DECENT ALGEBRAIC SPACES

Contents

1. Introduction

In this chapter we talk study "local" properties of general algebraic spaces, i.e., those algebraic spaces which aren't quasi-separated. Quasi-separated algebraic spaces are studied in [Knu71]. It turns out that essentially new phenomena happen, especially regarding points and specializations of points, on more general algebraic spaces. On the other hand, for most basic results on algebraic spaces, one needn't worry about these phenomena, which is why we have decided to have this material in a separate chapter following the standard development of the theory.

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

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Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Universally bounded fibres

We briefly discuss what it means for a morphism from a scheme to an algebraic space to have universally bounded fibres. Please refer to Morphisms, Section 50 for similar definitions and results on morphisms of schemes.

Definition 3.1. Let S be a scheme. Let X be an algebraic space over S, and let U be a scheme over S. Let $f : U \to X$ be a morphism over S. We say the *fibres of f are universally bounded*[1] if there exists an integer n such that for all fields k and all morphisms $\mathrm{Spec}(k) \to X$ the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite scheme over k whose degree over k is $\leq n$.

This definition makes sense because the fibre product $\mathrm{Spec}(k) \times_Y X$ is a scheme. Moreover, if Y is a scheme we recover the notion of Morphisms, Definition 50.1 by virtue of Morphisms, Lemma 50.2.

Lemma 3.2. *Let S be a scheme. Let X be an algebraic space over S. Let $V \to U$ be a morphism of schemes over S, and let $U \to X$ be a morphism from U to X. If the fibres of $V \to U$ and $U \to X$ are universally bounded, then so are the fibres of $V \to X$.*

Proof. Let n be an integer which works for $V \to U$, and let m be an integer which works for $U \to X$ in Definition 3.1. Let $\mathrm{Spec}(k) \to X$ be a morphism, where k is a field. Consider the morphisms

$$\mathrm{Spec}(k) \times_X V \longrightarrow \mathrm{Spec}(k) \times_X U \longrightarrow \mathrm{Spec}(k).$$

By assumption the scheme $\mathrm{Spec}(k) \times_X U$ is finite of degree at most m over k, and n is an integer which bounds the degree of the fibres of the first morphism. Hence by Morphisms, Lemma 50.3 we conclude that $\mathrm{Spec}(k) \times_X V$ is finite over k of degree at most nm. $\square$

Lemma 3.3. *Let S be a scheme. Let $Y \to X$ be a representable morphism of algebraic spaces over S. Let $U \to X$ be a morphism from a scheme to X. If the fibres of $U \to X$ are universally bounded, then the fibres of $U \times_X Y \to Y$ are universally bounded.*

Proof. This is clear from the definition, and properties of fibre products. (Note that $U \times_X Y$ is a scheme as we assumed $Y \to X$ representable, so the definition applies.) $\square$

Lemma 3.4. *Let S be a scheme. Let $g : Y \to X$ be a representable morphism of algebraic spaces over S. Let $f : U \to X$ be a morphism from a scheme towards X. Let $f' : U \times_X Y \to Y$ be the base change of f. If*

$$Im(|f| : |U| \to |X|) \subset Im(|g| : |Y| \to |X|)$$

and f' has universally bounded fibres, then f has universally bounded fibres.

[1] This is probably nonstandard notation.

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibre products $\mathrm{Spec}(k)\times_Y (U \times_X Y)$ as in Definition 3.1 for the morphism f'. We claim that n works for f also. Namely, suppose that $x : \mathrm{Spec}(k) \to X$ is a morphism from the spectrum of a field. Then either $\mathrm{Spec}(k) \times_X U$ is empty (and there is nothing to prove), or x is in the image of $|f|$. By Properties of Spaces, Lemma 4.3 and the assumption of the lemma we see that this means there exists a field extension $k \subset k'$ and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & X \end{tikzcd}$$

Hence we see that

$$\mathrm{Spec}(k') \times_Y (U \times_X Y) = \mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_X U)$$

Since the scheme $\mathrm{Spec}(k') \times_Y (U \times_X Y)$ is assumed finite of degree $\leq n$ over k' it follows that also $\mathrm{Spec}(k) \times_X U$ is finite of degree $\leq n$ over k as desired. (Some details omitted.) $\square$

Lemma 3.5. *Let S be a scheme. Let X be an algebraic space over S. Consider a commutative diagram*

$$\begin{tikzcd} U \arrow[rr, "f"'] \arrow[rd, "g"'] & & V \arrow[ld, "h"] \\ & X & \end{tikzcd}$$

where U and V are schemes. If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say $n \geq 0$ is an integer which bounds the degrees of the schemes $\mathrm{Spec}(k) \times_X U$ as in Definition 3.1. We claim n also works for h. Let $\mathrm{Spec}(k) \to X$ be a morphism from the spectrum of a field to X. Consider the morphism of schemes

$$\mathrm{Spec}(k) \times_X V \longrightarrow \mathrm{Spec}(k) \times_X U$$

It is flat and surjective. By assumption the scheme on the left is finite of degree $\leq n$ over $\mathrm{Spec}(k)$. It follows from Morphisms, Lemma 50.9 that the degree of the scheme on the right is also bounded by n as desired. $\square$

Lemma 3.6. *Let S be a scheme. Let X be an algebraic space over S, and let U be a scheme over S. Let $\varphi : U \to X$ be a morphism over S. If the fibres of φ are universally bounded, then there exists an integer n such that each fibre of $|U| \to |X|$ has at most n elements.*

Proof. The integer n of Definition 3.1 works. Namely, pick $x \in |X|$. Represent x by a morphism $x : \mathrm{Spec}(k) \to X$. Then we get a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \times_X U \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "x"] & X \end{tikzcd}$$

which shows (via Properties of Spaces, Lemma 4.3) that the inverse image of x in $|U|$ is the image of the top horizontal arrow. Since $\text{Spec}(k) \times_X U$ is finite of degree $\leq n$ over k it has at most n points. $\square$

4. Finiteness conditions and points

In this section we elaborate on the question of when points can be represented by monomorphisms from spectra of fields into the space.

Remark 4.1. Before we give the proof of the next lemma let us recall some facts about étale morphisms of schemes:

(1) An étale morphism is flat and hence generalizations lift along an étale morphism (Morphisms, Lemmas 37.12 and 26.8).
(2) An étale morphism is unramified, an unramified morphism is locally quasi-finite, hence fibres are discrete (Morphisms, Lemmas 37.16, 36.10, and 21.6).
(3) A quasi-compact étale morphism is quasi-finite and in particular has finite fibres (Morphisms, Lemmas 21.9 and 21.10).
(4) An étale scheme over a field k is a disjoint union of spectra of finite separable field extension of k (Morphisms, Lemma 37.7).

For a general discussion of étale morphisms, please see Étale Morphisms, Section 11.

Lemma 4.2. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *there exists a family of schemes U_i and étale morphisms $\varphi_i : U_i \to X$ such that $\coprod \varphi_i : \coprod U_i \to X$ is surjective, and such that for each i the fibre of $|U_i| \to |X|$ over x is finite, and*
(2) *for every affine scheme U and étale morphism $\varphi : U \to X$ the fibre of $|U| \to |X|$ over x is finite.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Let $\varphi_i : U_i \to X$ be a family of étale morphisms as in (1). Let $\varphi : U \to X$ be an étale morphism from an affine scheme towards X. Consider the fibre product diagrams

$$\begin{tikzcd} U \times_X U_i \arrow[r, "p_i"'] \arrow[d, "q_i"'] & U_i \arrow[d, "\varphi_i"] \\ U \arrow[r, "\varphi"] & X \end{tikzcd} \qquad \begin{tikzcd} \coprod U \times_X U_i \arrow[r, "\coprod p_i"'] \arrow[d, "\coprod q_i"'] & \coprod U_i \arrow[d, "\coprod \varphi_i"] \\ U \arrow[r, "\varphi"] & X \end{tikzcd}$$

Since q_i is étale it is open (see Remark 4.1). Moreover, the morphism $\coprod q_i$ is surjective. Hence there exist finitely many indices $i_1, \ldots, i_n$ and a quasi-compact opens $W_{i_j} \subset U \times_X U_{i_j}$ which surject onto U. The morphism p_i is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 50.8 to see the fibres of $p_{i_j}|_{W_{i_j}} : W_{i_j} \to U_i$ are finite. Hence by Properties of Spaces, Lemma 4.3 and the assumption on φ_i we conclude that the fibre of φ over x is finite. In other words (2) holds. $\square$

Lemma 4.3. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *there exists a scheme U, an étale morphism $\varphi : U \to X$, and points $u, u' \in U$ mapping to x such that setting $R = U \times_X U$ the fibre of*
$$|R| \to |U| \times_{|X|} |U|$$
over (u, u') is finite,
(2) *for every scheme U, étale morphism $\varphi : U \to X$ and any points $u, u' \in U$ mapping to x setting $R = U \times_X U$ the fibre of*
$$|R| \to |U| \times_{|X|} |U|$$
over (u, u') is finite,
(3) *there exists a morphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x such that the projections $\mathrm{Spec}(k) \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are étale and quasi-compact, and*
(4) *there exists a monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x.*

Proof. Assume (1), i.e., let $\varphi : U \to X$ be an étale morphism from a scheme towards X, and let u, u' be points of U lying over x such that the fibre of $|R| \to |U| \times_{|X|} |U|$ over (u, u') is a finite set. In this proof we think of a point $u = \mathrm{Spec}(\kappa(u))$ as a scheme. Note that $u \to U$, $u' \to U$ are monomorphisms (see Schemes, Lemma 23.6), hence $u \times_X u' \to R = U \times_X U$ is a monomorphism. In this language the assumption really means that $u \times_X u'$ is a scheme whose underlying topological space has finitely many points. Let $\psi : W \to X$ be an étale morphism from a scheme towards X. Let $w, w' \in W$ be points of W mapping to x. We have to show that $w \times_X w'$ is a scheme whose underlying topological space has finitely many points. Consider the fibre product diagram

$$\begin{tikzcd} W \times_X U \arrow[r, "p"] \arrow[d, "q"] & U \arrow[d, "\varphi"] \\ W \arrow[r, "\psi"] & X \end{tikzcd}$$

As x is the image of u and u' we may pick points $\tilde{w}, \tilde{w}'$ in $W \times_X U$ with $q(\tilde{w}) = w$, $q(\tilde{w}') = w'$, $u = p(\tilde{w})$ and $u' = p(\tilde{w}')$, see Properties of Spaces, Lemma 4.3. As p, q are étale the field extensions $\kappa(w) \subset \kappa(\tilde{w}) \supset \kappa(u)$ and $\kappa(w') \subset \kappa(\tilde{w}') \supset \kappa(u')$ are finite separable, see Remark 4.1. Then we get a commutative diagram

$$\begin{tikzcd} w \times_X w' \arrow[d] & \tilde{w} \times_X \tilde{w}' \arrow[l] \arrow[r] \arrow[d] & u \times_X u' \arrow[d] \\ w \times_X w' & \tilde{w} \times_S \tilde{w}' \arrow[l] \arrow[r] & u \times_S u' \end{tikzcd}$$

where the squares are fibre product squares. The lower horizontal morphisms are étale and quasi-compact, as any scheme of the form $\mathrm{Spec}(k) \times_S \mathrm{Spec}(k')$ is affine, and by our observations about the field extensions above. Thus we see that the top horizontal arrows are étale and quasi-compact and hence have finite fibres. We have seen above that $|u \times_X u'|$ is finite, so we conclude that $|w \times_X w'|$ is finite. In other words, (2) holds.

Assume (2). Let $U \to X$ be an étale morphism from a scheme U such that x is in the image of $|U| \to |X|$. Let $u \in U$ be a point mapping to x. Then we have seen in the previous paragraph that $u = \mathrm{Spec}(\kappa(u)) \to X$ has the property that $u \times_X u$

has a finite underlying topological space. On the other hand, the projection maps $u \times_X u \to u$ are the composition

$$u \times_X u \longrightarrow u \times_X U \longrightarrow u \times_X X = u,$$

i.e., the composition of a monomorphism (the base change of the monomorphism $u \to U$) by an étale morphism (the base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of fields finite separable over $\kappa(u)$ (see Remark 4.1). Since $u \times_X u$ is finite the image of it in $u \times_X U$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. By Schemes, Lemma 23.10 we conclude that $u \times_X u$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. In other words, we see that $u \times_X u \to u$ is quasi-compact and étale. This means that (3) holds.

Let us prove that (3) implies (4). Let $\mathrm{Spec}(k) \to X$ be a morphism from the spectrum of a field into X, in the equivalence class of x such that the two projections $t, s : R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are quasi-compact and étale. This means in particular that R is an étale equivalence relation on $\mathrm{Spec}(k)$. By Spaces, Theorem 10.5 we know that the quotient sheaf $X' = \mathrm{Spec}(k)/R$ is an algebraic space. By Groupoids, Lemma 18.6 the map $X' \to X$ is a monomorphism. Since s, t are quasi-compact, we see that R is quasi-compact and hence Properties of Spaces, Lemma 12.1 applies to X', and we see that $X' = \mathrm{Spec}(k')$ for some field k'. Hence we get a factorization

$$\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k') \longrightarrow X$$

which shows that $\mathrm{Spec}(k') \to X$ is a monomorphism mapping to $x \in |X|$. In other words (4) holds.

Finally, we prove that (4) implies (1). Let $\mathrm{Spec}(k) \to X$ be a monomorphism with k a field in the equivalence class of x. Let $U \to X$ be a surjective étale morphism from a scheme U to X. Let $u \in U$ be a point over x. Since $\mathrm{Spec}(k) \times_X u$ is nonempty, and since $\mathrm{Spec}(k) \times_X u \to u$ is a monomorphism we conclude that $\mathrm{Spec}(k) \times_X u = u$ (see Schemes, Lemma 23.10). Hence $u \to U \to X$ factors through $\mathrm{Spec}(k) \to X$, here is a picture

$$\begin{tikzcd} u \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & X \end{tikzcd}$$

Since the right vertical arrow is étale this implies that $k \subset \kappa(u)$ is a finite separable extension. Hence we conclude that

$$u \times_X u = u \times_{\mathrm{Spec}(k)} u$$

is a finite scheme, and we win by the discussion of the meaning of property (1) in the first paragraph of this proof. □

Lemma 4.4. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let U be a scheme and let $\varphi : U \to X$ be an étale morphism. The following are equivalent:*

(1) *x is in the image of $|U| \to |X|$, and setting $R = U \times_X U$ the fibres of both*

$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$

over x are finite,

(2) *there exists a monomorphism* $\mathrm{Spec}(k) \to X$ *with* k *a field in the equivalence class of* x*, and the fibre product* $\mathrm{Spec}(k) \times_X U$ *is a finite nonempty scheme over* k.

Proof. Assume (1). This clearly implies the first condition of Lemma 4.3 and hence we obtain a monomorphism $\mathrm{Spec}(k) \to X$ in the class of x. Taking the fibre product we see that $\mathrm{Spec}(k) \times_X U \to \mathrm{Spec}(k)$ is a scheme étale over $\mathrm{Spec}(k)$ with finitely many points, hence a finite nonempty scheme over k, i.e., (2) holds.

Assume (2). By assumption x is in the image of $|U| \to |X|$. The finiteness of the fibre of $|U| \to |X|$ over x is clear since this fibre is equal to $|\mathrm{Spec}(k) \times_X U|$ by Properties of Spaces, Lemma 4.3. The finiteness of the fibre of $|R| \to |X|$ above x is also clear since it is equal to the set underlying the scheme

$$(\mathrm{Spec}(k) \times_X U) \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_X U)$$

which is finite over k. Thus (1) holds. $\square$

Lemma 4.5. *Let* S *be a scheme. Let* X *be an algebraic space over* S. *Let* $x \in |X|$. *The following are equivalent:*

(1) *for every affine scheme* U*, any étale morphism* $\varphi : U \to X$ *setting* $R = U \times_X U$ *the fibres of both*
$$|U| \longrightarrow |X| \quad \textit{and} \quad |R| \longrightarrow |X|$$
over x *are finite,*

(2) *there exist schemes* U_i *and étale morphisms* $U_i \to X$ *such that* $\coprod U_i \to X$ *is surjective and for each* i*, setting* $R_i = U_i \times_X U_i$ *the fibres of both*
$$|U_i| \longrightarrow |X| \quad \textit{and} \quad |R_i| \longrightarrow |X|$$
over x *are finite,*

(3) *there exists a monomorphism* $\mathrm{Spec}(k) \to X$ *with* k *a field in the equivalence class of* x*, and for any affine scheme* U *and étale morphism* $U \to X$ *the fibre product* $\mathrm{Spec}(k) \times_X U$ *is a finite scheme over* k*, and*

(4) *there exists a quasi-compact monomorphism* $\mathrm{Spec}(k) \to X$ *with* k *a field in the equivalence class of* x.

Proof. The equivalence of (1) and (3) follows on applying Lemma 4.4 to every étale morphism $U \to X$ with U affine. It is clear that (3) implies (2). Assume $U_i \to X$ and R_i are as in (2). We conclude from Lemma 4.2 that for any affine scheme U and étale morphism $U \to X$ the fibre of $|U| \to |X|$ over x is finite. Say this fibre is $\{u_1, \ldots, u_n\}$. Then, as Lemma 4.3 (1) applies to $U_i \to X$ for some i such that x is in the image of $|U_i| \to |X|$, we see that the fibre of $|R = U \times_X U| \to |U| \times_{|X|} |U|$ is finite over (u_a, u_b), $a, b \in \{1, \ldots, n\}$. Hence the fibre of $|R| \to |X|$ over x is finite. In this way we see that (1) holds. At this point we know that (1), (2), and (3) are equivalent.

If (4) holds, then for any affine scheme U and étale morphism $U \to X$ the scheme $\mathrm{Spec}(k) \times_X U$ is on the one hand étale over k (hence a disjoint union of spectra of finite separable extensions of k by Remark 4.1) and on the other hand quasi-compact over U (hence quasi-compact). Thus we see that (3) holds. Conversely, if $U_i \to X$ is as in (2) and $\mathrm{Spec}(k) \to X$ is a monomorphism as in (3), then

$$\coprod \mathrm{Spec}(k) \times_X U_i \longrightarrow \coprod U_i$$

is quasi-compact (because over each U_i we see that $\mathrm{Spec}(k) \times_X U_i$ is a finite disjoint union spectra of fields). Thus $\mathrm{Spec}(k) \to X$ is quasi-compact by Morphisms of Spaces, Lemma 8.7. □

Lemma 4.6. *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent:*

(1) *there exist schemes U_i and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and each $U_i \to X$ has universally bounded fibres, and*
(2) *for every affine scheme U and étale morphism $\varphi : U \to X$ the fibres of $U \to X$ are universally bounded.*

Proof. The implication (2) ⇒ (1) is trivial. Assume (1). Let $(\varphi_i : U_i \to X)_{i \in I}$ be a collection of étale morphisms from schemes towards X, covering X, such that each φ_i has universally bounded fibres. Let $\psi : U \to X$ be an étale morphism from an affine scheme towards X. For each i consider the fibre product diagram

$$\begin{tikzcd} U \times_X U_i \arrow[r, "p_i"'] \arrow[d, "q_i"'] & U_i \arrow[d, "\varphi_i"] \\ U \arrow[r, "\psi"] & X \end{tikzcd}$$

Since q_i is étale it is open (see Remark 4.1). Moreover, we have $U = \bigcup \mathrm{Im}(q_i)$, since the family $(\varphi_i)_{i \in I}$ is surjective. Since U is affine, hence quasi-compact we can finite finitely many $i_1, \ldots, i_n \in I$ and quasi-compact opens $W_j \subset U \times_X U_{i_j}$ such that $U = \bigcup p_{i_j}(W_j)$. The morphism p_{i_j} is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 50.8 to see the fibres of $p_{i_j}|_{W_j} : W_j \to U_{i_j}$ are universally bounded. Hence by Lemma 3.2 we see that the fibres of $W_j \to X$ are universally bounded. Thus also $\coprod_{j=1,\ldots,n} W_j \to X$ has universally bounded fibres. Since $\coprod_{j=1,\ldots,n} W_j \to X$ factors through the surjective étale map $\coprod q_{i_j}|_{W_j} : \coprod_{j=1,\ldots,n} W_j \to U$ we see that the fibres of $U \to X$ are universally bounded by Lemma 3.5. In other words (2) holds. □

Lemma 4.7. *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent:*

(1) *there exists a Zariski covering $X = \bigcup X_i$ and for each i a scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$, and*
(2) *there exist schemes U_i and étale morphisms $U_i \to X$ such that the projections $U_i \times_X U_i \to U_i$ are quasi-compact and $\coprod U_i \to X$ is surjective.*

Proof. If (1) holds then the morphisms $U_i \to X_i \to X$ are étale (combine Morphisms, Lemma 37.3 and Spaces, Lemmas 5.4 and 5.3). Moreover, as $U_i \times_X U_i = U_i \times_{X_i} U_i$, both projections $U_i \times_X U_i \to U_i$ are quasi-compact.

If (2) holds then let $X_i \subset X$ be the open subspace corresponding to the image of the open map $|U_i| \to |X|$, see Properties of Spaces, Lemma 4.10. The morphisms $U_i \to X_i$ are surjective. Hence $U_i \to X_i$ is surjective étale, and the projections $U_i \times_{X_i} U_i \to U_i$ are quasi-compact, because $U_i \times_{X_i} U_i = U_i \times_X U_i$. Thus by Spaces, Lemma 11.2 the morphisms $U_i \to X_i$ are quasi-compact. □

5. Conditions on algebraic spaces

In this section we discuss the relationship between various natural conditions on algebraic spaces we have seen above. Please read Section 6 to get a feeling for the meaning of these conditions.

Lemma 5.1. *Let S be a scheme. Let X be an algebraic space over S. Consider the following conditions on X:*

(α) *For every $x \in |X|$, the equivalent conditions of Lemma 4.2 hold.*
(β) *For every $x \in |X|$, the equivalent conditions of Lemma 4.3 hold.*
(γ) *For every $x \in |X|$, the equivalent conditions of Lemma 4.5 hold.*
(δ) *The equivalent conditions of Lemma 4.6 hold.*
(ϵ) *The equivalent conditions of Lemma 4.7 hold.*
(ζ) *The space X is Zariski locally quasi-separated.*
(η) *The space X is quasi-separated*
(θ) *The space X is representable, i.e., X is a scheme.*
(ι) *The space X is a quasi-separated scheme.*

We have

$$
\begin{tikzcd}
 & (\theta) \arrow[dr, Rightarrow] & & & & \\
(\iota) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (\zeta) \arrow[r, Rightarrow] & (\epsilon) \arrow[r, Rightarrow] & (\delta) \arrow[r, Rightarrow] & (\gamma) \arrow[r, Leftrightarrow] & (\alpha) + (\beta) \\
 & (\eta) \arrow[ur, Rightarrow] & & & &
\end{tikzcd}
$$

Proof. The implication $(\gamma) \Leftrightarrow (\alpha) + (\beta)$ is immediate. The implications in the diamond on the left are clear from the definitions.

Assume (ζ), i.e., that X is Zariski locally quasi-separated. Then (ϵ) holds by Properties of Spaces, Lemma 6.6.

Assume (ϵ). By Lemma 4.7 there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists a scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$. Choose an i and an affine open subscheme $W \subset U_i$. It suffices to show that $W \to X$ has universally bounded fibres, since then the family of all these morphisms $W \to X$ covers X. To do this we consider the diagram

$$
\begin{tikzcd}
W \times_X U_i \arrow[r, "p"'] \arrow[d, "q"'] & U_i \arrow[d] \\
W \arrow[r] & X
\end{tikzcd}
$$

Since $W \to X$ factors through X_i we see that $W \times_X U_i = W \times_{X_i} U_i$, and hence q is quasi-compact. Since W is affine this implies that the scheme $W \times_X U_i$ is quasi-compact. Thus we may apply Morphisms, Lemma 50.8 and we conclude that p has universally bounded fibres. From Lemma 3.4 we conclude that $W \to X$ has universally bounded fibres as well.

Assume (δ). Let U be an affine scheme, and let $U \to X$ be an étale morphism. By assumption the fibres of the morphism $U \to X$ are universally bounded. Thus also the fibres of both projections $R = U \times_X U \to U$ are universally bounded, see

Lemma 3.3. And by Lemma 3.2 also the fibres of $R \to X$ are universally bounded. Hence for any $x \in X$ the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x are finite, see Lemma 3.6. In other words, the equivalent conditions of Lemma 4.5 hold. This proves that $(\delta) \Rightarrow (\gamma)$. □

Lemma 5.2. *Let S be a scheme. Let $\mathcal{P}$ be one of the properties (α), (β), (γ), (δ), (ϵ), (ζ), or (θ) of algebraic spaces listed in Lemma 5.1. Then if X is an algebraic space over S, and $X = \bigcup X_i$ is a Zariski open covering such that each X_i has $\mathcal{P}$, then X has $\mathcal{P}$.*

Proof. Let X be an algebraic space over S, and let $X = \bigcup X_i$ is a Zariski open covering such that each X_i has $\mathcal{P}$.

The case $\mathcal{P} = (\alpha)$. The condition (α) for X_i means that for every $x \in |X_i|$ and every affine scheme U, and étale morphism $\varphi : U \to X_i$ the fibre of $\varphi : |U| \to |X_i|$ over x is finite. Consider $x \in X$, an affine scheme U and an étale morphism $U \to X$. Since $X = \bigcup X_i$ is a Zariski open covering there exits a finite affine open covering $U = U_1 \cup \ldots \cup U_n$ such that each $U_j \to X$ factors through some X_{i_j}. By assumption the fibres of $|U_j| \to |X_{i_j}|$ over x are finite for $j = 1, \ldots, n$. Clearly this means that the fibre of $|U| \to |X|$ over x is finite. This proves the result for (α).

The case $\mathcal{P} = (\beta)$. The condition (β) for X_i means that every $x \in |X_i|$ is represented by a monomorphism from the spectrum of a field towards X_i. Hence the same follows for X as $X_i \to X$ is a monomorphism and $X = \bigcup X_i$.

The case $\mathcal{P} = (\gamma)$. Note that $(\gamma) = (\alpha) + (\beta)$ by Lemma 5.1 hence the lemma for (γ) follows from the cases treated above.

The case $\mathcal{P} = (\delta)$. The condition (δ) for X_i means there exist schemes U_{ij} and étale morphisms $U_{ij} \to X_i$ with universally bounded fibres which cover X_i. These schemes also give an étale surjective morphism $\coprod U_{ij} \to X$ and $U_{ij} \to X$ still has universally bounded fibres.

The case $\mathcal{P} = (\epsilon)$. The condition (ϵ) for X_i means we can find a set J_i and morphisms $\varphi_{ij} : U_{ij} \to X_i$ such that each φ_{ij} is étale, both projections $U_{ij} \times_{X_i} U_{ij} \to U_{ij}$ are quasi-compact, and $\coprod_{j \in J_i} U_{ij} \to X_i$ is surjective. In this case the compositions $U_{ij} \to X_i \to X$ are étale (combine Morphisms, Lemmas 37.3 and 37.9 and Spaces, Lemmas 5.4 and 5.3). Since $X_i \subset X$ is a subspace we see that $U_{ij} \times_{X_i} U_{ij} = U_{ij} \times_X U_{ij}$, and hence the condition on fibre products is preserved. And clearly $\coprod_{i,j} U_{ij} \to X$ is surjective. Hence X satisfies (ϵ).

The case $\mathcal{P} = (\zeta)$. The condition (ζ) for X_i means that X_i is Zariski locally quasi-separated. It is immediately clear that this means X is Zariski locally quasi-separated.

For (θ), see Properties of Spaces, Lemma 10.1. □

Lemma 5.3. *Let S be a scheme. Let $\mathcal{P}$ be one of the properties (β), (γ), (δ), (ϵ), or (θ) of algebraic spaces listed in Lemma 5.1. Let X, Y be algebraic spaces over S. Let $X \to Y$ be a representable morphism. If Y has property $\mathcal{P}$, so does X.*

Proof. Assume $f : X \to Y$ is a representable morphism of algebraic spaces, and assume that Y has $\mathcal{P}$. Let $x \in |X|$, and set $y = f(x) \in |Y|$.

The case $\mathcal{P} = (\beta)$. Condition (β) for Y means there exists a monomorphism $\mathrm{Spec}(k) \to Y$ representing y. The fibre product $X_y = \mathrm{Spec}(k) \times_Y X$ is a scheme, and x corresponds to a point of X_y, i.e., to a monomorphism $\mathrm{Spec}(k') \to X_y$. As $X_y \to X$ is a monomorphism also we see that x is represented by the monomorphism $\mathrm{Spec}(k') \to X_y \to X$. In other words (β) holds for X.

The case $\mathcal{P} = (\gamma)$. Since $(\gamma) \Rightarrow (\beta)$ we have seen in the preceding paragraph that y and x can be represented by monomorphisms as in the following diagram

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r, "x"'] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "y"] & Y \end{tikzcd}$$

Also, by definition of property (γ) via Lemma 4.5 (2) there exist schemes V_i and étale morphisms $V_i \to Y$ such that $\coprod V_i \to Y$ is surjective and for each i, setting $R_i = V_i \times_Y V_i$ the fibres of both

$$|V_i| \longrightarrow |Y| \quad \text{and} \quad |R_i| \longrightarrow |Y|$$

over y are finite. This means that the schemes $(V_i)_y$ and $(R_i)_y$ are finite schemes over $y = \mathrm{Spec}(k)$. As $X \to Y$ is representable, the fibre products $U_i = V_i \times_Y X$ are schemes. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Finally, for each i we have

$$(U_i)_x = (V_i \times_Y X)_x = (V_i)_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$$

and

$$(U_i \times_X U_i)_x = ((V_i \times_Y X) \times_X (V_i \times_Y X))_x = (R_i)_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$$

hence these are finite over k' as base changes of the finite schemes $(V_i)_y$ and $(R_i)_y$. This implies that (γ) holds for X, again via the second condition of Lemma 4.5.

The case $\mathcal{P} = (\delta)$. Let $V \to Y$ be an étale morphism with V an affine scheme. Since Y has property (δ) this morphism has universally bounded fibres. By Lemma 3.3 the base change $V \times_Y X \to X$ also has universally bounded fibres. Hence the first part of Lemma 4.6 applies and we see that Y also has property (δ).

The case $\mathcal{P} = (\epsilon)$. We will repeatedly use Spaces, Lemma 5.5. Let $V_i \to Y$ be as in Lemma 4.7 (2). Set $U_i = X \times_Y V_i$. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Because $U_i \times_X U_i = X \times_Y (V_i \times_Y V_i)$ we see that the projections $U_i \times_Y U_i \to U_i$ are base changes of the projections $V_i \times_Y V_i \to V_i$, and so quasi-compact as well. Hence X satisfies Lemma 4.7 (2).

The case $\mathcal{P} = (\theta)$. In this case the result is Categories, Lemma 8.3. $\square$

6. Reasonable and decent algebraic spaces

In Lemma 5.1 we have seen a number of conditions on algebraic spaces related to the behaviour of étale morphisms from affine schemes into X and related to the existence of special étale coverings of X by schemes. We tabulate the different types of conditions here:

(α)	fibres of etale morphisms from affines are finite
(β)	points come from monomorphisms of spectra of fields
(γ)	points come from quasi-compact monomorphisms of spectra of fields
(δ)	fibres of etale morphisms from affines are universally bounded
(ϵ)	cover by etale morphisms from schemes quasi-compact onto their image

The conditions in the following definition are not exactly conditions on the diagonal of X, but they are in some sense separation conditions on X.

Definition 6.1. Let S be a scheme. Let X be an algebraic space over S.

(1) We say X is *decent* if for every point $x \in X$ the equivalent conditions of Lemma 4.5 hold, in other words property (γ) of Lemma 5.1 holds.
(2) We say X is *reasonable* if the equivalent conditions of Lemma 4.6 hold, in other words property (δ) of Lemma 5.1 holds.
(3) We say X is *very reasonable* if the equivalent conditions of Lemma 4.7 hold, i.e., property (ϵ) of Lemma 5.1 holds.

We have the following implications among these conditions on algebraic spaces:

$$\begin{tikzcd}
\text{representable} \arrow[dr, Rightarrow] & & & \\
& \text{very reasonable} \arrow[r, Rightarrow] & \text{reasonable} \arrow[r, Rightarrow] & \text{decent} \\
\text{quasi-separated} \arrow[ur, Rightarrow] & & &
\end{tikzcd}$$

The notion of a very reasonable algebraic space is obsolete. It was introduced because the assumption was needed to prove some results which are now proven for the class of decent spaces. The class of decent spaces is the largest class of spaces X where one has a good relationship between the topology of $|X|$ and properties of X itself.

Example 6.2. The algebraic space $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ constructed in Spaces, Example 14.8 is not decent as its "generic point" cannot be represented by a monomorphism from the spectrum of a point.

Remark 6.3. Reasonable algebraic spaces are technically easier to work with than very reasonable algebraic spaces. For example, if $X \to Y$ is a quasi-compact étale surjective morphism of algebraic spaces and X is reasonable, then so is Y, see Lemma 15.8 but we don't know if this is true for the property "very reasonable". Below we give another technical property enjoyed by reasonable algebraic spaces.

Lemma 6.4. *Let S be a scheme. Let X be a quasi-compact reasonable algebraic space. Then there exists a directed system of quasi-compact and quasi-separated algebraic spaces X_i such that $X = \operatorname{colim}_i X_i$ (colimit in the category of sheaves).*

Proof. We sketch the proof. By Properties of Spaces, Lemma 6.3 we have $X = U/R$ with U affine. In this case, reasonable means $U \to X$ is universally bounded. Hence there exists an integer N such that the "fibres" of $U \to X$ have degree at most N, see Definition 3.1. Denote $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ the groupoid structural maps.

Claim: for every quasi-compact open $A \subset R$ there exists an open $R' \subset R$ such that

(1) $A \subset R'$,
(2) R' is quasi-compact, and
(3) $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$ is a groupoid scheme.

Note that $e : U \to R$ is open as it is a section of the étale morphism $s : R \to U$, see Étale Morphisms, Proposition 6.1. Moreover U is affine hence quasi-compact. Hence we may replace A by $A \cup e(U) \subset R$, and assume that A contains $e(U)$. Next, we define inductively $A^1 = A$, and

$$A^n = c(A^{n-1} \times_{s,U,t} A) \subset R$$

for $n \geq 2$. Arguing inductively, we see that A^n is quasi-compact for all $n \geq 2$, as the image of the quasi-compact fibre product $A^{n-1} \times_{s,U,t} A$. If k is an algebraically closed field over S, and we consider k-points then

$$A^n(k) = \left\{ (u, u') \in U(k) : \begin{matrix} \text{there exist } u = u_1, u_2, \ldots, u_n \in U(k) \text{ with} \\ (u_i, u_{i+1}) \in A \text{ for all } i = 1, \ldots, n-1. \end{matrix} \right\}$$

But as the fibres of $U(k) \to X(k)$ have size at most N we see that if $n > N$ then we get a repeat in the sequence above, and we can shorten it proving $A^N = A^n$ for all $n \geq N$. This implies that $R' = A^N$ gives a groupoid scheme $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$, proving the claim above.

Consider the map of sheaves on $(Sch/S)_{fppf}$

$$\mathrm{colim}_{R' \subset R}\, U/R' \longrightarrow U/R$$

where $R' \subset R$ runs over the quasi-compact open subschemes of R which give étale equivalence relations as above. Each of the quotients U/R' is an algebraic space (see Spaces, Theorem 10.5). Since R' is quasi-compact, and U affine the morphism $R' \to U \times_{\mathrm{Spec}(\mathbf{Z})} U$ is quasi-compact, and hence U/R' is quasi-separated. Finally, if T is a quasi-compact scheme, then

$$\mathrm{colim}_{R' \subset R}\, U(T)/R'(T) \longrightarrow U(T)/R(T)$$

is a bijection, since every morphism from T into R ends up in one of the open subrelations R' by the claim above. This clearly implies that the colimit of the sheaves U/R' is U/R. In other words the algebraic space $X = U/R$ is the colimit of the quasi-separated algebraic spaces U/R'. □

Lemma 6.5. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $X \to Y$ be a representable morphism. If Y is decent (resp. reasonable), then so is X.*

Proof. Translation of Lemma 5.3. □

Lemma 6.6. *Let S be a scheme. Let $X \to Y$ be an étale morphism of algebraic spaces over S. If Y is decent, resp. reasonable, then so is X.*

Proof. Let U be an affine scheme and $U \to X$ an étale morphism. Set $R = U \times_X U$ and $R' = U \times_Y U$. Note that $R \to R'$ is a monomorphism.

Let $x \in |X|$. To show that X is decent, we have to show that the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x are finite. But if Y is decent, then the fibres of $|U| \to |Y|$ and $|R'| \to |Y|$ are finite. Hence the result for "decent".

To show that X is reasonable, we have to show that the fibres of $U \to X$ are universally bounded. However, if Y is reasonable, then the fibres of $U \to Y$ are universally bounded, which immediately implies the same thing for the fibres of $U \to X$. Hence the result for "reasonable". □

7. Points and specializations

There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specializations between points in a fibre of $|f| : |X| \to |Y|$, see Examples, Lemma 42.1. If the source of the morphism is a scheme we can avoid this by imposing condition (α) on Y.

Lemma 7.1. *Let S be a scheme. Let X be an algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. Assume $u, u' \in |U|$ map to the same point x of $|X|$, and $u' \leadsto u$. If the pair (X, x) satisfies the equivalent conditions of Lemma 4.2 then $u = u'$.*

Proof. Assume the pair (X, x) satisfies the equivalent conditions for Lemma 4.2. Let U be a scheme, $U \to X$ étale, and let $u, u' \in |U|$ map to x of $|X|$, and $u' \leadsto u$. We may and do replace U by an affine neighbourhood of u. Let $t, s : R = U \times_X U \to U$ be the étale projection maps.

Pick a point $r \in R$ with $t(r) = u$ and $s(r) = u'$. This is possible by Properties of Spaces, Lemma 4.5. Because generalizations lift along the étale morphism t (Remark 4.1) we can find a specialization $r' \leadsto r$ with $t(r') = u'$. Set $u'' = s(r')$. Then $u'' \leadsto u'$. Thus we may repeat and find $r'' \leadsto r'$ with $t(r'') = u''$. Set $u''' = s(r'')$, and so on. Here is a picture:

$$
\begin{tikzcd}
 & r'' \arrow[dl, "t"'] \arrow[d, squiggly] \arrow[dr, "s"] & \\
u'' \arrow[d, squiggly] & r' \arrow[dl, "t"'] \arrow[d, squiggly] \arrow[dr, "s"] & u''' \arrow[d, squiggly] \\
u' \arrow[d, squiggly] & r \arrow[dl, "t"'] \arrow[dr, "s"] & u'' \arrow[d, squiggly] \\
u & & u'
\end{tikzcd}
$$

In Remark 4.1 we have seen that there are no specializations among points in the fibres of the étale morphism s. Hence if $u^{(n+1)} = u^{(n)}$ for some n, then also $r^{(n)} = r^{(n-1)}$ and hence also (by taking t) $u^{(n)} = u^{(n-1)}$. This then forces the whole tower to collapse, in particular $u = u'$. Thus we see that if $u \neq u'$, then all the specializations are strict and $\{u, u', u'', \ldots\}$ is an infinite set of points in U which map to the point x in $|X|$. As we chose U affine this contradicts the second part of Lemma 4.2, as desired. □

Lemma 7.2. *Let S be an algebraic space. Let X be an algebraic space over S. Let $x, x' \in |X|$ and assume $x' \leadsto x$, i.e., x is a specialization of x'. Assume the pair (X, x') satisfies the equivalent conditions of Lemma 4.5. Then for every étale morphism $\varphi : U \to X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \leadsto u$ with $\varphi(u') = x'$.*

Proof. We may replace U by an affine open neighbourhood of u. Hence we may assume that U is affine. As x is in the image of the open map $|U| \to |X|$, so is x'. Thus we may replace X by the Zariski open subspace corresponding to the image of $|U| \to |X|$, see Properties of Spaces, Lemma 4.10. In other words we may assume that $U \to X$ is surjective and étale. Let $s, t : R = U \times_X U \to U$ be the projections. By our assumption that (X, x') satisfies the equivalent conditions of Lemma 4.5 we see that the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x' are finite. Say $\{u'_1, \ldots, u'_n\} \subset U$ and $\{r'_1, \ldots, r'_m\} \subset R$ form the complete inverse image of $\{x'\}$. Consider the closed sets

$$T = \overline{\{u'_1\}} \cup \ldots \cup \overline{\{u'_n\}} \subset |U|, \quad T' = \overline{\{r'_1\}} \cup \ldots \cup \overline{\{r'_m\}} \subset |R|.$$

Trivially we have $s(T') \subset T$. Because R is an equivalence relation we also have $t(T') = s(T')$ as the set $\{r'_j\}$ is invariant under the inverse of R by construction. Let $w \in T$ be any point. Then $u'_i \rightsquigarrow w$ for some i. Choose $r \in R$ with $s(r) = w$. Since generalizations lift along $s : R \to U$, see Remark 4.1, we can find $r' \rightsquigarrow r$ with $s(r') = u'_i$. Then $r' = r'_j$ for some j and we conclude that $w \in s(T')$. Hence $T = s(T') = t(T')$ is an $|R|$-invariant closed set in $|U|$. This means T is the inverse image of a closed (!) subset $T'' = \varphi(T)$ of $|X|$, see Properties of Spaces, Lemmas 4.5 and 4.6. Hence $T'' = \overline{\{x'\}}$. Thus T contains some point u_1 mapping to x as $x \in T''$. I.e., we see that for some i there exists a specialization $u'_i \rightsquigarrow u_1$ which maps to the given specialization $x' \rightsquigarrow x$.

To finish the proof, choose a point $r \in R$ such that $s(r) = u$ and $t(r) = u_1$ (using Properties of Spaces, Lemma 4.3). As generalizations lift along t, and $u'_i \rightsquigarrow u_1$ we can find a specialization $r' \rightsquigarrow r$ such that $t(r') = u'_i$. Set $u' = s(r')$. Then $u' \rightsquigarrow u$ and $\varphi(u') = x'$ as desired. $\square$

8. Stratifying algebraic spaces by schemes

In this section we prove that a quasi-compact and quasi-separated algebraic space has a finite stratification by locally closed subspaces each of which is a scheme and such that the glueing of the parts is by elementary distinguihsed squares. We first prove a slightly weaker result for reasonable algebraic spaces.

Lemma 8.1. *Let S be a scheme. Let $W \to X$ be a morphism of a scheme W to an algebraic space X which is flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. There exist reduced closed subspaces*

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = X$$

such that with $X_r = Z_r \setminus Z_{r-1}$ the stratification $X = \coprod_{r=0,\ldots,n} X_r$ is characterized by the following universal property: Given $g : T \to X$ the projection $W \times_X T \to T$ is finite locally free of degree r if and only if $g(|T|) \subset |X_r|$.

Proof. Let n be an integer bounding the degrees of the fibres of $W \to X$. Choose a scheme U and a surjective étale morphism $U \to X$. Apply More on Morphisms, Lemma 31.9 to $W \times_X U \to U$. We obtain closed subsets

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset Y_2 \subset \ldots \subset Y_n = U$$

characterized by the property stated in the lemma for the morphism $W \times_X U \to U$. Clearly, the formation of these closed subsets commutes with base change. Setting $R = U \times_X U$ with projection maps $s, t : R \to U$ we conclude that

$$s^{-1}(Y_r) = t^{-1}(Y_r)$$

as closed subsets of R. In other words the closed subsets $Y_r \subset U$ are R-invariant. This means that $|Y_r|$ is the inverse image of a closed subset $Z_r \subset |X|$. Denote $Z_r \subset X$ also the reduced induced algebraic space structure, see Properties of Spaces, Definition 9.5.

Let $g : T \to X$ be a morphism of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to T$. To prove the final assertion of the lemma it suffices to prove the assertion for the composition $V \to X$ (by our definition of finite locally free morphisms, see Morphisms of Spaces, Section 42). Similarly, the morphism of schemes $W \times_X V \to V$ is finite locally free of degree r if and only if the morphism of schemes

$$W \times_X (U \times_X V) \longrightarrow U \times_X V$$

is finite locally free of degree r (see Descent, Lemma 19.28). By construction this happens if and only if $|U \times_X V| \to |U|$ maps into $|Y_r|$, which is true if and only if $|V| \to |X|$ maps into $|Z_r|$. $\square$

Lemma 8.2. *Let S be a scheme. Let $W \to X$ be a morphism of a scheme W to an algebraic space X which is flat, locally of finite presentation, separated, and locally quasi-finite. Then there exist open subspaces*

$$X = X_0 \supset X_1 \supset X_2 \supset \ldots$$

such that a morphism $\mathrm{Spec}(k) \to X$ factors through X_d if and only if $W \times_X \mathrm{Spec}(k)$ has degree $\geq d$ over k.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Apply More on Morphisms, Lemma 31.11 to $W \times_X U \to U$. We obtain open subschemes

$$U = U_0 \supset U_1 \supset U_2 \supset \ldots$$

characterized by the property stated in the lemma for the morphism $W \times_X U \to U$. Clearly, the formation of these closed subsets commutes with base change. Setting $R = U \times_X U$ with projection maps $s, t : R \to U$ we conclude that

$$s^{-1}(U_d) = t^{-1}(U_d)$$

as open subschemes of R. In other words the open subschemes $U_d \subset U$ are R-invariant. This means that U_d is the inverse image of an open subspace $X_d \subset X$ (Properties of Spaces, Lemma 9.2). $\square$

Lemma 8.3. *Let S be a scheme. Let X be a quasi-compact, reasonable algebraic space over S. There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

Proof. By Properties of Spaces, Lemma 6.3 we can choose an affine scheme U and a surjective étale morphism $U \to X$. Let n be an integer bounding the degrees of the fibres of $U \to X$ which exists as X is reasonable, see Definition 6.1. For $p \geq 0$ set

$$W_p = U \times_X \ldots \times_X U \setminus \text{all diagonals}$$

where the fibre product has p factors. Since U is separated, the morphism $U \to X$ is separated and all fibre products $U \times_X \ldots \times_X U$ are separated schemes. Since $U \to X$ is separated the diagonal $U \to U \times_X U$ is a closed immersion. Since $U \to X$ is étale the diagonal $U \to U \times_X U$ is an open immersion, see Morphisms of Spaces, Lemmas 36.10 and 35.9. Similarly, all the diagonal morphisms are open and closed immersions and W_p is an open and closed subscheme of $U \times_X \ldots \times_X U$. Moreover, the morphism

$$U \times_X \ldots \times_X U \longrightarrow U \times_{\mathrm{Spec}(\mathbf{Z})} \ldots \times_{\mathrm{Spec}(\mathbf{Z})} U$$

is locally quasi-finite and separated (Morphisms of Spaces, Lemma 4.5) and its target is an affine scheme. Hence every finite set of points of $U \times_X \ldots \times_X U$ is contained in an affine open, see More on Morphisms, Lemma 31.13. Therefore, the same is true for W_p. There is a free action of the symmetric group S_p on W_p over X (because we threw out the fix point locus from $U \times_X \ldots \times_X U$). By the above and Properties of Spaces, Proposition 11.1 the quotient $V_p = W_p/S_p$ is a scheme. Since the action of S_p on W_p was over X, there is a morphism $V_p \to X$. Since $W_p \to X$ is étale and since $W_p \to V_p$ is surjective étale, it follows that also $V_p \to X$ is étale, see Properties of Spaces, Lemma 13.3.

We let $U_p \subset X$ be the open subspace which is the image of $V_p \to X$. By construction morphism $\mathrm{Spec}(k) \to X$ with k algebraically closed, factors through U_p if and only if $U \times_X \mathrm{Spec}(k)$ has $\geq p$ points. It follows that the U_p give a filtration of X as stated in the lemma. Moreover, $\mathrm{Spec}(k) \to X$ factors through T_p if and only if $U \times_X \mathrm{Spec}(k)$ has exactly p points. In this case we see that $V_p \times_X \mathrm{Spec}(k)$ has exactly one point. Set $Z_p = f_p^{-1}(T_p) \subset V_p$. This is a closed subscheme of V_p. Then $Z_p \to T_p$ is an étale morphism between algebraic spaces which induces a bijection on k-valued points for any algebraically closed field k. To be sure this implies that $Z_p \to T_p$ is universally injective, whence an open immersion by Morphisms of Spaces, Lemma 45.2 hence an isomorphism and we win. □

Lemma 8.4. *Let S be a scheme. Let X be a quasi-compact, reasonable algebraic space over S. There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

such that each $T_p = U_p \backslash U_{p+1}$ (with reduced induced subspace structure) is a scheme.

Proof. Immediate consequence of Lemma 8.3. □

The following result is almost identical to [GR71, Proposition 5.7.8].

Lemma 8.5. *Let X be a quasi-compact and quasi-separated algebraic space over* $\mathrm{Spec}(\mathbf{Z})$. *There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a quasi-compact separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

Proof. The proof of this lemma is identical to the proof of Lemma 8.3. Observe that a quasi-separated space is reasonable, see Decent Spaces, Lemma 5.1 and Decent Spaces, Definition 6.1. At the end of the argument we add that since X is quasi-separated the schemes $V \times_X \ldots \times_X V$ are all quasi-compact. Hence the schemes W_p are quasi-compact. Hence the schemes $V_p = W_p/S_p$ are quasi-compact. □

9. Schematic locus

In this section we prove that a decent algebraic space has a dense open subspace which is a scheme. We first prove this for reasonable algebraic spaces.

Proposition 9.1. *Let S be a scheme. Let X be an algebraic space over S. If X is reasonable, then there exists a dense open subspace of X which is a scheme.*

Proof. By Properties of Spaces, Lemma 10.1 the question is local on X. Hence we may assume there exists an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 6.1). Let n be an integer bounding the degrees of the fibres of $U \to X$ which exists as X is reasonable, see Definition 6.1. We will argue by induction on n that whenever

(1) $U \to X$ is a surjective étale morphism whose fibres have degree $\leq n$, and
(2) U is isomorphic to a locally closed subscheme of an affine scheme

then the schematic locus is dense in X.

Let $X_n \subset X$ be the open subspace which is the complement of the closed subspace $Z_{n-1} \subset X$ constructed in Lemma 8.1 using the morphism $U \to X$. Let $U_n \subset U$ be the inverse image of X_n. Then $U_n \to X_n$ is finite locally free of degree n. Hence X_n is a scheme by Properties of Spaces, Proposition 11.1 (and the fact that any finite set of points of U_n is contained in an affine open of U_n, see Properties, Lemma 27.5).

Let $X' \subset X$ be the open subspace such that $|X'|$ is the interior of $|Z_{n-1}|$ in $|X|$ (see Topology, Definition 20.1). Let $U' \subset U$ be the inverse image. Then $U' \to X'$ is surjective étale and has degrees of fibres bounded by $n - 1$. By induction we see that the schematic locus of X' is an open dense $X'' \subset X'$. By elementary topology we see that $X'' \cup X_n \subset X$ is open and dense and we win. ☐

Theorem 9.2 (David Rydh)**.** *Let S be a scheme. Let X be an algebraic space over S. If X is decent, then there exists a dense open subspace of X which is a scheme.*

Proof. Assume X is a decent algebraic space for which the theorem is false. By Properties of Spaces, Lemma 10.1 there exists a largest open subspace $X' \subset X$ which is a scheme. Since X' is not dense in X, there exists an open subspace $X'' \subset X$ such that $|X''| \cap |X'| = \emptyset$. Replacing X by X'' we get a nonempty decent algebraic space X which does not contain *any* open subspace which is a scheme.

Choose a nonempty affine scheme U and an étale morphism $U \to X$. We may and do replace X by the open subscheme corresponding to the image of $|U| \to |X|$. Consider the sequence of open subspaces

$$X = X_0 \supset X_1 \supset X_2 \ldots$$

constructed in Lemma 8.2 for the morphism $U \to X$. Note that $X_0 = X_1$ as $U \to X$ is surjective. Let $U = U_0 = U_1 \supset U_2 \ldots$ be the induced sequence of open subschemes of U.

Choose a nonempty open affine $V_1 \subset U_1$ (for example $V_1 = U_1$). By induction we will construct a sequence of nonempty affine opens $V_1 \supset V_2 \supset \ldots$ with $V_n \subset U_n$. Namely, having constructed $V_1, \ldots, V_{n-1}$ we can always choose V_n unless $V_{n-1} \cap U_n = \emptyset$. But if $V_{n-1} \cap U_n = \emptyset$, then the open subspace $X' \subset X$ with $|X'| = \text{Im}(|V_{n-1}| \to |X|)$ is contained in $|X| \setminus |X_n|$. Hence $V_{n-1} \to X'$ is an étale morphism whose fibres have degree bounded by $n - 1$. In other words, X' is reasonable (by definition), hence X' contains a nonempty open subscheme by Proposition 9.1. This is a contradiction which shows that we can pick V_n.

By Limits, Lemma 3.4 the limit $V_\infty = \lim V_n$ is a nonempty scheme. Pick a morphism $\mathrm{Spec}(k) \to V_\infty$. The composition $\mathrm{Spec}(k) \to V_\infty \to U \to X$ has image contained in all X_d by construction. In other words, the fibred $U \times_X \mathrm{Spec}(k)$ has infinite degree which contradicts the definition of a decent space. This contradiction finishes the proof of the theorem. $\square$

10. Points on spaces

In this section we prove some properties of points on decent algebraic spaces.

Lemma 10.1. *Let S be a scheme. Let X be an algebraic space over S. Consider the map*

$$\{\mathrm{Spec}(k) \to X \textit{ monomorphism}\} \longrightarrow |X|$$

This map is always injective. If X is decent then this map is a bijection.

Proof. We have seen in Properties of Spaces, Lemma 4.11 that the map is an injection in general. By Lemma 5.1 it is surjective when X is decent (actually one can say this is part of the definition of being decent). $\square$

The following lemma is a tiny bit stronger than Properties of Spaces, Lemma 12.1. We will improve this lemma in Lemma 12.1.

Lemma 10.2. *Let S be a scheme. Let k be a field. Let X be an algebraic space over S and assume that there exists a surjective étale morphism $\mathrm{Spec}(k) \to X$. If X is decent, then $X \cong \mathrm{Spec}(k')$ where $k' \subset k$ is a finite separable extension.*

Proof. The assumption implies that $|X| = \{x\}$ is a singleton. Since X is decent we can find a quasi-compact monomorphism $\mathrm{Spec}(k') \to X$ whose image is x. Then the projection $U = \mathrm{Spec}(k') \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is a monomorphism, whence $U = \mathrm{Spec}(k)$, see Schemes, Lemma 23.10. Hence the projection $\mathrm{Spec}(k) = U \to \mathrm{Spec}(k')$ is étale and we win. $\square$

The following lemma shows that specialization of points behaves well on decent algebraic spaces. Spaces, Example 14.9 shows that this is **not** true in general.

Lemma 10.3. *Let S be a scheme. Let X be a decent algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. If $u, u' \in |U|$ map to the same point of $|X|$, and $u' \leadsto u$, then $u = u'$.*

Proof. Combine Lemmas 5.1 and 7.1. $\square$

Lemma 10.4. *Let S be an algebraic space. Let X be a decent algebraic space over S. Let $x, x' \in |X|$ and assume $x' \leadsto x$, i.e., x is a specialization of x'. Then for every étale morphism $\varphi : U \to X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \leadsto u$ with $\varphi(u') = x'$.*

Proof. Combine Lemmas 5.1 and 7.2. $\square$

Lemma 10.5. *Let S be a scheme. Let X be a decent algebraic space over S. Then $|X|$ is Kolmogorov (see Topology, Definition 7.4).*

Proof. Let $x_1, x_2 \in |X|$ with $x_1 \leadsto x_2$ and $x_2 \leadsto x_1$. We have to show that $x_1 = x_2$. Pick a scheme U and an étale morphism $U \to X$ such that x_1, x_2 are both in the image of $|U| \to |X|$. By Lemma 10.4 we can find a specialization $u_1 \leadsto u_2$ in U mapping to $x_1 \leadsto x_2$. By Lemma 10.4 we can find $u_2' \leadsto u_1$ mapping to $x_2 \leadsto x_1$. This means that $u_2' \leadsto u_2$ is a specialization between points of U mapping to the same point of X, namely x_2. This is not possible, unless $u_2' = u_2$, see Lemma 10.3. Hence also $u_1 = u_2$ as desired. □

Proposition 10.6. *Let S be a scheme. Let X be a decent algebraic space over S. Then the topological space $|X|$ is sober (see Topology, Definition 7.4).*

Proof. We have seen in Lemma 10.5 that $|X|$ is Kolmogorov. Hence it remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Properties of Spaces, Lemma 9.3 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. By definition this means that $Z \to X$ is a representable morphism of algebraic spaces. Hence Z is a decent algebraic space by Lemma 5.3. By Theorem 9.2 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that $|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that Z' is an irreducible scheme. By Schemes, Lemma 11.1 we conclude that $|Z'|$ is the closure of a single point $\eta \in T$ and hence also $T = \overline{\{\eta\}}$, and we win. □

For decent algebraic spaces dimension works as expected.

Lemma 10.7. *Let S be a scheme. Dimension as defined in Properties of Spaces, Section 8 behaves well on decent algebraic spaces X over S.*

(1) *If $x \in |X|$, then $\dim_x(|X|) = \dim_x(X)$, and*
(2) *$\dim(|X|) = \dim(X)$.*

Proof. Proof of (1). Choose a scheme U with a point $u \in U$ and an étale morphism $h : U \to X$ mapping u to x. By definition the dimension of X at x is $\dim_u(|U|)$. Thus we may pick U such that $\dim_x(X) = \dim(|U|)$. Let d be an integer. If $\dim(U) \geq d$, then there exists a sequence of nontrivial specializations $u_d \leadsto \ldots \leadsto u_0$ in U. Taking the image we find a corresponding sequence $h(u_d) \leadsto \ldots \leadsto h(u_0)$ each of which is nontrivial by Lemma 10.3. Hence we see that the image of $|U|$ in $|X|$ has dimension at least d. Conversely, suppose that $x_d \leadsto \ldots \leadsto x_0$ is a sequence of specializations in $|X|$ with x_0 in the image of $|U| \to |X|$. Then we can lift this to a sequence of specializations in U by Lemma 10.4.

Part (2) is an immediate consequence of part (1) and the definitions. □

Lemma 10.8. *Let S be a scheme. Let X be a decent algebraic space over S. Let $x \in |X|$. The following are equivalent*

(1) *x is a generic point of an irreducible component of $|X|$,*
(2) *for any étale morphism $(Y, y) \to (X, x)$ of pointed algebraic spaces, y is a generic point of an irreducible component of $|Y|$,*
(3) *the dimension of the local ring of X at x is zero (Properties of Spaces, Definition 20.2).*

Proof. Observe that any Y as in (2) is decent by Lemma 6.6. Thus it suffices to prove the equivalence of (1) and (3) as then the equivalence with (2) follows since the dimension of the local ring of Y at y is equal to the dimension of the local ring of X at x. Let $f : U \to X$ be an étale morphism from an affine scheme and let $u \in U$ be a point mapping to x.

Assume (1). Let $u' \rightsquigarrow u$ be a specialization in U. Then $f(u') = f(u) = x$. By Lemma 10.3 we see that $u' = u$. Hence u is a generic point of an irreducible component of U. Thus $\dim(\mathcal{O}_{U,u}) = 0$ and we see that (2) holds.

Assume (2). The point x is contained in an irreducible component $T \subset |X|$. Since $|X|$ is sober (Proposition 10.6) we T has a generic point x'. Of course $x' \rightsquigarrow x$. Then we can lift this specialization to $u' \rightsquigarrow u$ in U (Lemma 10.4). This contradicts the assumption that $\dim(\mathcal{O}_{U,u}) = 0$ unless $u' = u$, i.e., $x' = x$. $\square$

Lemma 10.9. *Let S be a scheme. Let $X \to Y$ be a locally quasi-finite morphism of algebraic spaces over S. Let $x \in |X|$ with image $y \in |Y|$. Then the dimension of the local ring of Y at y is $\geq$ to the dimension of the local ring of X at x.*

Proof. The definition of the dimension of the local ring of a point on an algebraic space is given in Properties of Spaces, Definition 20.2. Choose an étale morphism $(V, v) \to (Y, y)$ where V is a scheme. Choose an étale morphism $U \to V \times_Y X$ and a point $u \in U$ mapping to $x \in |X|$ and $v \in V$. Then $U \to V$ is locally quasi-finite and we have to prove that

$$\dim(\mathcal{O}_{V,v}) \geq \dim(\mathcal{O}_{U,u})$$

This is Algebra, Lemma 121.4. $\square$

11. Reduced singleton spaces

A *singleton* space is an algebraic space X such that $|X|$ is a singleton. It turns out that these can be more interesting than just being the spectrum of a field, see Spaces, Example 14.7. We develop a tiny bit of machinery to be able to talk about these.

Lemma 11.1. *Let S be a scheme. Let Z be an algebraic space over S. Let k be a field and let $\mathrm{Spec}(k) \to Z$ be surjective and flat. Then any morphism $\mathrm{Spec}(k') \to Z$ where k' is a field is surjective and flat.*

Proof. Consider the fibre square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d] \\ \mathrm{Spec}(k') \arrow[r] & Z \end{tikzcd}$$

Note that $T \to \mathrm{Spec}(k')$ is flat and surjective hence T is not empty. On the other hand $T \to \mathrm{Spec}(k)$ is flat as k is a field. Hence $T \to Z$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 29.5 that $\mathrm{Spec}(k') \to Z$ is flat. It is surjective as by assumption $|Z|$ is a singleton. $\square$

Lemma 11.2. *Let S be a scheme. Let Z be an algebraic space over S. The following are equivalent*

1. *Z is reduced and $|Z|$ is a singleton,*
2. *there exists a surjective flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field, and*
3. *there exists a locally of finite type, surjective, flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field.*

Proof. Assume (1). Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W. Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = \mathrm{Spec}(\kappa(\eta)) \to W$ is flat. We see that the composition $\eta \to Z$ is flat (see Morphisms of Spaces, Lemma 28.3). It is also surjective as $|Z|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field k and a surjective flat morphism $\mathrm{Spec}(k) \to Z$. Then $W \times_Z \mathrm{Spec}(k)$ is a scheme étale over k. Hence $W \times_Z \mathrm{Spec}(k)$ is a disjoint union of spectra of fields (see Remark 4.1), in particular reduced. Since $W \times_Z \mathrm{Spec}(k) \to W$ is surjective and flat we conclude that W is reduced (Descent, Lemma 15.1). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite type by Morphisms of Spaces, Lemmas 23.2 and 36.9. It is also flat and surjective by Lemma 11.1. Hence (3) holds. □

The following lemma singles out a slightly better class of singleton algebraic spaces than the preceding lemma.

Lemma 11.3. *Let S be a scheme. Let Z be an algebraic space over S. The following are equivalent*

(1) *Z is reduced, locally Noetherian, and $|Z|$ is a singleton, and*
(2) *there exists a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field.*

Proof. Assume (2) holds. By Lemma 11.2 we see that Z is reduced and $|Z|$ is a singleton. Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to Z$. Then $W \times_Z \mathrm{Spec}(k)$ is a scheme étale over k, hence a disjoint union of spectra of fields (see Remark 4.1), hence locally Noetherian. Since $W \times_Z \mathrm{Spec}(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_Z \mathrm{Spec}(k) \to W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent, Lemma 12.1). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism $w : \mathrm{Spec}(k) \to W$ is of finite presentation, see Morphisms, Lemma 22.7. Hence the composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 27.2 and 36.8. It is also flat and surjective by Lemma 11.1. Hence (2) holds. □

Lemma 11.4. *Let S be a scheme. Let $Z' \to Z$ be a monomorphism of algebraic spaces over S. Assume there exists a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to Z$. Then either Z' is empty or $Z' = Z$.*

Proof. We may assume that Z' is nonempty. In this case the fibre product $T = Z' \times_Z \mathrm{Spec}(k)$ is nonempty, see Properties of Spaces, Lemma 4.3. Now T is an algebraic space and the projection $T \to \mathrm{Spec}(k)$ is a monomorphism. Hence $T = \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemma 10.8. We conclude that $\mathrm{Spec}(k) \to Z$ factors through Z'. But as $\mathrm{Spec}(k) \to Z$ is surjective, flat and locally of finite presentation, we see that $\mathrm{Spec}(k) \to Z$ is surjective as a map of sheaves on $(Sch/S)_{fppf}$ (see Spaces, Remark 5.2) and we conclude that $Z' = Z$. $\square$

The following lemma says that to each point of an algebraic space we can associate a canonical reduced, locally Noetherian singleton algebraic space.

Lemma 11.5. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Then there exists a unique monomorphism $Z \to X$ of algebraic spaces over S such that Z is an algebraic space which satisfies the equivalent conditions of Lemma 11.3 and such that the image of $|Z| \to |X|$ is $\{x\}$.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Set $R = U \times_X U$ so that $X = U/R$ is a presentation (see Spaces, Section 9). Set

$$U' = \coprod\nolimits_{u \in U \text{ lying over } x} \mathrm{Spec}(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_X U' = R \times_{(U \times_S U)} (U' \times_S U').$$

Because $U' \to U$ is a monomorphism we see that the projections $s', t' : R' \to U'$ factor as a monomorphism followed by an étale morphism. Hence, as U' is a disjoint union of spectra of fields, using Remark 4.1, and using Schemes, Lemma 23.10 we conclude that R' is a disjoint union of spectra of fields and that the morphisms $s', t' : R' \to U'$ are étale. Hence $Z = U'/R'$ is an algebraic space by Spaces, Theorem 10.5. As R' is the restriction of R by $U' \to U$ we see $Z \to X$ is a monomorphism by Groupoids, Lemma 18.6. Since $Z \to X$ is a monomorphism we see that $|Z| \to |X|$ is injective, see Morphisms of Spaces, Lemma 10.9. By Properties of Spaces, Lemma 4.3 we see that

$$|U'| = |Z \times_X U'| \to |Z| \times_{|X|} |U'|$$

is surjective which implies (by our choice of U') that $|Z| \to |X|$ has image $\{x\}$. We conclude that $|Z|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., we see that Z satisfies the equivalent conditions of Lemma 11.3.

Let us prove uniqueness of $Z \to X$. Suppose that $Z' \to X$ is a second such monomorphism of algebraic spaces. Then the projections

$$Z' \longleftarrow Z' \times_X Z \longrightarrow Z$$

are monomorphisms. The algebraic space in the middle is nonempty by Properties of Spaces, Lemma 4.3. Hence the two projections are isomorphisms by Lemma 11.4 and we win. $\square$

We introduce the following terminology which foreshadows the residual gerbes we will introduce later, see Properties of Stacks, Definition 11.8.

Definition 11.6. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The *residual space of X at x*[2] is the monomorphism $Z_x \to X$ constructed in Lemma 11.5.

In particular we know that Z_x is a locally Noetherian, reduced, singleton algebraic space and that there exists a field and a surjective, flat, locally finitely presented morphism

$$\mathrm{Spec}(k) \longrightarrow Z_x.$$

It turns out that Z_x is a regular algebraic space as follows from the following lemma.

Lemma 11.7. *A reduced, locally Noetherian singleton algebraic space Z is regular.*

Proof. Let Z be a reduced, locally Noetherian singleton algebraic space over a scheme S. Let $W \to Z$ be a surjective étale morphism where W is a scheme. Let k be a field and let $\mathrm{Spec}(k) \to Z$ be surjective, flat, and locally of finite presentation (see Lemma 11.3). The scheme $T = W \times_Z \mathrm{Spec}(k)$ is étale over k in particular regular, see Remark 4.1. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent, Lemma 15.2. By definition this means that Z is regular. □

12. Decent spaces

In this section we collect some useful facts on decent spaces.

Lemma 12.1. *Let S be a scheme. Let X be a decent algebraic space over S.*

(1) *If $|X|$ is a singleton then X is a scheme.*
(2) *If $|X|$ is a singleton and X is reduced, then $X \cong \mathrm{Spec}(k)$ for some field k.*

Proof. Assume $|X|$ is a singleton. It follows immediately from Theorem 9.2 that X is a scheme, but we can also argue directly as follows. Choose an affine scheme U and a surjective étale morphism $U \to X$. Set $R = U \times_X U$. Then U and R have finitely many points by Lemma 4.5 (and the definition of a decent space). All of these points are closed in U and R by Lemma 10.3. It follows that U and R are affine schemes. We may shrink U to a singleton space. Then U is the spectrum of a henselian local ring, see Algebra, Lemma 145.11. The projections $R \to U$ are étale, hence finite étale because U is the spectrum of a 0-dimensional henselian local ring, see Algebra, Lemma 145.3. It follows that X is a scheme by Groupoids, Proposition 21.8.

Part (2) follows from (1) and the fact that a reduced singleton scheme is the spectrum of a field. □

Remark 12.2. We will see in Limits of Spaces, Lemma 15.3 that an algebraic space whose reduction is a scheme is a scheme.

Lemma 12.3. *Let S be a scheme. Let X be a decent algebraic space over S. Consider a commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

[2] This is nonstandard notation.

Assume that the image point $s \in S$ of $\mathrm{Spec}(k) \to S$ *is a closed point and that* $\kappa(s) \subset k$ *is algebraic. Then the image x of* $\mathrm{Spec}(k) \to X$ *is a closed point of* $|X|$.

Proof. Suppose that $x \rightsquigarrow x'$ for some $x' \in |X|$. Choose an étale morphism $U \to X$ where U is a scheme and a point $u' \in U'$ mapping to x'. Choose a specialization $u \rightsquigarrow u'$ in U with u mapping to x in X, see Lemma 10.4. Then u is the image of a point w of the scheme $W = \mathrm{Spec}(k) \times_X U$. Since the projection $W \to \mathrm{Spec}(k)$ is étale we see that $\kappa(w) \supset k$ is finite. Hence $\kappa(w) \supset \kappa(s)$ is algebraic. Hence $\kappa(u) \supset \kappa(s)$ is algebraic. Thus u is a closed point of U by Morphisms, Lemma 21.2. Thus $u = u'$, whence $x = x'$. $\square$

Lemma 12.4. *Let S be a scheme. Let X be a decent algebraic space over S. Consider a commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

Assume that the image point $s \in S$ of $\mathrm{Spec}(k) \to S$ *is a closed point and that* $\kappa(s) \subset k$ *is finite. Then* $\mathrm{Spec}(k) \to X$ *is finite morphism. If* $\kappa(s) = k$ *then* $\mathrm{Spec}(k) \to X$ *is closed immersion.*

Proof. By Lemma 12.3 the image point $x \in |X|$ is closed. Let $Z \subset X$ be the reduced closed subspace with $|Z| = \{x\}$ (Properties of Spaces, Lemma 9.3). Note that Z is a decent algebraic space by Lemma 6.5. By Lemma 12.1 we see that $Z = \mathrm{Spec}(k')$ for some field k'. Of course $k \supset k' \supset \kappa(s)$. Then $\mathrm{Spec}(k) \to Z$ is a finite morphism of schemes and $Z \to X$ is a finite morphism as it is a closed immersion. Hence $\mathrm{Spec}(k) \to X$ is finite (Morphisms of Spaces, Lemma 41.4). If $k = \kappa(s)$, then $\mathrm{Spec}(k) = Z$ and $\mathrm{Spec}(k) \to X$ is a closed immersion. $\square$

Lemma 12.5. *Let S be a scheme. Suppose X is a decent algebraic space over S. Let $x \in |X|$ be a closed point. Then x can be represented by a closed immersion* $i : \mathrm{Spec}(k) \to X$ *from the spectrum of a field.*

Proof. We know that x can be represented by a quasi-compact monomorphism $i : \mathrm{Spec}(k) \to X$ where k is a field (Definition 6.1). Let $U \to X$ be an étale morphism where U is an affine scheme. As x is closed and X decent, the fibre F of $|U| \to |X|$ over x consists of closed points (Lemma 10.3). As i is a monomorphism, so is $U_k = U \times_X \mathrm{Spec}(k) \to U$. In particular, the map $|U_k| \to F$ is injective. Since U_k is quasi-compact and étale over a field, we see that U_k is a finite disjoint union of spectra of fields (Remark 4.1). Say $U_k = \mathrm{Spec}(k_1) \amalg \ldots \amalg \mathrm{Spec}(k_r)$. Since $\mathrm{Spec}(k_i) \to U$ is a monomorphism, we see that its image u_i has residue field $\kappa(u_i) = k_i$. Since $u_i \in F$ is a closed point we conclude the morphism $\mathrm{Spec}(k_i) \to U$ is a closed immersion. As the u_i are pairwise distinct, $U_k \to U$ is a closed immersion. Hence i is a closed immersion (Morphisms of Spaces, Lemma 12.1). This finishes the proof. $\square$

13. Locally separated spaces

It turns out that a locally separated algebraic space is decent.

Lemma 13.1. *Let A be a ring. Let k be a field. Let $\mathfrak{p}_n$, $n \geq 1$ be a sequence of pairwise distinct primes of A. Moreover, for each n let $k \to \kappa(\mathfrak{p})$ be an embedding. Then the closure of the image of*

$$\coprod\nolimits_{n \neq m} \mathrm{Spec}(\kappa(\mathfrak{p}_n) \otimes_k \kappa(\mathfrak{p}_m)) \longrightarrow \mathrm{Spec}(A \otimes A)$$

meets the diagonal.

Proof. Set $k_n = \kappa(\mathfrak{p}_n)$. We may assume that $A = \prod k_n$. Denote $x_n = \mathrm{Spec}(k_n)$ the open and closed point corresponding to $A \to k_n$. Then $\mathrm{Spec}(A) = Z \amalg \{x_n\}$ where Z is a nonempty closed subset. Namely, $Z = V(e_n; n \geq 1)$ where e_n is the idempotent of A corresponding to the factor k_n and Z is nonempty as the ideal generated by the e_n is not equal to A. We will show that the closure of the image contains $\Delta(Z)$. The kernel of the map

$$(\prod k_n) \otimes_k (\prod k_m) \longrightarrow \prod\nolimits_{n \neq m} k_n \otimes_k k_m$$

is the ideal generated by $e_n \otimes e_n$, $n \geq 1$. Hence the closure of the image of the map on spectra is $V(e_n \otimes e_n; n \geq 1)$ whose intersection with $\Delta(\mathrm{Spec}(A))$ is $\Delta(Z)$. Thus it suffices to show that

$$\coprod\nolimits_{n \neq m} \mathrm{Spec}(k_n \otimes_k k_m) \longrightarrow \mathrm{Spec}(\prod\nolimits_{n \neq m} k_n \otimes_k k_m)$$

has dense image. This follows as the family of ring maps $\prod_{n \neq m} k_n \otimes_k k_m \to k_n \otimes_k k_m$ is jointly injective. $\square$

Lemma 13.2 (David Rydh)**.** *A locally separated algebraic space is decent.*

Proof. Let S be a base scheme. Let X be a locally separated algebraic space over S. Let $x \in |X|$. Choose a scheme U, an étale morphism $U \to X$, and a point $u \in U$ mapping to x in $|X|$. As usual we identify $u = \mathrm{Spec}(\kappa(u))$. As X is locally separated the morphism

$$u \times_X u \to u \times u$$

is an immersion (Morphisms of Spaces, Lemma 4.5). Hence More on Groupoids, Lemma 10.5 tells us that it is a closed immersion (use Schemes, Lemma 10.4). As $u \times_X u \to u \times_X U$ is a monomorphism (base change of $u \to U$) and as $u \times_X U \to u$ is étale we conclude that $u \times_X u$ is a disjoint union of spectra of fields (see Remark 4.1 and Schemes, Lemma 23.10). Since it is also closed in the affine scheme $u \times u$ we conclude $u \times_X u$ is a finite disjoint union of spectra of fields. Thus x can be represented by a monomorphism $\mathrm{Spec}(k) \to X$ where k is a field, see Lemma 4.3.

Next, let $U = \mathrm{Spec}(A)$ be an affine scheme and let $U \to X$ be an étale morphism. To finish the proof it suffices to show that $F = U \times_X \mathrm{Spec}(k)$ is finite. Write $F = \coprod_{i \in I} \mathrm{Spec}(k_i)$ as the disjoint union of finite separable extensions of k. We have to show that I is finite. Set $R = U \times_X U$. As X is locally separated, the morphism $j : R \to U \times U$ is an immersion. Let $e : U \to R$ be the diagonal map. Using that e is a morphism between étale schemes over U such that $\Delta = j \circ e$ is a closed immersion, we conclude that $R = e(U) \coprod W$ for some open and closed subscheme $W \subset R$. Since j is an immersion and $j|_{e(U)}$ is a closed immersion we conclude that $\overline{j(W)} \cap \Delta(U) = \emptyset$ in $U \times U$. Note that W contains $\mathrm{Spec}(k_i \otimes_k k_{i'})$ for all $i \neq i'$, $i, i' \in I$. By Lemma 13.1 we conclude that I is finite as desired. $\square$

14. Valuative criterion

For a quasi-compact morphism from a decent space the valuative criterion is necessary in order for the morphism to be universally closed.

Proposition 14.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact, and X is decent. Then f is universally closed if and only if the existence part of the valuative criterion holds.*

Proof. In Morphisms of Spaces, Lemma 39.1 we have seen one of the implications. To prove the other, assume that f is universally closed. Let

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

be a diagram as in Morphisms of Spaces, Definition 38.1. Let $X_A = \mathrm{Spec}(A) \times_Y X$, so that we have

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[rd] & X_A \arrow[d] \\ & \mathrm{Spec}(A) \end{tikzcd}$$

By Morphisms of Spaces, Lemma 8.3 we see that $X_A \to \mathrm{Spec}(A)$ is quasi-compact. Since $X_A \to X$ is representable, we see that X_A is decent also, see Lemma 5.3. Moreover, as f is universally closed, we see that $X_A \to \mathrm{Spec}(A)$ is universally closed. Hence we may and do replace X by X_A and Y by $\mathrm{Spec}(A)$.

Let $x' \in |X|$ be the equivalence class of $\mathrm{Spec}(K) \to X$. Let $y \in |Y| = |\mathrm{Spec}(A)|$ be the closed point. Set $y' = f(x')$; it is the generic point of $\mathrm{Spec}(A)$. Since f is universally closed we see that $f(\overline{\{x'\}})$ contains $\overline{\{y'\}}$, and hence contains y. Let $x \in \overline{\{x'\}}$ be a point such that $f(x) = y$. Let U be a scheme, and $\varphi : U \to X$ an étale morphism such that there exists a $u \in U$ with $\varphi(u) = x$. By Lemma 7.2 and our assumption that X is decent there exists a specialization $u' \rightsquigarrow u$ on U with $\varphi(u') = x'$. This means that there exists a common field extension $K \subset K' \supset \kappa(u')$ such that

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] \arrow[rd] & X \arrow[d] \\ & \mathrm{Spec}(A) \end{tikzcd}$$

is commutative. This gives the following commutative diagram of rings

$$\begin{tikzcd} K' & \mathcal{O}_{U,u} \arrow[l] \\ K \arrow[u] & \\ & A \arrow[ul] \arrow[uu] \end{tikzcd}$$

By Algebra, Lemma 48.2 we can find a valuation ring $A' \subset K'$ dominating the image of $\mathcal{O}_{U,u}$ in K'. Since by construction $\mathcal{O}_{U,u}$ dominates A we see that A' dominates A also. Hence we obtain a diagram resembling the second diagram of Morphisms of Spaces, Definition 38.1 and the proposition is proved. □

The following lemma is a special case of the more general Lemma 15.11.

Lemma 14.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and quasi-separated. Then f is universally closed if and only if the existence part of the valuative criterion holds (Morphisms of Spaces, Definition 38.1).*

Proof. This is a combination of Morphisms of Spaces, Lemma 39.1 and Proposition 14.1. Namely, the implication in one direction is given by Morphisms of Spaces, Lemma 39.1. For the converse, assume f is quasi-separated, quasi-compact, and universally closed and assume given a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

as in Morphisms of Spaces, Definition 38.1. A formal argument shows that the existence of the desired diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] \arrow[urr] & \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

can be reduced to the case of the morphism $X_A \to \mathrm{Spec}(A)$. In this case the algebraic space X_A is quasi-separated, hence decent (property (γ) of Lemma 5.1). Hence the existence of $A \subset A'$ and the arrow $\mathrm{Spec}(A') \to X_A$ follows from Proposition 14.1. □

Lemma 14.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and separated. Then the following are equivalent*

1. *f is universally closed,*
2. *the existence part of the valuative criterion holds as in Morphisms of Spaces, Definition 38.1, and*

(3) *given any commutative solid diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 20.3.

Proof. Since f is separated parts (2) and (3) are equivalent by Morphisms of Spaces, Lemma 38.5. The equivalence of (3) and (1) follows from Lemma 14.2. $\square$

Lemma 14.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and quasi-separated. Then the following are equivalent*

(1) *f is separated and universally closed,*
(2) *the valuative criterion holds as in Morphisms of Spaces, Definition 38.1,*
(3) *given any commutative solid diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow, i.e., f satisfies the valuative criterion as in Schemes, Definition 20.3.

Proof. Since f is quasi-separated, the uniqueness part of the valutative criterion implies f is separated (Morphisms of Spaces, Lemma 40.2). Conversely, if f is separated, then it satisfies the uniqueness part of the valuative criterion (Morphisms of Spaces, Lemma 40.1). Having said this, we see that in each of the three cases the morphism f is separated and satisfies the uniqueness part of the valuative criterion. In this case the lemma is a formal consequence of Lemma 14.3. $\square$

Lemma 14.5 (Valuative criterion for properness)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is of finite type and quasi-separated. Then the following are equivalent*

(1) *f is proper,*
(2) *the valuative criterion holds as in Morphisms of Spaces, Definition 38.1,*
(3) *given any commutative solid diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow, i.e., f satisfies the valuative criterion as in Schemes, Definition 20.3.

Proof. Formal consequence of Lemma 14.4 and the definitions. □

15. Relative conditions

This is a (yet another) technical section dealing with conditions on algebraic spaces having to do with points. It is probably a good idea to skip this section.

Definition 15.1. Let S be a scheme. We say an algebraic space X over S *has property* (β) if X has the corresponding property of Lemma 5.1. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

1. (1) We say f *has property* (β) if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ has property (β).
2. (2) We say f is *decent* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a decent algebraic space.
3. (3) We say f is *reasonable* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a reasonable algebraic space.
4. (4) We say f is *very reasonable* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a very reasonable algebraic space.

We refer to Remark 15.10 for an informal discussion. It will turn out that the class of very reasonable morphisms is not so useful, but that the classes of decent and reasonable morphisms are useful.

Lemma 15.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We have the following implications among the conditions on f:*

$$\begin{tikzcd}
\textit{representable} \arrow[dr, Rightarrow] & & & & \\
& \textit{very reasonable} \arrow[r, Rightarrow] & \textit{reasonable} \arrow[r, Rightarrow] & \textit{decent} \arrow[r, Rightarrow] & (\beta) \\
\textit{quasi-separated} \arrow[ur, Rightarrow] & & & &
\end{tikzcd}$$

Proof. This is clear from the definitions, Lemma 5.1 and Morphisms of Spaces, Lemma 4.12. □

Here is another sanity check.

Lemma 15.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If X is decent (resp. is reasonable, resp. has property (β) of Lemma 5.1), then f is decent (resp. reasonable, resp. has property (β)).*

Proof. Let T be a scheme and let $T \to Y$ be a morphism. Then $T \to Y$ is representable, hence the base change $T \times_Y X \to X$ is represenble. Hence if X is decent (or reasonable), then so is $T \times_Y X$, see Lemma 6.5. Similarly, for property (β), see Lemma 5.3. □

Lemma 15.4. *Having property (β), being decent, or being reasonable is preserved under arbitrary base change.*

Proof. This is immediate from the definition. □

Lemma 15.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\omega \in \{\beta, decent, reasonable\}$. Suppose that Y has property (ω) and $f : X \to Y$ has (ω). Then X has (ω).*

Proof. Let us prove the lemma in case $\omega = \beta$. In this case we have to show that any $x \in |X|$ is represented by a monomorphism from the spectrum of a field into X. Let $y = f(x) \in |Y|$. By assumption there exists a field k and a monomorphism $\mathrm{Spec}(k) \to Y$ representing y. Then x corresponds to a point x' of $\mathrm{Spec}(k) \times_Y X$. By assumption x' is represented by a monomorphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \times_Y X$. Clearly the composition $\mathrm{Spec}(k') \to X$ is a monomorphism representing x.

Let us prove the lemma in case $\omega = decent$. Let $x \in |X|$ and $y = f(x) \in |Y|$. By the result of the preceding paragraph we can choose a diagram

$$\begin{tikzcd} \mathrm{Spec}(k') \arrow[r, "x"'] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[r, "y"] & Y \end{tikzcd}$$

whose horizontal arrows monomorphisms. As Y is decent the morphism y is quasi-compact. As f is decent the algebraic space $\mathrm{Spec}(k) \times_Y X$ is decent. Hence the monomorphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \times_Y X$ is quasi-compact. Then the monomorphism $x : \mathrm{Spec}(k') \to X$ is quasi-compact as a composition of quasi-compact morphisms (use Morphisms of Spaces, Lemmas 8.3 and 8.4). As the point x was arbitrary this implies X is decent.

Let us prove the lemma in case $\omega = reasonable$. Choose $V \to Y$ étale with V an affine scheme. Choose $U \to V \times_Y X$ étale with U an affine scheme. By assumption $V \to Y$ has universally bounded fibres. By Lemma 3.3 the morphism $V \times_Y X \to X$ has universally bounded fibres. By assumption on f we see that $U \to V \times_Y X$ has universally bounded fibres. By Lemma 3.2 the composition $U \to X$ has universally bounded fibres. Hence there exists sufficiently many étale morphisms $U \to X$ from schemes with universally bounded fibres, and we conclude that X is reasonable. $\square$

Lemma 15.6. *Having property (β), being decent, or being reasonable is preserved under compositions.*

Proof. Let $\omega \in \{\beta, decent, reasonable\}$. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over the scheme S. Assume f and g both have property (ω). Then we have to show that for any scheme T and morphism $T \to Z$ the space $T \times_Z X$ has (ω). By Lemma 15.4 this reduces us to the following claim: Suppose that Y is an algebraic space having property (ω), and that $f : X \to Y$ is a morphism with (ω). Then X has (ω). This is the content of Lemma 15.5. $\square$

Lemma 15.7. *Let S be a scheme. Let $f : X \to Y$, $g : Z \to Y$ be morphisms of algebraic spaces over S. If X and Y are decent (resp. reasonable, resp. have property (β) of Lemma 5.1), then so does $X \times_Y Z$.*

Proof. Namely, by Lemma 15.3 the morphism $X \to Y$ has the property. Then the base change $X \times_Y Z \to Z$ has the property by Lemma 15.4. And finally this implies $X \times_Y Z$ has the property by Lemma 15.5. $\square$

Lemma 15.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P} \in \{(\beta), decent, reasonable\}$. Assume*

(1) *f is quasi-compact,*
(2) *f is étale,*
(3) *$|f| : |X| \to |Y|$ is surjective, and*
(4) *the algebraic space X has property $\mathcal{P}$.*

Then Y has property $\mathcal{P}$.

Proof. Let us prove this in case $\mathcal{P} = (\beta)$. Let $y \in |Y|$ be a point. We have to show that y can be represented by a monomorphism from a field. Choose a point $x \in |X|$ with $f(x) = y$. By assumption we may represent x by a monomorphism $\mathrm{Spec}(k) \to X$, with k a field. By Lemma 4.3 it suffices to show that the projections $\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are étale and quasi-compact. We can factor the first projection as

$$\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k) \times_Y X \longrightarrow \mathrm{Spec}(k)$$

The first morphism is a monomorphism, and the second is étale and quasi-compact. By Properties of Spaces, Lemma 13.8 we see that $\mathrm{Spec}(k) \times_Y X$ is a scheme. Hence it is a finite disjoint union of spectra of finite separable field extensions of k. By Schemes, Lemma 23.10 we see that the first arrow identifies $\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k)$ with a finite disjoint union of spectra of finite separable field extensions of k. Hence the projection morphism is étale and quasi-compact.

Let us prove this in case $\mathcal{P} = $ *decent*. We have already seen in the first paragraph of the proof that this implies that every $y \in |Y|$ can be represented by a monomorphism $y : \mathrm{Spec}(k) \to Y$. Pick such a y. Pick an affine scheme U and an étale morphism $U \to X$ such that the image of $|U| \to |Y|$ contains y. By Lemma 4.5 it suffices to show that U_y is a finite scheme over k. The fibre product $X_y = \mathrm{Spec}(k) \times_Y X$ is a quasi-compact étale algebraic space over k. Hence by Properties of Spaces, Lemma 13.8 it is a scheme. So it is a finite disjoint union of spectra of finite separable extensions of k. Say $X_y = \{x_1, \ldots, x_n\}$ so x_i is given by $x_i : \mathrm{Spec}(k_i) \to X$ with $[k_i : k] < \infty$. By assumption X is decent, so the schemes $U_{x_i} = \mathrm{Spec}(k_i) \times_X U$ are finite over k_i. Finally, we note that $U_y = \coprod U_{x_i}$ as a scheme and we conclude that U_y is finite over k as desired.

Let us prove this in case $\mathcal{P} = $ *reasonable*. Pick an affine scheme V and an étale morphism $V \to Y$. We have the show the fibres of $V \to Y$ are universally bounded. The algebraic space $V \times_Y X$ is quasi-compact. Thus we can find an affine scheme W and a surjective étale morphism $W \to V \times_Y X$, see Properties of Spaces, Lemma 6.3. Here is a picture (solid diagram)

$$\begin{tikzcd}
W \arrow[r] \arrow[dr] & V \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"'] & \mathrm{Spec}(k) \arrow[l, dotted, "x"'] \arrow[dl, dotted, "y"] \\
 & V \arrow[r] & Y &
\end{tikzcd}$$

The morphism $W \to X$ is universally bounded by our assumption that the space X is reasonable. Let n be an integer bounding the degrees of the fibres of $W \to X$. We claim that the same integer works for bounding the fibres of $V \to Y$. Namely, suppose $y \in |Y|$ is a point. Then there exists a $x \in |X|$ with $f(x) = y$ (see above). This means we can find a field k and morphisms x, y given as dotted arrows in the diagram above. In particular we get a surjective étale morphism

$$\mathrm{Spec}(k) \times_{x,X} W \to \mathrm{Spec}(k) \times_{x,X} (V \times_Y X) = \mathrm{Spec}(k) \times_{y,Y} V$$

which shows that the degree of $\mathrm{Spec}(k) \times_{y,Y} V$ over k is less than or equal to the degree of $\mathrm{Spec}(k) \times_{x,X} W$ over k, i.e., $\leq n$, and we win. (This last part of the argument is the same as the argument in the proof of Lemma 3.4. Unfortunately that lemma is not general enough because it only applies to representable morphisms.) $\square$

Lemma 15.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P} \in \{(\beta), decent, reasonable, very\ reasonable\}$. The following are equivalent*

1. *f is $\mathcal{P}$,*
2. *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
3. *for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is $\mathcal{P}$, and*
4. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \to Y_i$ has $\mathcal{P}$.*

If $\mathcal{P} \in \{(\beta), decent, reasonable\}$, then this is also equivalent to

5. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. The implication (3) $\Rightarrow$ (1) can be seen as follows. Let $Z \to Y$ be a morphism whose source is a scheme over S. Consider the algebraic space $Z \times_Y X$. If we assume (3), then for any affine open $W \subset Z$, the open subspace $W \times_Y X$ of $Z \times_Y X$ has property $\mathcal{P}$. Hence by Lemma 5.2 the space $Z \times_Y X$ has property $\mathcal{P}$, i.e., (1) holds. A similar argument (omitted) shows that (4) implies (1).

The implication (1) $\Rightarrow$ (5) is trivial. Let $V \to Y$ be an étale morphism from a scheme as in (5). Let Z be an affine scheme, and let $Z \to Y$ be a morphism. Consider the diagram

$$\begin{tikzcd} Z \times_Y V \arrow[r, "q"'] \arrow[d, "p"'] & V \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

Since p is étale, and hence open, we can choose finitely many affine open subschemes $W_i \subset Z \times_Y V$ such that $Z = \bigcup p(W_i)$. Consider the commutative diagram

$$\begin{tikzcd} V \times_Y X \arrow[d] & (\coprod W_i) \times_Y X \arrow[l] \arrow[r] \arrow[d] & Z \times_Y X \arrow[d] \\ V & \coprod W_i \arrow[l] \arrow[r] & Z \end{tikzcd}$$

We know $V \times_Y X$ has property $\mathcal{P}$. By Lemma 5.3 we see that $(\coprod W_i) \times_Y X$ has property $\mathcal{P}$. Note that the morphism $(\coprod W_i) \times_Y X \to Z \times_Y X$ is étale and quasi-compact as the base change of $\coprod W_i \to Z$. Hence by Lemma 15.8 we conclude that $Z \times_Y X$ has property $\mathcal{P}$. $\square$

Remark 15.10. An informal description of the properties (β), decent, reasonable, very reasonable was given in Section 6. A morphism has one of these properties if (very) loosely speaking the fibres of the morphism have the corresponding properties. Being decent is useful to prove things about specializations of points on $|X|$. Being reasonable is a bit stronger and technically quite easy to work with.

Here is a lemma we promised earlier which uses decent morphisms.

Lemma 15.11. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and decent. (For example if f is representable, or quasi-separated, see Lemma 15.2.) Then f is universally closed if and only if the existence part of the valuative criterion holds.*

Proof. In Morphisms of Spaces, Lemma 39.1 we proved that any quasi-compact morphism which satisfies the existence part of the valuative criterion is universally closed. To prove the other, assume that f is universally closed. In the proof of Proposition 14.1 we have seen that it suffices to show, for any valuation ring A, and any morphism $\text{Spec}(A) \to Y$, that the base change $f_A : X_A \to \text{Spec}(A)$ satisfies the existence part of the valuative criterion. By definition the algebraic space X_A has property (γ) and hence Proposition 14.1 applies to the morphism f_A and we win. $\square$

16. Points of fibres

Let S be a scheme. Consider a cartesian diagram

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{16.0.1}$$

of algebraic spaces over S. Let $x \in |X|$ and $z \in |Z|$ be points mapping to the same point $y \in |Y|$. We may ask: When is the set

$$F_{x,z} = \{w \in |W| \text{ such that } p(w) = x \text{ and } q(w) = z\} \tag{16.0.2}$$

finite?

Example 16.1. If X, Y, Z are schemes, then the set $F_{x,z}$ is equal to the spectrum of $\kappa(x) \otimes_{\kappa(y)} \kappa(z)$ (Schemes, Lemma 17.5). Thus we obtain a finite set if either $\kappa(y) \subset \kappa(x)$ is finite or if $\kappa(y) \subset \kappa(z)$ is finite. In particular, this is always the case if g is quasi-finite at z (Morphisms, Lemma 21.5).

Example 16.2. Let K be a characteristic 0 field endowed with an automorphism σ of infinite order. Set $Y = \text{Spec}(K)/\mathbf{Z}$ and $X = \mathbf{A}^1_K/\mathbf{Z}$ where $\mathbf{Z}$ acts on K via σ and on $\mathbf{A}^1_K = \text{Spec}(K[t])$ via $t \mapsto t+1$. Let $Z = \text{Spec}(K)$. Then $W = \mathbf{A}^1_K$. Picture

$$\begin{tikzcd} \mathbf{A}^1_K \arrow[r, "q"] \arrow[d, "p"] & \text{Spec}(K) \arrow[d, "g"] \\ \mathbf{A}^1_K/\mathbf{Z} \arrow[r, "f"] & \text{Spec}(K)/\mathbf{Z} \end{tikzcd}$$

Take x corresponding to $t = 0$ and z the unique point of $\text{Spec}(K)$. Then we see that $F_{x,z} = \mathbf{Z}$ as a set.

Lemma 16.3. *In the situation of (16.0.1) if $Z' \to Z$ is a morphism and $z' \in |Z'|$ maps to z, then the induced map $F_{x,z'} \to F_{x,z}$ is surjective.*

Proof. Set $W' = X \times_Y Z' = W \times_Z Z'$. Then $|W'| \to |W| \times_{|Z|} |Z'|$ is surjective by Properties of Spaces, Lemma 4.3. Hence the surjectivity of $F_{x,z'} \to F_{x,z}$. $\square$

Lemma 16.4. *In diagram (16.0.1) the set (16.0.2) is finite if f is of finite type and f is quasi-finite at x.*

Proof. The morphism p is quasi-finite at every $w \in F_{x,z}$, see Morphisms of Spaces, Lemma 26.2. Hence the lemma follows from Morphisms of Spaces, Lemma 26.9. $\square$

Lemma 16.5. *In diagram (16.0.1) the set (16.0.2) is finite if y can be represented by a monomorphism $\mathrm{Spec}(k) \to Y$ where k is a field and g is quasi-finite at z. (Special case: Y is decent and g is étale.)*

Proof. By Lemma 16.3 applied twice we may replace Z by $Z_k = \mathrm{Spec}(k) \times_Y Z$ and X by $X_k = \mathrm{Spec}(k) \times_Y X$. We may and do replace Y by $\mathrm{Spec}(k)$ as well. Note that $Z_k \to \mathrm{Spec}(k)$ is quasi-finite at z by Morphisms of Spaces, Lemma 26.2. Choose a scheme V, a point $v \in V$, and an étale morphism $V \to Z_k$ mapping v to z. Choose a scheme U, a point $u \in U$, and an étale morphism $U \to X_k$ mapping u to x. Again by Lemma 16.3 it suffices to show $F_{u,v}$ is finite for the diagram

$$\begin{tikzcd} U \times_{\mathrm{Spec}(k)} V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

The morphism $V \to \mathrm{Spec}(k)$ is quasi-finite at v (follows from the general discussion in Morphisms of Spaces, Section 22 and the definition of being quasi-finite at a point). At this point the finiteness follows from Example 16.1. The parenthetical remark of the statement of the lemma follows from the fact that on decent spaces points are represented by monomorphisms from fields and from the fact that an étale morphism of algebraic spaces is quasi-finite. $\square$

Lemma 16.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $y \in |Y|$ and assume that y is represented by a quasi-compact monomorphism $\mathrm{Spec}(k) \to Y$. Then $|X_k| \to |X|$ is a homeomorphism onto $f^{-1}(\{y\}) \subset |X|$ with induced topology.*

Proof. We will use Properties of Spaces, Lemma 13.7 and Morphisms of Spaces, Lemma 10.9 without further mention. Let $V \to Y$ be an étale morphism with V affine such that there exists a $v \in V$ mapping to y. Since $\mathrm{Spec}(k) \to Y$ is quasi-compact there are a finite number of points of V mapping to y (Lemma 4.5). After shrinking V we may assume v is the only one. Choose a scheme U and a surjective étale morphism $U \to X$. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[d] & U_V \arrow[l] \arrow[d] & U_v \arrow[l] \arrow[d] \\ X \arrow[d] & X_V \arrow[l] \arrow[d] & X_v \arrow[l] \arrow[d] \\ Y & V \arrow[l] & v \arrow[l] \end{tikzcd}$$

Since $U_v \to U_V$ identifies U_v with a subset of U_V with the induced topology (Schemes, Lemma 18.5), and since $|U_V| \to |X_V|$ and $|U_v| \to |X_v|$ are surjective and open, we see that $|X_v| \to |X_V|$ is a homeomorphism onto its image (with induced topology). On the other hand, the inverse image of $f^{-1}(\{y\})$ under the open map $|X_V| \to |X|$ is equal to $|X_v|$. We conclude that $|X_v| \to f^{-1}(\{y\})$ is open. The morphism $X_v \to X$ factors through X_k and $|X_k| \to |X|$ is injective with image $f^{-1}(\{y\})$ by Properties of Spaces, Lemma 4.3. Using $|X_v| \to |X_k| \to f^{-1}(\{y\})$ the lemma follows because $X_v \to X_k$ is surjective. □

Lemma 16.7. *Let X be an algebraic space locally of finite type over a field k. Let $x \in |X|$. Consider the conditions*

1. $\dim_x(|X|) = 0$,
2. *x is closed in $|X|$ and if $x' \rightsquigarrow x$ in $|X|$ then $x' = x$,*
3. *x is an isolated point of $|X|$,*
4. $\dim_x(X) = 0$,
5. *$X \to \mathrm{Spec}(k)$ is quasi-finite at x.*

Then (2), (3), (4), and (5) are equivalent. If X is decent, then (1) is equivalent to the others.

Proof. Parts (4) and (5) are equivalent for example by Morphisms of Spaces, Lemmas 32.7 and 32.8.

Let $U \to X$ be an étale morphism where U is an affine scheme and let $u \in U$ be a point mapping to x. Moreover, if x is a closed point, e.g., in case (2) or (3), then we may and do assume that u is a closed point. Observe that $\dim_u(U) = \dim_x(X)$ by definition and that this is equal to $\dim(\mathcal{O}_{U,u})$ if u is a closed point, see Algebra, Lemma 110.6.

If $\dim_x(X) > 0$ and u is closed, by the arguments above we can choose a nontrivial specialization $u' \rightsquigarrow u$ in U. Then the transcendence degree of $\kappa(u')$ over k exceeds the transcendence degree of $\kappa(u)$ over k. It follows that the images x and x' in X are distinct, because the transcendence degree of x/k and x'/k are well defined, see Morphisms of Spaces, Definition 31.1. This applies in particular in cases (2) and (3) and we conclude that (2) and (3) imply (4).

Conversely, if $X \to \mathrm{Spec}(k)$ is locally quasi-finite at x, then $U \to \mathrm{Spec}(k)$ is locally quasi-finite at u, hence u is an isolated point of U (Morphisms, Lemma 21.6). It follows that (5) implies (2) and (3) as $|U| \to |X|$ is continuous and open.

Assume X is decent and (1) holds. Then $\dim_x(X) = \dim_x(|X|)$ by Lemma 10.7 and the proof is complete. □

Lemma 16.8. *Let X be an algebraic space locally of finite type over a field k. Consider the conditions*

1. *$|X|$ is a finite set,*
2. *$|X|$ is a discrete space,*
3. $\dim(|X|) = 0$,
4. $\dim(X) = 0$,
5. *$X \to \mathrm{Spec}(k)$ is locally quasi-finite,*

Then (2), (3), (4), and (5) are equivalent. If X is decent, then (1) implies the others.

Proof. Parts (4) and (5) are equivalent for example by Morphisms of Spaces, Lemma 32.7.

Let $U \to X$ be a surjective étale morphism where U is a scheme.

If $\dim(U) > 0$, then choose a nontrivial specialization $u \rightsquigarrow u'$ in U and the transcendence degree of $\kappa(u)$ over k exceeds the transcendence degree of $\kappa(u')$ over k. It follows that the images x and x' in X are distinct, because the transcendence degree of x/k and x'/k is well defined, see Morphisms of Spaces, Definition 31.1. We conclude that (2) and (3) imply (4).

Conversely, if $X \to \mathrm{Spec}(k)$ is locally quasi-finite, then U is locally Noetherian (Morphisms, Lemma 16.6) of dimension 0 (Morphisms, Lemma 30.5) and hence is a disjoint union of spectra of Artinian local rings (Properties, Lemma 10.3). Hence U is a discrete topological space, and since $|U| \to |X|$ is continuous and open, the same is true for $|X|$. In other words, (4) implies (2) and (3).

Assume X is decent and (1) holds. Then we may choose U above to be affine. The fibres of $|U| \to |X|$ are finite (this is a part of the defining property of decent spaces). Hence U is a finite type scheme over k with finitely many points. Hence U is quasi-finite over k (Morphisms, Lemma 21.7) which by definition means that $X \to \mathrm{Spec}(k)$ is locally quasi-finite. $\square$

Lemma 16.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $x \in |X|$ with image $y \in |Y|$. Let $F = f^{-1}(\{y\})$ with induced topology from $|X|$. Let k be a field and let $\mathrm{Spec}(k) \to Y$ be in the equivalence class defining y. Set $X_k = \mathrm{Spec}(k) \times_Y X$. Let $\tilde{x} \in |X_k|$ map to $x \in |X|$. Consider the following conditions*

1. $\dim_x(F) = 0$,
2. *x is isolated in F,*
3. *x is closed in F and if $x' \rightsquigarrow x$ in F, then $x = x'$,*
4. $\dim_{\tilde{x}}(|X_k|) = 0$,
5. *$\tilde{x}$ is isolated in $|X_k|$,*
6. *$\tilde{x}$ is closed in $|X_k|$ and if $\tilde{x}' \rightsquigarrow \tilde{x}$ in $|X_k|$, then $\tilde{x} = \tilde{x}'$,*
7. $\dim_{\tilde{x}}(X_k) = 0$,
8. *f is quasi-finite at x.*

Then we have

$$(4) \underset{f\ decent}{\Longrightarrow} (5) \Longleftrightarrow (6) \Longleftrightarrow (7) \Longleftrightarrow (8)$$

If Y is decent, then conditions (2) and (3) are equivalent to each other and to conditions (5), (6), (7), and (8). If Y and X are decent, then all conditions are equivalent.

Proof. By Lemma 16.7 conditions (5), (6), and (7) are equivalent to each other and to the condition that $X_k \to \mathrm{Spec}(k)$ is quasi-finite at $\tilde{x}$. Thus by Morphisms of Spaces, Lemma 26.2 they are also equivalent to (8). If f is decent, then X_k is a decent algebraic space and Lemma 16.7 shows that (4) implies (5).

If Y is decent, then we can pick a quasi-compact monomorphism $\mathrm{Spec}(k') \to Y$ in the equivalence class of y. In this case Lemma 16.6 tells us that $|X_{k'}| \to F$ is a homeomorphism. Combined with the arguments given above this implies the remaining statements of the lemma; details omitted. $\square$

Lemma 16.10. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $y \in |Y|$. Let k be a field and let $\mathrm{Spec}(k) \to Y$ be in the equivalence class defining y. Set $X_k = \mathrm{Spec}(k) \times_Y X$*

and let $F = f^{-1}(\{y\})$ with the induced topology from $|X|$. Consider the following conditions

1. *F is finite,*
2. *F is a discrete topological space,*
3. *$\dim(F) = 0$,*
4. *$|X_k|$ is a finite set,*
5. *$|X_k|$ is a discrete space,*
6. *$\dim(|X_k|) = 0$,*
7. *$\dim(X_k) = 0$,*
8. *f is quasi-finite at all points of $|X|$ lying over y.*

Then we have

$$(1) \Longleftarrow (4) \underset{f\ decent}{\Longrightarrow} (5) \Longleftrightarrow (6) \Longleftrightarrow (7) \Longleftrightarrow (8)$$

If Y is decent, then conditions (2) and (3) are equivalent to each other and to conditions (5), (6), (7), and (8). If Y and X are decent, then (1) implies all the other conditions.

Proof. By Lemma 16.8 conditions (5), (6), and (7) are equivalent to each other and to the condition that $X_k \to \mathrm{Spec}(k)$ is locally quasi-finite. Thus by Morphisms of Spaces, Lemma 26.2 they are also equivalent to (8). If f is decent, then X_k is a decent algebraic space and Lemma 16.8 shows that (4) implies (5).

The map $|X_k| \to F$ is surjective by Properties of Spaces, Lemma 4.3 and we see (4) $\Rightarrow$ (1).

If Y is decent, then we can pick a quasi-compact monomorphism $\mathrm{Spec}(k') \to Y$ in the equivalence class of y. In this case Lemma 16.6 tells us that $|X_{k'}| \to F$ is a homeomorphism. Combined with the arguments given above this implies the remaining statements of the lemma; details omitted. □

17. Monomorphisms

Here is another case where monomorphisms are representable.

Lemma 17.1. *Let S be a scheme. Let Y be a disjoint union of spectra of zero dimensional local rings over S. Let $f : X \to Y$ be a monomorphism of algebraic spaces over S. Then f is representable, i.e., X is a scheme.*

Proof. This immediately reduces to the case $Y = \mathrm{Spec}(A)$ where A is a zero dimensional local ring, i.e., $\mathrm{Spec}(A) = \{\mathfrak{m}_A\}$ is a singleton. If $X = \emptyset$, then there is nothing to prove. If not, choose a nonempty affine scheme $U = \mathrm{Spec}(B)$ and an étale morphism $U \to X$. As $|X|$ is a singleton (as a subset of $|Y|$, see Morphisms of Spaces, Lemma 10.9) we see that $U \to X$ is surjective. Note that $U \times_X U = U \times_Y U = \mathrm{Spec}(B \otimes_A B)$. Thus we see that the ring maps $B \to B \otimes_A B$ are étale. Since

$$(B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B) = (B/\mathfrak{m}_A B) \otimes_{A/\mathfrak{m}_A} (B/\mathfrak{m}_A B)$$

we see that $B/\mathfrak{m}_A B \to (B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B)$ is flat and in fact free of rank equal to the dimension of $B/\mathfrak{m}_A B$ as a $A/\mathfrak{m}_A$-vector space. Since $B \to B \otimes_A B$ is étale, this can only happen if this dimension is finite (see for example Morphisms, Lemmas 50.7 and 50.8). Every prime of B lies over $\mathfrak{m}_A$ (the unique prime of A). Hence $\mathrm{Spec}(B) = \mathrm{Spec}(B/\mathfrak{m}_A)$ as a topological space, and this space is a finite discrete set as $B/\mathfrak{m}_A B$ is an Artinian ring, see Algebra, Lemmas 51.2 and 51.6. Hence all prime ideals of B are maximal and $B = B_1 \times \ldots \times B_n$ is a product of finitely many local rings of dimension zero, see Algebra, Lemma 51.5. Thus $B \to B \otimes_A B$ is finite étale as all the local rings B_i are henselian by Algebra, Lemma 145.11. Thus X is an affine scheme by Groupoids, Proposition 21.8. □

18. Birational morphisms

The following definition of a birational morphism of algebraic spaces seems to be the closest to our definition (Morphisms, Definition 9.1) of a birational morphism of schemes.

Definition 18.1. Let S be a scheme. Let X and Y algebraic spaces over S. Assume X and Y are decent and that $|X|$ and $|Y|$ have finitely many irreducible components. We say a morphism $f : X \to Y$ is *birational* if

1. $|f|$ induces a bijection between the set of generic points of irreducible components of $|X|$ and the set of generic points of the irreducible components of $|Y|$, and
2. for every generic point $x \in |X|$ of an irreducible component the local ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism (see clarification below).

Clarification: Since X and Y are decent the topological spaces $|X|$ and $|Y|$ are sober (Proposition 10.6). Hence condition (1) makes sense. Moreover, because we have assumed that $|X|$ and $|Y|$ have finitely many irreducible components, we see that the generic points $x_1, \ldots, x_n \in |X|$, resp. $y_1, \ldots, y_n \in |Y|$ are contained in any dense open of $|X|$, resp. $|Y|$. In particular, they are contained in the schematic locus of X, resp. Y by Theorem 9.2. Thus we can define $\mathcal{O}_{X,x_i}$, resp. $\mathcal{O}_{Y,y_i}$ to be the local ring of this scheme at x_i, resp. y_i.

Another and perhaps better way to say all of this is that the morphism $f : X \to Y$ is birational if there exist dense open subspaces $X' \subset X$ and $Y' \subset Y$ such that

1. $f(X') \subset Y'$,
2. X' and Y' are representable, and
3. $f|_{X'} : X' \to Y'$ is birational in the sense of Morphisms, Definition 9.1.

However, we do insist that X and Y are decent with finitely many irreducible components.

Lemma 18.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which are decent and have finitely many irreducible components. If f is birational then f is dominant.*

Proof. Follows immediately from the definitions. See Morphisms of Spaces, Definition 18.1. □

19. Other chapters

Preliminaries

1. Introduction
2. Conventions
3. Set Theory
4. Categories
5. Topology
6. Sheaves on Spaces
7. Sites and Sheaves
8. Stacks
9. Fields
10. Commutative Algebra

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