DESIRABLES

Contents

1. Introduction

This is basically just a list of things that we want to put in the stacks project. As we add material to the project continuously this is always somewhat behind the current state of the project. In fact, it may have been a mistake to try and list things we should add, because it seems impossible to keep it up to date.

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2. Conventions

We should have a chapter with a short list of conventions used in the document. This chapter already exists, see Conventions, Section 1, but a lot more could be added there. Especially useful would be to find "hidden" conventions and tacit assumptions and put those there.

3. Sites and Topoi

We have a chapter on sites and sheaves, see Sites, Section 1. We have a chapter on ringed sites (and topoi) and modules on them, see Modules on Sites, Section 1. We have a chapter on cohomology in this setting, see Cohomology on Sites, Section 1. But a lot more could be added, especially in the chapter on cohomology.

4. Stacks

We have a chapter on (abstract) stacks, see Stacks, Section 1. It would be nice if

(1) improve the discussion on "stackyfication",
(2) give examples of stackyfication,
(3) more examples in general,
(4) improve the discussion of gerbes.

Example result which has not been added yet: Given a sheaf of abelian groups $\mathcal{F}$ over $\mathcal{C}$ the set of equivalence classes of gerbes with "group" $\mathcal{F}$ is bijective to $H^2(\mathcal{C}, \mathcal{F})$.

5. Simplicial methods

We have a chapter on simplicial methods, see Simplicial, Section 1. This has to be reviewed and improved. The discussion of the relationship between simplicial homotopy (also known as combinatorial homotopy) and Kan complexes should be improved upon. Moreover, there should be a chapter on "simplicial algebraic geometry", where we discuss simplicial schemes, and how to think of their geometry, cohomology, etc. Then this should be tied into the chapter on hypercoverings to "explain" the results of this chapter in the new language.

6. Cohomology of schemes

There is already a chapter on cohomology of quasi-coherent sheaves, see Cohomology of Schemes, Section 1. We also have chapters on étale cohomology of schemes, crystalline cohomology of schemes, derived categories of schemes. But most of the material is very basic and a lot more could be added here.

7. Deformation theory à la Schlessinger

We have a chapter on this material, see Formal Deformation Theory, Section 1. What is needed is worked out examples of the general theory, for example the case of representations of a fixed abstract group.

8. Definition of algebraic stacks

An algebraic stack is a stack in groupoids over the category of schemes with the fppf topology that has a diagonal representable by algebraic spaces and is the target of a surjective smooth morphism from a scheme. The notion "Deligne-Mumford stack" will be reserved for a stack as in [DM69]. We will reserve the term "Artin stack" for a stack such as in the papers by Artin [Art69], and [Art74]. (See also [CdJ02].) In other words, and Artin stack will be an algebraic stack with some reasonable finiteness and separatedness conditions.

9. Examples of schemes, algebraic spaces, algebraic stacks

It really is not that hard to show that $\mathcal{M}_g$ is an algebraic stack for $g \geq 2$. We should really have a long list of moduli problems here and prove they are all algebraic stacks. Some of them we can prove are algebraic using Artin approximation. For example the Kontsevich moduli space in characteristic $p > 0$.

Here are some items for the list of moduli problems mentioned above.

(1) $\mathcal{M}_g$, i.e., moduli of smooth projective curves of genus g,
(2) $\overline{\mathcal{M}}_g$, i.e., moduli of stable genus g curves,
(3) $\mathcal{A}_g$, i.e., principally polarized abelian schemes of genus g,
(4) $\mathcal{M}_{1,1}$, i.e., 1-pointed smooth projective genus 1 curves,
(5) $\mathcal{M}_{g,n}$, i.e., smooth projective genus g-curves with n pairwise distinct labeled points,
(6) $\overline{\mathcal{M}}_{g,n}$, i.e., stable n-pointed nodal projective genus g-curves,
(7) $\mathcal{H}om_S(\mathcal{X}, \mathcal{Y})$, moduli of morphisms (with suitable conditions on the stacks $\mathcal{X}$, $\mathcal{Y}$ and the base scheme S),
(8) $Bun_G(X) = \mathcal{H}om_S(X, BG)$, the stack of G-bundles of the geometric Langlands programme (with suitable conditions on the scheme X, the group scheme G, and the base scheme S),
(9) $Pic_{\mathcal{X}/S}$, i.e., the Picard stack associated to an algebraic stack over a base scheme (or space).

How about the algebraic space you get from the deformation theory of a general surface in $\mathbf{P}^3$ with a node? (I mean where you deform it to a general smooth surface in $\mathbf{P}^3$.) Perhaps we can talk about some small dimensional examples here too. For example the stack where you have $\mathbf{A}^1$ with a $B(\mathbf{Z}/2)$ sitting at 0. Bugeyed covers. And so on.

10. Properties of algebraic stacks

This is perhaps one of the easier projects to work on, as most of the basic theory is there now. An interesting project is discussing the various ways of defining what a proper algebraic stack is. Of course these things are really properties of morphisms of stacks. We can define singularities (up to smooth factors) etc. Prove that a connected normal stack is irreducible, etc.

11. Lisse étale site of an algebraic stack

This has been introduced in Cohomology of Stacks, Section 11. An example to show that it is not functorial with respect to 1-morphisms of algebraic stacks is discussed in Examples, Section 50. Of course a lot more could be said about this, but it turns out to be very useful to prove things using the "big" étale site as much as possible.

12. Things you always wanted to know but were afraid to ask

There are going to be lots of lemmas that you use over and over again that are useful but aren't really mentioned specifically in the literature, or it isn't easy to find references for. Bag of tricks.

Example: Given two groupoids in schemes $R \rightrightarrows U$ and $R' \rightrightarrows U'$ what does it mean to have a 1-morphism $[U/R] \to [U'/R']$ purely in terms of groupoids in schemes.

13. Quasi-coherent sheaves on stacks

These are defined and discussed in the chapter Cohomology of Stacks, Section 1.
Derived categories of modules are discussed in the chapter Derived Categories of
Stacks, Section 1. A lot more could be added to these chapters.

14. Flat and smooth

Artin's theorem that having a flat surjection from a scheme is a replacement for the
smooth surjective condition. This is now available as Criteria for Representability,
Theorem 16.1.

15. Artin's representability theorem

This is discussed in the chapter Artin's Axioms, Section 1. We also have an appli-
cation, see Quot, Theorem 5.12. There should be a lot more applications and the
chapter itself has to be cleaned up as well.

16. DM stacks are finitely covered by schemes

This all begins with Gabber's lemma I think. Somewhere in Asterisque about
Faltings proof of Mordell?

17. Martin Olsson's paper on properness

This proves two notions of proper are the same. We can also discuss Faltings result
that it suffices to use DVR's in certain cases.

18. Proper pushforward of coherent sheaves

No comments yet.

19. Keel and Mori

See [KM97]. This material has been incorporated throughout the Stacks project.
See for example More on Groupoids, Section 12 and More on Groupoids in Spaces,
Section 12.

20. Add more here

Please.

21. Other chapters

Preliminaries

- (1) Introduction
- (2) Conventions
- (3) Set Theory
- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces
- (7) Sites and Sheaves
- (8) Stacks
- (9) Fields
- (10) Commutative Algebra
- (11) Brauer Groups
- (12) Homological Algebra
- (13) Derived Categories
- (14) Simplicial Methods
- (15) More on Algebra
- (16) Smoothing Ring Maps
- (17) Sheaves of Modules
- (18) Modules on Sites
- (19) Injectives
- (20) Cohomology of Sheaves
- (21) Cohomology on Sites

References

[Art69] Michael Artin, *Algebraization of formal moduli: I*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 21–71.

[Art74] _____, *Versal deformations and algebraic stacks*, Inventiones Mathematics **27** (1974), 165–189.

[CdJ02] Brian Conrad and Aise Johan de Jong, *Approximation of versal deformations*, J. Algebra **255** (2002), no. 2, 489–515.

[DM69] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHES **36** (1969), 75–110.

[KM97] Sean Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), 193–213.