcal{O}_{\mathcal{D}'}$  and  $(h, h^\#)$  is the identity.

**Proof.** This follows from Sites, Lemma 28.6, and Sites, Remarks 28.7 and 28.8. You just have to carry along the sheaves of rings. Some details omitted.  $\square$

## 8. 2-morphisms of ringed topoi

This is a brief section concerning the notion of a 2-morphism of ringed topoi.

**Definition 8.1.** Let  $f, g : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be two morphisms of ringed topoi. A 2-morphism from  $f$  to  $g$  is given by a transformation of functors  $t : f_* \rightarrow g_*$  such that

$$\begin{array}{ccc} & \mathcal{O}_{\mathcal{D}} & \\ f^\# \swarrow & & \searrow g^\# \\ f_*\mathcal{O}_{\mathcal{C}} & \xrightarrow{t} & g_*\mathcal{O}_{\mathcal{C}} \end{array}$$

is commutative.

Pictorially we sometimes represent  $t$  as follows:

$$\begin{array}{ccc} (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) & \xrightarrow{f} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \\ & \Downarrow t & \\ & \xrightarrow{g} & \end{array}$$

As in Sites, Section 35 giving a 2-morphism  $t : f_* \rightarrow g_*$  is equivalent to giving  $t : g^{-1} \rightarrow f^{-1}$  (usually denoted by the same symbol) such that the diagram

$$\begin{array}{ccc} f^{-1}\mathcal{O}_{\mathcal{D}} & \xleftarrow{t} & g^{-1}\mathcal{O}_{\mathcal{D}} \\ & \searrow f^\# & \swarrow g^\# \\ & \mathcal{O}_{\mathcal{C}} & \end{array}$$

is commutative. As in Sites, Section 35 the axioms of a strict 2-category hold with horizontal and vertical compositions defined as explained in loc. cit.

### 9. Presheaves of modules

Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings on  $\mathcal{C}$ . At this point we have not yet defined a presheaf of  $\mathcal{O}$ -modules. Thus we do so right now.

**Definition 9.1.** Let  $\mathcal{C}$  be a category, and let  $\mathcal{O}$  be a presheaf of rings on  $\mathcal{C}$ .

- (1) A *presheaf of  $\mathcal{O}$ -modules* is given by an abelian presheaf  $\mathcal{F}$  together with a map of presheaves of sets

$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$

such that for every object  $U$  of  $\mathcal{C}$  the map  $\mathcal{O}(U) \times \mathcal{F}(U) \rightarrow \mathcal{F}(U)$  defines the structure of an  $\mathcal{O}(U)$ -module structure on the abelian group  $\mathcal{F}(U)$ .

- (2) A *morphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  of presheaves of  $\mathcal{O}$ -modules* is a morphism of abelian presheaves  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  such that the diagram

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ \text{id} \times \varphi \downarrow & & \downarrow \varphi \\ \mathcal{O} \times \mathcal{G} & \longrightarrow & \mathcal{G} \end{array}$$

commutes.

- (3) The set of  $\mathcal{O}$ -module morphisms as above is denoted  $\text{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ .  
(4) The category of presheaves of  $\mathcal{O}$ -modules is denoted  $\text{PMod}(\mathcal{O})$ .

Suppose that  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$  is a morphism of presheaves of rings on the category  $\mathcal{C}$ . In this case, if  $\mathcal{F}$  is a presheaf of  $\mathcal{O}_2$ -modules then we can think of  $\mathcal{F}$  as a presheaf of  $\mathcal{O}_1$ -modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \rightarrow \mathcal{O}_2 \times \mathcal{F} \rightarrow \mathcal{F}.$$

We sometimes denote this by  $\mathcal{F}_{\mathcal{O}_1}$  to indicate the restriction of rings. We call this the *restriction of  $\mathcal{F}$* . We obtain the restriction functor

$$\text{PMod}(\mathcal{O}_2) \longrightarrow \text{PMod}(\mathcal{O}_1)$$

On the other hand, given a presheaf of  $\mathcal{O}_1$ -modules  $\mathcal{G}$  we can construct a presheaf of  $\mathcal{O}_2$ -modules  $\mathcal{O}_2 \otimes_{p, \mathcal{O}_1} \mathcal{G}$  by the rule

$$U \longmapsto (\mathcal{O}_2 \otimes_{p, \mathcal{O}_1} \mathcal{G})(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

where  $U \in \text{Ob}(\mathcal{C})$ , with obvious restriction mappings. The index  $p$  stands for “presheaf” and not “point”. This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor

$$\text{PMod}(\mathcal{O}_1) \longrightarrow \text{PMod}(\mathcal{O}_2)$$

**Lemma 9.2.** *With  $\mathcal{C}$ ,  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$ ,  $\mathcal{F}$  and  $\mathcal{G}$  as above there exists a canonical bijection*

$$\text{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \text{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p, \mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

*In other words, the restriction and change of rings functors defined above are adjoint to each other.*

**Proof.** This follows from the fact that for a ring map  $A \rightarrow B$  the restriction functor and the change of ring functor are adjoint to each other.  $\square$



## 10. Sheaves of modules

**Definition 10.1.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ .

- (1) A *sheaf of  $\mathcal{O}$ -modules* is a presheaf of  $\mathcal{O}$ -modules  $\mathcal{F}$ , see Definition 9.1, such that the underlying presheaf of abelian groups  $\mathcal{F}$  is a sheaf.
- (2) A *morphism of sheaves of  $\mathcal{O}$ -modules* is a morphism of presheaves of  $\mathcal{O}$ -modules.
- (3) Given sheaves of  $\mathcal{O}$ -modules  $\mathcal{F}$  and  $\mathcal{G}$  we denote  $\text{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$  the set of morphism of sheaves of  $\mathcal{O}$ -modules.
- (4) The category of sheaves of  $\mathcal{O}$ -modules is denoted  $\text{Mod}(\mathcal{O})$ .

This definition kind of makes sense even if  $\mathcal{O}$  is just a presheaf of rings, although we do not know any examples where this is useful, and we will avoid using the terminology “sheaves of  $\mathcal{O}$ -modules” in case  $\mathcal{O}$  is not a sheaf of rings.

## 11. Sheafification of presheaves of modules

**Lemma 11.1.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}$  be a presheaf of rings on  $\mathcal{C}$ . Let  $\mathcal{F}$  be a presheaf  $\mathcal{O}$ -modules. Let  $\mathcal{O}^\#$  be the sheafification of  $\mathcal{O}$  as a presheaf of rings, see Sites, Section 43. Let  $\mathcal{F}^\#$  be the sheafification of  $\mathcal{F}$  as a presheaf of abelian groups. There exists a map of sheaves of sets*

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{O}^\# \times \mathcal{F}^\# & \longrightarrow & \mathcal{F}^\# \end{array}$$

commute and which makes  $\mathcal{F}^\#$  into a sheaf of  $\mathcal{O}^\#$ -modules. In addition, if  $\mathcal{G}$  is a sheaf of  $\mathcal{O}^\#$ -modules, then any morphism of presheaves of  $\mathcal{O}$ -modules  $\mathcal{F} \rightarrow \mathcal{G}$  (into the restriction of  $\mathcal{G}$  to a  $\mathcal{O}$ -module) factors uniquely as  $\mathcal{F} \rightarrow \mathcal{F}^\# \rightarrow \mathcal{G}$  where  $\mathcal{F}^\# \rightarrow \mathcal{G}$  is a morphism of  $\mathcal{O}^\#$ -modules.

**Proof.** Omitted. □

This actually means that the functor  $i : \text{Mod}(\mathcal{O}^\#) \rightarrow \text{PMod}(\mathcal{O})$  (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma  $\# : \text{PMod}(\mathcal{O}) \rightarrow \text{Mod}(\mathcal{O}^\#)$  are adjoint. In a formula

$$\text{Mor}_{\text{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \text{Mor}_{\text{Mod}(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

An important case happens when  $\mathcal{O}$  is already a sheaf of rings. In this case the formula reads

$$\text{Mor}_{\text{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \text{Mor}_{\text{Mod}(\mathcal{O})}(\mathcal{F}^\#, \mathcal{G})$$

because  $\mathcal{O} = \mathcal{O}^\#$  in this case.

**Lemma 11.2.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}$  be a presheaf of rings on  $\mathcal{C}$ . The sheafification functor*

$$\text{PMod}(\mathcal{O}) \longrightarrow \text{Mod}(\mathcal{O}^\#), \quad \mathcal{F} \longmapsto \mathcal{F}^\#$$

*is exact.*

**Proof.** This is true because it holds for sheafification  $\text{PAb}(\mathcal{C}) \rightarrow \text{Ab}(\mathcal{C})$ . See the discussion in Section 3. □

Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a morphism of sheaves of rings on  $\mathcal{C}$ . In Section 9 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If  $\mathcal{F}$  is a sheaf of  $\mathcal{O}_2$ -modules then the restriction  $\mathcal{F}_{\mathcal{O}_1}$  of  $\mathcal{F}$  is clearly a sheaf of  $\mathcal{O}_1$ -modules. We obtain the restriction functor

$$\text{Mod}(\mathcal{O}_2) \longrightarrow \text{Mod}(\mathcal{O}_1)$$

On the other hand, given a sheaf of  $\mathcal{O}_1$ -modules  $\mathcal{G}$  the presheaf of  $\mathcal{O}_2$ -modules  $\mathcal{O}_2 \otimes_{p, \mathcal{O}_1} \mathcal{G}$  is in general not a sheaf. Hence we define the *tensor product sheaf*  $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$  by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p, \mathcal{O}_1} \mathcal{G})^\#$$

as the sheafification of our construction for presheaves. We obtain the *change of rings* functor

$$\text{Mod}(\mathcal{O}_1) \longrightarrow \text{Mod}(\mathcal{O}_2)$$

**Lemma 11.3.** *With  $X, \mathcal{O}_1, \mathcal{O}_2, \mathcal{F}$  and  $\mathcal{G}$  as above there exists a canonical bijection*

$$\text{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \text{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

*In other words, the restriction and change of rings functors are adjoint to each other.*

**Proof.** This follows from Lemma 9.2 and the fact that  $\text{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \text{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p, \mathcal{O}_1} \mathcal{G}, \mathcal{F})$  because  $\mathcal{F}$  is a sheaf.  $\square$

**Lemma 11.4.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{O} \rightarrow \mathcal{O}'$  be an epimorphism of sheaves of rings. Let  $\mathcal{G}_1, \mathcal{G}_2$  be  $\mathcal{O}'$ -modules. Then*

$$\text{Hom}_{\mathcal{O}'}(\mathcal{G}_1, \mathcal{G}_2) = \text{Hom}_{\mathcal{O}}(\mathcal{G}_1, \mathcal{G}_2).$$

*In other words, the restriction functor  $\text{Mod}(\mathcal{O}') \rightarrow \text{Mod}(\mathcal{O})$  is fully faithful.*

**Proof.** This is the sheaf version of Algebra, Lemma 103.14 and is proved in exactly the same way.  $\square$

## 12. Morphisms of topoi and sheaves of modules

All of this material is completely straightforward. We formulate everything in the case of morphisms of topoi, but of course the results also hold in the case of morphisms of sites.

**Lemma 12.1.** *Let  $\mathcal{C}, \mathcal{D}$  be sites. Let  $f : \text{Sh}(\mathcal{C}) \rightarrow \text{Sh}(\mathcal{D})$  be a morphism of topoi. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ . Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules. There is a natural map of sheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

*which turns  $f_*\mathcal{F}$  into a sheaf of  $f_*\mathcal{O}$ -modules. This construction is functorial in  $\mathcal{F}$ .*

**Proof.** Denote  $\mu : \mathcal{O} \times \mathcal{F} \rightarrow \mathcal{F}$  the multiplication map. Recall that  $f_*$  (on sheaves of sets) is left exact and hence commutes with products. Hence  $f_*\mu$  is a map as indicated. This proves the lemma.  $\square$

**Lemma 12.2.** *Let  $\mathcal{C}, \mathcal{D}$  be sites. Let  $f : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be a morphism of topoi. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{D}$ . Let  $\mathcal{G}$  be a sheaf of  $\mathcal{O}$ -modules. There is a natural map of sheaves of sets*

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

*which turns  $f^{-1}\mathcal{G}$  into a sheaf of  $f^{-1}\mathcal{O}$ -modules. This construction is functorial in  $\mathcal{G}$ .*

**Proof.** Denote  $\mu : \mathcal{O} \times \mathcal{G} \rightarrow \mathcal{G}$  the multiplication map. Recall that  $f^{-1}$  (on sheaves of sets) is exact and hence commutes with products. Hence  $f^{-1}\mu$  is a map as indicated. This proves the lemma.  $\square$

**Lemma 12.3.** *Let  $\mathcal{C}, \mathcal{D}$  be sites. Let  $f : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be a morphism of topoi. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{D}$ . Let  $\mathcal{G}$  be a sheaf of  $\mathcal{O}$ -modules. Let  $\mathcal{F}$  be a sheaf of  $f^{-1}\mathcal{O}$ -modules. Then*

$$\text{Mor}_{\text{Mod}(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \text{Mor}_{\text{Mod}(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

*Here we use Lemmas 12.2 and 12.1, and we think of  $f_*\mathcal{F}$  as an  $\mathcal{O}$ -module by restriction via  $\mathcal{O} \rightarrow f_*f^{-1}\mathcal{O}$ .*

**Proof.** First we note that we have

$$\text{Mor}_{\text{Ab}(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) = \text{Mor}_{\text{Ab}(\mathcal{D})}(\mathcal{G}, f_*\mathcal{F}).$$

by Sites, Proposition 43.3. Suppose that  $\alpha : f^{-1}\mathcal{G} \rightarrow \mathcal{F}$  and  $\beta : \mathcal{G} \rightarrow f_*\mathcal{F}$  are morphisms of abelian sheaves which correspond via the formula above. We have to show that  $\alpha$  is  $f^{-1}\mathcal{O}$ -linear if and only if  $\beta$  is  $\mathcal{O}$ -linear. For example, suppose  $\alpha$  is  $f^{-1}\mathcal{O}$ -linear, then clearly  $f_*\alpha$  is  $f_*f^{-1}\mathcal{O}$ -linear, and hence (as restriction is a functor) is  $\mathcal{O}$ -linear. Hence it suffices to prove that the adjunction map  $\mathcal{G} \rightarrow f_*f^{-1}\mathcal{G}$  is  $\mathcal{O}$ -linear. Using that both  $f_*$  and  $f^{-1}$  commute with products (on sheaves of sets) this comes down to showing that

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{G} & \longrightarrow & f_*f^{-1}(\mathcal{O} \times \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & f_*f^{-1}\mathcal{G} \end{array}$$

is commutative. This holds because the adjunction mapping  $\text{id}_{Sh(\mathcal{D})} \rightarrow f_*f^{-1}$  is a transformation of functors. We omit the proof of the implication  $\beta$  linear  $\Rightarrow \alpha$  linear.  $\square$

**Lemma 12.4.** *Let  $\mathcal{C}, \mathcal{D}$  be sites. Let  $f : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be a morphism of topoi. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ . Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules. Let  $\mathcal{G}$  be a sheaf of  $f_*\mathcal{O}$ -modules. Then*

$$\text{Mor}_{\text{Mod}(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \text{Mor}_{\text{Mod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

*Here we use Lemmas 12.2 and 12.1, and we use the canonical map  $f^{-1}f_*\mathcal{O} \rightarrow \mathcal{O}$  in the definition of the tensor product.*

**Proof.** Note that we have

$$\text{Mor}_{\text{Mod}(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \text{Mor}_{\text{Mod}(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}})$$

by Lemma 11.3. Hence the result follows from Lemma 12.3.  $\square$

### 13. Morphisms of ringed topoi and modules

We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed topoi.

**Definition 13.1.** Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi or ringed sites.

- (1) Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_{\mathcal{C}}$ -modules. We define the *pushforward* of  $\mathcal{F}$  as the sheaf of  $\mathcal{O}_{\mathcal{D}}$ -modules which as a sheaf of abelian groups equals  $f_*\mathcal{F}$  and with module structure given by the restriction via  $f^\# : \mathcal{O}_{\mathcal{D}} \rightarrow f_*\mathcal{O}_{\mathcal{C}}$  of the module structure

$$f_*\mathcal{O}_{\mathcal{C}} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

from Lemma 12.1.

- (2) Let  $\mathcal{G}$  be a sheaf of  $\mathcal{O}_{\mathcal{D}}$ -modules. We define the *pullback*  $f^*\mathcal{G}$  to be the sheaf of  $\mathcal{O}_{\mathcal{C}}$ -modules defined by the formula

$$f^*\mathcal{F} = \mathcal{O}_{\mathcal{C}} \otimes_{f^{-1}\mathcal{O}_{\mathcal{D}}} f^{-1}\mathcal{F}$$

where the ring map  $f^{-1}\mathcal{O}_{\mathcal{D}} \rightarrow \mathcal{O}_{\mathcal{C}}$  is  $f^\#$ , and where the module structure is given by Lemma 12.2.

Thus we have defined functors

$$\begin{aligned} f_* : Mod(\mathcal{O}_{\mathcal{C}}) &\longrightarrow Mod(\mathcal{O}_{\mathcal{D}}) \\ f^* : Mod(\mathcal{O}_{\mathcal{D}}) &\longrightarrow Mod(\mathcal{O}_{\mathcal{C}}) \end{aligned}$$

The final result on these functors is that they are indeed adjoint as expected.

**Lemma 13.2.** *Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi or ringed sites. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_{\mathcal{C}}$ -modules. Let  $\mathcal{G}$  be a sheaf of  $\mathcal{O}_{\mathcal{D}}$ -modules. There is a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{C}}}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{D}}}(\mathcal{G}, f_*\mathcal{F}).$$

*In other words: the functor  $f^*$  is the left adjoint to  $f_*$ .*

**Proof.** This follows from the work we did before:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{\mathcal{C}}}(f^*\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{Mod(\mathcal{O}_{\mathcal{C}})}(\mathcal{O}_{\mathcal{C}} \otimes_{f^{-1}\mathcal{O}_{\mathcal{D}}} f^{-1}\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{Mod(f^{-1}\mathcal{O}_{\mathcal{D}})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_{\mathcal{D}}}) \\ &= \mathrm{Hom}_{\mathcal{O}_{\mathcal{D}}}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

Here we use Lemmas 11.3 and 12.3. □

**Lemma 13.3.**  *$(f, f^\#) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \rightarrow (Sh(\mathcal{C}_2), \mathcal{O}_2)$  and  $(g, g^\#) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \rightarrow (Sh(\mathcal{C}_3), \mathcal{O}_3)$  be morphisms of ringed topoi. There are canonical isomorphisms of functors  $(g \circ f)_* \cong g_* \circ f_*$  and  $(g \circ f)^* \cong f^* \circ g^*$ .*

**Proof.** This is clear from the definitions. □

#### 14. The abelian category of sheaves of modules

Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a ringed topos. Let  $\mathcal{F}, \mathcal{G}$  be sheaves of  $\mathcal{O}$ -modules, see Sheaves, Definition 10.1. Let  $\varphi, \psi : \mathcal{F} \rightarrow \mathcal{G}$  be morphisms of sheaves of  $\mathcal{O}$ -modules. We define  $\varphi + \psi : \mathcal{F} \rightarrow \mathcal{G}$  to be the sum of  $\varphi$  and  $\psi$  as morphisms of abelian sheaves. This is clearly again a map of  $\mathcal{O}$ -modules. It is also clear that composition of maps of  $\mathcal{O}$ -modules is bilinear with respect to this addition. Thus  $Mod(\mathcal{O})$  is a pre-additive category, see Homology, Definition 3.1.

We will denote  $0$  the sheaf of  $\mathcal{O}$ -modules which has constant value  $\{0\}$  for all objects  $U$  of  $\mathcal{C}$ . Clearly this is both a final and an initial object of  $Mod(\mathcal{O})$ . Given a morphism of  $\mathcal{O}$ -modules  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  the following are equivalent: (a)  $\varphi$  is zero, (b)  $\varphi$  factors through  $0$ , (c)  $\varphi$  is zero on sections over each object  $U$ .

Moreover, given a pair  $\mathcal{F}, \mathcal{G}$  of sheaves of  $\mathcal{O}$ -modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps  $(i, j, p, q)$  as in Homology, Definition 3.5. Thus  $Mod(\mathcal{O})$  is an additive category, see Homology, Definition 3.8.

Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of  $\mathcal{O}$ -modules. We may define  $\text{Ker}(\varphi)$  to be the kernel of  $\varphi$  as a map of abelian sheaves. By Section 3 this is the subsheaf of  $\mathcal{F}$  with sections

$$\text{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all objects  $U$  of  $\mathcal{C}$ . It is easy to see that this is indeed a kernel in the category of  $\mathcal{O}$ -modules. In other words, a morphism  $\alpha : \mathcal{H} \rightarrow \mathcal{F}$  factors through  $\text{Ker}(\varphi)$  if and only if  $\varphi \circ \alpha = 0$ .

Similarly, we define  $\text{Coker}(\varphi)$  as the cokernel of  $\varphi$  as a map of abelian sheaves. There is a unique multiplication map

$$\mathcal{O} \times \text{Coker}(\varphi) \longrightarrow \text{Coker}(\varphi)$$

such that the map  $\mathcal{G} \rightarrow \text{Coker}(\varphi)$  becomes a morphism of  $\mathcal{O}$ -modules (verification omitted). The map  $\mathcal{G} \rightarrow \text{Coker}(\varphi)$  is surjective (as a map of sheaves of sets, see Section 3). To show that  $\text{Coker}(\varphi)$  is a cokernel in  $Mod(\mathcal{O})$ , note that if  $\beta : \mathcal{G} \rightarrow \mathcal{H}$  is a morphism of  $\mathcal{O}$ -modules such that  $\beta \circ \varphi$  is zero, then you get for every object  $U$  of  $\mathcal{C}$  a map induced by  $\beta$  from  $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$  into  $\mathcal{H}(U)$ . By the universal property of sheafification (see Sheaves, Lemma 20.1) we obtain a canonical map  $\text{Coker}(\varphi) \rightarrow \mathcal{H}$  such that the original  $\beta$  is equal to the composition  $\mathcal{G} \rightarrow \text{Coker}(\varphi) \rightarrow \mathcal{H}$ . The morphism  $\text{Coker}(\varphi) \rightarrow \mathcal{H}$  is unique because of the surjectivity mentioned above.

**Lemma 14.1.** *Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a ringed topos. The category  $Mod(\mathcal{O})$  is an abelian category. The forgetful functor  $Mod(\mathcal{O}) \rightarrow Ab(\mathcal{C})$  is exact, hence kernels, cokernels and exactness of  $\mathcal{O}$ -modules, correspond to the corresponding notions for abelian sheaves.*

**Proof.** Above we have seen that  $Mod(\mathcal{O})$  is an additive category, with kernels and cokernels and that  $Mod(\mathcal{O}) \rightarrow Ab(\mathcal{C})$  preserves kernels and cokernels. By Homology, Definition 5.1 we have to show that image and coimage agree. This is clear because it is true in  $Ab(\mathcal{C})$ . The lemma follows.  $\square$

**Lemma 14.2.** *Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a ringed topos. All limits and colimits exist in  $Mod(\mathcal{O})$  and the forgetful functor  $Mod(\mathcal{O}) \rightarrow Ab(\mathcal{C})$  commutes with them. Moreover, filtered colimits are exact.*

**Proof.** The final statement follows from the first as filtered colimits are exact in  $Ab(\mathcal{C})$  by Lemma 3.2. Let  $\mathcal{I} \rightarrow Mod(\mathcal{C})$ ,  $i \mapsto \mathcal{F}_i$  be a diagram. Let  $\lim_i \mathcal{F}_i$  be the limit of the diagram in  $Ab(\mathcal{C})$ . By the description of this limit in Lemma 3.2 we see immediately that there exists a multiplication

$$\mathcal{O} \times \lim_i \mathcal{F}_i \longrightarrow \lim_i \mathcal{F}_i$$

which turns  $\lim_i \mathcal{F}_i$  into a sheaf of  $\mathcal{O}$ -modules. It is easy to see that this is the limit of the diagram in  $Mod(\mathcal{C})$ . Let  $\text{colim}_i \mathcal{F}_i$  be the colimit of the diagram in  $PAb(\mathcal{C})$ . By the description of this colimit in the proof of Lemma 2.1 we see immediately that there exists a multiplication

$$\mathcal{O} \times \text{colim}_i \mathcal{F}_i \longrightarrow \text{colim}_i \mathcal{F}_i$$

which turns  $\text{colim}_i \mathcal{F}_i$  into a presheaf of  $\mathcal{O}$ -modules. Applying sheafification we get a sheaf of  $\mathcal{O}$ -modules  $(\text{colim}_i \mathcal{F}_i)^\#$ , see Lemma 11.1. It is easy to see that  $(\text{colim}_i \mathcal{F}_i)^\#$  is the colimit of the diagram in  $Mod(\mathcal{O})$ , and by Lemma 3.2 forgetting the  $\mathcal{O}$ -module structure is the colimit in  $Ab(\mathcal{C})$ .  $\square$

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of  $\mathcal{O}$ -modules in terms of limits and colimits, as in Categories, Section 23. See Homology, Lemma 7.1 for a description of exactness properties in terms of short exact sequences.

**Lemma 14.3.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi.*

- (1) *The functor  $f_*$  is left exact. In fact it commutes with all limits.*
- (2) *The functor  $f^*$  is right exact. In fact it commutes with all colimits.*

**Proof.** This is true because  $(f^*, f_*)$  is an adjoint pair of functors, see Lemma 13.2. See Categories, Section 24.  $\square$

**Lemma 14.4.** *Let  $\mathcal{C}$  be a site. If  $\{p_i\}_{i \in I}$  is a conservative family of points, then we may check exactness of a sequence of abelian sheaves on the stalks at the points  $p_i$ ,  $i \in I$ . If  $\mathcal{C}$  has enough points, then exactness of a sequence of abelian sheaves may be checked on stalks.*

**Proof.** This is immediate from Sites, Lemma 37.2.  $\square$

## 15. Exactness of pushforward

Some technical lemmas concerning exactness properties of pushforward.

**Lemma 15.1.** *Let  $f : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be a morphism of topoi. The following are equivalent:*

- (1)  *$f^{-1}f_*\mathcal{F} \rightarrow \mathcal{F}$  is surjective for all  $\mathcal{F}$  in  $Ab(\mathcal{C})$ , and*
- (2)  *$f_* : Ab(\mathcal{C}) \rightarrow Ab(\mathcal{D})$  reflects surjections.*

*In this case the functor  $f_* : Ab(\mathcal{C}) \rightarrow Ab(\mathcal{D})$  is faithful.*

**Proof.** Assume (1). Suppose that  $a : \mathcal{F} \rightarrow \mathcal{F}'$  is a map of abelian sheaves on  $\mathcal{C}$  such that  $f_*a$  is surjective. As  $f^{-1}$  is exact this implies that  $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \rightarrow f^{-1}f_*\mathcal{F}'$  is surjective. Combined with (1) this implies that  $a$  is surjective. This means that (2) holds.

Assume (2). Let  $\mathcal{F}$  be an abelian sheaf on  $\mathcal{C}$ . We have to show that the map  $f^{-1}f_*\mathcal{F} \rightarrow \mathcal{F}$  is surjective. By (2) it suffices to show that  $f_*f^{-1}f_*\mathcal{F} \rightarrow f_*\mathcal{F}$  is surjective. And this is true because there is a canonical map  $f_*\mathcal{F} \rightarrow f_*f^{-1}f_*\mathcal{F}$  which is a one-sided inverse.

We omit the proof of the final assertion.  $\square$

**Lemma 15.2.** *Let  $f : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be a morphism of topoi. Assume at least one of the following properties holds*

- (1)  *$f_*$  transforms surjections of sheaves of sets into surjections,*
- (2)  *$f_*$  transforms surjections of abelian sheaves into surjections,*
- (3)  *$f_*$  commutes with coequalizers on sheaves of sets,*
- (4)  *$f_*$  commutes with pushouts on sheaves of sets,*

*Then  $f_* : Ab(\mathcal{C}) \rightarrow Ab(\mathcal{D})$  is exact.*

**Proof.** Since  $f_* : Ab(\mathcal{C}) \rightarrow Ab(\mathcal{D})$  is a right adjoint we already know that it transforms a short exact sequence  $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$  of abelian sheaves on  $\mathcal{C}$  into an exact sequence

$$0 \rightarrow f_*\mathcal{F}_1 \rightarrow f_*\mathcal{F}_2 \rightarrow f_*\mathcal{F}_3$$

see Categories, Sections 23 and 24 and Homology, Section 7. Hence it suffices to prove that the map  $f_*\mathcal{F}_2 \rightarrow f_*\mathcal{F}_3$  is surjective. If (1), (2) holds, then this is clear from the definitions. By Sites, Lemma 40.1 we see that either (3) or (4) formally implies (1), hence in these cases we are done also.  $\square$

**Lemma 15.3.** *Let  $f : \mathcal{D} \rightarrow \mathcal{C}$  be a morphism of sites associated to the continuous functor  $u : \mathcal{C} \rightarrow \mathcal{D}$ . Assume  $u$  is almost cocontinuous. Then*

- (1)  *$f_* : Ab(\mathcal{D}) \rightarrow Ab(\mathcal{C})$  is exact.*
- (2) *if  $f^\sharp : f^{-1}\mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{O}_{\mathcal{D}}$  is given so that  $f$  becomes a morphism of ringed sites, then  $f_* : Mod(\mathcal{O}_{\mathcal{D}}) \rightarrow Mod(\mathcal{O}_{\mathcal{C}})$  is exact.*

**Proof.** Part (2) follows from part (1) by Lemma 14.2. Part (1) follows from Sites, Lemmas 41.6 and 40.1.  $\square$

## 16. Exactness of lower shriek

Let  $u : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between sites. Assume that

- (a)  $u$  is cocontinuous, and
- (b)  $u$  is continuous.

Let  $g : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be the morphism of topoi associated with  $u$ , see Sites, Lemma 20.1. Recall that  $g^{-1} = u^p$ , i.e.,  $g^{-1}$  is given by the simple formula  $(g^{-1}\mathcal{G})(U) = \mathcal{G}(u(U))$ , see Sites, Lemma 20.5. We would like to show that  $g^{-1} : Ab(\mathcal{D}) \rightarrow Ab(\mathcal{C})$  has a left adjoint  $g_!$ . By Sites, Lemma 20.5 the functor  $g_!^{Sh} = (u_p)^\#$  is a left adjoint on sheaves of sets. Moreover, we know that  $g_!^{Sh}\mathcal{F}$  is the sheaf associated to the presheaf

$$V \mapsto \operatorname{colim}_{V \rightarrow u(U)} \mathcal{F}(U)$$

where the colimit is over  $(\mathcal{I}_V^u)^{opp}$  and is taken in the category of sets. Hence the following definition is natural.

**Definition 16.1.** With  $u : \mathcal{C} \rightarrow \mathcal{D}$  satisfying (a), (b) above. For  $\mathcal{F} \in PAb(\mathcal{C})$  we define  $g_{p!}\mathcal{F}$  as the presheaf

$$V \longmapsto \operatorname{colim}_{V \rightarrow u(U)} \mathcal{F}(U)$$

with colimits over  $(\mathcal{I}_V^u)^{opp}$  taken in  $Ab$ . For  $\mathcal{F} \in PAb(\mathcal{C})$  we set  $g_!\mathcal{F} = (g_{p!}\mathcal{F})^\#$ .

The reason for being so explicit with this is that the functors  $g_!^{Sh}$  and  $g_!$  are different. Whenever we use both we have to be careful to make the distinction clear.

**Lemma 16.2.** *The functor  $g_{p!}$  is a left adjoint to the functor  $u^p$ . The functor  $g_!$  is a left adjoint to the functor  $g^{-1}$ . In other words the formulas*

$$\begin{aligned} \operatorname{Mor}_{PAb(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) &= \operatorname{Mor}_{PAb(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}), \\ \operatorname{Mor}_{Ab(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \operatorname{Mor}_{Ab(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G}) \end{aligned}$$

hold bifunctorially in  $\mathcal{F}$  and  $\mathcal{G}$ .

**Proof.** The second formula follows formally from the first, since if  $\mathcal{F}$  and  $\mathcal{G}$  are abelian sheaves then

$$\begin{aligned} \operatorname{Mor}_{Ab(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \operatorname{Mor}_{PAb(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}) \\ &= \operatorname{Mor}_{Ab(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G}) \end{aligned}$$

by the universal property of sheafification.

To prove the first formula, let  $\mathcal{F}, \mathcal{G}$  be abelian presheaves. To prove the lemma we will construct maps from the group on the left to the group on the right and omit the verification that these are mutually inverse.

Note that there is a canonical map of abelian presheaves  $\mathcal{F} \rightarrow u^p g_{p!}\mathcal{F}$  which on sections over  $U$  is the natural map  $\mathcal{F}(U) \rightarrow \operatorname{colim}_{u(U) \rightarrow u(U')} \mathcal{F}(U')$ , see Sites, Lemma 5.3. Given a map  $\alpha : g_{p!}\mathcal{F} \rightarrow \mathcal{G}$  we get  $u^p\alpha : u^p g_{p!}\mathcal{F} \rightarrow u^p\mathcal{G}$  which we can precompose by the map  $\mathcal{F} \rightarrow u^p g_{p!}\mathcal{F}$ .

Note that there is a canonical map of abelian presheaves  $g_{p!}u^p\mathcal{G} \rightarrow \mathcal{G}$  which on sections over  $V$  is the natural map  $\operatorname{colim}_{V \rightarrow u(U)} \mathcal{G}(u(U)) \rightarrow \mathcal{G}(V)$ . It maps a section  $s \in u(U)$  in the summand corresponding to  $t : V \rightarrow u(U)$  to  $t^*s \in \mathcal{G}(V)$ . Hence, given a map  $\beta : \mathcal{F} \rightarrow u^p\mathcal{G}$  we get a map  $g_{p!}\beta : g_{p!}\mathcal{F} \rightarrow g_{p!}u^p\mathcal{G}$  which we can postcompose with the map  $g_{p!}u^p\mathcal{G} \rightarrow \mathcal{G}$  above.  $\square$

**Lemma 16.3.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be sites. Let  $u : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Assume that*

- (a)  *$u$  is cocontinuous,*
- (b)  *$u$  is continuous, and*
- (c) *fibre products and equalizers exist in  $\mathcal{C}$  and  $u$  commutes with them.*

*In this case the functor  $g_! : Ab(\mathcal{C}) \rightarrow Ab(\mathcal{D})$  is exact.*

**Proof.** Compare with Sites, Lemma 20.6. Assume (a), (b), and (c). We already know that  $g_!$  is right exact as it is a left adjoint, see Categories, Lemma 24.5 and Homology, Section 7. We have  $g_! = (g_{p!})^\#$ . We have to show that  $g_!$  transforms injective maps of abelian sheaves into injective maps of abelian presheaves. Recall that sheafification of abelian presheaves is exact, see Lemma 3.2. Thus it suffices to show that  $g_{p!}$  transforms injective maps of abelian presheaves into injective maps of abelian presheaves. To do this it suffices that colimits over the categories  $(\mathcal{I}_V^u)^{opp}$  of Sites, Section 5 transform injective maps between diagrams into injections. This follows from Sites, Lemma 5.1 and Algebra, Lemma 8.11.  $\square$



**Lemma 16.4.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be sites. Let  $u : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Assume that*

- (a)  *$u$  is cocontinuous,*
- (b)  *$u$  is continuous, and*
- (c)  *$u$  is fully faithful.*

*For  $g$  and  $g_!$  as above the canonical map  $\mathcal{F} \rightarrow g^{-1}g_!\mathcal{F}$  is an isomorphism for all abelian sheaves  $\mathcal{F}$  on  $\mathcal{C}$ .*

**Proof.** Pick  $U \in \text{Ob}(\mathcal{C})$ . We will show that  $g^{-1}g_!\mathcal{F}(U) = \mathcal{F}(U)$ . First, note that  $g^{-1}g_!\mathcal{F}(U) = g_!\mathcal{F}(u(U))$ . Hence it suffices to show that  $g_!\mathcal{F}(u(U)) = \mathcal{F}(U)$ . We know that  $g_!\mathcal{F}$  is the (abelian) sheaf associated to the presheaf  $g_{p!}\mathcal{F}$  which is defined by the rule

$$V \mapsto \text{colim}_{V \rightarrow u(U')} \mathcal{F}(U')$$

with colimit taken in  $Ab$ . If  $V = u(U)$ , then, as  $u$  is fully faithful this colimit is over  $U \rightarrow U'$ . Hence we conclude that  $g_{p!}\mathcal{F}(u(U)) = \mathcal{F}(U)$ . Since  $u$  is cocontinuous and continuous any covering of  $u(U)$  in  $\mathcal{D}$  can be refined by a covering (!)  $\{u(U_i) \rightarrow u(U)\}$  of  $\mathcal{D}$  where  $\{U_i \rightarrow U\}$  is a covering in  $\mathcal{C}$ . This implies that  $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$  also, since in the colimit defining the value of  $(g_{p!}\mathcal{F})^+$  on  $u(U)$  we may restrict to the cofinal system of coverings  $\{u(U_i) \rightarrow u(U)\}$  as above. Hence we see that  $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$  for all objects  $U$  of  $\mathcal{C}$  as well. Repeating this argument one more time gives the equality  $(g_{p!}\mathcal{F})^\#(u(U)) = \mathcal{F}(U)$  for all objects  $U$  of  $\mathcal{C}$ . This produces the desired equality  $g^{-1}g_!\mathcal{F} = \mathcal{F}$ .  $\square$

**Remark 16.5.** In general the functor  $g_!$  cannot be extended to categories of modules in case  $g$  is (part of) a morphism of ringed topoi. Namely, given any ring map  $A \rightarrow B$  the functor  $M \mapsto B \otimes_A M$  has a right adjoint (restriction) but not in general a left adjoint (because its existence would imply that  $A \rightarrow B$  is flat). We will see in Section 19 below that it is possible to define  $j_!$  on sheaves of modules in the case of a localization of sites. We will discuss this in greater generality in Section 40 below.

**Lemma 16.6.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be sites. Let  $g : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be the morphism of topoi associated to a continuous and cocontinuous functor  $u : \mathcal{C} \rightarrow \mathcal{D}$ .*

- (1) *If  $u$  has a left adjoint  $w$ , then  $g_!$  agrees with  $g_!^{Sh}$  on underlying sheaves of sets and  $g_!$  is exact.*
- (2) *If in addition  $w$  is cocontinuous, then  $g_! = h^{-1}$  and  $g^{-1} = h_*$  where  $h : Sh(\mathcal{D}) \rightarrow Sh(\mathcal{C})$  is the morphism of topoi associated to  $w$ .*

**Proof.** This Lemma is the analogue of Sites, Lemma 22.1. From Sites, Lemma 18.3 we see that the categories  $\mathcal{I}_V^u$  have an initial object. Thus the underlying set of a colimit of a system of abelian groups over  $(\mathcal{I}_V^u)^{opp}$  is the colimit of the underlying sets. Whence the agreement of  $g_!^{Sh}$  and  $g_!$  by our construction of  $g_!$  in Definition 16.1. The exactness and (2) follow immediately from the corresponding statements of Sites, Lemma 22.1.  $\square$

## 17. Global types of modules

**Definition 17.1.** Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a ringed topos. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules.

- (1) We say  $\mathcal{F}$  is a *free  $\mathcal{O}$ -module* if  $\mathcal{F}$  is isomorphic as an  $\mathcal{O}$ -module to a sheaf of the form  $\bigoplus_{i \in I} \mathcal{O}$ .

- (2) We say  $\mathcal{F}$  is *finite free* if  $\mathcal{F}$  is isomorphic as an  $\mathcal{O}$ -module to a sheaf of the form  $\bigoplus_{i \in I} \mathcal{O}$  with a finite index set  $I$ .
- (3) We say  $\mathcal{F}$  is *generated by global sections* if there exists a surjection

$$\bigoplus_{i \in I} \mathcal{O} \longrightarrow \mathcal{F}$$

from a free  $\mathcal{O}$ -module onto  $\mathcal{F}$ .

- (4) Given  $r \geq 0$  we say  $\mathcal{F}$  is *generated by  $r$  global sections* if there exists a surjection  $\mathcal{O}^{\oplus r} \rightarrow \mathcal{F}$ .
- (5) We say  $\mathcal{F}$  is *generated by finitely many global sections* if it is generated by  $r$  global sections for some  $r \geq 0$ .
- (6) We say  $\mathcal{F}$  has a *global presentation* if there exists an exact sequence

$$\bigoplus_{j \in J} \mathcal{O} \longrightarrow \bigoplus_{i \in I} \mathcal{O} \longrightarrow \mathcal{F}$$

of  $\mathcal{O}$ -modules.

- (7) We say  $\mathcal{F}$  has a *global finite presentation* if there exists an exact sequence

$$\bigoplus_{j \in J} \mathcal{O} \longrightarrow \bigoplus_{i \in I} \mathcal{O} \longrightarrow \mathcal{F}$$

of  $\mathcal{O}$ -modules with  $I$  and  $J$  finite sets.

Note that for any set  $I$  the direct sum  $\bigoplus_{i \in I} \mathcal{O}$  exists (Lemma 14.2) and is the sheafification of the presheaf  $U \mapsto \bigoplus_{i \in I} \mathcal{O}(U)$ . This module is called the *free  $\mathcal{O}$ -module on the set  $I$* .

**Lemma 17.2.** *Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{F}$  be an  $\mathcal{O}_{\mathcal{D}}$ -module.*

- (1) *If  $\mathcal{F}$  is free then  $f^*\mathcal{F}$  is free.*
- (2) *If  $\mathcal{F}$  is finite free then  $f^*\mathcal{F}$  is finite free.*
- (3) *If  $\mathcal{F}$  is generated by global sections then  $f^*\mathcal{F}$  is generated by global sections.*
- (4) *Given  $r \geq 0$  if  $\mathcal{F}$  is generated by  $r$  global sections, then  $f^*\mathcal{F}$  is generated by  $r$  global sections.*
- (5) *If  $\mathcal{F}$  is generated by finitely many global sections then  $f^*\mathcal{F}$  is generated by finitely many global sections.*
- (6) *If  $\mathcal{F}$  has a global presentation then  $f^*\mathcal{F}$  has a global presentation.*
- (7) *If  $\mathcal{F}$  has a finite global presentation then  $f^*\mathcal{F}$  has a finite global presentation.*

**Proof.** This is true because  $f^*$  commutes with arbitrary colimits (Lemma 14.3) and  $f^*\mathcal{O}_{\mathcal{D}} = \mathcal{O}_{\mathcal{C}}$ .  $\square$

## 18. Intrinsic properties of modules

Let  $\mathcal{P}$  be a property of sheaves of modules on ringed topoi. We say  $\mathcal{P}$  is an *intrinsic property* if we have  $\mathcal{P}(\mathcal{F}) \Leftrightarrow \mathcal{P}(f^*\mathcal{F})$  whenever  $(f, f^\#) : (Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{C}), \mathcal{O})$  is an equivalence of ringed topoi. For example, the property of being free is intrinsic. Indeed, the free  $\mathcal{O}$ -module on the set  $I$  is characterized by the property that

$$\text{Mor}_{\text{Mod}(\mathcal{O})}(\bigoplus_{i \in I} \mathcal{O}, \mathcal{F}) = \prod_{i \in I} \text{Mor}_{Sh(\mathcal{C})}(\{*\}, \mathcal{F})$$

for a variable  $\mathcal{F}$  in  $\text{Mod}(\mathcal{O})$ . Alternatively, we can also use Lemma 17.2 to see that being free is intrinsic. In fact, each of the properties defined in Definition

17.1 is intrinsic for the same reason. How will we go about defining other intrinsic properties of  $\mathcal{O}$ -modules?

The upshot of Lemma 7.2 is the following: Suppose you want to define an intrinsic property  $\mathcal{P}$  of an  $\mathcal{O}$ -module on a topos. Then you can proceed as follows:

- (1) Given any site  $\mathcal{C}$ , any sheaf of rings  $\mathcal{O}$  on  $\mathcal{C}$  and any  $\mathcal{O}$ -module  $\mathcal{F}$  define the corresponding property  $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$ .
- (2) For any pair of sites  $\mathcal{C}, \mathcal{C}'$ , any special cocontinuous functor  $u : \mathcal{C} \rightarrow \mathcal{C}'$ , any sheaf of rings  $\mathcal{O}$  on  $\mathcal{C}$  any  $\mathcal{O}$ -module  $\mathcal{F}$ , show that

$$\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F}) \Leftrightarrow \mathcal{P}(\mathcal{C}', g_*\mathcal{O}, g_*\mathcal{F})$$

where  $g : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{C}')$  is the equivalence of topoi associated to  $u$ .

In this case, given any ringed topos  $(Sh(\mathcal{C}), \mathcal{O})$  and any sheaf of  $\mathcal{O}$ -modules  $\mathcal{F}$  we simply say that  $\mathcal{F}$  has property  $\mathcal{P}$  if  $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$  is true. And Lemma 7.2 combined with (2) above guarantees that this is well defined.

Moreover, the same Lemma 7.2 also guarantees that if in addition

- (3) For any morphism of ringed sites  $(f, f^\#) : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  such that  $f$  is given by a functor  $u : \mathcal{D} \rightarrow \mathcal{C}$  satisfying the assumptions of Sites, Proposition 15.6, and any  $\mathcal{O}_{\mathcal{D}}$ -module  $\mathcal{G}$  we have

$$\mathcal{P}(\mathcal{D}, \mathcal{O}_{\mathcal{D}}, \mathcal{G}) \Rightarrow \mathcal{P}(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, f^*\mathcal{G})$$

then it is true that  $\mathcal{P}$  is preserved under pullback of modules w.r.t. arbitrary morphisms of ringed topoi.

We will use this method in the following sections to see that: locally free, locally generated by sections, locally generated by  $r$  sections, finite type, finite presentation, quasi-coherent, and coherent are intrinsic properties of modules.

Perhaps a more satisfying method would be to find an intrinsic definition of these notions, rather than the laborious process sketched here. On the other hand, in many geometric situations where we want to apply these definitions we are given a definite ringed site, and a definite sheaf of modules, and it is nice to have a definition already adapted to this language.

## 19. Localization of ringed sites

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U \in \text{Ob}(\mathcal{C})$ . We explain the counterparts of the results in Sites, Section 24 in this setting.

Denote  $\mathcal{O}_U = j_U^{-1}\mathcal{O}$  the restriction of  $\mathcal{O}$  to the site  $\mathcal{C}/U$ . It is described by the simple rule  $\mathcal{O}_U(V/U) = \mathcal{O}(V)$ . With this notation the localization morphism  $j_U$  becomes a morphism of ringed topoi

$$(j_U, j_U^\#) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

namely, we take  $j_U^\# : j_U^{-1}\mathcal{O} \rightarrow \mathcal{O}_U$  the identity map. Moreover, we obtain the following descriptions for pushforward and pullback of modules.

**Definition 19.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U \in \text{Ob}(\mathcal{C})$ .

- (1) The ringed site  $(\mathcal{C}/U, \mathcal{O}_U)$  is called the *localization of the ringed site  $(\mathcal{C}, \mathcal{O})$  at the object  $U$* .
- (2) The morphism of ringed topoi  $(j_U, j_U^\#) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \rightarrow (Sh(\mathcal{C}), \mathcal{O})$  is called the *localization morphism*.

- (3) The functor  $j_{U*} : \text{Mod}(\mathcal{O}_U) \rightarrow \text{Mod}(\mathcal{O})$  is called the *direct image functor*.
- (4) For a sheaf of  $\mathcal{O}$ -modules  $\mathcal{F}$  on  $\mathcal{C}$  the sheaf  $j_U^* \mathcal{F}$  is called the *restriction of  $\mathcal{F}$  to  $\mathcal{C}/U$* . We will sometimes denote it by  $\mathcal{F}|_{\mathcal{C}/U}$  or even  $\mathcal{F}|_U$ . It is described by the simple rule  $j_U^*(\mathcal{F})(X/U) = \mathcal{F}(X)$ .
- (5) The left adjoint  $j_{U!} : \text{Mod}(\mathcal{O}_U) \rightarrow \text{Mod}(\mathcal{O})$  of restriction is called *extension by zero*. It exists and is exact by Lemmas 19.2 and 19.3.

As in the topological case, see Sheaves, Section 31, the extension by zero  $j_{U!}$  functor is different from extension by the empty set  $j_{U!}$  defined on sheaves of sets. Here is the lemma defining extension by zero.

**Lemma 19.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U \in \text{Ob}(\mathcal{C})$ . The restriction functor  $j_U^* : \text{Mod}(\mathcal{O}) \rightarrow \text{Mod}(\mathcal{O}_U)$  has a left adjoint  $j_{U!} : \text{Mod}(\mathcal{O}_U) \rightarrow \text{Mod}(\mathcal{O})$ . So*

$$\text{Mor}_{\text{Mod}(\mathcal{O}_U)}(\mathcal{G}, j_U^* \mathcal{F}) = \text{Mor}_{\text{Mod}(\mathcal{O})}(j_{U!} \mathcal{G}, \mathcal{F})$$

for  $\mathcal{F} \in \text{Ob}(\text{Mod}(\mathcal{O}))$  and  $\mathcal{G} \in \text{Ob}(\text{Mod}(\mathcal{O}_U))$ . Moreover, the extension by zero  $j_{U!} \mathcal{G}$  of  $\mathcal{G}$  is the sheaf associated to the presheaf

$$V \mapsto \bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings and an obvious  $\mathcal{O}$ -module structure.

**Proof.** The  $\mathcal{O}$ -module structure on the presheaf is defined as follows. If  $f \in \mathcal{O}(V)$  and  $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$ , then we define  $f \cdot s = fs$  where  $f \in \mathcal{O}_U(\varphi : V \rightarrow U) = \mathcal{O}(V)$  (because  $\mathcal{O}_U$  is the restriction of  $\mathcal{O}$  to  $\mathcal{C}/U$ ).

Similarly, let  $\alpha : \mathcal{G} \rightarrow \mathcal{F}|_U$  be a morphism of  $\mathcal{O}_U$ -modules. In this case we can define a map from the presheaf of the lemma into  $\mathcal{F}$  by mapping

$$\bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{G}(V \xrightarrow{\varphi} U) \longrightarrow \mathcal{F}(V)$$

by the rule that  $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$  maps to  $\alpha(s) \in \mathcal{F}(V)$ . It is clear that this is  $\mathcal{O}$ -linear, and hence induces a morphism of  $\mathcal{O}$ -modules  $\alpha' : j_{U!} \mathcal{G} \rightarrow \mathcal{F}$  by the properties of sheafification of modules (Lemma 11.1).

Conversely, let  $\beta : j_{U!} \mathcal{G} \rightarrow \mathcal{F}$  be a map of  $\mathcal{O}$ -modules. Recall from Sites, Section 24 that there exists an extension by the empty set  $j_{U!}^{Sh} : \text{Sh}(\mathcal{C}/U) \rightarrow \text{Sh}(\mathcal{C})$  on sheaves of sets which is left adjoint to  $j_U^{-1}$ . Moreover,  $j_{U!}^{Sh} \mathcal{G}$  is the sheaf associated to the presheaf

$$V \mapsto \coprod_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

Hence there is a natural map  $j_{U!}^{Sh} \mathcal{G} \rightarrow j_{U!} \mathcal{G}$  of sheaves of sets. Hence precomposing  $\beta$  by this map we get a map of sheaves of sets  $j_{U!}^{Sh} \mathcal{G} \rightarrow \mathcal{F}$  which by adjunction corresponds to a map of sheaves of sets  $\beta' : \mathcal{G} \rightarrow \mathcal{F}|_U$ . We claim that  $\beta'$  is  $\mathcal{O}_U$ -linear. Namely, suppose that  $\varphi : V \rightarrow U$  is an object of  $\mathcal{C}/U$  and that  $s, s' \in \mathcal{G}(\varphi : V \rightarrow U)$ , and  $f \in \mathcal{O}(V) = \mathcal{O}_U(\varphi : V \rightarrow U)$ . Then by the discussion above we see that  $\beta'(s + s')$ , resp.  $\beta'(fs)$  in  $\mathcal{F}|_U(\varphi : V \rightarrow U)$  correspond to  $\beta(s + s')$ , resp.  $\beta(fs)$  in  $\mathcal{F}(V)$ . Since  $\beta$  is a homomorphism we conclude.

To conclude the proof of the lemma we have to show that the constructions  $\alpha \mapsto \alpha'$  and  $\beta \mapsto \beta'$  are mutually inverse. We omit the verifications.  $\square$

**Lemma 19.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U \in \text{Ob}(\mathcal{C})$ . The functor  $j_{U!} : \text{Mod}(\mathcal{O}_U) \rightarrow \text{Mod}(\mathcal{O})$  is exact.*

**Proof.** Since  $j_{U!}$  is a left adjoint to  $j_U^*$  we see that it is right exact (see Categories, Lemma 24.5 and Homology, Section 7). Hence it suffices to show that if  $\mathcal{G}_1 \rightarrow \mathcal{G}_2$  is an injective map of  $\mathcal{O}_U$ -modules, then  $j_{U!}\mathcal{G}_1 \rightarrow j_{U!}\mathcal{G}_2$  is injective. The map on sections of presheaves over an object  $V$  (as in Lemma 19.2) is the map

$$\bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{G}_1(V \xrightarrow{\varphi} U) \longrightarrow \bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{G}_2(V \xrightarrow{\varphi} U)$$

which is injective by assumption. Since sheafification is exact by Lemma 11.2 we conclude  $j_{U!}\mathcal{G}_1 \rightarrow j_{U!}\mathcal{G}_2$  is injective and we win.  $\square$

**Lemma 19.4.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $f : V \rightarrow U$  be a morphism of  $\mathcal{C}$ . Then there exists a commutative diagram*

$$\begin{array}{ccc} (Sh(\mathcal{C}/V), \mathcal{O}_V) & \xrightarrow{(j, j^\#)} & (Sh(\mathcal{C}/U), \mathcal{O}_U) \\ & \searrow (j_V, j_V^\#) \quad \swarrow (j_U, j_U^\#) & \\ & (Sh(\mathcal{C}), \mathcal{O}) & \end{array}$$

of ringed topoi. Here  $(j, j^\#)$  is the localization morphism associated to the object  $U/V$  of the ringed site  $(\mathcal{C}/V, \mathcal{O}_V)$ .

**Proof.** The only thing to check is that  $j_V^\# = j^\# \circ j^{-1}(j_U^\#)$ , since everything else follows directly from Sites, Lemma 24.7 and Sites, Equation (24.7.1). We omit the verification of the equality.  $\square$

**Remark 19.5.** In the situation of Lemma 19.2 the diagram

$$\begin{array}{ccc} Mod(\mathcal{O}_U) & \xrightarrow{j_{U!}} & Mod(\mathcal{O}_{\mathcal{C}}) \\ \text{forget} \downarrow & & \downarrow \text{forget} \\ Ab(\mathcal{C}/U) & \xrightarrow{j_{U!}^{Ab}} & Ab(\mathcal{C}) \end{array}$$

commutes. This is clear from the explicit description of the functor  $j_{U!}$  in the lemma.

**Remark 19.6.** Localization and presheaves of modules; see Sites, Remark 24.9. Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings. Let  $U$  be an object of  $\mathcal{C}$ . Strictly speaking the functors  $j_U^*$ ,  $j_{U*}$  and  $j_{U!}$  have not been defined for presheaves of  $\mathcal{O}$ -modules. But of course, we can think of a presheaf as a sheaf for the chaotic topology on  $\mathcal{C}$  (see Sites, Examples 6.6). Hence we also obtain a functor

$$j_U^* : PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}_U)$$

and functors

$$j_{U*}, j_{U!} : PMod(\mathcal{O}_U) \longrightarrow PMod(\mathcal{O})$$

which are right, left adjoint to  $j_U^*$ . Inspecting the proof of Lemma 19.2 we see that  $j_{U!}\mathcal{G}$  is the presheaf

$$V \longmapsto \bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor  $j_{U!}$  is exact (by Lemma 19.3 in the case of the discrete topologies). Moreover, if  $\mathcal{C}$  is actually a site, and  $\mathcal{O}$  is actually a sheaf of rings,

then the diagram

$$\begin{array}{ccc} \text{Mod}(\mathcal{O}_U) & \xrightarrow{j_{U!}} & \text{Mod}(\mathcal{O}) \\ \text{forget} \downarrow & & \uparrow (\ )^\# \\ \text{PMod}(\mathcal{O}_U) & \xrightarrow{j_{U!}} & \text{PMod}(\mathcal{O}) \end{array}$$

commutes.

**Remark 19.7** (Map from lower shriek to pushforward). Let  $U$  be an object of  $\mathcal{C}$ . For any abelian sheaf  $\mathcal{G}$  on  $\mathcal{C}/U$  there is a canonical map

$$c : j_{U!}\mathcal{G} \longrightarrow j_{U*}\mathcal{G}$$

Namely, this is the same thing as a map  $j_U^{-1}j_{U!}\mathcal{G} \rightarrow \mathcal{G}$ . Note that restriction commutes with sheafification. Thus we can use the presheaf of Lemma 19.2. Hence it suffices to define for  $V/U$  a map

$$\bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{G}(V) \longrightarrow \mathcal{G}(V)$$

compatible with restrictions. We simply take the map which is zero on all summands except for the one where  $\varphi$  is the structure morphism  $V \rightarrow U$  where we take 1. Moreover, if  $\mathcal{O}$  is a sheaf of rings on  $\mathcal{C}$  and  $\mathcal{G}$  is an  $\mathcal{O}_U$ -module, then the displayed map above is a map of  $\mathcal{O}$ -modules.

## 20. Localization of morphisms of ringed sites

This section is the analogue of Sites, Section 27.

**Lemma 20.1.** *Let  $(f, f^\#) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$  be a morphism of ringed sites where  $f$  is given by the continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$ . Let  $V$  be an object of  $\mathcal{D}$  and set  $U = u(V)$ . Then there is a canonical map of sheaves of rings  $(f')^\#$  such that the diagram of Sites, Lemma 27.1 is turned into a commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C}/U), \mathcal{O}_U) & \xrightarrow{(j_U, j_U^\#)} & (Sh(\mathcal{C}), \mathcal{O}) \\ (f', (f')^\#) \downarrow & & \downarrow (f, f^\#) \\ (Sh(\mathcal{D}/V), \mathcal{O}'_V) & \xrightarrow{(j_V, j_V^\#)} & (Sh(\mathcal{D}), \mathcal{O}'). \end{array}$$

Moreover, in this situation we have  $f'_*j_U^{-1} = j_V^{-1}f_*$  and  $f'_*j_U^* = j_V^*f_*$ .

**Proof.** Just take  $(f')^\#$  to be

$$(f')^{-1}\mathcal{O}'_V = (f')^{-1}j_V^{-1}\mathcal{O}' = j_U^{-1}f^{-1}\mathcal{O}' \xrightarrow{j_U^{-1}f^\#} j_U^{-1}\mathcal{O} = \mathcal{O}_U$$

and everything else follows from Sites, Lemma 27.1. (Note that  $j^{-1} = j^*$  on sheaves of modules if  $j$  is a localization morphism, hence the first equality of functors implies the second.)  $\square$

**Lemma 20.2.** *Let  $(f, f^\#) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$  be a morphism of ringed sites where  $f$  is given by the continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$ . Let  $V \in \text{Ob}(\mathcal{D})$ ,  $U \in \text{Ob}(\mathcal{C})$  and*

$c : U \rightarrow u(V)$  a morphism of  $\mathcal{C}$ . There exists a commutative diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}/U), \mathcal{O}_U) & \xrightarrow{(j_U, j_U^\#)} & (Sh(\mathcal{C}), \mathcal{O}) \\ (f_c, f_c^\#) \downarrow & & \downarrow (f, f^\#) \\ (Sh(\mathcal{D}/V), \mathcal{O}'_V) & \xrightarrow{(j_V, j_V^\#)} & (Sh(\mathcal{D}), \mathcal{O}). \end{array}$$

The morphism  $(f_c, f_c^\#)$  is equal to the composition of the morphism

$$(f', (f')^\#) : (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)}) \rightarrow (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 20.1 and the morphism

$$(j, j^\#) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \rightarrow (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)})$$

of Lemma 19.4. Given any morphisms  $b : V' \rightarrow V$ ,  $a : U' \rightarrow U$  and  $c' : U' \rightarrow u(V')$  such that

$$\begin{array}{ccc} U' & \xrightarrow{c'} & u(V') \\ a \downarrow & & \downarrow u(b) \\ U & \xrightarrow{c} & u(V) \end{array}$$

commutes, then the following diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}/U'), \mathcal{O}_{U'}) & \xrightarrow{(j_{U'/U}, j_{U'/U}^\#)} & (Sh(\mathcal{C}/U), \mathcal{O}_U) \\ (f_{c'}, f_{c'}^\#) \downarrow & & \downarrow (f_c, f_c^\#) \\ (Sh(\mathcal{D}/V'), \mathcal{O}'_{V'}) & \xrightarrow{(j_{V'/V}, j_{V'/V}^\#)} & (Sh(\mathcal{D}/V), \mathcal{O}'_V) \end{array}$$

commutes.

**Proof.** On the level of morphisms of topoi this is Sites, Lemma 27.3. To check that the diagrams commute as morphisms of ringed topoi use Lemmas 19.4 and 20.1 exactly as in the proof of Sites, Lemma 27.3.  $\square$

## 21. Localization of ringed topoi

This section is the analogue of Sites, Section 29 in the setting of ringed topoi.

**Lemma 21.1.** *Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a ringed topos. Let  $\mathcal{F} \in Sh(\mathcal{C})$  be a sheaf. For a sheaf  $\mathcal{H}$  on  $\mathcal{C}$  denote  $\mathcal{H}_{\mathcal{F}}$  the sheaf  $\mathcal{H} \times \mathcal{F}$  seen as an object of the category  $Sh(\mathcal{C})/\mathcal{F}$ . The pair  $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}})$  is a ringed topos and there is a canonical morphism of ringed topoi*

$$(j_{\mathcal{F}}, j_{\mathcal{F}}^\#) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \rightarrow (Sh(\mathcal{C}), \mathcal{O})$$

which is a localization as in Section 19 such that

- (1) the functor  $j_{\mathcal{F}}^{-1}$  is the functor  $\mathcal{H} \mapsto \mathcal{H}_{\mathcal{F}}$ ,
- (2) the functor  $j_{\mathcal{F}}^*$  is the functor  $\mathcal{H} \mapsto \mathcal{H}_{\mathcal{F}}$ ,
- (3) the functor  $j_{\mathcal{F}!}$  on sheaves of sets is the forgetful functor  $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$ ,
- (4) the functor  $j_{\mathcal{F}!}$  on sheaves of modules associates to the  $\mathcal{O}_{\mathcal{F}}$ -module  $\varphi : \mathcal{G} \rightarrow \mathcal{F}$  the  $\mathcal{O}$ -module which is the sheafification of the presheaf

$$V \mapsto \bigoplus_{s \in \mathcal{F}(V)} \{\sigma \in \mathcal{G}(V) \mid \varphi(\sigma) = s\}$$

for  $V \in \text{Ob}(\mathcal{C})$ .

**Proof.** By Sites, Lemma 29.1 we see that  $Sh(\mathcal{C})/\mathcal{F}$  is a topos and that (1) and (3) are true. In particular this shows that  $j_{\mathcal{F}}^{-1}\mathcal{O} = \mathcal{O}_{\mathcal{F}}$  and shows that  $\mathcal{O}_{\mathcal{F}}$  is a sheaf of rings. Thus we may choose the map  $j_{\mathcal{F}}^{\sharp}$  to be the identity, in particular we see that (2) is true. Moreover, the proof of Sites, Lemma 29.1 shows that we may assume  $\mathcal{C}$  is a site with all finite limits and a subcanonical topology and that  $\mathcal{F} = h_U$  for some object  $U$  of  $\mathcal{C}$ . Then (4) follows from the description of  $j_{\mathcal{F}!}$  in Lemma 19.2. Alternatively one could show directly that the functor described in (4) is a left adjoint to  $j_{\mathcal{F}}^*$ .  $\square$

**Definition 21.2.** Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a ringed topos. Let  $\mathcal{F} \in Sh(\mathcal{C})$ .

- (1) The ringed topos  $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}})$  is called the *localization of the ringed topos*  $(Sh(\mathcal{C}), \mathcal{O})$  at  $\mathcal{F}$ .
- (2) The morphism of ringed topoi  $(j_{\mathcal{F}}, j_{\mathcal{F}}^{\sharp}) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \rightarrow (Sh(\mathcal{C}), \mathcal{O})$  of Lemma 21.1 is called the *localization morphism*.

We continue the tradition, established in the chapter on sites, that we check the localization constructions on topoi are compatible with the constructions of localization on sites, whenever this makes sense.

**Lemma 21.3.** *With  $(Sh(\mathcal{C}), \mathcal{O})$  and  $\mathcal{F} \in Sh(\mathcal{C})$  as in Lemma 21.1. If  $\mathcal{F} = h_U^{\sharp}$  for some object  $U$  of  $\mathcal{C}$  then via the identification  $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^{\sharp}$  of Sites, Lemma 24.4 we have*

- (1) *canonically  $\mathcal{O}_U = \mathcal{O}_{\mathcal{F}}$ , and*
- (2) *with these identifications we have  $(j_{\mathcal{F}}, j_{\mathcal{F}}^{\sharp}) = (j_U, j_U^{\sharp})$ .*

**Proof.** The assertion for underlying topoi is Sites, Lemma 29.5. Note that  $\mathcal{O}_U$  is the restriction of  $\mathcal{O}$  which by Sites, Lemma 24.6 corresponds to  $\mathcal{O} \times h_U^{\sharp}$  under the equivalence of Sites, Lemma 24.4. By definition of  $\mathcal{O}_{\mathcal{F}}$  we get (1). What's left is to prove that  $j_{\mathcal{F}}^{\sharp} = j_U^{\sharp}$  under this identification. We omit the verification.  $\square$

Localization is functorial in the following two ways: We can “relocalize” a localization (see Lemma 21.4) or we can given a morphism of ringed topoi, localize upstairs at the inverse image of a sheaf downstairs and get a commutative diagram of locally ringed spaces (see Lemma 22.1).

**Lemma 21.4.** *Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a ringed topos. If  $s : \mathcal{G} \rightarrow \mathcal{F}$  is a morphism of sheaves on  $\mathcal{C}$  then there exists a natural commutative diagram of morphisms of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_{\mathcal{G}}) & \xrightarrow{(j, j^{\sharp})} & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \\ & \searrow (j_{\mathcal{G}}, j_{\mathcal{G}}^{\sharp}) \quad \swarrow (j_{\mathcal{F}}, j_{\mathcal{F}}^{\sharp}) & \\ & (Sh(\mathcal{C}), \mathcal{O}) & \end{array}$$

where  $(j, j^{\sharp})$  is the localization morphism of the ringed topos  $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}})$  at the object  $\mathcal{G}/\mathcal{F}$ .

**Proof.** All assertions follow from Sites, Lemma 29.6 except the assertion that  $j_{\mathcal{G}}^{\sharp} = j^{\sharp} \circ j^{-1}(j_{\mathcal{F}}^{\sharp})$ . We omit the verification.  $\square$

**Lemma 21.5.** *With  $(Sh(\mathcal{C}), \mathcal{O})$ ,  $s : \mathcal{G} \rightarrow \mathcal{F}$  as in Lemma 21.4. If there exist a morphism  $f : V \rightarrow U$  of  $\mathcal{C}$  such that  $\mathcal{G} = h_V^{\sharp}$  and  $\mathcal{F} = h_U^{\sharp}$  and  $s$  is induced by*



$f$ , then the diagrams of Lemma 19.4 and Lemma 21.4 agree via the identifications  $(j_{\mathcal{F}}, j_{\mathcal{F}}^{\sharp}) = (j_U, j_U^{\sharp})$  and  $(j_{\mathcal{G}}, j_{\mathcal{G}}^{\sharp}) = (j_V, j_V^{\sharp})$  of Lemma 21.3.

**Proof.** All assertions follow from Sites, Lemma 29.7 except for the assertion that the two maps  $j^{\sharp}$  agree. This holds since in both cases the map  $j^{\sharp}$  is simply the identity. Some details omitted.  $\square$

## 22. Localization of morphisms of ringed topoi

This section is the analogue of Sites, Section 30.

**Lemma 22.1.** *Let*

$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$

*be a morphism of ringed topoi. Let  $\mathcal{G}$  be a sheaf on  $\mathcal{D}$ . Set  $\mathcal{F} = f^{-1}\mathcal{G}$ . Then there exists a commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) & \xrightarrow{(j_{\mathcal{F}}, j_{\mathcal{F}}^{\sharp})} & (Sh(\mathcal{C}), \mathcal{O}) \\ (f', (f')^{\sharp}) \downarrow & & \downarrow (f, f^{\sharp}) \\ (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}}) & \xrightarrow{(j_{\mathcal{G}}, j_{\mathcal{G}}^{\sharp})} & (Sh(\mathcal{D}), \mathcal{O}') \end{array}$$

*We have  $f'_* j_{\mathcal{F}}^{-1} = j_{\mathcal{G}}^{-1} f_*$  and  $f'_* j_{\mathcal{F}}^* = j_{\mathcal{G}}^* f_*$ . Moreover, the morphism  $f'$  is characterized by the rule*

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F}).$$

**Proof.** By Sites, Lemma 30.1 we have the diagram of underlying topoi, the equality  $f'_* j_{\mathcal{F}}^{-1} = j_{\mathcal{G}}^{-1} f_*$ , and the description of  $(f')^{-1}$ . To define  $(f')^{\sharp}$  we use the map

$$(f')^{\sharp} : \mathcal{O}'_{\mathcal{G}} = j_{\mathcal{G}}^{-1} \mathcal{O}' \xrightarrow{j_{\mathcal{G}}^{-1} f^{\sharp}} j_{\mathcal{G}}^{-1} f_* \mathcal{O} = f'_* j_{\mathcal{F}}^{-1} \mathcal{O} = f'_* \mathcal{O}_{\mathcal{F}}$$

or equivalently the map

$$(f')^{\sharp} : (f')^{-1} \mathcal{O}'_{\mathcal{G}} = (f')^{-1} j_{\mathcal{G}}^{-1} \mathcal{O}' = j_{\mathcal{F}}^{-1} f^{-1} \mathcal{O}' \xrightarrow{j_{\mathcal{F}}^{-1} f^{\sharp}} j_{\mathcal{F}}^{-1} \mathcal{O} = \mathcal{O}_{\mathcal{F}}.$$

We omit the verification that these two maps are indeed adjoint to each other. The second construction of  $(f')^{\sharp}$  shows that the diagram commutes in the 2-category of ringed topoi (as the maps  $j_{\mathcal{F}}^{\sharp}$  and  $j_{\mathcal{G}}^{\sharp}$  are identities). Finally, the equality  $f'_* j_{\mathcal{F}}^* = j_{\mathcal{G}}^* f_*$  follows from the equality  $f'_* j_{\mathcal{F}}^{-1} = j_{\mathcal{G}}^{-1} f_*$  and the fact that pullbacks of sheaves of modules and sheaves of sets agree, see Lemma 21.1.  $\square$

**Lemma 22.2.** *Let*

$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$

*be a morphism of ringed topoi. Let  $\mathcal{G}$  be a sheaf on  $\mathcal{D}$ . Set  $\mathcal{F} = f^{-1}\mathcal{G}$ . If  $f$  is given by a continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$  and  $\mathcal{G} = h_V^{\sharp}$ , then the commutative diagrams of Lemma 20.1 and Lemma 22.1 agree via the identifications of Lemma 21.3.*

**Proof.** At the level of morphisms of topoi this is Sites, Lemma 30.2. This works also on the level of morphisms of ringed topoi since the formulas defining  $(f')^{\sharp}$  in the proofs of Lemma 20.1 and Lemma 22.1 agree.  $\square$

**Lemma 22.3.** *Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}')$  be a morphism of ringed topoi. Let  $\mathcal{G}$  be a sheaf on  $\mathcal{D}$ , let  $\mathcal{F}$  be a sheaf on  $\mathcal{C}$ , and let  $s : \mathcal{F} \rightarrow f^{-1}\mathcal{G}$  a morphism of sheaves. There exists a commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) & \xrightarrow{(j_{\mathcal{F}}, j_{\mathcal{F}}^\#)} & (Sh(\mathcal{C}), \mathcal{O}) \\ (f_c, f_c^\#) \downarrow & & \downarrow (f, f^\#) \\ (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}}) & \xrightarrow{(j_{\mathcal{G}}, j_{\mathcal{G}}^\#)} & (Sh(\mathcal{D}), \mathcal{O}'). \end{array}$$

The morphism  $(f_s, f_s^\#)$  is equal to the composition of the morphism

$$(f', (f')^\#) : (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}}) \rightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}})$$

of Lemma 22.1 and the morphism

$$(j, j^\#) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \rightarrow (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}})$$

of Lemma 21.4. Given any morphisms  $b : \mathcal{G}' \rightarrow \mathcal{G}$ ,  $a : \mathcal{F}' \rightarrow \mathcal{F}$ , and  $s' : \mathcal{F}' \rightarrow f^{-1}\mathcal{G}'$  such that

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{s'} & f^{-1}\mathcal{G}' \\ a \downarrow & & \downarrow f^{-1}b \\ \mathcal{F} & \xrightarrow{s} & f^{-1}\mathcal{G} \end{array}$$

commutes, then the following diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{F}', \mathcal{O}_{\mathcal{F}'}) & \xrightarrow{(j_{\mathcal{F}'/\mathcal{F}}, j_{\mathcal{F}'/\mathcal{F}}^\#)} & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \\ (f_{s'}, f_{s'}^\#) \downarrow & & \downarrow (f_s, f_s^\#) \\ (Sh(\mathcal{D})/\mathcal{G}', \mathcal{O}'_{\mathcal{G}'}) & \xrightarrow{(j_{\mathcal{G}'/\mathcal{G}}, j_{\mathcal{G}'/\mathcal{G}}^\#)} & (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}}) \end{array}$$

commutes.

**Proof.** On the level of morphisms of topoi this is Sites, Lemma 30.3. To check that the diagrams commute as morphisms of ringed topoi use the commutative diagrams of Lemmas 21.4 and 22.1.  $\square$

**Lemma 22.4.** *Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}')$ ,  $s : \mathcal{F} \rightarrow f^{-1}\mathcal{G}$  be as in Lemma 22.3. If  $f$  is given by a continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$  and  $\mathcal{G} = h_V^\#$ ,  $\mathcal{F} = h_U^\#$  and  $s$  comes from a morphism  $c : U \rightarrow u(V)$ , then the commutative diagrams of Lemma 20.2 and Lemma 22.3 agree via the identifications of Lemma 21.3.*

**Proof.** This is formal using Lemmas 21.5 and 22.2.  $\square$

### 23. Local types of modules

According to our general strategy explained in Section 18 we first define the local types for sheaves of modules on a ringed site, and then we immediately show that these types are intrinsic, hence make sense for sheaves of modules on ringed topoi.

**Definition 23.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules. We will freely use the notions defined in Definition 17.1.

- (1) We say  $\mathcal{F}$  is *locally free* if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}_{i \in I}$  of  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/U_i}$  is a free  $\mathcal{O}_{U_i}$ -module.

- (2) We say  $\mathcal{F}$  is *finite locally free* if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}_{i \in I}$  of  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/U_i}$  is a finite free  $\mathcal{O}_{U_i}$ -module.
- (3) We say  $\mathcal{F}$  is *locally generated by sections* if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}_{i \in I}$  of  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/U_i}$  is an  $\mathcal{O}_{U_i}$ -module generated by global sections.
- (4) Given  $r \geq 0$  we say  $\mathcal{F}$  is *locally generated by  $r$  sections* if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}_{i \in I}$  of  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/U_i}$  is an  $\mathcal{O}_{U_i}$ -module generated by  $r$  global sections.
- (5) We say  $\mathcal{F}$  is *of finite type* if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}_{i \in I}$  of  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/U_i}$  is an  $\mathcal{O}_{U_i}$ -module generated by finitely many global sections.
- (6) We say  $\mathcal{F}$  is *quasi-coherent* if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}_{i \in I}$  of  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/U_i}$  is an  $\mathcal{O}_{U_i}$ -module which has a global presentation.
- (7) We say  $\mathcal{F}$  is *of finite presentation* if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}_{i \in I}$  of  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/U_i}$  is an  $\mathcal{O}_{U_i}$ -module which has a finite global presentation.
- (8) We say  $\mathcal{F}$  is *coherent* if and only if  $\mathcal{F}$  is of finite type, and for every object  $U$  of  $\mathcal{C}$  and any  $s_1, \dots, s_n \in \mathcal{F}(U)$  the kernel of the map  $\bigoplus_{i=1, \dots, n} \mathcal{O}_U \rightarrow \mathcal{F}|_U$  is of finite type on  $(\mathcal{C}/U, \mathcal{O}_U)$ .

**Lemma 23.2.** *Any of the properties (1) – (8) of Definition 23.1 is intrinsic (see discussion in Section 18).*

**Proof.** Let  $\mathcal{C}, \mathcal{D}$  be sites. Let  $u : \mathcal{C} \rightarrow \mathcal{D}$  be a special cocontinuous functor. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ . Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules on  $\mathcal{C}$ . Let  $g : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be the equivalence of topoi associated to  $u$ . Set  $\mathcal{O}' = g_*\mathcal{O}$ , and let  $g^\# : \mathcal{O}' \rightarrow g_*\mathcal{O}$  be the identity. Finally, set  $\mathcal{F}' = g_*\mathcal{F}$ . Let  $\mathcal{P}_l$  be one of the properties (1) – (7) listed in Definition 23.1. (We will discuss the coherent case at the end of the proof.) Let  $\mathcal{P}_g$  denote the corresponding property listed in Definition 17.1. We have already seen that  $\mathcal{P}_g$  is intrinsic. We have to show that  $\mathcal{P}_l(\mathcal{C}, \mathcal{O}, \mathcal{F})$  holds if and only if  $\mathcal{P}_l(\mathcal{D}, \mathcal{O}', \mathcal{F}')$  holds.

Assume that  $\mathcal{F}$  has  $\mathcal{P}_l$ . Let  $V$  be an object of  $\mathcal{D}$ . One of the properties of a special cocontinuous functor is that there exists a covering  $\{u(U_i) \rightarrow V\}_{i \in I}$  in the site  $\mathcal{D}$ . By assumption, for each  $i$  there exists a covering  $\{U_{ij} \rightarrow U_i\}_{j \in J_i}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{U_{ij}}$  is  $\mathcal{P}_g$ . By Sites, Lemma 28.3 we have commutative diagrams of ringed topoi

$$\begin{array}{ccc}
 (Sh(\mathcal{C}/U_{ij}), \mathcal{O}_{U_{ij}}) & \longrightarrow & (Sh(\mathcal{C}), \mathcal{O}) \\
 \downarrow & & \downarrow \\
 (Sh(\mathcal{D}/u(U_{ij})), \mathcal{O}'_{u(U_{ij})}) & \longrightarrow & (Sh(\mathcal{D}), \mathcal{O}')
 \end{array}$$

where the vertical arrows are equivalences. Hence we conclude that  $\mathcal{F}'|_{u(U_{ij})}$  has property  $\mathcal{P}_g$  also. And moreover,  $\{u(U_{ij}) \rightarrow V\}_{i \in I, j \in J_i}$  is a covering of the site  $\mathcal{D}$ . Hence  $\mathcal{F}'$  has property  $\mathcal{P}_l$ .

Assume that  $\mathcal{F}'$  has  $\mathcal{P}_l$ . Let  $U$  be an object of  $\mathcal{C}$ . By assumption, there exists a covering  $\{V_i \rightarrow u(U)\}_{i \in I}$  such that  $\mathcal{F}'|_{V_i}$  has property  $\mathcal{P}_g$ . Because  $u$  is cocontinuous we can refine this covering by a family  $\{u(U_j) \rightarrow u(U)\}_{j \in J}$  where  $\{U_j \rightarrow U\}_{j \in J}$

is a covering in  $\mathcal{C}$ . Say the refinement is given by  $\alpha : J \rightarrow I$  and  $u(U_j) \rightarrow V_{\alpha(j)}$ . Restricting is transitive, i.e.,  $(\mathcal{F}'|_{V_{\alpha(j)}})|_{u(U_j)} = \mathcal{F}'|_{u(U_j)}$ . Hence by Lemma 17.2 we see that  $\mathcal{F}'|_{u(U_j)}$  has property  $\mathcal{P}_g$ . Hence the diagram

$$\begin{array}{ccc} (Sh(\mathcal{C}/U_j), \mathcal{O}_{U_j}) & \longrightarrow & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow & & \downarrow \\ (Sh(\mathcal{D}/u(U_j)), \mathcal{O}'_{u(U_j)}) & \longrightarrow & (Sh(\mathcal{D}), \mathcal{O}') \end{array}$$

where the vertical arrows are equivalences shows that  $\mathcal{F}|_{U_j}$  has property  $\mathcal{P}_g$  also. Thus  $\mathcal{F}$  has property  $\mathcal{P}_l$  as desired.

Finally, we prove the lemma in case  $\mathcal{P}_l = \text{coherent}^2$ . Assume  $\mathcal{F}$  is coherent. This implies that  $\mathcal{F}$  is of finite type and hence  $\mathcal{F}'$  is of finite type also by the first part of the proof. Let  $V$  be an object of  $\mathcal{D}$  and let  $s_1, \dots, s_n \in \mathcal{F}'(V)$ . We have to show that the kernel  $\mathcal{K}'$  of  $\bigoplus_{j=1, \dots, n} \mathcal{O}_V \rightarrow \mathcal{F}'|_V$  is of finite type on  $\mathcal{D}/V$ . This means we have to show that for any  $V'/V$  there exists a covering  $\{V'_i \rightarrow V'\}$  such that  $\mathcal{F}'|_{V'_i}$  is generated by finitely many sections. Replacing  $V$  by  $V'$  (and restricting the sections  $s_j$  to  $V'$ ) we reduce to the case where  $V' = V$ . Since  $u$  is a special cocontinuous functor, there exists a covering  $\{u(U_i) \rightarrow V\}_{i \in I}$  in the site  $\mathcal{D}$ . Using the isomorphism of topoi  $Sh(\mathcal{C}/U_i) = Sh(\mathcal{D}/u(U_i))$  we see that  $\mathcal{K}'|_{u(U_i)}$  corresponds to the kernel  $\mathcal{K}_i$  of a map  $\bigoplus_{j=1, \dots, n} \mathcal{O}_{U_i} \rightarrow \mathcal{F}|_{U_i}$ . Since  $\mathcal{F}$  is coherent we see that  $\mathcal{K}_i$  is of finite type. Hence we conclude (by the first part of the proof again) that  $\mathcal{K}|_{u(U_i)}$  is of finite type. Thus there exist coverings  $\{V_{il} \rightarrow u(U_i)\}$  such that  $\mathcal{K}|_{V_{il}}$  is generated by finitely many global sections. Since  $\{V_{il} \rightarrow V\}$  is a covering of  $\mathcal{D}$  we conclude that  $\mathcal{K}$  is of finite type as desired.

Assume  $\mathcal{F}'$  is coherent. This implies that  $\mathcal{F}'$  is of finite type and hence  $\mathcal{F}$  is of finite type also by the first part of the proof. Let  $U$  be an object of  $\mathcal{C}$ , and let  $s_1, \dots, s_n \in \mathcal{F}(U)$ . We have to show that the kernel  $\mathcal{K}$  of  $\bigoplus_{j=1, \dots, n} \mathcal{O}_U \rightarrow \mathcal{F}|_U$  is of finite type on  $\mathcal{C}/U$ . Using the isomorphism of topoi  $Sh(\mathcal{C}/U) = Sh(\mathcal{D}/u(U))$  we see that  $\mathcal{K}|_U$  corresponds to the kernel  $\mathcal{K}'$  of a map  $\bigoplus_{j=1, \dots, n} \mathcal{O}_{u(U)} \rightarrow \mathcal{F}'|_{u(U)}$ . As  $\mathcal{F}'$  is coherent, we see that  $\mathcal{K}'$  is of finite type. Hence, by the first part of the proof again, we conclude that  $\mathcal{K}$  is of finite type.  $\square$

Hence from now on we may refer to the properties of  $\mathcal{O}$ -modules defined in Definition 23.1 without specifying a site.

**Lemma 23.3.** *Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a ringed topos. Let  $\mathcal{F}$  be an  $\mathcal{O}$ -module. Assume that the site  $\mathcal{C}$  has a final object  $X$ . Then*

- (1) *The following are equivalent*
  - (a)  $\mathcal{F}$  is locally free,
  - (b) there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is a locally free  $\mathcal{O}_{X_i}$ -module, and
  - (c) there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is a free  $\mathcal{O}_{X_i}$ -module.
- (2) *The following are equivalent*
  - (a)  $\mathcal{F}$  is finite locally free,

<sup>2</sup>The mechanics of this are a bit awkward, and we suggest the reader skip this part of the proof.

- (b) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is a finite locally free  $\mathcal{O}_{X_i}$ -module, and*
- (c) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is a finite free  $\mathcal{O}_{X_i}$ -module.*
- (3) *The following are equivalent*
  - (a)  *$\mathcal{F}$  is locally generated by sections,*
  - (b) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is an  $\mathcal{O}_{X_i}$ -module locally generated by sections, and*
  - (c) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is an  $\mathcal{O}_{X_i}$ -module globally generated by sections.*
- (4) *Given  $r \geq 0$ , the following are equivalent*
  - (a)  *$\mathcal{F}$  is locally generated by  $r$  sections,*
  - (b) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is an  $\mathcal{O}_{X_i}$ -module locally generated by  $r$  sections, and*
  - (c) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is an  $\mathcal{O}_{X_i}$ -module globally generated by  $r$  sections.*
- (5) *The following are equivalent*
  - (a)  *$\mathcal{F}$  is of finite type,*
  - (b) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is an  $\mathcal{O}_{X_i}$ -module of finite type, and*
  - (c) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is an  $\mathcal{O}_{X_i}$ -module globally generated by finitely many sections.*
- (6) *The following are equivalent*
  - (a)  *$\mathcal{F}$  is quasi-coherent,*
  - (b) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is a quasi-coherent  $\mathcal{O}_{X_i}$ -module, and*
  - (c) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is an  $\mathcal{O}_{X_i}$ -module which has a global presentation.*
- (7) *The following are equivalent*
  - (a)  *$\mathcal{F}$  is of finite presentation,*
  - (b) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is an  $\mathcal{O}_{X_i}$ -module of finite presentation, and*
  - (c) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is an  $\mathcal{O}_{X_i}$ -module has a finite global presentation.*
- (8) *The following are equivalent*
  - (a)  *$\mathcal{F}$  is coherent, and*
  - (b) *there exists a covering  $\{X_i \rightarrow X\}$  in  $\mathcal{C}$  such that each restriction  $\mathcal{F}|_{\mathcal{C}/X_i}$  is a coherent  $\mathcal{O}_{X_i}$ -module.*

**Proof.** In each case we have (a)  $\Rightarrow$  (b). In each of the cases (1) - (6) condition (b) implies condition (c) by axiom (2) of a site (see Sites, Definition 6.2) and the definition of the local types of modules. Suppose  $\{X_i \rightarrow X\}$  is a covering. Then for every object  $U$  of  $\mathcal{C}$  we get an induced covering  $\{X_i \times_X U \rightarrow U\}$ . Moreover, the global property for  $\mathcal{F}|_{\mathcal{C}/X_i}$  in part (c) implies the corresponding global property for  $\mathcal{F}|_{\mathcal{C}/X_i \times_X U}$  by Lemma 17.2, hence the sheaf has property (a) by definition. We omit the proof of (b)  $\Rightarrow$  (a) in case (7).  $\square$

**Lemma 23.4.** *Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{F}$  be an  $\mathcal{O}_{\mathcal{D}}$ -module.*

- (1) If  $\mathcal{F}$  is locally free then  $f^*\mathcal{F}$  is locally free.
- (2) If  $\mathcal{F}$  is finite locally free then  $f^*\mathcal{F}$  is finite locally free.
- (3) If  $\mathcal{F}$  is locally generated by sections then  $f^*\mathcal{F}$  is locally generated by sections.
- (4) If  $\mathcal{F}$  is locally generated by  $r$  sections then  $f^*\mathcal{F}$  is locally generated by  $r$  sections.
- (5) If  $\mathcal{F}$  is of finite type then  $f^*\mathcal{F}$  is of finite type.
- (6) If  $\mathcal{F}$  is quasi-coherent then  $f^*\mathcal{F}$  is quasi-coherent.
- (7) If  $\mathcal{F}$  is of finite presentation then  $f^*\mathcal{F}$  is of finite presentation.

**Proof.** According to the discussion in Section 18 we need only check preservation under pullback for a morphism of ringed sites  $(f, f^\#) : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  such that  $f$  is given by a left exact, continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$  between sites which have all finite limits. Let  $\mathcal{G}$  be a sheaf of  $\mathcal{O}_{\mathcal{D}}$ -modules which has one of the properties (1) – (6) of Definition 23.1. We know  $\mathcal{D}$  has a final object  $Y$  and  $X = u(Y)$  is a final object for  $\mathcal{C}$ . By assumption we have a covering  $\{Y_i \rightarrow Y\}$  such that  $\mathcal{G}|_{\mathcal{D}/Y_i}$  has the corresponding global property. Set  $X_i = u(Y_i)$  so that  $\{X_i \rightarrow X\}$  is a covering in  $\mathcal{C}$ . We get a commutative diagram of morphisms ringed sites

$$\begin{array}{ccc} (\mathcal{C}/X_i, \mathcal{O}_{\mathcal{C}}|_{X_i}) & \longrightarrow & (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \\ \downarrow & & \downarrow \\ (\mathcal{D}/Y_i, \mathcal{O}_{\mathcal{D}}|_{Y_i}) & \longrightarrow & (\mathcal{D}, \mathcal{O}_{\mathcal{D}}) \end{array}$$

by Sites, Lemma 27.2. Hence by Lemma 17.2 that  $f^*\mathcal{G}|_{X_i}$  has the corresponding global property. Hence we conclude that  $\mathcal{G}$  has the local property we started out with by Lemma 23.3.  $\square$

## 24. Basic results on local types of modules

Basic lemmas related to the definitions made above.

**Lemma 24.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\theta : \mathcal{G} \rightarrow \mathcal{F}$  be a surjective  $\mathcal{O}$ -module map with  $\mathcal{F}$  of finite presentation and  $\mathcal{G}$  of finite type. Then  $\text{Ker}(\theta)$  is of finite type.*

**Proof.** Omitted. Hint: See Modules, Lemma 11.3.  $\square$

## 25. Closed immersions of ringed topoi

When do we declare a morphism of ringed topoi  $i : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}')$  to be a closed immersion? By analogy with the discussion in Modules, Section 13 it seems natural to assume at least:

- (1) The functor  $i$  is a closed immersion of topoi (Sites, Definition 42.7).
- (2) The associated map  $\mathcal{O}' \rightarrow i_*\mathcal{O}$  is surjective.

These conditions already imply a number of pleasing results which we discuss in this section. However, it seems prudent to not actually define the notion of a closed immersion of ringed topoi as there are many different definitions we could use.

**Lemma 25.1.** *Let  $i : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}')$  be a morphism of ringed topoi. Assume  $i$  is a closed immersion of topoi and  $i^\# : \mathcal{O}' \rightarrow i_*\mathcal{O}$  is surjective. Denote  $\mathcal{I} \subset \mathcal{O}'$  the kernel of  $i^\#$ . The functor*

$$i_* : \text{Mod}(\mathcal{O}) \longrightarrow \text{Mod}(\mathcal{O}')$$

is exact, fully faithful, with essential image those  $\mathcal{O}'$ -modules  $\mathcal{G}$  such that  $\mathcal{I}\mathcal{G} = 0$ .

**Proof.** By Lemma 15.2 and Sites, Lemma 42.8 we see that  $i_*$  is exact. From the fact that  $i_*$  is fully faithful on sheaves of sets, and the fact that  $i^\#$  is surjective it follows that  $i_*$  is fully faithful as a functor  $Mod(\mathcal{O}) \rightarrow Mod(\mathcal{O}')$ . Namely, suppose that  $\alpha : i_*\mathcal{F}_1 \rightarrow i_*\mathcal{F}_2$  is an  $\mathcal{O}'$ -module map. By the fully faithfulness of  $i_*$  we obtain a map  $\beta : \mathcal{F}_1 \rightarrow \mathcal{F}_2$  of sheaves of sets. To prove  $\beta$  is a map of modules we have to show that

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F}_1 & \longrightarrow & \mathcal{F}_1 \\ \downarrow & & \downarrow \\ \mathcal{O} \times \mathcal{F}_2 & \longrightarrow & \mathcal{F}_2 \end{array}$$

commutes. It suffices to prove commutativity after applying  $i_*$ . Consider

$$\begin{array}{ccccc} \mathcal{O}' \times i_*\mathcal{F}_1 & \longrightarrow & i_*\mathcal{O} \times i_*\mathcal{F}_1 & \longrightarrow & i_*\mathcal{F}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}' \times i_*\mathcal{F}_2 & \longrightarrow & i_*\mathcal{O} \times i_*\mathcal{F}_2 & \longrightarrow & i_*\mathcal{F}_2 \end{array}$$

We know the outer rectangle commutes. Since  $i^\#$  is surjective we conclude.

To finish the proof we have to prove the statement on the essential image of  $i_*$ . It is clear that  $i_*\mathcal{F}$  is annihilated by  $\mathcal{I}$  for any  $\mathcal{O}$ -module  $\mathcal{F}$ . Conversely, let  $\mathcal{G}$  be a  $\mathcal{O}'$ -module with  $\mathcal{I}\mathcal{G} = 0$ . By definition of a closed subtopos there exists a subsheaf  $\mathcal{U}$  of the final object of  $\mathcal{D}$  such that the essential image of  $i_*$  on sheaves of sets is the class of sheaves of sets  $\mathcal{H}$  such that  $\mathcal{H} \times \mathcal{U} \rightarrow \mathcal{U}$  is an isomorphism. In particular,  $i_*\mathcal{O} \times \mathcal{U} = \mathcal{U}$ . This implies that  $\mathcal{I} \times \mathcal{U} = \mathcal{O} \times \mathcal{U}$ . Hence our module  $\mathcal{G}$  satisfies  $\mathcal{G} \times \mathcal{U} = \{0\} \times \mathcal{U} = \mathcal{U}$  (because the zero module is isomorphic to the final object of sheaves of sets). Thus there exists a sheaf of sets  $\mathcal{F}$  on  $\mathcal{C}$  with  $i_*\mathcal{F} = \mathcal{G}$ . Since  $i_*$  is fully faithful on sheaves of sets, we see that in order to define the addition  $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$  and the multiplication  $\mathcal{O} \times \mathcal{F} \rightarrow \mathcal{F}$  it suffices to use the addition

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$$

(given to us as  $\mathcal{G}$  is a  $\mathcal{O}'$ -module) and the multiplication

$$i_*\mathcal{O} \times \mathcal{G} \rightarrow \mathcal{G}$$

which is given to us as we have the multiplication by  $\mathcal{O}'$  which annihilates  $\mathcal{I}$  by assumption and  $i_*\mathcal{O} = \mathcal{O}'/\mathcal{I}$ . By construction  $\mathcal{G}$  is isomorphic to the pushforward of the  $\mathcal{O}$ -module  $\mathcal{F}$  so constructed.  $\square$

## 26. Tensor product

In Sections 9 and 11 we defined the change of rings functor by a tensor product construction. To be sure this construction makes sense also to define the tensor product of presheaves of  $\mathcal{O}$ -modules. To be precise, suppose  $\mathcal{C}$  is a category,  $\mathcal{O}$  is a presheaf of rings, and  $\mathcal{F}, \mathcal{G}$  are presheaves of  $\mathcal{O}$ -modules. In this case we define  $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$  to be the presheaf

$$U \longmapsto (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$$

If  $\mathcal{C}$  is a site,  $\mathcal{O}$  is a sheaf of rings and  $\mathcal{F}, \mathcal{G}$  are sheaves of  $\mathcal{O}$ -modules then we define

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})^\#$$

to be the sheaf of  $\mathcal{O}$ -modules associated to the presheaf  $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$ .

Here are some formulas which we will use below without further mention:

$$(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}) \otimes_{p,\mathcal{O}} \mathcal{H} = \mathcal{F} \otimes_{p,\mathcal{O}} (\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{H}),$$

and similarly for sheaves. If  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$  is a map of presheaves of rings, then

$$(\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{G}) \otimes_{p,\mathcal{O}_1} \mathcal{O}_2 = (\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2) \otimes_{p,\mathcal{O}_2} (\mathcal{G} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2),$$

and similarly for sheaves. These follow from their algebraic counterparts and sheafification.

Let  $\mathcal{C}$  be a site, let  $\mathcal{O}$  be a sheaf of rings and let  $\mathcal{F}, \mathcal{G}, \mathcal{H}$  be sheaves of  $\mathcal{O}$ -modules. In this case we define

$$\text{Bilin}_{\mathcal{O}}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) = \{\varphi \in \text{Mor}_{\text{Sh}(\mathcal{C})}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) \mid \varphi \text{ is } \mathcal{O}\text{-bilinear}\}.$$

With this definition we have

$$\text{Hom}_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = \text{Bilin}_{\mathcal{O}}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

In other words  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$  represents the functor which associates to  $\mathcal{H}$  the set of bilinear maps  $\mathcal{F} \times \mathcal{G} \rightarrow \mathcal{H}$ . In particular, since the notion of a bilinear map makes sense for a pair of modules on a ringed topos, we see that the tensor product of sheaves of modules is intrinsic to the topos (compare the discussion in Section 18). In fact we have the following.

**Lemma 26.1.** *Let  $f : (\text{Sh}(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (\text{Sh}(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{F}, \mathcal{G}$  be  $\mathcal{O}_{\mathcal{D}}$ -modules. Then  $f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{D}}} \mathcal{G}) = f^* \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{C}}} f^* \mathcal{G}$  functorially in  $\mathcal{F}, \mathcal{G}$ .*

**Proof.** For a sheaf  $\mathcal{H}$  of  $\mathcal{O}_{\mathcal{C}}$  modules we have

$$\begin{aligned} \text{Hom}_{\mathcal{O}_{\mathcal{C}}}(f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{D}}} \mathcal{G}), \mathcal{H}) &= \text{Hom}_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{D}}} \mathcal{G}, f_* \mathcal{H}) \\ &= \text{Bilin}_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F} \times \mathcal{G}, f_* \mathcal{H}) \\ &= \text{Bilin}_{f^{-1}\mathcal{O}_{\mathcal{D}}}(f^{-1}\mathcal{F} \times f^{-1}\mathcal{G}, \mathcal{H}) \\ &= \text{Hom}_{f^{-1}\mathcal{O}_{\mathcal{D}}}(f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_{\mathcal{D}}} f^{-1}\mathcal{G}, \mathcal{H}) \\ &= \text{Hom}_{\mathcal{O}_{\mathcal{C}}}(f^* \mathcal{F} \otimes_{f^* \mathcal{O}_{\mathcal{D}}} f^* \mathcal{G}, \mathcal{H}) \end{aligned}$$

The interesting “=” in this sequence of equalities is the third equality. It follows from the definition and adjointness of  $f_*$  and  $f^{-1}$  (as discussed in previous sections) in a straightforward manner.  $\square$

**Lemma 26.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}, \mathcal{G}$  be sheaves of  $\mathcal{O}$ -modules.*

- (1) *If  $\mathcal{F}, \mathcal{G}$  are locally free, so is  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ .*
- (2) *If  $\mathcal{F}, \mathcal{G}$  are finite locally free, so is  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ .*
- (3) *If  $\mathcal{F}, \mathcal{G}$  are locally generated by sections, so is  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ .*
- (4) *If  $\mathcal{F}, \mathcal{G}$  are of finite type, so is  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ .*
- (5) *If  $\mathcal{F}, \mathcal{G}$  are quasi-coherent, so is  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ .*
- (6) *If  $\mathcal{F}, \mathcal{G}$  are of finite presentation, so is  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ .*
- (7) *If  $\mathcal{F}$  is of finite presentation and  $\mathcal{G}$  is coherent, then  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$  is coherent.*
- (8) *If  $\mathcal{F}, \mathcal{G}$  are coherent, so is  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ .*

**Proof.** Omitted. Hint: Compare with Sheaves of Modules, Lemma 15.5.  $\square$



## 27. Internal Hom

Let  $\mathcal{C}$  be a category and let  $\mathcal{O}$  be a presheaf of rings. Let  $\mathcal{F}, \mathcal{G}$  be presheaves of  $\mathcal{O}$ -modules. Consider the rule

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

For  $\varphi : V \rightarrow U$  in  $\mathcal{C}$  we define a restriction mapping

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{G}|_V)$$

by restricting via the relocalization morphism  $j : \mathcal{C}/V \rightarrow \mathcal{C}/U$ , see Sites, Lemma 24.7. Hence this defines a presheaf  $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ . In addition, given an element  $\varphi \in \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$  and a section  $f \in \mathcal{O}(U)$  then we can define  $f\varphi \in \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$  by either precomposing with multiplication by  $f$  on  $\mathcal{F}|_U$  or postcomposing with multiplication by  $f$  on  $\mathcal{G}|_U$  (it gives the same result). Hence we in fact get a presheaf of  $\mathcal{O}$ -modules. There is a canonical “evaluation” morphism

$$\mathcal{F} \otimes_{p, \mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

**Lemma 27.1.** *If  $\mathcal{C}$  is a site,  $\mathcal{O}$  is a sheaf of rings,  $\mathcal{F}$  is a presheaf of  $\mathcal{O}$ -modules, and  $\mathcal{G}$  is a sheaf of  $\mathcal{O}$ -modules, then  $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$  is a sheaf of  $\mathcal{O}$ -modules.*

**Proof.** Omitted. Hints: Note first that  $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}^\#, \mathcal{G})$ , which reduces the question to the case where both  $\mathcal{F}$  and  $\mathcal{G}$  are sheaves. The result for sheaves of sets is Sites, Lemma 25.1.  $\square$

In the situation of the lemma the “evaluation” morphism factors through the tensor product of sheaves of modules

$$\mathcal{F} \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

**Lemma 27.2.** *Internal hom and (co)limits. Let  $\mathcal{C}$  be a category and let  $\mathcal{O}$  be a presheaf of rings.*

- (1) *For any presheaf of  $\mathcal{O}$ -modules  $\mathcal{F}$  the functor*

$$P\mathrm{Mod}(\mathcal{O}) \longrightarrow P\mathrm{Mod}(\mathcal{O}), \quad \mathcal{G} \mapsto \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

*commutes with arbitrary limits.*

- (2) *For any presheaf of  $\mathcal{O}$ -modules  $\mathcal{G}$  the functor*

$$P\mathrm{Mod}(\mathcal{O}) \longrightarrow P\mathrm{Mod}(\mathcal{O})^{\mathrm{opp}}, \quad \mathcal{F} \mapsto \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

*commutes with arbitrary colimits, in a formula*

$$\mathrm{Hom}_{\mathcal{O}}(\mathrm{colim}_i \mathcal{F}_i, \mathcal{G}) = \lim_i \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}_i, \mathcal{G}).$$

*Suppose that  $\mathcal{C}$  is a site, and  $\mathcal{O}$  is a sheaf of rings.*

- (3) *For any sheaf of  $\mathcal{O}$ -modules  $\mathcal{F}$  the functor*

$$\mathrm{Mod}(\mathcal{O}) \longrightarrow \mathrm{Mod}(\mathcal{O}), \quad \mathcal{G} \mapsto \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

*commutes with arbitrary limits.*

- (4) *For any sheaf of  $\mathcal{O}$ -modules  $\mathcal{G}$  the functor*

$$\mathrm{Mod}(\mathcal{O}) \longrightarrow \mathrm{Mod}(\mathcal{O})^{\mathrm{opp}}, \quad \mathcal{F} \mapsto \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

*commutes with arbitrary colimits, in a formula*

$$\mathrm{Hom}_{\mathcal{O}}(\mathrm{colim}_i \mathcal{F}_i, \mathcal{G}) = \lim_i \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}_i, \mathcal{G}).$$

**Proof.** Let  $\mathcal{I} \rightarrow PMod(\mathcal{O})$ ,  $i \mapsto \mathcal{G}_i$  be a diagram. Let  $U$  be an object of the category  $\mathcal{C}$ . As  $j_U^*$  is both a left and a right adjoint we see that  $\lim_i j_U^* \mathcal{G}_i = j_U^* \lim_i \mathcal{G}_i$ . Hence we have

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \lim_i \mathcal{G}_i)(U) &= \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U, \lim_i \mathcal{G}_i|_U) \\ &= \lim_i \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}_i|_U) \\ &= \lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_i)(U) \end{aligned}$$

by definition of a limit. This proves (1). Part (2) is proved in exactly the same way. Part (3) follows from (1) because the limit of a diagram of sheaves is the same as the limit in the category of presheaves. Finally, (4) follow because, in the formula we have

$$\text{Mor}_{Mod(\mathcal{O})}(\text{colim}_i \mathcal{F}_i, \mathcal{G}) = \text{Mor}_{PMod(\mathcal{O})}(\text{colim}_i^{PSh} \mathcal{F}_i, \mathcal{G})$$

as the colimit  $\text{colim}_i \mathcal{F}_i$  is the sheafification of the colimit  $\text{colim}_i^{PSh} \mathcal{F}_i$  in  $PMod(\mathcal{O})$ . Hence (4) follows from (2) (by the remark on limits above again).  $\square$

**Lemma 27.3.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings.*

- (1) *Let  $\mathcal{F}, \mathcal{G}, \mathcal{H}$  be presheaves of  $\mathcal{O}$ -modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes_{p, \mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

*which is functorial in all three entries (sheaf Hom in all three spots). In particular,*

$$\text{Mor}_{PMod(\mathcal{O})}(\mathcal{F} \otimes_{p, \mathcal{O}} \mathcal{G}, \mathcal{H}) = \text{Mor}_{PMod(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

- (2) *Suppose that  $\mathcal{C}$  is a site,  $\mathcal{O}$  is a sheaf of rings, and  $\mathcal{F}, \mathcal{G}, \mathcal{H}$  are sheaves of  $\mathcal{O}$ -modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

*which is functorial in all three entries (sheaf Hom in all three spots). In particular,*

$$\text{Mor}_{Mod(\mathcal{O})}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = \text{Mor}_{Mod(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

**Proof.** This is the analogue of Algebra, Lemma 11.8. The proof is the same, and is omitted.  $\square$

**Lemma 27.4.** *Tensor product and (co)limits. Let  $\mathcal{C}$  be a category and let  $\mathcal{O}$  be a presheaf of rings.*

- (1) *For any presheaf of  $\mathcal{O}$ -modules  $\mathcal{F}$  the functor*

$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{p, \mathcal{O}} \mathcal{G}$$

*commutes with arbitrary colimits.*

- (2) *Suppose that  $\mathcal{C}$  is a site, and  $\mathcal{O}$  is a sheaf of rings. For any sheaf of  $\mathcal{O}$ -modules  $\mathcal{F}$  the functor*

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$$

*commutes with arbitrary colimits.*

**Proof.** This is because tensor product is adjoint to internal hom according to Lemma 27.3. See Categories, Lemma 24.4.  $\square$

**Lemma 27.5.** *Let  $\mathcal{C}$  be a category, resp. a site. Let  $\mathcal{O} \rightarrow \mathcal{O}'$  be a map of presheaves, resp. sheaves of rings. Then*

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}'}(\mathcal{G}, \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}', \mathcal{F}))$$

for any  $\mathcal{O}'$ -module  $\mathcal{G}$  and  $\mathcal{O}$ -module  $\mathcal{F}$ .

**Proof.** This is the analogue of Algebra, Lemma 13.4. The proof is the same, and is omitted.  $\square$

## 28. Flat modules

We can define flat modules exactly as in the case of modules over rings.

**Definition 28.1.** Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings.

- (1) A presheaf  $\mathcal{F}$  of  $\mathcal{O}$ -modules is called *flat* if the functor

$$P\mathrm{Mod}(\mathcal{O}) \longrightarrow P\mathrm{Mod}(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{p, \mathcal{O}} \mathcal{F}$$

is exact.

- (2) A map  $\mathcal{O} \rightarrow \mathcal{O}'$  of presheaves of rings is called *flat* if  $\mathcal{O}'$  is flat as a presheaf of  $\mathcal{O}$ -modules.
- (3) If  $\mathcal{C}$  is a site,  $\mathcal{O}$  is a sheaf of rings and  $\mathcal{F}$  is a sheaf of  $\mathcal{O}$ -modules, then we say  $\mathcal{F}$  is *flat* if the functor

$$\mathrm{Mod}(\mathcal{O}) \longrightarrow \mathrm{Mod}(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$$

is exact.

- (4) A map  $\mathcal{O} \rightarrow \mathcal{O}'$  of sheaves of rings on a site is called *flat* if  $\mathcal{O}'$  is flat as a sheaf of  $\mathcal{O}$ -modules.

The notion of a flat module or flat ring map is intrinsic (Section 18).

**Lemma 28.2.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings. Let  $\mathcal{F}$  be a presheaf of  $\mathcal{O}$ -modules. If each  $\mathcal{F}(U)$  is a flat  $\mathcal{O}(U)$ -module, then  $\mathcal{F}$  is flat.*

**Proof.** This is immediate from the definitions.  $\square$

**Lemma 28.3.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings. Let  $\mathcal{F}$  be a presheaf of  $\mathcal{O}$ -modules. If  $\mathcal{F}$  is a flat  $\mathcal{O}$ -module, then  $\mathcal{F}^\#$  is a flat  $\mathcal{O}^\#$ -module.*

**Proof.** Omitted. (Hint: Sheafification is exact.)  $\square$

**Lemma 28.4.** *Colimits and tensor product.*

- (1) *A filtered colimit of flat presheaves of modules is flat. A direct sum of flat presheaves of modules is flat.*
- (2) *A filtered colimit of flat sheaves of modules is flat. A direct sum of flat sheaves of modules is flat.*

**Proof.** Part (1) follows from Lemma 27.4 and Algebra, Lemma 8.9 by looking at sections over objects. To see part (2), use Lemma 27.4 and the fact that a filtered colimit of exact complexes is an exact complex (this uses that sheafification is exact and commutes with colimits). Some details omitted.  $\square$

**Lemma 28.5.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings. Let  $U$  be an object of  $\mathcal{C}$ . Consider the functor  $j_U : \mathcal{C}/U \rightarrow \mathcal{C}$ .*

- (1) *The presheaf of  $\mathcal{O}$ -modules  $j_{U!}\mathcal{O}_U$  (see Remark 19.6) is flat.*
- (2) *If  $\mathcal{C}$  is a site,  $\mathcal{O}$  is a sheaf of rings,  $j_{U!}\mathcal{O}_U$  is a flat sheaf of  $\mathcal{O}$ -modules.*

**Proof.** Proof of (1). By the discussion in Remark 19.6 we see that

$$j_{U!}\mathcal{O}_U(V) = \bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{O}(V)$$

which is a flat  $\mathcal{O}(V)$ -module. Hence (1) follows from Lemma 28.2. Then (2) follows as  $j_{U!}\mathcal{O}_U = (j_{U!}\mathcal{O}_U)^\#$  (the first  $j_{U!}$  on sheaves, the second on presheaves) and Lemma 28.3.  $\square$

**Lemma 28.6.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings.*

- (1) *Any presheaf of  $\mathcal{O}$ -modules is a quotient of a direct sum  $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$ .*
- (2) *Any presheaf of  $\mathcal{O}$ -modules is a quotient of a flat presheaf of  $\mathcal{O}$ -modules.*
- (3) *If  $\mathcal{C}$  is a site,  $\mathcal{O}$  is a sheaf of rings, then any sheaf of  $\mathcal{O}$ -modules is a quotient of a direct sum  $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$ .*
- (4) *If  $\mathcal{C}$  is a site,  $\mathcal{O}$  is a sheaf of rings, then any sheaf of  $\mathcal{O}$ -modules is a quotient of a flat sheaf of  $\mathcal{O}$ -modules.*

**Proof.** Proof of (1). For every object  $U$  of  $\mathcal{C}$  and every  $s \in \mathcal{F}(U)$  we get a morphism  $j_{U!}\mathcal{O}_U \rightarrow \mathcal{F}$ , namely the adjoint to the morphism  $\mathcal{O}_U \rightarrow \mathcal{F}|_U$ ,  $1 \mapsto s$ . Clearly the map

$$\bigoplus_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective. The source is flat by combining Lemmas 28.4 and 28.5 which proves (2). The sheaf case follows from this either by sheafifying or repeating the same argument.  $\square$

**Lemma 28.7.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings. Let*

$$0 \rightarrow \mathcal{F}'' \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow 0$$

*be a short exact sequence of presheaves of  $\mathcal{O}$ -modules. Assume  $\mathcal{F}$  is flat. Then*

- (1) *For any presheaf  $\mathcal{G}$  of  $\mathcal{O}$ -modules, the sequence*

$$0 \rightarrow \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} \rightarrow \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} \rightarrow \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} \rightarrow 0$$

*is exact.*

- (2) *If  $\mathcal{C}$  is a site, and  $\mathcal{O}$ ,  $\mathcal{F}$ ,  $\mathcal{F}'$ ,  $\mathcal{F}''$ , and  $\mathcal{G}$  are all sheaves, the sequence*

$$0 \rightarrow \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \rightarrow \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \rightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \rightarrow 0$$

*is exact.*

**Proof.** Choose a flat presheaf of  $\mathcal{O}$ -modules  $\mathcal{G}'$  which surjects onto  $\mathcal{G}$ . This is possible by Lemma 28.6. Let  $\mathcal{G}'' = \text{Ker}(\mathcal{G}' \rightarrow \mathcal{G})$ . The lemma follows by applying the snake lemma to the following diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} & \rightarrow & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} & \rightarrow & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 \rightarrow & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}' & \rightarrow & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}' & \rightarrow & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}' & \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \rightarrow & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \rightarrow & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}'' \rightarrow 0 \\
& & & & & & \uparrow \\
& & & & & & 0
\end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the flat module  $\mathcal{G}'$  is exact. The sheaf case follows from the presheaf case as sheafification is exact.  $\square$

**Lemma 28.8.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings. Let*

$$0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow 0$$

*be a short exact sequence of presheaves of  $\mathcal{O}$ -modules.*

- (1) *If  $\mathcal{F}_2$  and  $\mathcal{F}_0$  are flat so is  $\mathcal{F}_1$ .*
- (2) *If  $\mathcal{F}_1$  and  $\mathcal{F}_0$  are flat so is  $\mathcal{F}_2$ .*

*If  $\mathcal{C}$  is a site and  $\mathcal{O}$  is a sheaf of rings then the same result holds  $\text{Mod}(\mathcal{O})$ .*

**Proof.** Let  $\mathcal{G}^\bullet$  be an arbitrary exact complex of presheaves of  $\mathcal{O}$ -modules. Assume that  $\mathcal{F}_0$  is flat. By Lemma 28.7 we see that

$$0 \rightarrow \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_2 \rightarrow \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_1 \rightarrow \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_0 \rightarrow 0$$

is a short exact sequence of complexes of presheaves of  $\mathcal{O}$ -modules. Hence (1) and (2) follow from the snake lemma. The case of sheaves of modules is proved in the same way.  $\square$

**Lemma 28.9.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{O}$  be a presheaf of rings. Let*

$$\dots \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow \mathcal{Q} \rightarrow 0$$

*be an exact complex of presheaves of  $\mathcal{O}$ -modules. If  $\mathcal{Q}$  and all  $\mathcal{F}_i$  are flat  $\mathcal{O}$ -modules, then for any presheaf  $\mathcal{G}$  of  $\mathcal{O}$ -modules the complex*

$$\dots \rightarrow \mathcal{F}_2 \otimes_{p,\mathcal{O}} \mathcal{G} \rightarrow \mathcal{F}_1 \otimes_{p,\mathcal{O}} \mathcal{G} \rightarrow \mathcal{F}_0 \otimes_{p,\mathcal{O}} \mathcal{G} \rightarrow \mathcal{Q} \otimes_{p,\mathcal{O}} \mathcal{G} \rightarrow 0$$

*is exact also. If  $\mathcal{C}$  is a site and  $\mathcal{O}$  is a sheaf of rings then the same result holds  $\text{Mod}(\mathcal{O})$ .*

**Proof.** Follows from Lemma 28.7 by splitting the complex into short exact sequences and using Lemma 28.8 to prove inductively that  $\text{Im}(\mathcal{F}_{i+1} \rightarrow \mathcal{F}_i)$  is flat.  $\square$

**Lemma 28.10.** *Let  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a map of sheaves of rings on a site  $\mathcal{C}$ . If  $\mathcal{G}$  is a flat  $\mathcal{O}_1$ -module, then  $\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2$  is a flat  $\mathcal{O}_2$ -module.*

**Proof.** This is true because

$$(\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2) \otimes_{\mathcal{O}_2} \mathcal{H} = \mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{F}$$

(as sheaves of abelian groups for example).  $\square$

The following lemma gives one direction of the equational criterion of flatness (Algebra, Lemma 38.10).

**Lemma 28.11.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}$  be a flat  $\mathcal{O}$ -module. Let  $U$  be an object of  $\mathcal{C}$  and let*

$$\mathcal{O}_U \xrightarrow{(f_1, \dots, f_n)} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \dots, s_n)} \mathcal{F}|_U$$

*be a complex of  $\mathcal{O}_U$ -modules. There exists a covering  $\{U_i \rightarrow U\}$  and for each  $i$  a factorization*

$$\mathcal{O}_{U_i}^{\oplus n} \xrightarrow{A} \mathcal{O}_{U_i}^{\oplus m} \xrightarrow{(t_1, \dots, t_m)} \mathcal{F}|_{U_i}$$

*of  $(s_1, \dots, s_n)|_{U_i}$  such that  $A \circ (f_1, \dots, f_n)|_{U_i} = 0$ .*

**Proof.** Let  $\mathcal{I} \subset \mathcal{O}_U$  be the sheaf of ideals generated by  $f_1, \dots, f_n$ . Then  $\sum f_j \otimes s_j$  is a section of  $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$  which maps to zero in  $\mathcal{F}|_U$ . As  $\mathcal{F}|_U$  is flat the map  $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U \rightarrow \mathcal{F}|_U$  is injective. Since  $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$  is the sheaf associated to the presheaf tensor product, we see there exists a covering  $\{U_i \rightarrow U\}$  such that  $\sum f_j|_{U_i} \otimes s_j|_{U_i}$  is zero in  $\mathcal{I}(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{F}(U_i)$ . Unwinding the definitions using Algebra, Lemma 103.10 we find  $t_1, \dots, t_m \in \mathcal{F}(U_i)$  and  $a_{jk} \in \mathcal{O}(U_i)$  such that  $\sum a_{jk} f_j|_{U_i} = 0$  and  $s_j|_{U_i} = \sum a_{jk} t_k$ .  $\square$

**Lemma 28.12.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}$  be locally of finite presentation and flat. Then given an object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}$  such that  $\mathcal{F}|_{U_i}$  is a direct summand of a finite free  $\mathcal{O}_{U_i}$ -module.*

**Proof.** Choose an object  $U$  of  $\mathcal{C}$ . After replacing  $U$  by the members of a covering, we may assume there exists a presentation

$$\mathcal{O}_U^{\oplus r} \rightarrow \mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F} \rightarrow 0$$

By Lemma 28.11 we may assume, after replacing  $U$  by the members of a covering, assume there exists a factorization

$$\mathcal{O}_U^{\oplus n} \rightarrow \mathcal{O}_U^{\oplus n_1} \rightarrow \mathcal{F}$$

such that the composition  $\mathcal{O}_U^{\oplus r} \rightarrow \mathcal{O}_U^{\oplus n} \rightarrow \mathcal{O}_U^{\oplus n_1}$  annihilates the first summand of  $\mathcal{O}_U^{\oplus r}$ . Repeating this argument  $r - 1$  more times we obtain a factorization

$$\mathcal{O}_U^{\oplus n} \rightarrow \mathcal{O}_U^{\oplus n_r} \rightarrow \mathcal{F}$$

such that the composition  $\mathcal{O}_U^{\oplus r} \rightarrow \mathcal{O}_U^{\oplus n} \rightarrow \mathcal{O}_U^{\oplus n_r}$  is zero. This means that the surjection  $\mathcal{O}_U^{\oplus n_r} \rightarrow \mathcal{F}$  has a section and we win.  $\square$

**Lemma 28.13.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}' \rightarrow \mathcal{O}$  be a surjection of sheaves of rings whose kernel  $\mathcal{I}$  is an ideal of square zero. Let  $\mathcal{F}'$  be an  $\mathcal{O}'$ -module and set  $\mathcal{F} = \mathcal{F}'/\mathcal{I}\mathcal{F}'$ . The following are equivalent*

- (1)  $\mathcal{F}'$  is a flat  $\mathcal{O}'$ -module, and
- (2)  $\mathcal{F}$  is a flat  $\mathcal{O}$ -module and  $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \rightarrow \mathcal{F}'$  is injective.

**Proof.** If (1) holds, then  $\mathcal{F} = \mathcal{F}' \otimes_{\mathcal{O}'} \mathcal{O}$  is flat over  $\mathcal{O}$  by Lemma 28.10 and we see the map  $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \rightarrow \mathcal{F}'$  is injective by applying  $-\otimes_{\mathcal{O}'} \mathcal{F}'$  to the exact sequence  $0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}' \rightarrow \mathcal{O} \rightarrow 0$ , see Lemma 28.7. Assume (2). In the rest of the proof we will use without further mention that  $\mathcal{K} \otimes_{\mathcal{O}'} \mathcal{F}' = \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}$  for any  $\mathcal{O}'$ -module  $\mathcal{K}$  annihilated by  $\mathcal{I}$ . Let  $\alpha : \mathcal{G}' \rightarrow \mathcal{H}'$  be an injective map of  $\mathcal{O}'$ -modules. Let  $\mathcal{G} \subset \mathcal{G}'$ , resp.  $\mathcal{H} \subset \mathcal{H}'$  be the subsheaf of sections annihilated by  $\mathcal{I}$ . Consider the diagram

$$\begin{array}{ccccccc} \mathcal{G} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{G}' \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{G}'/\mathcal{G} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{H} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{H}' \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{H}'/\mathcal{H} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & 0 \end{array}$$

Note that  $\mathcal{G}'/\mathcal{G}$  and  $\mathcal{H}'/\mathcal{H}$  are annihilated by  $\mathcal{I}$  and that  $\mathcal{G}'/\mathcal{G} \rightarrow \mathcal{H}'/\mathcal{H}$  is injective. Thus the right vertical arrow is injective as  $\mathcal{F}$  is flat over  $\mathcal{O}$ . The same is true for the left vertical arrow. Hence the middle vertical arrow is injective and  $\mathcal{F}'$  is flat.  $\square$

## 29. Towards constructible modules

Recall that a quasi-compact object of a site is one such that every covering of it can be refined by a finite covering. It turns out that if every object of a site has a covering by quasi-compact objects, then the modules  $j_! \mathcal{O}_U$  with  $U$  quasi-compact form a particularly nice set of generators for the category of all modules.

**Lemma 29.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\{U_i \rightarrow U\}$  be a covering of  $\mathcal{C}$ . Then the sequence*

$$\bigoplus j_{U_i \times_U U_j!} \mathcal{O}_{U_i \times_U U_j} \rightarrow \bigoplus j_{U_i!} \mathcal{O}_{U_i} \rightarrow j_! \mathcal{O}_U \rightarrow 0$$

*is exact.*

**Proof.** This holds because for any  $\mathcal{O}$ -module  $\mathcal{F}$  the functor  $\mathrm{Hom}_{\mathcal{O}}(-, \mathcal{F})$  turns our sequence into the exact sequence  $0 \rightarrow \mathcal{F}(U) \rightarrow \prod \mathcal{F}(U_i) \rightarrow \prod \mathcal{F}(U_i \times_U U_j)$ . Then the lemma follows from Homology, Lemma 5.8.  $\square$

**Lemma 29.2.** *Let  $\mathcal{C}$  be a site. Let  $W$  be a quasi-compact object of  $\mathcal{C}$ .*

- (1) *The functor  $\mathrm{Sh}(\mathcal{C}) \rightarrow \mathrm{Sets}, \mathcal{F} \mapsto \mathcal{F}(W)$  commutes with coproducts.*
- (2) *Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ . The functor  $\mathrm{Mod}(\mathcal{O}) \rightarrow \mathrm{Ab}, \mathcal{F} \mapsto \mathcal{F}(W)$  commutes with direct sums.*

**Proof.** Proof of (1). Taking sections over  $W$  commutes with filtered colimits with injective transition maps by Sites, Lemma 11.2. If  $\mathcal{F}_i$  is a family of sheaves of sets indexed by a set  $I$ . Then  $\coprod \mathcal{F}_i$  is the filtered colimit over the partially ordered set of finite subsets  $E \subset I$  of the coproducts  $\mathcal{F}_E = \coprod_{i \in E} \mathcal{F}_i$ . Since the transition maps are injective we conclude.

Proof of (2). Let  $\mathcal{F}_i$  be a family of sheaves of  $\mathcal{O}$ -modules indexed by a set  $I$ . Then  $\bigoplus \mathcal{F}_i$  is the filtered colimit over the partially ordered set of finite subsets  $E \subset I$  of the direct sums  $\mathcal{F}_E = \bigoplus_{i \in E} \mathcal{F}_i$ . A filtered colimit of abelian sheaves can be computed in the category of sheaves of sets. Moreover, for  $E \subset E'$  the transition map  $\mathcal{F}_E \rightarrow \mathcal{F}_{E'}$  is injective (as sheafification is exact and the injectivity is clear on underlying presheaves). Hence it suffices to show the result for a finite index set by Sites, Lemma 11.2. The finite case is dealt with in Lemma 3.2 (it holds over any object of  $\mathcal{C}$ ).  $\square$

**Lemma 29.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be a quasi-compact object of  $\mathcal{C}$ . Then the functor  $\mathrm{Hom}_{\mathcal{O}}(j_! \mathcal{O}_U, -)$  commutes with direct sums.*

**Proof.** This is true because  $\mathrm{Hom}_{\mathcal{O}}(j_! \mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$  and because the functor  $\mathcal{F} \mapsto \mathcal{F}(U)$  commutes with direct sums by Lemma 29.2.  $\square$

In order to state the sharpest possible results in the following we introduce some notation.

**Situation 29.4.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$  be a set of objects. We consider the following conditions

- (1) Every object of  $\mathcal{C}$  has a covering by elements of  $\mathcal{B}$ .
- (2) Every  $U \in \mathcal{B}$  is quasi-compact.
- (3) For a finite covering  $\{U_i \rightarrow U\}$  with  $U_i, U \in \mathcal{B}$  the fibre products  $U_i \times_U U_j$  are quasi-compact.

**Lemma 29.5.** *In Situation 29.4 assume (1) holds.*

- (1) Every sheaf of sets is the target of a surjective map whose source is a co-product  $\coprod h_{U_i}^\#$  with  $U_i$  in  $\mathcal{B}$ .
- (2) If  $\mathcal{O}$  is a sheaf of rings, then every  $\mathcal{O}$ -module is a quotient of a direct sum  $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$  with  $U_i$  in  $\mathcal{B}$ .

**Proof.** Follows immediately from Lemmas 28.6 and 29.1.  $\square$

**Lemma 29.6.** *In Situation 29.4 assume (1) and (2) hold.*

- (1) Every sheaf of sets is a filtered colimit of sheaves of the form

$$(29.6.1) \quad \text{Coequalizer} \left( \coprod_{j=1, \dots, m} h_{V_j}^\# \rightrightarrows \coprod_{i=1, \dots, n} h_{U_i}^\# \right)$$

with  $U_i$  and  $V_j$  in  $\mathcal{B}$ .

- (2) If  $\mathcal{O}$  is a sheaf of rings, then every  $\mathcal{O}$ -module is a filtered colimit of sheaves of the form

$$(29.6.2) \quad \text{Coker} \left( \bigoplus_{j=1, \dots, m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus_{i=1, \dots, n} j_{U_i!}\mathcal{O}_{U_i} \right)$$

with  $U_i$  and  $V_j$  in  $\mathcal{B}$ .

**Proof.** Proof of (1). By Lemma 29.5 every sheaf of sets  $\mathcal{F}$  is the target of a surjection whose source is a coprod  $\mathcal{F}_0$  of sheaves the form  $h_U^\#$  with  $U \in \mathcal{B}$ . Applying this to  $\mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$  we find that  $\mathcal{F}$  is a coequalizer of a pair of maps

$$\coprod_{j \in J} h_{V_j}^\# \rightrightarrows \coprod_{i \in I} h_{U_i}^\#$$

for some index sets  $I, J$  and  $V_j$  and  $U_i$  in  $\mathcal{B}$ . For every finite subset  $J' \subset J$  there is a finite subset  $I' \subset I$  such that the coproduct over  $j \in J'$  maps into the coprod over  $i \in I'$  via both maps, see Lemma 29.3. Thus our sheaf is the colimit of the cokernels of these maps between finite coproducts.

Proof of (2). By Lemma 29.5 every module is a quotient of a direct sum of modules of the form  $j_U!\mathcal{O}_U$  with  $U \in \mathcal{B}$ . Thus every module is a cokernel

$$\text{Coker} \left( \bigoplus_{j \in J} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus_{i \in I} j_{U_i!}\mathcal{O}_{U_i} \right)$$

for some index sets  $I, J$  and  $V_j$  and  $U_i$  in  $\mathcal{B}$ . For every finite subset  $J' \subset J$  there is a finite subset  $I' \subset I$  such that the direct sum over  $j \in J'$  maps into the direct sum over  $i \in I'$ , see Lemma 29.3. Thus our module is the colimit of the cokernels of these maps between finite direct sums.  $\square$

**Lemma 29.7.** *In Situation 29.4 assume (1) and (2) hold. Let  $\mathcal{O}$  be a sheaf of rings. Then a cokernel of a map between modules as in (29.6.2) is another module as in (29.6.2).*

**Proof.** Let  $\mathcal{F} = \text{Coker}(\bigoplus j_{V_j!}\mathcal{O}_{V_j} \rightarrow \bigoplus j_{U_i!}\mathcal{O}_{U_i})$  as in (29.6.2). It suffices to show that the cokernel of a map  $\varphi : j_W!\mathcal{O}_W \rightarrow \mathcal{F}$  with  $W \in \mathcal{B}$  is another module of the same type. The map  $\varphi$  corresponds to  $s \in \mathcal{F}(W)$ . Since  $W$  is quasi-compact we can find a finite covering  $\{W_k \rightarrow W\}$  with  $W_k \in \mathcal{B}$  such that  $s|_{W_k}$  comes from a section  $\sum s_{ki}$  of  $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$ . This determines maps  $j_{W_k!}\mathcal{O}_{W_k} \rightarrow \bigoplus j_{U_i!}\mathcal{O}_{U_i}$ . Since  $\bigoplus j_{W_k!}\mathcal{O}_{W_k} \rightarrow j_W!\mathcal{O}_W$  is surjective (Lemma 29.1) we see that  $\text{Coker}(\varphi)$  is equal to

$$\text{Coker} \left( \bigoplus j_{W_k!}\mathcal{O}_{W_k} \oplus \bigoplus j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus j_{U_i!}\mathcal{O}_{U_i} \right)$$

as desired.  $\square$



**Lemma 29.8.** *In Situation 29.4 assume (1), (2), and (3) hold. Let  $\mathcal{O}$  be a sheaf of rings. Then given a map*

$$\bigoplus_{j=1,\dots,m} j_{V_j!} \mathcal{O}_{V_j} \longrightarrow \bigoplus_{i=1,\dots,n} j_{U_i!} \mathcal{O}_{U_i}$$

*with  $U_i$  and  $V_j$  in  $\mathcal{B}$ , and finite coverings  $\{U_{ik} \rightarrow U_i\}$  by  $U_{ik} \in \mathcal{B}$ , there exist a finite set of  $W_l \in \mathcal{B}$  and a commutative diagram*

$$\begin{array}{ccc} \bigoplus j_{W_l!} \mathcal{O}_{W_l} & \longrightarrow & \bigoplus j_{U_i!} \mathcal{O}_{U_{ik}} \\ \downarrow & & \downarrow \\ \bigoplus j_{V_j!} \mathcal{O}_{V_j} & \longrightarrow & \bigoplus j_{U_i!} \mathcal{O}_{U_i} \end{array}$$

*inducing an isomorphism on cokernels of the horizontal maps.*

**Proof.** Since  $\bigoplus j_{U_{ik}!} \mathcal{O}_{U_{ik}} \rightarrow \bigoplus j_{U_i!} \mathcal{O}_{U_i}$  is surjective (Lemma 29.1), we can find finite coverings  $\{V_{jm} \rightarrow V_j\}$  with  $V_{jm} \in \mathcal{B}$  such that we can find a commutative diagram

$$\begin{array}{ccc} \bigoplus j_{V_{jm}!} \mathcal{O}_{V_{jm}} & \longrightarrow & \bigoplus j_{U_i!} \mathcal{O}_{U_{ik}} \\ \downarrow & & \downarrow \\ \bigoplus j_{V_j!} \mathcal{O}_{V_j} & \longrightarrow & \bigoplus j_{U_i!} \mathcal{O}_{U_i} \end{array}$$

Adding

$$\bigoplus j_{U_{ik} \times_{U_i} U_{ik'}!} \mathcal{O}_{U_{ik} \times_{U_i} U_{ik'}}$$

to the upper left corner finishes the proof by Lemma 29.1.  $\square$

**Lemma 29.9.** *In Situation 29.4 assume (1), (2), and (3) hold. Let  $\mathcal{O}$  be a sheaf of rings. Then an extension of modules as in (29.6.2) is another module as in (29.6.2).*

**Proof.** Let  $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$  be a short exact sequence of  $\mathcal{O}$ -modules with  $\mathcal{F}_1$  and  $\mathcal{F}_3$  as in (29.6.2). Choose presentations

$$\bigoplus A_{V_j} \rightarrow \bigoplus A_{U_i} \rightarrow \mathcal{F}_1 \rightarrow 0 \quad \text{and} \quad \bigoplus A_{T_j} \rightarrow \bigoplus A_{W_i} \rightarrow \mathcal{F}_3 \rightarrow 0$$

In this proof the direct sums are always finite, and we write  $A_U = j_{U!} \mathcal{O}_U$  for  $U \in \mathcal{B}$ . By Lemma 29.8 we may replace  $W_i$  by finite coverings  $\{W_{ik} \rightarrow W_i\}$  with  $W_{ik} \in \mathcal{B}$ . Thus we may assume the map  $\bigoplus A_{W_i} \rightarrow \mathcal{F}_3$  lifts to a map into  $\mathcal{F}_2$ . Consider the kernel

$$\mathcal{K}_2 = \text{Ker}(\bigoplus A_{U_i} \oplus \bigoplus A_{W_i} \rightarrow \mathcal{F}_2)$$

By the snake lemma this kernel surjections onto  $\mathcal{K}_3 = \text{Ker}(\bigoplus A_{W_i} \rightarrow \mathcal{F}_3)$ . Thus after replacing each  $T_j$  by a finite covering with elements of  $\mathcal{B}$  (permissible by Lemma 29.1) we may assume there is a map  $\bigoplus A_{T_j} \rightarrow \mathcal{K}_2$  lifting the given map  $\bigoplus A_{T_j} \rightarrow \mathcal{K}_3$ . Then  $\bigoplus A_{V_j} \oplus \bigoplus A_{T_j} \rightarrow \mathcal{K}_2$  is surjective which finishes the proof.  $\square$

**Lemma 29.10.** *In Situation 29.4 assume (1), (2), and (3) hold. Let  $\mathcal{O}$  be a sheaf of rings. Let  $\mathcal{A} \subset \text{Mod}(\mathcal{O})$  be the full subcategory of modules isomorphic to a cokernel as in (29.6.2). If the kernel of every map of  $\mathcal{O}$ -modules of the form*

$$\bigoplus_{j=1,\dots,m} j_{V_j!} \mathcal{O}_{V_j} \longrightarrow \bigoplus_{i=1,\dots,n} j_{U_i!} \mathcal{O}_{U_i}$$

*with  $U_i$  and  $V_j$  in  $\mathcal{B}$ , is in  $\mathcal{A}$ , then  $\mathcal{A}$  is weak Serre subcategory of  $\text{Mod}(\mathcal{O})$ .*

**Proof.** We will use the criterion of Homology, Lemma 9.3. By the results of Lemmas 29.7 and 29.9 it suffices to see that the kernel of a map  $\mathcal{F} \rightarrow \mathcal{G}$  between objects of  $\mathcal{A}$  is in  $\mathcal{A}$ . To prove this choose presentations

$$\bigoplus A_{V_j} \rightarrow \bigoplus A_{U_i} \rightarrow \mathcal{F} \rightarrow 0 \quad \text{and} \quad \bigoplus A_{T_j} \rightarrow \bigoplus A_{W_i} \rightarrow \mathcal{G} \rightarrow 0$$

In this proof the direct sums are always finite, and we write  $A_U = j_{U!}\mathcal{O}_U$  for  $U \in \mathcal{B}$ . Using Lemmas 29.1 and 29.8 and arguing as in the proof of Lemma 29.9 we may assume that the map  $\mathcal{F} \rightarrow \mathcal{G}$  lifts to a map of presentations

$$\begin{array}{ccccccc} \bigoplus A_{V_j} & \longrightarrow & \bigoplus A_{U_i} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus A_{T_j} & \longrightarrow & \bigoplus A_{W_i} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

Then we see that

$$\text{Ker}(\mathcal{F} \rightarrow \mathcal{G}) = \text{Coker}\left(\bigoplus A_{V_j} \rightarrow \text{Ker}\left(\bigoplus A_{T_j} \oplus \bigoplus A_{U_i} \rightarrow \bigoplus A_{W_i}\right)\right)$$

and the lemma follows from the assumption and Lemma 29.7.  $\square$

### 30. Flat morphisms

**Definition 30.1.** Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  be a morphism of ringed topoi. We say  $(f, f^\#)$  is *flat* if the ring map  $f^\# : f^{-1}\mathcal{O}' \rightarrow \mathcal{O}$  is flat. We say a morphism of ringed sites is *flat* if the associated morphism of ringed topoi is flat.

**Lemma 30.2.** Let  $f : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{C}')$  be a morphism of ringed topoi. Then

$$f^{-1} : Ab(\mathcal{C}') \rightarrow Ab(\mathcal{C}), \quad \mathcal{F} \mapsto f^{-1}\mathcal{F}$$

is exact. If  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  is a flat morphism of ringed topoi then

$$f^* : Mod(\mathcal{O}') \rightarrow Mod(\mathcal{O}), \quad \mathcal{F} \mapsto f^*\mathcal{F}$$

is exact.

**Proof.** Given an abelian sheaf  $\mathcal{G}$  on  $\mathcal{C}'$  the underlying sheaf of sets of  $f^{-1}\mathcal{G}$  is the same as  $f^{-1}$  of the underlying sheaf of sets of  $\mathcal{G}$ , see Sites, Section 43. Hence the exactness of  $f^{-1}$  for sheaves of sets (required in the definition of a morphism of topoi, see Sites, Definition 16.1) implies the exactness of  $f^{-1}$  as a functor on abelian sheaves.

To see the statement on modules recall that  $f^*\mathcal{F}$  is defined as the tensor product  $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}', f^\#} \mathcal{O}$ . Hence  $f^*$  is a composition of functors both of which are exact.  $\square$

**Definition 30.3.** Let  $f : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}')$  be a morphism of ringed topoi. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules. We say that  $\mathcal{F}$  is *flat over*  $(Sh(\mathcal{D}), \mathcal{O}')$  if  $\mathcal{F}$  is flat as an  $f^{-1}\mathcal{O}'$ -module.

This is compatible with the notion as defined for morphisms of ringed spaces, see Modules, Definition 17.3 and the discussion following.

### 31. Invertible modules

Here is the definition.

**Definition 31.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site.

- (1) A finite locally free  $\mathcal{O}$ -module  $\mathcal{F}$  is said to have *rank*  $r$  if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}$  of  $U$  such that  $\mathcal{F}|_{U_i}$  is isomorphic to  $\mathcal{O}_{U_i}^{\oplus r}$  as an  $\mathcal{O}_{U_i}$ -module.
- (2) An *invertible  $\mathcal{O}$ -module* is a finite locally free  $\mathcal{O}$ -module of rank 1.
- (3) The sheaf  $\mathcal{O}^*$  is the subsheaf of  $\mathcal{O}$  defined by the rule

$$U \mapsto \mathcal{O}^*(U) = \{f \in \mathcal{O}(U) \mid \exists g \in \mathcal{O}(U) \text{ such that } fg = 1\}$$

It is a sheaf of abelian groups with multiplication as the group law.

**Lemma 31.2.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed space.

- (1) If  $\mathcal{L}, \mathcal{N}$  are invertible  $\mathcal{O}$ -modules, then so is  $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{N}$ .
- (2) If  $\mathcal{L}$  is an invertible  $\mathcal{O}$ -modules, then so is  $\mathcal{L}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$ .
- (3) If  $\mathcal{L}$  is an invertible  $\mathcal{O}$ -module, then the evaluation map  $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{L}^{\otimes -1} \rightarrow \mathcal{O}$  is an isomorphism.

**Proof.** Omitted. □

**Lemma 31.3.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed space. There exists a set of invertible modules  $\{\mathcal{L}_i\}_{i \in I}$  such that each invertible module on  $(\mathcal{C}, \mathcal{O})$  is isomorphic to exactly one of the  $\mathcal{L}_i$ .

**Proof.** Omitted, but see Sheaves of Modules, Lemma 21.5. □

This lemma says roughly speaking that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 31.2 says that tensor product defines the structure of an abelian group on this set.

**Definition 31.4.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. The *Picard group*  $\text{Pic}(\mathcal{O})$  the ringed site is the abelian group whose elements are isomorphism classes of invertible  $\mathcal{O}$ -modules, with addition corresponding to tensor product.

### 32. Modules of differentials

In this section we briefly explain how to define the module of relative differentials for a morphism of ringed topoi. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 127).

**Definition 32.1.** Let  $\mathcal{C}$  be a site. Let  $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a homomorphism of sheaves of rings. Let  $\mathcal{F}$  be an  $\mathcal{O}_2$ -module. A  $\mathcal{O}_1$ -*derivation* or more precisely a  $\varphi$ -*derivation* into  $\mathcal{F}$  is a map  $D : \mathcal{O}_2 \rightarrow \mathcal{F}$  which is additive, annihilates the image of  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$ , and satisfies the *Leibniz rule*

$$D(ab) = aD(b) + D(a)b$$

for all  $a, b$  local sections of  $\mathcal{O}_2$  (wherever they are both defined). We denote  $\text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$  the set of  $\varphi$ -derivations into  $\mathcal{F}$ .

This is the sheaf theoretic analogue of Algebra, Definition 32.1. Given a derivation  $D : \mathcal{O}_2 \rightarrow \mathcal{F}$  as in the definition the map on global sections

$$D : \Gamma(\mathcal{O}_2) \longrightarrow \Gamma(\mathcal{F})$$

clearly is a  $\Gamma(\mathcal{O}_1)$ -derivation as in the algebra definition. Note that if  $\alpha : \mathcal{F} \rightarrow \mathcal{G}$  is a map of  $\mathcal{O}_2$ -modules, then there is an induced map

$$\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule  $D \mapsto \alpha \circ D$ . In other words we obtain a functor.

**Lemma 32.2.** *Let  $\mathcal{C}$  be a site. Let  $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a homomorphism of sheaves of rings. The functor*

$$\mathrm{Mod}(\mathcal{O}_2) \longrightarrow \mathrm{Ab}, \quad \mathcal{F} \longmapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

*is representable.*

**Proof.** This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets  $\mathcal{F}$  on  $\mathcal{C}$ , let us denote  $\mathcal{O}_2[\mathcal{F}]$  the sheafification of the presheaf  $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$  where this denotes the free  $\mathcal{O}_1(U)$ -module on the set  $\mathcal{F}(U)$ . For  $s \in \mathcal{F}(U)$  we denote  $[s]$  the corresponding section of  $\mathcal{O}_2[\mathcal{F}]$  over  $U$ . If  $\mathcal{F}$  is a sheaf of  $\mathcal{O}_2$ -modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which on the presheaf level is given by the rule  $\sum f_s[s] \mapsto \sum f_s s$ . We will employ the short hand  $[s] \mapsto s$  to describe this map and similarly for other maps below. Consider the map of  $\mathcal{O}_2$ -modules

$$(32.2.1) \quad \begin{array}{ccc} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a, b)] \oplus [(f, g)] \oplus [h] & \longmapsto & [a + b] - [a] - [b] + \\ & & [fg] - g[f] - f[g] + \\ & & [\varphi(h)] \end{array}$$

with short hand notation as above. Set  $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$  equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$d : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section  $f$  to the image of  $[f]$  in  $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ . By construction  $d$  is a  $\mathcal{O}_1$ -derivation. Next, let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_2$ -modules and let  $D : \mathcal{O}_2 \rightarrow \mathcal{F}$  be a  $\mathcal{O}_1$ -derivation. Then we can consider the  $\mathcal{O}_2$ -linear map  $\mathcal{O}_2[\mathcal{O}_2] \rightarrow \mathcal{F}$  which sends  $[g]$  to  $D(g)$ . It follows from the definition of a derivation that this map annihilates sections in the image of the map (32.2.1) and hence defines a map

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

Since it is clear that  $D = \alpha_D \circ d$  the lemma is proved.  $\square$

**Definition 32.3.** Let  $\mathcal{C}$  be a site. Let  $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a homomorphism of sheaves of rings. The *module of differentials* of the ring map  $\varphi$  is the object representing the functor  $\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$  which exists by Lemma 32.2. It is denoted  $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ , and the *universal  $\varphi$ -derivation* is denoted  $d : \mathcal{O}_2 \rightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$ .

Since this module and the derivation form the universal object representing a functor, this notion is clearly intrinsic (i.e., does not depend on the choice of the site underlying the ringed topos, see Section 18). Note that  $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$  is the cokernel of the map (32.2.1) of  $\mathcal{O}_2$ -modules. Moreover the map  $d$  is described by the rule that  $df$  is the image of the local section  $[f]$ .

**Lemma 32.4.** *Let  $\mathcal{C}$  be a site. Let  $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a homomorphism of presheaves of rings. Then  $\Omega_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}$  is the sheaf associated to the presheaf  $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ .*

**Proof.** Consider the map (32.2.1). There is a similar map of presheaves whose value on  $U \in \text{Ob}(\mathcal{C})$  is

$$\mathcal{O}_2(U)[\mathcal{O}_2(U) \times \mathcal{O}_2(U)] \oplus \mathcal{O}_2(U)[\mathcal{O}_2(U) \times \mathcal{O}_2(U)] \oplus \mathcal{O}_2(U)[\mathcal{O}_1(U)] \longrightarrow \mathcal{O}_2(U)[\mathcal{O}_2(U)]$$

The cokernel of this map has value  $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$  over  $U$  by the construction of the module of differentials in Algebra, Definition 127.2. On the other hand, the sheaves in (32.2.1) are the sheafifications of the presheaves above. Thus the result follows as sheafification is exact.  $\square$

**Lemma 32.5.** *Let  $f : \text{Sh}(\mathcal{D}) \rightarrow \text{Sh}(\mathcal{C})$  be a morphism of topoi. Let  $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a homomorphism of sheaves of rings on  $\mathcal{C}$ . Then there is a canonical identification  $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$  compatible with universal derivations.*

**Proof.** This holds because the sheaf  $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$  is the cokernel of the map (32.2.1) and a similar statement holds for  $\Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ , because the functor  $f^{-1}$  is exact, and because  $f^{-1}(\mathcal{O}_2[\mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2]$ ,  $f^{-1}(\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2 \times f^{-1}\mathcal{O}_2]$ , and  $f^{-1}(\mathcal{O}_2[\mathcal{O}_1]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_1]$ .  $\square$

**Lemma 32.6.** *Let  $\mathcal{C}$  be a site. Let  $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a homomorphism of sheaves of rings. For any object  $U$  of  $\mathcal{C}$  there is a canonical isomorphism*

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

*compatible with universal derivations.*

**Proof.** This is a special case of Lemma 32.5.  $\square$

**Lemma 32.7.** *Let  $\mathcal{C}$  be a site. Let*

$$\begin{array}{ccc} \mathcal{O}_2 & \xrightarrow{\varphi} & \mathcal{O}'_2 \\ \uparrow & & \uparrow \\ \mathcal{O}_1 & \longrightarrow & \mathcal{O}'_1 \end{array}$$

*be a commutative diagram of sheaves of rings on  $\mathcal{C}$ . The map  $\mathcal{O}_2 \rightarrow \mathcal{O}'_2$  composed with the map  $d : \mathcal{O}'_2 \rightarrow \Omega_{\mathcal{O}'_2/\mathcal{O}'_1}$  is a  $\mathcal{O}_1$ -derivation. Hence we obtain a canonical map of  $\mathcal{O}_2$ -modules  $\Omega_{\mathcal{O}_2/\mathcal{O}_1} \rightarrow \Omega_{\mathcal{O}'_2/\mathcal{O}'_1}$ . It is uniquely characterized by the property that  $d(f)$  maps to  $d(\varphi(f))$  for any local section  $f$  of  $\mathcal{O}_2$ . In this way  $\Omega_{-/-}$  becomes a functor on the category of arrows of sheaves of rings.*

**Proof.** This lemma proves itself.  $\square$

**Lemma 32.8.** *In Lemma 32.7 suppose that  $\mathcal{O}_2 \rightarrow \mathcal{O}'_2$  is surjective with kernel  $\mathcal{I} \subset \mathcal{O}_2$  and assume that  $\mathcal{O}_1 = \mathcal{O}'_1$ . Then there is a canonical exact sequence of  $\mathcal{O}'_2$ -modules*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}'_2 \longrightarrow \Omega_{\mathcal{O}'_2/\mathcal{O}_1} \longrightarrow 0$$

*The leftmost map is characterized by the rule that a local section  $f$  of  $\mathcal{I}$  maps to  $df \otimes 1$ .*

**Proof.** For a local section  $f$  of  $\mathcal{I}$  denote  $\bar{f}$  the image of  $f$  in  $\mathcal{I}/\mathcal{I}^2$ . To show that the map  $\bar{f} \mapsto df \otimes 1$  is well defined we just have to check that  $df_1 f_2 \otimes 1 = 0$  if  $f_1, f_2$  are local sections of  $\mathcal{I}$ . And this is clear from the Leibniz rule  $df_1 f_2 \otimes 1 = (f_1 df_2 + f_2 df_1) \otimes 1 = df_2 \otimes f_1 + df_1 \otimes f_2 = 0$ . A similar computation shows this map is  $\mathcal{O}'_2 = \mathcal{O}_2/\mathcal{I}$ -linear. The map on the right is the one from Lemma 32.7.

To see that the sequence is exact, we argue as follows. Let  $\mathcal{O}_2'' \subset \mathcal{O}_2'$  be the presheaf of  $\mathcal{O}_1$ -algebras whose value on  $U$  is the image of  $\mathcal{O}_2(U) \rightarrow \mathcal{O}_2'(U)$ . By Algebra, Lemma 127.9 the sequences

$$\mathcal{I}(U)/\mathcal{I}(U)^2 \rightarrow \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)} \otimes_{\mathcal{O}_2(U)} \mathcal{O}_2''(U) \rightarrow \Omega_{\mathcal{O}_2''(U)/\mathcal{O}_1(U)} \rightarrow 0$$

are exact for all objects  $U$  of  $\mathcal{C}$ . Since sheafification is exact this gives an exact sequence of sheaves of  $(\mathcal{O}_2')^\#$ -modules. By Lemma 32.4 and the fact that  $(\mathcal{O}_2'')^\# = \mathcal{O}_2'$  we conclude.  $\square$

Here is a particular situation where derivations come up naturally.

**Lemma 32.9.** *Let  $\mathcal{C}$  be a site. Let  $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a homomorphism of sheaves of rings. Consider a short exact sequence*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{A} \rightarrow \mathcal{O}_2 \rightarrow 0$$

*Here  $\mathcal{A}$  is a sheaf of  $\mathcal{O}_1$ -algebras,  $\pi : \mathcal{A} \rightarrow \mathcal{O}_2$  is a surjection of sheaves of  $\mathcal{O}_1$ -algebras, and  $\mathcal{F} = \text{Ker}(\pi)$  is its kernel. Assume  $\mathcal{F}$  an ideal sheaf with square zero in  $\mathcal{A}$ . So  $\mathcal{F}$  has a natural structure of an  $\mathcal{O}_2$ -module. A section  $s : \mathcal{O}_2 \rightarrow \mathcal{A}$  of  $\pi$  is a  $\mathcal{O}_1$ -algebra map such that  $\pi \circ s = \text{id}$ . Given any section  $s : \mathcal{O}_2 \rightarrow \mathcal{F}$  of  $\pi$  and any  $\varphi$ -derivation  $D : \mathcal{O}_1 \rightarrow \mathcal{F}$  the map*

$$s + D : \mathcal{O}_1 \rightarrow \mathcal{A}$$

*is a section of  $\pi$  and every section  $s'$  is of the form  $s + D$  for a unique  $\varphi$ -derivation  $D$ .*

**Proof.** Recall that the  $\mathcal{O}_2$ -module structure on  $\mathcal{F}$  is given by  $h\tau = \tilde{h}\tau$  (multiplication in  $\mathcal{A}$ ) where  $h$  is a local section of  $\mathcal{O}_2$ , and  $\tilde{h}$  is a local lift of  $h$  to a local section of  $\mathcal{A}$ , and  $\tau$  is a local section of  $\mathcal{F}$ . In particular, given  $s$ , we may use  $\tilde{h} = s(h)$ . To verify that  $s + D$  is a homomorphism of sheaves of rings we compute

$$\begin{aligned} (s + D)(ab) &= s(ab) + D(ab) \\ &= s(a)s(b) + aD(b) + D(a)b \\ &= s(a)s(b) + s(a)D(b) + D(a)s(b) \\ &= (s(a) + D(a))(s(b) + D(b)) \end{aligned}$$

by the Leibniz rule. In the same manner one shows  $s + D$  is a  $\mathcal{O}_1$ -algebra map because  $D$  is an  $\mathcal{O}_1$ -derivation. Conversely, given  $s'$  we set  $D = s' - s$ . Details omitted.  $\square$

**Definition 32.10.** Let  $X = (Sh(\mathcal{C}), \mathcal{O})$  and  $Y = (Sh(\mathcal{C}'), \mathcal{O}')$  be ringed topoi. Let  $(f, f^\#) : X \rightarrow Y$  be a morphism of ringed topoi. In this situation

- (1) for a sheaf  $\mathcal{F}$  of  $\mathcal{O}$ -modules a  $Y$ -derivation  $D : \mathcal{O} \rightarrow \mathcal{F}$  is just a  $f^\#$ -derivation, and
- (2) the sheaf of differentials  $\Omega_{X/Y}$  of  $X$  over  $Y$  is the module of differentials of  $f^\# : f^{-1}\mathcal{O}' \rightarrow \mathcal{O}$ , see Definition 32.3.

Thus  $\Omega_{X/Y}$  comes equipped with a universal  $Y$ -derivation  $d_{X/Y} : \mathcal{O} \rightarrow \Omega_{X/Y}$ . We sometimes write  $\Omega_{X/Y} = \Omega_f$ .

Recall that  $f^\# : f^{-1}\mathcal{O}' \rightarrow \mathcal{O}$  so that this definition makes sense.

**Lemma 32.11.** *Let  $X = (Sh(\mathcal{C}_X), \mathcal{O}_X)$ ,  $Y = (Sh(\mathcal{C}_Y), \mathcal{O}_Y)$ ,  $X' = (Sh(\mathcal{C}_{X'}), \mathcal{O}_{X'})$ , and  $Y' = (Sh(\mathcal{C}_{Y'}), \mathcal{O}_{Y'})$  be ringed topoi. Let*

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{\quad} & Y \end{array}$$

*be a commutative diagram of morphisms of ringed topoi. The map  $f^\# : \mathcal{O}_X \rightarrow f_*\mathcal{O}_{X'}$  composed with the map  $f_*d_{X'/Y'} : f_*\mathcal{O}_{X'} \rightarrow f_*\Omega_{X'/Y'}$  is a  $Y$ -derivation. Hence we obtain a canonical map of  $\mathcal{O}_X$ -modules  $\Omega_{X/Y} \rightarrow f_*\Omega_{X'/Y'}$ , and by adjointness of  $f_*$  and  $f^*$  a canonical  $\mathcal{O}_{X'}$ -module homomorphism*

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

*It is uniquely characterized by the property that  $f^*d_{X/Y}(t)$  maps to  $d_{X'/Y'}(f^*t)$  for any local section  $t$  of  $\mathcal{O}_X$ .*

**Proof.** This is clear except for the last assertion. Let us explain the meaning of this. Let  $U \in \text{Ob}(\mathcal{C}_X)$  and let  $t \in \mathcal{O}_X(U)$ . This is what it means for  $t$  to be a local section of  $\mathcal{O}_X$ . Now, we may think of  $t$  as a map of sheaves of sets  $t : h_U^\# \rightarrow \mathcal{O}_X$ . Then  $f^{-1}t : f^{-1}h_U^\# \rightarrow f^{-1}\mathcal{O}_X$ . By  $f^*t$  we mean the composition

$$\begin{array}{ccccc} & & f^*t & & \\ & \searrow & \text{---} & \searrow & \\ f^{-1}h_U^\# & \xrightarrow{f^{-1}t} & f^{-1}\mathcal{O}_X & \xrightarrow{f^\#} & \mathcal{O}_{X'} \end{array}$$

Note that  $d_{X/Y}(t) \in \Omega_{X/Y}(U)$ . Hence we may think of  $d_{X/Y}(t)$  as a map  $d_{X/Y}(t) : h_U^\# \rightarrow \Omega_{X/Y}$ . Then  $f^{-1}d_{X/Y}(t) : f^{-1}h_U^\# \rightarrow f^{-1}\Omega_{X/Y}$ . By  $f^*d_{X/Y}(t)$  we mean the composition

$$\begin{array}{ccccc} & & f^*d_{X/Y}(t) & & \\ & \searrow & \text{---} & \searrow & \\ f^{-1}h_U^\# & \xrightarrow{f^{-1}d_{X/Y}(t)} & f^{-1}\Omega_{X/Y} & \xrightarrow{1 \otimes \text{id}} & f^*\Omega_{X/Y} \end{array}$$

OK, and now the statement of the lemma means that we have

$$c_f \circ f^*t = f^*d_{X/Y}(t)$$

as maps from  $f^{-1}h_U^\#$  to  $\Omega_{X'/Y'}$ . We omit the verification that this property holds for  $c_f$  as defined in the lemma. (Hint: The first map  $c'_f : \Omega_{X/Y} \rightarrow f_*\Omega_{X'/Y'}$  satisfies  $c'_f(d_{X/Y}(t)) = f_*d_{X'/Y'}(f^\#(t))$  as sections of  $f_*\Omega_{X'/Y'}$  over  $U$ , and you have to turn this into the equality above by using adjunction.) The reason that this uniquely characterizes  $c_f$  is that the images of  $f^*d_{X/Y}(t)$  generate the  $\mathcal{O}_{X'}$ -module  $f^*\Omega_{X/Y}$  simply because the local sections  $d_{X/Y}(t)$  generate the  $\mathcal{O}_X$ -module  $\Omega_{X/Y}$ .  $\square$

### 33. Finite order differential operators

In this section we introduce differential operators of finite order. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 128).

**Definition 33.1.** Let  $\mathcal{C}$  be a site. Let  $\varphi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a homomorphism of sheaves of rings. Let  $k \geq 0$  be an integer. Let  $\mathcal{F}, \mathcal{G}$  be sheaves of  $\mathcal{O}_2$ -modules. A *differential operator*  $D : \mathcal{F} \rightarrow \mathcal{G}$  of order  $k$  is an  $\mathcal{O}_1$ -linear map such that for all local

sections  $g$  of  $\mathcal{O}_2$  the map  $s \mapsto D(gs) - gD(s)$  is a differential operator of order  $k - 1$ . For the base case  $k = 0$  we define a differential operator of order 0 to be an  $\mathcal{O}_2$ -linear map.

If  $D : \mathcal{F} \rightarrow \mathcal{G}$  is a differential operator of order  $k$ , then for all local sections  $g$  of  $\mathcal{O}_2$  the map  $gD$  is a differential operator of order  $k$ . The sum of two differential operators of order  $k$  is another. Hence the set of all these

$$\text{Diff}^k(\mathcal{F}, \mathcal{G}) = \text{Diff}_{\mathcal{O}_2/\mathcal{O}_1}^k(\mathcal{F}, \mathcal{G})$$

is a  $\Gamma(\mathcal{C}, \mathcal{O}_2)$ -module. We have

$$\text{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \dots$$

The rule which maps  $U \in \text{Ob}(\mathcal{C})$  to the module of differential operators  $D : \mathcal{F}|_U \rightarrow \mathcal{G}|_U$  of order  $k$  is a sheaf of  $\mathcal{O}_2$ -modules on the site  $\mathcal{C}$ . Thus we obtain a sheaf of differential operators (if we ever need this we will add a definition here).

**Lemma 33.2.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a map of sheaves of rings. Let  $\mathcal{E}, \mathcal{F}, \mathcal{G}$  be sheaves of  $\mathcal{O}_2$ -modules. If  $D : \mathcal{E} \rightarrow \mathcal{F}$  and  $D' : \mathcal{F} \rightarrow \mathcal{G}$  are differential operators of order  $k$  and  $k'$ , then  $D' \circ D$  is a differential operator of order  $k + k'$ .*

**Proof.** Let  $g$  be a local section of  $\mathcal{O}_2$ . Then the map which sends a local section  $x$  of  $\mathcal{E}$  to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on  $k + k'$ .  $\square$

**Lemma 33.3.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a map of sheaves of rings. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_2$ -modules. Let  $k \geq 0$ . There exists a sheaf of  $\mathcal{O}_2$ -modules  $\mathcal{P}_{\mathcal{O}_2/\mathcal{O}_1}^k(\mathcal{F})$  and a canonical isomorphism*

$$\text{Diff}_{\mathcal{O}_2/\mathcal{O}_1}^k(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}_2}(\mathcal{P}_{\mathcal{O}_2/\mathcal{O}_1}^k(\mathcal{F}), \mathcal{G})$$

*functorial in the  $\mathcal{O}_2$ -module  $\mathcal{G}$ .*

**Proof.** The existence follows from general category theoretic arguments (insert future reference here), but we will also give a direct construction as this construction will be useful in the future proofs. We will freely use the notation introduced in the proof of Lemma 32.2. Given any differential operator  $D : \mathcal{F} \rightarrow \mathcal{G}$  we obtain an  $\mathcal{O}_2$ -linear map  $L_D : \mathcal{O}_2[\mathcal{F}] \rightarrow \mathcal{G}$  sending  $[m]$  to  $D(m)$ . If  $D$  has order 0 then  $L_D$  annihilates the local sections

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0m]$$

where  $g_0$  is a local section of  $\mathcal{O}_2$  and  $m, m'$  are local sections of  $\mathcal{F}$ . If  $D$  has order 1, then  $L_D$  annihilates the local sections

$$[m + m'] - [m] - [m'], \quad f[m] - [fm], \quad g_0g_1[m] - g_0[g_1m] - g_1[g_0m] + [g_1g_0m]$$

where  $f$  is a local section of  $\mathcal{O}_1$ ,  $g_0, g_1$  are local sections of  $\mathcal{O}_2$ , and  $m, m'$  are local sections of  $\mathcal{F}$ . If  $D$  has order  $k$ , then  $L_D$  annihilates the local sections  $[m + m'] - [m] - [m']$ ,  $f[m] - [fm]$ , and the local sections

$$g_0g_1 \dots g_k[m] - \sum g_0 \dots \hat{g}_i \dots g_k[g_im] + \dots + (-1)^{k+1}[g_0 \dots g_km]$$

Conversely, if  $L : \mathcal{O}_2[\mathcal{F}] \rightarrow \mathcal{G}$  is an  $\mathcal{O}_2$ -linear map annihilating all the local sections listed in the previous sentence, then  $m \mapsto L([m])$  is a differential operator of order



$k$ . Thus we see that  $\mathcal{P}_{\mathcal{O}_2/\mathcal{O}_1}^k(\mathcal{F})$  is the quotient of  $\mathcal{O}_2[\mathcal{F}]$  by the  $\mathcal{O}_2$ -submodule generated by these local sections.  $\square$

**Definition 33.4.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a map of sheaves of rings. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_2$ -modules. The module  $\mathcal{P}_{\mathcal{O}_2/\mathcal{O}_1}^k(\mathcal{F})$  constructed in Lemma 33.3 is called the *module of principal parts of order  $k$*  of  $\mathcal{F}$ .

Note that the inclusions

$$\mathrm{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \dots$$

correspond via Yoneda's lemma (Categories, Lemma 3.5) to surjections

$$\dots \rightarrow \mathcal{P}_{\mathcal{O}_2/\mathcal{O}_1}^2(\mathcal{F}) \rightarrow \mathcal{P}_{\mathcal{O}_2/\mathcal{O}_1}^1(\mathcal{F}) \rightarrow \mathcal{P}_{\mathcal{O}_2/\mathcal{O}_1}^0(\mathcal{F}) = \mathcal{F}$$

**Lemma 33.5.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a homomorphism of presheaves of rings. Let  $\mathcal{F}$  be a presheaf of  $\mathcal{O}_2$ -modules. Then  $\mathcal{P}_{\mathcal{O}_2/\mathcal{O}_1}^k(\mathcal{F}^\#)$  is the sheaf associated to the presheaf  $U \mapsto \mathcal{P}_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}^k(\mathcal{F}(U))$ .

**Proof.** This can be proved in exactly the same way as is done for the sheaf of differentials in Lemma 32.4. Perhaps a more pleasing approach is to use the universal property of Lemma 33.3 directly to see the equality. We omit the details.  $\square$

**Lemma 33.6.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a homomorphism of presheaves of rings. Let  $\mathcal{F}$  be a presheaf of  $\mathcal{O}_2$ -modules. There is a canonical short exact sequence

$$0 \rightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{F} \rightarrow \mathcal{P}_{\mathcal{O}_2/\mathcal{O}_1}^1(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow 0$$

functorial in  $\mathcal{F}$  called the *sequence of principal parts*.

**Proof.** Follows from the commutative algebra version (Algebra, Lemma 128.6) and Lemmas 32.4 and 33.5.  $\square$

**Remark 33.7.** Let  $\mathcal{C}$  be a site. Suppose given a commutative diagram of sheaves of rings

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

a  $\mathcal{B}$ -module  $\mathcal{F}$ , a  $\mathcal{B}'$ -module  $\mathcal{F}'$ , and a  $\mathcal{B}$ -linear map  $\mathcal{F} \rightarrow \mathcal{F}'$ . Then we get a compatible system of module maps

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathcal{P}_{\mathcal{B}'/\mathcal{A}'}^2(\mathcal{F}') & \longrightarrow & \mathcal{P}_{\mathcal{B}'/\mathcal{A}'}^1(\mathcal{F}') & \longrightarrow & \mathcal{P}_{\mathcal{B}'/\mathcal{A}'}^0(\mathcal{F}') \\ & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & \mathcal{P}_{\mathcal{B}/\mathcal{A}}^2(\mathcal{F}) & \longrightarrow & \mathcal{P}_{\mathcal{B}/\mathcal{A}}^1(\mathcal{F}) & \longrightarrow & \mathcal{P}_{\mathcal{B}/\mathcal{A}}^0(\mathcal{F}) \end{array}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules  $\mathcal{P}_{\mathcal{B}/\mathcal{A}}^k(\mathcal{M})$  in terms of (local) generators and relations in the proof of Lemma 33.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 33.6.

### 34. The naive cotangent complex

This section is the analogue of Algebra, Section 129 and Modules, Section 24. We advise the reader to read those sections first.

Let  $\mathcal{C}$  be a site. Let  $\mathcal{A} \rightarrow \mathcal{B}$  be a homomorphism of sheaves of rings on  $\mathcal{C}$ . In this section, for any sheaf of sets  $\mathcal{E}$  on  $\mathcal{C}$  we denote  $\mathcal{A}[\mathcal{E}]$  the sheafification of the presheaf  $U \mapsto \mathcal{A}(U)[\mathcal{E}(U)]$ . Here  $\mathcal{A}(U)[\mathcal{E}(U)]$  denotes the polynomial algebra over  $\mathcal{A}(U)$  whose variables correspond to the elements of  $\mathcal{E}(U)$ . We denote  $[e] \in \mathcal{A}(U)[\mathcal{E}(U)]$  the variable corresponding to  $e \in \mathcal{E}(U)$ . There is a canonical surjection of  $\mathcal{A}$ -algebras

$$(34.0.1) \quad \mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{B}, \quad [b] \longmapsto b$$

whose kernel we denote  $\mathcal{I} \subset \mathcal{A}[\mathcal{B}]$ . It is a simple observation that  $\mathcal{I}$  is generated by the local sections  $[b][b'] - [bb']$  and  $[a] - a$ . According to Lemma 32.8 there is a canonical map

$$(34.0.2) \quad \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$$

whose cokernel is canonically isomorphic to  $\Omega_{\mathcal{B}/\mathcal{A}}$ .

**Definition 34.1.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{A} \rightarrow \mathcal{B}$  be a homomorphism of sheaves of rings on  $\mathcal{C}$ . The *naive cotangent complex*  $NL_{\mathcal{B}/\mathcal{A}}$  is the chain complex (34.0.2)

$$NL_{\mathcal{B}/\mathcal{A}} = (\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B})$$

with  $\mathcal{I}/\mathcal{I}^2$  placed in (homological) degree 1 and  $\Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$  placed in degree 0.

This construction satisfies a functoriality similar to that discussed in Lemma 32.7 for modules of differentials. Namely, given a commutative diagram

$$(34.1.1) \quad \begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

of sheaves of rings on  $\mathcal{C}$  there is a canonical  $\mathcal{B}$ -linear map of complexes

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'}$$

Namely, the maps in the commutative diagram give rise to a canonical map  $\mathcal{A}[\mathcal{B}] \rightarrow \mathcal{A}'[\mathcal{B}']$  which maps  $\mathcal{I}$  into  $\mathcal{I}' = \text{Ker}(\mathcal{A}'[\mathcal{B}'] \rightarrow \mathcal{B}')$ . Thus a map  $\mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{I}'/(\mathcal{I}')^2$  and a map between modules of differentials, which together give the desired map between the naive cotangent complexes.

We can choose a different presentation of  $\mathcal{B}$  as a quotient of a polynomial algebra over  $\mathcal{A}$  and still obtain the same object of  $D(\mathcal{B})$ . To explain this, suppose that  $\mathcal{E}$  is a sheaf of sets on  $\mathcal{C}$  and  $\alpha : \mathcal{E} \rightarrow \mathcal{B}$  a map of sheaves of sets. Then we obtain an  $\mathcal{A}$ -algebra homomorphism  $\mathcal{A}[\mathcal{E}] \rightarrow \mathcal{B}$ . Assume this map is surjective, and let  $\mathcal{J} \subset \mathcal{A}[\mathcal{E}]$  be the kernel. Set

$$NL(\alpha) = (\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$$

Here is the result.

**Lemma 34.2.** *In the situation above there is a canonical isomorphism  $NL(\alpha) = NL_{\mathcal{B}/\mathcal{A}}$  in  $D(\mathcal{B})$ .*

**Proof.** Observe that  $NL_{\mathcal{B}/\mathcal{A}} = NL(\text{id}_{\mathcal{B}})$ . Thus it suffices to show that given two maps  $\alpha_i : \mathcal{E}_i \rightarrow \mathcal{B}$  as above, there is a canonical quasi-isomorphism  $NL(\alpha_1) = NL(\alpha_2)$  in  $D(\mathcal{B})$ . To see this set  $\mathcal{E} = \mathcal{E}_1 \amalg \mathcal{E}_2$  and  $\alpha = \alpha_1 \amalg \alpha_2 : \mathcal{E} \rightarrow \mathcal{B}$ . Set  $\mathcal{J}_i = \text{Ker}(\mathcal{A}[\mathcal{E}_i] \rightarrow \mathcal{B})$  and  $\mathcal{J} = \text{Ker}(\mathcal{A}[\mathcal{E}] \rightarrow \mathcal{B})$ . We obtain maps  $\mathcal{A}[\mathcal{E}_i] \rightarrow \mathcal{A}[\mathcal{E}]$  which send  $\mathcal{J}_i$  into  $\mathcal{J}$ . Thus we obtain canonical maps of complexes

$$NL(\alpha_i) \longrightarrow NL(\alpha)$$

and it suffices to show these maps are quasi-isomorphism. To see this we argue as follows. First, observe that  $H_0(NL(\alpha_i)) = \Omega_{\mathcal{B}/\mathcal{A}}$  and  $H_0(NL(\alpha)) = \Omega_{\mathcal{B}/\mathcal{A}}$  by Lemma 32.8 hence the map is an isomorphism on 0th homology sheaves. Similarly, we claim that  $H_1(NL(\alpha_i))$  and  $H_1(NL(\alpha))$  are the sheaves associated to the presheaf  $U \mapsto H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$ . If the claim holds, then the proof is finished.

Proof of the claim. Let  $\alpha : \mathcal{E} \rightarrow \mathcal{B}$  be as above. Let  $\mathcal{B}' \subset \mathcal{B}$  be the subpresheaf of  $\mathcal{A}$ -algebras whose value on  $U$  is the image of  $\mathcal{A}(U)[\mathcal{E}(U)] \rightarrow \mathcal{B}(U)$ . Let  $\mathcal{I}'$  be the presheaf whose value on  $U$  is the kernel of  $\mathcal{A}(U)[\mathcal{E}(U)] \rightarrow \mathcal{B}(U)$ . Then  $\mathcal{I}$  is the sheafification of  $\mathcal{I}'$  and  $\mathcal{B}$  is the sheafification of  $\mathcal{B}'$ . Similarly,  $H_1(NL(\alpha))$  is the sheafification of the presheaf

$$U \longmapsto \text{Ker}(\mathcal{I}'(U)/\mathcal{I}'(U)^2 \rightarrow \Omega_{\mathcal{A}(U)[\mathcal{E}(U)]/\mathcal{A}(U)} \otimes_{\mathcal{A}(U)[\mathcal{E}(U)]} \mathcal{B}'(U))$$

by Lemma 32.4. By Algebra, Lemma 129.2 we conclude  $H_1(NL(\alpha))$  is the sheaf associated to the presheaf  $U \mapsto H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$ . Thus we have to show that the maps  $H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)}) \rightarrow H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$  induce an isomorphism  $\mathcal{H}'_1 \rightarrow \mathcal{H}_1$  of sheafifications.

Injectivity of  $\mathcal{H}'_1 \rightarrow \mathcal{H}_1$ . Let  $f \in H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$  map to zero in  $\mathcal{H}_1(U)$ . To show:  $f$  maps to zero in  $\mathcal{H}'_1(U)$ . The assumption means there is a covering  $\{U_i \rightarrow U\}$  such that  $f$  maps to zero in  $H_1(L_{\mathcal{B}(U_i)/\mathcal{A}(U_i)})$  for all  $i$ . Replace  $U$  by  $U_i$  to get to the point where  $f$  maps to zero in  $H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$ . By Algebra, Lemma 129.9 we can find a finitely generated subalgebra  $\mathcal{B}'(U) \subset \mathcal{B} \subset \mathcal{B}(U)$  such that  $f$  maps to zero in  $H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$ . Since  $\mathcal{B} = (\mathcal{B}')^\#$  we can find a covering  $\{U_i \rightarrow U\}$  such that  $\mathcal{B} \rightarrow \mathcal{B}(U_i)$  factors through  $\mathcal{B}'(U_i)$ . Hence  $f$  maps to zero in  $H_1(L_{\mathcal{B}'(U_i)/\mathcal{A}(U_i)})$  as desired.

The surjectivity of  $\mathcal{H}'_1 \rightarrow \mathcal{H}_1$  is proved in exactly the same way.  $\square$

**Lemma 34.3.** *Let  $f : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be morphism of topoi. Let  $\mathcal{A} \rightarrow \mathcal{B}$  be a homomorphism of sheaves of rings on  $\mathcal{D}$ . Then  $f^{-1} NL_{\mathcal{B}/\mathcal{A}} = NL_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$ .*

**Proof.** Omitted. Hint: Use Lemma 32.5.  $\square$

The cotangent complex of a morphism of ringed topoi is defined in terms of the cotangent complex we defined above.

**Definition 34.4.** Let  $X = (Sh(\mathcal{C}), \mathcal{O})$  and  $Y = (Sh(\mathcal{C}'), \mathcal{O}')$  be ringed topoi. Let  $(f, f^\#) : X \rightarrow Y$  be a morphism of ringed topoi. The *naive cotangent complex*  $NL_f = NL_{X/Y}$  of the given morphism of ringed topoi is  $NL_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$ . We sometimes write  $NL_{X/Y} = NL_{\mathcal{O}_X/\mathcal{O}_Y}$ .

### 35. Stalks of modules

We have to be a bit careful when taking stalks at points, since the colimit defining a stalk (see Sites, Equation 31.1.1) may not be filtered<sup>3</sup>. On the other hand, by definition of a point of a site the stalk functor is exact and commutes with arbitrary colimits. In other words, it behaves exactly as if the colimit were filtered.

**Lemma 35.1.** *Let  $\mathcal{C}$  be a site. Let  $p$  be a point of  $\mathcal{C}$ .*

- (1) *We have  $(\mathcal{F}^\#)_p = \mathcal{F}_p$  for any presheaf of sets on  $\mathcal{C}$ .*
- (2) *The stalk functor  $Sh(\mathcal{C}) \rightarrow Sets, \mathcal{F} \mapsto \mathcal{F}_p$  is exact (see Categories, Definition 23.1) and commutes with arbitrary colimits.*
- (3) *The stalk functor  $PSh(\mathcal{C}) \rightarrow Sets, \mathcal{F} \mapsto \mathcal{F}_p$  is exact (see Categories, Definition 23.1) and commutes with arbitrary colimits.*

**Proof.** By Sites, Lemma 31.5 we have (1). By Sites, Lemmas 31.4 we see that  $PSh(\mathcal{C}) \rightarrow Sets, \mathcal{F} \mapsto \mathcal{F}_p$  is a left adjoint, and by Sites, Lemma 31.5 we see the same thing for  $Sh(\mathcal{C}) \rightarrow Sets, \mathcal{F} \mapsto \mathcal{F}_p$ . Hence the stalk functor commutes with arbitrary colimits (see Categories, Lemma 24.4). It follows from the definition of a point of a site, see Sites, Definition 31.2 that  $Sh(\mathcal{C}) \rightarrow Sets, \mathcal{F} \mapsto \mathcal{F}_p$  is exact. Since sheafification is exact (Sites, Lemma 10.14) it follows that  $PSh(\mathcal{C}) \rightarrow Sets, \mathcal{F} \mapsto \mathcal{F}_p$  is exact.  $\square$

In particular, since the stalk functor  $\mathcal{F} \mapsto \mathcal{F}_p$  on presheaves commutes with all finite limits and colimits we may apply the reasoning of the proof of Sites, Proposition 43.3. The result of such an argument is that if  $\mathcal{F}$  is a (pre)sheaf of algebraic structures listed in Sites, Proposition 43.3 then the stalk  $\mathcal{F}_p$  is naturally an algebraic structure of the same kind. Let us explain this in detail when  $\mathcal{F}$  is an abelian presheaf. In this case the addition map  $+: \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$  induces a map

$$+: \mathcal{F}_p \times \mathcal{F}_p = (\mathcal{F} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

where the equal sign uses that stalk functor on presheaves of sets commutes with finite limits. This defines a group structure on the stalk  $\mathcal{F}_p$ . In this way we obtain our stalk functor

$$PAb(\mathcal{C}) \longrightarrow Ab, \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of  $\mathcal{F}_p$  is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of abelian groups by precomposing with the inclusion  $Ab(\mathcal{C}) \subset PAb(\mathcal{C})$ .

**Lemma 35.2.** *Let  $\mathcal{C}$  be a site. Let  $p$  be a point of  $\mathcal{C}$ .*

- (1) *The functor  $Ab(\mathcal{C}) \rightarrow Ab, \mathcal{F} \mapsto \mathcal{F}_p$  is exact.*
- (2) *The stalk functor  $PAb(\mathcal{C}) \rightarrow Ab, \mathcal{F} \mapsto \mathcal{F}_p$  is exact.*
- (3) *For  $\mathcal{F} \in \text{Ob}(PAb(\mathcal{C}))$  we have  $\mathcal{F}_p = \mathcal{F}_p^\#$ .*

**Proof.** This is formal from the results of Lemma 35.1 and the construction of the stalk functor above.  $\square$

Next, we turn to the case of sheaves of modules. Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. (It suffices for the discussion that  $\mathcal{O}$  be a presheaf of rings.) Let  $\mathcal{F}$  be a presheaf of  $\mathcal{O}$ -modules. Let  $p$  be a point of  $\mathcal{C}$ . In this case we get a map

$$\cdot : \mathcal{O}_p \times \mathcal{O}_p = (\mathcal{O} \times \mathcal{O})_p \longrightarrow \mathcal{O}_p$$

<sup>3</sup>Of course in almost any naturally occurring case the colimit is filtered and some of the discussion in this section may be simplified.

which is the stalk of the multiplication map and

$$\cdot : \mathcal{O}_p \times \mathcal{F}_p = (\mathcal{O} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

which is the stalk of the multiplication map. We omit the verification that this defines a ring structure on  $\mathcal{O}_p$  and an  $\mathcal{O}_p$ -module structure on  $\mathcal{F}_p$ . In this way we obtain a functor

$$PMod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}_p), \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of  $\mathcal{F}_p$  is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of  $\mathcal{O}$ -modules by precomposing with the inclusion  $Mod(\mathcal{O}) \subset PMod(\mathcal{O})$ .

**Lemma 35.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $p$  be a point of  $\mathcal{C}$ .*

- (1) *The functor  $Mod(\mathcal{O}) \rightarrow Mod(\mathcal{O}_p)$ ,  $\mathcal{F} \mapsto \mathcal{F}_p$  is exact.*
- (2) *The stalk functor  $PMod(\mathcal{O}) \rightarrow Mod(\mathcal{O}_p)$ ,  $\mathcal{F} \mapsto \mathcal{F}_p$  is exact.*
- (3) *For  $\mathcal{F} \in \text{Ob}(PMod(\mathcal{O}))$  we have  $\mathcal{F}_p = \mathcal{F}_p^\#$ .*

**Proof.** This is formal from the results of Lemma 35.2, the construction of the stalk functor above, and Lemma 14.1.  $\square$

**Lemma 35.4.** *Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi or ringed sites. Let  $p$  be a point of  $\mathcal{C}$  or  $Sh(\mathcal{C})$  and set  $q = f \circ p$ . Then*

$$(f^* \mathcal{F})_p = \mathcal{F}_q \otimes_{\mathcal{O}_{\mathcal{D}, q}} \mathcal{O}_{\mathcal{C}, p}$$

for any  $\mathcal{O}_{\mathcal{D}}$ -module  $\mathcal{F}$ .

**Proof.** We have

$$f^* \mathcal{F} = f^{-1} \mathcal{F} \otimes_{f^{-1} \mathcal{O}_{\mathcal{D}}} \mathcal{O}_{\mathcal{C}}$$

by definition. Since taking stalks at  $p$  (i.e., applying  $p^{-1}$ ) commutes with  $\otimes$  by Lemma 26.1 we win by the relation between the stalk of pullbacks at  $p$  and stalks at  $q$  explained in Sites, Lemma 33.1 or Sites, Lemma 33.2.  $\square$

### 36. Skyscraper sheaves

Let  $p$  be a point of a site  $\mathcal{C}$  or a topos  $Sh(\mathcal{C})$ . In this section we study the exactness properties of the functor which associates to an abelian group  $A$  the skyscraper sheaf  $p_* A$ . First, recall that  $p_* : Sets \rightarrow Sh(\mathcal{C})$  has a lot of exactness properties, see Sites, Lemmas 31.9 and 31.10.

**Lemma 36.1.** *Let  $\mathcal{C}$  be a site. Let  $p$  be a point of  $\mathcal{C}$  or of its associated topos.*

- (1) *The functor  $p_* : Ab \rightarrow Ab(\mathcal{C})$ ,  $A \mapsto p_* A$  is exact.*
- (2) *There is a functorial direct sum decomposition*

$$p^{-1} p_* A = A \oplus I(A)$$

for  $A \in \text{Ob}(Ab)$ .

**Proof.** By Sites, Lemma 31.9 there are functorial maps  $A \rightarrow p^{-1} p_* A \rightarrow A$  whose composition equals  $\text{id}_A$ . Hence a functorial direct sum decomposition as in (2) with  $I(A)$  the kernel of the adjunction map  $p^{-1} p_* A \rightarrow A$ . The functor  $p_*$  is left exact by Lemma 14.3. The functor  $p_*$  transforms surjections into surjections by Sites, Lemma 31.10. Hence (1) holds.  $\square$

To do the same thing for sheaves of modules, suppose given a point  $p$  of a ringed topos  $(Sh(\mathcal{C}), \mathcal{O})$ . Recall that  $p^{-1}$  is just the stalk functor. Hence we can think of  $p$  as a morphism of ringed topoi

$$(p, \text{id}_{\mathcal{O}_p}) : (Sh(pt), \mathcal{O}_p) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}).$$

Thus we get a pullback functor  $p^* : Mod(\mathcal{O}) \rightarrow Mod(\mathcal{O}_p)$  which equals the stalk functor, and which we discussed in Lemma 35.3. In this section we consider the functor  $p_* : Mod(\mathcal{O}_p) \rightarrow Mod(\mathcal{O})$ .

**Lemma 36.2.** *Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a ringed topos. Let  $p$  be a point of the topos  $Sh(\mathcal{C})$ .*

- (1) *The functor  $p_* : Mod(\mathcal{O}_p) \rightarrow Mod(\mathcal{O})$ ,  $M \mapsto p_*M$  is exact.*
- (2) *There is a functorial direct sum decomposition of  $\mathcal{O}_p$ -modules*

$$p^{-1}p_*M = M \oplus I(M)$$

*for  $M$  a  $\mathcal{O}_p$ -module.*

**Proof.** This follows immediately from the corresponding result for abelian sheaves in Lemma 36.1.  $\square$

**Example 36.3.** Let  $G$  be a group. Consider the site  $\mathcal{T}_G$  and its point  $p$ , see Sites, Example 32.6. Let  $R$  be a ring with a  $G$ -action which corresponds to a sheaf of rings  $\mathcal{O}$  on  $\mathcal{T}_G$ . Then  $\mathcal{O}_p = R$  where we forget the  $G$ -action. In this case  $p^{-1}p_*M = \text{Map}(G, M)$  and  $I(M) = \{f : G \rightarrow M \mid f(1_G) = 0\}$  and  $M \rightarrow \text{Map}(G, M)$  assigns to  $m \in M$  the constant function with value  $m$ .

### 37. Localization and points

**Lemma 37.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $p$  be a point of  $\mathcal{C}$ . Let  $U$  be an object of  $\mathcal{C}$ . For  $\mathcal{G}$  in  $Mod(\mathcal{O}_U)$  we have*

$$(j_{U!}\mathcal{G})_p = \bigoplus_q \mathcal{G}_q$$

*where the coproduct is over the points  $q$  of  $\mathcal{C}/U$  lying over  $p$ , see Sites, Lemma 34.2.*

**Proof.** We use the description of  $j_{U!}\mathcal{G}$  as the sheaf associated to the presheaf  $V \mapsto \bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{G}(V/\varphi U)$  of Lemma 19.2. The stalk of  $j_{U!}\mathcal{G}$  at  $p$  is equal to the stalk of this presheaf, see Lemma 35.3. Let  $u : \mathcal{C} \rightarrow \text{Sets}$  be the functor corresponding to  $p$  (see Sites, Section 31). Hence we see that

$$(j_{U!}\mathcal{G})_p = \text{colim}_{(V, y)} \bigoplus_{\varphi : V \rightarrow U} \mathcal{G}(V/\varphi U)$$

where the colimit is taken in the category of abelian groups. To a quadruple  $(V, y, \varphi, s)$  occurring in this colimit, we can assign  $x = u(\varphi)(y) \in u(U)$ . Hence we obtain

$$(j_{U!}\mathcal{G})_p = \bigoplus_{x \in u(U)} \text{colim}_{(\varphi : V \rightarrow U, y, u(\varphi)(y)=x)} \mathcal{G}(V/\varphi U).$$

This is equal to the expression of the lemma by the description of the points  $q$  lying over  $x$  in Sites, Lemma 34.2.  $\square$

**Remark 37.2.** Warning: The result of Lemma 37.1 has no analogue for  $j_{U,*}$ .

### 38. Pullbacks of flat modules

The pullback of a flat module along a morphism of ringed topoi is flat. This is quite tricky to prove, except when there are enough points. Here we prove it only in this case and we will add the general case if we ever need it.

**Lemma 38.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $p$  be a point of  $\mathcal{C}$ . If  $\mathcal{F}$  is a flat  $\mathcal{O}$ -module, then  $\mathcal{F}_p$  is a flat  $\mathcal{O}_p$ -module.*

**Proof.** Let  $M$  be an  $\mathcal{O}_p$ -module. Then

$$\begin{aligned} (p_* M \otimes_{\mathcal{O}} \mathcal{F})_p &= p^{-1}(p_* M \otimes_{\mathcal{O}} \mathcal{F}) \\ &= p^{-1} p_* M \otimes_{\mathcal{O}_p} \mathcal{F}_p \\ &= M \otimes_{\mathcal{O}_p} \mathcal{F}_p \oplus I(M) \otimes_{\mathcal{O}_p} \mathcal{F}_p \end{aligned}$$

where we have used the description of the stalk functor as a pullback, Lemma 26.1, and Lemma 36.2. Since  $p_*$  is exact by Lemma 36.2, it is clear that if  $\mathcal{F}$  is flat, then also the functor  $M \mapsto M \otimes_{\mathcal{O}_p} \mathcal{F}_p$  is exact, i.e.,  $\mathcal{F}_p$  is flat.  $\square$

**Lemma 38.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules. Let  $\{p_i\}_{i \in I}$  be a conservative family of points of  $\mathcal{C}$ . Then  $\mathcal{F}$  is flat if and only if  $\mathcal{F}_{p_i}$  is a flat  $\mathcal{O}_{p_i}$ -module for all  $i \in I$ .*

**Proof.** By Lemma 38.1 we see one of the implications. For the converse, use that  $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_p = \mathcal{F}_p \otimes_{\mathcal{O}_p} \mathcal{G}_p$  by Lemma 26.1 (as taking stalks at  $p$  is given by  $p^{-1}$ ) and Lemma 14.4.  $\square$

**Lemma 38.3.** *Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi or ringed sites. Assume  $\mathcal{C}$  has enough points<sup>4</sup>. Then  $f^* \mathcal{F}$  is a flat  $\mathcal{O}_{\mathcal{C}}$ -module whenever  $\mathcal{F}$  is a flat  $\mathcal{O}_{\mathcal{D}}$ -module.*

**Proof.** Let  $p$  be a point of  $\mathcal{C}$  and set  $q = f \circ p$ . Then

$$(f^* \mathcal{F})_p = \mathcal{F}_q \otimes_{\mathcal{O}_{\mathcal{D}, q}} \mathcal{O}_{\mathcal{C}, p}$$

by Lemma 35.4. Hence if  $\mathcal{F}$  is flat, then  $\mathcal{F}_q$  is a flat  $\mathcal{O}_{\mathcal{D}, q}$ -module by Lemma 38.1 and hence by Algebra, Lemma 38.6 we see that  $(f^* \mathcal{F})_p$  is a flat  $\mathcal{O}_{\mathcal{C}, p}$ -module. This implies that  $f^* \mathcal{F}$  is a flat  $\mathcal{O}_{\mathcal{C}}$ -module by Lemma 38.2.  $\square$

### 39. Locally ringed topoi

A reference for this section is [AGV71, Exposé IV, Exercice 13.9].

**Lemma 39.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. The following are equivalent*

- (1) *For every object  $U$  of  $\mathcal{C}$  and  $f \in \mathcal{O}(U)$  there exists a covering  $\{U_j \rightarrow U\}$  such that for each  $j$  either  $f|_{U_j}$  is invertible or  $(1 - f)|_{U_j}$  is invertible.*
- (2) *For every object  $U$  of  $\mathcal{C}$  and  $f_1, \dots, f_n \in \mathcal{O}(U)$  which generate the unit ideal in  $\mathcal{O}(U)$  there exists a covering  $\{U_j \rightarrow U\}$  such that for each  $j$  there exists an  $i$  such that  $f_i|_{U_j}$  is invertible.*
- (3) *The map of sheaves of sets*

$$(\mathcal{O} \times \mathcal{O}) \amalg (\mathcal{O} \times \mathcal{O}) \longrightarrow \mathcal{O} \times \mathcal{O}$$

*which maps  $(f, a)$  in the first component to  $(f, af)$  and  $(f, b)$  in the second component to  $(f, b(1 - f))$  is surjective.*

<sup>4</sup>This assumption is not necessary, see introduction to this section.

**Proof.** It is clear that (2) implies (1). To show that (1) implies (2) we argue by induction on  $n$ . The first case is  $n = 2$  (since  $n = 1$  is trivial). In this case we have  $a_1f_1 + a_2f_2 = 1$  for some  $a_1, a_2 \in \mathcal{O}(U)$ . By assumption we can find a covering  $\{U_j \rightarrow U\}$  such that for each  $j$  either  $a_1f_1|_{U_j}$  is invertible or  $a_2f_2|_{U_j}$  is invertible. Hence either  $f_1|_{U_j}$  is invertible or  $f_2|_{U_j}$  is invertible as desired. For  $n > 2$  we have  $a_1f_1 + \dots + a_nf_n = 1$  for some  $a_1, \dots, a_n \in \mathcal{O}(U)$ . By the case  $n = 2$  we see that we have some covering  $\{U_j \rightarrow U\}_{j \in J}$  such that for each  $j$  either  $f_n|_{U_j}$  is invertible or  $a_1f_1 + \dots + a_{n-1}f_{n-1}|_{U_j}$  is invertible. Say the first case happens for  $j \in J_n$ . Set  $J' = J \setminus J_n$ . By induction hypothesis, for each  $j \in J'$  we can find a covering  $\{U_{jk} \rightarrow U_j\}_{k \in K_j}$  such that for each  $k \in K_j$  there exists an  $i \in \{1, \dots, n-1\}$  such that  $f_i|_{U_{jk}}$  is invertible. By the axioms of a site the family of morphisms  $\{U_j \rightarrow U\}_{j \in J_n} \cup \{U_{jk} \rightarrow U\}_{j \in J', k \in K_j}$  is a covering which has the desired property.

Assume (1). To see that the map in (3) is surjective, let  $(f, c)$  be a section of  $\mathcal{O} \times \mathcal{O}$  over  $U$ . By assumption there exists a covering  $\{U_j \rightarrow U\}$  such that for each  $j$  either  $f$  or  $1 - f$  restricts to an invertible section. In the first case we can take  $a = c|_{U_j}(f|_{U_j})^{-1}$ , and in the second case we can take  $b = c|_{U_j}(1 - f|_{U_j})^{-1}$ . Hence  $(f, c)$  is in the image of the map on each of the members. Conversely, assume (3) holds. For any  $U$  and  $f \in \mathcal{O}(U)$  there exists a covering  $\{U_j \rightarrow U\}$  of  $U$  such that the section  $(f, 1)|_{U_j}$  is in the image of the map in (3) on sections over  $U_j$ . This means precisely that either  $f$  or  $1 - f$  restricts to an invertible section over  $U_j$ , and we see that (1) holds.  $\square$

**Lemma 39.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Consider the following conditions*

- (1) *For every object  $U$  of  $\mathcal{C}$  and  $f \in \mathcal{O}(U)$  there exists a covering  $\{U_j \rightarrow U\}$  such that for each  $j$  either  $f|_{U_j}$  is invertible or  $(1 - f)|_{U_j}$  is invertible.*
- (2) *For every point  $p$  of  $\mathcal{C}$  the stalk  $\mathcal{O}_p$  is either the zero ring or a local ring.*

*We always have (1)  $\Rightarrow$  (2). If  $\mathcal{C}$  has enough points then (1) and (2) are equivalent.*

**Proof.** Assume (1). Let  $p$  be a point of  $\mathcal{C}$  given by a functor  $u : \mathcal{C} \rightarrow \text{Sets}$ . Let  $f_p \in \mathcal{O}_p$ . Since  $\mathcal{O}_p$  is computed by Sites, Equation (31.1.1) we may represent  $f_p$  by a triple  $(U, x, f)$  where  $x \in u(U)$  and  $f \in \mathcal{O}(U)$ . By assumption there exists a covering  $\{U_i \rightarrow U\}$  such that for each  $i$  either  $f$  or  $1 - f$  is invertible on  $U_i$ . Because  $u$  defines a point of the site we see that for some  $i$  there exists an  $x_i \in u(U_i)$  which maps to  $x \in u(U)$ . By the discussion surrounding Sites, Equation (31.1.1) we see that  $(U, x, f)$  and  $(U_i, x_i, f|_{U_i})$  define the same element of  $\mathcal{O}_p$ . Hence we conclude that either  $f_p$  or  $1 - f_p$  is invertible. Thus  $\mathcal{O}_p$  is a ring such that for every element  $a$  either  $a$  or  $1 - a$  is invertible. This means that  $\mathcal{O}_p$  is either zero or a local ring, see Algebra, Lemma 17.2.

Assume (2) and assume that  $\mathcal{C}$  has enough points. Consider the map of sheaves of sets

$$\mathcal{O} \times \mathcal{O} \amalg \mathcal{O} \times \mathcal{O} \longrightarrow \mathcal{O} \times \mathcal{O}$$

of Lemma 39.1 part (3). For any local ring  $R$  the corresponding map  $(R \times R) \amalg (R \times R) \rightarrow R \times R$  is surjective, see for example Algebra, Lemma 17.2. Since each  $\mathcal{O}_p$  is a local ring or zero the map is surjective on stalks. Hence, by our assumption that  $\mathcal{C}$  has enough points it is surjective and we win.  $\square$

In Modules, Section 2 we pointed out how in a ringed space  $(X, \mathcal{O}_X)$  there can be an open subspace over which the structure sheaf is zero. To prevent this we can



require the sections 1 and 0 to have different values in every stalk of the space  $X$ . In the setting of ringed topoi and ringed sites the condition is that

$$(39.2.1) \quad \emptyset^\# \longrightarrow \text{Equalizer}(0, 1 : * \longrightarrow \mathcal{O})$$

is an isomorphism of sheaves. Here  $*$  is the singleton sheaf, resp.  $\emptyset^\#$  is the “empty sheaf”, i.e., the final, resp. initial object in the category of sheaves, see Sites, Example 10.2, resp. Section 41. In other words, the condition is that whenever  $U \in \text{Ob}(\mathcal{C})$  is not sheaf theoretically empty, then  $1, 0 \in \mathcal{O}(U)$  are not equal. Let us state the obligatory lemma.

**Lemma 39.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Consider the statements*

- (1) *(39.2.1) is an isomorphism, and*
- (2) *for every point  $p$  of  $\mathcal{C}$  the stalk  $\mathcal{O}_p$  is not the zero ring.*

*We always have (1)  $\Rightarrow$  (2) and if  $\mathcal{C}$  has enough points then (1)  $\Leftrightarrow$  (2).*

**Proof.** Omitted. □

Lemmas 39.1, 39.2, and 39.3 motivate the following definition.

**Definition 39.4.** A ringed site  $(\mathcal{C}, \mathcal{O})$  is said to be *locally ringed site* if (39.2.1) is an isomorphism, and the equivalent properties of Lemma 39.1 are satisfied.

In [AGV71, Exposé IV, Exercice 13.9] the condition that (39.2.1) be an isomorphism is missing leading to a slightly different notion of a locally ringed site and locally ringed topoi. As we are motivated by the notion of a locally ringed space we decided to add this condition (see explanation above).

**Lemma 39.5.** *Being a locally ringed site is an intrinsic property. More precisely,*

- (1) *if  $f : \text{Sh}(\mathcal{C}') \rightarrow \text{Sh}(\mathcal{C})$  is a morphism of topoi and  $(\mathcal{C}, \mathcal{O})$  is a locally ringed site, then  $(\mathcal{C}', f^{-1}\mathcal{O})$  is a locally ringed site, and*
- (2) *if  $(f, f^\#) : (\text{Sh}(\mathcal{C}'), \mathcal{O}') \rightarrow (\text{Sh}(\mathcal{C}), \mathcal{O})$  is an equivalence of ringed topoi, then  $(\mathcal{C}, \mathcal{O})$  is locally ringed if and only if  $(\mathcal{C}', \mathcal{O}')$  is locally ringed.*

**Proof.** It is clear that (2) follows from (1). To prove (1) note that as  $f^{-1}$  is exact we have  $f^{-1}* = *$ ,  $f^{-1}\emptyset^\# = \emptyset^\#$ , and  $f^{-1}$  commutes with products, equalizers and transforms isomorphisms and surjections into isomorphisms and surjections. Thus  $f^{-1}$  transforms the isomorphism (39.2.1) into its analogue for  $f^{-1}\mathcal{O}$  and transforms the surjection of Lemma 39.1 part (3) into the corresponding surjection for  $f^{-1}\mathcal{O}$ . □

In fact Lemma 39.5 part (2) is the analogue of Schemes, Lemma 2.2. It assures us that the following definition makes sense.

**Definition 39.6.** A ringed topoi  $(\text{Sh}(\mathcal{C}), \mathcal{O})$  is said to be *locally ringed* if the underlying ringed site  $(\mathcal{C}, \mathcal{O})$  is locally ringed.

Next, we want to work out what it means to have a morphism of locally ringed spaces. In order to do this we have the following lemma.

**Lemma 39.7.** *Let  $(f, f^\#) : (\text{Sh}(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (\text{Sh}(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Consider the following conditions*

(1) *The diagram of sheaves*

$$\begin{array}{ccc} f^{-1}(\mathcal{O}_{\mathcal{D}}^*) & \xrightarrow{f^\#} & \mathcal{O}_{\mathcal{C}}^* \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{O}_{\mathcal{D}}) & \xrightarrow{f^\#} & \mathcal{O}_{\mathcal{C}} \end{array}$$

*is cartesian.*

(2) *For any point  $p$  of  $\mathcal{C}$ , setting  $q = f \circ p$ , the diagram*

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{D},q}^* & \longrightarrow & \mathcal{O}_{\mathcal{C},p}^* \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{D},q} & \longrightarrow & \mathcal{O}_{\mathcal{C},p} \end{array}$$

*of sets is cartesian.*

We always have (1)  $\Rightarrow$  (2). If  $\mathcal{C}$  has enough points then (1) and (2) are equivalent. If  $(Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}})$  and  $(Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  are locally ringed topoi then (2) is equivalent to

(3) *For any point  $p$  of  $\mathcal{C}$ , setting  $q = f \circ p$ , the ring map  $\mathcal{O}_{\mathcal{D},q} \rightarrow \mathcal{O}_{\mathcal{C},p}$  is a local ring map.*

In fact, properties (2), or (3) for a conservative family of points implies (1).

**Proof.** This lemma proves itself, in other words, it follows by unwinding the definitions.  $\square$

**Definition 39.8.** Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Assume  $(Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}})$  and  $(Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  are locally ringed topoi. We say that  $(f, f^\#)$  is a *morphism of locally ringed topoi* if and only if the diagram of sheaves

$$\begin{array}{ccc} f^{-1}(\mathcal{O}_{\mathcal{D}}^*) & \xrightarrow{f^\#} & \mathcal{O}_{\mathcal{C}}^* \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{O}_{\mathcal{D}}) & \xrightarrow{f^\#} & \mathcal{O}_{\mathcal{C}} \end{array}$$

(see Lemma 39.7) is cartesian. If  $(f, f^\#)$  is a morphism of ringed sites, then we say that it is a *morphism of locally ringed sites* if the associated morphism of ringed topoi is a morphism of locally ringed topoi.

It is clear that an isomorphism of ringed topoi between locally ringed topoi is automatically an isomorphism of locally ringed topoi.

**Lemma 39.9.** *Let  $(f, f^\#) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \rightarrow (Sh(\mathcal{C}_2), \mathcal{O}_2)$  and  $(g, g^\#) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \rightarrow (Sh(\mathcal{C}_3), \mathcal{O}_3)$  be morphisms of locally ringed topoi. Then the composition  $(g, g^\#) \circ (f, f^\#)$  (see Definition 7.1) is also a morphism of locally ringed topoi.*

**Proof.** Omitted.  $\square$

**Lemma 39.10.** *If  $f : Sh(\mathcal{C}') \rightarrow Sh(\mathcal{C})$  is a morphism of topoi. If  $\mathcal{O}$  is a sheaf of rings on  $\mathcal{C}$ , then*

$$f^{-1}(\mathcal{O}^*) = (f^{-1}\mathcal{O})^*.$$

In particular, if  $\mathcal{O}$  turns  $\mathcal{C}$  into a locally ringed site, then setting  $f^\sharp = \text{id}$  the morphism of ringed topoi

$$(f, f^\sharp) : (Sh(\mathcal{C}'), f^{-1}\mathcal{O}) \rightarrow (Sh(\mathcal{C}, \mathcal{O}))$$

is a morphism of locally ringed topoi.

**Proof.** Note that the diagram

$$\begin{array}{ccc} \mathcal{O}^* & \xrightarrow{\quad} & * \\ u \mapsto (u, u^{-1}) \downarrow & & \downarrow 1 \\ \mathcal{O} \times \mathcal{O} & \xrightarrow{(a,b) \mapsto ab} & \mathcal{O} \end{array}$$

is cartesian. Since  $f^{-1}$  is exact we conclude that

$$\begin{array}{ccc} f^{-1}(\mathcal{O}^*) & \xrightarrow{\quad} & * \\ u \mapsto (u, u^{-1}) \downarrow & & \downarrow 1 \\ f^{-1}\mathcal{O} \times f^{-1}\mathcal{O} & \xrightarrow{(a,b) \mapsto ab} & f^{-1}\mathcal{O} \end{array}$$

is cartesian which implies the first assertion. For the second, note that  $(\mathcal{C}', f^{-1}\mathcal{O})$  is a locally ringed site by Lemma 39.5 so that the assertion makes sense. Now the first part implies that the morphism is a morphism of locally ringed topoi.  $\square$

**Lemma 39.11.** *Localization of locally ringed sites and topoi.*

- (1) Let  $(\mathcal{C}, \mathcal{O})$  be a locally ringed site. Let  $U$  be an object of  $\mathcal{C}$ . Then the localization  $(\mathcal{C}/U, \mathcal{O}_U)$  is a locally ringed site, and the localization morphism

$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \rightarrow (Sh(\mathcal{C}), \mathcal{O})$$

is a morphism of locally ringed topoi.

- (2) Let  $(\mathcal{C}, \mathcal{O})$  be a locally ringed site. Let  $f : V \rightarrow U$  be a morphism of  $\mathcal{C}$ . Then the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C}/V), \mathcal{O}_V) \rightarrow (Sh(\mathcal{C}/U), \mathcal{O}_U)$$

of Lemma 19.4 is a morphism of locally ringed topoi.

- (3) Let  $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \rightarrow (\mathcal{D}, \mathcal{O}')$  be a morphism of locally ringed sites where  $f$  is given by the continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$ . Let  $V$  be an object of  $\mathcal{D}$  and let  $U = u(V)$ . Then the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \rightarrow (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 20.1 is a morphism of locally ringed sites.

- (4) Let  $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \rightarrow (\mathcal{D}, \mathcal{O}')$  be a morphism of locally ringed sites where  $f$  is given by the continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$ . Let  $V \in \text{Ob}(\mathcal{D})$ ,  $U \in \text{Ob}(\mathcal{C})$ , and  $c : U \rightarrow u(V)$ . Then the morphism

$$(f_c, (f_c)^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \rightarrow (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 20.2 is a morphism of locally ringed topoi.

- (5) Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a locally ringed topos. Let  $\mathcal{F}$  be a sheaf on  $\mathcal{C}$ . Then the localization  $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}})$  is a locally ringed topos and the localization morphism

$$(j_{\mathcal{F}}, j_{\mathcal{F}}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \rightarrow (Sh(\mathcal{C}), \mathcal{O})$$

is a morphism of locally ringed topoi.

- (6) Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a locally ringed topos. Let  $s : \mathcal{G} \rightarrow \mathcal{F}$  be a map of sheaves on  $\mathcal{C}$ . Then the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_{\mathcal{G}}) \longrightarrow (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}})$$

of Lemma 21.4 is a morphism of locally ringed topoi.

- (7) Let  $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$  be a morphism of locally ringed topoi. Let  $\mathcal{G}$  be a sheaf on  $\mathcal{D}$ . Set  $\mathcal{F} = f^{-1}\mathcal{G}$ . Then the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}})$$

of Lemma 22.1 is a morphism of locally ringed topoi.

- (8) Let  $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$  be a morphism of locally ringed topoi. Let  $\mathcal{G}$  be a sheaf on  $\mathcal{D}$ , let  $\mathcal{F}$  be a sheaf on  $\mathcal{C}$ , and let  $s : \mathcal{F} \rightarrow f^{-1}\mathcal{G}$  be a morphism of sheaves. Then the morphism

$$(f_s, (f_s)^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}})$$

of Lemma 22.3 is a morphism of locally ringed topoi.

**Proof.** Part (1) is clear since  $\mathcal{O}_U$  is just the restriction of  $\mathcal{O}$ , so Lemmas 39.5 and 39.10 apply. Part (2) is clear as the morphism  $(j, j^\sharp)$  is actually a localization of a locally ringed site so (1) applies. Part (3) is clear also since  $(f')^\sharp$  is just the restriction of  $f^\sharp$  to the topos  $Sh(\mathcal{C})/\mathcal{F}$ , see proof of Lemma 22.1 (hence the diagram of Definition 39.8 for the morphism  $f'$  is just the restriction of the corresponding diagram for  $f$ , and restriction is an exact functor). Part (4) follows formally on combining (2) and (3). Parts (5), (6), (7), and (8) follow from their counterparts (1), (2), (3), and (4) by enlarging the sites as in Lemma 7.2 and translating everything in terms of sites and morphisms of sites using the comparisons of Lemmas 21.3, 21.5, 22.2, and 22.4. (Alternatively one could use the same arguments as in the proofs of (1), (2), (3), and (4) to prove (5), (6), (7), and (8) directly.)  $\square$

#### 40. Lower shriek for modules

In this section we extend the construction of  $g_!$  discussed in Section 16 to the case of sheaves of modules.

**Lemma 40.1.** *Let  $u : \mathcal{C} \rightarrow \mathcal{D}$  be a continuous and cocontinuous functor between sites. Denote  $g : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{O}_{\mathcal{D}})$  the associated morphism of topoi. Let  $\mathcal{O}_{\mathcal{D}}$  be a sheaf of rings on  $\mathcal{D}$ . Set  $\mathcal{O}_{\mathcal{C}} = g^{-1}\mathcal{O}_{\mathcal{D}}$ . Hence  $g$  becomes a morphism of ringed topoi with  $g^* = g^{-1}$ . In this case there exists a functor*

$$g_! : Mod(\mathcal{O}_{\mathcal{C}}) \longrightarrow Mod(\mathcal{O}_{\mathcal{D}})$$

*which is left adjoint to  $g^*$ .*

**Proof.** Let  $U$  be an object of  $\mathcal{C}$ . For any  $\mathcal{O}_{\mathcal{D}}$ -module  $\mathcal{G}$  we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{\mathcal{C}}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{G}) &= g^{-1}\mathcal{G}(U) \\ &= \mathcal{G}(u(U)) \\ &= \mathrm{Hom}_{\mathcal{O}_{\mathcal{D}}}(j_{u(U)!}\mathcal{O}_{u(U)}, \mathcal{G}) \end{aligned}$$

because  $g^{-1}$  is described by restriction, see Sites, Lemma 20.5. Of course a similar formula holds a direct sum of modules of the form  $j_{U!}\mathcal{O}_U$ . By Homology, Lemma 25.6 and Lemma 28.6 we see that  $g_!$  exists.  $\square$

**Remark 40.2.** Warning! Let  $u : \mathcal{C} \rightarrow \mathcal{D}$ ,  $g$ ,  $\mathcal{O}_{\mathcal{D}}$ , and  $\mathcal{O}_{\mathcal{C}}$  be as in Lemma 40.1. In general it is **not** the case that the diagram

$$\begin{array}{ccc} \text{Mod}(\mathcal{O}_{\mathcal{C}}) & \xrightarrow{g_!} & \text{Mod}(\mathcal{O}_{\mathcal{D}}) \\ \text{forget} \downarrow & & \downarrow \text{forget} \\ \text{Ab}(\mathcal{C}) & \xrightarrow{g_!^{Ab}} & \text{Ab}(\mathcal{D}) \end{array}$$

commutes (here  $g_!^{Ab}$  is the one from Lemma 16.2). There is a transformation of functors

$$g_!^{Ab} \circ \text{forget} \longrightarrow \text{forget} \circ g_!$$

From the proof of Lemma 40.1 we see that this is an isomorphism if and only if  $g_! j_{U!} \mathcal{O}_U = g_!^{Ab} j_{U!} \mathcal{O}_U$  for all objects  $U$  of  $\mathcal{C}$ , in other words, if and only if

$$g_!^{Ab} j_{U!} \mathcal{O}_U = j_{u(U)!} \mathcal{O}_{u(U)}$$

for all objects  $U$  of  $\mathcal{C}$ . Note that for such a  $U$  we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow{u'} & \mathcal{D}/u(U) \\ j_U \downarrow & & \downarrow j_{u(U)} \\ \mathcal{C} & \xrightarrow{u} & \mathcal{D} \end{array}$$

of cocontinuous functors of sites, see Sites, Lemma 27.4. Hence we see that  $g_! = g_!^{Ab}$  if the canonical map

$$(40.2.1) \quad (g')_!^{Ab} \mathcal{O}_U \longrightarrow \mathcal{O}_{u(U)}$$

is an isomorphism for all objects  $U$  of  $\mathcal{C}$ . Here  $g' : \text{Sh}(\mathcal{C}/U) \rightarrow \text{Sh}(\mathcal{D}/u(U))$  is the morphism of topoi induced by the cocontinuous functor  $u'$ .

#### 41. Constant sheaves

Let  $E$  be a set and let  $\mathcal{C}$  be a site. We will denote  $\underline{E}$  the constant sheaf with value  $E$  on  $\mathcal{C}$ . If  $E$  is an abelian group, ring, module, etc, then  $\underline{E}$  is a sheaf of abelian groups, rings, modules, etc.

**Lemma 41.1.** *Let  $\mathcal{C}$  be a site. If  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence of abelian groups, then  $0 \rightarrow \underline{A} \rightarrow \underline{B} \rightarrow \underline{C} \rightarrow 0$  is an exact sequence of abelian sheaves and in fact it is even exact as a sequence of abelian presheaves.*

**Proof.** Since sheafification is exact it is clear that  $0 \rightarrow \underline{A} \rightarrow \underline{B} \rightarrow \underline{C} \rightarrow 0$  is an exact sequence of abelian sheaves. Thus  $0 \rightarrow \underline{A} \rightarrow \underline{B} \rightarrow \underline{C}$  is an exact sequence of abelian presheaves. To see that  $\underline{B} \rightarrow \underline{C}$  is surjective, pick a set theoretical section  $s : C \rightarrow B$ . This induces a section  $\underline{s} : \underline{C} \rightarrow \underline{B}$  of sheaves of sets left inverse to the surjection  $\underline{B} \rightarrow \underline{C}$ .  $\square$

**Lemma 41.2.** *Let  $\mathcal{C}$  be a site. Let  $\Lambda$  be a ring and let  $M$  and  $Q$  be  $\Lambda$ -modules. If  $Q$  is a finitely presented  $\Lambda$ -module, then we have  $\underline{M \otimes_{\Lambda} Q}(U) = \underline{M}(U) \otimes_{\Lambda} Q$  for all  $U \in \text{Ob}(\mathcal{C})$ .*

**Proof.** Choose a presentation  $\Lambda^{\oplus m} \rightarrow \Lambda^{\oplus n} \rightarrow Q \rightarrow 0$ . This gives an exact sequence  $M^{\oplus m} \rightarrow M^{\oplus n} \rightarrow M \otimes Q \rightarrow 0$ . By Lemma 41.1 we obtain an exact sequence

$$\underline{M}(U)^{\oplus m} \rightarrow \underline{M}(U)^{\oplus n} \rightarrow \underline{M \otimes Q}(U) \rightarrow 0$$

which proves the lemma. (Note that taking sections over  $U$  always commutes with finite direct sums, but not arbitrary direct sums.)  $\square$

**Lemma 41.3.** *Let  $\mathcal{C}$  be a site. Let  $\Lambda$  be a coherent ring. Let  $M$  be a flat  $\Lambda$ -module. For  $U \in \text{Ob}(\mathcal{C})$  the module  $\underline{M}(U)$  is a flat  $\Lambda$ -module.*

**Proof.** Let  $I \subset \Lambda$  be a finitely generated ideal. By Algebra, Lemma 38.4 it suffices to show that  $\underline{M}(U) \otimes_{\Lambda} I \rightarrow \underline{M}(U)$  is injective. As  $\Lambda$  is coherent  $I$  is finitely presented as a  $\Lambda$ -module. By Lemma 41.2 we see that  $\underline{M}(U) \otimes I = \underline{M} \otimes I$ . Since  $M$  is flat the map  $M \otimes I \rightarrow M$  is injective, whence  $\underline{M} \otimes I \rightarrow \underline{M}$  is injective.  $\square$

**Lemma 41.4.** *Let  $\mathcal{C}$  be a site. Let  $\Lambda$  be a Noetherian ring. Let  $I \subset \Lambda$  be an ideal. The sheaf  $\underline{\Lambda}^{\wedge} = \lim \underline{\Lambda}/I^n$  is a flat  $\underline{\Lambda}$ -algebra. Moreover we have canonical identifications*

$$\underline{\Lambda}/I\underline{\Lambda} = \underline{\Lambda}/\underline{I} = \underline{\Lambda}^{\wedge}/I\underline{\Lambda}^{\wedge} = \underline{\Lambda}^{\wedge}/\underline{I} \cdot \underline{\Lambda}^{\wedge} = \underline{\Lambda}^{\wedge}/\underline{I}^{\wedge} = \underline{\Lambda}/\underline{I}$$

where  $\underline{I}^{\wedge} = \lim \underline{I}/I^n$ .

**Proof.** To prove  $\underline{\Lambda}^{\wedge}$  is flat, it suffices to show that  $\underline{\Lambda}^{\wedge}(U)$  is flat as a  $\Lambda$ -module for each  $U \in \text{Ob}(\mathcal{C})$ , see Lemmas 28.2 and 28.3. By Lemma 41.3 we see that

$$\underline{\Lambda}^{\wedge}(U) = \lim \underline{\Lambda}/I^n(U)$$

is a limit of a system of flat  $\Lambda/I^n$ -modules. By Lemma 41.1 we see that the transition maps are surjective. We conclude by More on Algebra, Lemma 19.4.

To see the equalities, note that  $\underline{\Lambda}(U)/I\underline{\Lambda}(U) = \underline{\Lambda}/I(U)$  by Lemma 41.2. It follows that  $\underline{\Lambda}/I\underline{\Lambda} = \underline{\Lambda}/\underline{I} = \underline{\Lambda}/\underline{I}$ . The system of short exact sequences

$$0 \rightarrow \underline{I}/I^n(U) \rightarrow \underline{\Lambda}/I^n(U) \rightarrow \underline{\Lambda}/I(U) \rightarrow 0$$

has surjective transition maps, hence gives a short exact sequence

$$0 \rightarrow \lim \underline{I}/I^n(U) \rightarrow \lim \underline{\Lambda}/I^n(U) \rightarrow \lim \underline{\Lambda}/I(U) \rightarrow 0$$

see Homology, Lemma 27.3. Thus we see that  $\underline{\Lambda}^{\wedge}/\underline{I}^{\wedge} = \underline{\Lambda}/\underline{I}$ . Since

$$I\underline{\Lambda}^{\wedge} \subset \underline{I} \cdot \underline{\Lambda}^{\wedge} \subset \underline{I}^{\wedge}$$

it suffices to show that  $I\underline{\Lambda}^{\wedge}(U) = \underline{I}^{\wedge}(U)$  for all  $U$ . Choose generators  $I = (f_1, \dots, f_r)$ . This gives a short exact sequence  $0 \rightarrow K \rightarrow \Lambda^{\oplus r} \rightarrow I \rightarrow 0$ . We obtain short exact sequences

$$0 \rightarrow (K \cap I^n)/I^n K(U) \rightarrow (\Lambda/I^n)^{\oplus r}(U) \rightarrow \underline{I}/I^n(U) \rightarrow 0$$

By Artin-Rees (Algebra, Lemma 49.2) the system of modules on the left hand side has ML. (It is zero as a pro-object.) Thus we see that  $(\underline{\Lambda}^{\wedge})^{\oplus r}(U) \rightarrow \underline{I}^{\wedge}(U)$  is surjective by Homology, Lemma 27.3 which is what we wanted to show.  $\square$

**Lemma 41.5.** *Let  $\mathcal{C}$  be a site. Let  $\Lambda$  be a ring and let  $M$  be a  $\Lambda$ -module. Assume  $\text{Sh}(\mathcal{C})$  is not the empty topos. Then*

- (1)  $\underline{M}$  is a finite type sheaf of  $\underline{\Lambda}$ -modules if and only if  $M$  is a finite  $\Lambda$ -module, and
- (2)  $\underline{M}$  is a finitely presented sheaf of  $\underline{\Lambda}$ -modules if and only if  $M$  is a finitely presented  $\Lambda$ -module.

**Proof.** Proof of (1). If  $M$  is generated by  $x_1, \dots, x_r$  then  $x_1, \dots, x_r$  define global sections of  $\underline{M}$  which generate it, hence  $\underline{M}$  is of finite type. Conversely, assume  $\underline{M}$  is of finite type. Let  $U \in \mathcal{C}$  be an object which is not sheaf theoretically empty (Sites, Definition 41.1). Such an object exists as we assumed  $Sh(\mathcal{C})$  is not the empty topos. Then there exists a covering  $\{U_i \rightarrow U\}$  and finitely many sections  $s_{ij} \in \underline{M}(U_i)$  generating  $\underline{M}|_{U_i}$ . After refining the covering we may assume that  $s_{ij}$  come from elements  $x_{ij}$  of  $M$ . Then  $x_{ij}$  define global sections of  $\underline{M}$  whose restriction to  $U$  generate  $\underline{M}$ .

Assume there exist elements  $x_1, \dots, x_r$  of  $M$  which define global sections of  $\underline{M}$  generating  $\underline{M}$  as a sheaf of  $\underline{\Lambda}$ -modules. We will show that  $x_1, \dots, x_r$  generate  $M$  as a  $\Lambda$ -module. Let  $x \in M$ . We can find a covering  $\{U_i \rightarrow U\}_{i \in I}$  and  $f_{i,j} \in \underline{\Lambda}(U_i)$  such that  $x|_{U_i} = \sum f_{i,j} x_j|_{U_i}$ . After refining the covering we may assume  $f_{i,j} \in \underline{\Lambda}$ . Since  $U$  is not sheaf theoretically empty we see that  $I \neq \emptyset$ . Thus we can pick  $i \in I$  and we see that  $x = \sum f_{i,j} x_j$  in  $M$  as desired.

Proof of (2). Assume  $\underline{M}$  is a  $\underline{\Lambda}$ -module of finite presentation. By (1) we see that  $M$  is of finite type. Choose generators  $x_1, \dots, x_r$  of  $M$  as a  $\Lambda$ -module. This determines a short exact sequence  $0 \rightarrow K \rightarrow \Lambda^{\oplus r} \rightarrow M \rightarrow 0$  which turns into a short exact sequence

$$0 \rightarrow \underline{K} \rightarrow \underline{\Lambda}^{\oplus r} \rightarrow \underline{M} \rightarrow 0$$

by Lemma 41.1. By Lemma 24.1 we see that  $\underline{K}$  is of finite type. Hence  $K$  is a finite  $\Lambda$ -module by (1). Thus  $M$  is a  $\Lambda$ -module of finite presentation.  $\square$

## 42. Locally constant sheaves

Here is the general definition.

**Definition 42.1.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{F}$  be a sheaf of sets, groups, abelian groups, rings, modules over a fixed ring  $\Lambda$ , etc.

- (1) We say  $\mathcal{F}$  is a *constant sheaf* of sets, groups, abelian groups, rings, modules over a fixed ring  $\Lambda$ , etc if it is isomorphic as a sheaf of sets, groups, abelian groups, rings, modules over a fixed ring  $\Lambda$ , etc to a constant sheaf  $\underline{E}$  as in Section 41.
- (2) We say  $\mathcal{F}$  is *locally constant* if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}$  such that  $\mathcal{F}|_{U_i}$  is a constant sheaf.
- (3) If  $\mathcal{F}$  is a sheaf of sets or groups, then we say  $\mathcal{F}$  is *finite locally constant* if the constant values are finite sets or finite groups.

**Lemma 42.2.** Let  $f : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be a morphism of topoi. If  $\mathcal{G}$  is a locally constant sheaf of sets, groups, abelian groups, rings, modules over a fixed ring  $\Lambda$ , etc on  $\mathcal{D}$ , the same is true for  $f^{-1}\mathcal{G}$  on  $\mathcal{C}$ .

**Proof.** Omitted.  $\square$

**Lemma 42.3.** Let  $\mathcal{C}$  be a site with a final object  $X$ .

- (1) Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a map of locally constant sheaves of sets on  $\mathcal{C}$ . If  $\mathcal{F}$  is finite locally constant, there exists a covering  $\{U_i \rightarrow X\}$  such that  $\varphi|_{U_i}$  is the map of constant sheaves associated to a map of sets.
- (2) Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a map of locally constant sheaves of abelian groups on  $\mathcal{C}$ . If  $\mathcal{F}$  is finite locally constant, there exists a covering  $\{U_i \rightarrow X\}$  such that

$\varphi|_{U_i}$  is the map of constant abelian sheaves associated to a map of abelian groups.

- (3) Let  $\Lambda$  be a ring. Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a map of locally constant sheaves of  $\Lambda$ -modules on  $\mathcal{C}$ . If  $\mathcal{F}$  is of finite type, then there exists a covering  $\{U_i \rightarrow X\}$  such that  $\varphi|_{U_i}$  is the map of constant sheaves of  $\Lambda$ -modules associated to a map of  $\Lambda$ -modules.

**Proof.** Proof omitted.  $\square$

**Lemma 42.4.** Let  $\mathcal{C}$  be a site. Let  $\Lambda$  be a ring. Let  $M, N$  be  $\Lambda$ -modules. Let  $\mathcal{F}, \mathcal{G}$  be a locally constant sheaves of  $\Lambda$ -modules.

- (1) If  $M$  is of finite presentation, then

$$\underline{\mathrm{Hom}}_{\Lambda}(M, N) = \mathcal{H}om_{\Lambda}(\underline{M}, \underline{N})$$

- (2) If  $M$  and  $N$  are both of finite presentation, then

$$\underline{\mathrm{Isom}}_{\Lambda}(M, N) = \underline{\mathrm{Isom}}_{\Lambda}(\underline{M}, \underline{N})$$

- (3) If  $\mathcal{F}$  is of finite presentation, then  $\mathcal{H}om_{\Lambda}(\mathcal{F}, \mathcal{G})$  is a locally constant sheaf of  $\Lambda$ -modules.  
(4) If  $\mathcal{F}$  and  $\mathcal{G}$  are both of finite presentation, then  $\underline{\mathrm{Isom}}_{\Lambda}(\mathcal{F}, \mathcal{G})$  is a locally constant sheaf of sets.

**Proof.** Proof of (1). Set  $E = \mathrm{Hom}_{\Lambda}(M, N)$ . We want to show the canonical map

$$\underline{E} \longrightarrow \mathcal{H}om_{\Lambda}(\underline{M}, \underline{N})$$

is an isomorphism. The module  $M$  has a presentation  $\Lambda^{\oplus s} \rightarrow \Lambda^{\oplus t} \rightarrow M \rightarrow 0$ . Then  $E$  sits in an exact sequence

$$0 \rightarrow E \rightarrow \mathrm{Hom}_{\Lambda}(\Lambda^{\oplus t}, N) \rightarrow \mathrm{Hom}_{\Lambda}(\Lambda^{\oplus s}, N)$$

and we have similarly

$$0 \rightarrow \mathcal{H}om_{\Lambda}(\underline{M}, \underline{N}) \rightarrow \mathcal{H}om_{\Lambda}(\underline{\Lambda}^{\oplus t}, \underline{N}) \rightarrow \mathcal{H}om_{\Lambda}(\underline{\Lambda}^{\oplus s}, \underline{N})$$

This reduces the question to the case where  $M$  is a finite free module where the result is clear.

Proof of (3). The question is local on  $\mathcal{C}$ , hence we may assume  $\mathcal{F} = \underline{M}$  and  $\mathcal{G} = \underline{N}$  for some  $\Lambda$ -modules  $M$  and  $N$ . By Lemma 41.5 the module  $M$  is of finite presentation. Thus the result follows from (1).

Parts (2) and (4) follow from parts (1) and (3) and the fact that  $\underline{\mathrm{Isom}}$  can be viewed as the subsheaf of sections of  $\mathcal{H}om_{\Lambda}(\mathcal{F}, \mathcal{G})$  which have an inverse in  $\mathcal{H}om_{\Lambda}(\mathcal{G}, \mathcal{F})$ .  $\square$

**Lemma 42.5.** Let  $\mathcal{C}$  be a site.

- (1) The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside  $\mathrm{Sh}(\mathcal{C})$ .  
(2) The category of finite locally constant abelian sheaves is a weak Serre subcategory of  $\mathrm{Ab}(\mathcal{C})$ .  
(3) Let  $\Lambda$  be a Noetherian ring. The category of finite type, locally constant sheaves of  $\Lambda$ -modules on  $\mathcal{C}$  is a weak Serre subcategory of  $\mathrm{Mod}(\mathcal{C}, \Lambda)$ .



**Proof.** Proof of (1). We may work locally on  $\mathcal{C}$ . Hence by Lemma 42.3 we may assume we are given a finite diagram of finite sets such that our diagram of sheaves is the associated diagram of constant sheaves. Then we just take the limit or colimit in the category of sets and take the associated constant sheaf. Some details omitted.

To prove (2) and (3) we use the criterion of Homology, Lemma 9.3. Existence of kernels and cokernels is argued in the same way as above. Of course, the reason for using a Noetherian ring in (3) is to assure us that the kernel of a map of finite  $\Lambda$ -modules is a finite  $\Lambda$ -module. To see that the category is closed under extensions (in the case of sheaves  $\Lambda$ -modules), assume given an extension of sheaves of  $\Lambda$ -modules

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0$$

on  $\mathcal{C}$  with  $\mathcal{F}, \mathcal{G}$  finite type and locally constant. Localizing on  $\mathcal{C}$  we may assume  $\mathcal{F}$  and  $\mathcal{G}$  are constant, i.e., we get

$$0 \rightarrow \underline{M} \rightarrow \mathcal{E} \rightarrow \underline{N} \rightarrow 0$$

for some  $\Lambda$ -modules  $M, N$ . Choose generators  $y_1, \dots, y_m$  of  $N$ , so that we get a short exact sequence  $0 \rightarrow K \rightarrow \Lambda^{\oplus m} \rightarrow N \rightarrow 0$  of  $\Lambda$ -modules. Localizing further we may assume  $y_j$  lifts to a section  $s_j$  of  $\mathcal{E}$ . Thus we see that  $\mathcal{E}$  is a pushout as in the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underline{K} & \longrightarrow & \underline{\Lambda^{\oplus m}} & \longrightarrow & \underline{N} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{M} & \longrightarrow & \mathcal{E} & \longrightarrow & \underline{N} \longrightarrow 0 \end{array}$$

By Lemma 42.3 again (and the fact that  $K$  is a finite  $\Lambda$ -module as  $\Lambda$  is Noetherian) we see that the map  $\underline{K} \rightarrow \underline{M}$  is locally constant, hence we conclude.  $\square$

**Lemma 42.6.** *Let  $\mathcal{C}$  be a site. Let  $\Lambda$  be a ring. The tensor product of two locally constant sheaves of  $\Lambda$ -modules on  $\mathcal{C}$  is a locally constant sheaf of  $\Lambda$ -modules.*

**Proof.** Omitted.  $\square$

### 43. Other chapters

#### Preliminaries

- (1) Introduction
- (2) Conventions
- (3) Set Theory
- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces
- (7) Sites and Sheaves
- (8) Stacks
- (9) Fields
- (10) Commutative Algebra
- (11) Brauer Groups
- (12) Homological Algebra
- (13) Derived Categories
- (14) Simplicial Methods

- (15) More on Algebra
- (16) Smoothing Ring Maps
- (17) Sheaves of Modules
- (18) Modules on Sites
- (19) Injectives
- (20) Cohomology of Sheaves
- (21) Cohomology on Sites
- (22) Differential Graded Algebra
- (23) Divided Power Algebra
- (24) Hypercoverings

#### Schemes

- (25) Schemes
- (26) Constructions of Schemes
- (27) Properties of Schemes
- (28) Morphisms of Schemes

- |                                     |                                     |
|-------------------------------------|-------------------------------------|
| (29) Cohomology of Schemes          | (61) More on Groupoids in Spaces    |
| (30) Divisors                       | (62) Bootstrap                      |
| (31) Limits of Schemes              | Topics in Geometry                  |
| (32) Varieties                      | (63) Quotients of Groupoids         |
| (33) Topologies on Schemes          | (64) Simplicial Spaces              |
| (34) Descent                        | (65) Formal Algebraic Spaces        |
| (35) Derived Categories of Schemes  | (66) Restricted Power Series        |
| (36) More on Morphisms              | (67) Resolution of Surfaces         |
| (37) More on Flatness               | Deformation Theory                  |
| (38) Groupoid Schemes               | (68) Formal Deformation Theory      |
| (39) More on Groupoid Schemes       | (69) Deformation Theory             |
| (40) Étale Morphisms of Schemes     | (70) The Cotangent Complex          |
| Topics in Scheme Theory             | Algebraic Stacks                    |
| (41) Chow Homology                  | (71) Algebraic Stacks               |
| (42) Adequate Modules               | (72) Examples of Stacks             |
| (43) Dualizing Complexes            | (73) Sheaves on Algebraic Stacks    |
| (44) Étale Cohomology               | (74) Criteria for Representability  |
| (45) Crystalline Cohomology         | (75) Artin's Axioms                 |
| (46) Pro-étale Cohomology           | (76) Quot and Hilbert Spaces        |
| Algebraic Spaces                    | (77) Properties of Algebraic Stacks |
| (47) Algebraic Spaces               | (78) Morphisms of Algebraic Stacks  |
| (48) Properties of Algebraic Spaces | (79) Cohomology of Algebraic Stacks |
| (49) Morphisms of Algebraic Spaces  | (80) Derived Categories of Stacks   |
| (50) Decent Algebraic Spaces        | (81) Introducing Algebraic Stacks   |
| (51) Cohomology of Algebraic Spaces | Miscellany                          |
| (52) Limits of Algebraic Spaces     | (82) Examples                       |
| (53) Divisors on Algebraic Spaces   | (83) Exercises                      |
| (54) Algebraic Spaces over Fields   | (84) Guide to Literature            |
| (55) Topologies on Algebraic Spaces | (85) Desirables                     |
| (56) Descent and Algebraic Spaces   | (86) Coding Style                   |
| (57) Derived Categories of Spaces   | (87) Obsolete                       |
| (58) More on Morphisms of Spaces    | (88) GNU Free Documentation License |
| (59) Pushouts of Algebraic Spaces   | (89) Auto Generated Index           |
| (60) Groupoids in Algebraic Spaces  |                                     |

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