DUALIZING COMPLEXES

Contents

1. Introduction

A reference is the book [Har66].

The goals of this chapter are the following:

(1) Define what it means to have a dualizing complex $\omega_A^\bullet$ over a Noetherian ring A, namely
   (a) we have $\omega_A^\bullet \in D^+(A)$,
   (b) the cohomology modules $H^i(\omega_A^\bullet)$ are all finite A-modules,
   (c) $\omega_A^\bullet$ has finite injective dimension, and
   (d) we have $A \to R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ is a quasi-isomorphism.
(2) List elementary properties of dualizing complexes.
(3) Show a dualizing complex gives rise to a dimension function.
(4) Show a dualizing complex gives rise to a good notion of a reflexive hull.
(5) Prove the finiteness theorem when a dualizing complex exists.

This is a chapter of the Stacks Project, version 714e994, compiled on Oct 28, 2014.

2. Essential surjections and injections

We will mostly work in categories of modules, but we may as well make the definition in general.

Definition 2.1. Let $\mathcal{A}$ be an abelian category.

(1) An injection $A \subset B$ of $\mathcal{A}$ is *essential*, or we say that B is an *essential extension of* A, if every nonzero subobject $B' \subset B$ has nonzero intersection with A.
(2) A surjection $f : A \to B$ of $\mathcal{A}$ is *essential* if for every proper subobject $A' \subset A$ we have $f(A') \neq B$.

Some lemmas about this notion.

Lemma 2.2. *Let $\mathcal{A}$ be an abelian category.*

(1) *If $A \subset B$ and $B \subset C$ are essential extensions, then $A \subset C$ is an essential extension.*
(2) *If $A \subset B$ is an essential extension and $C \subset B$ is a subobject, then $A \cap C \subset C$ is an essential extension.*
(3) *If $A \to B$ and $B \to C$ are essential surjections, then $A \to C$ is an essential surjection.*
(4) *Given an essential surjection $f : A \to B$ and a surjection $A \to C$ with kernel K, the morphism $C \to B/f(K)$ is an essential surjection.*

Proof. Omitted. $\square$

Lemma 2.3. *Let R be a ring. Let M be an R-module. Let $E = \operatorname{colim} E_i$ be a filtered colimit of R-modules. Suppose given a compatible system of essential injections $M \to E_i$ of R-modules. Then $M \to E$ is an essential extension of M.*

Proof. Immediate from the definitions and the fact that filtered colimits are exact (Algebra, Lemma 8.9). $\square$

Lemma 2.4. *Let R be a ring. Let $M \subset N$ be R-modules. The following are equivalent*

(1) *$M \subset N$ is an essential extension,*
(2) *for all $x \in N$ there exists an $f \in R$ such that $fx \in M$ and $fx \neq 0$.*

Proof. Assume (1) and let $x \in N$ be a nonzero element. By (1) we have $Rx \cap M \neq 0$. This implies (2).

Assume (2). Let $N' \subset N$ be a nonzero submodule. Pick $x \in N'$ nonzero. By (2) we can find $f \in$ with $fx \in N$ and $fx \neq 0$. Thus $N' \cap M \neq 0$. $\square$

3. Injective modules

Some results about injective modules over rings.

Lemma 3.1. *Let R be a ring. Any product of injective R-modules is injective.*

Proof. Special case of Homology, Lemma 23.3. $\square$

Lemma 3.2. *Let $R \to S$ be a flat ring map. If E is an injective S-module, then E is injective as an R-module.*

Proof. This is true because $\mathrm{Hom}_R(M, E) = \mathrm{Hom}_S(M \otimes_R S, E)$ by Algebra, Lemma 13.3 and the fact that tensoring with S is exact. $\square$

Lemma 3.3. *Let $R \to S$ be an epimorphism of rings. Let E be an S-module. If E is injective as an R-module, then E is an injective S-module.*

Proof. This is true because $\mathrm{Hom}_R(N, E) = \mathrm{Hom}_S(N, E)$ for any S-module N, see Algebra, Lemma 103.14. $\square$

Lemma 3.4. *Let $R \to S$ be a ring map. If E is an injective R-module, then $\mathrm{Hom}_R(S, E)$ is an injective S-module.*

Proof. This is true because $\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, E)) = \mathrm{Hom}_R(N, E)$ by Algebra, Lemma 13.4. $\square$

Lemma 3.5. *Let R be a ring. Let I be an injective R-module. Let $E \subset I$ be a submodule. The following are equivalent*

1. *E is injective, and*
2. *for all $E \subset E' \subset I$ with $E \subset E'$ essential we have $E = E'$.*

In particular, an R-module is injective if and only if every essential extension is trivial.

Proof. The final assertion follows from the first and the fact that the category of R-modules has enough injectives (More on Algebra, Section 42).

Assume (1). Let $E \subset E' \subset I$ as in (2). Then the map $\mathrm{id}_E : E \to E$ can be extended to a map $\alpha : E' \to E$. The kernel of α has to be zero because it intersects E trivially and E' is an essential extension. Hence $E = E'$.

Assume (2). Let $M \subset N$ be R-modules and let $\varphi : M \to E$ be an R-module map. In order to prove (1) we have to show that φ extends to a morphism $N \to E$. Consider the set $\mathcal{S}$ of pairs (M', φ') where $M \subset M' \subset N$ and $\varphi' : M' \to E$ is an R-module map agreeing with φ on M. We define an ordering on $\mathcal{S}$ by the rule $(M', \varphi') \leq (M'', \varphi'')$ if and only if $M' \subset M''$ and $\varphi''|_{M'} = \varphi'$. It is clear that we can take the maximum of a totally ordered subset of $\mathcal{S}$. Hence by Zorn's lemma we may assume (M, φ) is a maximal element.

Choose an extension $\psi : N \to I$ of φ composed with the inclusion $E \to I$. This is possible as I is injective. If $\psi(N) \subset E$, then ψ is the desired extension. If $\psi(N)$ is not contained in E, then by (2) the inclusion $E \subset E + \psi(N)$ is not essential. hence we can find a nonzero submodule $K \subset E + \psi(N)$ meeting E in 0. This means that $M' = \psi^{-1}(E + K)$ strictly contains M. Thus we can extend φ to M' using

$$M' \xrightarrow{\psi|_{M'}} E + K \to (E + K)/K = E$$

This contradicts the maximality of (M, φ). $\square$

Example 3.6. Let R be a reduced ring. Let $\mathfrak{p} \subset R$ be a minimal prime so that $K = R_\mathfrak{p}$ is a field (Algebra, Lemma 24.1). Then K is an injective R-module. Namely, we have $\mathrm{Hom}_R(M, K) = \mathrm{Hom}_K(M_\mathfrak{p}, K)$ for any R-module M. Since localization is an exact functor and taking duals is an exact functor on K-vector spaces we conclude $\mathrm{Hom}_R(-, K)$ is an exact functor, i.e., K is an injective R-module.

Lemma 3.7. *Let R be a ring. Let E be an R-module. The following are equivalent*

(1) *E is an injective R-module, and*
(2) *given an ideal $I \subset R$ and a module map $\varphi : I \to E$ there exists an extension of φ to and R-module map $R \to E$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from the definitions. Thus we assume (2) holds and we prove (1). First proof: Since R is a generator for the category of R-modules, this follows from Injectives, Lemma 11.5.

Second proof: We have to show that every essential extension $E \subset E'$ is trivial. Pick $x \in E'$ and set $I = \{f \in R \mid fx \in E\}$. The map $I \to E$, $f \mapsto fx$ extends to $\psi : R \to E$ by (2). Then $x' = x - \psi(1)$ is an element of E' whose annihilator in E'/E is I and which is annihilated by I as an element of E'. Thus $Rx' = (R/I)x'$ does not intersect E. Since $E \subset E'$ is an essential extension it follows that $x' \in E$ as desired. $\square$

Lemma 3.8. *Let R be a Noetherian ring. A direct sum of injective modules is injective.*

Proof. Let E_i be a family of injective modules parametrized by a set I. Set $E = \bigcup E_i$. To show that E is injective we use Lemma 3.7. Thus let $\varphi : I \to E$ be a module map from an ideal of R into E. As I is a finite R-module (because R is Noetherian) we can find finitely many elements $i_1, \ldots, i_r \in I$ such that φ maps into $\bigcup_{j=1,\ldots,r} E_{i_j}$. Then we can extend φ into $\bigcup_{j=1,\ldots,r} E_{i_j}$ using the injectivity of the modules E_{i_j}. $\square$

Lemma 3.9. *Let R be a Noetherian ring. Let $S \subset R$ be a multiplicative subset. If E is an injective R-module, then $S^{-1}E$ is an injective $S^{-1}R$-module.*

Proof. Since $R \to S^{-1}R$ is an epimorphism of rings, it suffices to show that $S^{-1}E$ is injective as an R-module, see Lemma 3.3. To show this we use Lemma 3.7. Thus let $I \subset R$ be an ideal and let $\varphi : I \to S^{-1}E$ be an R-module map. As I is a finitely presented R-module (because R is Noetherian) we can find find an $f \in S$ and an R-module map $I \to E$ such that $f\varphi$ is the composition $I \to E \to S^{-1}E$ (Algebra, Lemma 10.2). Then we can extend $I \to E$ to a homomorphism $R \to E$. Then the composition

$$R \to E \to S^{-1}E \xrightarrow{f^{-1}} S^{-1}E$$

is the desired extension of φ to R. $\square$

Lemma 3.10. *Let R be a Noetherian ring. Let I be an injective R-module.*

(1) *Let $f \in R$. Then $E = \bigcup I[f^n] = I[f^\infty]$ is an injective submodule of I.*
(2) *Let $J \subset R$ be an ideal. Then the J-power torsion submodule $I[J^\infty]$ is an injective submodule of I.*

Proof. We will use Lemma 3.5 to prove (1). Suppose that $E \subset E' \subset I$ and that E' is an essential extension of E. We will show that $E' = E$. If not, then we can find $x \in E'$ and $x \notin E$. Let $J = \{a \in R \mid ax \in E'\}$. Since R is Noetherian we can choose x with J maximal. Since R is Noetherian we can write $J = (g_1, \ldots, g_t)$ for some $g_i \in R$. Say f^{n_i} annihilates $g_i x$. Set $n = \max\{n_i\}$. Then $x' = f^n x$ is an element of E' not in E and is annihilated by J. By maximality of J we see that $Rx' = (R/J)x' \cap E = (0)$. Hence E' is not an essential extension of E a contradiction.

To prove (2) write $J = (f_1, \ldots, f_t)$. Then $I[J^\infty]$ is equal to

$$(\ldots((I[f_1^\infty])[f_2^\infty])\ldots)[f_t^\infty]$$

and the result follows from (1) and induction. $\square$

Lemma 3.11. *Let A be a Noetherian ring. Let E be an injective A-module. Then $E \otimes_A A[x]$ has injective-amplitude $[0,1]$ as an object of $D(A[x])$. In particular, $E \otimes_A A[x]$ has finite injective dimension as an $A[x]$-module.*

Proof. Let us write $E[x] = E \otimes_A A[x]$. Consider the short exact sequence of $A[x]$-modules

$$0 \to E[x] \to \mathrm{Hom}_A(A[x], E[x]) \to \mathrm{Hom}_A(A[x], E[x]) \to 0$$

where the first map sends $p \in E[x]$ to $f \mapsto fp$ and the second map sends φ to $f \mapsto \varphi(xf) - x\varphi(f)$. The second map is surjective because $\mathrm{Hom}_A(A[x], E[x]) = \prod_{n \geq 0} E[x]$ as an abelian group and the map sends (e_n) to $(e_{n+1} - xe_n)$ which is surjective. As an A-module we have $E[x] \cong \bigoplus_{n \geq 0} E$ which is injective by Lemma 3.8. Hence the $A[x]$-module $\mathrm{Hom}_A(A[x], I[x])$ is injective by Lemma 3.4 and the proof is complete. $\square$

4. Projective covers

In this section we briefly discuss projective covers.

Definition 4.1. Let R be a ring. A surjection $P \to M$ of R-modules is said to be a *projective cover*, or sometimes a *projective envelope*, if P is a projective R-module and $P \to M$ is an essential surjection.

Projective covers do not always exist. For example, if k is a field and $R = k[x]$ is the polynomial ring over k, then the module $M = R/(x)$ does not have a projective cover. Namely, for any surjection $f : P \to M$ with P projective over R, the proper submodule $(x-1)P$ surjects onto M. Hence f is not essential.

Lemma 4.2. *Let R be a ring and let M be an R-module. If a projective cover of M exists, then it is unique up to isomorphism.*

Proof. Let $P \to M$ and $P' \to M$ be projective covers. Because P is a projective R-module and $P' \to M$ is surjective, we can find an R-module map $\alpha : P \to P'$ compatible with the maps to M. Since $P' \to M$ is essential, we see that α is surjective. As P' is a projective R-module we can choose a direct sum decomposition $P = \mathrm{Ker}(\alpha) \oplus P'$. Since $P' \to M$ is surjective and since $P \to M$ is essential we conclude that $\mathrm{Ker}(\alpha)$ is zero as desired. $\square$

Here is an example where projective covers exist.

Lemma 4.3. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Any finite R-module has a projective cover.*

Proof. Let M be a finite R-module. Let $r = \dim_\kappa(M/\mathfrak{m}M)$. Choose $x_1, \ldots, x_r \in M$ mapping to a basis of $M/\mathfrak{m}M$. Consider the map $f : R^{\oplus r} \to M$. By Nakayama's lemma this is a surjection (Algebra, Lemma 19.1). If $N \subset R^{\oplus R}$ is a proper submodule, then $N/\mathfrak{m}N \to \kappa^{\oplus r}$ is not surjective (by Nakayama's lemma again) hence $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is not surjective. Thus f is an essential surjection. $\square$

5. Injective hulls

In this section we briefly discuss injective hulls.

Definition 5.1. Let R be a ring. A injection $M \to I$ of R-modules is said to be an *injective hull* if I is a injective R-module and $M \to I$ is an essential injection.

Injective hulls always exist.

Lemma 5.2. *Let R be a ring. Any R-module has an injective hull.*

Proof. Let M be an R-module. By More on Algebra, Section 42 the category of R-modules has enough injectives. Choose an injection $M \to I$ with I an injective R-module. Consider the set $\mathcal{S}$ of submodules $M \subset E \subset I$ such that E is an essential extension of M. We order $\mathcal{S}$ by inclusion. If $\{E_\alpha\}$ is a totally ordered subset of $\mathcal{S}$, then $\bigcup E_\alpha$ is an essential extension of M too (Lemma 2.3). Thus we can apply Zorn's lemma and find a maximal element $E \in \mathcal{S}$. We claim $M \subset E$ is an injective hull, i.e., E is an injective R-module. This follows from Lemma 3.5. $\square$

Lemma 5.3. *Let R be a ring. Let M, N be R-modules and let $M \to E$ and $N \to E'$ be injective hulls. Then*

(1) *for any R-module map $\varphi : M \to N$ there exists an R-module map $\psi : E \to E'$ such that*
$$\begin{tikzcd} M \arrow[r] \arrow[d, "\varphi"'] & E \arrow[d, "\psi"] \\ N \arrow[r] & E' \end{tikzcd}$$
commutes,
(2) *if φ is injective, then ψ is injective,*
(3) *if φ is an essential injection, then ψ is an isomorphism,*
(4) *if φ is an isomorphism, then ψ is an isomorphism,*
(5) *if $M \to I$ is an embedding of M into an injective R-module, then there is an isomorphism $I \cong E \oplus I'$ compatible with the embeddings of M,*

In particular, the injective hull E of M is unique up to isomorphism.

Proof. Part (1) follows from the fact that E' is an injective R-module. Part (2) follows as $\mathrm{Ker}(\psi) \cap M = 0$ and E is an essential extension of M. Assume φ is an essential injection. Then $E \cong \psi(E) \subset E'$ by (2) which implies $E' = \psi(E) \oplus E''$ because E is injective. Since E' is an essential extension of M (Lemma 2.2) we get $E'' = 0$. Part (4) is a special case of (3). Assume $M \to I$ as in (5). Choose a map $\alpha : E \to I$ extending the map $M \to I$. Arguing as before we see that α is injective. Thus as before $\alpha(E)$ splits off from I. This proves (5). $\square$

Example 5.4. Let R be a domain with fraction field K. Then $R \subset K$ is an injective hull of R. Namely, by Example 3.6 we see that K is an injective R-module and by Lemma 2.4 we see that $R \subset K$ is an essential extension.

Definition 5.5. An object X of an additive category is called *indecomposable* if it is nonzero and if $X = Y \oplus Z$, then either $Y = 0$ or $Z = 0$.

Lemma 5.6. *Let R be a ring. Let E be an indecomposable injective R-module. Then*

(1) *E is the injective hull of any nonzero submodule of E,*

(2) *the intersection of any two nonzero submodules of E is nonzero,*
(3) $End_R(E, E)$ *is a noncommutative local ring with maximal ideal those* $\varphi : E \to E$ *whose kernel is nonzero, and*
(4) *the set of zerodivisors on E is a prime ideal $\mathfrak{p}$ of R and E is an injective $R_{\mathfrak{p}}$-module.*

Proof. Part (1) follows from Lemma 5.3. Part (2) follows from part (1) and the definition of injective hulls.

Proof of (3). Set $A = \text{End}_R(E, E)$ and $I = \{\varphi \in A \mid \text{Ker}(f) \neq 0\}$. The statement means that I is a two sided ideal and that any $\varphi \in A$, $\varphi \notin I$ is invertible. Suppose φ and ψ are not injective. Then $\text{Ker}(\varphi) \cap \text{Ker}(\psi)$ is nonzero by (2). Hence $\varphi + \psi \in I$. It follows that I is a two sided ideal. If $\varphi \in A$, $\varphi \notin I$, then $E \cong \varphi(E) \subset E$ is an injective submodule, hence $E = \varphi(E)$ because E is indecomposable.

Proof of (4). Consider the ring map $R \to A$ and let $\mathfrak{p} \subset R$ be the inverse image of the maximal ideal I. Then it is clear that $\mathfrak{p}$ is a prime ideal and that $R \to A$ extends to $R_{\mathfrak{p}} \to A$. Thus E is an $R_{\mathfrak{p}}$-module. It follows from Lemma 3.3 that E is injective as an $R_{\mathfrak{p}}$-module. □

Lemma 5.7. *Let $\mathfrak{p} \subset R$ be a prime of a ring R. Let E be the injective hull of $R/\mathfrak{p}$. Then*
(1) *E is indecomposable,*
(2) *E is the injective hull of $\kappa(\mathfrak{p})$,*
(3) *E is the injective hull of $\kappa(\mathfrak{p})$ over the ring $R_{\mathfrak{p}}$.*

Proof. As $R/\mathfrak{p} \subset \kappa(\mathfrak{p})$ we can extend the embedding to a map $\kappa(\mathfrak{p}) \to E$. Hence (2) holds. For $f \in R$, $f \notin \mathfrak{p}$ the map $f : \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p})$ is an isomorphism hence the map $f : E \to E$ is an isomorphism, see Lemma 5.3. Thus E is an $R_{\mathfrak{p}}$-module. It is injective as an $R_{\mathfrak{p}}$-module by Lemma 3.3. Finally, let $E' \subset E$ be a nonzero injective R-submodule. Then $J = (R/\mathfrak{p}) \cap E'$ is nonzero. After shrinking E' we may assume that E' is the injective hull of J (see Lemma 5.3 for example). Observe that $R/\mathfrak{p}$ is an essential extension of J for example by Lemma 2.4. Hence $E' \to E$ is an isomorphism by Lemma 5.3 part (3). Hence E is indecomposable. □

Lemma 5.8. *Let R be a Noetherian ring. Let E be an indecomposable injective R-module. Then there exists a prime ideal $\mathfrak{p}$ of R such that E is the injective hull of $\kappa(\mathfrak{p})$.*

Proof. Let $\mathfrak{p}$ be the prime ideal found in Lemma 5.6. Say $\mathfrak{p} = (f_1, \ldots, f_r)$. Pick a nonzero element $x \in \bigcap \text{Ker}(f_i : E \to E)$, see Lemma 5.6. Then $(R_{\mathfrak{p}})x$ is a module isomorphic to $\kappa(\mathfrak{p})$ inside E. We conclude by Lemma 5.6. □

Proposition 5.9 (Structure injective modules over Noetherian rings)**.** *Let R be a Noetherian ring. Every injective module is a direct sum of indecomposable injective modules. Every indecomposable injective module is the injective hull of the residue field at a prime.*

Proof. The second statement is Lemma 5.8. For the second statement, let I be an injective R-module. We will use transfinite induction to construct $I_\alpha \subset I$ for ordinals α which are direct sums of indecomposable injective R-modules $E_{\beta+1}$ for $\beta < \alpha$. For $\alpha = 0$ we let $I_0 = 0$. Suppose given an ordinal α such that I_α has been constructed. Then I_α is an injective R-module by Lemma 3.8. Hence $I \cong I_\alpha \oplus I'$.

If $I' = 0$ we are done. If not, then I' has an associated prime by Algebra, Lemma 62.7. Thus I' contains a copy of $R/\mathfrak{p}$ for some prime $\mathfrak{p}$. Hence I' contains an indecomposable submodule E by Lemmas 5.3 and 5.7. Set $I_{\alpha+1} = I_\alpha \oplus E_\alpha$. If α is a limit ordinal and I_β has been constructed for $\beta < \alpha$, then we set $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$. Observe that $I_\alpha = \bigoplus_{\beta<\alpha} E_{\beta+1}$. This concludes the proof. $\square$

6. Duality over Artinian local rings

Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Recall that this implies R is Noetherian and that R has finite length as an R-module. Moreover an R-module is finite if and only if it has finite length. We will use these facts without further mention in this section. Please see Algebra, Sections 50 and 51 and Algebra, Proposition 59.6 for more details.

Lemma 6.1. *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. For every finite R-module M we have*

$$\mathit{length}_R(M) = \mathit{length}_R(\mathrm{Hom}_R(M, E))$$

In particular, the injective hull E of κ is a finite R-module.

Proof. Because E is an essential extension of κ we have $\kappa = E[\mathfrak{m}]$ where $E[\mathfrak{m}]$ is the $\mathfrak{m}$-torsion in E (notation as in More on Algebra, Section 63). Hence $\mathrm{Hom}_R(\kappa, E) \cong \kappa$ and the equality of lengths holds for $M = \kappa$. We prove the displayed equality of the lemma by induction on the length of M. If M is nonzero there exists a surjection $M \to \kappa$ with kernel M'. Since the functor $M \mapsto \mathrm{Hom}_R(M, E)$ is exact we obtain a short exact sequence

$$0 \to \mathrm{Hom}_R(\kappa, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(M', E) \to 0.$$

Additivity of length for this sequence and the sequence $0 \to M' \to M \to \kappa \to 0$ and the equality for M' (induction hypothesis) and κ implies the equality for M. The final statement of the lemma follows as $E = \mathrm{Hom}_R(R, E)$. $\square$

Lemma 6.2. *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. For any finite R-module M the evaluation map*

$$M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$$

is an isomorphism. In particular $R = \mathrm{Hom}_R(E, E)$.

Proof. Observe that the displayed arrow is injective. Namely, if $x \in M$ is a nonzero element, then there is a nonzero map $Rx \to \kappa$ which we can extend to a map $\varphi : M \to E$ that doesn't vanish on x. Since the source and target of the arrow have the same length by Lemma 6.1 we conclude it is an isomorphism. The final statement follows on taking $M = R$. $\square$

To state the next lemma, denote Mod_R^{fg} the category of finite R-modules over a ring R.

Lemma 6.3. *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. The functor $D(-) = \mathrm{Hom}_R(-, E)$ induces an exact anti-equivalence $Mod_R^{fg} \to Mod_R^{fg}$ and $D \circ D \cong id$.*

Proof. We have seen that $D \circ D = \mathrm{id}$ on Mod_R^{fg} in Lemma 6.2. It follows immediately that D is an anti-equivalence. $\square$

Lemma 6.4. *Assumptions and notation as in Lemma 6.3. Let $I \subset R$ be an ideal and M a finite R-module. Then*

$$D(M[I]) = D(M)/ID(M) \quad \textit{and} \quad D(M/IM) = D(M)[I]$$

Proof. Say $I = (f_1, \ldots, f_t)$. Consider the map

$$M^{\oplus t} \xrightarrow{f_1, \ldots, f_t} M$$

with cokernel M/IM. Applying the exact functor D we conclude that $D(M/IM)$ is $D(M)[I]$. The other case is proved in the same way. $\square$

7. Injective hull of the residue field

Most of our results will be for Noetherian local rings in this section.

Lemma 7.1. *Let $R \to S$ be a surjective map of local rings with kernel I. Let E be the injective hull of the residue field of R over R. Then $E[I]$ is the injective hull of the residue field of S over S.*

Proof. Observe that $E[I] = \mathrm{Hom}_R(S, E)$ as $S = R/I$. Hence $E[I]$ is an injective S-module by Lemma 3.4. Since E is an essential extension of $\kappa = R/\mathfrak{m}_R$ it follows that $E[I]$ is an essential extension of κ as well. The result follows. $\square$

Lemma 7.2. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let E be the injective hull of κ. Let M be a $\mathfrak{m}$-power torsion R-module with $n = \dim_\kappa(M[\mathfrak{m}]) < \infty$. Then M is isomorphic to a submodule of $E^{\oplus n}$.*

Proof. Observe that $E^{\oplus n}$ is the injective hull of $\kappa^{\oplus n} = M[\mathfrak{m}]$. Thus there is an R-module map $M \to E^{\oplus n}$ which is injective on $M[\mathfrak{m}]$. Since M is $\mathfrak{m}$-power torsion the inclusion $M[\mathfrak{m}] \subset M$ is an essential extension (for example by Lemma 2.4) we conclude that the kernel of $M \to E^{\oplus n}$ is zero. $\square$

Lemma 7.3. *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Let E_n be an injective hull of κ over $R/\mathfrak{m}^n$. Then $E = \bigcup E_n$ and $E_n = E[\mathfrak{m}^n]$.*

Proof. We have $E_n = E[\mathfrak{m}^n]$ by Lemma 7.1. We have $E = \bigcup E_n$ because $\bigcup E_n = E[\mathfrak{m}^\infty]$ is an injective R-submodule which contains κ, see Lemma 3.10. $\square$

The following lemma tells us the injective hull of the residue field of a Noetherian local ring only depends on the completion.

Lemma 7.4. *Let $R \to S$ be a flat local homomorphism of local Noetherian rings such that $R/\mathfrak{m}_R \cong S/\mathfrak{m}_R S$. Then the injective hull of the residue field of R is the injective hull of the residue field of S.*

Proof. Set $\kappa = R/\mathfrak{m}_R = S/\mathfrak{m}_S$. Let E_R be the injective hull of κ over R. Let E_S be the injective hull of κ over S. Observe that E_S is an injective R-module by Lemma 3.2. Choose an extension $E_R \to E_S$ of the identification of residue fields. This map is an isomorphism by Lemma 7.3 because $R \to S$ induces an isomorphism $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ for all n. $\square$

Lemma 7.5. *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Then $\mathrm{Hom}_R(E, E)$ is canonically isomorphic to the completion of R.*

Proof. Write $E = \bigcup E_n$ with $E_n = E[\mathfrak{m}^n]$ as in Lemma 7.3. Any endomorphism of E preserves this filtration. Hence

$$\mathrm{Hom}_R(E, E) = \lim \mathrm{Hom}_R(E_n, E_n)$$

The lemma follows as $\mathrm{Hom}_R(E_n, E_n) = \mathrm{Hom}_{R/\mathfrak{m}^n}(E_n, E_n) = R/\mathfrak{m}^n$ by Lemma 6.2. $\square$

Lemma 7.6. *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Then E satisfies the descending chain condition.*

Proof. If $E \subset M_1 \subset M_2 \ldots$ is a sequence of submodules, then

$$\mathrm{Hom}_R(E, E) \to \mathrm{Hom}_R(M_1, E) \to \mathrm{Hom}_R(M_2, E) \to \ldots$$

is sequence of surjections. By Lemma 7.5 each of these is a module over the completion $R^\wedge = \mathrm{Hom}_R(E, E)$. Since $R^\wedge$ is Noetherian (Algebra, Lemma 93.10) the sequence stabilizes: $\mathrm{Hom}_R(M_n, E) = \mathrm{Hom}_R(M_{n+1}, E) = \ldots$. Since E is injective, this can only happen if $\mathrm{Hom}_R(M_n/M_{n+1}, E)$ is zero. However, if M_n/M_{n+1} is nonzero, then it contains a nonzero element annihilated by $\mathfrak{m}$, because E is $\mathfrak{m}$-power torsion by Lemma 7.3. In this case M_n/M_{n+1} has a nonzero map into E, contradicting the assumed vanishing. This finishes the proof. $\square$

Lemma 7.7. *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ.*

1. *For an R-module M the following are equivalent:*
   - (a) *M satisfies the ascending chain condition,*
   - (b) *M is a finite R-module, and*
   - (c) *there exist n, m and an exact sequence $R^{\oplus m} \to R^{\oplus n} \to M \to 0$.*
2. *For an R-module M the following are equivalent:*
   - (a) *M satisfies the descending chain condition,*
   - (b) *M is $\mathfrak{m}$-power torsion and $\dim_\kappa(M[\mathfrak{m}]) < \infty$, and*
   - (c) *there exist n, m and an exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.*

Proof. We omit the proof of (1).

Let M be an R-module with the descending chain condition. Let $x \in M$. Then $\mathfrak{m}^n x$ is a descending chain of submodules, hence stabilizes. Thus $\mathfrak{m}^n x = \mathfrak{m}^{n+1} x$ for some n. By Nakayama's lemma (Algebra, Lemma 19.1) this implies $\mathfrak{m}^n x = 0$, i.e., x is $\mathfrak{m}$-power torsion. Since $M[\mathfrak{m}]$ is a vector space over κ it has to be finite dimensional in order to have the descending chain condition.

Assume that M is $\mathfrak{m}$-power torsion and has a finite dimensional $\mathfrak{m}$-torsion submodule $M[\mathfrak{m}]$. By Lemma 7.2 we see that M is a submodule of $E^{\oplus n}$ for some n. Consider the quotient $N = E^{\oplus n}/M$. By Lemma 7.6 the module E has the descending chain condition hence so do $E^{\oplus n}$ and N. Therefore N satisfies (2)(a) which implies N satisfies (2)(b) by the second paragraph of the proof. Thus by Lemma 7.2 again we see that N is a submodule of $E^{\oplus m}$ for some m. Thus we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.

Assume we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$. Since E satisfies the descending chain condition by Lemma 7.6 so does M. $\square$

Proposition 7.8 (Matlis duality). *Let $(R, \mathfrak{m}, \kappa)$ be a complete local Noetherian ring. Let E be an injective hull of κ over R. The functor $D(-) = \mathrm{Hom}_R(-, E)$ induces an anti-equivalence*

$$\left\{\begin{matrix} R\text{-modules with the} \\ \text{descending chain condition} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} R\text{-modules with the} \\ \text{ascending chain condition} \end{matrix}\right\}$$

and we have $D \circ D = id$ on either side of the equivalence.

Proof. By Lemma 7.5 we have $R = \mathrm{Hom}_R(E, E) = D(E)$. Of course we have $E = \mathrm{Hom}_R(R, E) = D(R)$. Since E is injective the functor D is exact. The result now follows immediately from the description of the categories in Lemma 7.7. $\square$

8. Local cohomology

Let A be a ring and let $I \subset A$ be a finitely generated ideal (if I is not finitely generated perhaps a different definition should be used). Let $Z = V(I) \subset \mathrm{Spec}(A)$. Recall that the category I^∞-torsion of I-power torsion modules only depends on the closed subset Z and not on the choice of the finitely generated ideal I such that $Z = V(I)$, see More on Algebra, Lemma 62.6. In this section we will consider the functor

$$H^0_I : \mathrm{Mod}_A \longrightarrow I^\infty\text{-torsion}, \quad M \longmapsto M[I^\infty] = \bigcup M[I^n]$$

which sends M to the submodule of I-power torsion as well as its relationship with the functors

$$\mathcal{H}_Z : Ab(X) \longrightarrow Ab(Z)$$

and $\Gamma_Z(-) = \Gamma(Z, \mathcal{H}_Z(-))$ of Cohomology, Section 22.

Let A be a ring and let I be a finitely generated ideal. Note that I^∞-torsion is a Grothendieck abelian category (direct sums exist, filtered colimits are exact, and $\bigoplus A/I^n$ is a generator by More on Algebra, Lemma 62.2). Hence the derived category $D(I^\infty\text{-torsion})$ exists, see Injectives, Remark 13.3. Our functor H^0_I is left exact and has a derived extension which we will denote

$$R\Gamma_I : D(A) \longrightarrow D(I^\infty\text{-torsion}).$$

Warning: this functor does not deserve the name local cohomology unless the ring A is Noetherian. The functors H^0_I, $R\Gamma_I$, and the satellites H^p_I only depend on the closed subset $Z \subset \mathrm{Spec}(A)$ and not on the choice of the finitely generated ideal I such that $V(I) = Z$. However, we insist on using the subscript I for the functors above as the notation $R\Gamma_Z$ is going to be used for a different functor, see (8.4.1), which agrees with the functor $R\Gamma_I$ only (as far as we know) in case A is Noetherian (see Lemma 8.9).

Lemma 8.1. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functor $R\Gamma_I$ is right adjoint to the functor $D(I^\infty\text{-torsion}) \to D(A)$.*

Proof. This follows from the fact that taking I-power torsion submodules is the right adjoint to the inclusion functor I^∞-torsion $\to \mathrm{Mod}_A$. See Derived Categories, Lemma 28.4. $\square$

Lemma 8.2. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. For any object K of $D(A)$ we have*

$$R\Gamma_I(K) = hocolim\ R\mathrm{Hom}(A/I^n, K)$$

in $D(A)$ and

$$R^q\Gamma_I(K) = \mathrm{colim}_n\, Ext^q_A(A/I^n, K)$$

as modules for all $q \in \mathbf{Z}$.

Proof. Let $J^\bullet$ be a K-injective complex representing K. Then

$$R\Gamma_I(K) = J^\bullet[I^\infty] = \mathrm{colim}\, J^\bullet[I^n] = \mathrm{colim}\, \mathrm{Hom}_A(A/I^n, J^\bullet)$$

By Derived Categories, Lemma 31.4 we obtain the first equality. The second equality is clear because $H^q(\mathrm{Hom}_A(A/I^n, J^\bullet)) = \mathrm{Ext}^q_A(A/I^n, K)$ and because filtered colimits are exact in the category of abelian groups. $\square$

Lemma 8.3. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then $R\Gamma_I(K^\bullet) = 0$.*

Proof. Namely, in this case the cohomology modules of $R\Gamma_I(K^\bullet)$ are both f-power torsion and f acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero. $\square$

Let A be a ring and $I \subset A$ a finitely generated ideal. By More on Algebra, Lemma 62.5 the category of I-power torsion modules is a Serre subcategory of the category of all A-modules, hence there is a functor

$$D(I^\infty\text{-torsion}) \to D_{I^\infty\text{-torsion}}(A) \tag{8.3.1}$$

see Derived Categories, Section 13.

Lemma 8.4. *Let A be a ring and let I be a finitely generated ideal. Let M and N be I-power torsion modules.*

1. $\mathrm{Hom}_{D(A)}(M, N) = \mathrm{Hom}_{D(I^\infty\textit{-torsion})}(M, N)$,
2. $Ext^1_{D(A)}(M, N) = Ext^1_{D(I^\infty\textit{-torsion})}(M, N)$,
3. $Ext^2_{D(I^\infty\textit{-torsion})}(M, N) \to Ext^2_{D(A)}(M, N)$ *is not surjective in general,*
4. *(8.3.1) is not an equivalence in general.*

Proof. Parts (1) and (2) follow immediately from the fact that I-power torsion forms a Serre subcategory of Mod_A. Part (4) follows from part (3).

For part (3) let A be a ring with an element $f \in A$ such that $A[f]$ contains a nonzero element x and A contains elements x_n with $f^n x_n = x$. Such a ring A exists because we can take

$$A = \mathbf{Z}[f, x, x_n]/(fx, f^n x_n - x)$$

Given A set $I = (f)$. Then the exact sequence

$$0 \to A[f] \to A \xrightarrow{f} A \to A/fA \to 0$$

defines an element in $\mathrm{Ext}^2_A(A/fA, A[f])$. We claim this element does not come from an element of $\mathrm{Ext}^2_{D(f^\infty\text{-torsion})}(A/fA, A[f])$. Namely, if it did, then there would be an exact sequence

$$0 \to A[f] \to M \to N \to A/fA \to 0$$

where M and N are f-power torsion modules defining the same 2 extension class. Since $A \to A$ is a complex of free modules and since the 2 extension classes are the same we would be able to find a map

$$\begin{tikzcd}
0 \arrow[r] & A[f] \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\varphi"'] & A \arrow[r] \arrow[d, "\psi"'] & A/fA \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A[f] \arrow[r] & M \arrow[r] & N \arrow[r] & A/fA \arrow[r] & 0
\end{tikzcd}$$

(some details omitted). Then we could replace M by the image of φ and N by the image of ψ. Then M would be a cyclic module, hence $f^n M = 0$ for some n. Considering $\varphi(x_{n+1})$ we get a contradiction with the fact that $f^{n+1}x_n = x$ is nonzero in $A[f]$. □

Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $Z = V(I) \subset \mathrm{Spec}(A)$. We will construct a functor

$$R\Gamma_Z : D(A) \longrightarrow D_{I^\infty\text{-torsion}}(A). \tag{8.4.1}$$

which is right adjoint to the inclusion functor. The cohomology modules of $R\Gamma_Z(K)$ are the *local cohomology groups of K with respect to Z*. In fact, we will show $R\Gamma_Z$ computes cohomology with support in Z for the assocated complex of quasi-comherent sheaves on $\mathrm{Spec}(A)$. By Lemma 8.4 this functor will in general **not** be equal to $R\Gamma_I(-)$ even viewed as functors into $D(A)$.

Lemma 8.5. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. There exists a right adjoint $R\Gamma_Z$ (8.4.1) to the inclusion functor $D_{I^\infty\text{-torsion}}(A) \to D(A)$. In fact, if I is generated by $f_1, \ldots, f_r \in A$, then we have*

$$R\Gamma_Z(K) = (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \otimes_A^{\mathbf{L}} K$$

functorially in K.

Proof. Say $I = (f_1, \ldots, f_r)$ be an ideal. Let $K^\bullet$ be a complex of A-modules. There is a canonical map of complexes

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \longrightarrow A.$$

from the extended Čech complex to A. Tensoring with $K^\bullet$, taking associated total complex, we get a map

$$\mathrm{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r})\right) \longrightarrow K^\bullet$$

in $D(A)$. We claim the cohomology modules of the complex on the left are I-power torsion, i.e., the LHS is an object of $D_{I^\infty\text{-torsion}}(A)$. Namely, we have

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) = \mathrm{colim}\, K(A, f_1^n, \ldots, f_r^n)$$

by More on Algebra, Lemma 20.13. Moreover, multiplication by f_i^n on the complex $K(A, f_1^n, \ldots, f_r^n)$ is homotopic to zero by More on Algebra, Lemma 20.6. Since

$$H^q(LHS) = \mathrm{colim}\, H^q(\mathrm{Tot}(K^\bullet \otimes_A K(A, f_1^n, \ldots, f_r^n)))$$

we obtain our claim. On the other hand, if $K^\bullet$ is an object of $D_{I^\infty\text{-torsion}}(A)$, then the complexes $K^\bullet \otimes_A A_{f_{i_0}\ldots f_{i_p}}$ have vanishing cohomology. Hence in this case the map $LHS \to K^\bullet$ is an isomorphism in $D(A)$. The construction

$$R\Gamma_Z(K^\bullet) = \text{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r})\right)$$

is functorial in $K^\bullet$ and defines an exact functor $D(A) \to D_{I^\infty\text{-torsion}}(A)$ between triangulated categories. It follows formally from the existence of the natural transformation $R\Gamma_Z \to \text{id}$ given above and the fact that this evaluates to an isomorphism on $K^\bullet$ in the subcategory, that $R\Gamma_Z$ is the desired right adjoint. □

Lemma 8.6. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then $R\Gamma_Z(K^\bullet) = 0$.*

Proof. Namely, in this case the cohomology modules of $R\Gamma_Z(K^\bullet)$ are both f-power torsion and f acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero. □

Lemma 8.7. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. For $K, L \in D(A)$ general we have*

$$R\Gamma_Z(K \otimes_A^{\mathbf{L}} L) = K \otimes_A^{\mathbf{L}} R\Gamma_Z(L) = R\Gamma_Z(K) \otimes_A^{\mathbf{L}} L = R\Gamma_Z(K) \otimes_A^{\mathbf{L}} R\Gamma_Z(L)$$

If K or L is in $D_{I^\infty\text{-torsion}}(A)$ then so is $K \otimes_A^{\mathbf{L}} L$.

Proof. By Lemma 8.5 we know that $R\Gamma_Z$ is given by $C \otimes^{\mathbf{L}} -$ for some $C \in D(A)$. Hence, for $K, L \in D(A)$ general we have

$$R\Gamma_Z(K \otimes_A^{\mathbf{L}} L) = K \otimes^{\mathbf{L}} L \otimes_A^{\mathbf{L}} C = K \otimes_A^{\mathbf{L}} R\Gamma_Z(L)$$

The other equalities follow formally from this one. This also implies the last statement of the lemma. □

The following lemma tells us that the functor $R\Gamma_Z$ is related to local cohomology.

Lemma 8.8. *Let A be a ring and let I be a finitely generated ideal. With $Z = V(I) \subset X = \text{Spec}(A)$ there is a functorial isomorphism*

$$R\Gamma_Z(K^\bullet) = R\Gamma_Z(\widetilde{K^\bullet})$$

where on the left we have (8.4.1) and on the right we have the functor of Cohomology, Section 22.

Proof. Denote $\mathcal{F}^\bullet = \widetilde{K^\bullet}$ be the complex of quasi-coherent $\mathcal{O}_X$-modules on X associated to $K^\bullet$. By Cohomology, Section 22 there exists a distinguished triangle

$$R\Gamma_Z(X, \mathcal{F}^\bullet) \to R\Gamma(X, \mathcal{F}^\bullet) \to R\Gamma(U, \mathcal{F}^\bullet) \to R\Gamma_Z(X, \mathcal{F}^\bullet)[1]$$

where $U = X \setminus Z$. We know that $R\Gamma(X, \mathcal{F}^\bullet) = K^\bullet$ for example by Derived Categories of Schemes, Lemma 3.4. Say $I = (f_1, \ldots, f_r)$. Then we obtain a finite affine open covering $\mathcal{U} : U = D(f_1) \cup \ldots \cup D(f_r)$. By Derived Categories of Schemes, Lemma 8.4 the alternating Čech complex

$$\text{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$$

computes $R\Gamma(U, \mathcal{F}^\bullet)$. Working through the definitions we find

$$R\Gamma(U, \mathcal{F}^\bullet) = \text{Tot}\left(K^\bullet \otimes_A (\prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r})\right)$$

It is clear that $R\Gamma(X, \mathcal{F}^\bullet) \to R\Gamma(U, \mathcal{F}^\bullet)$ is given by the map from A into $\prod A_{f_i}$. Hence we conclude that

$$R\Gamma_Z(X, \mathcal{F}^\bullet) = \text{Tot}\left(K^\bullet \otimes_A (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r})\right)$$

By Lemma 8.5 this complex computes $R\Gamma_Z(K^\bullet)$ and we see the lemma holds. $\square$

Let A be a ring and let $I \subset A$ be a finitely generated ideal. Set $Z = V(I) \subset \text{Spec}(A)$. There is a natural transformation of functors

$$(8.3.1) \circ R\Gamma_I(-) \longrightarrow R\Gamma_Z(-) \tag{8.8.1}$$

Namely, given a complex of A-modules $K^\bullet$ the canonical map $R\Gamma_I(K^\bullet) \to K^\bullet$ in $D(A)$ factors (uniquely) through $R\Gamma_Z(K^\bullet)$ as $R\Gamma_I(K^\bullet)$ has I-power torsion cohomology modules (see Lemma 8.1). In general this map is not an isomorphism (we've seen this above).

Lemma 8.9. *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Denote $j : D(I^\infty\text{-torsion}) \to D_{I^\infty\text{-torsion}}(A)$ the functor (8.3.1).*

1. *the adjunction $j(R\Gamma_I(K)) \to K$ is an isomorphism for $K \in D_{I^\infty\text{-torsion}}(A)$,*
2. *the functor j is an equivalence,*
3. *the transformation of functors (8.8.1) is an isomorphism,*

Proof. A formal argument, which we omit, shows that it suffices to prove (1).

Let M be an I-power torsion A-module. Choose an embedding $M \to J$ into an injective A-module. Then $J[I^\infty]$ is an injective A-module, see Lemma 3.10, and we obtain an embedding $M \to J[I^\infty]$. Thus every I-power torsion module has an injective resolution $M \to J^\bullet$ with J^n also I-power torsion. It follows that $R\Gamma_I(M) = M$ (this is not a triviality and this is not true in general if A is not Noetherian). Next, suppose that $K \in D^+_{I^\infty\text{-torsion}}(A)$. Then the spectral sequence

$$R^q\Gamma_I(H^p(K)) \Rightarrow R^{p+q}\Gamma_I(K)$$

(Derived Categories, Lemma 21.3) converges and above we have seen that only the terms with $q = 0$ are nonzero. Thus we see that $R\Gamma_I(K) \to K$ is an isomorphism.

Suppose K is an arbitrary object of $D_{I^\infty\text{-torsion}}(A)$. We have

$$R^q\Gamma_I(K) = \text{colim}\, \text{Ext}^q_A(A/I^n, K)$$

by Lemma 8.2. Choose $f_1, \ldots, f_r \in A$ generating I. Let $K_n^\bullet = K(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex with terms in degrees $-r, \ldots, 0$. Since the pro-objects $\{A/I^n\}$ and $\{K_n^\bullet\}$ in $D(A)$ are the same by More on Algebra, Lemma 64.18, we see that

$$R^q\Gamma_I(K) = \text{colim}\, \text{Ext}^q_A(K_n^\bullet, K)$$

Pick any complex $K^\bullet$ of A-modules representing K. Since $K_n^\bullet$ is a finite complex of finite free modules we see that

$$\text{Ext}^q_A(K_n, K) = H^q(\text{Tot}((K_n^\bullet)^\vee \otimes_A K^\bullet))$$

where $(K_n^\bullet)^\vee$ is the dual of the complex $K_n^\bullet$. See More on Algebra, Lemma 55.2. As $(K_n^\bullet)^\vee$ is a complex of finite free A-modules sitting in degrees $0, \ldots, r$ we see that the terms of the complex $\text{Tot}((K_n^\bullet)^\vee \otimes_A K^\bullet))$ are the same as the terms of the complex $\text{Tot}((K_n^\bullet)^\vee \otimes_A \tau_{\geq q-r-2} K^\bullet))$ in degrees $q - 1$ and higher. Hence we see that

$$\text{Ext}^q_A(K_n, K) = \text{Ext}^q_A(K_n, \tau_{\geq q-r-2} K)$$

for all n. It follows that

$$R^q\Gamma_I(K) = R^q\Gamma_I(\tau_{\geq q-r-2}K) = H^q(\tau_{\geq q-r-2}K) = H^q(K)$$

Thus we see that the map $R\Gamma_I(K) \to K$ is an isomorphism. $\square$

Lemma 8.10. *If A is a Noetherian ring and $I = (f_1, \ldots, f_r)$ an ideal. There are canonical isomorphisms*

$$R\Gamma_I(A) \to (A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \to R\Gamma_Z(A)$$

in $D(A)$.

Proof. This follows from Lemma 8.9 and the computation of the functor $R\Gamma_Z$ in Lemma 8.5. $\square$

Lemma 8.11. *Let $A \to B$ be a ring map. Let $I \subset A$ be a finitely generated ideal. Let $Z = V(I) \subset \text{Spec}(A)$ and $Y = V(IB) \subset \text{Spec}(B)$. For K in $D(A)$ we have $R\Gamma_Z(K) \otimes_A^{\mathbf{L}} B = R\Gamma_Y(K \otimes_A^{\mathbf{L}} B)$.*

Proof. This follows from uniquess of adjoint functors as both $R\Gamma_Z(-) \otimes_A^{\mathbf{L}} B$ and $R\Gamma_Y(- \otimes_A^{\mathbf{L}} B)$ are right adjoint to the functor $D_{(IB)^\infty\text{torsion}}(B) \to D(A)$. Alternatively, one can use the description of $R\Gamma_Z$ and $R\Gamma_Y$ in terms of alternating Čech complexes (Lemma 8.5) and use that formation of the extended Čech complex commutes with base change. $\square$

Lemma 8.12. *If $A \to B$ is a homomorphism of Noetherian rings and $I \subset A$ is an ideal, then in $D(B)$ we have*

$$R\Gamma_I(A) \otimes_A^{\mathbf{L}} B = R\Gamma_Z(A) \otimes_A^{\mathbf{L}} B = R\Gamma_Y(B) = R\Gamma_{IB}(B)$$

where $Y = V(IB) \subset \text{Spec}(B)$.

Proof. Combine Lemmas 8.10 and 8.11. $\square$

The following lemma is the analogue of More on Algebra, Lemma 64.26 for complexes with torsion cohomologies.

Lemma 8.13. *Let $A \to B$ be a flat ring map and let $I \subset A$ be a finitely generated ideal such that $A/I = B/IB$. Then base change and restriction induce quasi-inverse equivalences $D_{I^\infty\text{-torsion}}(A) = D_{(IB)^\infty\text{-torsion}}(B)$.*

Proof. More precisely the functors are $K \mapsto K \otimes_A^{\mathbf{L}} B$ for K in $D_{I^\infty\text{-torsion}}(A)$ and $M \mapsto M_A$ for M in $D_{(IB)^\infty\text{-torsion}}(B)$. The reason this works is that $H^i(K \otimes_A^{\mathbf{L}} B) = H^i(K) \otimes_A B = H^i(K)$. The first equality holds as $A \to B$ is flat and the second by More on Algebra, Lemma 63.2. $\square$

The following lemma was shown for Hom and Ext^1 of modules in More on Algebra, Lemmas 63.3 and 63.8.

Lemma 8.14. *Let $A \to B$ be a flat ring map and let $I \subset A$ be a finitely generated ideal such that $A/I \to B/IB$ is an isomorphism. For $K \in D_{I^\infty\text{-torsion}}(A)$ and $L \in D(A)$ the map*

$$R\operatorname{Hom}_A(K, L) \longrightarrow R\operatorname{Hom}_B(K \otimes_A B, L \otimes_A B)$$

is a quasi-isomorphism. In particular, if M, N are A-modules and M is I-power torsion, then the canonical map

$$Ext^i_A(M, N) \longrightarrow Ext^i_B(M \otimes_A B, N \otimes_A B)$$

is an isomorphism for all i.

Proof. Let $Z = V(I) \subset \mathrm{Spec}(A)$ and $Y = V(IB) \subset \mathrm{Spec}(B)$. Since the cohomology modules of K are I power torsion, the canonical map $R\Gamma_Z(L) \to L$ induces an isomorphism

$$R\,\mathrm{Hom}_A(K, R\Gamma_Z(L)) \to R\,\mathrm{Hom}_A(K, L)$$

in $D(A)$. Similarly, the cohomology modules of $K \otimes_A B$ are IB power torsion and we have an isomorphism

$$R\,\mathrm{Hom}_B(K \otimes_A B, R\Gamma_Y(L \otimes_A B)) \to R\,\mathrm{Hom}_B(K \otimes_A B, L \otimes_A B)$$

in $D(B)$. By Lemma 8.11 we have $R\Gamma_Z(L) \otimes_A B = R\Gamma_Y(L \otimes_A B)$. Hence it suffices to show that the map

$$R\,\mathrm{Hom}_A(K, R\Gamma_Z(L)) \to R\,\mathrm{Hom}_B(K \otimes_A B, R\Gamma_Z(L) \otimes_A B)$$

is a quasi-isommorphism. This follows from Lemma 8.13. $\square$

9. Torsion versus complete modules

Let A be a ring and let I be a finitely generated ideal. In this case we can consider the derived category $D_{I^\infty\text{-torsion}}(A)$ of complexes with I-power torsion cohomology modules (Section 8) and the derived category $D_{comp}(A, I)$ of derived complete complexes (More on Algebra, Section 64). In this section we show these categories are equivalent. A more general statement can be found in [DG02].

Lemma 9.1. *Let A be a ring and let I be a finitely generated ideal. Let $R\Gamma_Z$ be as in Lemma 8.5. Let $^\wedge$ denote derived completion as in More on Algebra, Lemma 64.9. For an object K in $D(A)$ we have*

$$R\Gamma_Z(K^\wedge) = R\Gamma_Z(K) \quad \text{and} \quad (R\Gamma_Z(K))^\wedge = K^\wedge$$

in $D(A)$.

Proof. Choose $f_1, \ldots, f_r \in A$ generating I. Recall that

$$K^\wedge = R\,\mathrm{Hom}\Big((A \to \prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1 \ldots f_r}), K\Big)$$

by More on Algebra, Lemma 64.9. Hence the cone $C = \mathrm{Cone}(K \to K^\wedge)$ is given by

$$R\,\mathrm{Hom}\Big((\prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1 \ldots f_r}), K\Big)$$

which can be represented by a complex endowed with a finite filtration whose succesive quotients are isomorphic to

$$R\,\mathrm{Hom}(A_{f_{i_0} \ldots f_{i_p}}, K), \quad p > 0$$

These complexes vanish on applying $R\Gamma_Z$, see Lemma 8.6. Applying $R\Gamma_Z$ to the distinguished triangle $K \to K^\wedge \to C \to K[1]$ we see that the first formula of the lemma is correct.

Recall that

$$R\Gamma_Z(K) = K \otimes^{\mathbf{L}} (A \to \prod A_{f_{i_0}} \to \prod A_{f_{i_0 i_1}} \to \ldots \to A_{f_1 \ldots f_r})$$

by Lemma 8.5. Hence the cone $C = \mathrm{Cone}(R\Gamma_Z(K) \to K)$ can be represented by a complex endowed with a finite filtration whose succesive quotients are isomorphic to

$$K \otimes_A A_{f_{i_0} \ldots f_{i_p}}, \quad p > 0$$

These complexes vanish on applying $^\wedge$, see More on Algebra, Lemma 64.10. Applying derived completion to the distinguished triangle $R\Gamma_Z(K) \to K \to C \to R\Gamma_Z(K)[1]$ we see that the second formula of the lemma is correct. $\square$

The following result is a special case of a very general phenomenon concerning admissible subcategories of a triangulated category.

Proposition 9.2. *Let A be a ring and let $I \subset A$ be a finitely generated ideal. The functors $R\Gamma_Z$ and $^\wedge$ define quasi-inverse equivalences of categories*

$$D_{I^\infty\text{-}torsion}(A) \leftrightarrow D_{comp}(A, I)$$

Proof. Follows immediately from Lemma 9.1. $\square$

The following addendum of the proposition above makes the correspondence on morphisms more precise.

Lemma 9.3. *With notation as in Lemma 9.1. For objects K, L in $D(A)$ there is a canonical isomorphism*

$$R\operatorname{Hom}(K^\wedge, L^\wedge) \longrightarrow R\operatorname{Hom}(R\Gamma_Z(K), R\Gamma_Z(L))$$

in $D(A)$.

Proof. Say $I = (f_1, \ldots, f_r)$. Denote $C = (A \to \prod A_{f_i} \to \ldots \to A_{f_1\ldots f_r})$ the alternating Čech complex. Then derived completion is given by $R\operatorname{Hom}(C, -)$ and local cohomology by $C \otimes^{\mathbf{L}} -$. Combinging the isomorphism

$$R\operatorname{Hom}(K \otimes^{\mathbf{L}} C, L \otimes^{\mathbf{L}} C) = R\operatorname{Hom}(K, R\operatorname{Hom}(C, L \otimes^{\mathbf{L}} C))$$

(More on Algebra, Lemma 55.1) and the map

$$L \to R\operatorname{Hom}(C, L \otimes^{\mathbf{L}} C)$$

(More on Algebra, Lemma 55.8) we obtain a map

$$\gamma : R\operatorname{Hom}(K, L) \to R\operatorname{Hom}(K \otimes^{\mathbf{L}} C, L \otimes^{\mathbf{L}} C)$$

On the other hand, the right hand side is derived complete as it is equal to

$$R\operatorname{Hom}(C, R\operatorname{Hom}(K, L \otimes^{\mathbf{L}} C)).$$

Thus γ factors through the derived completion of $R\operatorname{Hom}(K, L)$ by the universal property of derived completion. However, the derived completion goes inside the $R\operatorname{Hom}$ by More on Algebra, Lemma 64.11 and we obtain the desired map.

To show that the map of the lemma is an isomorphism we may assume that K and L are derived complete, i.e., $K = K^\wedge$ and $L = L^\wedge$. In this case we are looking at the map

$$\gamma : R\operatorname{Hom}(K, L) \longrightarrow R\operatorname{Hom}(R\Gamma_Z(K), R\Gamma_Z(L))$$

By Proposition 9.2 we know that the cohomology groups of the left and the right hand side coincide. In other words, we have to check that the map γ sends a morphism $\alpha : K \to L$ in $D(A)$ to the morphism $R\Gamma_Z(\alpha) : R\Gamma_Z(K) \to R\Gamma_Z(L)$. We omit the verification (hint: note that $R\Gamma_Z(\alpha)$ is just the map $\alpha \otimes \text{id}_C : K \otimes^{\mathbf{L}} C \to L \otimes^{\mathbf{L}} C$ which is almost the same as the construction of the map in More on Algebra, Lemma 55.8). $\square$

10. Trivial duality for a ring map

Let $A \to B$ be a ring homomorphism. Consider the functor

$$\mathrm{Hom}(B, -) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B, \quad M \longmapsto \mathrm{Hom}_A(B, M)$$

This functor is left exact and has a derived extension $R\,\mathrm{Hom}(B, -) : D(A) \to D(B)$. Note that for every $K \in D(A)$ there is a canonical map $i_*R\,\mathrm{Hom}(B, K) \to K$ where $i_* : D(B) \to D(A)$ is the obvious functor.

Lemma 10.1. *With notation as above. The functor $R\,\mathrm{Hom}(B, -)$ is the right adjoint to the functor $i_* : D(B) \to D(A)$.*

Proof. This is a consequence of the fact that i_* and $\mathrm{Hom}_A(B, -)$ are adjoint functors by Algebra, Lemma 13.3. See Derived Categories, Lemma 28.4. ☐

Lemma 10.2. *With notation as above. For K in $D(A)$ we have $R^q\,\mathrm{Hom}(B, K) = \mathrm{Ext}^q_A(B, K)$ as A-modules (the left hand side starts out as a B-module).*

Proof. Omitted. ☐

Let A be a Noetherian ring. We will denote

$$D_{Coh}(A) \subset D(A)$$

the full subcategory consisting of those objects K of $D(A)$ whose cohomology modules are all finite A-modules. This makes sense by Derived Categories, Section 13 because as A is Noetherian, the subcategory of finite A-modules is a Serre subcategory of Mod_A.

Lemma 10.3. *With notation as above, assume $A \to B$ is a finite ring map of Noetherian rings. Then $R\,\mathrm{Hom}(B, -)$ maps $D^+_{Coh}(A)$ into $D^+_{Coh}(B)$.*

Proof. We have to show: if $K \in D^+(A)$ has finite cohomology modules, then the complex $R\,\mathrm{Hom}(B, K)$ has finite cohomology modules too. This follows for example from Lemma 10.2 if we can show the ext modules $\mathrm{Ext}^i_A(B, K)$ are finite A-modules. Since K is bounded below there is a convergent spectral sequence

$$\mathrm{Ext}^p_A(B, H^q(K)) \Rightarrow \mathrm{Ext}^{p+q}_A(B, K)$$

This finishes the proof as the modules $\mathrm{Ext}^p_A(B, H^q(K))$ are finite by Algebra, Lemma 69.9. ☐

Remark 10.4. Let A be a ring and let $I \subset A$ be an ideal. Set $B = A/I$. In this case the functor $\mathrm{Hom}_A(B, -)$ is equal to the functor

$$\mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B, \quad M \longmapsto M[I]$$

which sends M to the submodule of I-torsion.

11. Sections with support in a closed subscheme

Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces such that i is a homomorphism onto a closed subset and such that $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. (For example a closed immersion of schemes.) Let $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})$$

a sheaf of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$. Hence by Modules, Lemma 13.4 there is a sheaf of $\mathcal{O}_Z$-modules, which we will denote $\mathcal{H}om(\mathcal{O}_Z, \mathcal{F})$, such that

$$i_* \mathcal{H}om(\mathcal{O}_Z, \mathcal{F}) = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})$$

as $\mathcal{O}_X$-modules. We spell out what this means.

Lemma 11.1. *With notation as above. The functor $\mathcal{H}om(\mathcal{O}_Z, -)$ is a right adjoint to the functor $i_* : Mod(\mathcal{O}_Z) \to Mod(\mathcal{O}_X)$. For $V \subset Z$ open we have*

$$\Gamma(V, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})) = \{s \in \Gamma(U, \mathcal{F}) \mid \mathcal{I}s = 0\}$$

where $U \subset X$ is an open whose intersection with Z is V.

Proof. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Z$-modules. Then

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{i_*\mathcal{O}_Z}(i_*\mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{F})) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{G}, \mathcal{H}om(\mathcal{O}_Z, \mathcal{F}))$$

The first equality by Modules, Lemma 19.5 and the second by the fully faithfulness of i_*, see Modules, Lemma 13.4. The description of sections is left to the reader. $\square$

The functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Z), \quad \mathcal{F} \longmapsto \mathcal{H}om(\mathcal{O}_Z, \mathcal{F})$$

is left exact and has a derived extension

$$R\mathcal{H}om(\mathcal{O}_Z, -) : D(\mathcal{O}_X) \to D(\mathcal{O}_Z).$$

Lemma 11.2. *With notation as above. The functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ is the right adjoint of the functor $i_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$.*

Proof. This is a consequence of the fact that i_* and $\mathcal{H}om(\mathcal{O}_Z, -)$ are adjoint functors by Lemma 11.1. See Derived Categories, Lemma 28.4. $\square$

Lemma 11.3. *With notation as above. For any $\mathcal{O}_X$-module $\mathcal{F}$ we have*

$$i_*R\mathcal{H}om(\mathcal{O}_Z, \mathcal{F}) = R\mathcal{H}om(i_*\mathcal{O}_Z, \mathcal{F})$$

in $D(\mathcal{O}_X)$.

Proof. Omitted. $\square$

Lemma 11.4. *In the situation above, assume $i : Z \to X$ is a pseudo-coherent morphism of schemes (for example if X is locally Noetherian). Then*

1. *$R\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(\mathcal{O}_Z)$, and*
2. *if $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$, then the diagram*

$$\begin{tikzcd} D^+(B) \arrow[r] & D^+_{QCoh}(\mathcal{O}_Z) \\ D^+(A) \arrow[r] \arrow[u, "R\,\mathrm{Hom}(B,-)"] & D^+_{QCoh}(\mathcal{O}_X) \arrow[u, "R\mathcal{H}om(\mathcal{O}_Z,-)"'] \end{tikzcd}$$

is commutative.

Proof. To explain the parenthetical remark, if X is locally Noetherian, then i is pseudo-coherent by More on Morphisms, Lemma 40.8.

Let K be an object of $D^+_{QCoh}(\mathcal{O}_X)$. To prove (1), by Morphisms, Lemma 4.1 it suffices to show that i_* applied to $H^n(R\mathcal{H}om(\mathcal{O}_Z, K))$ produces a quasi-coherent module on X. By Lemma 11.3 this means we have to show that $R\mathcal{H}om(i_*\mathcal{O}_Z, K)$

is in $D_{QCoh}(\mathcal{O}_X)$. Since i is pseudo-coherent the sheaf $\mathcal{O}_Z$ is a pseudo-coherent $\mathcal{O}_X$-module. Hence the result follows from Derived Categories of Schemes, Lemma 9.8.

Assume $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ as in (2). Let $I^\bullet$ be a bounded below complex of injective A-modules representing an object K of $D^+(A)$. Then we know that $R\,\mathrm{Hom}(B, K) = \mathrm{Hom}_A(B, I^\bullet)$ viewed as a complex of B-modules. Choose a quasi-isomorphism

$$\widetilde{I^\bullet} \longrightarrow \mathcal{I}^\bullet$$

where $\mathcal{I}^\bullet$ is a bounded below complex of injective $\mathcal{O}_X$-modules. It follows from the description of the functor $\mathcal{H}om(\mathcal{O}_Z, -)$ in Lemma 11.1 that there is a map

$$\mathrm{Hom}_A(B, I^\bullet) \longrightarrow \Gamma(Z, \mathcal{H}om(\mathcal{O}_Z, \mathcal{I}^\bullet))$$

Observe that $\mathcal{H}om(\mathcal{O}_Z, \mathcal{I}^\bullet)$ represents $R\,\mathcal{H}om(\mathcal{O}_Z, \widetilde{K})$. Applying the universal property of the $\widetilde{\ }$ functor we obtain a map

$$\widetilde{\mathrm{Hom}_A(B, I^\bullet)} \longrightarrow R\,\mathcal{H}om(\mathcal{O}_Z, \widetilde{K})$$

in $D(\mathcal{O}_Z)$. We may check that this map is an isomorphism in $D(\mathcal{O}_Z)$ after applying i_*. However, once we apply i_* we obtain the isomorphism of Derived Categories of Schemes, Lemma 9.8 via the identification of Lemma 11.3. $\square$

Lemma 11.5. *In this situation above. Assume X is a locally Noetherian scheme. Then $R\,\mathcal{H}om(\mathcal{O}_Z, -)$ maps $D^+_{Coh}(\mathcal{O}_X)$ into $D^+_{Coh}(\mathcal{O}_Z)$.*

Proof. The question is local on X, hence we may assume that X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ with A Noetherian and $A \to B$ surjective. In this case, we can apply Lemma 11.4 to translate the question into algebra. The corresponding algebra result is a consequence of Lemma 10.3. $\square$

Lemma 11.6. *Let $i : D \to X$ be the inclusion of an effective Cartier divisor. Denote $\mathcal{N} = i^*\mathcal{O}_X(D)$ the normal sheaf of i (Morphisms, Section 33). Then for a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ we have $R\,\mathcal{H}om(\mathcal{O}_D, \mathcal{E}) = i^*\mathcal{E} \otimes_{\mathcal{O}_D} \mathcal{N}[-1]$.*

Proof. Omitted. This lemma can be significantly generalized. $\square$

12. Dualizing complexes

In this section we define dualizing complexes for Noetherian rings.

Definition 12.1. Let A be a Noetherian ring. A *dualizing complex* is a complex of A-modules $\omega_A^\bullet$ such that

1. $\omega_A^\bullet$ has finite injective dimension,
2. $H^i(\omega_A^\bullet)$ is a finite A-module for all i, and
3. $A \to R\,\mathrm{Hom}(\omega_A^\bullet, \omega_A^\bullet)$ is a quasi-isomorphism.

This definition takes some time getting used to. It is perhaps a good idea to prove some of the following lemmas yourself without reading the proofs.

Lemma 12.2. *Let A be a Noetherian ring. If $\omega_A^\bullet$ is a dualizing complex, then the functor*

$$D : K \longmapsto R\,\mathrm{Hom}(K, \omega_A^\bullet)$$

is an anti-equivalence $D_{Coh}(A) \to D_{Coh}(A)$ which exchanges $D^+_{Coh}(A)$ and $D^-_{Coh}(A)$ and induces an equivalence $D^b_{Coh}(A) \to D^b_{Coh}(A)$. Moreover $D \circ D$ is isomorphic to the identity functor.

Proof. Let K be an object of $D_{Coh}(A)$. Pick an integer n and consider the distinguihsed triangle

$$\tau_{\leq n}K \to K \to \tau_{\geq n+1}K \to \tau_{\leq n}K[1]$$

see Derived Categories, Remark 12.4. Since $\omega_A^\bullet$ has finite injective dimension we see that $R\operatorname{Hom}(\tau_{\geq n+1}K, \omega_A^\bullet)$ has vanishing cohomology in degrees $\geq n-c$ for some constant c. On the other hand, we obtain a spectral sequence

$$\operatorname{Ext}^p_A(H^{-q}(\tau_{\leq n}K, \omega_A^\bullet) \Rightarrow \operatorname{Ext}^{p+q}_A(\tau_{\leq n}K, \omega_A^\bullet) = H^{p+q}(R\operatorname{Hom}(\tau_{\leq n}K, \omega_A^\bullet))$$

which shows that these cohomology modules are finite. Since for $n > p+q+c$ this is equal to $H^{p+q}(R\operatorname{Hom}(K, \omega_A^\bullet))$ we see that $R\operatorname{Hom}(K, \omega_A^\bullet)$ is indeed an object of $D_{Coh}(A)$. By More on Algebra, Lemma 55.6 and the assumptions on the dualizing complex we obtain a canonical isomorphism

$$K = R\operatorname{Hom}(\omega_A^\bullet, \omega_A^\bullet) \otimes_A^{\mathbf{L}} K \longrightarrow R\operatorname{Hom}(R\operatorname{Hom}(K, \omega_A^\bullet), \omega_A^\bullet)$$

Thus our functor has a quasi-inverse and the proof is complete. □

Lemma 12.3. *Let A be a Noetherian ring. Let $K \in D^b_{Coh}(A)$. Let $\mathfrak m$ be a maximal ideal of A. If $H^i(K)/\mathfrak m H^i(K) \neq 0$, then there exists a finite A-module E annihilated by a power of $\mathfrak m$ and a map $K \to E[-i]$ which is nonzero on $H^i(K)$.*

Proof. Let I be the injective hull of the residue field of $\mathfrak m$. If $H^i(K)/\mathfrak m H^i(K) \neq 0$, then there exists a nonzero map $H^i(K) \to I$. Since I is injective, we can lift this to a nonzero map $K \to I[-i]$. Recall that $I = \bigcup I[\mathfrak m^n]$, see Lemma 7.2 and that each of the modules $E = I[\mathfrak m^n]$ is of the desired type. Thus it suffices to prove that

$$\operatorname{Hom}_{D(A)}(K, I) = \operatorname{colim} \operatorname{Hom}_{D(A)}(K, I[\mathfrak m^n])$$

This would be immediate if K where a compact object (or a perfect object) of $D(A)$. This is not the case, but K is a pseudo-coherent object which is enough here. Namely, we can represent K by a bounded above complex of finite free R-modules $K^\bullet$. In this case the Hom groups above are computed by using $\operatorname{Hom}_{K(A)}(K^\bullet, -)$. As each K^n is finite free the limit statement holds and the proof is complete. □

Let R be a ring. We will say that an object L of $D(R)$ is *invertible* if there is an open covering $\operatorname{Spec}(R) = \bigcup D(f_i)$ such that $L \otimes_R R_{f_i} \cong R_{f_i}[-n_i]$ for some integers n_i. In this case, the function

$$\mathfrak p \mapsto n_\mathfrak p, \quad \text{where } n_\mathfrak p \text{ is the unique integer such that } H^{n_\mathfrak p}(L \otimes \kappa(\mathfrak p)) \neq 0$$

is locally constant on $\operatorname{Spec}(R)$. In particular, it follows that $L = \bigoplus H^n(L)[-n]$ which gives a well defined complex of R-modules (with zero differentials) representing L. Since each $H^n(L)$ is finite projective and nonzero for only a finite number of n we also see that L is a perfect object of $D(R)$.

Lemma 12.4. *Let A be a Noetherian ring. Let $F : D^b_{Coh}(A) \to D^b_{Coh}(A)$ be an A-linear equivalence of categories. Then $F(A)$ is an invertible object of $D(A)$.*

Proof. Let $\mathfrak m \subset A$ be a maximal ideal with residue field κ. Consider the object $F(\kappa)$. Since $\kappa = \operatorname{Hom}_{D(A)}(\kappa, \kappa)$ we find that all cohomology groups of $F(\kappa)$ are annihilated by $\mathfrak m$. We also see that

$$\operatorname{Ext}^i_A(\kappa, \kappa) = \operatorname{Ext}^i_A(F(\kappa), F(\kappa)) = \operatorname{Hom}_{D(A)}(F(\kappa), F(\kappa)[-i])$$

is zero for $i < 0$. Say $H^a(F(\kappa)) \neq 0$ and $H^b(F(\kappa)) \neq 0$ with a minimal and b maximal (so in particular $a \leq b$). Then there is a nonzero map

$$F(\kappa) \to H^b(F(\kappa))[-b] \to H^a(F(\kappa))[-b] \to F(\kappa)[a-b]$$

in $D(A)$ (nonzero because it induces a nonzero map on cohomology). This proves that $b = a$. We conclude that $F(\kappa) = \kappa[-a]$.

Let G be a quasi-inverse to our functor F. Arguing as above we find an integer b such that $G(\kappa) = \kappa[-b]$. On composing we find $a + b = 0$. Let E be a finite A-module wich is annihilated by a power of $\mathfrak{m}$. Arguing by induction on the length of E we find that $G(E) = E'[-b]$ for some finite A-module E' annihilated by a power of $\mathfrak{m}$. Then $E[-a] = F(E')$. Next, we consider the groups

$$\mathrm{Ext}^i_A(A, E') = \mathrm{Ext}^i_A(F(A), F(E')) = \mathrm{Hom}_{D(A)}(F(A), E[-a+i])$$

The left hand side is nonzero if and only if $i = 0$ and then we get E'. Applying this with $E = E' = \kappa$ and using Nakayama's lemma this implies that $H^j(F(A))$ is zero for $j > a$ and generated by 1 element for $j = a$. On the other hand, if $H^j(F(A))_{\mathfrak{m}}$ is not zero for some $j < a$, then there is a map $F(A) \to E[-a+i]$ for some $i < 0$ and some E (Lemma 12.3) Thus we see that $F(A)_{\mathfrak{m}} = M[-a]$ for some $A_{\mathfrak{m}}$-module M generated by 1 element. However, since

$$A_{\mathfrak{m}} = \mathrm{Hom}_{D(A)}(A, A)_{\mathfrak{m}} = \mathrm{Hom}_{D(A)}(F(A), F(A))_{\mathfrak{m}} = \mathrm{Hom}_{A_{\mathfrak{m}}}(M, M)$$

we see that $M \cong A_{\mathfrak{m}}$. We conclude that there exists an element $f \in A$, $f \notin \mathfrak{m}$ such that $F(A)_f$ is isomorphic to $A_f[-a]$. This finishes the proof. □

Lemma 12.5. *Let A be a Noetherian ring. If $\omega_A^\bullet$ and $(\omega'_A)^\bullet$ are dualizing complexes, then $(\omega'_A)^\bullet$ is quasi-isomorphic to $\omega_A^\bullet \otimes_A^{\mathbf{L}} L$ for some invertible object L of $D(A)$.*

Proof. By Lemmas 12.2 and 12.4 the functor $K \mapsto R\,\mathrm{Hom}(R\,\mathrm{Hom}(K, \omega_A^\bullet), (\omega'_A)^\bullet)$ maps A to an invertible object L. In other words, there is an isomorphism

$$L \longrightarrow R\,\mathrm{Hom}(\omega_A^\bullet, (\omega'_A)^\bullet)$$

Since L has finite tor dimension, this means that we can apply More on Algebra, Lemma 55.6 to see that

$$R\,\mathrm{Hom}(\omega_A^\bullet, (\omega'_A)^\bullet) \otimes_A^{\mathbf{L}} K \longrightarrow R\,\mathrm{Hom}(R\,\mathrm{Hom}(K, \omega_A^\bullet), (\omega'_A)^\bullet)$$

is an isomorphism for K in $D^b_{Coh}(A)$. In particular, setting $K = \omega_A^\bullet$ finishes the proof. □

Lemma 12.6. *Let A be a Noetherian ring. Let $B = S^{-1}A$ be a localization. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A B$ is a dualizing complex for B.*

Proof. Let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $S^{-1}I^\bullet$ is a bounded complex of injective $B = S^{-1}A$-modules (Lemma 3.9) representing $\omega_A^\bullet \otimes_A B$. Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension. Since $H^i(\omega_A^\bullet \otimes_A B) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

$$B \longrightarrow R\,\mathrm{Hom}(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B)$$

is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case, see More on Algebra, Lemma 55.7. □

Lemma 12.7. *Let A be a Noetherian ring. Let $f_1, \ldots, f_n \in A$ generate the unit ideal. If $\omega_A^\bullet$ is a complex of A-modules such that $(\omega_A^\bullet)_{f_i}$ is a dualizing complex for A_{f_i} for all i, then $\omega_A^\bullet$ is a dualizing complex for A.*

Proof. Consider the double complex

$$\prod\nolimits_{i_0} (\omega_A^\bullet)_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} (\omega_A^\bullet)_{f_{i_0} f_{i_1}} \to \ldots$$

The associated total complex is quasi-isomorphic to $\omega_A^\bullet$ for example by Descent, Remark 3.10 or by Derived Categories of Schemes, Lemma 8.4. By assumption the complexes $(\omega_A^\bullet)_{f_i}$ have finite injective dimension as complexes of A_{f_i}-modules. This implies that each of the complexes $(\omega_A^\bullet)_{f_{i_0} \ldots f_{i_p}}$, $p > 0$ has finite injective dimension over $A_{f_{i_0} \ldots f_{i_p}}$, see Lemma 3.9. This in turn implies that each of the complexes $(\omega_A^\bullet)_{f_{i_0} \ldots f_{i_p}}$, $p > 0$ has finite injective dimension over A, see Lemma 3.2. Hence $\omega_A^\bullet$ has finite injective dimension as a complex of A-modules (as it can be represented by a complex endowed with a finite filtration whose graded parts have finite injective dimension). Since $H^n(\omega_A^\bullet)_{f_i}$ is a finite A_{f_i} module for each i we see that $H^i(\omega_A^\bullet)$ is a finite A-module, see Algebra, Lemma 23.2. Finally, the (derived) base change of the map $A \to R\operatorname{Hom}(\omega_A^\bullet, \omega_A^\bullet)$ to A_{f_i} is the map $A_{f_i} \to R\operatorname{Hom}((\omega_A^\bullet)_{f_i}, (\omega_A^\bullet)_{f_i})$ by More on Algebra, Lemma 55.7. Hence we deduce that $A \to R\operatorname{Hom}(\omega_A^\bullet, \omega_A^\bullet)$ is an isomorphism and the proof is complete. $\square$

Lemma 12.8. *Let $A \to B$ be a surjective homomorphism of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex. Then $R\operatorname{Hom}(B, \omega_A^\bullet)$ is a dualizing complex for B.*

Proof. Let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $\operatorname{Hom}_A(B, I^\bullet)$ is a bounded complex of injective B-modules (Lemma 3.4) representing $R\operatorname{Hom}(B, \omega_A^\bullet)$. Thus $R\operatorname{Hom}(B, \omega_A^\bullet)$ has finite injective dimension. By Lemma 10.3 it is an object of $D_{Coh}(B)$. Finally, we compute

$$\operatorname{Hom}_{D(B)}(R\operatorname{Hom}(B, \omega_A^\bullet), R\operatorname{Hom}(B, \omega_A^\bullet)) = \operatorname{Hom}_{D(A)}(R\operatorname{Hom}(B, \omega_A^\bullet), \omega_A^\bullet) = B$$

and for $n \neq 0$ we compute

$$\operatorname{Hom}_{D(B)}(R\operatorname{Hom}(B, \omega_A^\bullet), R\operatorname{Hom}(B, \omega_A^\bullet)[n]) = \operatorname{Hom}_{D(A)}(R\operatorname{Hom}(B, \omega_A^\bullet), \omega_A^\bullet[n]) = 0$$

which proves the last property of a dualizing complex. In the displayed equations, the first equality holds by Lemma 10.1 and the second equality holds by Lemma 12.2. $\square$

Lemma 12.9. *Let A be a Noetherian ring. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A A[x]$ is a dualizing complex for $A[x]$.*

Proof. Set $B = A[x]$ and $\omega_B^\bullet = \omega_A^\bullet \otimes_A B$. It follows from Lemma 3.11 and More on Algebra, Lemma 53.4 that $\omega_B^\bullet$ has finite injective dimension. Since $H^i(\omega_B^\bullet) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

$$B \longrightarrow R\operatorname{Hom}(\omega_B^\bullet, \omega_B^\bullet)$$

is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case, see More on Algebra, Lemma 55.7. $\square$

Proposition 12.10. *Let A be a Noetherian ring which has a dualizing complex. Then any A-algebra essentially of finite type over A has a dualixing complex.*

Proof. This follows from a combination of Lemmas 12.6, 12.8, and 12.9. $\square$

Lemma 12.11. *Let A be a Noetherian ring. Let $\omega_A^\bullet$ be a dualizing complex. Let $\mathfrak{m} \subset A$ be a maximal ideal and set $\kappa = A/\mathfrak{m}$. Then $R\operatorname{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[n]$ for some $n \in \mathbf{Z}$.*

Proof. This is true because $R\operatorname{Hom}_A(\kappa, \omega_A^\bullet)$ is a dualizing complex over κ (Lemma 12.8), because dualizing complexes over κ are unique up to shifts (Lemma 12.5), and because κ is a dualizing complex over κ. □

13. Dualizing complexes over local rings

In this section $(A, \mathfrak{m}, \kappa)$ will be a Noetherian local ring endowed with a dualizing complex $\omega_A^\bullet$ such that the integer n of Lemma 12.11 is zero. More precisely, we assume that $R\operatorname{Hom}_A(\kappa, \omega_A^\bullet) = \kappa[0]$. In this case we will say that the dualizing complex is *normalized*. Observe that a normalized dualizing complex is unique up to isomorphism and that any other dualizing complex for A is isomorphic to a shift of a normalized one (Lemma 12.5).

Lemma 13.1. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $A \to B$ be surjective. Then $\omega_B^\bullet = R\operatorname{Hom}_A(B, \omega_A^\bullet)$ is a normalized dualizing complex for B.*

Proof. By Lemma 12.8 the complex $\omega_B^\bullet$ is dualizing for B. We compute

$$R\operatorname{Hom}_B(\kappa, R\operatorname{Hom}_A(B, \omega_A^\bullet)) = R\operatorname{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[0]$$

The first equality by Lemma 10.1. □

Lemma 13.2. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let F be an A-linear self-equivalence of the category of finite length A-modules. Then F is isomorphic to the identity functor.*

Proof. Since κ is the unique simple object of the category we have $F(\kappa) \cong \kappa$. Since our category is abelian, we find that F is exact. Hence $F(E)$ has the same length as E for all finite length modules E. Since $\operatorname{Hom}(E, \kappa) = \operatorname{Hom}(F(E), F(\kappa)) \cong \operatorname{Hom}(F(E), \kappa)$ we conclude from Nakayama's lemma that E and $F(E)$ have the same number of generators. Hence $F(A/\mathfrak{m}^n)$ is a cyclic A-module. Pick a generator $e \in F(A/\mathfrak{m}^n)$. Since F is A-linear we conclude that $\mathfrak{m}^n e = 0$. The map $A/\mathfrak{m}^n \to F(A/\mathfrak{m}^n)$ has to be an isomorphism as the lengths are equal. Pick an element

$$e \in \lim F(A/\mathfrak{m}^n)$$

which maps to a generator for all n (small argument omitted). Then we obtain a system of isomorphisms $A/\mathfrak{m}^n \to F(A/\mathfrak{m}^n)$ compatible with all A-module maps $A/\mathfrak{m}^n \to A/\mathfrak{m}^{n'}$ (by A-linearity of F again). Since any finite lenghth module is a cokernel of a map between direct sums of cyclic modules, we obtain the isomorphism of the lemma. □

Lemma 13.3. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let E be an injective hull of κ. Then there exists a functorial isomorphism*

$$R\operatorname{Hom}(N, \omega_A^\bullet) = \operatorname{Hom}_A(N, E)[0]$$

for N running through the finite length A-modules.

Proof. By induction on the length of N we see that $R\operatorname{Hom}(N, \omega_A^\bullet)$ is a module of finite length sitting in degree 0. Thus $R\operatorname{Hom}_A(-, \omega_A^\bullet)$ induces an anti-equivalence on the category of finite length modules. Since the same is true for $\operatorname{Hom}_A(-, E)$ by Proposition 7.8 we see that

$$N \longmapsto \operatorname{Hom}_A(R\operatorname{Hom}(N, \omega_A^\bullet), E)$$

is an equivalence as in Lemma 13.2. Hence it is isommorphic to the identity functor. Since $\operatorname{Hom}_A(-, E)$ applied twice is the identity (Proposition 7.8) we obtain the statement of the lemma. ☐

Lemma 13.4. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. If $\dim(A) = 0$, then $\omega_A^\bullet \cong E[0]$ where E is an injective hull of the residue field.*

Proof. Immediate from Lemma 13.3. ☐

Lemma 13.5. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex. Let $I \subset \mathfrak{m}$ be an ideal of finite length. Set $B = A/I$. Then there is a distinguished triangle*

$$\omega_B^\bullet \to \omega_A^\bullet \to \operatorname{Hom}_A(I, E)[0] \to \omega_B^\bullet[1]$$

in $D(A)$ where E is an injective hull of κ and $\omega_B^\bullet$ is a normalized dualizing complex for B.

Proof. Use the short exact sequence $0 \to I \to A \to B \to 0$ and Lemmas 13.3 and 13.1. ☐

Lemma 13.6. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $f \in \mathfrak{m}$ be a nonzerodivisor. Set $B = A/(f)$. Then there is a distinguished triangle*

$$\omega_B^\bullet \to \omega_A^\bullet \to \omega_A^\bullet \to \omega_B^\bullet[1]$$

in $D(A)$ where $\omega_B^\bullet$ is a normalized dualizing complex for B.

Proof. Use the short exact sequence $0 \to A \to A \to B \to 0$ and Lemma 13.1. ☐

Lemma 13.7. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $d = \dim(A)$. Then*

1. *if $H^i(\omega_A^\bullet)$ is nonzero, then $i \in \{-d, \ldots, 0\}$,*
2. *the dimension of the support of $H^i(\omega_A^\bullet)$ is at most $-i$,*

Proof. We prove this by induction on the dimension of A. If $\dim(A) = 0$ this follows immediately from Lemma 13.4.

Assume that the result holds for rings of dimension $< d$ and that A has depth at least 1. Then we can find a nonzero divisor f and apply Lemma 13.6 and the induction hypothesis to B. It follows that multiplication by f is surjective on $H^i(\omega_A^\bullet)$ for $i > 0$ and $i < d$. By Nakayama we conclude these cohomology modules are zero, i.e., (1) holds. If the dimension of the support of $H^i(\omega_A^\bullet)$ is e, then the dimension of the support of $H^i(\omega_A^\bullet)/fH^i(\omega_A^\bullet) \subset H^{i+1}(\omega_B^\bullet)$ is at least $e - 1$. Hence our induction assumption gives that $e \leq -i$.

If A has depth 0, then we let $I = A[\mathfrak{m}^\infty]$ be the maximal ideal of A having finite length. Then $B = A/I$ has depth ≥ 1 so we know the result for B. Applying Lemma 13.5 we obtain the result for A. ☐

Lemma 13.8. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $\mathfrak{p}$ be a minimal prime of A with $\dim(A/\mathfrak{p}) = e$. Then $H^i(\omega_A^\bullet)_\mathfrak{p}$ is nonzero if and only if $i = -e$.*

Proof. Since $A_\mathfrak{p}$ has dimension zero, there exists an integer $n > 0$ such that $\mathfrak{p}^n A_\mathfrak{p}$ is zero. Set $B = A/\mathfrak{p}^n$ and $\omega_B^\bullet = R\operatorname{Hom}_A(B, \omega_A^\bullet)$. Since $B_\mathfrak{p} = A_\mathfrak{p}$ we see that $(\omega_B^\bullet)_\mathfrak{p} \cong (\omega_A^\bullet)_\mathfrak{p}$ by using More on Algebra, Lemma 55.7. By Lemma 13.1 we may replace A by B. After doing so, we see that $\dim(A) = e$. Then we see that $H^i(\omega_A^\bullet)_\mathfrak{p}$ can only be nonzero if $i = -e$ by Lemma 13.7. On the other hand, since $(\omega_A^\bullet)_\mathfrak{p}$ is a dualizing complex for the nonzero ring $A_\mathfrak{p}$ (Lemma 12.6) we see that the remaining module has to be nonzero. $\square$

14. The dimension function of a dualizing complex

Our results in the local setting have the following consequence: a Noetherian ring with has a dualizing complex is a universally catenary ring of finite dimension.

Lemma 14.1. *Let A be a Noetherian ring. Let $\mathfrak{p}$ be a minimal prime of A. Then $H^i(\omega_A^\bullet)_\mathfrak{p}$ is nonzero for exactly one i.*

Proof. The complex $\omega_A^\bullet \otimes_A A_\mathfrak{p}$ is a dualizing complex for $A_\mathfrak{p}$ (Lemma 12.6). The dimension of $A_\mathfrak{p}$ is zero as $\mathfrak{p}$ is minimal. Hence the result follows from Lemma 13.4. $\square$

Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Lemma 12.11 allows us to define a function

$$\delta = \delta_{\omega_A^\bullet} : \operatorname{Spec}(A) \longrightarrow \mathbf{Z}$$

by mapping $\mathfrak{p}$ to the integer of Lemma 12.11 for the dualizing complex $(\omega_A^\bullet)_\mathfrak{p}$ over $A_\mathfrak{p}$ (Lemma 12.6) and the residue field $\kappa(\mathfrak{p})$. To be precise, we define $\delta(\mathfrak{p})$ to be the unique integer such that

$$(\omega_A^\bullet)_\mathfrak{p}[-\delta(\mathfrak{p})]$$

is a normalized dualizing complex over the Noetherian local ring $A_\mathfrak{p}$.

Lemma 14.2. *Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Let $A \to B$ be a surjective ring map and let $\omega_B^\bullet = R\operatorname{Hom}(B, \omega_A^\bullet)$ be the dualizing complex for B of Lemma 12.8. Then we have*

$$\delta_{\omega_B^\bullet} = \delta_{\omega_A^\bullet}|_{\operatorname{Spec}(B)}$$

Proof. This follows from the definition of the functions and Lemma 13.1. $\square$

Lemma 14.3. *Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. The function $\delta = \delta_{\omega_A^\bullet}$ defined above is a dimension function (Topology, Definition 19.1).*

Proof. Let $\mathfrak{p} \subset \mathfrak{q}$ be an immediate specialization. We have to show that $\delta(\mathfrak{p}) = \delta(\mathfrak{q}) + 1$. We may replace A by $A/\mathfrak{p}$, the complex $\omega_A^\bullet$ by $\omega_{A/\mathfrak{p}}^\bullet = R\operatorname{Hom}(A/\mathfrak{p}, \omega_A^\bullet)$, the prime $\mathfrak{p}$ by (0), and the prime $\mathfrak{q}$ by $\mathfrak{q}/\mathfrak{p}$, see Lemma 14.2. Thus we may assume that A is a domain, $\mathfrak{p} = (0)$, and $\mathfrak{q}$ is a prime ideal of height 1.

Then $H^i(\omega_A^\bullet)_{(0)}$ is nonzero for exactly one i, say i_0, by Lemma 14.1. In fact $i_0 = -\delta((0))$ because $(\omega_A^\bullet)_{(0)}[-\delta((0))]$ is a normalized dualizing complex over the field $A_{(0)}$.

On the other hand $(\omega_A^\bullet)_\mathfrak{q}[-\delta(\mathfrak{q})]$ is a normalized dualizing complex for $A_\mathfrak{q}$. By Lemma 13.8 we see that

$$H^e((\omega_A^\bullet)_\mathfrak{q}[-\delta(\mathfrak{q})])_{(0)} = H^{e-\delta(\mathfrak{q})}(\omega_A^\bullet)_{(0)}$$

is nonzero only for $e = -\dim(A_\mathfrak{q}) = -1$. We conclude

$$-\delta((0)) = -1 - \delta(\mathfrak{p})$$

as desired. $\square$

Lemma 14.4. *Let A be a Noetherian ring which has a dualizing complex. Then A is universally catenary of finite dimension.*

Proof. Because $\mathrm{Spec}(A)$ has a dimension function by Lemma 14.3 it is catenary, see Topology, Lemma 19.2. Hence A is catenary, see Algebra, Lemma 101.2. It follows from Proposition 12.10 that A is universally catenary.

Because any dualizing complex $\omega_A^\bullet$ is in $D^b_{Coh}(A)$ the values of the function $\delta_{\omega_A^\bullet}$ in minimal primes are bounded by Lemma 14.1. On the other hand, for a maximal ideal $\mathfrak{m}$ with residue field κ the integer $i = -\delta(\mathfrak{m})$ is the unique integer such that $\mathrm{Ext}^i_A(\kappa, \omega_A^\bullet)$ is nonzero (Lemma 12.11). Since $\omega_A^\bullet$ has finite injective dimension these values are bounded too. Since the dimension of A is the maximal value of $\delta(\mathfrak{p}) - \delta(\mathfrak{m})$ where $\mathfrak{p} \subset \mathfrak{m}$ are a pair consisting of a minimal prime and a maximal prime we find that the dimension of $\mathrm{Spec}(A)$ is bounded. $\square$

15. The local duality theorem

The main result in this section is due to Grothendieck.

Lemma 15.1. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let $Z = V(\mathfrak{m}) \subset \mathrm{Spec}(A)$. Then $E = R^0\Gamma_Z(\omega_A^\bullet)$ is an injective hull of κ and $R\Gamma_Z(\omega_A^\bullet) = E[0]$.*

Proof. By Lemma 8.9 we have $R\Gamma_\mathfrak{m} = R\Gamma_Z$. Thus

$$R\Gamma_Z(\omega_A^\bullet) = R\Gamma_\mathfrak{m}(\omega_A^\bullet) = \mathrm{hocolim}\ R\,\mathrm{Hom}(A/\mathfrak{m}^n, \omega_A^\bullet)$$

by Lemma 8.2. Let E' be an injective hull of the residue field. By Lemma 13.3 we can find isomorphisms

$$R\,\mathrm{Hom}(A/\mathfrak{m}^n, \omega_A^\bullet) \cong \mathrm{Hom}_A(A/I^n, E')[0]$$

compatible with transition maps. Since $E' = \bigcup E'[\mathfrak{m}^n] = \mathrm{colim}\,\mathrm{Hom}_A(A/I^n, E')$ by Lemma 7.3 we conclude that $E \cong E'$ and that all other cohomology groups of the complex $R\Gamma_Z(\omega_A^\bullet)$ are zero. $\square$

Remark 15.2. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with a normalized dualizing complex $\omega_A^\bullet$. By Lemma 15.1 above we see that $R\Gamma_Z(\omega_A^\bullet)$ is an injective hull of the residue field placed in degree 0. In fact, this gives a “construction” or “realization” of the injective hull which is slightly more canonical than just picking any old injective hull. Namely, a normalized dualizing complex is unique up to isomorphism, with group of automorphisms the group of units of A, whereas an injective hull of κ is unique up to isomorphism, with group of automorphisms the group of units of the completion $A^\wedge$ of A with respect to $\mathfrak{m}$.

Here is the main result of this section.

Theorem 15.3. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let E be an injective hull of the residue field. Let $Z = V(\mathfrak{m}) \subset \mathrm{Spec}(A)$. Denote $^\wedge$ derived completion with respect to $\mathfrak{m}$. Then*

$$R\,\mathrm{Hom}(K, \omega_A^\bullet)^\wedge \cong R\,\mathrm{Hom}(R\Gamma_Z(K), E[0])$$

for K in $D(A)$.

Proof. Observe that $E[0] \cong R\Gamma_Z(\omega_A^\bullet)$ by Lemma 15.1. By More on Algebra, Lemma 64.11 completion on the left hand side goes inside. Thus we have to prove

$$R\,\mathrm{Hom}(K^\wedge, (\omega_A^\bullet)^\wedge) = R\,\mathrm{Hom}(R\Gamma_Z(K), R\Gamma_Z(\omega_A^\bullet))$$

This follows from the equivalence between $D_{comp}(A, \mathfrak{m})$ and $D_{\mathfrak{m}^\infty\text{-torsion}}(A)$ given in Proposition 9.2. More precisely, it is a special case of Lemma 9.3. □

Here is a special case of the theorem above.

Lemma 15.4. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let E be an injective hull of the residue field. Let $K \in D_{Coh}(A)$. Then*

$$Ext^i_A(K, \omega_A^\bullet)^\wedge = \mathrm{Hom}_A(H^i_{\mathfrak{m}}(K), E)$$

where $^\wedge$ denotes $\mathfrak{m}$-adic completion.

Proof. By Lemma 12.2 we see that $R\,\mathrm{Hom}(K, \omega_A^\bullet)$ is an object of $D_{Coh}(A)$. It follows that the cohomology modules of the derived completion of $R\,\mathrm{Hom}(K, \omega_A^\bullet)$ are equal to the usual completions $\mathrm{Ext}^i_A(K, \omega_A^\bullet)^\wedge$ by More on Algebra, Lemma 64.20. On the other hand, we have $R\Gamma_{\mathfrak{m}} = R\Gamma_Z$ for $Z = V(\mathfrak{m})$ by Lemma 8.9. Moreover, the functor $\mathrm{Hom}_A(-, E)$ is exact hence factors through cohomology. Hence the lemma is consequence of Theorem 15.3. □

16. Dualizing complexes on schemes

We define a dualizing complex on a locally Noetherian scheme to be a complex which affine locally comes from a dualizing complex on the corresponding ring. This is not completely standard but agrees with all definitions in the literature on Noetherian schemes of finite dimension.

Lemma 16.1. *Let X be a locally Noetherian scheme. Let K be an object of $D(\mathcal{O}_X)$. The following are equivalent*

1. *For every affine open $U = \mathrm{Spec}(A) \subset X$ there exists a dualizing complex $\omega_A^\bullet$ for A such that $K|_U$ is isomorphic to the image of $\omega_A^\bullet$ by the functor $\widetilde{\ }: D(A) \to D(\mathcal{O}_U)$.*
2. *There is an affine open covering $X = \bigcup U_i$, $U_i = \mathrm{Spec}(A_i)$ such that for each i there exists a dualizing complex $\omega_i^\bullet$ for A_i such that $K|_U$ is isomorphic to the image of $\omega_i^\bullet$ by the functor $\widetilde{\ }: D(A_i) \to D(\mathcal{O}_{U_i})$.*

Proof. Assume (2) and let $U = \mathrm{Spec}(A)$ be an affine open of X. Since condition (2) implies that K is in $D_{QCoh}(\mathcal{O}_X)$ we find an object $\omega_A^\bullet$ in $D(A)$ whose associated complex of quasi-coherent sheaves is isomorphic to $K|_U$, see Derived Categories of Schemes, Lemma 3.4. We will show that $\omega_A^\bullet$ is a dualizing complex for A which will finish the proof.

Since $X = \bigcup U_i$ is an open covering, we can find a standard open covering $U = D(f_1) \cup \ldots \cup D(f_m)$ such that each $D(f_j)$ is a standard open in one of the affine opens U_i, see Schemes, Lemma 11.5. Say $D(f_j) = D(g_j)$ for $g_j \in A_{i_j}$. Then $A_{f_j} \cong (A_{i_j})_{g_j}$ and we have

$$(\omega_A^\bullet)_{f_j} \cong (\omega_i^\bullet)_{g_j}$$

in the derived category by Derived Categories of Schemes, Lemma 3.4. By Lemma 12.6 we find that the complex $(\omega_A^\bullet)_{f_j}$ is a dualizing complex over A_{f_j} for $j = 1, \ldots, m$. This implies that $\omega_A^\bullet$ is dualizing by Lemma 12.7. □

Definition 16.2. Let X be a locally Noetherian scheme. An object K of $D(\mathcal{O}_X)$ is called a *dualizing complex* if K satisfies the equivalent conditions of Lemma 16.1.

Please see remarks made at the beginning of this section.

Lemma 16.3. *Let A be a Noetherian ring and let $X = \text{Spec}(A)$. Let K, L be objects of $D(A)$. If $K \in D_{Coh}(A)$ and L has finite injective dimension, then*

$$R\,\mathcal{H}om(\widetilde{K}, \widetilde{L}) = R\,\widetilde{\text{Hom}(K, L)}$$

in $D(\mathcal{O}_X)$.

Proof. We may assume that L is given by a finite complex $I^\bullet$ of injective A-modules. By induction on the length of $I^\bullet$ and compatibility of the constructions with distinguished triangles, we reduce to the case that $L = I[0]$ where I is an injective A-module. In this case, Derived Categories of Schemes, Lemma 9.8, tells us that the nth cohomology sheaf of $R\,\mathcal{H}om(\widetilde{K}, \widetilde{L})$ is the sheaf associated to the presheaf

$$D(f) \longmapsto \text{Ext}^n_{A_f}(K \otimes_A A_f, I \otimes_A A_f)$$

Since A is Noetherian, the A_f-module $I \otimes_A A_f$ is injective (Lemma 3.9). Hence we see that

$$\begin{aligned} \text{Ext}^n_{A_f}(K \otimes_A A_f, I \otimes_A A_f) &= \text{Hom}_{A_f}(H^{-n}(K \otimes_A A_f), I \otimes_A A_f) \\ &= \text{Hom}_{A_f}(H^{-n}(K) \otimes_A A_f, I \otimes_A A_f) \\ &= \text{Hom}_A(H^{-n}(K), I) \otimes_A A_f \end{aligned}$$

The last equality because $H^{-n}(K)$ is a finite A-module. This proves that the canonical map

$$R\,\widetilde{\text{Hom}(K, L)} \longrightarrow R\,\mathcal{H}om(\widetilde{K}, \widetilde{L})$$

is a quasi-isomorphism in this case and the proof is done. □

Lemma 16.4. *Let K be a dualizing complex on a locally Noetherian scheme X. Then K is an object of $D_{Coh}(\mathcal{O}_X)$ and $D = R\,\mathcal{H}om(-, K)$ induces an anti-equivalence*

$$D : D_{Coh}(\mathcal{O}_X) \longrightarrow D_{Coh}(\mathcal{O}_X)$$

such that $D \circ D \cong id$. If X is quasi-compact, then D exchanges $D^+_{Coh}(\mathcal{O}_X)$ and $D^-_{Coh}(\mathcal{O}_X)$ and induces an equivalence $D^b_{Coh}(\mathcal{O}_X) \to D^b_{Coh}(\mathcal{O}_X)$.

Proof. Let $U \subset X$ be an affine open. Say $U = \text{Spec}(A)$ and let $\omega_A^\bullet$ be a dualizing complex for A corresponding to $K|_U$ as in Lemma 16.1. By Lemma 16.3 the diagram

$$\begin{tikzcd} D_{Coh}(A) \arrow[r] \arrow[d, "R\,\text{Hom}(-, \omega_A^\bullet)"'] & D_{Coh}(\mathcal{O}_U) \arrow[d, "R\,\mathcal{H}om(-, K|_U)"] \\ D_{Coh}(A) \arrow[r] & D(\mathcal{O}_U) \end{tikzcd}$$

commutes. We conclude that D sends $D_{Coh}(\mathcal{O}_X)$ into $D_{Coh}(\mathcal{O}_X)$. Moreover, the canonical map

$$L \longrightarrow R\mathcal{H}om(R\mathcal{H}om(L, K), K)$$

(Cohomology on Sites, Lemma 26.5) is an isomorphism for all L because this is true on affines by Lemma 12.2. The statement on boundedness properties of the functor D in the quasi-compact case also folow from the corresponding statements of Lemma 12.2. $\square$

17. Twisted inverse image

References for this section are [Nee96] and [LN07]. Let $f : X \to Y$ be a morphism of schemes. In some papers, a *twisted inverse image* for f is defined to be a right adjoint to the functor $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_X)$. However, this terminology is not universally accepted and we refrain from giving a formal definition. We also will not use the notation $f^!$ for such a functor, as this would clash (for general morphisms f) with the notation in [Har66].

Lemma 17.1. *Let $f : X \to Y$ be a morphism between quasi-separated and quasi-compact schemes. The functor $Rf_* : D_{QCoh}(X) \to D_{QCoh}(Y)$ has a right adjoint.*

Proof. We will prove a right adjoint exists by verifying the hypotheses of Derived Categories, Proposition 35.2. First off, the category $D_{QCoh}(\mathcal{O}_X)$ has direct sums, see Derived Categories of Schemes, Lemma 3.1. The category $D_{QCoh}(\mathcal{O}_X)$ is compactly generated by Derived Categories of Schemes, Theorem 13.3. Since X and Y are quasi-compact and quasi-separated, so is f, see Schemes, Lemmas 21.14 and 21.15. Hence the functor Rf_* commutes with direct sums, see Derived Categories of Schemes, Lemma 4.2. This finishes the proof. $\square$

Example 17.2. Let $A \to B$ be a ring map. Let $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and $f : X \to Y$ the morphism corresponding to $A \to B$. Then Rf_* corresponds to restriction $D(B) \to D(A)$ via the equivalences $D(B) \to D_{QCoh}(\mathcal{O}_X)$ and $D(A) \to D_{QCoh}(\mathcal{O}_Y)$. Hence the right adjoint corresponds to the functor $K \longmapsto R\,\mathrm{Hom}(B, K)$ of Section 10.

Example 17.3. If $f : X \to Y$ is a separated finite type morphism of Noetherian schemes, then twisted inverse image does not map $D_{Coh}(\mathcal{O}_Y)$ into $D_{Coh}(\mathcal{O}_X)$. Namely, let k be a field and consider the morphism $f : \mathbf{A}^1_k \to \mathrm{Spec}(k)$. By Example 17.2 this corresponds to the question of whether $R\,\mathrm{Hom}(B, -)$ maps $D_{Coh}(A)$ into $D_{Coh}(B)$ where $A = k$ and $B = k[x]$. This is not true because

$$R\,\mathrm{Hom}(k[x], k) = \left(\prod\nolimits_{n \geq 0} k\right)[0]$$

which is not a finite $k[x]$-module. Hence $a(\mathcal{O}_Y)$ does not have coherent cohomology sheaves.

Example 17.4. If $f : X \to Y$ is a proper or even finite morphism of Noetherian schemes, then twisted inverse image does not map $D^-_{QCoh}(\mathcal{O}_Y)$ into $D^-_{QCoh}(\mathcal{O}_X)$. Namely, let k be a field, let $k[\epsilon]$ be the dual numbers over k, let $X = \mathrm{Spec}(k)$, and let $Y = \mathrm{Spec}(k[\epsilon])$. Then $\mathrm{Ext}^i_{k[\epsilon]}(k, k)$ is nonzero for all $i \geq 0$. Hence $a(\mathcal{O}_Y)$ is not bounded above by Example 17.2.

Lemma 17.5. *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X)$ be the right adjoint to Rf_* of Lemma 17.1. Then a maps $D^+_{QCoh}(\mathcal{O}_Y)$ into $D^+_{QCoh}(\mathcal{O}_X)$.*

Proof. By Derived Categories of Schemes, Lemma 4.1 the functor Rf_* has finite cohomological dimension. In other words, there exist an integer N such that $H^i(Rf_*L) = 0$ for $i \geq N + c$ if $H^j(L) = 0$ for $j \geq c$. Say $K \in D^+_{QCoh}(\mathcal{O}_Y)$ has $H^k(K) = 0$ for $k \geq c$. Then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\tau_{\leq c-N}a(K), a(K)) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\tau_{\leq c-N}a(K), K) = 0$$

by what we said above. Clearly, this implies that $a(K)$ is bounded below. $\square$

We often want to know whether the twisted inverse image commutes with base change. Thus we consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \tag{17.5.1}$$

of quasi-compact and quasi-separated schemes. Denote

$$\begin{aligned} a &: D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_X), \\ a' &: D_{QCoh}(\mathcal{O}_{Y'}) \to D_{QCoh}(\mathcal{O}_{X'}), \\ b &: D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{X'}), \\ b' &: D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{Y'}) \end{aligned}$$

the right adjoints to Rf_*, Rf'_*, Rg_*, and Rg'_* (Lemma 17.1). Since $Rf_* \circ Rg'_* = Rg_* \circ Rf'_*$ we get

$$b' \circ a = a' \circ b.$$

Another compatibility comes from the base change map of Cohomology, Remark 29.2. It induces a transformation of functors

$$Lg^* \circ Rf_* \longrightarrow Rf'_* \circ L(g')^*$$

on derived categories of sheaves with quasi-coherent cohomology. Hence a transformation between the right adjoints in the opposite direction

$$a \circ Rg_* \longleftarrow Rg'_* \circ a'$$

Lemma 17.6. *In diagram (17.5.1) assume that g is flat or more generally that f and g are Tor independent. Then $a \circ Rg_* \longleftarrow Rg'_* \circ a'$ is an isomorphism.*

Proof. In this case the base change map $Lg^* \circ Rf_*K \longrightarrow Rf'_* \circ L(g')^*K$ is an isomorphism for every K in $D_{QCoh}(\mathcal{O}_X)$ by Derived Categories of Schemes, Lemma 16.3. Thus the corresponding transformation between adjoint functors is an isomorphism as well. $\square$

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be a quasi-compact open subscheme and set $U = f^{-1}(V)$. This gives a cartesian square

$$\begin{tikzcd} U \arrow[r, "j'"] \arrow[d, "f|_U"'] & X \arrow[d, "f"] \\ V \arrow[r, "j"] & Y \end{tikzcd}$$

as in (17.5.1). By Lemma 17.6 we have $a \circ Rj_* = Rj'_* \circ a'$ where a and a' are the twisted inverse images corresponding to f and $f|_U$. Let $K \in D_{QCoh}(\mathcal{O}_Y)$. Then we get a map

$$(17.6.1) \qquad a(K)|_U \longrightarrow a(Rj_*K|_V)|_U = (Rj'_*a'(K|_V))|_U = a'(K|_V)$$

where the first arrow comes from the adjunction map $K \to Rj_*K|_V$.

Example 17.7. There is a finite morphsm $f : X \to Y$ of Noetherian schemes such that (17.6.1) is not an isomorphism for some $K \in D_{Coh}(\mathcal{O}_Y)$. Namely, let $X = \text{Spec}(B) \to Y = \text{Spec}(A)$ with $A = k[x, \epsilon]$ where k is a field and $\epsilon^2 = 0$ and $B = k[x] = A/(\epsilon)$. For $n \in \mathbf{N}$ set $M_n = A/(\epsilon, x^n)$. Observe that

$$\text{Ext}^i_A(B, M_n) = M_n, \quad i \geq 0$$

because B has the free periodic resolution $\ldots \to A \to A \to A$ with maps given by multiplication by ϵ. Consider the object $K = \bigoplus K_n[n] = \prod K_n[n]$ of $D_{Coh}(A)$ (equality in $D(A)$ by Derived Categories, Lemmas 31.2 and 32.2). Then we see that $a(K)$ correspnds to $R\,\text{Hom}(B, K)$ by Example 17.2 and

$$H^0(R\,\text{Hom}(B, K)) = \text{Ext}^0_A(B, K) = \prod\nolimits_{n \geq 1} \text{Ext}^n_A(B.M_n) = \prod\nolimits_{n \geq 1} M_n$$

by the above. But this module has elements which are not annihilated by any power of x, whereas the complex K does have every element of its cohomology annihilated by a power of x. In other words, for the map (17.6.1) with $V = D(x)$ and $U = D(x)$ and the complex K, the left hand side is nonzero and the right hand side is zero.

Lemma 17.8. *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be quasi-compact open with inverse image $U \subset X$. If for every $Q \in D^+_{QCoh}(\mathcal{O}_Y)$ supported on $Y \setminus V$ the twisted inverse image $a(Q)$ is supported on $X \setminus U$, then (17.6.1) is an isomorphism for all K in $D^+_{QCoh}(\mathcal{O}_Y)$.*

Proof. Choose a distinguished triangle

$$K \to Rj_*K|_V \to Q \to K[1]$$

Observe that Q is supported on $Y \setminus V$ (Derived Categories of Schemes, Definition 6.4). Applying the twisted inverse image a we obtain a distinguished triangle

$$a(K) \to a(Rj_*K|_V) \to a(Q) \to a(K)[1]$$

on X. If $a(Q)$ is supported on $X \setminus U$, then restricting to U the map $a(K)|_U \to a(Rj_*K|_V)|_U$ is an isomorphism, i.e., (17.6.1) is an isomorphism. □

Lemma 17.9. *Let $f : X \to Y$ be a proper[1] morphism of Noetherian schemes. The assumption and hence the conclusion of Lemma 17.8 holds for all opens V of Y.*

Proof. Let $Q \in D^+_{QCoh}(\mathcal{O}_Y)$ be supported on $Y \setminus V$. To get a contradiction, assume that $a(Q)$ is not supported on $X \setminus U$. Then we can find a perfect complex P_U on U and a nonzero map $P_U \to a(Q)|_U$ (follows from Derived Categories of Schemes, Theorem 13.3). Then using Derived Categories of Schemes, Lemma 11.9 we may assume there is a perfect complex P on X and a map $P \to a(Q)$ whose restriction

[1]This proof works for those morphisms of quasi-compact and quasi-separated schemes such that Rf_*P is pseudo-coherent for all P perfect on X. It follows easily from a theorem of Kiehl [Kie72] that this holds if f is proper and pseudo-coherent. This is the correct generality for this lemma and some of the other results in this section.

to U is nonzero. By definition of the twisted inverse image this map is adjoint to a map $Rf_*P \to Q$.

Because f is proper and X and Y Noetherian, the complex Rf_*P is pseudo-coherent, see Derived Categories of Schemes, Lemmas 5.1 and 9.4. Thus we may apply Derived Categories of Schemes, Lemma 14.3 and get a map $I \to \mathcal{O}_Y$ of perfect complexes whose restriction to V is an isomorphism such that the composition $I \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*P \to Rf_*P \to K$ is zero. By Derived Categories of Schemes, Lemma 16.1 we have $I \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*P = Rf_*(Lf^*I \otimes^{\mathbf{L}}_{\mathcal{O}_X} P)$. We conclude that the composition

$$Lf^*I \otimes^{\mathbf{L}}_{\mathcal{O}_X} P \to P \to a(K)$$

is zero. However, the restriction to U is the map $P|_U \to a(K)|_U$ which we assumed to be nonzero. This contradiction finishes the proof. $\square$

Lemma 17.10. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let a be the twisted inverse image. Then the canonical map*

$$Rf_*R\mathcal{H}om(L, a(K)) \longrightarrow R\mathcal{H}om(Rf_*L, K)$$

is an isomorphism for all $L \in D_{QCoh}(\mathcal{O}_X)$ and all $K \in D^+_{QCoh}(\mathcal{O}_Y)$.

Proof. Since a is the right adjoint to Rf_* there is a adjunction map $Rf_*a(K) \to K$. On the other hand, there is a canonical map

$$Rf_*R\mathcal{H}om(L, a(K)) \to R\mathcal{H}om(Rf_*L, Rf_*a(K))$$

which works on the level of complexes. Combining these we obtain the map of the lemma. Taking $H^0(V, -)$ for an open V of Y with inverse image U in X we get

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, a(K)|_U) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_V)}(Rf_*L|_V, K|_V)$$

Since we've shown above that $a(K)|_U$ is the twisted inverse image of $K|_V$ (Lemma 17.9) the two sides of this arrow are isomorphic. We omit the verification that the two maps agree. A similar argument works for $H^n(V, -)$. Thus the map defined above is an isomorphism on cohomology and hence an isomorphism in the derived category. $\square$

Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let a be the twisted inverse image (Lemma 17.1). There is a canonical map

$$Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y) \longrightarrow a(K) \tag{17.10.1}$$

functorial in K and compatible with distinguished triangles. Namely, this map is adjoint to a map

$$Rf_*(Lf^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} a(\mathcal{O}_Y)) = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Rf_*(a(\mathcal{O}_Y)) \longrightarrow K$$

(equality by Derived Categories of Schemes, Lemma 16.1) for which we use the adjunction map $Rf_*a(\mathcal{O}_Y) \to \mathcal{O}_Y$ and the indentity on K.

Lemma 17.11. *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. The map (17.10.1) is an isomorphism for every perfect object K of $D(\mathcal{O}_Y)$.*

Proof. For a perfect object K on Y and $L \in D_{QCoh}(\mathcal{O}_X)$ we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*L, K) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K^\wedge, \mathcal{O}_Y) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(L \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K^\wedge, a(\mathcal{O}_Y)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(L, a(\mathcal{O}_Y) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K) \end{aligned}$$

Hence the result by the Yoneda lemma. □

Lemma 17.12. *Let $i : Z \to X$ be a closed immersion of quasi-compact and quasi-separated schemes. Let $a : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Z)$ be the twisted inverse image, i.e., the right adjoint to Ri_*. Then there is a functorial isomorphism*

$$a(K) = R\mathcal{H}om(\mathcal{O}_Z, K)$$

for $K \in D^+_{QCoh}(\mathcal{O}_X)$.

Proof. By Lemma 11.2 the functor $R\mathcal{H}om(\mathcal{O}_Z, -)$ is a right adjoint to $Ri_* : D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$. Moreover, by Lemma 11.4 and Lemma 17.5 both $R\mathcal{H}om(\mathcal{O}_Z, -)$ and a map $D^+_{QCoh}(\mathcal{O}_X)$ into $D^+_{QCoh}(\mathcal{O}_Z)$. Hence we obtain the isomorphism by uniqueness of adjoint functors. □

Remark 17.13. The map (17.6.1) is a special case of a base change map. Namely, suppose that we have a diagram (17.5.1) where f and g are Tor independent. Let $K \in D_{QCoh}(\mathcal{O}_X)$. Then we can consider the composition

$$L(g')^*a(K) \to L(g')^*a(Rg_*Lg^*K) = L(g')^*Rg'_*a'(Lg^*K) \to a'(Lg^*K) \tag{17.13.1}$$

We need the assumption on Tor independence to get the equality sign (the canonical map goes the other way). The two arrows come from the adjunction maps $\mathrm{id} \to Rg_*Lg^*$ and $L(g')^*Rg'_* \to \mathrm{id}$. Alternatively, we can think of (17.5.1) by adjointness of $L(g')^*$ and $R(g')_*$ as the map

$$a(K) \to a(Rg_*Lg^*K) = Rg'_*a'(Lg^*K)$$

If $M \in D_{QCoh}(\mathcal{O}_X)$ then on Yoneda functors this map is given by

$$\begin{aligned} \mathrm{Hom}_X(M, a(K)) &= \mathrm{Hom}_Y(Rf_*M, K) \\ &\to \mathrm{Hom}_Y(Rf_*M, Rg_*Lg^*K) \\ &= \mathrm{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K) \\ &= \mathrm{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K) \\ &= \mathrm{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) \\ &= \mathrm{Hom}_X(M, Rg'_*a'(Lg^*K)) \end{aligned}$$

which makes things a little bit more explicit.

Lemma 17.14. *Let $A \to A'$ be a ring map. Let X be a quasi-compact and quasi-separated scheme over A. Let $h : X' = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A') \to X$ be the projection. Assume X and $\mathrm{Spec}(B)$ are Tor independent over $\mathrm{Spec}(A)$. We have*

$$\mathrm{Hom}_X(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} h_*\mathcal{O}_{X'}) = H^0(R\Gamma(X, R\mathcal{H}om(M, K)) \otimes^{\mathbf{L}}_A A')$$

in the following two cases

1. *$M \in D(\mathcal{O}_X)$ is perfect and $K \in D_{QCoh}(X)$, or*
2. *$M \in D(\mathcal{O}_X)$ is pseudo-coherent, $K \in D^+_{QCoh}(X)$, and A' has finite tor dimension over A.*

Proof. The complex $R\mathcal{H}om(M, K)$ is an object of $D_{QCoh}(\mathcal{O}_X)$ and we have

$$R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} h_*\mathcal{O}_{X'}) = R\mathcal{H}om(M, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} h_*\mathcal{O}_{X'}$$

in both cases (details omitted; hints: you can check this when X is affine and use Derived Categories of Schemes, Lemma 9.8 to identify the $R\mathcal{H}om$ complexes). Hence, by replacing K by $R\mathcal{H}om(M, K)$ we reduce to proving

$$H^0(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} h_*\mathcal{O}_{X'}) = H^0(R\Gamma(X, K) \otimes^{\mathbf{L}}_A A')$$

Note that the left hand side is equal to $H^0(X', Lh^*K)$ by Derived Categories of Schemes, Lemma 4.4. Hence the lemma is an example of by Derived Categories of Schemes, Lemma 16.3. $\square$

Lemma 17.15. *In diagram (17.5.1) assume*

1. *$g : Y' \to Y$ is a morphism of affine schemes,*
2. *$f : X \to Y$ is proper,*
3. *Y Noetherian, and*
4. *f or g is flat.*

Then the base change map (17.13.1) is an isomorphism for all $K \in D_{QCoh}(\mathcal{O}_X)$ if f is flat and for $K \in D^+_{QCoh}(\mathcal{O}_X)$ if g is flat.

Proof. Write $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(A')$. As a base change of an affine morphism, the morphism g' is affine. Hence Rg'_* reflects isomorphisms, see Derived Categories of Schemes, Lemma 4.3. Thus (17.13.1) is an isomorphism for $K \in D_{QCoh}(\mathcal{O}_X)$ if and only if the map $a(K) \to a(Rg_*Lg^*K) = Rg'_*a'(Lg^*K)$ induces an isomorphism

$$a(K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'} \to a(Rg_*Lg^*K)$$

(see Derived Categories of Schemes, Lemma 4.4). As $D_{QCoh}(\mathcal{O}_X)$ is generated by perfect objects (see Derived Categories of Schemes, Theorem 13.3), it suffices to check we obtain an isomorphism after applying the functor $\mathrm{Hom}_X(M, -)$ where M is perfect on X. On the left hand side we get

$$\begin{aligned}\mathrm{Hom}_X(M, a(K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'}) &= H^0(R\Gamma(X, R\mathcal{H}om(M, a(K))) \otimes^{\mathbf{L}}_A A') \\ &= H^0(R\Gamma(Y, R\mathcal{H}om(Rf_*M, K)) \otimes^{\mathbf{L}}_A A')\end{aligned}$$

The first equality by Lemma 17.14. Observe that $R\Gamma(X, R\mathcal{H}om(M, a(K)))$ is the complex of A-modules whose cohomology groups are $\mathrm{Hom}_X(M, a(K)[n])$ and similary for $R\Gamma(Y, R\mathcal{H}om(Rf_*M, K))$, see Cohomology, Lemma 34.1. Hence the second equality follows from the definition of a. In the case that f is flat the complex Rf_*M is perfect on Y (Derived Categories of Schemes, Lemma 17.1) and in general the complex Rf_*M is pseudo-coherent on Y (Derived Categories of Schemes, Lemmas 5.1 and 9.4). Thus we get on the right hand side

$$\begin{aligned}\mathrm{Hom}_X(M, a(Rg_*Lg^*K)) &= \mathrm{Hom}_Y(Rf_*M, Rg_*Lg^*K) \\ &= \mathrm{Hom}_Y(Rf_*M, K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g_*\mathcal{O}_{Y'}) \\ &= H^0(R\Gamma(Y, R\mathcal{H}om(Rf_*M, K)) \otimes^{\mathbf{L}}_A A')\end{aligned}$$

by the same arguments. Thus we get the same outcome as before. We omit the verification that our map induces the given identifications. $\square$

18. Flat and proper morphisms

The correct generality for this section would be to consider proper perfect morphisms of quasi-compact and quasi-separated schemes, see [LN07].

Lemma 18.1. *Let $f : X \to Y$ be a flat and proper morphism of Noetherian schemes. Let a be the twisted inverse image. Then a commutes with direct sums.*

Proof. Let P be a perfect object of $D(\mathcal{O}_X)$. By Derived Categories of Schemes, Lemma 17.1 the complex Rf_*P is perfect on Y. Let K_i be a family of objects of $D_{QCoh}(\mathcal{O}_Y)$. Then

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(K_i)) \end{aligned}$$

because a perfect object is compact (Derived Categories of Schemes, Proposition 14.1). Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Derived Categories of Schemes, Theorem 13.3) we conclude that the map $\bigoplus a(K_i) \to a(\bigoplus K_i)$ is an isomorphism, i.e., a commutes with direct sums. □

Lemma 18.2. *Let $f : X \to Y$ be a flat and proper morphism of Noetherian schemes. Let a be the twisted inverse image. Let $T \subset Y$ be closed. Then*

1. *if $Q \in D_{QCoh}(Y)$ is supported on T, then $a(Q)$ is supported on $f^{-1}(T)$,*
2. *the map (17.6.1) is an isomorphism for $K \in D_{QCoh}(\mathcal{O}_Y)$, and*
3. *the canonical map*

$$Rf_*R\mathcal{H}om(L, a(K)) \longrightarrow R\mathcal{H}om(Rf_*L, K)$$

is an isomorphism for all $L \in D_{QCoh}(\mathcal{O}_X)$ and all $K \in D_{QCoh}(\mathcal{O}_Y)$.

Proof. Arguing exactly as in the proof of Lemma 17.10 we see that (2) implies (3). Arguing exactly as in the proof of Lemma 17.8 we see that (1) implies (2).

Proof of (1). We will use the notation $D_{QCoh,T}(\mathcal{O}_Y)$ and $D_{QCoh,f^{-1}(T)}(\mathcal{O}_X)$ to denote complexes whose cohomology sheaves are supported on T and $f^{-1}(T)$. By Lemma 18.1 the functor a commutes with direct sums. Hence the strictly full, saturated, triangulated subcategory $\mathcal{D}$ with objects

$$\{Q \in D_{QCoh,T}(\mathcal{O}_Y) \mid a(Q) \in D_{QCoh,f^{-1}(T)}(\mathcal{O}_X)\}$$

is preserved by direct sums (and hence derived colimits). On the other hand, the category $D_{QCoh,T}(\mathcal{O}_Y)$ is generated by a perfect object E (see Derived Categories of Schemes, Lemma 13.5). By Lemma 17.9 we see that $E \in \mathcal{D}$. By Derived Categories, Lemma 34.3 every object Q of $D_{QCoh,T}(\mathcal{O}_Y)$ is a derived colimit of a system $Q_1 \to Q_2 \to Q_3 \to \ldots$ such that the cones of the transition maps are direct sums of shifts of E. Arguing by induction we see that $E_n \in \mathcal{D}$ for all n and finally that Q is in $\mathcal{D}$. Thus (1) is true. □

Lemma 18.3. *Let $f : X \to Y$ be a proper flat morphism of Noetherian schemes. The map (17.10.1) is an isomorphism for every object K of $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. By Lemma 18.1 we know that a commutes with direct sums. Hence the collection of objects of $D_{QCoh}(\mathcal{O}_Y)$ for which (17.10.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of $D_{QCoh}(\mathcal{O}_Y)$ which is moreover preserved under taking direct sums. Since $D_{QCoh}(\mathcal{O}_Y)$ is a module category (Derived Categories of Schemes, Theorem 15.3) generated by a single perfect object (Derived Categories of Schemes, Theorem 13.3) we can argue as in More on Algebra, Remark 45.11 to see that it suffices to prove (17.10.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 17.11. □

Lemma 18.4. *Let $f : X \to Y$ be a proper flat morphism of Noetherian schemes. Let $g : Y' \to Y$ be a morphism of finite type. Then the base change map (17.13.1) is an isomorphism for all $K \in D_{QCoh}(\mathcal{O}_X)$.*

Proof. By Lemma 18.2 formation of the functors a and a' commutes with restriction to opens of Y and Y'. Hence we may assume $Y' \to Y$ is a morphism of affine schemes. In this case the statement follows from Lemma 17.15. □

Lemma 18.5. *Let $f : X = \mathbf{P}^1_Y \to Y$ be the projection where Y is a Noetherian scheme. Let a be the twisted inverse image. Then $a(\mathcal{O}_Y)$ is isomorphic to $\mathcal{O}_X(-2)[1]$.*

Proof. Recall that there is an identification $Rf_*(\mathcal{O}_X(-2)[1]) = \mathcal{O}_Y$, see Cohomology of Schemes, Lemma 8.3 or 8.4. This determines a map $\mathcal{O}_X(-2)[1] \to a(\mathcal{O}_Y)$. By Lemma 17.9 construction of the twisted inverse image is local on the base. In particular, to check that $\mathcal{O}_X(-2)[1] \to a(\mathcal{O}_Y)$ is an isomorphism, we may work locally on Y. In other words, we may assume Y is affine. In the affine case the sheaves $\mathcal{O}_X$ and $\mathcal{O}_X(-1)$ generate $D_{QCoh}(X)$, see Derived Categories of Schemes, Lemma 13.4. Hence it suffices to show that the maps

$$\begin{aligned} H^{-n+1}(X, \mathcal{O}(-2)) &= \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[n], \mathcal{O}_X(-2)[1]) \\ &\to \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[n], a(\mathcal{O}_Y)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*(\mathcal{O}_X)[n], \mathcal{O}_Y) \\ &= H^{-n}(Y, \mathcal{O}_Y) \end{aligned}$$

and

$$\begin{aligned} H^{-n+1}(X, \mathcal{O}_X(-1)) &= \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(-1)[n], \mathcal{O}_X(-2)[1]) \\ &\to \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X(-1)[n], a(\mathcal{O}_Y)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*(\mathcal{O}_X(-1))[n], \mathcal{O}_Y) \\ &= 0 \end{aligned}$$

(where we used Cohomology of Schemes, Lemma 8.1) are isomorphisms for all $n \in \mathbf{Z}$. This is clear from the explicit computation of cohomology in Cohomology of Schemes, Lemma 8.1. □

Example 18.6. The base change map (17.13.1) is not an isomorphism if f is proper and perfect and g is perfect. Let k be a field. Let $Y = \mathbf{A}^2_k$ and let $f : X \to Y$ be the blow up of Y in the origin. Denote $E \subset X$ the exceptional divisor. Then we can factor f as

$$X \xrightarrow{i} \mathbf{P}^1_Y \xrightarrow{p} Y$$

This gives a factorization $a = c \circ b$ where b is the twisted inverse image for p and c is the twisted inverse image for i. Denote $\mathcal{O}(n)$ the Serre twist of the structure sheaf on $\mathbf{P}^1_Y$ and denote $\mathcal{O}_X(n)$ its restriction to X. Note that $X \subset \mathbf{P}^1_Y$ is cut out by a degree one equation, hence $\mathcal{O}(X) = \mathcal{O}(1)$. By Lemma 18.5 we have $b(\mathcal{O}_Y) = \mathcal{O}(-2)[1]$. By Lemma 17.12 we have

$$a(\mathcal{O}_Y) = c(b(\mathcal{O}_Y)) = c(\mathcal{O}(-2)[1]) = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}(-2)[1]) = \mathcal{O}_X(-1)$$

Last equality by Lemma 11.6. Hence the restriction of $a(\mathcal{O}_Y)$ to $E = \mathbf{P}^1_k$ is an invertible sheaf of degree -1 placed in cohomological degree 0. But on the other hand, $a'(\mathcal{O}_{\mathrm{Spec}(k)}) = \mathcal{O}_E(-2)[1]$ which is an invertible sheaf of degree -2 placed in cohomological degree -1, so different.

19. Upper shriek functors

In this section all schemes are Noetherian (but we will make sure all the assumptions are mentioned explicitly in the statements).

Let S be a Noetherian scheme. We will say a scheme X over S *has a compactification over S* if there exists an open immersion $X \to \overline{X}$ into a scheme $\overline{X}$ proper over S. If X has a compactification over S, then $X \to S$ is separated and of finite type. It is a theorem of Nagata (see [Con07], [Nag56], [Nag57], [Nag62], and [Nag63]) that the converse is true as well (we will give a precise statement and a proof if we ever need this result).

Lemma 19.1. *Let S be a Noetherian scheme. Let X be a scheme over S which has a compactification over S.*

1. *Any two compactifications of X/S can be dominated by a third.*
2. *If $X \to Y \to S$ is a factorization with $Y \to S$ of separated of finite type, then X has a compactification over Y.*

Proof. Omitted. □

Given a morphism $f : X \to Y$ of compactifyable schemes over a Noetherian base scheme S, we will define an exact functor

$$f^! : D^+_{QCoh}(\mathcal{O}_Y) \to D^+_{QCoh}(\mathcal{O}_X)$$

of triangulated categories. Namely, we choose a compactification $X \to \overline{X}$ over Y which is possible by Lemma 19.1. Denote $\overline{f} : \overline{X} \to Y$ the structure morphism. We let $\overline{a} : D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{\overline{X}})$ be the twisted inverse image, i.e., the right adjoint of $R\overline{f}_*$ constructed in Lemma 17.1. Then we set

$$f^!K = \overline{a}(K)|_X$$

for $K \in D^+_{QCoh}(\mathcal{O}_Y)$. The result is an object of $D^+_{QCoh}(\mathcal{O}_X)$ by Lemma 17.5.

Lemma 19.2. *The functor $f^!$ is independent of the choice of the compactification (up to canonical isomorphism). If f is an open immersion, then $f^! = f^*$. Moreover, if $f : X \to Y$, $g : Y \to Z$ are morphisms of compactifyable schemes over S, then there is a canonical isomorphism $f^! \circ g^! = (g \circ f)^!$.*

Proof. We first prove the last statement. Choose a compactification $Y \to \overline{Y} \to Z$ over Z and then choose a compactification $X \to \overline{X} \to \overline{Y}$ over $\overline{Y}$ using Lemma 19.1. Let $\overline{a}$ be the twisted inverse image for $\overline{X} \to \overline{Y}$ and let $\overline{b}$ be the twisted inverse image for $\overline{Y} \to Z$. Then $\overline{a} \circ \overline{b}$ is the twisted inverse image for the composition $\overline{X} \to Z$. Let

$\overline{a}'$ be the twisted inverse image for $\overline{X} \times_{\overline{Y}} Y \to Y$. Let K be an object of $D^+_{QCoh}(\mathcal{O}_Z$. To prove the statement on compositions it suffices to find a functorial isomorphism between

$$\overline{a}'(\overline{b}(K)|_Y) \quad \text{and} \quad \overline{a}(\overline{b}(K))|_{\overline{X} \times_{\overline{Y}} Y}$$

in $D(\mathcal{O}_{\overline{X}})$. The canonical map (17.6.1) from left to right is an isomorphism by Lemma 17.9.

Independence of the choice of the compactification is a special case of the argument above where $X \to Y$ is an isomorphism. The statement on open immersions is immediate from the construction (once it is shown to be independent of choices). $\square$

Lemma 19.3. *Let S be a Noetherian scheme. Let Y be a compactifyable scheme over S and let $f : X = \mathbf{A}^1_Y \to Y$ be the projection. Then there is a (noncanonical) isomorphism $f^!(-) \cong Lf^*(-)[1]$ of functors.*

Proof. Since $X = \mathbf{A}^1_Y \subset \mathbf{P}^1_Y$ and since $\mathcal{O}_{\mathbf{P}^1_Y}(-2)|_X \cong \mathcal{O}_X$ this follows from Lemmas 18.5 and 18.3. $\square$

Lemma 19.4. *Let S be a Noetherian scheme. Let Y be a compactifyable scheme over S and let $i : X \to Y$ be a closed immersion. Then there is a canonical isomorphism $i^!(-) = R\mathcal{H}om(\mathcal{O}_X, -)$ of functors.*

Proof. This is a restatement of Lemma 17.12. $\square$

Lemma 19.5. *Let S be a Noetherian scheme. Let $f : X \to Y$ be a morphism of compactifyable schemes over S. If K is a dualizing complex for Y, then $f^!K$ is a dualizing complex for X.*

Proof. The question is local on X hence we may assume that X and Y are affine schemes mapping into an affine open of S. In this case we can factor $f : X \to Y$ as

$$X \xrightarrow{i} \mathbf{A}^n_Y \to \mathbf{A}^{n-1}_Y \to \ldots \to \mathbf{A}^1_Y \to Y$$

where i is a closed immersion. By Lemmas 19.3 and 12.9 and induction we see that the $p^!K$ is a dualizing complex on $\mathbf{A}^n_Y$ where $p : \mathbf{A}^n_Y \to Y$ is the projection. Similarly, by Lemmas 12.8, 11.4, and 19.4 we see that $i^!$ transforms dualizing complexes into dualizing complexes. $\square$

20. Other chapters

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