SIMPLICIAL METHODS

Contents

1. Introduction

This is a minimal introduction to simplicial methods. We just add here whenever something is needed later on. A general reference to this material is perhaps [GJ99]. An example of the things you can do is the paper by Quillen on Homotopical Algebra, see [Qui67] or the paper on Étale Homotopy by Artin and Mazur, see [AM69].

2. The category of finite ordered sets

The category Δ is the category with

  (1) objects $[0], [1], [2], \ldots$ with $[n] = \{0, 1, 2, \ldots, n\}$ and
  (2) a morphism $[n] \to [m]$ is a nondecreasing map $\{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, m\}$ between the corresponding sets.

Here *nondecreasing* for a map $\varphi : [n] \to [m]$ means by definition that $\varphi(i) \geq \varphi(j)$ if $i \geq j$. In other words, Δ is a category equivalent to the "big" category of finite totally ordered sets and nondecreasing maps. There are exactly $n + 1$ morphisms $[0] \to [n]$ and there is exactly 1 morphism $[n] \to [0]$. There are exactly $(n + 1)(n + 2)/2$ morphisms $[1] \to [n]$ and there are exactly $n + 2$ morphisms $[n] \to [1]$. And so on and so forth.

Definition 2.1. For any integer $n \geq 1$, and any $0 \leq j \leq n$ we let $\delta_j^n : [n-1] \to [n]$ denote the injective order preserving map skipping j. For any integer $n \geq 0$, and any $0 \leq j \leq n$ we denote $\sigma_j^n : [n+1] \to [n]$ the surjective order preserving map with $(\sigma_j^n)^{-1}(\{j\}) = \{j, j+1\}$.

Lemma 2.2. *Any morphism in Δ can be written as a composition of the morphisms δ_j^n and σ_j^n.*

Proof. Let $\varphi : [n] \to [m]$ be a morphism of Δ. If $j \notin \mathrm{Im}(\varphi)$, then we can write φ as $\delta_j^m \circ \psi$ for some morphism $\psi : [n] \to [m-1]$. If $\varphi(j) = \varphi(j+1)$ then we can write φ as $\psi \circ \sigma_j^{n-1}$ for some morphism $\psi : [n-1] \to [m]$. The result follows because each replacement as above lowers $n+m$ and hence at some point φ is both injective and surjective, hence an identity morphism. $\square$

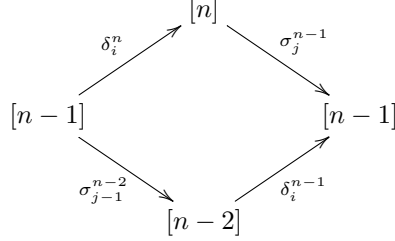**Lemma 2.3.** *The morphisms δ_j^n and σ_j^n satisfy the following relations.*

  (1) *If $0 \leq i < j \leq n+1$, then $\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n$. In other words the diagram*

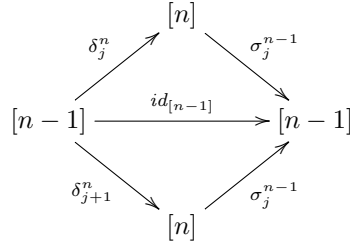 *commutes.*

(2) If $0 \leq i < j \leq n-1$, then $\sigma_j^{n-1} \circ \delta_i^n = \delta_i^{n-1} \circ \sigma_{j-1}^{n-2}$. In other words the diagram

$$
\begin{array}{ccc}
 & [n] & \\
\delta_i^n \nearrow & & \searrow \sigma_j^{n-1} \\
[n-1] & & [n-1] \\
\sigma_{j-1}^{n-2} \searrow & & \nearrow \delta_i^{n-1} \\
 & [n-2] &
\end{array}
$$

commutes.

(3) If $0 \leq j \leq n-1$, then $\sigma_j^{n-1} \circ \delta_j^n = id_{[n-1]}$ and $\sigma_j^{n-1} \circ \delta_{j+1}^n = id_{[n-1]}$. In other words the diagram

$$
\begin{array}{ccc}
 & [n] & \\
\delta_j^n \nearrow & & \searrow \sigma_j^{n-1} \\
[n-1] & \xrightarrow{\ id_{[n-1]}\ } & [n-1] \\
\delta_{j+1}^n \searrow & & \nearrow \sigma_j^{n-1} \\
 & [n] &
\end{array}
$$

commutes.

(4) If $0 < j+1 < i \leq n$, then $\sigma_j^{n-1} \circ \delta_i^n = \delta_{i-1}^{n-1} \circ \sigma_j^{n-2}$. In other words the diagram

$$
\begin{array}{ccc}
 & [n] & \\
\delta_i^n \nearrow & & \searrow \sigma_j^{n-1} \\
[n-1] & & [n-1] \\
\sigma_j^{n-2} \searrow & & \nearrow \delta_{i-1}^{n-1} \\
 & [n-2] &
\end{array}
$$
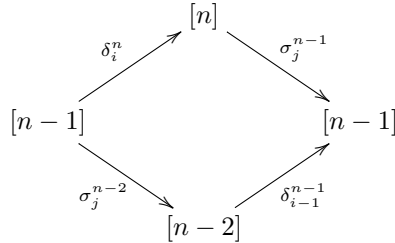
commutes.

(5) If $0 \leq i \leq j \leq n-1$, then $\sigma_j^{n-1} \circ \sigma_i^n = \sigma_i^{n-1} \circ \sigma_{j+1}^n$. In other words the diagram

$$
\begin{array}{ccc}
 & [n] & \\
\sigma_i^n \nearrow & & \searrow \sigma_j^{n-1} \\
[n+1] & & [n-1] \\
\sigma_{j+1}^n \searrow & & \nearrow \sigma_i^{n-1} \\
 & [n] &
\end{array}
$$
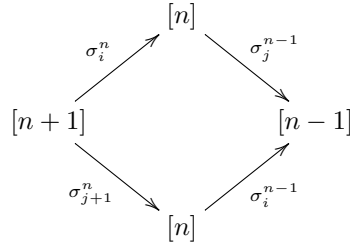
commutes.

Proof. Omitted. $\square$

Lemma 2.4. *The category Δ is the universal category with objects $[n]$, $n \geq 0$ and morphisms δ_j^n and σ_j^n such that (a) every morphism is a composition of these*

morphisms, (b) the relations listed in Lemma 2.3 are satisfied, and (c) any relation among the morphisms is a consequence of those relations.

Proof. Omitted. $\square$

3. Simplicial objects

Definition 3.1. Let $\mathcal{C}$ be a category.

(1) A *simplicial object U of $\mathcal{C}$* is a contravariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta^{opp} \longrightarrow \mathcal{C}$$
(2) If $\mathcal{C}$ is the category of sets, then we call U a *simplicial set*.
(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a *simplicial abelian group*.
(4) A *morphism of simplicial objects $U \to U'$* is a transformation of functors.
(5) The *category of simplicial objects of $\mathcal{C}$* is denoted $\mathrm{Simp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([n]) \to U([m])$, satisfying $U(\varphi \circ \psi) = U(\psi) \circ U(\varphi)$.

In particular there is a unique morphism $U([0]) \to U([n])$ and there are exactly $n+1$ morphisms $U([n]) \to U([0])$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 2 above.

Lemma 3.2. *Let $\mathcal{C}$ be a category.*

(1) *Given a simplicial object U in $\mathcal{C}$ we obtain a sequence of objects $U_n = U([n])$ endowed with the morphisms $d_j^n = U(\delta_j^n) : U_n \to U_{n-1}$ and $s_j^n = U(\sigma_j^n) : U_n \to U_{n+1}$. These morphisms satisfy the opposites of the relations displayed in Lemma 2.3.*
(2) *Conversely, given a sequence of objects U_n and morphisms d_j^n, s_j^n satisfying these relations there exists a unique simplicial object U in $\mathcal{C}$ such that $U_n = U([n])$, $d_j^n = U(\delta_j^n)$, and $s_j^n = U(\sigma_j^n)$.*
(3) *A morphism between simplicial objects U and U' is given by a family of morphisms $U_n \to U_n'$ commuting with the morphisms d_j^n and s_j^n.*

Proof. This follows from Lemma 2.4. $\square$

Remark 3.3. By abuse of notation we sometimes write $d_i : U_n \to U_{n-1}$ instead of d_i^n, and similarly for $s_i : U_n \to U_{n+1}$. The relations among the morphisms d_i^n and s_i^n may be expressed as follows:

(1) If $i < j$, then $d_i \circ d_j = d_{j-1} \circ d_i$.
(2) If $i < j$, then $d_i \circ s_j = s_{j-1} \circ d_i$.
(3) We have $\mathrm{id} = d_j \circ s_j = d_{j+1} \circ s_j$.
(4) If $i > j + 1$, then $d_i \circ s_j = s_j \circ d_{i-1}$.
(5) If $i \leq j$, then $s_i \circ s_j = s_{j+1} \circ s_i$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $s_0^0 = U(\sigma_0^0) : U_0 \to U_1$ and two morphisms $d_0^1 = U(\delta_0^1)$, and $d_1^1 = U(\delta_1^1)$ which are morphisms $U_1 \to U_0$. There are two morphisms $s_0^1 = U(\sigma_0^1)$, $s_1^1 = U(\sigma_1^1)$ which are morphisms $U_1 \to U_2$. Three morphisms $d_0^2 = U(\delta_0^2)$, $d_1^2 = U(\delta_1^2)$, $d_2^2 = U(\delta_2^2)$ which are morphisms $U_3 \to U_2$. And so on.

Pictorially we think of U as follows:

$$U_2 \underset{\longleftarrow}{\overset{\longrightarrow}{\rightleftarrows}} U_1 \overset{\longleftarrow}{\underset{\longrightarrow}{\longleftarrow}} U_0$$

Here the d-morphisms are the arrows pointing right and the s-morphisms are the arrows pointing left.

Example 3.4. The simplest example is the *constant* simplicial object with value $X \in \mathrm{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

Example 3.5. Suppose that $Y \to X$ is a morphism of $\mathcal{C}$ such that all the fibred products $Y \times_X Y \times_X \ldots \times_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold fibre product, and we let $\varphi : [n] \to [m]$ correspond to the map (on "coordinates") $(y_0, \ldots, y_m) \mapsto (y_{\varphi(0)}, \ldots, y_{\varphi(n)})$. In other words, the map $U_0 = Y \to U_1 = Y \times_X Y$ is the diagonal map. The two maps $U_1 = Y \times_X Y \to U_0 = Y$ are the projection maps.

Geometrically Example 3.5 above is an important example. It tells us that it is a good idea to think of the maps $d_j^n : U_n \to U_{n-1}$ as projection maps (forgetting the jth component), and to think of the maps $s_j^n : U_n \to U_{n+1}$ as diagonal maps (repeating the jth coordinate). We will return to this in the sections below.

Lemma 3.6. *Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. Each of the morphisms $s_i^n : U_n \to U_{n+1}$ has a left inverse. In particular s_i^n is a monomorphism.*

Proof. This is true because $d_i^{n+1} \circ s_i^n = \mathrm{id}_{U_n}$. $\qquad\square$

4. Simplicial objects as presheaves

Another observation is that we may think of a simplicial object of $\mathcal{C}$ as a presheaf with values in $\mathcal{C}$ over Δ. See Sites, Definition 2.2. And in fact, if U, U' are simplicial objects of $\mathcal{C}$, then we have

(4.0.1) $$\mathrm{Mor}(U, U') = \mathrm{Mor}_{PSh(\Delta)}(U, U').$$

Some of the material below could be replaced by the more general constructions in the chapter on sites. However, it seems a clearer picture arises from the arguments specific to simplicial objects.

5. Cosimplicial objects

A cosimplicial object of a category $\mathcal{C}$ could be defined simply as a simplicial object of the opposite category $\mathcal{C}^{opp}$. This is not really how the human brain works, so we introduce them separately here and point out some simple properties.

Definition 5.1. Let $\mathcal{C}$ be a category.
  (1) A *cosimplicial object U of $\mathcal{C}$* is a covariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta \longrightarrow \mathcal{C}$$
  (2) If $\mathcal{C}$ is the category of sets, then we call U a *cosimplicial set*.

(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a *cosimplicial abelian group*.
(4) A *morphism of cosimplicial objects* $U \to U'$ is a transformation of functors.
(5) The *category of cosimplicial objects of* $\mathcal{C}$ is denoted $\mathrm{CoSimp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([m]) \to U([n])$, satisfying $U(\varphi \circ \psi) = U(\varphi) \circ U(\psi)$.

In particular there is a unique morphism $U([n]) \to U([0])$ and there are exactly $n+1$ morphisms $U([0]) \to U([n])$ corresponding to the $n + 1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 2 above.

Lemma 5.2. *Let $\mathcal{C}$ be a category.*

(1) *Given a cosimplicial object U in $\mathcal{C}$ we obtain a sequence of objects $U_n = U([n])$ endowed with the morphisms $\delta_j^n = U(\delta_j^n) : U_{n-1} \to U_n$ and $\sigma_j^n = U(\sigma_j^n) : U_{n+1} \to U_n$. These morphisms satisfy the relations displayed in Lemma 2.3.*
(2) *Conversely, given a sequence of objects U_n and morphisms δ_j^n, σ_j^n satisfying these relations there exists a unique cosimplicial object U in $\mathcal{C}$ such that $U_n = U([n])$, $\delta_j^n = U(\delta_j^n)$, and $\sigma_j^n = U(\sigma_j^n)$.*
(3) *A morphism between cosimplicial objects U and U' is given by a family of morphisms $U_n \to U'_n$ commuting with the morphisms δ_j^n and σ_j^n.*

Proof. This follows from Lemma 2.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 5.3. By abuse of notation we sometimes write $\delta_i : U_{n-1} \to U_n$ instead of δ_i^n, and similarly for $\sigma_i : U_{n+1} \to U_n$. The relations among the morphisms δ_i^n and σ_i^n may be expressed as follows:

(1) If $i < j$, then $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$.
(2) If $i < j$, then $\sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1}$.
(3) We have $\mathrm{id} = \sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1}$.
(4) If $i > j + 1$, then $\sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j$.
(5) If $i \leq j$, then $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $\sigma_0^0 = U(\sigma_0^0) : U_1 \to U_0$ and two morphisms $\delta_0^1 = U(\delta_0^1)$, and $\delta_1^1 = U(\delta_1^1)$ which are morphisms $U_0 \to U_1$. There are two morphisms $\sigma_0^1 = U(\sigma_0^1)$, $\sigma_1^1 = U(\sigma_1^1)$ which are morphisms $U_2 \to U_1$. Three morphisms $\delta_0^2 = U(\delta_0^2)$, $\delta_1^2 = U(\delta_1^2)$, $\delta_2^2 = U(\delta_2^2)$ which are morphisms $U_2 \to U_3$. And so on.

Pictorially we think of U as follows:

$$U_0 \rightrightarrows U_1 \rightrightarrows U_2$$

Here the δ-morphisms are the arrows pointing right and the σ-morphisms are the arrows pointing left.

Example 5.4. The simplest example is the *constant* cosimplicial object with value $X \in \mathrm{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

Example 5.5. Suppose that $Y \to X$ is a morphism of C such that all the pushouts $Y \coprod_X Y \coprod_X \ldots \coprod_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold pushout, and we let $\varphi : [n] \to [m]$ correspond to the map

$$(y \text{ in } i\text{th component}) \mapsto (y \text{ in } \varphi(i)\text{th component})$$

on "coordinates". In other words, the map $U_1 = Y \coprod_X Y \to U_0 = Y$ is the identity on each component. The two maps $U_0 = Y \to U_1 = Y \coprod_X Y$ are the two natural maps.

Lemma 5.6. *Let $\mathcal{C}$ be a category. Let U be a cosimplicial object of $\mathcal{C}$. Each of the morphisms $\delta_i^n : U_{n-1} \to U_n$ has a left inverse. In particular δ_i^n is a monomorphism.*

Proof. This is true because $\sigma_i^{n-1} \circ \delta_i^n = \mathrm{id}_{U_n}$ for $j < n$. $\qquad\square$

6. Products of simplicial objects

Of course we should define the product of simplicial objects as the product in the category of simplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

Definition 6.1. Let $\mathcal{C}$ be a category. Let U and V be simplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The *product of U and V* is the simplicial object $U \times V$ defined as follows:

  (1) $(U \times V)_n = U_n \times V_n$,
  (2) $d_i^n = (d_i^n, d_i^n)$, and
  (3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $U \times V$ is the product of the presheaves U and V on Δ.

Lemma 6.2. *If U and V are simplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have*

$$\mathrm{Mor}(W, U \times V) = \mathrm{Mor}(W, U) \times \mathrm{Mor}(W, V)$$

for any third simplicial object W of $\mathcal{C}$.

Proof. Omitted. $\qquad\square$

7. Fibre products of simplicial objects

Of course we should define the fibre product of simplicial objects as the fibre product in the category of simplicial objects. This may lead to the potentially confusing situation where the fibre product exists but is not described as below. To avoid this we define the fibre product directly as follows.

Definition 7.1. Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : V \to U$, $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The *fibre product of V and W over U* is the simplicial object $V \times_U W$ defined as follows:

  (1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
  (2) $d_i^n = (d_i^n, d_i^n)$, and
  (3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \times_U W$ is the fibre product of the presheaves V and W over the presheaf U on Δ.

Lemma 7.2. *If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have*

$$\mathrm{Mor}(T, V \times_U W) = \mathrm{Mor}(T, V) \times_{\mathrm{Mor}(T,U)} \mathrm{Mor}(T, W)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. □

8. Pushouts of simplicial objects

Of course we should define the pushout of simplicial objects as the pushout in the category of simplicial objects. This may lead to the potentially confusing situation where the pushouts exist but are not as described below. To avoid this we define the pushout directly as follows.

Definition 8.1. Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : U \to V$, $b : U \to W$ be morphisms. Assume the pushouts $V_n \amalg_{U_n} W_n$ exist in $\mathcal{C}$. The *pushout of V and W over U* is the simplicial object $V \amalg_U W$ defined as follows:

(1) $(V \amalg_U W)_n = V_n \amalg_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \amalg_U W$ is the pushout of the presheaves V and W over the presheaf U on Δ.

Lemma 8.2. *If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : U \to V$, $b : U \to W$ are morphisms and if $V \amalg_U W$ exists, then we have*

$$\mathrm{Mor}(V \amalg_U W, T) = \mathrm{Mor}(V, T) \times_{\mathrm{Mor}(U,T)} \mathrm{Mor}(W, T)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. □

9. Products of cosimplicial objects

Of course we should define the product of cosimplicial objects as the product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

Definition 9.1. Let $\mathcal{C}$ be a category. Let U and V be cosimplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The *product of U and V* is the cosimplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(U \times V)(\varphi) : U_n \times V_n \to U_m \times V_m$ is the product $U(\varphi) \times V(\varphi)$.

Lemma 9.2. *If U and V are cosimplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have*

$$\mathrm{Mor}(W, U \times V) = \mathrm{Mor}(W, U) \times \mathrm{Mor}(W, V)$$

for any third cosimplicial object W of $\mathcal{C}$.

Proof. Omitted. □

10. Fibre products of cosimplicial objects

Of course we should define the fibre product of cosimplicial objects as the fibre product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the fibre product directly as follows.

Definition 10.1. Let $\mathcal{C}$ be a category. Let U, V, W be cosimplicial objects of $\mathcal{C}$. Let $a : V \to U$ and $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The *fibre product of V and W over U* is the cosimplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(V \times_U W)(\varphi) : V_n \times_{U_n} W_n \to V_m \times_{U_m} W_m$ is the product $V(\varphi) \times_{U(\varphi)} W(\varphi)$.

Lemma 10.2. *If U, V, W are cosimplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have*

$$\mathrm{Mor}(T, V \times_U W) = \mathrm{Mor}(T, V) \times_{\mathrm{Mor}(T,U)} \mathrm{Mor}(T, W)$$

for any fourth cosimplicial object T of $\mathcal{C}$.

Proof. Omitted. $\square$

11. Simplicial sets

Let U be a simplicial set. It is a good idea to think of U_0 as the 0-*simplices*, the set U_1 as the 1-*simplices*, the set U_2 as the 2-*simplices*, and so on.

We think of the maps $s_j^n : U_n \to U_{n+1}$ as the map that associates to an n-simplex A the degenerate $(n+1)$-simplex B whose $(j, j+1)$-edge is collapsed to the vertex j of A. We think of the map $d_j^n : U_n \to U_{n-1}$ as the map that associates to an n-simplex A one of the faces, namely the face that omits the vertex j. In this way it become possible to visualize the relations among the maps s_j^n and d_j^n geometrically.

Definition 11.1. Let U be a simplicial set. We say x is an *n-simplex of U* to signify that x is an element of U_n. We say that y is the $j$the *face of x* to signify that $d_j^n x = y$. We say that z is the jth *degeneracy of x* if $z = s_j^n x$. A simplex is called *degenerate* if it is the degeneracy of another simplex.

Here are a few fundamental examples.

Example 11.2. For every $n \geq 0$ we denote $\Delta[n]$ the simplicial set

$$\Delta^{opp} \longrightarrow Sets$$

$$[k] \longmapsto \mathrm{Mor}_\Delta([k], [n])$$

We leave it to the reader to verify the following statements. Every m-simplex of $\Delta[n]$ with $m > n$ is degenerate. There is a unique nondegenerate n-simplex of $\Delta[n]$, namely $\mathrm{id}_{[n]}$.

Lemma 11.3. *Let U be a simplicial set. Let $n \geq 0$ be an integer. There is a canonical bijection*

$$\mathrm{Mor}(\Delta[n], U) \longrightarrow U_n$$

which maps a morphism φ to the value of φ on the unique nondegenerate n-simplex of $\Delta[n]$.

Proof. Omitted. $\qquad \square$

Example 11.4. Consider the category $\Delta/[n]$ of objects over $[n]$ in Δ, see Categories, Example 2.13. There is a functor $p : \Delta/[n] \to \Delta$. The fibre category of p over $[k]$, see Categories, Section 33, has as objects the set $\Delta[n]_k$ of k-simplices in $\Delta[n]$, and as morphisms only identities. For every morphism $\varphi : [k] \to [l]$ of Δ, and every object $\psi : [l] \to [n]$ in the fibre category over $[l]$ there is a unique object over $[k]$ with a morphism covering φ, namely $\psi \circ \varphi : [k] \to [n]$. Thus $\Delta/[n]$ is fibred in sets over Δ. In other words, we may think of $\Delta/[n]$ as a presheaf of sets over Δ. See also, Categories, Example 36.7. And this presheaf of sets agrees with the simplicial set $\Delta[n]$. In particular, from Equation (4.0.1) and Lemma 11.3 above we get the formula

$$\mathrm{Mor}_{PSh(\Delta)}(\Delta/[n], U) = U_n$$

for any simplicial set U.

Lemma 11.5. *Let U, V be simplicial sets. Let $a, b \geq 0$ be integers. Assume every n-simplex of U is degenerate if $n > a$. Assume every n-simplex of V is degenerate if $n > b$. Then every n-simplex of $U \times V$ is degenerate if $n > a + b$.*

Proof. Suppose $n > a + b$. Let $(u, v) \in (U \times V)_n = U_n \times V_n$. By assumption, there exists a $\alpha : [n] \to [a]$ and a $u' \in U_a$ and a $\beta : [n] \to [b]$ and a $v' \in V_b$ such that $u = U(\alpha)(u')$ and $v = V(\beta)(v')$. Because $n > a + b$, there exists an $0 \leq i \leq a + b$ such that $\alpha(i) = \alpha(i+1)$ and $\beta(i) = \beta(i+1)$. It follows immediately that (u, v) is in the image of s_i^{n-1}. $\qquad \square$

12. Truncated simplicial objects and skeleton functors

Let $\Delta_{\leq n}$ denote the full subcategory of Δ with objects $[0], [1], [2], \ldots, [n]$. Let $\mathcal{C}$ be a category.

Definition 12.1. An *n-truncated simplicial object of $\mathcal{C}$* is a contravariant functor from $\Delta_{\leq n}$ to $\mathcal{C}$. A *morphism of n-truncated simplicial objects* is a transformation of functors. We denote the category of n-truncated simplicial objects of $\mathcal{C}$ by the symbol $\mathrm{Simp}_n(\mathcal{C})$.

Given a simplicial object U of $\mathcal{C}$ the truncation $\mathrm{sk}_n U$ is the restriction of U to the subcategory $\Delta_{\leq n}$. This defines a *skeleton functor*

$$\mathrm{sk}_n : \mathrm{Simp}(\mathcal{C}) \longrightarrow \mathrm{Simp}_n(\mathcal{C})$$

from the category of simplicial objects of $\mathcal{C}$ to the category of n-truncated simplicial objects of $\mathcal{C}$. See Remark 20.6 to avoid possible confusion with other functors in the literature.

13. Products with simplicial sets

Let $\mathcal{C}$ be a category. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. We can consider the covariant functor which associates to a simplicial object W of $\mathcal{C}$ the set

(13.0.1)
$$\left\{ (f_{n,u} : V_n \to W_n)_{n \geq 0, u \in U_n} \text{ such that } \begin{array}{c} \forall \varphi : [m] \to [n] \\ f_{m, U(\varphi)(u)} \circ V(\varphi) = W(\varphi) \circ f_{n,u} \end{array} \right\}$$

If this functor is of the form $\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(Q, -)$ then we can think of Q as the product of U with V. Instead of formalizing this in this way we just directly define the product as follows.

Definition 13.1. Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. In this case we define the *product $U \times V$ of U and V* to be the simplicial object of $\mathcal{C}$ whose nth term is the object

$$(U \times V)_n = \coprod_{u \in U_n} V_n$$

with maps for $\varphi : [m] \to [n]$ given by the morphism

$$\coprod_{u \in U_n} V_n \longrightarrow \coprod_{u' \in U_m} V_m$$

which maps the component V_n corresponding to u to the component V_m corresponding to $u' = U(\varphi)(u)$ via the morphism $V(\varphi)$. More loosely, if all of the coproducts displayed above exist (without assuming anything about $\mathcal{C}$) we will say that the *product $U \times V$ exists.*

Lemma 13.2. *Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. The functor $W \mapsto \mathrm{Mor}_{Simp(\mathcal{C})}(U \times V, W)$ is canonically isomorphic to the functor which maps W to the set in Equation (13.0.1).*

Proof. Omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 13.3. *Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let us temporarily denote FSSets the category of simplicial sets all of whose components are finite nonempty.*

(1) *The rule $(U, V) \mapsto U \times V$ defines a functor $FSSets \times Simp(\mathcal{C}) \to Simp(\mathcal{C})$.*
(2) *For every U, V as above there is a canonical map of simplicial objects*

$$U \times V \longrightarrow V$$

defined by taking the identity on each component of $(U \times V)_n = \coprod_u V_n$.

Proof. Omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We briefly study a special case of the construction above. Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $k \geq 0$ be an integer. If all coproducts $X \coprod \ldots \coprod X$ exist then according to the definition above the product

$$X \times \Delta[k]$$

exists, where we think of X as the corresponding constant simplicial object.

Lemma 13.4. *With X and k as above. For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection*

$$\mathrm{Mor}_{Simp(\mathcal{C})}(X \times \Delta[k], V) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, V_k).$$

wich maps γ to the restriction of the morphism γ_k to the component corresponding to $\mathrm{id}_{[k]}$. Similarly, for any $n \geq k$, if W is an n-truncated simplicial object of $\mathcal{C}$, then we have

$$\mathrm{Mor}_{Simp_n(\mathcal{C})}(sk_n(X \times \Delta[k]), W) = \mathrm{Mor}_{\mathcal{C}}(X, W_k).$$

Proof. A morphism $\gamma : X \times \Delta[k] \to V$ is given by a family of morphisms $\gamma_\alpha :$ $X \to V_n$ where $\alpha : [n] \to [k]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{\ \gamma_\alpha\ } & V_n \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & \big\downarrow{\scriptstyle V(\varphi)} \\
X & \xrightarrow{\ \gamma_{\alpha\circ\varphi}\ } & V_m
\end{array}
$$

commute. Taking $\alpha = \mathrm{id}_{[k]}$, we see that for any $\varphi : [m] \to [k]$ we have $\gamma_\varphi = V(\varphi)\circ\gamma_{\mathrm{id}_{[k]}}$. Thus the morphism γ is determined by the value of γ on the component corresponding to $\mathrm{id}_{[k]}$. Conversely, given such a morphism $f : X \to V_k$ we easily construct a morphism γ by putting $\gamma_\alpha = V(\alpha)\circ f$.

The truncated case is similar, and left to the reader. $\qquad\qquad\qquad\qquad\square$

A particular example of this is the case $k = 0$. In this case the formula of the lemma just says that

$$
\mathrm{Mor}_\mathcal{C}(X, V_0) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X, V)
$$

where on the right hand side X indicates the constant simplicial object with value X. We will use this formula without further mention in the following.

14. Hom from simplicial sets into cosimplicial objects

Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$, and let V be a cosimplicial object of $\mathcal{C}$. Then we get a cosimplicial set $\mathrm{Hom}_\mathcal{C}(U, V)$ as follows:

(1) we set $\mathrm{Hom}_\mathcal{C}(U, V)_n = \mathrm{Mor}_\mathcal{C}(U_n, V_n)$, and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}_\mathcal{C}(U, V)_m \to \mathrm{Hom}_\mathcal{C}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.

This is our motivation for the following definition.

Definition 14.1. Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object of $\mathcal{C}$. Let U be a simplicial set such that each U_n is finite nonempty. We define $\mathrm{Hom}(U, V)$ to be the cosimplicial object of $\mathcal{C}$ defined as follows:

(1) we set $\mathrm{Hom}(U, V)_n = \prod_{u \in U_n} V_n$, in other words the unique object of $\mathcal{C}$ such that its X-valued points satisfy

$$
\mathrm{Mor}_\mathcal{C}(X, \mathrm{Hom}(U, V)_n) = \mathrm{Map}(U_n, \mathrm{Mor}_\mathcal{C}(X, V_n))
$$

and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}(U, V)_m \to \mathrm{Hom}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$ on X-valued points as above.

We leave it to the reader to spell out the definition in terms of maps between products. We also point out that the construction is functorial in both U (contravariantly) and V (covariantly), exactly as in Lemma 13.3 in the case of products of simplicial sets with simplicial objects.

15. Internal Hom

Let $\mathcal{C}$ be a category with finite nonempty products. Let U, V be simplicial objects $\mathcal{C}$. In some cases the functor

$$\begin{aligned}
\mathrm{Simp}(\mathcal{C})^{opp} &\longrightarrow Sets \\
W &\longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times V, U)
\end{aligned}$$

is representable. In this case we denote $\mathcal{H}om(V, U)$ the resulting simplicial object of $\mathcal{C}$, and we say that the *internal hom of V into U exists.* Moreover, in this case we would have

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{C}}(X, \mathcal{H}om(V, U)_n) &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times \Delta[n], \mathcal{H}om(V, U)) \\
&= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times \Delta[n] \times V, U) \\
&= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X, \mathcal{H}om(\Delta[n] \times V, U)) \\
&= \mathrm{Mor}_{\mathcal{C}}(X, \mathcal{H}om(\Delta[n] \times V, U)_0)
\end{aligned}$$

provided that $\mathcal{H}om(\Delta[n] \times V, U)$ exists also. Here we have used the material from Section 13.

The lesson we learn from this is that, given U and V, if we want to construct the internal hom then we should try to construct the objects

$$\mathcal{H}om(\Delta[n] \times V, U)_0$$

because these should be the nth term of $\mathcal{H}om(V, U)$. In the next section we study a construction of simplicial objects "$\mathrm{Hom}(\Delta[n], U)$".

16. Hom from simplicial sets into simplicial objects

Motivated by the discussion on internal hom we define what should be the simplicial object classifying morphisms from a simplicial set into a given simplicial object of the category $\mathcal{C}$.

Definition 16.1. Let $\mathcal{C}$ be a category such that the coproduct of any two objects exists. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. We denote $\mathrm{Hom}(U, V)$ any simplicial object of $\mathcal{C}$ such that

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W, \mathrm{Hom}(U, V)) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times U, V)$$

functorially in the simplicial object W of $\mathcal{C}$.

Of course $\mathrm{Hom}(U, V)$ need not exist. Also, by the discussion in Section 15 we expect that if it does exist, then $\mathrm{Hom}(U, V)_n = \mathrm{Hom}(U \times \Delta[n], V)_0$. We do not use the italic notation for these Hom objects since $\mathrm{Hom}(U, V)$ is not an internal hom.

Lemma 16.2. *Assume the category $\mathcal{C}$ has coproducts of any two objects and countable limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor*

$$\begin{aligned}
\mathcal{C}^{opp} &\longrightarrow Sets \\
X &\longmapsto \mathrm{Mor}_{Simp(\mathcal{C})}(X \times U, V)
\end{aligned}$$

is representable.

Proof. A morphism from $X \times U$ into V is given by a collection of morphisms $f_u : X \to V_n$ with $n \geq 0$ and $u \in U_n$. And such a collection actually defines a morphism if and only if for all $\varphi : [m] \to [n]$ all the diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{f_u} & V_n \\
\mathrm{id}_X \downarrow & & \downarrow V(\varphi) \\
X & \xrightarrow{f_{U(\varphi)(u)}} & V_m
\end{array}
$$

commute. Thus it is natural to introduce a category $\mathcal{U}$ and a functor $\mathcal{V} : \mathcal{U}^{opp} \to \mathcal{C}$ as follows:

(1) The set of objects of $\mathcal{U}$ is $\coprod_{n \geq 0} U_n$,
(2) a morphism from $u' \in U_m$ to $u \in U_n$ is a $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$
(3) for $u \in U_n$ we set $\mathcal{V}(u) = V_n$, and
(4) for $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$ we set $\mathcal{V}(\varphi) = V(\varphi) : V_n \to V_m$.

At this point it is clear that our functor is nothing but the functor defining

$$\lim_{\mathcal{U}^{opp}} \mathcal{V}$$

Thus if $\mathcal{C}$ has countable limits then this limit and hence an object representing the functor of the lemma exist. $\square$

Lemma 16.3. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor*

$$
\begin{array}{ccc}
\mathcal{C}^{opp} & \longrightarrow & Sets \\
X & \longmapsto & \mathrm{Mor}_{Simp(\mathcal{C})}(X \times U, V)
\end{array}
$$

is representable.

Proof. We have to show that the category $\mathcal{U}$ described in the proof of Lemma 16.2 has a finite subcategory $\mathcal{U}'$ such that the limit of $\mathcal{V}$ over $\mathcal{U}'$ is the same as the limit of $\mathcal{V}$ over $\mathcal{U}$. We will use Categories, Lemma 17.4. For $m > 0$ let $\mathcal{U}_{\leq m}$ denote the full subcategory with objects $\coprod_{0 \leq n \leq m} U_m$. Let m_0 be an integer such that every n-simplex of the simplicial set U is degenerate if $n > m_0$. For any $m \geq m_0$ large enough, the subcategory $\mathcal{U}_{\leq m}$ satisfies property (1) of Categories, Definition 17.3.

Suppose that $u \in U_n$ and $u' \in U_{n'}$ with $n, n' \leq m_0$ and suppose that $\varphi : [k] \to [n]$, $\varphi' : [k] \to [n']$ are morphisms such that $U(\varphi)(u) = U(\varphi')(u')$. A simple combinatorial argument shows that if $k > 2m_0$, then there exists an index $0 \leq i \leq 2m_0$ such that $\varphi(i) = \varphi(i+1)$ and $\varphi'(i) = \varphi'(i+1)$. (The pigeon hole principle would tell you this works if $k > m_0^2$ which is good enough for the argument below anyways.) Hence, if $k > 2m_0$, we may write $\varphi = \psi \circ \sigma_i^{k-1}$ and $\varphi' = \psi' \circ \sigma_i^{k-1}$ for some $\psi : [k-1] \to [n]$ and some $\psi' : [k-1] \to [n']$. Since $s_i^{k-1} : U_{k-1} \to U_k$ is injective, see Lemma 3.6, we conclude that $U(\psi)(u) = U(\psi')(u')$ also. Continuing in this fashion we conclude that given morphisms $u \to z$ and $u' \to z$ of $\mathcal{U}$ with

$u, u' \in \mathcal{U}_{\leq m_0}$, there exists a commutative diagram
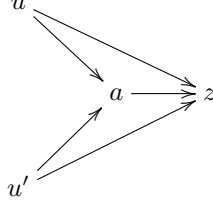
with $a \in \mathcal{U}_{\leq 2m_0}$.

It is easy to deduce from this that the finite subcategory $\mathcal{U}_{\leq 2m_0}$ works. Namely, suppose given $x' \in U_n$ and $x'' \in U_{n'}$ with $n, n' \leq 2m_0$ as well as morphisms $x' \to x$ and $x'' \to x$ of $\mathcal{U}$ with the same target. By our choice of m_0 we can find objects u, u' of $\mathcal{U}_{\leq m_0}$ and morphisms $u \to x'$, $u' \to x''$. By the above we can find $a \in \mathcal{U}_{\leq 2m_0}$ and morphisms $u \to a$, $u' \to a$ such that
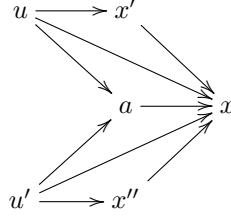
is commutative. Turning this diagram 90 degrees clockwise we get the desired diagram as in (2) of Categories, Definition 17.3. $\square$

Lemma 16.4. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then $\mathrm{Hom}(U, V)$ exists, moreover we have the expected equalities*

$$\mathrm{Hom}(U, V)_n = \mathrm{Hom}(U \times \Delta[n], V)_0.$$

Proof. We construct this simplicial object as follows. For $n \geq 0$ let $\mathrm{Hom}(U, V)_n$ denote the object of $\mathcal{C}$ representing the functor

$$X \longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times U \times \Delta[n], V)$$

This exists by Lemma 16.3 because $U \times \Delta[n]$ is a simplicial set with finite sets of simplices and no nondegenerate simplices in high enough degree, see Lemma 11.5. For $\varphi : [m] \to [n]$ we obtain an induced map of simplicial sets $\varphi : \Delta[m] \to \Delta[n]$. Hence we obtain a morphism $X \times U \times \Delta[m] \to X \times U \times \Delta[n]$ functorial in X, and hence a transformation of functors, which in turn gives

$$\mathrm{Hom}(U, V)(\varphi) : \mathrm{Hom}(U, V)_n \longrightarrow \mathrm{Hom}(U, V)_m.$$

Clearly this defines a contravariant functor $\mathrm{Hom}(U, V)$ from Δ into the category $\mathcal{C}$. In other words, we have a simplicial object of $\mathcal{C}$.

We have to show that $\mathrm{Hom}(U, V)$ satisfies the desired universal property

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W, \mathrm{Hom}(U, V)) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times U, V)$$

To see this, let $f : W \to \mathrm{Hom}(U, V)$ be given. We want to construct the element $f' : W \times U \to V$ of the right hand side. By construction, each $f_n : W_n \to \mathrm{Hom}(U, V)_n$

corresponds to a morphism $f_n : W_n \times U \times \Delta[n] \to V$. Further, for every morphism $\varphi : [m] \to [n]$ the diagram

$$
\begin{array}{ccc}
W_n \times U \times \Delta[m] & \xrightarrow{\ \ W(\varphi) \times \mathrm{id} \times \mathrm{id}\ \ } & W_m \times U \times \Delta[m] \\
{\scriptstyle \mathrm{id} \times \mathrm{id} \times \varphi} \downarrow & & \downarrow {\scriptstyle f_m} \\
W_n \times U \times \Delta[n] & \xrightarrow{\qquad f_n \qquad} & V
\end{array}
$$

is commutative. For $\psi : [n] \to [k]$ in $(\Delta[n])_k$ we denote $(f_n)_{k,\psi} : W_n \times U_k \to V_k$ the component of $(f_n)_k$ corresponding to the element ψ. We define $f_n' : W_n \times U_n \to V_n$ as $f_n' = (f_n)_{n,\mathrm{id}}$, in other words, as the restriction of $(f_n)_n : W_n \times U_n \times (\Delta[n])_n \to V_n$ to $W_n \times U_n \times \mathrm{id}_{[n]}$. To see that the collection (f_n') defines a morphism of simplicial objects, we have to show for any $\varphi : [m] \to [n]$ that $V(\varphi) \circ f_n' = f_m' \circ W(\varphi) \times U(\varphi)$. The commutative diagram above says that $(f_n)_{m,\varphi} : W_n \times U_m \to V_m$ is equal to $(f_m)_{m,\mathrm{id}} \circ W(\varphi) : W_n \times U_m \to V_m$. But then the fact that f_n is a morphism of simplicial objects implies that the diagram

$$
\begin{array}{ccc}
W_n \times U_n \times (\Delta[n])_n & \xrightarrow{\ (f_n)_n\ } & V_n \\
{\scriptstyle \mathrm{id} \times U(\varphi) \times \varphi} \downarrow & & \downarrow {\scriptstyle V(\varphi)} \\
W_n \times U_m \times (\Delta[n])_m & \xrightarrow{\ (f_n)_m\ } & V_m
\end{array}
$$

is commutative. And this implies that $(f_n)_{m,\varphi} \circ U(\varphi)$ is equal to $V(\varphi) \circ (f_n)_{n,\mathrm{id}}$. Altogether we obtain $V(\varphi) \circ (f_n)_{n,\mathrm{id}} = (f_n)_{m,\varphi} \circ U(\varphi) = (f_m)_{m,\mathrm{id}} \circ W(\varphi) \circ U(\varphi) = (f_m)_{m,\mathrm{id}} \circ W(\varphi) \times U(\varphi)$ as desired.

On the other hand, given a morphism $f' : W \times U \to V$ we define a morphism $f : W \to \mathrm{Hom}(U, V)$ as follows. By Lemma 13.4 the morphisms $\mathrm{id} : W_n \to W_n$ corresponds to a unique morphism $c_n : W_n \times \Delta[n] \to W$. Hence we can consider the composition

$$
W_n \times \Delta[n] \times U \xrightarrow{\ c_n\ } W \times U \xrightarrow{\ f'\ } V.
$$

By construction this corresponds to a unique morphism $f_n : W_n \to \mathrm{Hom}(U, V)_n$. We leave it to the reader to see that these define a morphism of simplicial sets as desired.

We also leave it to the reader to see that $f \mapsto f'$ and $f' \mapsto f$ are mutually inverse operations. $\qquad\square$

We spell out the construction above in a special case. Let X be an object of a category $\mathcal{C}$. Assume that self products $X \times \ldots \times X$ exist. Let k be an integer. Consider the simplicial object U with terms

$$
U_n = \prod\nolimits_{\alpha \in \mathrm{Mor}([k],[n])} X
$$

and maps given $\varphi : [m] \to [n]$

$$
\begin{array}{rccc}
U(\varphi) : \prod_{\alpha \in \mathrm{Mor}([k],[n])} X & \longrightarrow & \prod_{\alpha' \in \mathrm{Mor}([k],[m])} X \\
(f_\alpha)_\alpha & \longmapsto & (f_{\varphi \circ \alpha'})_{\alpha'}
\end{array}
$$

In terms of "coordinates", the element $(x_\alpha)_\alpha$ is mapped to the element $(x_{\varphi \circ \alpha'})_{\alpha'}$. We claim this object is equal to

$$
\mathrm{Hom}(\Delta[k], X)
$$

where we think of X as the constant simplicial object X.

Lemma 16.5. *With X, k and U as above.*

(1) *For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection*

$$\operatorname{Mor}_{Simp(\mathcal{C})}(V, U) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(V_k, X).$$

wich maps γ to the morphism γ_k composed with the projection onto the factor corresponding to $\operatorname{id}_{[k]}$.

(2) *Similarly, if W is an k-truncated simplicial object of $\mathcal{C}$, then we have*

$$\operatorname{Mor}_{Simp_k(\mathcal{C})}(W, sk_k U) = \operatorname{Mor}_{\mathcal{C}}(W_k, X).$$

(3) *The object U constructed above is an incarnation of $\operatorname{Hom}(\Delta[k], X)$.*

Proof. We first prove (1). Suppose that $\gamma : V \to U$ is a morphism. This is given by a family of morphisms $\gamma_\alpha : V_n \to X$ for $\gamma : [k] \to [n]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams

$$
\begin{array}{ccc}
X & \xleftarrow{\ \gamma_{\varphi \circ \alpha'}\ } & V_n \\
{\scriptstyle \operatorname{id}_X} \downarrow & & \downarrow {\scriptstyle V(\varphi)} \\
X & \xleftarrow{\ \gamma_{\alpha'}\ } & V_m
\end{array}
$$

commute for all $\alpha' : [k] \to [m]$. Taking $\alpha' = \operatorname{id}_{[k]}$, we see that for any $\varphi : [k] \to [n]$ we have $\gamma_\varphi = \gamma_{\operatorname{id}_{[k]}} \circ V(\varphi)$. Thus the morphism γ is determined by the component of γ_k corresponding to $\operatorname{id}_{[k]}$. Conversely, given such a morphism $f : V_k \to X$ we easily construct a morphism γ by putting $\gamma_\alpha = f \circ V(\alpha)$.

The truncated case is similar, and left to the reader.

To see (3) we argue as follows:

$$
\begin{array}{rcl}
\operatorname{Mor}(V, \operatorname{Hom}(\Delta[k], X)) & = & \operatorname{Mor}(V \times \Delta[k], X) \\
& = & \{(f_n : V_n \times \Delta[k]_n \to X) \mid f_n \text{ compatible}\} \\
& = & \{(f_n : V_n \to \prod_{\Delta[k]_n} X) \mid f_n \text{ compatible}\} \\
& = & \operatorname{Mor}(V, U)
\end{array}
$$

Thus U and $\operatorname{Hom}(\Delta[k], X)$ define the same functor on the category of simplicial objects and hence are canonically isomorphic. $\square$

Lemma 16.6. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let $a : U \to V$, $b : U \to W$ be morphisms of simplicial sets. Assume U_n, V_n, W_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U, V, W are degenerate for all $n \gg 0$. Let T be a simplicial object of $\mathcal{C}$. Then*

$$\operatorname{Hom}(V, T) \times_{\operatorname{Hom}(U, T)} \operatorname{Hom}(W, T) = \operatorname{Hom}(V \amalg_U W, T)$$

In other words, the fibre product on the left hand side is represented by the Hom object on the right hand side.

Proof. By Lemma 16.4 all the required Hom objects exist and satisfy the correct functorial properties. Now we can identify the nth term on the left hand side as the

object representing the functor that associates to X the first set of the following sequence of functorial equalities

$\mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(V, T) \times_{\mathrm{Hom}(U,T)} \mathrm{Hom}(W, T))$

$= \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(V, T)) \times_{\mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(U,T))} \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(W, T))$

$= \mathrm{Mor}(X \times \Delta[n] \times V, T) \times_{\mathrm{Mor}(X \times \Delta[n] \times U, T)} \mathrm{Mor}(X \times \Delta[n] \times W, T)$

$= \mathrm{Mor}(X \times \Delta[n] \times (V \amalg_U W), T))$

Here we have used the fact that

$$(X \times \Delta[n] \times V) \times_{X \times \Delta[n] \times U} (X \times \Delta[n] \times W) = X \times \Delta[n] \times (V \amalg_U W)$$

which is easy to verify term by term. The result of the lemma follows as the last term in the displayed sequence of equalities corresponds to $\mathrm{Hom}(V \amalg_U W, T)_n$. $\square$

17. Splitting simplicial objects

A subobject N of an object X of the category $\mathcal{C}$ is an object N of $\mathcal{C}$ together with a monomorphism $N \to X$. Of course we say (by abuse of notation) that the subobjects N, N' are equal if there exists an isomorphism $N \to N'$ compatible with the morphisms to X. The collection of subobjects forms a partially ordered set. (Because of our conventions on categories; not true for category of spaces up to homotopy for example.)

Definition 17.1. Let $\mathcal{C}$ be a category which admits finite nonempty coproducts. We say a simplicial object U of $\mathcal{C}$ is *split* if there exist subobjects $N(U_m)$ of U_m, $m \geq 0$ with the property that

(17.1.1) $$\coprod_{\varphi:[n] \to [m] \text{ surjective}} N(U_m) \longrightarrow U_n$$

is an isomorphism for all $n \geq 0$.

If this is the case, then $N(U_0) = U_0$. Next, we have $U_1 = U_0 \coprod N(U_1)$. Second we have

$$U_2 = U_0 \coprod N(U_1) \coprod N(U_1) \coprod N(U_2).$$

It turns out that in many categories $\mathcal{C}$ every simplicial object is split.

Lemma 17.2. *Let U be a simplicial set. Then U has a splitting with $N(U_m)$ equal to the set of nondegenerate m-simplices.*

Proof. Let $x \in U_n$. Suppose that there are surjections $\varphi : [n] \to [k]$ and $\psi : [n] \to [l]$ and nondegenerate simplices $y \in U_k$, $z \in U_l$ such that $x = U(\varphi)(y)$ and $x = U(\psi)(z)$. Choose a right inverse $\xi : [l] \to [n]$ of ψ, i.e., $\psi \circ \xi = \mathrm{id}_{[l]}$. Then $z = U(\xi)(x)$. Hence $z = U(\xi)(x) = U(\varphi \circ \xi)(y)$. Since z is nondegenerate we conclude that $\varphi \circ \xi : [l] \to [k]$ is surjective, and hence $l \geq k$. Similarly $k \geq l$. Hence we see that $\varphi \circ \xi : [l] \to [k]$ has to be the identity map for any choice of right inverse ξ of ψ. This easily implies that $\psi = \varphi$. $\square$

Of course it can happen that a map of simplicial sets maps a nondegenerate n-simplex to a degenerate n-simplex. Thus the splitting of Lemma 17.2 is not functorial. Here is a case where it is functorial.

Lemma 17.3. *Let $f : U \to V$ be a morphism of simplicial sets. Suppose that (a) the image of every nondegenerate simplex of U is a nondegenerate simplex of V and (b) no two nondegenerate simplices of U are mapped to the same simplex of V. Then f_n is injective for all n. Same holds with "injective" replaced by "surjective" or "bijective".*

Proof. Under hypothesis (a) we see that the map f preserves the disjoint union decompositions of the splitting of Lemma 17.2, in other words that we get commutative diagrams

$$
\begin{array}{ccc}
\coprod_{\varphi:[n]\to[m] \text{ surjective}} N(U_m) & \longrightarrow & U_n \\
\downarrow & & \downarrow \\
\coprod_{\varphi:[n]\to[m] \text{ surjective}} N(V_m) & \longrightarrow & V_n.
\end{array}
$$

And then (b) clearly shows that the left vertical arrow is injective (resp. surjective, resp. bijective). $\square$

Lemma 17.4. *Let U be a simplicial set. Let $n \geq 0$ be an integer. The rule*

$$
U'_m = \bigcup_{\varphi:[m]\to[i], \ i \leq n} Im(U(\varphi))
$$

defines a sub simplicial set $U' \subset U$ with $U'_i = U_i$ for $i \leq n$. Moreover, all m-simplices of U' are degenerate for all $m > n$.

Proof. If $x \in U_m$ and $x = U(\varphi)(y)$ for some $y \in U_i$, $i \leq n$ and some $\varphi : [m] \to [i]$ then any image $U(\psi)(x)$ for any $\psi : [m'] \to [m]$ is equal to $U(\varphi \circ \psi)(y)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial set. By construction all simplices in dimension $n + 1$ and higher are degenerate. $\square$

Lemma 17.5. *Let U be a simplicial abelian group. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \geq 1$ taking*

$$
N(U_m) = \bigcap_{i=0}^{m-1} Ker(d_i^m).
$$

Moreover, this splitting is functorial on the category of simplicial abelian groups.

Proof. By induction on n we will show that the choice of $N(U_m)$ in the lemma guarantees that (17.1.1) is an isomorphism for $m \leq n$. This is clear for $n = 0$. In the rest of this proof we are going to drop the superscripts from the maps d_i and s_i in order to improve readability. We will also repeatedly use the relations from Remark 3.3.

First we make a general remark. For $0 \leq i \leq m$ and $z \in U_m$ we have $d_i(s_i(z)) = z$. Hence we can write any $x \in U_{m+1}$ uniquely as $x = x' + x''$ with $d_i(x') = 0$ and $x'' \in Im(s_i)$ by taking $x' = (x - s_i(d_i(x)))$ and $x'' = s_i(d_i(x))$. Moreover, the element $z \in U_m$ such that $x'' = s_i(z)$ is unique because s_i is injective.

Here is a procedure for decomposing any $x \in U_{n+1}$. First, write $x = x_0 + s_0(z_0)$ with $d_0(x_0) = 0$. Next, write $x_0 = x_1 + s_1(z_1)$ with $d_n(x_1) = 0$. Continue like this

to get

$$\begin{aligned}
x &= x_0 + s_0(z_0),\\
x_0 &= x_1 + s_1(z_1),\\
x_1 &= x_2 + s_2(z_2),\\
\ldots\ &\ldots\ \ldots\\
x_{n-1} &= x_n + s_n(z_n)
\end{aligned}$$

where $d_i(x_i) = 0$ for all $i = n, \ldots, 0$. By our general remark above all of the x_i and z_i are determined uniquely by x. We claim that $x_i \in \mathrm{Ker}(d_0) \cap \mathrm{Ker}(d_1) \cap \ldots \cap \mathrm{Ker}(d_i)$ and $z_i \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$ for $i = n, \ldots, 0$. Here and in the following an empty intersection of kernels indicates the whole space; i.e., the notation $z_0 \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$ when $i = 0$ means $z_0 \in U_n$ with no restriction.

We prove this by ascending induction on i. It is clear for $i = 0$ by construction of x_0 and z_0. Let us prove it for $0 < i \le n$ assuming the result for $i - 1$. First of all we have $d_i(x_i) = 0$ by construction. So pick a j with $0 \le j < i$. We have $d_j(x_{i-1}) = 0$ by induction. Hence

$$0 = d_j(x_{i-1}) = d_j(x_i) + d_j(s_i(z_i)) = d_j(x_i) + s_{i-1}(d_j(z_i)).$$

The last equality by the relations of Remark 3.3. These relations also imply that $d_{i-1}(d_j(x_i)) = d_j(d_i(x_i)) = 0$ because $d_i(x_i) = 0$ by construction. Then the uniqueness in the general remark above shows the equality $0 = x' + x'' = d_j(x_i) + s_{i-1}(d_j(z_i))$ can only hold if both terms are zero. We conclude that $d_j(x_i) = 0$ and by injectivity of s_{i-1} we also conclude that $d_j(z_i) = 0$. This proves the claim.

The claim implies we can uniquely write

$$x = s_0(z_0) + s_1(z_1) + \ldots + s_n(z_n) + x_0$$

with $x_0 \in N(U_{n+1})$ and $z_i \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$. We can reformulate this as saying that we have found a direct sum decomposition

$$U_{n+1} = N(U_{n+1}) \oplus \bigoplus_{i=0}^{i=n} s_i\Big(\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})\Big)$$

with the property that

$$\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_j) = N(U_{n+1}) \oplus \bigoplus_{i=j+1}^{i=n} s_i\Big(\mathrm{Ker}(d_n) \cap \ldots \cap \mathrm{Ker}(d_{i-1})\Big)$$

for $j = 0, \ldots, n$. The result follows from this statement as follows. Each of the z_i in the expression for x can be written uniquely as

$$z_i = s_i(z'_{i,i}) + \ldots + s_{n-1}(z'_{i,n-1}) + z_{i,0}$$

with $z_{i,0} \in N(U_n)$ and $z'_{i,j} \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{j-1})$. The first few steps in the decomposition of z_i are zero because z_i already is in the kernel of $d_0, \ldots, d_i$. This in turn uniquely gives

$$x = x_0 + s_0(z_{0,0}) + s_1(z_{1,0}) + \ldots + s_n(z_{n,0}) + \sum_{0 \le i \le j \le n-1} s_i(s_j(z'_{i,j})).$$

Continuing in this fashion we see that we in the end obtain a decomposition of x as a sum of terms of the form

$$s_{i_1} s_{i_2} \ldots s_{i_k}(z)$$

with $0 \le i_1 \le i_2 \le \ldots \le i_k \le n - k + 1$ and $z \in N(U_{n+1-k})$. This is exactly the required decomposition, because any surjective map $[n+1] \to [n+1-k]$ can be uniquely expressed in the form

$$\sigma_{i_k}^{n-k} \ldots \sigma_{i_2}^{n-1} \sigma_{i_1}^{n}$$

with $0 \le i_1 \le i_2 \le \ldots \le i_k \le n - k + 1$. $\qquad\square$

Lemma 17.6. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \ge 1$ taking*

$$N(U_m) = \bigcap_{i=0}^{m-1} Ker(d_i^m).$$

Moreover, this splitting is functorial on the category of simplicial objects of $\mathcal{A}$.

Proof. For any object A of $\mathcal{A}$ we obtain a simplicial abelian group $\mathrm{Mor}_{\mathcal{A}}(A, U)$. Each of these are canonically split by Lemma 17.5. Moreover,

$$N(\mathrm{Mor}_{\mathcal{A}}(A, U_m)) = \bigcap_{i=0}^{m-1} \mathrm{Ker}(d_i^m) = \mathrm{Mor}_{\mathcal{A}}(A, N(U_m)).$$

Hence we see that the morphism (17.1.1) becomes an isomorphism after applying the functor $\mathrm{Mor}_{\mathcal{A}}(A, -)$ for any object of $\mathcal{A}$. Hence it is an isomorphism by the Yoneda lemma. $\qquad\square$

Lemma 17.7. *Let $\mathcal{A}$ be an abelian category. Let $f : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If the induced morphisms $N(f)_i : N(U)_i \to N(V)_i$ are injective for all i, then f_i is injective for all i. Same holds with "injective" replaced with "surjective", or "isomorphism".*

Proof. This is clear from Lemma 17.6 and the definition of a splitting. $\qquad\square$

Lemma 17.8. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Let $N(U_m)$ as in Lemma 17.6 above. Then $d_m^m(N(U_m)) \subset N(U_{m-1})$.*

Proof. For $j = 0, \ldots, m-2$ we have $d_j^{m-1} d_m^m = d_{m-1}^{m-1} d_j^m$ by the relations in Remark 3.3. The result follows. $\qquad\square$

Lemma 17.9. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \ge 0$ be an integer. The rule*

$$U_m' = \sum_{\varphi:[m]\to[i],\ i\le n} Im(U(\varphi))$$

defines a sub simplicial object $U' \subset U$ with $U_i' = U_i$ for $i \le n$. Moreover, $N(U_m') = 0$ for all $m > n$.

Proof. Pick $m, i \le n$ and some $\varphi : [m] \to [i]$. The image under $U(\psi)$ of $\mathrm{Im}(U(\varphi))$ for any $\psi : [m'] \to [m]$ is equal to the image of $U(\varphi \circ \psi)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial object. Pick $m > n$. We have to show $N(U_m') = 0$. By definition of $N(U_m)$ and $N(U_m')$ we have $N(U_m') = U_m' \cap N(U_m)$ (intersection of subobjects). Since U is split by Lemma 17.6, it suffices to show that U_m' is contained in the sum

$$\sum_{\varphi:[m]\to[m']\ \text{surjective},\ m'<m} \mathrm{Im}(U(\varphi)|_{N(U_{m'})}).$$

By the splitting each $U_{m'}$ is the sum of images of $N(U_{m''})$ via $U(\psi)$ for surjective maps $\psi : [m'] \to [m'']$. Hence the displayed sum above is the same as

$$\sum_{\varphi:[m]\to[m']\ \text{surjective},\ m'<m} \mathrm{Im}(U(\varphi)).$$

Clearly U'_m is contained in this by the simple fact that any $\varphi : [m] \to [i]$, $i \le n$ occurring in the definition of U'_m may be factored as $[m] \to [m'] \to [i]$ with $[m] \to [m']$ surjective and $m' < m$ as in the last displayed sum above. $\qquad\square$

18. Coskeleton functors

Let $\mathcal{C}$ be a category. The *coskeleton functor* (if it exists) is a functor

$$\mathrm{cosk}_n : \mathrm{Simp}_n(\mathcal{C}) \longrightarrow \mathrm{Simp}(\mathcal{C})$$

which is right adjoint to the skeleton functor. In a formula

(18.0.1) $$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(U, \mathrm{cosk}_n V) = \mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(\mathrm{sk}_n U, V)$$

Given a n-truncated simplicial object V we say that $cosk_n V$ *exists* if there exists a $\mathrm{cosk}_n V \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))$ and a morphism $\mathrm{sk}_n \mathrm{cosk}_n V \to V$ such that the displayed formula holds, in other words if the functor $U \mapsto \mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(\mathrm{sk}_n U, V)$ is representable. If it exists it is unique up to unique isomorphism by the Yoneda lemma. See Categories, Section 3.

Example 18.1. Suppose the category $\mathcal{C}$ has finite nonempty self products. A 0-truncated simplicial object of $\mathcal{C}$ is the same as an object X of $\mathcal{C}$. In this case we claim that $\mathrm{cosk}_0(X)$ is the simplicial object U with $U_n = X^{n+1}$ the $(n+1)$-fold self product of X, and structure of simplicial object as in Example 3.5. Namely, a morphism $V \to U$ where V is a simplicial object is given by morphisms $V_n \to X^{n+1}$, such that all the diagrams

$$
\begin{array}{ccc}
V_n & \longrightarrow & X^{n+1} \\
{\scriptstyle V([0]\to[n],0\mapsto i)}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_i} \\
V_0 & \longrightarrow & X
\end{array}
$$

commute. Clearly this means that the map determines and is determined by a unique morphism $V_0 \to X$. This proves that formula (18.0.1) holds.

Recall the category $\Delta/[n]$, see Example 11.4. We let $(\Delta/[n])_{\le m}$ denote the full subcategory of $\Delta/[n]$ consisting of objects $[k] \to [n]$ of $\Delta/[n]$ with $k \le m$. In other words we have the following commutative diagram of categories and functors

$$
\begin{array}{ccc}
(\Delta/[n])_{\le m} & \longrightarrow & \Delta/[n] \\
\downarrow & & \downarrow \\
\Delta_{\le m} & \longrightarrow & \Delta
\end{array}
$$

Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : (\Delta/[n])_{\le m}^{opp} \longrightarrow \mathcal{C}$$

by the rules

$$
\begin{array}{rcl}
([k] \to [n]) & \longmapsto & U_k \\
\psi : ([k'] \to [n]) \to ([k] \to [n]) & \longmapsto & U(\psi) : U_k \to U_{k'}
\end{array}
$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$\overline{\varphi} : (\Delta/[n])_{\le m} \longrightarrow (\Delta/[n'])_{\le m}$$

which maps $\alpha : [k] \to [n]$ to $\varphi \circ \alpha : [k] \to [n']$. The composition $U(n') \circ \overline{\varphi}$ is equal to the functor $U(n)$.

Lemma 18.2. *If the category $\mathcal{C}$ has finite limits, then $cosk_m$ functors exist for all m. Moreover, for any m-truncated simplicial object U the simplicial object $cosk_m U$ is described by the formula*

$$(cosk_m U)_n = \lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $cosk_m U(\varphi)$ comes from the identification $U(n') \circ \overline{\varphi} = U(n)$ above via Categories, Lemma 14.8.

Proof. During the proof of this lemma we denote $cosk_m U$ the simplicial object with $(cosk_m U)_n$ equal to $\lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$. We will conclude at the end of the proof that it does satisfy the required mapping property.

Suppose that V is a simplicial object. A morphism $\gamma : V \to cosk_m U$ is given by a sequence of morphisms $\gamma_n : V_n \to (cosk_m U)_n$. By definition of a limit, this is given by a collection of morphisms $\gamma(\alpha) : V_n \to U_k$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$. These morphisms then also satisfy the rules that

$$
\begin{array}{ccc}
V_n & \xrightarrow{\gamma(\alpha)} & U_k \\
{\scriptstyle V(\varphi)}\big\uparrow & & \big\uparrow{\scriptstyle U(\psi)} \\
V_{n'} & \xrightarrow{\gamma(\alpha')} & U_{k'}
\end{array}
$$

are commutative, given any $0 \leq k, k' \leq m$, $0 \leq n, n'$ and any $\psi : [k] \to [k']$, $\varphi : [n] \to [n']$, $\alpha : [k] \to [n]$ and $\alpha' : [k'] \to [n']$ in Δ such that $\varphi \circ \alpha = \alpha' \circ \psi$. Taking $n = k$, $\varphi = \alpha'$, and $\alpha = \psi = \mathrm{id}_{[k]}$ we deduce that $\gamma(\alpha') = \gamma(\mathrm{id}_{[k]}) \circ V(\alpha')$. In other words, the morphisms $\gamma(\mathrm{id}_{[k]})$, $k \leq m$ determine the morphism γ. And it is easy to see that these morphisms form a morphism $sk_m V \to U$.

Conversely, given a morphism $\gamma : sk_m V \to U$, we obtain a family of morphisms $\gamma(\alpha)$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$ by setting $\gamma(\alpha) = \gamma(\mathrm{id}_{[k]}) \circ V(\alpha)$. These morphisms satisfy all the displayed commutativity restraints pictured above, and hence give rise to a morphism $V \to cosk_m U$. $\square$

Lemma 18.3. *Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For $n \leq m$ the limit $\lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$ exists and is canonically isomorphic to U_n.*

Proof. This is true because the category $(\Delta/[n])_{\leq m}$ has an final object in this case, namely the identity map $[n] \to [n]$. $\square$

Lemma 18.4. *Let $\mathcal{C}$ be a category with finite limits. Let U be an n-truncated simplicial object of $\mathcal{C}$. The morphism $sk_n cosk_n U \to U$ is an isomorphism.*

Proof. Combine Lemmas 18.2 and 18.3. $\square$

Let us describe a particular instance of the coskeleton functor in more detail. By abuse of notation we will denote sk_n also the restriction functor $\mathrm{Simp}_{n'}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$ for any $n' \geq n$. We are going to describe a right adjoint of the functor $sk_n : \mathrm{Simp}_{n+1}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$. For $n \geq 1$, $0 \leq i < j \leq n+1$ define $\delta^{n+1}_{i,j} : [n-1] \to [n+1]$ to be the increasing map omitting i and j. Note that $\delta^{n+1}_{i,j} = \delta^{n+1}_j \circ \delta^n_i = \delta^{n+1}_i \circ \delta^n_{j-1}$, see Lemma 2.3. This motivates the following lemma.

Lemma 18.5. *Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set*

$$\{(f_0, \ldots, f_{n+1}) \in \mathrm{Mor}_{\mathcal{C}}(T, U_n) \mid d_{j-1}^n \circ f_i = d_i^n \circ f_j \ \forall \ 0 \leq i < j \leq n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then

$$U_{n+1} = \lim_{(\Delta/[n+1])_{\leq n}^{opp}} U(n)$$

Proof. The limit, if it exists, represents the functor that associates to an object T the set

$$\{(f_\alpha)_{\alpha:[k]\to[n+1], k\leq n} \mid f_{\alpha\circ\psi} = U(\psi) \circ f_\alpha \ \forall \ \psi : [k'] \to [k], \alpha : [k] \to [n+1]\}.$$

In fact we will show this functor is isomorphic to the one displayed in the lemma. The map in one direction is given by the rule

$$(f_\alpha)_\alpha \longmapsto (f_{\delta_0^{n+1}}, \ldots, f_{\delta_{n+1}^{n+1}}).$$

This satisfies the conditions of the lemma because

$$d_{j-1}^n \circ f_{\delta_i^{n+1}} = f_{\delta_i^{n+1}\circ\delta_{j-1}^n} = f_{\delta_j^{n+1}\circ\delta_i^n} = d_i^n \circ f_{\delta_j^{n+1}}$$

by the relations we recalled above the lemma. To construct a map in the other direction we have to associate to a system $(f_0, \ldots, f_{n+1})$ as in the displayed formula of the lemma a system of maps f_α. Let $\alpha : [k] \to [n+1]$ be given. Since $k \leq n$ the map α is not surjective. Hence we can write $\alpha = \delta_i^{n+1} \circ \psi$ for some $0 \leq i \leq n+1$ and some $\psi : [k] \to [n]$. We have no choice but to define

$$f_\alpha = U(\psi) \circ f_i.$$

Of course we have to check that this is independent of the choice of the pair (i, ψ). First, observe that given i there is a unique ψ which works. Second, suppose that (j, ϕ) is another pair. Then $i \neq j$ and we may assume $i < j$. Since both i, j are not in the image of α we may actually write $\alpha = \delta_{i,j}^{n+1} \circ \xi$ and then we see that $\psi = \delta_{j-1}^n \circ \xi$ and $\phi = \delta_i^n \circ \xi$. Thus

$$
\begin{aligned}
U(\psi) \circ f_i &= U(\delta_{j-1}^n \circ \xi) \circ f_i \\
&= U(\xi) \circ d_{j-1}^n \circ f_i \\
&= U(\xi) \circ d_i^n \circ f_j \\
&= U(\delta_i^n \circ \xi) \circ f_j \\
&= U(\phi) \circ f_j
\end{aligned}
$$

as desired. We still have to verify that the maps f_α so defined satisfy the rules of a system of maps $(f_\alpha)_\alpha$. To see this suppose that $\psi : [k'] \to [k]$, $\alpha : [k] \to [n+1]$ with $k, k' \leq n$. Set $\alpha' = \alpha \circ \psi$. Choose i not in the image of α. Then clearly i is not in the image of α' also. Write $\alpha = \delta_i^{n+1} \circ \phi$ (we cannot use the letter ψ here because we've already used it). Then obviously $\alpha' = \delta_i^{n+1} \circ \phi \circ \psi$. By construction above we then have

$$U(\psi) \circ f_\alpha = U(\psi) \circ U(\phi) \circ f_i = U(\phi \circ \psi) \circ f_i = f_{\alpha\circ\psi} = f_{\alpha'}$$

as desired. We leave to the reader the pleasant task of verifying that our constructions are mutually inverse bijections, and are functorial in T. $\qquad\square$

Lemma 18.6. *Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set*

$$\{(f_0, \ldots, f_{n+1}) \in \mathrm{Mor}_{\mathcal{C}}(T, U_n) \mid d^n_{j-1} \circ f_i = d^n_i \circ f_j \; \forall \; 0 \leq i < j \leq n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then there exists an $(n+1)$-truncated simplicial object $\tilde{U}$, with $sk_n\tilde{U} = U$ and $\tilde{U}_{n+1} = U_{n+1}$ such that the following adjointness holds

$$\mathrm{Mor}_{Simp_{n+1}(\mathcal{C})}(V, \tilde{U}) = \mathrm{Mor}_{Simp_n(\mathcal{C})}(sk_n V, U)$$

Proof. By Lemma 18.3 there are identifications

$$U_i = \lim_{(\Delta/[i])^{opp}_{\leq n}} U(i)$$

for $0 \leq i \leq n$. By Lemma 18.5 we have

$$U_{n+1} = \lim_{(\Delta/[n+1])^{opp}_{\leq n}} U(n).$$

Thus we may define for any $\varphi : [i] \to [j]$ with $i, j \leq n+1$ the corresponding map $\tilde{U}(\varphi) : \tilde{U}_j \to \tilde{U}_i$ exactly as in Lemma 18.2. This defines an $(n+1)$-truncated simplicial object $\tilde{U}$ with $sk_n\tilde{U} = U$.

To see the adjointness we argue as follows. Given any element $\gamma : sk_n V \to U$ of the right hand side of the formula consider the morphisms $f_i = \gamma_n \circ d^{n+1}_i : V_{n+1} \to V_n \to U_n$. These clearly satisfy the relations $d^n_{j-1} \circ f_i = d^n_i \circ f_j$ and hence define a unique morphism $V_{n+1} \to U_{n+1}$ by our choice of U_{n+1}. Conversely, given a morphism $\gamma' : V \to \tilde{U}$ of the left hand side we can simply restrict to $\Delta_{\leq n}$ to get an element of the right hand side. We leave it to the reader to show these are mutually inverse constructions. $\square$

Remark 18.7. Let U, and U_{n+1} be as in Lemma 18.6. On T-valued points we can easily describe the face and degeneracy maps of $\tilde{U}$. Explicitly, the maps $d^{n+1}_i : U_{n+1} \to U_n$ are given by

$$(f_0, \ldots, f_{n+1}) \longmapsto f_i.$$

And the maps $s^n_j : U_n \to U_{n+1}$ are given by

$$\begin{aligned}
f \quad \longmapsto \quad & (s^{n-1}_{j-1} \circ d^{n-1}_0 \circ f, \\
& s^{n-1}_{j-1} \circ d^{n-1}_1 \circ f, \\
& \ldots \\
& s^{n-1}_{j-1} \circ d^{n-1}_{j-1} \circ f, \\
& f, \\
& f, \\
& s^{n-1}_j \circ d^{n-1}_{j+1} \circ f, \\
& s^{n-1}_j \circ d^{n-1}_{j+2} \circ f, \\
& \ldots \\
& s^{n-1}_j \circ d^{n-1}_n \circ f)
\end{aligned}$$

where we leave it to the reader to verify that the RHS is an element of the displayed set of Lemma 18.6. For $n = 0$ there is one map, namely $f \mapsto (f, f)$. For $n = 1$ there are two maps, namely $f \mapsto (f, f, s_0 d_1 f)$ and $f \mapsto (s_0 d_0 f, f, f)$. For $n = 2$

there are three maps, namely $f \mapsto (f, f, s_0 d_1 f, s_0 d_2 f)$, $f \mapsto (s_0 d_0 f, f, f, s_1 d_2 f)$, and $f \mapsto (s_1 d_0 f, s_1 d_1 f, f, f)$. And so on and so forth.

Remark 18.8. The construction of Lemma 18.6 above in the case of simplicial sets is the following. Given an n-truncated simplicial set U, we make a canonical $(n + 1)$-truncated simplicial set $\tilde{U}$ as follows. We add a set of $(n + 1)$-simplices U_{n+1} by the formula of the lemma. Namely, an element of U_{n+1} is a numbered collection of $(f_0, \ldots, f_{n+1})$ of n-simplices, with the property that they glue as they would in a $(n + 1)$-simplex. In other words, the ith face of f_j is the $(j - 1)$st face of f_i for $i < j$. Geometrically it is obvious how to define the face and degeneracy maps for $\tilde{U}$. If V is an $(n + 1)$-truncated simplicial set, then its $(n + 1)$-simplices give rise to compatible collections of n-simplices $(f_0, \ldots, f_{n+1})$ with $f_i \in V_n$. Hence there is a natural map $\mathrm{Mor}(\mathrm{sk}_n V, U) \to \mathrm{Mor}(V, \tilde{U})$ which is inverse to the canonical restriction mapping the other way.

Also, it is enough to do the combinatorics of the construction in the case of truncated simplicial sets. Namely, for any object T of the category $\mathcal{C}$, and any n-truncated simplicial object U of $\mathcal{C}$ we can consider the n-truncated simplicial set $\mathrm{Mor}(T, U)$. We may apply the construction to this, and take its set of $(n + 1)$-simplices, and require this to be representable. This is a good way to think about the result of Lemma 18.6.

Remark 18.9. *Inductive construction of coskeleta.* Suppose that $\mathcal{C}$ is a category with finite limits. Suppose that U is an m-truncated simplicial object in $\mathcal{C}$. Then we can inductively construct n-truncated objects U^n as follows:

(1) To start, set $U^m = U$.
(2) Given U^n for $n \geq m$ set $U^{n+1} = \tilde{U}^n$, where $\tilde{U}^n$ is constructed from U^n as in Lemma 18.6.

Since the construction of Lemma 18.6 has the property that it leaves the n-skeleton of U^n unchanged, we can then define $\mathrm{cosk}_m U$ to be the simplicial object with $(\mathrm{cosk}_m U)_n = U_n^n = U_n^{n+1} = \ldots$. And it follows formally from Lemma 18.6 that U^n satisfies the formula

$$\mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(V, U^n) = \mathrm{Mor}_{\mathrm{Simp}_m(\mathcal{C})}(\mathrm{sk}_m V, U)$$

for all $n \geq m$. It also then follows formally from this that

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(V, \mathrm{cosk}_m U) = \mathrm{Mor}_{\mathrm{Simp}_m(\mathcal{C})}(\mathrm{sk}_m V, U)$$

with $\mathrm{cosk}_m U$ chosen as above.

Lemma 18.10. *Let $\mathcal{C}$ be a category which has finite limits.*

(1) *For every n the functor $\mathrm{sk}_n : \mathrm{Simp}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$ has a right adjoint cosk_n.*
(2) *For every $n' \geq n$ the functor $\mathrm{sk}_n : \mathrm{Simp}_{n'}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$ has a right adjoint, namely $\mathrm{sk}_{n'} \mathrm{cosk}_n$.*
(3) *For every $m \geq n \geq 0$ and every n-truncated simplicial object U of $\mathcal{C}$ we have $\mathrm{cosk}_m \mathrm{sk}_m \mathrm{cosk}_n U = \mathrm{cosk}_n U$.*
(4) *If U is a simplicial object of $\mathcal{C}$ such that the canonical map $U \to \mathrm{cosk}_n \mathrm{sk}_n U$ is an isomorphism for some $n \geq 0$, then the canonical map $U \to \mathrm{cosk}_m \mathrm{sk}_m U$ is an isomorphism for all $m \geq n$.*

Proof. The existence in (1) follows from Lemma 18.2 above. Parts (2) and (3) follow from the discussion in Remark 18.9. After this (4) is obvious. $\square$

Remark 18.11. We do not need all finite limits in order to be able to define the coskeleton functors. Here are some remarks

(1) We have seen in Examples 18.1 that if $\mathcal{C}$ has products of pairs of objects then cosk_0 exists.
(2) For $k > 0$ the functor cosk_k exists if $\mathcal{C}$ has finite connected limits.

This is clear from the inductive procedure of constructing coskeleta (Remarks 18.8 and 18.9) but it also follows from the fact that the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k + 1$ used in Lemma 18.2 are connected. Observe that we do not need the categories for $n \leq k$ by Lemma 18.3 or Lemma 18.4. (As k gets higher the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k + 1$ are more and more connected in a topological sense.)

Lemma 18.12. *Let U, V be n-truncated simplicial objects of a category $\mathcal{C}$. Then*

$$cosk_n(U \times V) = cosk_n U \times cosk_n V$$

whenever the left and right hand sides exist.

Proof. Let W be a simplicial object. We have

$$
\begin{aligned}
\mathrm{Mor}(W, \mathrm{cosk}_n(U \times V)) &= \mathrm{Mor}(\mathrm{sk}_n W, U \times V) \\
&= \mathrm{Mor}(\mathrm{sk}_n W, U) \times \mathrm{Mor}(\mathrm{sk}_n W, V) \\
&= \mathrm{Mor}(W, \mathrm{cosk}_n U) \times \mathrm{Mor}(W, \mathrm{cosk}_n V) \\
&= \mathrm{Mor}(W, \mathrm{cosk}_n U \times \mathrm{cosk}_n V)
\end{aligned}
$$

The lemma follows. $\square$

Lemma 18.13. *Assume $\mathcal{C}$ has fibre products. Let U, V, W be n-truncated simplicial objects of the category $\mathcal{C}$. Then*

$$cosk_n(V \times_U W) = cosk_n U \times_{cosk_n U} cosk_n V$$

whenever the left and right hand side exist.

Proof. Omitted, but very similar to the proof of Lemma 18.12 above. $\square$

Lemma 18.14. *Let $\mathcal{C}$ be a category with finite limits. Let $X \in \mathrm{Ob}(\mathcal{C})$. The functor $\mathcal{C}/X \to \mathcal{C}$ commutes with the coskeleton functors $cosk_k$ for $k \geq 1$.*

Proof. The statement means that if U is a simplicial object of $\mathcal{C}/X$ which we can think of as a simplicial object of $\mathcal{C}$ with a morphism towards the constant simplicial object X, then $\mathrm{cosk}_k U$ computed in $\mathcal{C}/X$ is the same as computed in $\mathcal{C}$. This follows for example from Categories, Lemma 16.2 because the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k + 1$ used in Lemma 18.2 are connected. Observe that we do not need the categories for $n \leq k$ by Lemma 18.3 or Lemma 18.4. $\square$

Lemma 18.15. *The canonical map $\Delta[n] \to cosk_1 sk_1 \Delta[n]$ is an isomorphism.*

Proof. Consider a simplicial set U and a morphism $f : U \to \Delta[n]$. This is a rule that associates to each $u \in U_i$ a map $f_u : [i] \to [n]$ in Δ. Furthermore, these maps should have the property that $f_u \circ \varphi = f_{U(\varphi)(u)}$ for any $\varphi : [j] \to [i]$. Denote

$\epsilon_j^i : [0] \to [i]$ the map which maps 0 to j. Denote $F : U_0 \to [n]$ the map $u \mapsto f_u(0)$. Then we see that

$$f_u(j) = F(\epsilon_j^i(u))$$

for all $0 \leq j \leq i$ and $u \in U_i$. In particular, if we know the function F then we know the maps f_u for all $u \in U_i$ all i. Conversely, given a map $F : U_0 \to [n]$, we can set for any i, and any $u \in U_i$ and any $0 \leq j \leq i$

$$f_u(j) = F(\epsilon_j^i(u))$$

This does not in general define a morphism f of simplicial sets as above. Namely, the condition is that all the maps f_u are nondecreasing. This clearly is equivalent to the condition that $F(\epsilon_j^i(u)) \leq F(\epsilon_{j'}^i(u))$ whenever $0 \leq j \leq j' \leq i$ and $u \in U_i$. But in this case the morphisms

$$\epsilon_j^i, \epsilon_{j'}^i : [0] \to [i]$$

both factor through the map $\epsilon_{j,j'}^i : [1] \to [i]$ defined by the rules $0 \mapsto j$, $1 \mapsto j'$. In other words, it is enough to check the inequalities for $i = 1$ and $u \in X_1$. In other words, we have

$$\mathrm{Mor}(U, \Delta[n]) = \mathrm{Mor}(\mathrm{sk}_1 U, \mathrm{sk}_1 \Delta[n])$$

as desired. $\square$

19. Augmentations

Definition 19.1. Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. An *augmentation* $\epsilon : U \to X$ *of U towards an object X of $\mathcal{C}$* is a morphism from U into the constant simplicial object X.

Lemma 19.2. *Let $\mathcal{C}$ be a category. Let $X \in \mathrm{Ob}(\mathcal{C})$. Let U be a simplicial object of $\mathcal{C}$. To give an augmentation of U towards X is the same as giving a morphism $\epsilon_0 : U_0 \to X$ such that $\epsilon_0 \circ d_0^1 = \epsilon_0 \circ d_1^1$.*

Proof. Given a morphism $\epsilon : U \to X$ we certainly obtain an ϵ_0 as in the lemma. Conversely, given ϵ_0 as in the lemma, define $\epsilon_n : U_n \to X$ by choosing any morphism $\alpha : [0] \to [n]$ and taking $\epsilon_n = \epsilon_0 \circ U(\alpha)$. Namely, if $\beta : [0] \to [n]$ is another choice, then there exists a morphism $\gamma : [1] \to [n]$ such that α and β both factor as $[0] \to [1] \to [n]$. Hence the condition on ϵ_0 shows that ϵ_n is well defined. Then it is easy to show that $(\epsilon_n) : U \to X$ is a morphism of simplicial objects. $\square$

Lemma 19.3. *Let $\mathcal{C}$ be a category with fibred products. Let $f : Y \to X$ be a morphism of $\mathcal{C}$. Let U be the simplicial object of $\mathcal{C}$ whose nth term is the $(n+1)$fold fibred product $Y \times_X Y \times_X \ldots \times_X Y$. See Example 3.5. For any simplicial object V of $\mathcal{C}$ we have*

$$\begin{aligned}
\mathrm{Mor}_{Simp(\mathcal{C})}(V, U) &= \mathrm{Mor}_{Simp_1(\mathcal{C})}(sk_1 V, sk_1 U) \\
&= \{g_0 : V_0 \to Y \mid f \circ g_0 \circ d_0^1 = f \circ g_0 \circ d_1^1\}
\end{aligned}$$

In particular we have $U = cosk_1 sk_1 U$.

Proof. Suppose that $g : \mathrm{sk}_1 V \to \mathrm{sk}_1 U$ is a morphism of 1-truncated simplicial objects. Then the diagram

$$
\begin{array}{ccc}
V_1 & \underset{d_1^1}{\overset{d_0^1}{\rightrightarrows}} & V_0 \\
{\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_0} \\
Y \times_X Y & \underset{pr_0}{\overset{pr_1}{\rightrightarrows}} & Y \longrightarrow X
\end{array}
$$

is commutative, which proves that the relation shown in the lemma holds. We have to show that, conversely, given a morphism g_0 satisfying the relation $f \circ g_0 \circ d_0^1 = f \circ g_0 \circ d_1^1$ we get a unique morphism of simplicial objects $g : V \to U$. This is done as follows. For any $n \geq 1$ let $g_{n,i} = g_0 \circ V([0] \to [n], 0 \mapsto i) : V_n \to Y$. The equality above implies that $f \circ g_{n,i} = f \circ g_{n,i+1}$ because of the commutative diagram

Hence we get $(g_{n,0}, \dots, g_{n,n}) : V_n \to Y \times_X \dots \times_X Y = U_n$. We leave it to the reader to see that this is a morphism of simplicial objects. The last assertion of the lemma is equivalent to the first equality in the displayed formula of the lemma. $\qquad\square$

Remark 19.4. Let $\mathcal{C}$ be a category with fibre products. Let V be a simplicial object. Let $\epsilon : V \to X$ be an augmentation. Let U be the simplicial object whose nth term is the $(n+1)$st fibred product of V_0 over X. By a simple combination of Lemmas 19.2 and 19.3 we obtain a canonical morphism $V \to U$.

20. Left adjoints to the skeleton functors

In this section we construct a left adjoint $i_{m!}$ of the skeleton functor sk_m in certain cases. The adjointness formula is

$$
\mathrm{Mor}_{\mathrm{Simp}_m(\mathcal{C})}(U, \mathrm{sk}_m V) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(i_{m!} U, V).
$$

It turns out that this left adjoint exists when the category $\mathcal{C}$ has finite colimits.

We use a similar construction as in Section 12. Recall the category $[n]/\Delta$ of objects under $[n]$, see Categories, Example 2.14. Its objects are morphisms $\alpha : [n] \to [k]$ and its morphisms are commutative triangles. We let $([n]/\Delta)_{\leq m}$ denote the full subcategory of $[n]/\Delta$ consisting of objects $[n] \to [k]$ with $k \leq m$. Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$
U(n) : ([n]/\Delta)_{\leq m}^{opp} \longrightarrow \mathcal{C}
$$

by the rules

$$
\begin{aligned}
([n] \to [k]) &\longmapsto U_k \\
\psi : ([n] \to [k']) \to ([n] \to [k]) &\longmapsto U(\psi) : U_k \to U_{k'}
\end{aligned}
$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$\underline{\varphi} : ([n']/\Delta)_{\leq m} \longrightarrow ([n]/\Delta)_{\leq m}$$

which maps $\alpha : [n'] \to [k]$ to $\varphi \circ \alpha : [n] \to [k]$. The composition $U(n) \circ \underline{\varphi}$ is equal to the functor $U(n')$.

Lemma 20.1. *Let $\mathcal{C}$ be a category which has finite colimits. The functors $i_{m!}$ exist for all m. Let U be an m-truncated simplicial object of $\mathcal{C}$. The simplicial object $i_{m!}U$ is described by the formula*

$$(i_{m!}U)_n = \mathrm{colim}_{([n]/\Delta)^{opp}_{\leq m}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $i_{m!}U(\varphi)$ comes from the identification $U(n) \circ \underline{\varphi} = U(n')$ above via Categories, Lemma 14.7.

Proof. In this proof we denote $i_{m!}U$ the simplicial object whose nth term is given by the displayed formula of the lemma. We will show it satisfies the adjointness property.

Let V be a simplicial object of $\mathcal{C}$. Let $\gamma : U \to \mathrm{sk}_m V$ be given. A morphism

$$\mathrm{colim}_{([n]/\Delta)^{opp}_{\leq m}} U(n) \to T$$

is given by a compatible system of morphisms $f_\alpha : U_k \to T$ where $\alpha : [n] \to [k]$ with $k \leq m$. Certainly, we have such a system of morphisms by taking the compositions

$$U_k \xrightarrow{\gamma_k} V_k \xrightarrow{V(\alpha)} V_n.$$

Hence we get an induced morphism $(i_{m!}U)_n \to V_n$. We leave it to the reader to see that these form a morphism of simplicial objects $\gamma' : i_{m!}U \to V$.

Conversely, given a morphism $\gamma' : i_{m!}U \to V$ we obtain a morphism $\gamma : U \to \mathrm{sk}_m V$ by setting $\gamma_i : U_i \to V_i$ equal to the composition

$$U_i \xrightarrow{\mathrm{id}_{[i]}} \mathrm{colim}_{([i]/\Delta)^{opp}_{\leq m}} U(i) \xrightarrow{\gamma'_i} V_i$$

for $0 \leq i \leq n$. We leave it to the reader to see that this is the inverse of the construction above. $\square$

Lemma 20.2. *Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For any $n \leq m$ the colimit*

$$\mathrm{colim}_{([n]/\Delta)^{opp}_{\leq m}} U(n)$$

exists and is equal to U_n.

Proof. This is so because the category $([n]/\Delta)_{\leq m}$ has an initial object, namely $\mathrm{id} : [n] \to [n]$. $\square$

Lemma 20.3. *Let $\mathcal{C}$ be a category which has finite colimits. Let U be an m-truncated simplicial object of $\mathcal{C}$. The map $U \to sk_m i_{m!}U$ is an isomorphism.*

Proof. Combine Lemmas 20.1 and 20.2. $\square$

Lemma 20.4. *If U is an m-truncated simplicial set and $n > m$ then all n-simplices of $i_{m!}U$ are degenerate.*

Proof. This can be seen from the construction of $i_{m!}U$ in Lemma 20.1, but we can also argue directly as follows. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subset with $V'_i = V_i$ for $i \leq m$ and all i simplices degenerate for $i > m$, see Lemma 17.4. By the adjunction formula, since $\mathrm{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. $\qquad\square$

Lemma 20.5. *Let U be a simplicial set. Let $n \geq 0$ be an integer. The morphism $i_{n!} sk_n U \to U$ identifies $i_{n!} sk_n U$ with the simplicial set $U' \subset U$ defined in Lemma 17.4.*

Proof. By Lemma 20.4 the only nondegenerate simplices of $i_{n!}\mathrm{sk}_n U$ are in degrees $\leq n$. The map $i_{n!}\mathrm{sk}_n U \to U$ is an isomorphism in degrees $\leq n$. Combined we conclude that the map $i_{n!}\mathrm{sk}_n U \to U$ maps nondegenerate simplices to nondegenerate simplices and no two nondegenerate simplices have the same image. Hence Lemma 17.3 applies. Thus $i_{n!}\mathrm{sk}_n U \to U$ is injective. The result follows easily from this. $\qquad\square$

Remark 20.6. In some texts the composite functor

$$\mathrm{Simp}(\mathcal{C}) \xrightarrow{\mathrm{sk}_m} \mathrm{Simp}_m(\mathcal{C}) \xrightarrow{i_{m!}} \mathrm{Simp}(\mathcal{C})$$

is denoted sk_m. This makes sense for simplicial sets, because then Lemma 20.5 says that $i_{m!}\mathrm{sk}_m V$ is just the sub simplicial set of V consisting of all i-simplices of V, $i \leq m$ and their degeneracies. In those texts it is also customary to denote the composition

$$\mathrm{Simp}(\mathcal{C}) \xrightarrow{\mathrm{sk}_m} \mathrm{Simp}_m(\mathcal{C}) \xrightarrow{\mathrm{cosk}_m} \mathrm{Simp}(\mathcal{C})$$

by cosk_m.

Lemma 20.7. *Let $U \subset V$ be simplicial sets. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that*

(1) *$V_i = U_i$ for $i < n$,*
(2) *$V_n = U_n \cup \{x\}$,*
(3) *any $z \in V_j$, $z \notin U_j$ for $j > n$ is degenerate.*

Let $\Delta[n] \to V$ be the unique morphism mapping the nondegenerate n-simplex of $\Delta[n]$ to x. In this case the diagram

$$
\begin{array}{ccc}
\Delta[n] & \longrightarrow & V \\
\uparrow & & \uparrow \\
i_{(n-1)!} sk_{n-1}\Delta[n] & \longrightarrow & U
\end{array}
$$

is a pushout diagram.

Proof. Let us denote $\partial\Delta[n] = i_{(n-1)!}\mathrm{sk}_{n-1}\Delta[n]$ for convenience. There is a natural map $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$. We have to show that it is bijective in degree j for all j. This is clear for $j \leq n$. Let $j > n$. The third condition means that any $z \in V_j$, $z \notin U_j$ is a degenerate simplex, say $z = s_i^{j-1}(z')$. Of course $z' \notin U_{j-1}$. By induction it follows that z' is a degeneracy of x. Thus we conclude that all j-simplices of V are either in U or degeneracies of x. This implies that the map $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$ is surjective. Note that a nondegenerate simplex of $U \amalg_{\partial\Delta[n]} \Delta[n]$ is either the image of a nondegenerate simplex of U, or the image of the (unique) nondegenerate n-simplex of $\Delta[n]$. Since clearly x is nondegenerate we deduce that

$U \amalg_{\partial\Delta[n]} \Delta[n] \to V$ maps nondegenerate simplices to nondegenerate simplices and is injective on nondegenerate simplices. Hence it is injective, by Lemma 17.3. $\square$

Lemma 20.8. *Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U and V have finitely many nondegenerate simplices. Then there exists a sequence of sub simplicial sets*

$$U = W^0 \subset W^1 \subset W^2 \subset \dots W^r = V$$

such that Lemma 20.7 applies to each of the inclusions $W^i \subset W^{i+1}$.

Proof. Let n be the smallest integer such that V has a nondegenerate simplex that does not belong to U. Let $x \in V_n$, $x \notin U_n$ be such a nondegenerate simplex. Let $W \subset V$ be the set of elements which are either in U, or are a (repeated) degeneracy of x (in other words, are of the form $V(\varphi)(x)$ with $\varphi : [m] \to [n]$ surjective). It is easy to see that W is a simplicial set. The inclusion $U \subset W$ satisfies the conditions of Lemma 20.7. Moreover the number of nondegenerate simplices of V which are not contained in W is exactly one less than the number of nondegenerate simplices of V which are not contained in U. Hence we win by induction on this number. $\square$

Lemma 20.9. *Let $\mathcal{A}$ be an abelian category Let U be an m-truncated simplicial object of $\mathcal{A}$. For $n > m$ we have $N(i_{m!}U)_n = 0$.*

Proof. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subobject of V with $V'_i = V_i$ for $i \leq m$ and $N(V'_i) = 0$ for $i > m$, see Lemma 17.9. By the adjunction formula, since $\mathrm{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. $\square$

Lemma 20.10. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The morphism $i_{n!}sk_n U \to U$ identifies $i_{n!}sk_n U$ with the simplicial subobject $U' \subset U$ defined in Lemma 17.9.*

Proof. By Lemma 20.9 we have $N(i_{n!}\mathrm{sk}_n U)_i = 0$ for $i > n$. The map $i_{n!}\mathrm{sk}_n U \to U$ is an isomorphism in degrees $\leq n$, see Lemma 20.3. Combined we conclude that the map $i_{n!}\mathrm{sk}_n U \to U$ induces injective maps $N(i_{n!}\mathrm{sk}_n U)_i \to N(U)_i$ for all i. Hence Lemma 17.7 applies. Thus $i_{n!}\mathrm{sk}_n U \to U$ is injective. The result follows easily from this. $\square$

Here is another way to think about the coskeleton functor using the material above.

Lemma 20.11. *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let V be a simplicial object of $\mathcal{C}$. In this case*

$$(cosk_n sk_n V)_{n+1} = \mathrm{Hom}(i_{n!}sk_n \Delta[n+1], V)_0.$$

Proof. By Lemma 13.4 the object on the left represents the functor which assigns to X the first set of the following equalities

$$\begin{aligned}
\mathrm{Mor}(X \times \Delta[n+1], \mathrm{cosk}_n \mathrm{sk}_n V) &= \mathrm{Mor}(X \times \mathrm{sk}_n \Delta[n+1], \mathrm{sk}_n V) \\
&= \mathrm{Mor}(X \times i_{n!}\mathrm{sk}_n \Delta[n+1], V).
\end{aligned}$$

The object on the right in the formula of the lemma is represented by the functor which assigns to X the last set in the sequence of equalities. This proves the result.

In the sequence of equalities we have used that $\mathrm{sk}_n(X \times \Delta[n+1]) = X \times \mathrm{sk}_n \Delta[n+1]$ and that $i_{n!}(X \times \mathrm{sk}_n \Delta[n+1]) = X \times i_{n!}\mathrm{sk}_n \Delta[n+1]$. The first equality is obvious. For any (possibly truncated) simplicial object W of $\mathcal{C}$ and any object X of $\mathcal{C}$ denote temporarily $\mathrm{Mor}_{\mathcal{C}}(X, W)$ the (possibly truncated) simplicial set $[n] \mapsto \mathrm{Mor}_{\mathcal{C}}(X, W_n)$.

From the definitions it follows that $\mathrm{Mor}(U \times X, W) = \mathrm{Mor}(U, \mathrm{Mor}_{\mathcal{C}}(X, W))$ for any (possibly truncated) simplicial set U. Hence

$$
\begin{aligned}
\mathrm{Mor}(X \times i_{n!}\mathrm{sk}_n\Delta[n+1], W) &= \mathrm{Mor}(i_{n!}\mathrm{sk}_n\Delta[n+1], \mathrm{Mor}_{\mathcal{C}}(X, W)) \\
&= \mathrm{Mor}(\mathrm{sk}_n\Delta[n+1], \mathrm{sk}_n\,\mathrm{Mor}_{\mathcal{C}}(X, W)) \\
&= \mathrm{Mor}(X \times \mathrm{sk}_n\Delta[n+1], \mathrm{sk}_nW) \\
&= \mathrm{Mor}(i_{n!}(X \times \mathrm{sk}_n\Delta[n+1]), W).
\end{aligned}
$$

This proves the second equality used, and ends the proof of the lemma. $\square$

Lemma 20.12. *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let X be an object of $\mathcal{C}$. Let $k \geq 0$. The canonical map*

$$\mathrm{Hom}(\Delta[k], X) \longrightarrow cosk_1\,sk_1\,\mathrm{Hom}(\Delta[k], X)$$

is an isomorphism.

Proof. For any simplicial object V we have

$$
\begin{aligned}
\mathrm{Mor}(V, \mathrm{cosk}_1\mathrm{sk}_1\,\mathrm{Hom}(\Delta[k], X)) &= \mathrm{Mor}(\mathrm{sk}_1V, \mathrm{sk}_1\,\mathrm{Hom}(\Delta[k], X)) \\
&= \mathrm{Mor}(i_{1!}\mathrm{sk}_1V, \mathrm{Hom}(\Delta[k], X)) \\
&= \mathrm{Mor}(i_{1!}\mathrm{sk}_1V \times \Delta[k], X)
\end{aligned}
$$

The first equality by the adjointness of sk and cosk, the second equality by the adjointness of $i_{1!}$ and sk_1, and the first equality by Definition 16.1 where the last X denotes the constant simplicial object with value X. By Lemma 19.2 an element in this set depends only on the terms of degree 0 and 1 of $i_{1!}\mathrm{sk}_1V \times \Delta[k]$. These agree with the degree 0 and 1 terms of $V \times \Delta[k]$, see Lemma 20.3. Thus the set above is equal to $\mathrm{Mor}(V \times \Delta[k], X) = \mathrm{Mor}(V, \mathrm{Hom}(\Delta[k], X))$. $\square$

Lemma 20.13. *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let X be an object of $\mathcal{C}$. Let $k \geq 0$. The canonical map*

$$\mathrm{Hom}(\Delta[k], X)_1 \longrightarrow (cosk_0\,sk_0\,\mathrm{Hom}(\Delta[k], X))_1$$

is identified with the map

$$\prod_{\alpha:[k]\to[1]} X \longrightarrow X \times X$$

which is the projection onto the factors where α is a constant map.

Proof. It is shown in Example 18.1 that cosk_0Z equals $Z \times Z$ in degree 1. Moreover, it is true in general that the morphism $V_1 \to (\mathrm{cosk}_0\mathrm{sk}_0V)_1$ is the morphism $(d_0^1, d_1^1) : V_1 \to V_0 \times V_0$ (left to the reader). Thus we simply have to compute the 0th and 1st term of $\mathrm{Hom}(\Delta[k], X)$. According to Lemma 16.5 we have $\mathrm{Hom}(\Delta[k], X)_0 = \prod_{\alpha:[k]\to[0]} X = X$, and $\mathrm{Hom}(\Delta[k], X)_0 = \prod_{\alpha:[k]\to[1]} X$. The lemma follows from the description of the morphisms of the simplicial object just above Lemma 16.5. $\square$

21. Simplicial objects in abelian categories

Recall that an abelian category is defined in Homology, Section 5.

Lemma 21.1. *Let $\mathcal{A}$ be an abelian category.*

(1) *The categories $Simp(\mathcal{A})$ and $CoSimp(\mathcal{A})$ are abelian.*
(2) *A morphism of (co)simplicial objects $f : A \to B$ is injective if and only if each $f_n : A_n \to B_n$ is injective.*

(3) *A morphism of (co)simplicial objects $f : A \to B$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.*

(4) *A sequence of (co)simplicial objects*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B if and only if each sequence

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

is exact at B_i.

Proof. Pre-additivity is easy. A final object is given by $U_n = 0$ in all degrees. Existence of direct products we saw in Lemmas 6.2 and 9.2. Kernels and cokernels are obtained by taking termwise kernels and cokernels. $\square$

For an object A of $\mathcal{A}$ and an integer k consider the k-truncated simplicial object U with

(1) $U_i = 0$ for $i < k$,
(2) $U_k = A$,
(3) all morphisms $U(\varphi)$ equal to zero, except $U(\mathrm{id}_{[k]}) = \mathrm{id}_A$.

Since $\mathcal{A}$ has both finite limits and finite colimits we see that both $\mathrm{cosk}_k U$ and $i_{k!} U$ exist. We will describe both of these and the canonical map $i_{k!} U \to \mathrm{cosk}_k U$.

Lemma 21.2. *With A, k and U as above, so $U_i = 0$, $i < k$ and $U_k = A$.*

(1) *Given a k-truncated simplicial object V we have*

$$\mathrm{Mor}(U, V) = \{f : A \to V_k \mid d_i^k \circ f = 0, \ i = 0, \ldots, k\}$$

and

$$\mathrm{Mor}(V, U) = \{f : V_k \to A \mid f \circ s_i^{k-1} = 0, \ i = 0, \ldots, k-1\}.$$

(2) *The object $i_{k!} U$ has nth term equal to $\bigoplus_\alpha A$ where α runs over all surjective morphisms $\alpha : [n] \to [k]$.*

(3) *For any $\varphi : [m] \to [n]$ the map $i_{k!} U(\varphi)$ is described as the mapping $\bigoplus_\alpha A \to \bigoplus_{\alpha'} A$ which maps to component corresponding to $\alpha : [n] \to [k]$ to zero if $\alpha \circ \varphi$ is not surjective and by the identity to the component corresponding to $\alpha \circ \varphi$ if it is surjective.*

(4) *The object $\mathrm{cosk}_k U$ has nth term equal to $\bigoplus_\beta A$, where β runs over all injective morphisms $\beta : [k] \to [n]$.*

(5) *For any $\varphi : [m] \to [n]$ the map $\mathrm{cosk}_k U(\varphi)$ is described as the mapping $\bigoplus_\beta A \to \bigoplus_{\beta'} A$ which maps to component corresponding to $\beta : [k] \to [n]$ to zero if β does not factor through φ and by the identity to each of the components corresponding to β' such that $\beta = \varphi \circ \beta'$ if it does.*

(6) *The canonical map $c : i_{k!} U \to \mathrm{cosk}_k U$ in degree n has (α, β) coefficient $A \to A$ equal to zero if $\alpha \circ \beta$ is not the identity and equal to id_A if it is.*

(7) *The canonical map $c : i_{k!} U \to \mathrm{cosk}_k U$ is injective.*

Proof. The proof of (1) is left to the reader.

Let us take the rules of (2) and (3) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $i_{k!} U$. This will prove (2), (3) at the same time. We have to show that given a morphism $f : U \to \mathrm{sk}_k V$ there exists a unique morphism $\tilde{f} : \tilde{U} \to V$ which recovers f upon taking the k-skeleton. From

(1) we see that f corresponds with a morphism $f_k : A \to V_k$ which maps into the kernel of d_i^k for all i. For any surjective $\alpha : [n] \to [k]$ we set $\tilde{f}_\alpha : A \to V_n$ equal to the composition $\tilde{f}_\alpha = V(\alpha) \circ f_k : A \to V_n$. We define $\tilde{f}_n : \tilde{U}_n \to V_n$ as the sum of the $\tilde{f}_\alpha$ over $\alpha : [n] \to [k]$ surjective. Such a collection of $\tilde{f}_\alpha$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$
\begin{array}{ccc}
\bigoplus_{\alpha:[n]\to[k] \text{ surjective}} A & \xrightarrow{\ \tilde{f}_n\ } & V_n \\
{\scriptstyle (3)}\big\downarrow & & \big\downarrow{\scriptstyle V(\varphi)} \\
\bigoplus_{\alpha':[m]\to[k] \text{ surjective}} A & \xrightarrow{\ \tilde{f}_m\ } & V_m
\end{array}
$$

is commutative. Choosing $\varphi = \alpha$ shows our choice of $\tilde{f}_\alpha$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the left upper corner separately. It is clear for the summands corresponding to α where $\alpha \circ \varphi$ is surjective, because those get mapped by id_A to the summand with $\alpha' = \alpha \circ \varphi$, and we have $\tilde{f}_{\alpha'} = V(\alpha') \circ f_k = V(\alpha \circ \varphi) \circ f_k = V(\varphi) \circ \tilde{f}_\alpha$. For those where $\alpha \circ \varphi$ is not surjective, we have to show that $V(\varphi) \circ \tilde{f}_\alpha = 0$. By definition this is equal to $V(\varphi) \circ V(\alpha) \circ f_k = V(\alpha \circ \varphi) \circ f_k$. Since $\alpha \circ \varphi$ is not surjective we can write it as $\delta_i^k \circ \psi$, and we deduce that $V(\varphi) \circ V(\alpha) \circ f_k = V(\psi) \circ d_i^k \circ f_k = 0$ see above.

Let us take the rules of (4) and (5) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $\mathrm{cosk}_k U$. This will prove (4), (5) at the same time. The argument is completely dual to the proof of (2), (3) above, but we give it anyway. We have to show that given a morphism $f : \mathrm{sk}_k V \to U$ there exists a unique morphism $\tilde{f} : V \to \tilde{U}$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : V_k \to A$ which is zero on the image of s_i^{k-1} for all i. For any injective $\beta : [k] \to [n]$ we set $\tilde{f}_\beta : V_n \to A$ equal to the composition $\tilde{f}_\beta = f_k \circ V(\beta) : V_n \to A$. We define $\tilde{f}_n : V_n \to \tilde{U}_n$ as the sum of the $\tilde{f}_\beta$ over $\beta : [k] \to [n]$ injective. Such a collection of $\tilde{f}_\beta$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$
\begin{array}{ccc}
V_n & \xrightarrow{\ \tilde{f}_n\ } & \bigoplus_{\beta:[k]\to[n] \text{ injective}} A \\
{\scriptstyle V(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle (5)} \\
V_m & \xrightarrow{\ \tilde{f}_m\ } & \bigoplus_{\beta':[k]\to[m] \text{ injective}} A
\end{array}
$$

is commutative. Choosing $\varphi = \beta$ shows our choice of $\tilde{f}_\beta$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the right lower corner separately. It is clear for the summands corresponding to β' where $\varphi \circ \beta'$ is injective, because these summands get mapped into by exactly the summand with $\beta = \varphi \circ \beta'$ and we have in that case $\tilde{f}_{\beta'} \circ V(\varphi) = f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\beta) = \tilde{f}_\beta$. For those where $\varphi \circ \beta'$ is not injective, we have to show that $\tilde{f}_{\beta'} \circ V(\varphi) = 0$. By definition this is equal to $f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\varphi \circ \beta')$. Since $\varphi \circ \beta'$ is not injective we can write it as $\psi \circ \sigma_i^{k-1}$, and we deduce that $f_k \circ V(\beta') \circ V(\varphi) = f_k \circ s_i^{k-1} \circ V(\psi) = 0$ see above.

The composition $i_{k!} U \to \mathrm{cosk}_k U$ is the unique map of simplicial objects which is the identity on $A = U_k = (i_{k!} U)_k = (\mathrm{cosk}_k U)_k$. Hence it suffices to check that the

proposed rule defines a morphism of simplicial objects. To see this we have to show that for any $\varphi : [m] \to [n]$ the diagram

$$
\begin{array}{ccc}
\bigoplus_{\alpha:[n]\to[k] \text{ surjective}} A & \xrightarrow{\quad(6)\quad} & \bigoplus_{\beta:[k]\to[n] \text{ injective}} A \\
{\scriptstyle(3)}\Big\downarrow & & \Big\downarrow{\scriptstyle(5)} \\
\bigoplus_{\alpha':[m]\to[k] \text{ surjective}} A & \xrightarrow{\quad(6)\quad} & \bigoplus_{\beta':[k]\to[m] \text{ injective}} A
\end{array}
$$

is commutative. Now we can think of this in terms of matrices filled with only 0's and 1's as follows: The matrix of (3) has a nonzero (α', α) entry if and only if $\alpha' = \alpha \circ \varphi$. Likewise the matrix of (5) has a nonzero (β', β) entry if and only if $\beta = \varphi \circ \beta'$. The upper matrix of (6) has a nonzero (α, β) entry if and only if $\alpha \circ \beta = \mathrm{id}_{[k]}$. Similarly for the lower matrix of (6). The commutativity of the diagram then comes down to computing the (α, β') entry for both compositions and seeing they are equal. This comes down to the following equality

$$
\#\left\{\beta \mid \beta = \varphi \circ \beta' \wedge \alpha \circ \beta = \mathrm{id}_{[k]}\right\} = \#\left\{\alpha' \mid \alpha' = \alpha \circ \varphi \wedge \alpha' \circ \beta' = \mathrm{id}_{[k]}\right\}
$$

whose proof may safely be left to the reader.

Finally, we prove (7). This follows directly from Lemmas 17.7, 18.4, 20.3 and 20.9. $\qquad\square$

Definition 21.3. Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. The *Eilenberg-Maclane object* $K(A, k)$ is given by the object $K(A, k) = i_{k!}U$ which is described in Lemma 21.2 above.

Lemma 21.4. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. Consider the simplicial object E defined by the following rules*

(1) $E_n = \bigoplus_\alpha A$, *where the sum is over $\alpha : [n] \to [k+1]$ whose image is either $[k]$ or $[k+1]$.*
(2) *Given $\varphi : [m] \to [n]$ the map $E_n \to E_m$ maps the summand corresponding to α via id_A to the summand corresponding to $\alpha \circ \varphi$, provided $\mathrm{Im}(\alpha \circ \varphi)$ is equal to $[k]$ or $[k+1]$.*

Then there exists a short exact sequence

$$0 \to K(A, k) \to E \to K(A, k+1) \to 0$$

which is term by term split exact.

Proof. The maps $K(A, k)_n \to E_n$ resp. $E_n \to K(A, k+1)_n$ are given by the inclusion of direct sums, resp. projection of direct sums which is obvious from the inclusions of index sets. It is clear that these are maps of simplicial objects. $\qquad\square$

Lemma 21.5. *Let $\mathcal{A}$ be an abelian category. For any simplicial object V of $\mathcal{A}$ we have*

$$V = \mathrm{colim}_n \, i_{n!} sk_n V$$

where all the transition maps are injections.

Proof. This is true simply because each V_m is equal to $(i_{n!}\mathrm{sk}_n V)_m$ as soon as $n \geq m$. See also Lemma 20.10 for the transition maps. $\qquad\square$

22. Simplicial objects and chain complexes

Let $\mathcal{A}$ be an abelian category. See Homology, Section 12 for conventions and notation regarding chain complexes. Let U be a simplicial object of $\mathcal{A}$. The *associated chain complex $s(U)$ of U*, sometimes called the *Moore complex*, is the chain complex

$$\ldots \to U_2 \to U_1 \to U_0 \to 0 \to 0 \to \ldots$$

with boundary maps $d_n : U_n \to U_{n-1}$ given by the formula

$$d_n = \sum_{i=0}^{n}(-1)^i d_i^n.$$

This is a complex because, by the relations listed in Remark 3.3, we have

$$
\begin{aligned}
d_n \circ d_{n+1} &= \Big(\sum_{i=0}^{n}(-1)^i d_i^n\Big) \circ \Big(\sum_{j=0}^{n+1}(-1)^j d_j^{n+1}\Big) \\
&= \sum_{0 \le i < j \le n+1}(-1)^{i+j} d_{j-1}^n \circ d_i^{n+1} + \sum_{n \ge i \ge j \ge 0}(-1)^{i+j} d_i^n \circ d_j^{n+1} \\
&= 0.
\end{aligned}
$$

The signs cancel! We denote the associated chain complex $s(U)$. Clearly, the construction is functorial and hence defines a functor

$$s : \mathrm{Simp}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\ge 0}(\mathcal{A}).$$

Thus we have the confusing but correct formula $s(U)_n = U_n$.

Lemma 22.1. *The functor s is exact.*

Proof. Clear from Lemma 21.1. $\square$

Lemma 22.2. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. Let E be the object described in Lemma 21.4. Then the complex $s(E)$ is acyclic.*

Proof. For a morphism $\alpha : [n] \to [k+1]$ we define $\alpha' : [n+1] \to [k+1]$ to be the map such that $\alpha'|_{[n]} = \alpha$ and $\alpha'(n+1) = k+1$. Note that if the image of α is $[k]$ or $[k+1]$, then the image of α' is $[k+1]$. Consider the family of maps $h_n : E_n \to E_{n+1}$ which maps the summand corresponding to α to the summand corresponding to α' via the identity on A. Let us compute $d_{n+1} \circ h_n - h_{n-1} \circ d_n$. We will first do this in case the category $\mathcal{A}$ is the category of abelian groups. Let us use the notation x_α to indicate the element $x \in A$ in the summand of E_n corresponding to the map α occurring in the index set. Let us also adopt the convention that x_α designates the zero element of E_n whenever $\mathrm{Im}(\alpha)$ is not $[k]$ or $[k+1]$. With these conventions we see that

$$d_{n+1}(h_n(x_\alpha)) = \sum_{i=0}^{n+1}(-1)^i x_{\alpha' \circ \delta_i^{n+1}}$$

and

$$h_{n-1}(d_n(x_\alpha)) = \sum_{i=0}^{n}(-1)^i x_{(\alpha \circ \delta_i^n)'}$$

It is easy to see that $\alpha' \circ \delta_i^{n+1} = (\alpha \circ \delta_i^n)'$ for $i = 0, \ldots, n$. It is also easy to see that $\alpha' \circ \delta_{n+1}^{n+1} = \alpha$. Thus we see that

$$(d_{n+1} \circ h_n - h_{n-1} \circ d_n)(x_\alpha) = (-1)^{n+1} x_\alpha$$

These identities continue to hold if $\mathcal{A}$ is any abelian category because they hold in the simplicial abelian group $[n] \mapsto \mathrm{Hom}(A, E_n)$; details left to the reader. We conclude that the identity map on E is homotopic to zero, with homotopy given by

the system of maps $h'_n = (-1)^{n+1}h_n : E_n \to E_{n+1}$. Hence we see that E is acyclic, for example by Homology, Lemma 12.5. $\qquad\square$

Lemma 22.3. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $H_i(s(K(A,k))) = A$ if $i = k$ and 0 else.*

Proof. First, let us prove this if $k = 0$. In this case we have $K(A,0)_n = A$ for all n. Furthermore, all the maps in this simplicial abelian group are id_A, in other words $K(A,0)$ is the constant simplicial object with value A. The boundary maps $d_n = \sum_{i=0}^n (-1)^i \mathrm{id}_A = 0$ if n odd and $= \mathrm{id}_A$ if n is even. Thus $s(K(A,0))$ looks like this

$$\ldots \to A \xrightarrow{0} A \xrightarrow{1} A \xrightarrow{0} A \to 0$$

and the result is clear.

Next, we prove the result for all k by induction. Given the result for k consider the short exact sequence

$$0 \to K(A,k) \to E \to K(A,k+1) \to 0$$

from Lemma 21.4. By Lemma 21.1 the associated sequence of chain complexes is exact. By Lemma 22.2 we see that $s(E)$ is acyclic. Hence the result for $k+1$ follows from the long exact sequence of homology, see Homology, Lemma 12.6. $\qquad\square$

There is a second chain complex we can associate to a simplicial object of $\mathcal{A}$. Recall that by Lemma 17.6 any simplicial object U of $\mathcal{A}$ is canonically split with $N(U_m) = \bigcap_{i=0}^{m-1} \mathrm{Ker}(d_i^m)$. We define the *normalized chain complex $N(U)$* to be the chain complex

$$\ldots \to N(U_2) \to N(U_1) \to N(U_0) \to 0 \to 0 \to \ldots$$

with boundary map $d_n : N(U_n) \to N(U_{n-1})$ given by the restriction of $(-1)^n d_n^n$ to the direct summand $N(U_n)$ of U_n. Note that Lemma 17.8 implies that $d_n^n(N(U_n)) \subset N(U_{n-1})$. It is a complex because $d_n^n \circ d_{n+1}^{n+1} = d_n^n \circ d_n^{n+1}$ and d_n^{n+1} is zero on $N(U_{n+1})$ by definition. Thus we obtain a second functor

$$N : \mathrm{Simp}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A}).$$

Here is the reason for the sign in the differential.

Lemma 22.4. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical map $N(U_n) \to U_n$ gives rise to a morphism of complexes $N(U) \to s(U)$.*

Proof. This is clear because the differential on $s(U)_n = U_n$ is $\sum (-1)^i d_i^n$ and the maps d_i^n, $i < n$ are zero on $N(U_n)$, whereas the restriction of $(-1)^n d_n^n$ is the boundary map of $N(U)$ by definition. $\qquad\square$

Lemma 22.5. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $N(K(A,k))_i = A$ if $i = k$ and 0 else.*

Proof. It is clear that $N(K(A,k))_i = 0$ when $i < k$ because $K(A,k)_i = 0$ in that case. It is clear that $N(K(A,k))_k = A$ since $K(A,k)_{k-1} = 0$ and $K(A,k)_k = A$. For $i > k$ we have $N(K(A,k))_i = 0$ by Lemma 20.9 and the definition of $K(A,k)$, see Definition 21.3. $\qquad\square$

Lemma 22.6. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(U) \to s(U)$ is split. In fact,*

$$s(U) = N(U) \oplus A(U)$$

for some complex $A(U)$. The construction $U \mapsto A(U)$ is functorial.

Proof. Define $A(U)_n$ to be the image of

$$\bigoplus\nolimits_{\varphi:[n]\to[m] \text{ surjective, } m<n} N(U_m) \xrightarrow{\oplus U(\varphi)} U_n$$

which is a subobject of U_n complementary to $N(U_n)$ according to Lemma 17.6 and Definition 17.1. We show that $A(U)$ is a subcomplex. Pick a surjective map $\varphi : [n] \to [m]$ with $m < n$ and consider the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

This composition is the sum of the maps

$$N(U_m) \xrightarrow{U(\varphi \circ \delta_i^n)} U_{n-1}$$

with sign $(-1)^i$, $i = 0, \ldots, n$.

First we will prove by ascending induction on m, $0 \leq m < n-1$ that all the maps $U(\varphi \circ \delta_i^n)$ map $N(U_m)$ into $A(U)_{n-1}$. (The case $m = n-1$ is treated below.) Whenever the map $\varphi \circ \delta_i^n : [n-1] \to [m]$ is surjective then the image of $N(U_m)$ under $U(\varphi \circ \delta_i^n)$ is contained in $A(U)_{n-1}$ by definition. If $\varphi \circ \delta_i^n : [n-1] \to [m]$ is not surjective, set $j = \varphi(i)$ and observe that i is the unique index whose image under φ is j. We may write $\varphi \circ \delta_i^n = \delta_j^m \circ \psi \circ \delta_i^n$ for some $\psi : [n-1] \to [m-1]$. Hence $U(\varphi \circ \delta_i^n) = U(\psi \circ \delta_i^n) \circ d_j^m$ which is zero on $N(U_m)$ unless $j = m$. If $j = m$, then $d_m^m(N(U_m)) \subset N(U_{m-1})$ and hence $U(\varphi \circ \delta_i^n)(N(U_m)) \subset U(\psi \circ \delta_i^n)(N(U_{m-1}))$ and we win by induction hypothesis.

To finish proving that $A(U)$ is a subcomplex we still have to deal with the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

in case $m = n-1$. In this case $\varphi = \sigma_j^{n-1}$ for some $0 \leq j \leq n-1$ and $U(\varphi) = s_j^{n-1}$. Thus the composition is given by the sum

$$\sum (-1)^i d_i^n \circ s_j^{n-1}$$

Recall from Remark 3.3 that $d_j^n \circ s_j^{n-1} = d_{j+1}^n \circ s_j^{n-1} = \mathrm{id}$ and these drop out because the corresponding terms have opposite signs. The map $d_n^n \circ s_j^{n-1}$, if $j < n-1$, is equal to $s_j^{n-2} \circ d_{n-1}^{n-1}$. Since d_{n-1}^{n-1} maps $N(U_{n-1})$ into $N(U_{n-2})$, we see that the image $d_n^n(s_j^{n-1}(N(U_{n-1}))$ is contained in $s_j^{n-2}(N(U_{n-2}))$ which is contained in $A(U_{n-1})$ by definition. For all other combinations of (i, j) we have either $d_i^n \circ s_j^{n-1} = s_{j-1}^{n-2} \circ d_i^{n-1}$ (if $i < j$), or $d_i^n \circ s_j^{n-1} = s_j^{n-2} \circ d_{i-1}^{n-1}$ (if $n > i > j+1$) and in these cases the map is zero because of the definition of $N(U_{n-1})$. $\square$

Lemma 22.7. *The functor N is exact.*

Proof. By Lemma 22.1 and the functorial decomposition of Lemma 22.6. $\square$

Lemma 22.8. *Let $\mathcal{A}$ be an abelian category. Let V be a simplicial object of $\mathcal{A}$.*
The canonical morphism of chain complexes $N(V) \to s(V)$ is a quasi-isomorphism.
In other words, the complex $A(V)$ of Lemma 22.6 is acyclic.

Proof. Note that the result holds for $K(A, k)$ for any object A and any $k \geq 0$,
by Lemmas 22.3 and 22.5. Consider the hypothesis $IH_{n,m}$: for all V such that
$V_j = 0$ for $j \leq m$ and all $i \leq n$ the map $N(V) \to s(V)$ induces an isomorphism
$H_i(N(V)) \to H_i(s(V))$.

To start of the induction, note that $IH_{n,n}$ is trivially true, because in that case
$N(V)_n = 0$ and $s(V)_n = 0$.

Assume $IH_{n,m}$, with $m \leq n$. Pick a simplicial object V such that $V_j = 0$ for $j < m$.
By Lemma 21.2 and Definition 21.3 we have $K(V_m, m) = i_{m!}\mathrm{sk}_m V$. By Lemma
20.10 the natural morphism

$$K(V_m, m) = i_{m!}\mathrm{sk}_m V \to V$$

is injective. Thus we get a short exact sequence

$$0 \to K(V_m, m) \to V \to W \to 0$$

for some W with $W_i = 0$ for $i = 0, \ldots, m$. This short exact sequence induces a
morphism of short exact sequence of associated complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N(K(V_m, m)) & \longrightarrow & N(V) & \longrightarrow & N(W) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & s(K(V_m, m)) & \longrightarrow & s(V) & \longrightarrow & s(W) & \longrightarrow & 0
\end{array}
$$

see Lemmas 22.1 and 22.7. Hence we deduce the result for V from the result on
the ends. $\square$

23. Dold-Kan

Lemma 23.1. *Let $\mathcal{A}$ be an abelian category. The functor N is faithful, and reflects*
isomorphisms, injections and surjections.

Proof. The faithfulness is immediate from the canonical splitting of Lemma 17.6.
The statement on reflecting injections, surjections, and isomorphisms follows from
Lemma 17.7. $\square$

Lemma 23.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $N : \mathcal{A} \to \mathcal{B}$, and $S : \mathcal{B} \to \mathcal{A}$*
be functors. Suppose that
   (1) *the functors S and N are exact,*
   (2) *there is an isomorphism $g : N \circ S \to id_{\mathcal{B}}$ to the identity functor of $\mathcal{B}$,*
   (3) *N is faithful, and*
   (4) *S is essentially surjective.*
Then S and N are quasi-inverse equivalences of categories.

Proof. It suffices to construct a functorial isomorphism $S(N(A)) \cong A$. To do this
choose B and an isomorphism $f : A \to S(B)$. Consider the map

$$f^{-1} \circ g_{S(B)} \circ S(N(f)) : S(N(A)) \to S(N(S(B))) \to S(B) \to A.$$

It is easy to show this does not depend on the choice of f, B and gives the desired
isomorphism $S \circ N \to id_{\mathcal{A}}$. $\square$

Theorem 23.3. *Let $\mathcal{A}$ be an abelian category. The functor N induces an equivalence of categories*

$$N : Simp(\mathcal{A}) \longrightarrow Ch_{\geq 0}(\mathcal{A})$$

Proof. We will describe a functor in the reverse direction inspired by the construction of Lemma 21.4 (except that we throw in a sign to get the boundaries right). Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For each $n \geq 0$ denote

$$I_n = \Big\{ \alpha : [n] \to \{0, 1, 2, \ldots\} \mid \text{Im}(\alpha) = [k] \text{ for some } k \Big\}.$$

For $\alpha \in I_n$ we denote $k(\alpha)$ the unique integer such that $\text{Im}(\alpha) = [k]$. We define a simplicial object $S(A_\bullet)$ as follows:

(1) $S(A_\bullet)_n = \bigoplus_{\alpha \in I_n} A_{k(\alpha)}$, which we will write as $\bigoplus_{\alpha \in I_n} A_{k(\alpha)} \cdot \alpha$ to suggest thinking of "α" as a basis vector for the summand corresponding to it,
(2) given $\varphi : [m] \to [n]$ we define $S(A_\bullet)(\varphi)$ by its restriction to the direct summand $A_{k(\alpha)} \cdot \alpha$ of $S(A_\bullet)_n$ as follows
  (a) $\alpha \circ \varphi \notin I_m$ then we set it equal to zero,
  (b) $\alpha \circ \varphi \in I_m$ but $k(\alpha \circ \varphi)$ not equal to either $k(\alpha)$ or $k(\alpha) - 1$ then we set it equal to zero as well,
  (c) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha)$ then we use the identity map to the summand $A_{k(\alpha \circ \varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$, and
  (d) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha) - 1$ then we use $(-1)^{k(\alpha)} d_{A,k(\alpha)}$ to the summand $A_{k(\alpha \circ \varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$.

It is an exercise (FIXME) to show that this is a simplicial complex; one has to use in particular that the compositions $d_{A,k} \circ d_{A,k-1}$ are all zero.

Having verified this, the correct way to proceed with the proof would be to prove directly that N and S are quasi-inverse functors (FIXME). Instead we prove this by an indirect method using Eilenberg-Maclane objects and truncations. It is clear that $A_\bullet \mapsto S(A_\bullet)$ is an exact functor from chain complexes to simplicial objects. If $A_i = 0$ for $i = 0, \ldots, n$ then $S(A_\bullet)_i = 0$ for $i = 0, \ldots, n$. The objects $K(A, k)$, see Definition 21.3, are equal to $S(A[-k])$ where $A[-k]$ is the chain complex with A in degree k and zero elsewhere.

Moreover, for each integer k we get a sub simplicial object $S_{\leq k}(A_\bullet)$ by considering only those α with $k(\alpha) \leq k$. In fact this is nothing but $S(\sigma_{\leq k} A_\bullet)$, where $\sigma_{\leq k} A_\bullet$ is the "stupid" truncation of $A_\bullet$ at k (which simply replaces A_i by 0 for $i > k$). Also, by Lemma 20.10 we see that it is equal to $i_{k!} \text{sk}_k S(A_\bullet)$. Clearly, the quotient $S_{\leq k}(A_\bullet)/S_{\leq k-1}(A_\bullet) = K(A_k, k)$ and the quotient $S(A_\bullet)/S_{\leq k}(A_\bullet) = S(A/\sigma_{\leq k} A_\bullet)$ is a simplicial object whose ith term is zero for $i = 0, \ldots, k$. Since $S_{\leq k-1}(A_\bullet)$ is filtered with subquotients $K(A_i, i)$, $i < k$ we see that $N(S_{\leq k-1}(A_\bullet))_k = 0$ by exactness of the functor N, see Lemma 22.7. All in all we conclude that the maps

$$N(S(A_\bullet))_k \leftarrow N(S_{\leq k}(A_\bullet))_k \to N(S(A_k[-k])) = N(K(A_k, k))_k = A_k$$

are functorial isomorphisms.

It is actually easy to identify the map $A_k \to N(S(A_\bullet))_k$. Note that there is a unique map $A_k \to S(A_\bullet)_k$ corresponding to the summand $\alpha = \text{id}_{[k]}$. Note that $\text{Im}(\text{id}_{[k]} \circ \delta_i^k)$ has cardinality $k - 1$ but does not have image $[k-1]$ unless $i = k$. Hence d_i^k kills the summand $A_k \cdot \text{id}_{[k]}$ for $i = 0, \ldots, k-1$. From the abstract

computation of $N(S(A_\bullet))_k$ above we conclude that the summand $A_k \cdot \mathrm{id}_{[k]}$ is equal to $N(S(A_\bullet))_k$.

In order to show that $N \circ S$ is the identity functor on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$, the last thing we have to verify is that we recover the map $d_{A,k+1} : A_{k+1} \to A_k$ as the differential on the complex $N(S(A_\bullet))$ as follows

$$A_{k+1} = N(S(A_\bullet))_{k+1} \to N(S(A_\bullet))_k = A_k$$

By definition the map $N(S(A_\bullet))_{k+1} \to N(S(A_\bullet))_k$ corresponds to the restriction of $(-1)^{k+1} d_{k+1}^{k+1}$ to $N(S(A_\bullet))$ which is the summand $A_{k+1} \cdot \mathrm{id}_{[k+1]}$. And by the definition of $S(A_\bullet)$ above the map d_{k+1}^{k+1} maps $A_{k+1} \cdot \mathrm{id}_{[k+1]}$ into $A_k \cdot \mathrm{id}_{[k]}$ by $(-1)^{k+1} d_{A,k+1}$. The signs cancel and hence the desired equality.

We know that N is faithful, see Lemma 23.1. If we can show that S is essentially surjective, then it will follow that N is an equivalence, see Homology, Lemma 23.2. Note that if $A_\bullet$ is a chain complex then $S(A_\bullet) = \mathrm{colim}_n S_{\leq n}(A_\bullet) = \mathrm{colim}_n S(\sigma_{\leq n} A_\bullet) = \mathrm{colim}_n i_{n!} \mathrm{sk}_n S(A_\bullet)$ by construction of S. By Lemma 21.5 it suffices to show that $i_{n!} V$ is in the essential image for any n-truncated simplicial object V. By induction on n it suffices to show that any extension

$$0 \to S(A_\bullet) \to V \to K(A, n) \to 0$$

where $A_i = 0$ for $i \geq n$ is in the essential image of S. By Homology, Lemma 7.2 we have abelian group homomorphisms

$$\mathrm{Ext}_{\mathrm{Simp}(\mathcal{A})}(K(A, n), S(A_\bullet)) \xrightleftharpoons[S]{N} \mathrm{Ext}_{\mathrm{Ch}_{\geq 0}(\mathcal{A})}(A[-n], A_\bullet)$$

between ext groups (see Homology, Definition 6.2). We want to show that S is surjective. We know that $N \circ S = \mathrm{id}$. Hence it suffices to show that $\mathrm{Ker}(N) = 0$. Clearly an extension

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-2} & \longrightarrow & \ldots & \longrightarrow & A_0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
E: \quad 0 & \longrightarrow & A & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-2} & \longrightarrow & \ldots & \longrightarrow & A_0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

of $A_\bullet$ by $A[-n]$ in $\mathrm{Ch}(\mathcal{A})$ is zero if and only if the map $A \to A_{n-1}$ is zero. Thus we have to show that any extension

$$0 \to S(A_\bullet) \to V \to K(A, n) \to 0$$

such that $A = N(V)_n \to N(V)_{n-1}$ is zero is split. By Lemma 21.2 we have

$$\mathrm{Mor}(K(A, n), V) = \left\{ f : A \to \bigcap\nolimits_{i=0}^{n} \mathrm{Ker}(d_i^n : V_n \to V_{n-1}) \right\}$$

and if $A = N(V)_n \to N(V)_{n-1}$ is zero, then the intersection occurring in the formula above is equal to A. Let $i : K(A, n) \to V$ be the morphism corresponding to id_A on the right hand side of the displayed formula. Clearly this is a section to the map $V \to K(A, n)$ and the extension is split as desired. $\square$

24. Dold-Kan for cosimplicial objects

Let $\mathcal{A}$ be an abelian category. According to Homology, Lemma 5.2 also $\mathcal{A}^{opp}$ is abelian. It follows formally from the definitions that

$$\text{CoSimp}(\mathcal{A}) = \text{Simp}(\mathcal{A}^{opp})^{opp}.$$

Thus Dold-Kan (Theorem 23.3) implies that $\text{CoSimp}(\mathcal{A})$ is equivalent to the category $\text{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}$. And it follows formally from the definitions that

$$\text{CoCh}_{\geq 0}(\mathcal{A}) = \text{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}.$$

Putting these arrows together we obtain an equivalence

$$Q : \text{CoSimp}(\mathcal{A}) \longrightarrow \text{CoCh}_{\geq 0}(\mathcal{A}).$$

In this section we describe Q.

First we define the *cochain complex $s(U)$ associated to a cosimplicial object U*. It is the cochain complex with terms zero in negative degrees, and $s(U)^n = U_n$ for $n \geq 0$. As differentials we use the maps $d^n : s(U)^n \to s(U)^{n+1}$ defined by $d^n = \sum_{i=0}^{n+1} (-1)^i \delta_i^{n+1}$. In other words the complex $s(U)$ looks like

$$0 \longrightarrow U_0 \xrightarrow{\delta_0^1 - \delta_1^1} U_1 \xrightarrow{\delta_0^2 - \delta_1^2 + \delta_2^2} U_2 \longrightarrow \dots$$

This is sometimes also called the *Moore complex* associated to U.

On the other hand, given a cosimplicial object U of $\mathcal{A}$ set $Q(U)^0 = U_0$ and

$$Q(U)^n = \text{Coker}\left(\bigoplus_{i=0}^{n-1} U_{n-1} \xrightarrow{\delta_i^n} U_n \right).$$

The differential $d^n : Q(U)^n \to Q(U)^{n+1}$ is induced by $(-1)^{n+1} \delta_{n+1}^{n+1}$, i.e., by fitting the morphism $(-1)^{n+1} \delta_{n+1}^{n+1}$ into a commutative diagram

$$
\begin{array}{ccc}
U_n & \xrightarrow{(-1)^{n+1}\delta_{n+1}^{n+1}} & U_{n+1} \\
\downarrow & & \downarrow \\
Q(U)^n & \xrightarrow{\quad d_n \quad} & Q(U)^{n+1}.
\end{array}
$$

We leave it to the reader to show that this diagram makes sense, i.e., that the image of δ_i^n maps into the kernel of the right vertical arrow for $i = 0, \dots, n-1$. (This is dual to Lemma 17.8.) Thus our cochain complex $Q(U)$ looks like this

$$0 \to Q(U)^0 \to Q(U)^1 \to Q(U)^2 \to \dots$$

This is called the *normalized cochain complex associated to U*. The dual to the Dold-Kan Theorem 23.3 is the following.

Lemma 24.1. *Let $\mathcal{A}$ be an abelian category.*

(1) *The functor $s : CoSimp(\mathcal{A}) \to CoCh_{\geq 0}(\mathcal{A})$ is exact.*
(2) *The maps $s(U)^n \to Q(U)^n$ define a morphism of cochain complexes.*
(3) *There exists a functorial direct sum decomposition $s(U) = A(U) \oplus Q(U)$ in $CoCh_{\geq 0}(\mathcal{A})$.*
(4) *The functor Q is exact.*
(5) *The morphism of complexes $s(U) \to Q(U)$ is a quasi-isomorphism.*
(6) *The functor $U \mapsto Q(U)^\bullet$ defines an equivalence of categories $CoSimp(\mathcal{A}) \to CoCh_{\geq 0}(\mathcal{A})$.*

Proof. Omitted. But the results are the exact dual statements to Lemmas 22.1, 22.4, 22.6, 22.7, 22.8, and Theorem 23.3. □

25. Homotopies

Consider the simplicial sets $\Delta[0]$ and $\Delta[1]$. Recall that there are two morphisms

$$e_0, e_1 : \Delta[0] \longrightarrow \Delta[1],$$

coming from the morphisms $[0] \to [1]$ mapping 0 to an element of $[1] = \{0, 1\}$. Recall also that each set $\Delta[1]_k$ is finite. Hence, if the category $\mathcal{C}$ has finite coproducts, then we can form the product

$$U \times \Delta[1]$$

for any simplicial object U of $\mathcal{C}$, see Definition 13.1. Note that $\Delta[0]$ has the property that $\Delta[0]_k = \{*\}$ is a singleton for all $k \geq 0$. Hence $U \times \Delta[0] = U$. Thus e_0, e_1 above gives rise to morphisms

$$e_0, e_1 : U \to U \times \Delta[1].$$

Definition 25.1. Let $\mathcal{C}$ be a category having finite coproducts. Suppose that U and V are two simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be two morphisms.

(1) We say a morphism

$$h : U \times \Delta[1] \longrightarrow V$$

is a *homotopy connecting a to b* if $a = h \circ e_0$ and $b = h \circ e_1$.
(2) We say morphisms a and b are *homotopic* if there exists a homotopy connecting a to b or a homotopy connecting b to a.

Warning: Being homotopic is **not** an equivalence relation on the set of all morphisms from U to V! The relation "there exists a homotopy from a to b" is not symmetric.

It turns out we can define homotopies between pairs of maps of simplicial objects in any category. To do this you just work out what it means to have the morphisms $h_n : (U \times \Delta[1])_n \to V_n$ in terms of the mapping property of coproducts.

Let $\mathcal{C}$ be a category with finite coproducts. Let U, V be simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy connecting a to b. For every $n \geq 0$ let us write

$$\Delta[1]_n = \{\alpha_0^n, \dots, \alpha_{n+1}^n\}$$

where $\alpha_i^n : [n] \to [1]$ is the map such that

$$\alpha_i^n(j) = \begin{cases} 0 & \text{if} \quad j < i \\ 1 & \text{if} \quad j \geq i \end{cases}$$

Thus

$$h_n : (U \times \Delta[1])_n = \coprod U_n \cdot \alpha_i^n \longrightarrow V_n$$

has a component $h_{n,i} : U_n \to V_n$ which is the restriction to the summand corresponding to α_i^n for all $i = 0, \dots, n + 1$.

Lemma 25.2. *In the situation above, we have the following relations:*

(1) *We have $h_{n,0} = b_n$ and $h_{n,n+1} = a_n$.*
(2) *We have $d_j^n \circ h_{n,i} = h_{n-1,i-1} \circ d_j^n$ for $i > j$.*
(3) *We have $d_j^n \circ h_{n,i} = h_{n-1,i} \circ d_j^n$ for $i \leq j$.*

(4) *We have $s_j^n \circ h_{n,i} = h_{n+1,i+1} \circ s_j^n$ for $i > j$.*
(5) *We have $s_j^n \circ h_{n,i} = h_{n+1,i} \circ s_j^n$ for $i \leq j$.*

Conversely, given a system of maps $h_{n,i}$ satisfying the properties listed above, then these define a morphisms h which is a homotopy between a and b.

Proof. Omitted. You can prove the last statement using the fact, see Lemma 2.4 that to give a morphism of simplicial objects is the same as giving a sequence of morphisms h_n commuting with all d_j^n and s_j^n. $\square$

Example 25.3. Suppose in the situation above $a = b$. Then there is a *trivial* homotopy between a and b, namely the one with $h_{n,i} = a_n = b_n$.

Remark 25.4. Let $\mathcal{C}$ be any category (no assumptions whatsoever). We say that a pair of morphisms $a, b : U \to V$ of simplicial objects are *homotopic* if there exist morphisms[1] $h_{n,i} : U_n \to V_n$, for $n \geq 0$, $i = 0, \ldots, n+1$ satisfying the relations of Lemma 25.2 (potentially with the roles of a and b switched). This is a "better" definition, because it applies to any category. Also it has the following property: if $F : \mathcal{C} \to \mathcal{C}'$ is any functor then a homotopic to b implies trivially that $F(a)$ is homotopic to $F(b)$. Since the lemma says that the newer notion is the same as the old one in case finite coproduct exist, we deduce in particular that functors preserve the old notion whenever both categories have finite coproducts.

Remark 25.5. Let $\mathcal{C}$ be any category. Suppose two morphisms $a, a' : U \to V$ of simplicial objects are homotopic. Then for any morphism $b : V \to W$ the two maps $b \circ a, b \circ a' : U \to W$ are homotopic. Similarly, for any morphism $c : X \to U$ the two maps $a \circ c, a' \circ c : X \to V$ are homotopic. In fact the maps $b \circ a \circ c, b \circ a' \circ c : X \to W$ are homotopic. Namely, if the maps $h_{n,i} : U \to U$ define a homotopy between a and a' then the maps $b \circ h_{n,i} \circ c$ define a homotopy between $b \circ a \circ c$ and $b \circ a' \circ c$.

Definition 25.6. Let U and V be two simplicial objects of a category $\mathcal{C}$. We say a morphism $a : U \to V$ is a *homotopy equivalence* if there exists a morphism $b : V \to U$ such that $a \circ b$ is homotopic to id_V and $b \circ a$ is homotopic to id_U. If there exists such a morphism between U and V, then we say that U and V are *homotopy equivalent*[2].

Example 25.7. The simplicial set $\Delta[m]$ is homotopy equivalent to $\Delta[0]$. Namely, there is a unique morphism $f : \Delta[m] \to \Delta[0]$ and we take $g : \Delta[0] \to \Delta[m]$ to be given by the inclusion of the last 0-simplex of $\Delta[m]$. We have $f \circ g = \mathrm{id}$ and we will give a homotopy $h : \Delta[m] \times \Delta[1] \to \Delta[m]$ between $\mathrm{id}_{\Delta[m]}$ and $g \circ f$. Namely h given by the maps

$$\mathrm{Mor}_\Delta([n], [m]) \times \mathrm{Mor}_\Delta([n], [1]) \to \mathrm{Mor}_\Delta([n], [m])$$

which send (φ, α) to

$$k \mapsto \begin{cases} \varphi(k) & \text{if} \quad \alpha(k) = 0 \\ m & \text{if} \quad \alpha(k) = 1 \end{cases}$$

Note that this only works because we took g to be the inclusion of the last 0-simplex. If we took g to be the inclusion of the first 0-simplex we could find a homotopy from $g \circ f$ to $\mathrm{id}_{\Delta[m]}$. This is an illustration of the asymmetry inherent in homotopies in the category of simplicial sets.

---

[1]In the literature, often the maps $h_{n+1,i} \circ s_i : U_n \to V_{n+1}$ are used instead of the maps $h_{n,i}$. Of course the relations these maps satisfy are different from the ones in Lemma 25.2.

[2]Warning: This notion is **not** an equivalence relation on objects in general.

The following lemma says that $U \times \Delta[1]$ is homotopy equivalent to U.

Lemma 25.8. *Let $\mathcal{C}$ be a category with finite coproducts. Let U be a simplicial object of $\mathcal{C}$. Consider the maps $e_1, e_0 : U \to U \times \Delta[1]$, and $\pi : U \times \Delta[1] \to U$, see Lemma 13.3.*

(1) *We have $\pi \circ e_1 = \pi \circ e_0 = \mathrm{id}_U$, and*
(2) *The morphisms $\mathrm{id}_{U \times \Delta[1]}$, and $e_0 \circ \pi$ are homotopic.*
(3) *The morphisms $\mathrm{id}_{U \times \Delta[1]}$, and $e_1 \circ \pi$ are homotopic.*

Proof. The first assertion is trivial. For the second, consider the map of simplicial sets $\Delta[1] \times \Delta[1] \longrightarrow \Delta[1]$ which in degree n assigns to a pair (β_1, β_2), $\beta_i : [n] \to [1]$ the morphism $\beta : [n] \to [1]$ defined by the rule

$$\beta(i) = \max\{\beta_1(i), \beta_2(i)\}.$$

It is a morphism of simplicial sets, because the action $\Delta[1](\varphi) : \Delta[1]_n \to \Delta[1]_m$ of $\varphi : [m] \to [n]$ is by precomposing. Clearly, using notation from Section 25, we have $\beta = \beta_1$ if $\beta_2 = \alpha_0^n$ and $\beta = \alpha_{n+1}^n$ if $\beta_2 = \alpha_{n+1}^n$. This implies easily that the induced morphism

$$U \times \Delta[1] \times \Delta[1] \longrightarrow U \times \Delta[1]$$

of Lemma 13.3 is a homotopy between $\mathrm{id}_{U \times \Delta[1]}$ and $e_0 \circ \pi$. Similarly for $e_1 \circ \pi$ (use minimum instead of maximum). $\square$

Lemma 25.9. *Let $f : Y \to X$ be a morphism of a category $\mathcal{C}$ with fibre products. Assume f has a section s. Consider the simplicial object U constructed in Example 3.5 starting with f. The morphism $U \to U$ which in each degree is the self map $(s \circ f)^{n+1}$ of $Y \times_X \ldots \times_X Y$ given by $s \circ f$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant simplicial object X.*

Proof. Set $g^0 = \mathrm{id}_Y$ and $g^1 = s \circ f$. We use the morphisms

$$
\begin{aligned}
Y \times_X \ldots \times_X Y \times \mathrm{Mor}([n], [1]) &\quad \to \quad Y \times_X \ldots \times_X Y \\
(y_0, \ldots, y_n) \times \alpha &\quad \mapsto \quad (g^{\alpha(0)}(y_0), \ldots, g^{\alpha(n)}(y_n))
\end{aligned}
$$

where we use the functor of points point of view to define the maps. Another way to say this is to say that $h_{n,0} = \mathrm{id}$, $h_{n,n+1} = (s \circ f)^{n+1}$ and $h_{n,i} = \mathrm{id}_Y^{i+1} \times (s \circ f)^{n+1-i}$. We leave it to the reader to show that these satisfy the relations of Lemma 25.2. Hence they define the desired homotopy. See also Remark 25.4 which shows that we do not need to assume anything else on the category $\mathcal{C}$. $\square$

Lemma 25.10. *Let $\mathcal{C}$ be a category.*

(1) *If $a_t, b_t : X_t \to Y_t$, $t \in T$ are homotopic morphisms between simplicial objects of $\mathcal{C}$, then $\prod a_t, \prod b_t : \prod X_t \to \prod Y_t$ are homotopic morphisms between simplicial objects of $\mathcal{C}$, provided $\prod X_t$ and $\prod Y_t$ exist in $\mathrm{Simp}(\mathcal{C})$.*
(2) *If (X_t, Y_t), $t \in T$ are homotopy equivalent pairs of simplicial objects of $\mathcal{C}$, then $\prod X_t$ and $\prod Y_t$ are homotopy equivalent pairs of simplicial objects of $\mathcal{C}$, provided $\prod X_t$ and $\prod Y_t$ exist in $\mathrm{Simp}(\mathcal{C})$.*

Proof. If $h_t = (h_{t,n,i})$ are homotopies connecting a_t and b_t (see Remark 25.4), then $h = (\prod_t h_{t,n,i})$ is a homotopy connecting $\prod a_t$ and $\prod b_t$. This proves (1). Part (2) follows from part (1) and the definitions. $\square$

26. Homotopies in abelian categories

Let $\mathcal{A}$ be an abelian category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy connecting a and b. Consider the two morphisms of chain complexes $s(a), s(b) : s(U) \longrightarrow s(V)$. Using the notation introduced above Lemma 25.2 we define

$$s(h)_n : U_n \longrightarrow V_{n+1}$$

by the formula

(26.0.1) $$s(h)_n = \sum_{i=0}^{n} (-1)^{i+1} h_{n+1,i+1} \circ s_i^n.$$

Let us compute $d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n$. We first compute

$$
\begin{aligned}
d_{n+1} \circ s(h)_n &= \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{j+i+1} d_j^{n+1} \circ h_{n+1,i+1} \circ s_i^n \\
&= \sum_{1 \le i+1 \le j \le n+1} (-1)^{j+i+1} h_{n,i+1} \circ d_j^{n+1} \circ s_i^n \\
&\quad + \sum_{n \ge i \ge j \ge 0} (-1)^{i+j+1} h_{n,i} \circ d_j^{n+1} \circ s_i^n \\
&= \sum_{1 \le i+1 < j \le n+1} (-1)^{j+i+1} h_{n,i+1} \circ s_i^{n-1} \circ d_{j-1}^n \\
&\quad + \sum_{1 \le i+1 = j \le n+1} (-1)^{j+i+1} h_{n,i+1} \\
&\quad + \sum_{n \ge i = j \ge 0} (-1)^{i+j+1} h_{n,i} \\
&\quad + \sum_{n \ge i > j \ge 0} (-1)^{i+j+1} h_{n,i} \circ s_{i-1}^{n-1} \circ d_j^n
\end{aligned}
$$

We leave it to the reader to see that the first and the last of the four sums cancel exactly against all the terms of

$$s(h)_{n-1} \circ d_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+1+j} h_{n,i+1} \circ s_i^{n-1} \circ d_j^n.$$

Hence we obtain

$$
\begin{aligned}
d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n &= \sum_{j=1}^{n+1} (-1)^{2j} h_{n,j} + \sum_{i=0}^{n} (-1)^{2i+1} h_{n,i} \\
&= h_{n,n+1} - h_{n,0} \\
&= a_n - b_n
\end{aligned}
$$

Thus we've proved part of the following lemma.

Lemma 26.1. *Let $\mathcal{A}$ be an abelian category. Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$, and $N(a), N(b) : N(U) \to N(V)$ are homotopic maps of chain complexes.*

Proof. The part about $s(a)$ and $s(b)$ is clear from the calculation above the lemma. On the other hand, if follows from Lemma 22.6 that $N(a)$, $N(b)$ are compositions

$$N(U) \to s(U) \to s(V) \to N(V)$$

where we use $s(a)$, $s(b)$ in the middle. Hence the assertion follows from Homology, Lemma 12.1. $\square$

Lemma 26.2. *Let $\mathcal{A}$ be an abelian category. Let $a : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If a is a homotopy equivalence, then $s(a) : s(U) \to s(V)$, and $N(a) : N(U) \to N(V)$ are homotopy equivalences of chain complexes.*

Proof. Omitted. See Lemma 26.1 above. $\square$

27. Homotopies and cosimplicial objects

Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object and consider $\mathrm{Hom}(\Delta[1], V)$, see Section 14. The morphisms $e_0, e_1 : \Delta[0] \to \Delta[1]$ produce two morphisms $e_0, e_1 : \mathrm{Hom}(\Delta[1], V) \to V$.

Definition 27.1. Let $\mathcal{C}$ be a category having finite products. Suppose that U and V are two cosimplicial objects of $\mathcal{C}$. We say morphisms $a, b : U \to V$ are *homotopic* if there exists a morphism
$$h : U \longrightarrow \mathrm{Hom}(\Delta[1], V)$$
such that $a = e_0 \circ h$ and $b = e_1 \circ h$. In this case h is called a *homotopy connecting a and b*.

This is really exactly the same as the notion we introduced for simplicial objects earlier. In particular, recall that $\Delta[1]_n$ is a finite set, and that
$$h_n = (h_{n,\alpha}) : U \longrightarrow \prod\nolimits_{\alpha \in \Delta[1]_n} V_n$$
is given by a collection of maps $h_{n,\alpha} : U_n \to V_n$ parametrized by elements of $\Delta[1]_n = \mathrm{Mor}_\Delta([n], [1])$. As in Lemma 25.2 these morphisms satisfy some relations. Namely, for every $f : [n] \to [m]$ in Δ we should have

(27.1.1) $$h_{m,\alpha} \circ U(f) = V(f) \circ h_{n,\alpha \circ f}$$

The condition that $a = e_0 \circ h$ means that $a_n = h_{n,0:[n] \to [1]}$ where $0 : [n] \to [1]$ is the constant map with value zero. Similarly, we should have $b_n = h_{n,1:[n] \to [1]}$. In particular we deduce once more that the notion of homotopy can be formulated between cosimplicial objects of any category, i.e., existence of products is not necessary. Here is a precise formulation of why this is dual to the notion of a homotopy between morphisms of simplicial objects.

Lemma 27.2. *Let $\mathcal{C}$ be a category having finite products. Suppose that U and V are two cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of cosimplicial objects. Recall that U, V correspond to simplicial objects U', V' of $\mathcal{C}^{opp}$. Moreover a, b correspond to morphisms $a', b' : V' \to U'$. The following are equivalent*

(1) *The morphisms $a, b : U \to V$ of cosimplicial objects are homotopic.*
(2) *The morphisms $a', b' : V' \to U'$ of simplicial objects of $\mathcal{C}^{opp}$ are homotopic.*

Proof. If $\mathcal{C}$ has finite products, then $\mathcal{C}^{opp}$ has finite coproducts. And the contravariant functor $(-)' : \mathcal{C} \to \mathcal{C}^{opp}$ transforms products into coproducts. Then it is immediate from the definitions that $(\mathrm{Hom}(\Delta[1], V))' = V' \times \Delta[1]$. And so on and so forth. $\square$

Lemma 27.3. *Let $\mathcal{C}, \mathcal{C}', \mathcal{D}, \mathcal{D}'$ be categories such that $\mathcal{C}, \mathcal{C}'$ have finite products, and $\mathcal{D}, \mathcal{D}'$ have finite coproducts.*

(1) *Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{D}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of simplicial objects.*

(2) *Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of cosimplicial objects.*

(3) *Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{C}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of cosimplicial objects.*

(4) *Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of simplicial objects.*

Proof. By Lemma 27.2 above, we can turn F into a covariant functor between a pair of categories which have finite coproducts, and we have to show that the functor preserves homotopic pairs of maps. It is explained in Remark 25.4 how this is the case. Even if the functor does not commute with coproducts! $\square$

Lemma 27.4. *Let $f : Y \to X$ be a morphism of a category $\mathcal{C}$ with pushouts. Assume f has a section s. Consider the cosimplicial object U constructed in Example 5.5 starting with f. The morphism $U \to U$ which in each degree is the self map of $Y \amalg_X \ldots \amalg_X Y$ given by $s \circ f$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant cosimplicial object X.*

Proof. The dual statement which is Lemma 25.9. Hence this lemma follows on applying Lemma 27.2. $\square$

Lemma 27.5. *Let $\mathcal{A}$ be an abelian category. Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$, and $Q(a), Q(b) : Q(U) \to Q(V)$ are homotopic maps of cochain complexes.*

Proof. Let $(-)' : \mathcal{A} \to \mathcal{A}^{opp}$ be the contravariant functor $A \mapsto A$. By Lemma 27.4 the maps a' and b' are homotopic. By Lemma 26.1 we see that $s(a')$ and $s(b')$ are homotopic maps of chain complexes. Since $s(a') = (s(a))'$ and $s(b') = (s(b))'$ we conclude that also $s(a)$ and $s(b)$ are homotopic by applying the additive contravariant functor $(-)'' : \mathcal{A}^{opp} \to \mathcal{A}$. The result for the Q-complexes follows from the direct sum decomposition of Lemma 24.1 for example. $\square$

28. More homotopies in abelian categories

Let $\mathcal{A}$ be an abelian category. In this section we show that a homotopy between morphisms in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ always comes from a morphism $U \times \Delta[1] \to V$ in the category of simplicial objects. In some sense this will provide a converse to Lemma 26.1. We first develop some material on homotopies between morphisms of chain complexes.

Lemma 28.1. *Let $\mathcal{A}$ be an abelian category. Let A be a chain complex. Consider the covariant functor*

$$B \longmapsto \{(a, b, h) \mid a, b : A \to B \text{ and } h \text{ a homotopy between } a, b\}$$

There exists a chain complex $\diamond A$ such that $\mathrm{Mor}_{Ch(\mathcal{A})}(\diamond A, -)$ is isomorphic to the displayed functor. The construction $A \mapsto \diamond A$ is functorial.

Proof. We set $\diamond A_n = A_n \oplus A_n \oplus A_{n-1}$, and we define $d_{\diamond A, n}$ by the matrix

$$d_{\diamond A, n} = \begin{pmatrix} d_{A,n} & 0 & \mathrm{id}_{A_{n-1}} \\ 0 & d_{A,n} & -\mathrm{id}_{A_{n-1}} \\ 0 & 0 & -d_{A,n-1} \end{pmatrix} : A_n \oplus A_n \oplus A_{n-1} \to A_{n-1} \oplus A_{n-1} \oplus A_{n-2}$$

If $\mathcal{A}$ is the category of abelian groups, and $(x, y, z) \in A_n \oplus A_n \oplus A_{n-1}$ then $d_{\diamond A, n}(x, y, z) = (d_n(x) + z, d_n(y) - z, -d_{n-1}(z))$. It is easy to verify that $d^2 = 0$. Clearly, there are two maps $\diamond a, \diamond b : A \to \diamond A$ (first summand and second summand), and a map $\diamond A \to A[-1]$ which give a short exact sequence

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

which is termwise split. Moreover, there is a sequence of maps $\diamond h_n : A_n \to \diamond A_{n+1}$, namely the identity from A_n to the summand A_n of $\diamond A_{n+1}$, such that $\diamond h$ is a homotopy between $\diamond a$ and $\diamond b$.

We conclude that any morphism $f : \diamond A \to B$ gives rise to a triple (a, b, h) by setting $a = f \circ \diamond a$, $b = f \circ \diamond b$ and $h_n = f_{n+1} \circ \diamond h_n$. Conversely, given a triple (a, b, h) we get a morphism $f : \diamond A \to B$ by taking

$$f_n = (a_n, b_n, h_{n-1}).$$

To see that this is a morphism of chain complexes you have to do a calculation. We only do this in case $\mathcal{A}$ is the category of abelian groups: Say $(x, y, z) \in \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$. Then

$$\begin{aligned} f_{n-1}(d_n(x, y, z)) &= f_{n-1}(d_n(x) + z, d_n(y) - z, -d_{n-1}(z)) \\ &= a_n(d_n(x)) + a_n(z) + b_n(d_n(y)) - b_n(z) - h_{n-2}(d_{n-1}(z)) \end{aligned}$$

and

$$\begin{aligned} d_n(f_n(x, y, z) &= d_n(a_n(x) + b_n(y) + h_{n-1}(z)) \\ &= d_n(a_n(x)) + d_n(b_n(y)) + d_n(h_{n-1}(z)) \end{aligned}$$

which are the same by definition of a homotopy. $\square$

Note that the extension

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

comes with sections of the morphisms $\diamond A_n \to A[-1]_n$ with the property that the associated morphism $\delta : A[-1] \to (A \oplus A)[-1]$, see Homology, Lemma 14.4 equals the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$.

Lemma 28.2. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A \oplus A \to B \to C \to 0$$

be a short exact sequence of chain complexes of $\mathcal{A}$. Suppose given in addition morphisms $s_n : C_n \to B_n$ splitting the associated short exact sequence in degree n. Let $\delta(s) : C \to (A \oplus A)[-1] = A[-1] \oplus A[-1]$ be the associated morphism of complexes, see Homology, Lemma 14.4. If $\delta(s)$ factors through the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$, then there is a unique morphism $B \to \diamond A$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A \oplus A & \longrightarrow & \diamond A & \longrightarrow & A[-1] & \longrightarrow & 0 \end{array}$$

where the vertical maps are compatible with the splittings s_n and the splittings of $\diamond A_n \to A[-1]_n$ as well.

Proof. Denote $(p_n, q_n) : B_n \to A_n \oplus A_n$ the morphism π_n of Homology, Lemma 14.4. Also write $(a, b) : A \oplus A \to B$, and $r : B \to C$ for the maps in the short exact sequence. Write the factorization of $\delta(s)$ as $\delta(s) = (1, -1) \circ f$. This means that $p_{n-1} \circ d_{B,n} \circ s_n = f_n$, and $q_{n-1} \circ d_{B,n} \circ s_n = -f_n$, and Set $B_n \to \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$ equal to $(p_n, q_n, f_n \circ r_n)$.

Now we have to check that this actually defines a morphism of complexes. We will only do this in the case of abelian groups. Pick $x \in B_n$. Then $x = a_n(x_1) + b_n(x_2) + s_n(x_3)$ and it suffices to show that our definition commutes with differential for each term separately. For the term $a_n(x_1)$ we have $(p_n, q_n, f_n \circ r_n)(a_n(x_1)) = (x_1, 0, 0)$ and the result is obvious. Similarly for the term $b_n(x_2)$. For the term $s_n(x_3)$ we have

$$
\begin{aligned}
(p_n, q_n, f_n \circ r_n)(d_n(s_n(x_3))) &= (p_n, q_n, f_n \circ r_n)(\\
&\qquad a_n(f_n(x_3)) - b_n(f_n(x_3)) + s_n(d_n(x_3))) \\
&= (f_n(x_3), -f_n(x_3), f_n(d_n(x_3)))
\end{aligned}
$$

by definition of f_n. And

$$
\begin{aligned}
d_n(p_n, q_n, f_n \circ r_n)(s_n(x_3)) &= d_n(0, 0, f_n(x_3)) \\
&= (f_n(x_3), -f_n(x_3), d_{A[-1],n}(f_n(x_3)))
\end{aligned}
$$

The result follows as f is a morphism of complexes. $\qquad\square$

Lemma 28.3. *Let $\mathcal{A}$ be an abelian category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be a pair of morphisms. Assume the corresponding maps of chain complexes $N(a), N(b) : N(U) \to N(V)$ are homotopic by a homotopy $\{N_n : N(U)_n \to N(V)_{n+1}\}$. Then a, b are homotopic in the sense of Definition 25.1. Moreover, one can choose the homotopy $h : U \times \Delta[1] \to V$ such that $N_n = N(h)_n$ where $N(h)$ is the homotopy coming from h as in Section 26.*

Proof. Let $(\diamond N(U), \diamond a, \diamond b, \diamond h)$ be as in Lemma 28.1 and its proof. By that lemma there exists a morphism $\diamond N(U) \to N(V)$ representing the triple $(N(a), N(b), \{N_n\})$. We will show there exists a morphism $\psi : N(U \times \Delta[1]) \to \diamond N(U)$ such that $\diamond a = \psi \circ N(e_0)$, and $\diamond b = \psi \circ N(e_1)$. Moreover, we will show that the homotopy between $N(e_0), N(e_1) : N(U) \to N(U \times \Delta[1])$ coming from (26.0.1) and Lemma 26.1 with $h = \mathrm{id}_{U \times \Delta[1]}$ is mapped via ψ to the canonical homotopy $\diamond h$ between the two maps $\diamond a, \diamond b : N(U) \to \diamond N(U)$. Certainly this will imply the lemma.

Note that $N : \mathrm{Simp}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$ as a functor is a direct summand of the functor $N : \mathrm{Simp}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$. Also, the functor $\diamond$ is compatible with direct sums. Thus it suffices instead to construct a morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$ with the corresponding properties. This is what we do below.

By Definition 25.1 the morphisms $e_0 : U \to U \times \Delta[1]$ and $e_1 : U \to U \times \Delta[1]$ are homotopic with homotopy $\mathrm{id}_{U \times \Delta[1]}$. By Lemma 26.1 we get an explicit homotopy $\{h_n : s(U)_n \to s(U \times \Delta[1])_{n+1}\}$ between the morphisms of chain complexes $s(e_0) : s(U) \to s(U \times \Delta[1])$ and $s(e_1) : s(U) \to s(U \times \Delta[1])$. By Lemma 28.2 above we get a corresponding morphism

$$
\Phi : \diamond s(U) \to s(U \times \Delta[1])
$$

According to the construction, Φ_n restricted to the summand $s(U)[-1]_n = s(U)_{n-1}$ of $\diamond s(U)_n$ is equal to h_{n-1}. And

$$h_{n-1} = \sum_{i=0}^{n-1} (-1)^{i+1} s_i^n \cdot \alpha_{i+1}^n : U_{n-1} \to \bigoplus_j U_n \cdot \alpha_j^n.$$

with obvious notation.

On the other hand, the morphisms $e_i : U \to U \times \Delta[1]$ induce a morphism $(e_0, e_1) :$ $U \oplus U \to U \times \Delta[1]$. Denote W the cokernel. Note that, if we write $(U \times \Delta[1])_n = \bigoplus_{\alpha:[n] \to [1]} U_n \cdot \alpha$, then we may identify $W_n = \bigoplus_{i=1}^{n} U_n \cdot \alpha_i^n$ with α_i^n as in Section 25. We have a commutative diagram

$$0 \longrightarrow U \oplus U \longrightarrow U \times \Delta[1] \longrightarrow W \longrightarrow 0$$

$$(1,1) \searrow \qquad \downarrow \pi$$

$$U$$

This implies we have a similar commutative diagram after applying the functor s. Next, we choose the splittings $\sigma_n : s(W)_n \to s(U \times \Delta[1])_n$ by mapping the summand $U_n \cdot \alpha_i^n \subset W_n$ via $(-1, 1)$ to the summands $U_n \cdot \alpha_0^n \oplus U_n \cdot \alpha_i^n \subset (U \times \Delta[1])_n$. Note that $s(\pi)_n \circ \sigma_n = 0$. It follows that $(1,1) \circ \delta(\sigma)_n = 0$. Hence $\delta(\sigma)$ factors as in Lemma 28.2. By that lemma we obtain a canonical morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$.

To compute Ψ we first compute the morphism $\delta(\sigma) : s(W) \to s(U)[-1] \oplus s(U)[-1]$. According to Homology, Lemma 14.4 and its proof, to do this we have compute

$$d_{s(U \times \delta[1]), n} \circ \sigma_n - \sigma_{n-1} \circ d_{s(W), n}$$

and write it as a morphism into $U_{n-1} \cdot \alpha_0^{n-1} \oplus U_{n-1} \cdot \alpha_n^{n-1}$. We only do this in case $\mathcal{A}$ is the category of abelian groups. We use the short hand notation x_α for $x \in U_n$ to denote the element x in the summand $U_n \cdot \alpha$ of $(U \times \Delta[1])_n$. Recall that

$$d_{s(U \times \delta[1]), n} = \sum_{i=0}^{n} (-1)^i d_i^n$$

where d_i^n maps the summand $U_n \cdot \alpha$ to the summand $U_{n-1} \cdot (\alpha \circ \delta_i^n)$ via the morphism d_i^n of the simplicial object U. In terms of the notation above this means

$$d_{s(U \times \delta[1]), n}(x_\alpha) = \sum_{i=0}^{n} (-1)^i (d_i^n(x))_{\alpha \circ \delta_i^n}$$

Starting with $x_\alpha \in W_n$, in other words $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$, we see that $\sigma_n(x_\alpha) = x_\alpha - x_{\alpha_0^n}$ and hence

$$(d_{s(U \times \delta[1]), n} \circ \sigma_n)(x_\alpha) = \sum_{i=0}^{n} (-1)^i (d_i^n(x))_{\alpha \circ \delta_i^n} - \sum_{i=0}^{n} (-1)^i (d_i^n(x))_{\alpha_0^n \circ \delta_i^n}$$

To compute $d_{s(W), n}(x_\alpha)$, we have to omit all terms where $\alpha \circ \delta_i^n = \alpha_0^{n-1}, \alpha_n^{n-1}$. Hence we get

$$(\sigma_{n-1} \circ d_{s(W), n})(x_\alpha) =$$

$$\sum_{i=0,\ldots,n \text{ and } \alpha \circ \delta_i^n \neq \alpha_0^{n-1} \text{ or } \alpha_n^{n-1}} \left((-1)^i (d_i^n(x))_{\alpha \circ \delta_i^n} - (-1)^i (d_i^n(x))_{\alpha_0^{n-1}} \right)$$

Clearly the difference of the two terms is the sum

$$\sum_{i=0,\ldots,n \text{ and } \alpha \circ \delta_i^n = \alpha_0^{n-1} \text{ or } \alpha_n^{n-1}} \left((-1)^i (d_i^n(x))_{\alpha \circ \delta_i^n} - (-1)^i (d_i^n(x))_{\alpha_0^{n-1}} \right)$$

Of course, if $\alpha \circ \delta_i^n = \alpha_0^{n-1}$ then the term drops out. Recall that $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$. The only way $\alpha_j^n \circ \delta_i^n = \alpha_n^{n-1}$ is if $j = n$ and $i = n$. Thus we actually

get 0 unless $j = n$ and in that case we get $(-1)^n (d_n^n(x))_{\alpha_n^{n-1}} - (-1)^n (d_n^n(x))_{\alpha_0^{n-1}}$. In other words, we conclude the morphism

$$\delta(\sigma)_n : W_n \to (s(U)[-1] \oplus s(U)[-1])_n = U_{n-1} \oplus U_{n-1}$$

is zero on all summands except $U_n \cdot \alpha_n^n$ and on that summand it is equal to $((-1)^n d_n^n, -(-1)^n d_n^n)$. (Namely, the first summand of the two corresponds to the factor with α_n^{n-1} because that is the map $[n-1] \to [1]$ which maps everybody to 0, and hence corresponds to e_0.)

We obtain a canonical diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & \diamond s(U) & \longrightarrow & s(U)[-1] & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle \Phi} & & \downarrow & & \\
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & s(U \times \Delta[1]) & \longrightarrow & s(W) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle \Psi} & & \downarrow & & \\
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & \diamond s(U) & \longrightarrow & s(U)[-1] & \longrightarrow & 0
\end{array}
$$

We claim that $\Phi \circ \Psi$ is the identity. To see this it is enough to prove that the composition of Φ and $\delta(\sigma)$ as a map $s(U)[-1] \to s(W) \to s(U)[-1] \oplus s(U)[-1]$ is the identity in the first factor and minus identity in the second. By the computations above it is $((-1)^n d_0^n, -(-1)^n d_0^n) \circ (-1)^n s_n^n = (1, -1)$ as desired. $\qquad\square$

29. Trivial Kan fibrations

Recall that for $n \geq 0$ the simplicial set $\Delta[n]$ is given by the rule $[k] \mapsto \mathrm{Mor}_\Delta([k], [n])$, see Example 11.2. Recall that $\Delta[n]$ has a unique nondegenerate n-simplex and all nondegenerate simplices are faces of this n-simplex. In fact, the nondegenerate simplices of $\Delta[n]$ correspond exactly to injective morphisms $[k] \to [n]$, which we may identify with subsets of $[n]$. Moreover, recall that $\mathrm{Mor}(\Delta[n], X) = X_n$ for any simplicial set X (Lemma 11.3). We set

$$\partial \Delta[n] = i_{(n-1)!} \mathrm{sk}_{n-1} \Delta[n]$$

and we call it the *boundary* of $\Delta[n]$. From Lemma 20.5 we see that $\partial \Delta[n] \subset \Delta[n]$ is the simplicial subset having the same nondegenerate simplices in degrees $\leq n-1$ but not containing the nondegenerate n-simplex.

Definition 29.1. A map $X \to Y$ of simplicial sets is called a *trivial Kan fibration* if $X_0 \to Y_0$ is surjective and for all $n \geq 1$ and any commutative solid diagram

$$
\begin{array}{ccc}
\partial \Delta[n] & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\Delta[n] & \longrightarrow & Y
\end{array}
$$

a dotted arrow exists making the diagram commute.

A trivial Kan fibration satisfies a very general lifting property.

Lemma 29.2. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. For any solid commutative diagram*

$$
\begin{array}{ccc}
Z & \xrightarrow{\quad} & X \\
\downarrow & {}^{b}\nearrow & \downarrow \\
W & \xrightarrow{\quad a \quad} & Y
\end{array}
$$

of simplicial sets with $Z \to W$ (termwise) injective a dotted arrow exists making the diagram commute.

Proof. Suppose that $Z \neq W$. Let n be the smallest integer such that $Z_n \neq W_n$. Let $x \in W_n$, $x \notin Z_n$. Denote $Z' \subset W$ the simplicial subset containing Z, x, and all degeneracies of x. Let $\varphi : \Delta[n] \to Z'$ be the morphism corresponding to x (Lemma 11.3). Then $\varphi|_{\partial\Delta[n]}$ maps into Z as all the nondegenerate simplices of $\partial\Delta[n]$ end up in Z. By assumption we can extend $b \circ \varphi|_{\partial\Delta[n]}$ to $\beta : \Delta[n] \to X$. By Lemma 20.7 the simplicial set Z' is the pushout of $\Delta[n]$ and Z along $\partial\Delta[n]$. Hence b and β define a morphism $b' : Z' \to X$. In other words, we have extended the morphism b to a bigger simplicial subset of Z.

The proof is finished by an application of Zorn's lemma (omitted). $\qquad\square$

Lemma 29.3. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism of simplicial sets. Then $X \times_Y Y' \to Y'$ is a trivial Kan fibration.*

Proof. This follows immediately from the functorial properties of the fibre product (Lemma 7.2) and the definitions. $\qquad\square$

Lemma 29.4. *The composition of two trivial Kan fibrations is a trivial Kan fibration.*

Proof. Omitted. $\qquad\square$

Lemma 29.5. *Let $\ldots \to U^2 \to U^1 \to U^0$ be a sequence of trivial Kan fibrations. Let $U = \lim U^t$ defined by taking $U_n = \lim U_n^t$. Then $U \to U^0$ is a trivial Kan fibration.*

Proof. Omitted. Hint: use that for a countable sequence of surjections of sets the inverse limit is nonempty. $\qquad\square$

Lemma 29.6. *Let $X_i \to Y_i$ be a set of trivial Kan fibrations. Then $\prod X_i \to \prod Y_i$ is a trivial Kan fibration.*

Proof. Omitted. $\qquad\square$

Lemma 29.7. *A filtered colimit of trivial Kan fibrations is a trivial Kan fibration.*

Proof. Omitted. Hint: See description of filtered colimits of sets in Categories, Section 19. $\qquad\square$

Lemma 29.8. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Then f is a homotopy equivalence.*

Proof. By Lemma 29.2 we can choose an right inverse $g : Y \to X$ to f. Consider the diagram

$$
\begin{array}{ccc}
\partial\Delta[1] \times X & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\Delta[1] \times X & \longrightarrow & Y
\end{array}
$$

Here the top horizontal arrow is given by id_X and $g \circ f$ where we use that $(\partial\Delta[1] \times X)_n = X_n \amalg X_n$ for all $n \geq 0$. The bottom horizontal arrow is given by the map $\Delta[1] \to \Delta[0]$ and $f : X \to Y$. The diagram commutes as $f \circ g \circ f = f$. By Lemma 29.2 we can fill in the dotted arrow and we win. $\qquad\square$

30. Kan fibrations

Let n, k be integers with $0 \leq k \leq n$ and $1 \leq n$. Let $\sigma_0, \dots, \sigma_n$ be the $n + 1$ faces of the unique nondegenerate n-simplex σ of $\Delta[n]$, i.e., $\sigma_i = d_i \sigma$. We let

$$\Lambda_k[n] \subset \Delta[n]$$

be the *kth horn* of the n-simplex $\Delta[n]$. It is the simplicial subset of $\Delta[n]$ generated by $\sigma_0, \dots, \hat{\sigma}_k, \dots, \sigma_n$. In other words, the image of the displayed inclusion contains all the nondegenerate simplices of $\Delta[n]$ except for σ and σ_k.

Definition 30.1. A map $X \to Y$ of simplicial sets is called a *Kan fibration* if for all k, n with $1 \leq n$, $0 \leq k \leq n$ and any commutative solid diagram

$$
\begin{array}{ccc}
\Lambda_k[n] & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\Delta[n] & \longrightarrow & Y
\end{array}
$$

a dotted arrow exists making the diagram commute. A *Kan complex* is a simplicial set X such that $X \to *$ is a Kan fibration, where $*$ is the constant simplicial set on a singleton.

Note that $\Lambda_k[n]$ is always nonempty. This a morphism from the empty simplicial set to any simplicial set is always a Kan fibration. It follows from Lemma 29.2 that a trivial Kan fibration is a Kan fibration.

Lemma 30.2. *Let $f : X \to Y$ be a Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism of simplicial sets. Then $X \times_Y Y' \to Y'$ is a Kan fibration.*

Proof. This follows immediately from the functorial properties of the fibre product (Lemma 7.2) and the definitions. $\qquad\square$

Lemma 30.3. *The composition of two Kan fibrations is a Kan fibration.*

Proof. Omitted. $\qquad\square$

Lemma 30.4. *Let $\dots \to U^2 \to U^1 \to U^0$ be a sequence of Kan fibrations. Let $U = \lim U^t$ defined by taking $U_n = \lim U_n^t$. Then $U \to U^0$ is a Kan fibration.*

Proof. Omitted. Hint: use that for a countable sequence of surjections of sets the inverse limit is nonempty. $\qquad\square$

Lemma 30.5. *Let $X_i \to Y_i$ be a set of Kan fibrations. Then $\prod X_i \to \prod Y_i$ is a Kan fibration.*

Proof. Omitted. $\square$

The following lemma is due to J.C. Moore, see [Moo55].

Lemma 30.6. *Let X be a simplicial group. Then X is a Kan complex.*

Proof. The following proof is basically just a translation into English of the proof in the reference mentioned above. Using the terminology as explained in the introduction to this section, suppose $f : \Lambda_k[n] \to X$ is a morphism from a horn. Set $x_i = f(\sigma_i) \in X_{n-1}$ for $i = 0, \ldots, \hat{k}, \ldots, n$. This means that for $i < j$ we have $d_i x_j = d_{j-1} x_i$ whenever $i, j \neq k$. We have to find an $x \in X_n$ such that $x_i = d_i x$ for $i = 0, \ldots, \hat{k}, \ldots, n$.

We first prove there exists a $u \in X_n$ such that $d_i u = x_i$ for $i < k$. This is trivial for $k = 0$. If $k > 0$, one defines by induction an element $u^r \in X_n$ such that $d_i u^r = x_i$ for $0 \leq i \leq r$. Start with $u^0 = s_0 x_0$. If $r < k - 1$, we set

$$y^r = s_{r+1}((d_{r+1} u^r)^{-1} x_{r+1}), \quad u^{r+1} = u^r y^r.$$

An easy calculation shows that $d_i y^r = 1$ (unit element of the group X_{n-1}) for $i \leq r$ and $d_{r+1} y^r = (d_{r+1} u^r)^{-1} x_{r+1}$. It follows that $d_i u^{r+1} = x_i$ for $i \leq r + 1$. Finally, take $u = u^{k-1}$ to get u as promised.

Next we prove, by induction on the integer r, $0 \leq r \leq n - k$, there exists a $x^r \in X_n$ such that

$$d_i x^r = x_i \quad \text{for } i < k \text{ and } i > n - r.$$

Start with $x^0 = u$ for $r = 0$. Having defined x^r for $r \leq n - k - 1$ we set

$$z^r = s_{n-r-1}((d_{n-r} x^r)^{-1} x_{n-r}), \quad x^{r+1} = x^r z^r$$

A simple calculation, using the given relations, shows that $d_i z^r = 1$ for $i < k$ and $i > n - r$ and that $d_{n-r}(z^r) = (d_{n-r} x^r)^{-1} x_{n-r}$. It follows that $d_i x^{r+1} = x_i$ for $i < k$ and $i > n - r - 1$. Finally, we take $x = x^{n-k}$ which finishes the proof. $\square$

Lemma 30.7. *Let $f : X \to Y$ be a homomorphism of simplicial abelian groups which is termwise surjective. Then f is a Kan fibration of simplicial sets.*

Proof. Consider a commutative solid diagram

$$
\begin{array}{ccc}
\Lambda_k[n] & \xrightarrow{a} & X \\
\downarrow & \nearrow & \downarrow \\
\Delta[n] & \xrightarrow{b} & Y
\end{array}
$$

as in Definition 30.1. The map a corresponds to $x_0, \ldots, \hat{x}_k, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1} x_i$ for $i < j$, $i, j \neq k$. The map b corresponds to an element $y \in Y_n$ such that $d_i y = f(x_i)$ for $i \neq k$. Our task is to produce an $x \in X_n$ such that $d_i x = x_i$ for $i \neq k$ and $f(x) = y$.

Since f is termwise surjective we can find $x \in X_n$ with $f(x) = y$. Replace y by $0 = y - f(x)$ and x_i by $x_i - d_i x$ for $i \neq k$. Then we see that we may assume $y = 0$. In particular $f(x_i) = 0$. In other words, we can replace X by $\mathrm{Ker}(f) \subset X$ and Y by 0. In this case the statement become Lemma 30.6. $\square$

Lemma 30.8. *Let $f : X \to Y$ be a homomorphism of simplicial abelian groups which is termwise surjective and induces a quasi-isomorphism on associated chain complexes. Then f is a trivial Kan fibration of simplicial sets.*

Proof. Consider a commutative solid diagram

$$
\begin{array}{ccc}
\partial\Delta[n] & \overset{a}{\longrightarrow} & X \\
\downarrow & \nearrow & \downarrow \\
\Delta[n] & \overset{b}{\longrightarrow} & Y
\end{array}
$$

as in Definition 29.1. The map a corresponds to $x_0, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1} x_i$ for $i < j$. The map b corresponds to an element $y \in Y_n$ such that $d_i y = f(x_i)$. Our task is to produce an $x \in X_n$ such that $d_i x = x_i$ and $f(x) = y$.

Since f is termwise surjective we can find $x \in X_n$ with $f(x) = y$. Replace y by $0 = y - f(x)$ and x_i by $x_i - d_i x$. Then we see that we may assume $y = 0$. In particular $f(x_i) = 0$. In other words, we can replace X by $\mathrm{Ker}(f) \subset X$ and Y by 0. This works, because by Homology, Lemma 12.6 the homology of the chain complex associated to $\mathrm{Ker}(f)$ is zero and hence $\mathrm{Ker}(f) \to 0$ induces a quasi-isomorphism on associated chain complexes.

Since X is a Kan complex (Lemma 30.6) we can find $x \in X_n$ with $d_i x = x_i$ for $i = 0, \ldots, n-1$. After replacing x_i by $x_i - d_i x$ for $i = 0, \ldots, n$ we may assume that $x_0 = x_1 = \ldots = x_{n-1} = 0$. In this case we see that $d_i x_n = 0$ for $i = 0, \ldots, n-1$. Thus $x_n \in N(X)_{n-1}$ and lies in the kernel of the differential $N(X)_{n-1} \to N(X)_{n-2}$. Here $N(X)$ is the normalized chain complex associated to X, see Section 22. Since $N(X)$ is quasi-isomorphic to $s(X)$ (Lemma 22.8) and thus acyclic we find $x \in N(X_n)$ whose differential is x_n. This x answers the question posed by the lemma and we are done. $\square$

Lemma 30.9. *Let $f : X \to Y$ be a map of simplicial abelian groups. If f is termwise surjective[3] and a homotopy equivalence of simplicial sets, then f induces a quasi-isomorphism of associated chain complexes.*

Proof. By assumption there exists a map $g : Y \to X$ of simplicial sets, a homotopy $h : X \times \Delta[1] \to X$ between $g \circ f$ and id_X, and a homotopy $h' : Y \times \Delta[1] \to Y$ between $f \circ g$ and id_Y. During this proof we will write $H_n(X) = H_n(s(X)) = H_n(N(X))$, see Section 22.

Note that $H_0(X)$ is the cokernel of the difference map $d_1 - d_0 : X_1 \to X_0$. Observe that $x \in X_0$ corresponds to a morphism $\Delta[0] \to X$. Composing h with the induced map $\Delta[0] \times \Delta[1] \to X \times \Delta[1]$ we see that x and $g(f(x))$ are equal to $d_0 x'$ and $d_1 x'$ for some $x' \in X_1$. Similarly for $y \in Y_0$. We conclude that f defines a bijection $H_0(X) \to H_0(Y)$.

---

[3]This assumption is not necessary. Also the proof as currently given is not the right one. A better proof is to define the homotopy groups of Kan complex and show that these are equal to the homology groups of the associated complex for a simplicial abelian group.

Let $n \geq 1$. Consider the simplicial set S which is the pushout of

$$
\begin{array}{ccc}
\partial\Delta[n] & \longrightarrow & * \\
\downarrow & & \downarrow \\
\Delta[n] & \longrightarrow & S
\end{array}
$$

Concretely, we take

$$S_k = \{\varphi : [k] \to [n] \mid \varphi \text{ is surjective}\} \amalg \{*\}.$$

Denote $E = \mathbf{Z}[S]$ the free abelian group on S. The inclusion $\Delta[0] \to S$ coming from $* \in S_0$ determines an injection $K(\mathbf{Z}, 0) \to E$ whose cokernel is the object $K(\mathbf{Z}, n)$, i.e., we have a short exact sequence

$$0 \to K(\mathbf{Z}, 0) \to E \to K(\mathbf{Z}, n) \to 0$$

See Definition 21.3 and the description of the Eilenberg-Maclane objects in Lemma 21.2. Note that the extension above is split, for example because the element $\xi = [\mathrm{id}_{[n]}] - [*] \in E_n$ satisfies $d_i \xi = 0$ and maps to the "generator" of $K(\mathbf{Z}, n)$. We have

$$\mathrm{Mor}_{\mathrm{Simp}(Sets)}(S, X) = \mathrm{Mor}_{\mathrm{Simp}(Ab)}(E, X) = X_0 \times \bigcap\nolimits_{i=0,\ldots,n} \mathrm{Ker}(d_i : X_n \to X_{n-1})$$

This uses the choice of our splitting above and the description of morphisms out of Eilenberg-Maclane objects given in Lemma 21.2. Note that we can think of $\bigcap_{i=0,\ldots,n} \mathrm{Ker}(d_i : X_n \to X_{n-1})$ as the cycles in degree n in the normalized chain complex associated to X, see Section 22. If two maps $a, b : S \to X$ are homotopic (as maps of simplicial sets), then the corresponding maps $a', b' : E \to X$ are homotopic as maps of simplicial abelian groups (because taking the free abelian group on is a functor). Thus if a, resp. b correspond to (a_0, a_n), resp. (b_0, b_n) in the formula above, then a_0 and b_0 define the same element of $H_0(X)$ and a_n and b_n define the same class in $H_n(X)$. See Lemma 26.1.

We come the final arguments of the proof. An element y of $H_n(Y)$ can be represented by an element y_n in $\bigcap_{i=0,\ldots,n} \mathrm{Ker}(d_i : Y_n \to Y_{n-1})$. Let $a : S \to Y$ be the map of simplicial sets corresponding to $(0, y_n)$. Then $b = g \circ a$ corresponds to some (b_0, b_n) as above for X. Using the homotopy h' we see $(f(b_0), f(b_n))$ and $(0, y_n)$ come from homotopic maps $S \to Y$ and hence y_n and $f(b_n)$ define the same element of $H_n(Y)$. Clearly this shows that $H_n(f)$ is surjective. Conversely, suppose x_n in $\bigcap_{i=0,\ldots,n} \mathrm{Ker}(d_i : X_n \to X_{n-1})$ and $f(x_n) = d(y')$ with $y' \in N(Y_{n+1})$. Since f is termwise surjective so is the induced map $f : N(X_{n+1}) \to N(Y_{n+1})$ (see Lemma 22.6). Thus we can pick $x' \in N(X_{n+1})$ mapping to y'. After replacing x_n by $x_n - d(x')$ we reach the point where $f(x_n) = 0$. This means that the morphism $a : S \to X$ corresponding to $(0, x_n)$ has the property that $f \circ a$ is the constant morphism with value 0 in Y. Hence $g \circ f \circ a$ is also a constant morphism, i.e., corresponds to a pair $(b_0, 0)$. Since as before x_n and 0 represent the same element of $H_n(X)$ we conclude. $\qquad\square$

31. A homotopy equivalence

Suppose that A, B are sets, and that $f : A \to B$ is a map. Consider the associated map of simplicial sets

$$\mathrm{cosk}_0(A) = \Big(\ldots A \times A \times A \rightrightarrows A \times A \rightrightarrows A \Big)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\mathrm{cosk}_0(B) = \Big(\ldots B \times B \times B \rightrightarrows B \times B \rightrightarrows B \Big)$$

See Example 18.1. The case $n = 0$ of the following lemma says that this map of simplicial sets is a trivial Kan fibration if f is surjective.

Lemma 31.1. *Let $f : V \to U$ be a morphism of simplicial sets. Let $n \geq 0$ be an integer. Assume*

(1) *The map $f_i : V_i \to U_i$ is a bijection for $i < n$.*
(2) *The map $f_n : V_n \to U_n$ is a surjection.*
(3) *The canonical morphism $U \to \mathrm{cosk}_n \mathrm{sk}_n U$ is an isomorphism.*
(4) *The canonical morphism $V \to \mathrm{cosk}_n \mathrm{sk}_n V$ is an isomorphism.*

Then f is a trivial Kan fibration.

Proof. Consider a solid diagram

$$\begin{array}{ccc} \partial\Delta[k] & \longrightarrow & V \\ \downarrow & \nearrow & \downarrow \\ \Delta[k] & \longrightarrow & U \end{array}$$

as in Definition 29.1. Let $x \in U_k$ be the k-simplex corresponding to the lower horizontal arrow. If $k \leq n$ then the dotted arrow is the one corresponding to a lift $y \in V_k$ of x; the diagram will commute as the other nondegenerate simplices of $\Delta[k]$ are in degrees $< k$ where f is an isomorphism. If $k > n$, then by conditions (3) and (4) we have (using adjointness of skeleton and coskeleton functors)

$$\mathrm{Mor}(\Delta[k], U) = \mathrm{Mor}(\mathrm{sk}_n\Delta[k], \mathrm{sk}_n U) = \mathrm{Mor}(\mathrm{sk}_n\partial\Delta[k], \mathrm{sk}_n U) = \mathrm{Mor}(\partial\Delta[k], U)$$

and similarly for V because $\mathrm{sk}_n\Delta[k] = \mathrm{sk}_n\partial\Delta[k]$ for $k > n$. Thus we obtain a unique dotted arrow fitting into the diagram in this case also. $\square$

Let A, B be sets. Let $f^0, f^1 : A \to B$ be maps of sets. Consider the induced maps $f^0, f^1 : \mathrm{cosk}_0(A) \to \mathrm{cosk}_0(B)$ abusively denoted by the same symbols. The following lemma for $n = 0$ says that f^0 is homotopic to f^1. In fact, the homotopy is given by the map $h : \mathrm{cosk}_0(A) \times \Delta[1] \to \mathrm{cosk}_0(A)$ with components

$$h_m : A \times \ldots \times A \times \mathrm{Mor}_\Delta([m], [1]) \longrightarrow A \times \ldots \times A,$$
$$(a_0, \ldots, a_m, \alpha) \longmapsto (f^{\alpha(0)}(a_0), \ldots, f^{\alpha(m)}(a_m))$$

To check that this works, note that for a map $\varphi : [k] \to [m]$ the induced maps are $(a_0, \ldots, a_m) \mapsto (a_{\varphi(0)}, \ldots, a_{\varphi(k)})$ and $\alpha \mapsto \alpha \circ \varphi$. Thus $h = (h_m)_{m \geq 0}$ is clearly a map of simplicial sets as desired.

Lemma 31.2. *Let $f^0, f^1 : V \to U$ be maps of a simplicial sets. Let $n \geq 0$ be an integer. Assume*

(1) *The maps $f_i^j : V_i \to U_i$, $j = 0, 1$ are equal for $i < n$.*
(2) *The canonical morphism $U \to \mathrm{cosk}_n \mathrm{sk}_n U$ is an isomorphism.*
(3) *The canonical morphism $V \to \mathrm{cosk}_n \mathrm{sk}_n V$ is an isomorphism.*

Then f^0 is homotopic to f^1.

First proof. Let W be the n-truncated simplicial set with $W_i = U_i$ for $i < n$ and $W_n = U_n / \sim$ where $\sim$ is the equivalence relation generated by $f^0(y) \sim f^1(y)$ for $y \in V_n$. This makes sense as the morphisms $U(\varphi) : U_n \to U_i$ corresponding to $\varphi : [i] \to [n]$ for $i < n$ factor through the quotient map $U_n \to W_n$ because f^0 and f^1 are morphisms of simplicial sets and equal in degrees $< n$. Next, we upgrade W to a simplicial set by taking $\mathrm{cosk}_n W$. By Lemma 31.1 the morphism $g : U \to W$ is a trivial Kan fibration. Observe that $g \circ f^0 = g \circ f^1$ by construction and denote this morphism $f : V \to W$. Consider the diagram

$$
\begin{array}{ccc}
\partial\Delta[1] \times V & \xrightarrow{\ f^0, f^1\ } & U \\
\downarrow & \nearrow & \downarrow \\
\Delta[1] \times V & \xrightarrow{\quad f \quad} & W
\end{array}
$$

By Lemma 29.2 the dotted arrow exists and the proof is done. $\qquad\square$

Second proof. We have to construct a morphism of simplicial sets $h : V \times \Delta[1] \to U$ which recovers f^i on composing with e_i. The case $n = 0$ was dealt with above the lemma. Thus we may assume that $n \geq 1$. The map $\Delta[1] \to \mathrm{cosk}_1 \mathrm{sk}_1 \Delta[1]$ is an isomorphism, see Lemma 18.15. Thus we see that $\Delta[1] \to \mathrm{cosk}_n \mathrm{sk}_n \Delta[1]$ is an isomorphism as $n \geq 1$, see Lemma 18.10. And hence $V \times \Delta[1] \to \mathrm{cosk}_n \mathrm{sk}_n(V \times \Delta[1])$ is an isomorphism too, see Lemma 18.12. In other words, in order to construct the homotopy it suffices to construct a suitable morphism of n-truncated simplicial sets $h : \mathrm{sk}_n V \times \mathrm{sk}_n \Delta[1] \to \mathrm{sk}_n U$.

For $k = 0, \ldots, n-1$ we define h_k by the formula $h_k(v, \alpha) = f^0(v) = f^1(v)$. The map $h_n : V_n \times \mathrm{Mor}_\Delta([k], [1]) \to U_n$ is defined as follows. Pick $v \in V_n$ and $\alpha : [n] \to [1]$:

(1) If $\mathrm{Im}(\alpha) = \{0\}$, then we set $h_n(v, \alpha) = f^0(v)$.
(2) If $\mathrm{Im}(\alpha) = \{0, 1\}$, then we set $h_n(v, \alpha) = f^0(v)$.
(3) If $\mathrm{Im}(\alpha) = \{1\}$, then we set $h_n(v, \alpha) = f^1(v)$.

Let $\varphi : [k] \to [l]$ be a morphism of $\Delta_{\leq n}$. We will show that the diagram

$$
\begin{array}{ccc}
V_l \times \mathrm{Mor}([l], [1]) & \longrightarrow & U_l \\
\downarrow & & \downarrow \\
V_k \times \mathrm{Mor}([k], [1]) & \longrightarrow & U_k
\end{array}
$$

commutes. Pick $v \in V_l$ and $\alpha : [l] \to [1]$. The commutativity means that

$$
h_k(V(\varphi)(v), \alpha \circ \varphi) = U(\varphi)(h_l(v, \alpha)).
$$

In almost every case this holds because $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^0(V(\varphi)(v))$ and $U(\varphi)(h_l(v, \alpha)) = U(\varphi)(f^0(v))$, combined with the fact that f^0 is a morphism of simplicial sets. The only cases where this does not hold is when either (A) $\mathrm{Im}(\alpha) = \{1\}$ and $l = n$ or (B) $\mathrm{Im}(\alpha \circ \varphi) = \{1\}$ and $k = n$. Observe moreover that necessarily $f^0(v) = f^1(v)$ for any degenerate n-simplex of V. Thus we can narrow the cases

above down even further to the cases (A) $\text{Im}(\alpha) = \{1\}$, $l = n$ and v nondegenerate, and (B) $\text{Im}(\alpha \circ \varphi) = \{1\}$, $k = n$ and $V(\varphi)(v)$ nondegenerate.

In case (A), we see that also $\text{Im}(\alpha \circ \varphi) = \{1\}$. Hence we see that not only $h_l(v, \alpha) = f^1(v)$ but also $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^1(V(\varphi)(v))$. Thus we see that the relation holds because f^1 is a morphism of simplicial sets.

In case (B) we conclude that $l = k = n$ and φ is bijective, since otherwise $V(\varphi)(v)$ is degenerate. Thus $\varphi = \text{id}_{[n]}$, which is a trivial case. $\square$

Lemma 31.3. *Let A, B be sets, and that $f : A \to B$ is a map. Consider the simplicial set U with n-simplices*

$$A \times_B A \times_B \ldots \times_B A \ (n+1 \ \text{factors}).$$

see Example 3.5. If f is surjective, the morphism $U \to B$ where B indicates the constant simplicial set with value B is a trivial Kan fibration.

Proof. Observe that U fits into a cartesian square

$$
\begin{array}{ccc}
U & \longrightarrow & \text{cosk}_0(B) \\
\downarrow & & \downarrow \\
B & \longrightarrow & \text{cosk}_0(A)
\end{array}
$$

Since the right vertical arrow is a trivial Kan fibration by Lemma 31.1, so is the left by Lemma 29.3. $\square$

32. Standard resolutions

Some of the material in this section can be found in [God73, Appendix 1] and [Ill72, I 1.5].

Situation 32.1. Let $\mathcal{A}$, $\mathcal{S}$ be categories and let $i : \mathcal{A} \to \mathcal{S}$ be a functor with a left adjoint $F : \mathcal{S} \to \mathcal{A}$.

In this very general situation we will construct a simplicial object X in the category of functors from $\mathcal{A}$ to $\mathcal{A}$. Please keep the following example in mind while we do this.

Example 32.2. As an example of the above we can take $i : Rings \to Sets$ to be the forgetful functor and $F : Sets \to Rings$ to be the functor that associates to a set E the polynomial algebra $\mathbf{Z}[E]$ on E over $\mathbf{Z}$. The simplicial object X when evaluated on an ring A will give the simplicial ring

$$\mathbf{Z}[\mathbf{Z}[\mathbf{Z}[A]]] \mathrel{\substack{\longrightarrow \\[-0.6em] \longrightarrow \\[-0.6em] \longrightarrow}} \mathbf{Z}[\mathbf{Z}[A]] \mathrel{\substack{\longrightarrow \\[-0.6em] \longrightarrow}} \mathbf{Z}[A]$$

which comes with an augmentation towards A. We will also show this augmentation is a homotopy equivalence.

For the general construction we will use the horizontal composition as defined in Categories, Section 26. The definition of the adjunction morphisms $k : F \circ i \to \text{id}_{\mathcal{A}}$ and $t : \text{id}_{\mathcal{S}} \to i \circ F$ in Categories, Section 24 shows that the compositions

$$(32.2.1) \qquad i \xrightarrow{t \star 1_i} i \circ F \circ i \xrightarrow{1_i \star k} i \quad \text{and} \quad F \xrightarrow{1_F \star t} F \circ i \circ F \xrightarrow{k \star 1_F} F$$

are the identity morphisms. Here to define the morphism $t \star 1$ we silently identify i with $\mathrm{id}_{\mathcal{S}} \circ i$ and 1 stands for $\mathrm{id}_i : i \to i$. We will use this notation and these relations repeatedly in what follows. For $n \geq 0$ we set

$$X_n = (F \circ i)^{\circ(n+1)} = F \circ i \circ F \circ \ldots \circ i \circ F$$

In other words, X_n is the $(n+1)$-fold composition of $F \circ i$ with itself. We also set $X_{-1} = \mathrm{id}_{\mathcal{A}}$. We have $X_{n+m+1} = X_n \circ X_m$ for all $n, m \geq -1$. We will endow this sequence of functors with the structure of a simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ by constructing the morphisms of functors

$$d_j^n : X_n \to X_{n-1}, \quad s_j^n : X_n \to X_{n+1}$$

satisfying the relations displayed in Lemma 2.3. Namely, we set

$$d_j^n = 1_{X_{j-1}} \star k \star 1_{X_{n-j-1}} \quad \text{and} \quad s_j^n = 1_{X_{j-1} \circ F} \star t \star 1_{i \circ X_{n-j-1}}$$

Finally, write $\epsilon_0 = k : X_0 \to X_{-1}$.

Example 32.3. In Example 32.2 we have $X_n(A) = \mathbf{Z}[\mathbf{Z}[\ldots [A] \ldots]]$ with $n + 1$ brackets. We describe the maps constructed above using a typical element $\xi = \sum n_i[n_{ij}[a_{ij}]]$ of $X_1(A)$. The maps $d_0, d_1 : \mathbf{Z}[\mathbf{Z}[A]] \to \mathbf{Z}[A]$ are given by

$$d_0(\xi) = \sum n_i n_{ij}[a_{ij}] \quad \text{and} \quad d_1(\xi) = \sum n_i[n_{ij} a_{ij}].$$

The maps $s_0, s_1 : \mathbf{Z}[\mathbf{Z}[A]] \to \mathbf{Z}[\mathbf{Z}[\mathbf{Z}[A]]]$ are given by

$$s_0(\xi) = \sum n_i[[n_{ij}[a_{ij}]]] \quad \text{and} \quad s_1(\xi) = \sum n_i[n_{ij}[[a_{ij}]]].$$

Lemma 32.4. *In Situation 32.1 the system $X = (X_n, d_j^n, s_j^n)$ is a simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ and ϵ_0 defines an augmentation ϵ from X to the constant simplicial object with value $X_{-1} = \mathrm{id}_{\mathcal{A}}$.*

Proof. Suppose that we have shown that X is a simplicial object. Then to prove that $\epsilon_0 = k$ defines an augmentation we have to check that $\epsilon_0 \circ d_0^1 = \epsilon_0 \circ d_1^1$ as morphisms $X_1 \to X_{-1}$, see Lemma 19.2. In other words, we have to check that the diagram

$$
\begin{array}{ccc}
F \circ i \circ F \circ i & \xrightarrow{\ 1_{F \circ i} \star k\ } & F \circ i \\
{\scriptstyle k \star 1_{F \circ i}} \downarrow & & \downarrow {\scriptstyle k} \\
F \circ i & \xrightarrow{\quad k \quad} & \mathrm{id}_{\mathcal{A}}
\end{array}
$$

is commutative. More precisely we should write this as the equality

$$(k \star 1_{\mathrm{id}_{\mathcal{A}}}) \circ (1_{F \circ i} \star k) = (1_{\mathrm{id}_{\mathcal{A}}} \star k) \circ (k \star 1_{F \circ i})$$

as morphisms $(F \circ i) \circ (F \circ i) \to \mathrm{id}_{\mathcal{A}} \circ \mathrm{id}_{\mathcal{A}}$. Applying the general property of Categories, Lemma 26.2 both sides expand to $k \star k$ when equality holds.

To prove that X is a simplicial object we have to check (see Remark 3.3):

(1) If $i < j$, then $d_i \circ d_j = d_{j-1} \circ d_i$.
(2) If $i < j$, then $d_i \circ s_j = s_{j-1} \circ d_i$.
(3) We have $\mathrm{id} = d_j \circ s_j = d_{j+1} \circ s_j$.
(4) If $i > j + 1$, then $d_i \circ s_j = s_j \circ d_{i-1}$.
(5) If $i \leq j$, then $s_i \circ s_j = s_{j+1} \circ s_i$.

Relation (1) is proved in exactly the same manner as the proof of the equality $\epsilon_0 \circ d_0^1 = \epsilon_0 \circ d_1^1$ above.

The simplest case of equality (5) is the commutativity of the diagram

$$
\begin{array}{ccc}
F \circ i & \xrightarrow{\ 1_F \star t \star 1_i\ } & F \circ i \circ F \circ i \\
{\scriptstyle 1_F \star t \star 1_i}\Big\downarrow & & \Big\downarrow{\scriptstyle 1_F \star t \star 1_{i \circ F \circ i}} \\
F \circ i \circ F \circ i & \xrightarrow{\ 1_{F \circ i \circ F} \star t \star 1_i\ } & F \circ i \circ F \circ i \circ F \circ i
\end{array}
$$

which holds because both compositions expand to the morphism $1_F \star t \star t \star 1_i$ from $F \circ \mathrm{id}_{\mathcal{A}} \circ \mathrm{id}_{\mathcal{A}} \circ i$ to $F \circ (i \circ F) \circ (i \circ F) \circ i$. All other cases of (5) are proved in the same manner.

The simplest case of equalities (2) and (4) is the commutativity of the diagram

$$
\begin{array}{ccc}
F \circ i \circ F \circ i & \xrightarrow{\ 1_F \star k \star 1_i\ } & F \circ i \\
{\scriptstyle 1_{F \circ i \circ F \circ i} \star t}\Big\downarrow & & \Big\downarrow{\scriptstyle 1_{F \circ i} \star t} \\
F \circ i \circ F \circ i \circ F \circ i & \xrightarrow{\ 1_F \star k \star 1_{i \circ F \circ i \circ F}\ } & F \circ i \circ F \circ i
\end{array}
$$

which again holds because both compositions expand to give $1_F \star k \star 1_i \star t$ as maps from $F \circ (i \circ F) \circ i \circ \mathrm{id}_{\mathcal{A}}$ to $F \circ \mathrm{id}_{\mathcal{A}} \circ i \circ (F \circ i)$. All other cases of (2) and (4) are proved in the same manner.

The relations (3) are the only nontrivial ones and these are consequences of the fact that the compositions in (32.2.1) are the identity. For example, the simplest case of (3) states that the compositions

$$
\begin{array}{ccc}
F \circ i & \xrightarrow{\ 1_{F \circ i} \star t\ } & F \circ i \circ F \circ i \\
{\scriptstyle t \star 1_{F \circ i}}\Big\downarrow & & \Big\downarrow{\scriptstyle 1_F \star k \star 1_i} \\
F \circ i \circ F \circ i & \xrightarrow{\ 1_F \star k \star 1_i\ } & F \circ i
\end{array}
$$

go around the diagram either way evaluate out to the identity. Going around the top the composition evaluates to $1_F \star ((k \star 1_i) \circ (1_i \star t))$ which is the identity by what was said above. The other cases of (3) are proved in the same manner. $\quad\square$

Before reading the proof of the following lemma, we strongly urge the reader to look at the example discussed in Example 32.6 in order to understand the purpose of the lemma.

Lemma 32.5. *In Situation 32.1 the maps*

$$
1_i \star \epsilon : i \circ X \to i, \quad \text{and} \quad \epsilon \star 1_F : X \circ F \to F
$$

are homotopy equivalences.

Proof. Denote $\epsilon_n : X_n \to X_{-1}$ the components of the augmentation morphism. We observe that $\epsilon_n = k^{\star(n+1)}$, the $(n+1)$-fold $\star$-composition of k. Recall that $t : \mathrm{id}_{\mathcal{S}} \to i \circ F$ is the adjunction map. We have the morphisms

$$
t^{\star(n+1)} \star 1_i : i \longrightarrow i \circ (F \circ i)^{\circ(n+1)} = i \circ X_n
$$

which are right inverse to $1_i \star \epsilon_n$ and the morphisms

$$
1_F \star t^{\star(n+1)} : F \longrightarrow (F \circ i)^{\circ(n+1)} \circ F = X_n \circ F
$$

which are right inverse to $\epsilon_n \star 1_F$. These morphisms determine morphisms of simplicial objects $b : i \to i \circ X$ and $c : F \to X \circ F$ (proof omitted). To finish it suffices to construct a homotopy between the morphisms $1, b \circ (1_i \star \epsilon) : i \circ X \to i \circ X$ and between the two morphisms $1, c \circ (\epsilon \star 1_F) : X \circ F \to X \circ F$.

To show the morphisms $b \circ (1_i \star \epsilon), 1 : i \circ X \to i \circ X$ are homotopic we have to construct morphisms

$$h_{n,j} : i \circ X_n \to i \circ X_n$$

for $n \geq 0$ and $0 \leq j \leq n + 1$ satisfying the relations described in Lemma 25.2. See also Remark 25.4. We are forced to set $h_{n,0} = 1$ and

$$h_{n,n+1} = b_n \circ (1_i \star \epsilon_n) = (t^{\star(n+1)} \star 1_i) \circ (1_i \star k^{\star(n+1)})$$

Thus a logical choice is

$$h_{n,j} = (t^{\star(j)} \star 1) \circ (1_i \star k^{\star(j)} \star 1)$$

Here and in the rest of the proof we drop the subscript from 1 if it is clear by knowing the source and the target of the morphism what this subscript should be. Writing

$$i \circ X_n = i \circ F \circ i \circ \ldots \circ F \circ i$$

we can think of the morphism $h_{n,j}$ as collapsing the first j pairs $(F \circ i)$ to $\mathrm{id}_{\mathcal{S}}$ using $k^{\star(j)}$, then adding a $\mathrm{id}_{\mathcal{S}}$ in front and expanding this to j pairs $(i \circ F)$ using $t^{\star(j)}$. We have to prove

(1) We have $d_m^n \circ h_{n,j} = h_{n-1,j-1} \circ d_m^n$ for $j > m$.
(2) We have $d_m^n \circ h_{n,j} = h_{n-1,j} \circ d_m^n$ for $j \leq m$.
(3) We have $s_m^n \circ h_{n,j} = h_{n+1,j+1} \circ s_m^n$ for $j > m$.
(4) We have $s_m^n \circ h_{n,j} = h_{n+1,j} \circ s_m^n$ for $j \leq m$.

Recall that d_m^n is given by applying k to the $(m+1)$st pair $(F \circ i)$ in the functor $X_n = (F \circ i)^{\circ(n+1)}$. Thus it is clear that (2) holds (because k does $\star$-commute with k, but not with t). Similarly, s_m^n is given by applying $1_F \star t \star i_i$ to the $(m+1)$st pair $(F \circ i)$ in $X_n = (F \circ i)^{\circ(n+1)}$. Thus it is clear that (4) holds. In the two remaining cases one uses the fact that the compositions in (32.2.1) are the identity causes the drop in the index j. Some details omitted.

To show the morphisms $1, c \circ (\epsilon \star 1_F) : X \circ F \to X \circ F$ are homotopic we have to construct morphisms

$$h_{n,j} : X_n \circ F \longrightarrow X_n \circ F$$

for $n \geq 0$ and $0 \leq j \leq n + 1$ satisfying the relations described in Lemma 25.2. See also Remark 25.4. We are forced to set $h_{n,0} = 1$ and

$$h_{n,n+1} = c_n \circ (\epsilon_n \star 1_F) = (1_F \star t^{\star(n+1)}) \circ (k^{\star(n+1)} \star 1_F)$$

Thus a logical choice is

$$h_{n,j} = (1_F \star t^{\star(j)} \star 1) \circ (k^{\star(j)} \star 1)$$

Here and in the rest of the proof we drop the subscript from 1 if it is clear by knowing the source and the target of the morphism what this subscript should be. Writing

$$X_n \circ F = F \circ i \circ F \circ \ldots \circ i \circ F$$

we can think of the morphism $h_{n,j}$ as collapsing the first j pairs $(F \circ i)$ to $\mathrm{id}_{\mathcal{S}}$ using $k^{\star(j)}$, then inserting a $\mathrm{id}_{\mathcal{S}}$ just after the first F and expanding this to j pairs $(i \circ F)$ using $t^{\star(j)}$. We have to prove

(1) We have $d_m^n \circ h_{n,j} = h_{n-1,j-1} \circ d_m^n$ for $j > m$.
(2) We have $d_m^n \circ h_{n,j} = h_{n-1,j} \circ d_m^n$ for $j \leq m$.
(3) We have $s_m^n \circ h_{n,j} = h_{n+1,j+1} \circ s_m^n$ for $j > m$.
(4) We have $s_m^n \circ h_{n,j} = h_{n+1,j} \circ s_m^n$ for $j \leq m$.

Recall that d_m^n is given by applying k to the $(m+1)$st pair $(F \circ i)$ in the functor $X_n = (F \circ i)^{\circ(n+1)}$. Thus it is clear that (2) holds (because k does $\star$-commute with k, but not with t). Similarly, s_m^n is given by applying $1_F \star t \star i_i$ to the $(m+1)$st pair $(F \circ i)$ in $X_n = (F \circ i)^{\circ(n+1)}$. Thus it is clear that (4) holds. In the two remaining cases one uses the fact that the compositions in (32.2.1) are the identity causes the drop in the index j. Some details omitted. $\square$

Example 32.6. Going back to the example discussed in Example 32.2 our Lemma 32.5 signifies that for any ring A the map of simplicial rings

$$
\begin{array}{ccccc}
\mathbf{Z}[\mathbf{Z}[\mathbf{Z}[A]]] & \rightrightarrows & \mathbf{Z}[\mathbf{Z}[A]] & \rightrightarrows & \mathbf{Z}[A] \\
\downarrow & & \downarrow & & \downarrow \\
A & \rightrightarrows & A & \rightrightarrows & A
\end{array}
$$

is a homotopy equivalence on underlying simplicial sets. Moreover, the inverse map constructed in Lemma 32.5 is in degree n given by

$$ a \longmapsto [\ldots [a] \ldots] $$

with obvious notation. In the other direction the lemma tells us that for every set E there is a homotopy equivalence

$$
\begin{array}{ccccc}
\mathbf{Z}[\mathbf{Z}[\mathbf{Z}[\mathbf{Z}[E]]]] & \rightrightarrows & \mathbf{Z}[\mathbf{Z}[\mathbf{Z}[E]]] & \rightrightarrows & \mathbf{Z}[\mathbf{Z}[E]] \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Z}[E] & \rightrightarrows & \mathbf{Z}[E] & \rightrightarrows & \mathbf{Z}[E]
\end{array}
$$

of rings. The inverse map constructed in the lemma is in degree n given by the ring map

$$ \sum m_{e_1,\ldots,e_p}[e_1][e_2]\ldots[e_p] \longmapsto \sum m_{e_1,\ldots,e_p}[\ldots[e_1]\ldots][\ldots[e_2]\ldots]\ldots[\ldots[e_p]\ldots] $$

(with obvious notation).

33. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves

(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra

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