DIVISORS

Contents

1. Introduction

In this chapter we study some very basic questions related to defining divisors, etc. A basic reference is [DG67].

2. Associated points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is *associated* to M if there exists an element of M whose annihilator is $\mathfrak{p}$. See Algebra, Definition 62.1. Here is the definition of associated points for quasi-coherent sheaves on schemes as given in [DG67, IV Definition 3.1.1].

Definition 2.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) We say $x \in X$ is *associated* to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote $\mathrm{Ass}(\mathcal{F})$ or $\mathrm{Ass}_X(\mathcal{F})$ the set of associated points of $\mathcal{F}$.

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(3) The *associated points of* X are the associated points of $\mathcal{O}_X$.

These definitions are most useful when X is locally Noetherian and $\mathcal{F}$ of finite type. For example it may happen that a generic point of an irreducible component of X is not associated to X, see Example 2.7. In the non-Noetherian case it may be more convenient to use weakly associated points, see Section 5. Let us link the scheme theoretic notion with the algebraic notion on affine opens; note that this correspondence works perfectly only for locally Noetherian schemes.

Lemma 2.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime.*

(1) *If $\mathfrak{p}$ is associated to M, then x is associated to $\mathcal{F}$.*
(2) *If $\mathfrak{p}$ is finitely generated, then the converse holds as well.*

In particular, if X is locally Noetherian, then the equivalence

$$\mathfrak{p} \in Ass(M) \Leftrightarrow x \in Ass(\mathcal{F})$$

holds for all pairs $(\mathfrak{p}, x)$ as above.

Proof. This follows from Algebra, Lemma 62.14. But we can also argue directly as follows. Suppose $\mathfrak{p}$ is associated to M. Then there exists an $m \in M$ whose annihilator is $\mathfrak{p}$. Since localization is exact we see that $\mathfrak{p}A_{\mathfrak{p}}$ is the annihilator of $m/1 \in M_{\mathfrak{p}}$. Since $M_{\mathfrak{p}} = \mathcal{F}_x$ (Schemes, Lemma 5.4) we conclude that x is associated to $\mathcal{F}$.

Conversely, assume that x is associated to $\mathcal{F}$, and $\mathfrak{p}$ is finitely generated. As x is associated to $\mathcal{F}$ there exists an element $m' \in M_{\mathfrak{p}}$ whose annihilator is $\mathfrak{p}A_{\mathfrak{p}}$. Write $m' = m/f$ for some $f \in A$, $f \notin \mathfrak{p}$. The annihilator I of m is an ideal of A such that $IA_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. Hence $I \subset \mathfrak{p}$, and $(\mathfrak{p}/I)_{\mathfrak{p}} = 0$. Since $\mathfrak{p}$ is finitely generated, there exists a $g \in A$, $g \notin \mathfrak{p}$ such that $g(\mathfrak{p}/I) = 0$. Hence the annihilator of gm is $\mathfrak{p}$ and we win.

If X is locally Noetherian, then A is Noetherian (Properties, Lemma 5.2) and $\mathfrak{p}$ is always finitely generated. $\square$

Lemma 2.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \subset Supp(\mathcal{F})$.*

Proof. This is immediate from the definitions. $\square$

Lemma 2.4. *Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $Ass(\mathcal{F}_2) \subset Ass(\mathcal{F}_1) \cup Ass(\mathcal{F}_3)$ and $Ass(\mathcal{F}_1) \subset Ass(\mathcal{F}_2)$.*

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 62.3. $\square$

Lemma 2.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \cap U$ is finite for every quasi-compact open $U \subset X$.*

Proof. This is true because the set of associated primes of a finite module over a Noetherian ring is finite, see Algebra, Lemma 62.5. To translate from schemes to algebra use that U is a finite union of affine opens, each of these opens is the spectrum of a Noetherian ring (Properties, Lemma 5.2), $\mathcal{F}$ corresponds to a finite module over this ring (Cohomology of Schemes, Lemma 9.1), and finally use Lemma 2.2. $\square$

Lemma 2.6. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = 0 \Leftrightarrow Ass(\mathcal{F}) = \emptyset.$$

Proof. If $\mathcal{F} = 0$, then $\text{Ass}(\mathcal{F}) = \emptyset$ by definition. Conversely, if $\text{Ass}(\mathcal{F}) = \emptyset$, then $\mathcal{F} = 0$ by Algebra, Lemma 62.7. To translate from schemes to algebra, restrict to any affine and use Lemma 2.2. $\square$

Example 2.7. Let k be a field. The ring $R = R[x_1, x_2, x_3, \ldots]/(x_i^2)$ is local with locally nilpotent maximal ideal $\mathfrak{m}$. There exists no element of R which has annihilator $\mathfrak{m}$. Hence $\text{Ass}(R) = \emptyset$, and $X = \text{Spec}(R)$ is an example of a scheme which has no associated points.

Lemma 2.8. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in Supp(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $Supp(\mathcal{F})$. Then $x \in Ass(\mathcal{F})$. In particular, any generic point of an irreducible component of X is an associated point of X.*

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\text{Ass}(\mathcal{F}_x) \subset \text{Spec}(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 62.7. On the other hand, by assumption $\text{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\text{Ass}(\mathcal{F}_x) \subset \text{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 62.2) we see that $\mathfrak{m}_x$ is associated to $\mathcal{F}_x$ and we win. $\square$

3. Morphisms and associated points

Lemma 3.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have*

$$Ass_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \supset \bigcup\nolimits_{s \in Ass_S(\mathcal{G})} Ass_{X_s}(\mathcal{F}_s)$$

and equality holds if S is locally Noetherian.

Proof. Let $x \in X$ and let $s = f(x) \in S$. Set $B = \mathcal{O}_{X,x}$, $A = \mathcal{O}_{S,s}$, $N = \mathcal{F}_x$, and $M = \mathcal{G}_s$. Note that the stalk of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ at x is equal to the B-module $M \otimes_A N$. Hence $x \in \text{Ass}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if $\mathfrak{m}_B$ is in $\text{Ass}_B(M \otimes_A N)$. Similarly $s \in \text{Ass}_S(\mathcal{G})$ and $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $\mathfrak{m}_A \in \text{Ass}_A(M)$ and $\mathfrak{m}_B/\mathfrak{m}_A B \in \text{Ass}_{B \otimes \kappa(\mathfrak{m}_A)}(N \otimes \kappa(\mathfrak{m}_A))$. Thus the lemma follows from Algebra, Lemma 64.5. $\square$

4. Embedded points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is an *embedded associated* to M if it is an associated prime of M which is not minimal among the associated primes of M. See Algebra, Definition 66.1. Here is the definition of embedded associated points for quasi-coherent sheaves on schemes as given in [DG67, IV Definition 3.1.1].

Definition 4.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) An *embedded associated point* of $\mathcal{F}$ is an associated point which is not maximal among the associated points of $\mathcal{F}$, i.e., it is the specialization of another associated point of $\mathcal{F}$.

(2) A point x of X is called an *embedded point* if x is an embedded associated point of $\mathcal{O}_X$.
(3) An *embedded component* of X is an irreducible closed subset $Z = \overline{\{x\}}$ where x is an embedded point of X.

In the Noetherian case when $\mathcal{F}$ is coherent we have the following.

Lemma 4.2. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then*

(1) *the generic points of irreducible components of Supp$(\mathcal{F})$ are associated points of $\mathcal{F}$, and*
(2) *an associated point of $\mathcal{F}$ is embedded if and only if it is not a generic point of an irreducible component of Supp$(\mathcal{F})$.*

In particular an embedded point of X is an associated point of X which is not a generic point of an irreducible component of X.

Proof. Recall that in this case $Z = \text{Supp}(\mathcal{F})$ is closed, see Morphisms, Lemma 5.3 and that the generic points of irreducible components of Z are associated points of $\mathcal{F}$, see Lemma 2.8. Finally, we have $\text{Ass}(\mathcal{F}) \subset Z$, by Lemma 2.3. These results, combined with the fact that Z is a sober topological space and hence every point of Z is a specialization of a generic point of Z, imply (1) and (2). □

Lemma 4.3. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Then the following are equivalent:*

(1) *$\mathcal{F}$ has no embedded associated points, and*
(2) *$\mathcal{F}$ has property (S_1).*

Proof. This is Algebra, Lemma 146.2, combined with Lemma 2.2 above. □

Lemma 4.4. *Let X be a locally Noetherian scheme. Let $U \subset X$ be an open subscheme. The following are equivalent*

(1) *U is scheme theoretically dense in X (Morphisms, Definition 7.1),*
(2) *U is dense in X and U contains all embedded points of X.*

Proof. The question is local on X, hence we may assume that $X = \text{Spec}(A)$ where A is a Noetherian ring. Then U is quasi-compact (Properties, Lemma 5.3) hence $U = D(f_1) \cup \ldots \cup D(f_n)$ (Algebra, Lemma 28.1). In this situation U is scheme theoretically dense in X if and only if $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective, see Morphisms, Example 7.4. Condition (2) translated into algebra means that for every associated prime $\mathfrak{p}$ of A there exists an i with $f_i \not\in \mathfrak{p}$.

Assume (1), i.e., $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective. If $x \in A$ has annihilator a prime $\mathfrak{p}$, then x maps to a nonzero element of A_{f_i} for some i and hence $f_i \not\in \mathfrak{p}$. Thus (2) holds. Assume (2), i.e., every associated prime $\mathfrak{p}$ of A corresponds to a prime of A_{f_i} for some i. Then $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective because $A \to \prod_{\mathfrak{p} \in \text{Ass}(A)} A_{\mathfrak{p}}$ is injective by Algebra, Lemma 62.18. □

Lemma 4.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. The set of coherent subsheaves*

$$\{\mathcal{K} \subset \mathcal{F} \mid \textit{Supp}(\mathcal{K}) \textit{ is nowhere dense in } \textit{Supp}(\mathcal{F})\}$$

has a maximal element $\mathcal{K}$. Setting $\mathcal{F}' = \mathcal{F}/\mathcal{K}$ we have the following

(1) *Supp$(\mathcal{F}') = $ Supp$(\mathcal{F})$,*

(2) *$\mathcal{F}'$ has no embedded associated points, and*
(3) *there exists a dense open $U \subset X$ such that $U \cap Supp(\mathcal{F})$ is dense in $Supp(\mathcal{F})$ and $\mathcal{F}'|_U \cong \mathcal{F}|_U$.*

Proof. This follows from Algebra, Lemmas 66.2 and 66.3. Note that U can be taken as the complement of the closure of the set of embedded associated points of $\mathcal{F}$. $\square$

Lemma 4.6. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module without embedded associated points. Set*

$$\mathcal{I} = Ker(\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})).$$

This is a coherent sheaf of ideals which defines a closed subscheme $Z \subset X$ without embedded points. Moreover there exists a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{F} = (Z \to X)_*\mathcal{G}$, (b) $\mathcal{G}$ has no associated embedded points, and (c) $Supp(\mathcal{G}) = Z$ (as sets).

Proof. Some of the statements we have seen in the proof of Cohomology of Schemes, Lemma 9.7. The others follow from Algebra, Lemma 66.4. $\square$

5. Weakly associated points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is *weakly associated* to M if there exists an element m of M such that $\mathfrak{p}$ is minimal among the primes containing the annihilator of m. See Algebra, Definition 65.1. If R is a local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is associated to M if and only if there exists an element $m \in M$ whose annihilator has radical $\mathfrak{m}$, see Algebra, Lemma 65.2.

Definition 5.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.
(1) We say $x \in X$ is *weakly associated* to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is weakly associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote $\mathrm{WeakAss}(\mathcal{F})$ the set of weakly associated points of $\mathcal{F}$.
(3) The *weakly associated points of* X are the weakly associated points of $\mathcal{O}_X$.

In this case, on any affine open, this corresponds exactly to the weakly associated primes as defined above. Here is the precise statement.

Lemma 5.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent*
(1) *$\mathfrak{p}$ is weakly associated to M, and*
(2) *x is weakly associated to $\mathcal{F}$.*

Proof. This follows from Algebra, Lemma 65.2. $\square$

Lemma 5.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$Ass(\mathcal{F}) \subset WeakAss(\mathcal{F}) \subset Supp(\mathcal{F}).$$

Proof. This is immediate from the definitions. $\square$

Lemma 5.4. *Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $WeakAss(\mathcal{F}_2) \subset WeakAss(\mathcal{F}_1) \cup WeakAss(\mathcal{F}_3)$ and $WeakAss(\mathcal{F}_1) \subset WeakAss(\mathcal{F}_2)$.*

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 65.3. $\square$

Lemma 5.5. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = (0) \Leftrightarrow \mathit{WeakAss}(\mathcal{F}) = \emptyset$$

Proof. Follows from Lemma 5.2 and Algebra, Lemma 65.4 $\square$

Lemma 5.6. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in \mathit{Supp}(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $\mathit{Supp}(\mathcal{F})$. Then $x \in \mathit{WeakAss}(\mathcal{F})$. In particular, any generic point of an irreducible component of X is weakly associated to $\mathcal{O}_X$.*

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\text{WeakAss}(\mathcal{F}_x) \subset \text{Spec}(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 65.4. On the other hand, by assumption $\text{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\text{WeakAss}(\mathcal{F}_x) \subset \text{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 65.5) we see that $\mathfrak{m}_x$ is weakly associated to $\mathcal{F}_x$ and we win. $\square$

Lemma 5.7. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathfrak{m}_x$ is a finitely generated ideal of $\mathcal{O}_{X,x}$, then*

$$x \in \mathit{Ass}(\mathcal{F}) \Leftrightarrow x \in \mathit{WeakAss}(\mathcal{F}).$$

In particular, if X is locally Noetherian, then $\mathit{Ass}(\mathcal{F}) = \mathit{WeakAss}(\mathcal{F})$.

Proof. See Algebra, Lemma 65.8. $\square$

6. Morphisms and weakly associated points

Lemma 6.1. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we have*

$$\mathit{WeakAss}_S(f_*\mathcal{F}) \subset f(\mathit{WeakAss}_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. Then $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 65.10. $\square$

Lemma 6.2. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is locally Noetherian, then we have*

$$f(\mathit{Ass}_X(\mathcal{F})) = \mathit{Ass}_S(f_*\mathcal{F}) = \mathit{WeakAss}_S(f_*\mathcal{F}) = f(\mathit{WeakAss}_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. As X is locally Noetherian the ring B is Noetherian, see Properties, Lemma 5.2. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 2.2 the associated points of $\mathcal{F}$ correspond exactly to the associated primes of M, and any associated prime of M as an A-module is an associated points of $f_*\mathcal{F}$. Hence the inclusion

$$f(\text{Ass}_X(\mathcal{F})) \subset \text{Ass}_S(f_*\mathcal{F})$$

follows from Algebra, Lemma 62.12. We have the inclusion

$$\text{Ass}_S(f_*\mathcal{F}) \subset \text{WeakAss}_S(f_*\mathcal{F})$$

by Lemma 5.3. We have the inclusion

$$\mathrm{WeakAss}_S(f_*\mathcal{F}) \subset f(\mathrm{WeakAss}_X(\mathcal{F}))$$

by Lemma 6.1. The outer sets are equal by Lemma 5.7 hence we have equality everywhere. □

Lemma 6.3. *Let $f : X \to S$ be a finite morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathit{WeakAss}(f_*\mathcal{F}) = f(\mathit{WeakAss}(\mathcal{F}))$.*

Proof. We may assume X and S affine, so $X \to S$ comes from a finite ring map $A \to B$. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 65.12. □

Lemma 6.4. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $x \in X$ with $s = f(x)$. If f is flat at x, the point x is a generic point of the fibre X_s, and $s \in \mathit{WeakAss}_S(\mathcal{G})$, then $x \in \mathit{WeakAss}(f^*\mathcal{G})$.*

Proof. Let $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$, and $M = \mathcal{G}_s$. Let $m \in M$ be an element whose annihilator $I = \{a \in A \mid am = 0\}$ has radical $\mathfrak{m}_A$. Then $m \otimes 1$ has annihilator IB as $A \to B$ is faithfully flat. Thus it suffices to see that $\sqrt{IB} = \mathfrak{m}_B$. This follows from the fact that the maximal ideal of $B/\mathfrak{m}_A B$ is locally nilpotent (see Algebra, Lemma 24.1) and the assumption that $\sqrt{I} = \mathfrak{m}_A$. Some details omitted. □

7. Relative assassin

Definition 7.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative assassin of $\mathcal{F}$ in X over S* is the set

$$\mathrm{Ass}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \mathrm{Ass}_{X_s}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

Again there is a caveat that this is best used when the fibres of f are locally Noetherian and $\mathcal{F}$ is of finite type. In the general case we should probably use the relative weak assassin (defined in the next section).

Lemma 7.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $g : S' \to S$ be a morphism of schemes. Consider the base change diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

and set $\mathcal{F}' = (g')^*\mathcal{F}$. Let $x' \in X'$ be a point with images $x \in X$, $s' \in S'$ and $s \in S$. Assume f locally of finite type. Then $x' \in \mathit{Ass}_{X'/S'}(\mathcal{F}')$ if and only if $x \in \mathit{Ass}_{X/S}(\mathcal{F})$ and x' corresponds to a generic point of an irreducible component of $\mathrm{Spec}(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$.

Proof. Consider the morphism $X'_{s'} \to X_s$ of fibres. As $X_{s'} = X_s \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(s'))$ this is a flat morphism. Moreover $\mathcal{F}'_{s'}$ is the pullback of $\mathcal{F}_s$ via this morphism. As X_s is locally of finite type over the Noetherian scheme $\mathrm{Spec}(\kappa(s))$

we have that X_s is locally Noetherian, see Morphisms, Lemma 16.6. Thus we may apply Lemma 3.1 and we see that

$$\mathrm{Ass}_{X'_{s'}}(\mathcal{F}'_{s'}) = \bigcup\nolimits_{x \in \mathrm{Ass}(\mathcal{F}_s)} \mathrm{Ass}((X'_{s'})_x).$$

Thus to prove the lemma it suffices to show that the associated points of the fibre $(X'_{s'})_x$ of the morphism $X'_{s'} \to X_s$ over x are its generic points. Note that $(X'_{s'})_x = \mathrm{Spec}(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$ as schemes. By Algebra, Lemma 155.1 the ring $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ is a Noetherian Cohen-Macaulay ring. Hence its associated primes are its minimal primes, see Algebra, Proposition 62.6 (minimal primes are associated) and Algebra, Lemma 146.2 (no embedded primes). □

Remark 7.3. With notation and assumptions as in Lemma 7.2 we see that it is always the case that $(g')^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) \supset \mathrm{Ass}_{X'/S'}(\mathcal{F}')$. If the morphism $S' \to S$ is locally quasi-finite, then we actually have

$$(g')^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) = \mathrm{Ass}_{X'/S'}(\mathcal{F}')$$

because in this case the field extensions $\kappa(s) \subset \kappa(s')$ are always finite. In fact, this holds more generally for any morphism $g : S' \to S$ such that all the field extensions $\kappa(s) \subset \kappa(s')$ are algebraic, because in this case all prime ideals of $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ are maximal (and minimal) primes, see Algebra, Lemma 35.17.

8. Relative weak assassin

Definition 8.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative weak assassin of $\mathcal{F}$ in X over S* is the set

$$\mathrm{WeakAss}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \mathrm{WeakAss}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

Lemma 8.2. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathit{WeakAss}_{X/S}(\mathcal{F}) = \mathit{Ass}_{X/S}(\mathcal{F})$.*

Proof. This is true bacause the fibres of f are locally Noetherian schemes, and associated and weakly associated points agree on locally Noetherian schemes, see Lemma 5.7. □

9. Effective Cartier divisors

For some reason it seem convenient to define the notion of an effective Cartier divisor before anything else.

Definition 9.1. Let S be a scheme.

(1) A *locally principal closed subscheme* of S is a closed subscheme whose sheaf of ideals is locally generated by a single element.
(2) An *effective Cartier divisor* on S is a closed subscheme $D \subset S$ such that the ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

Thus an effective Cartier divisor is a locally principal closed subscheme, but the converse is not always true. Effective Cartier divisors are closed subschemes of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is not a zerodivisor. In particular they are nowhere dense.

Lemma 9.2. *Let S be a scheme. Let $D \subset S$ be a closed subscheme. The following are equivalent:*

1. *The subscheme D is an effective Cartier divisor on S.*
2. *For every $x \in D$ there exists an affine open neighbourhood $\text{Spec}(A) = U \subset S$ of x such that $U \cap D = \text{Spec}(A/(f))$ with $f \in A$ not a zerodivisor.*

Proof. Assume (1). For every $x \in D$ there exists an affine open neighbourhood $\text{Spec}(A) = U \subset S$ of x such that $\mathcal{I}_D|_U \cong \mathcal{O}_U$. In other words, there exists a section $f \in \Gamma(U, \mathcal{I}_D)$ which freely generates the restriction $\mathcal{I}_D|_U$. Hence $f \in A$, and the multiplication map $f : A \to A$ is injective. Also, since $\mathcal{I}_D$ is quasi-coherent we see that $D \cap U = \text{Spec}(A/(f))$.

Assume (2). Let $x \in D$. By assumption there exists an affine open neighbourhood $\text{Spec}(A) = U \subset S$ of x such that $U \cap D = \text{Spec}(A/(f))$ with $f \in A$ not a zerodivisor. Then $\mathcal{I}_D|_U \cong \mathcal{O}_U$ since it is equal to $\widetilde{(f)} \cong \widetilde{A} \cong \mathcal{O}_U$. Of course $\mathcal{I}_D$ restricted to the open subscheme $S \setminus D$ is isomorphic to $\mathcal{O}_{S \setminus D}$. Hence $\mathcal{I}_D$ is an invertible $\mathcal{O}_S$-module. □

Lemma 9.3. *Let S be a scheme. Let $Z \subset S$ be a locally principal closed subscheme. Let $U = S \setminus Z$. Then $U \to S$ is an affine morphism.*

Proof. The question is local on S, see Morphisms, Lemmas 13.3. Thus we may assume $S = \text{Spec}(A)$ and $Z = V(f)$ for some $f \in A$. In this case $U = D(f) = \text{Spec}(A_f)$ is affine hence $U \to S$ is affine. □

Lemma 9.4. *Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $U = S \setminus D$. Then $U \to S$ is an affine morphism and U is scheme theoretically dense in S.*

Proof. Affineness is Lemma 9.3. The density question is local on S, see Morphisms, Lemma 7.5. Thus we may assume $S = \text{Spec}(A)$ and D corresponding to the nonzerodivisor $f \in A$, see Lemma 9.2. Thus $A \subset A_f$ which implies that $U \subset S$ is scheme theoretically dense, see Morphisms, Example 7.4. □

Lemma 9.5. *Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $s \in D$. If $\dim_s(S) < \infty$, then $\dim_s(D) < \dim_s(S)$.*

Proof. Assume $\dim_s(S) < \infty$. Let $U = \text{Spec}(A) \subset S$ be an affine open neighbourhood of s such that $\dim(U) = \dim_s(S)$ and such that $D = V(f)$ for some nonzerodivisor $f \in A$ (see Lemma 9.2). Recall that $\dim(U)$ is the Krull dimension of the ring A and that $\dim(U \cap D)$ is the Krull dimension of the ring $A/(f)$. Then f is not contained in any minimal prime of A. Hence any maximal chain of primes in $A/(f)$, viewed as a chain of primes in A, can be extended by adding a minimal prime. □

Definition 9.6. Let S be a scheme. Given effective Cartier divisors D_1, D_2 on S we set $D = D_1 + D_2$ equal to the closed subscheme of S corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the *sum of the effective Cartier divisors D_1 and D_2*.

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i.

Lemma 9.7. *The sum of two effective Cartier divisors is an effective Cartier divisor.*

Proof. Omitted. Locally $f_1, f_2 \in A$ are nonzerodivisors, then also $f_1 f_2 \in A$ is a nonzerodivisor. $\square$

Lemma 9.8. *Let X be a scheme. Let D, D' be two effective Cartier divisors on X. If $D \subset D'$ (as closed subschemes of X), then there exists an effective Cartier divisor D'' such that $D' = D + D''$.*

Proof. Omitted. $\square$

Lemma 9.9. *Let X be a scheme. Let Z, Y be two closed subschemes of X with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$. If $\mathcal{I}\mathcal{J}$ defines an effective Cartier divisor $D \subset X$, then Z and Y are effective Cartier divisors and $D = Z + Y$.*

Proof. Applying Lemma 9.2 we obtain the following algebra situation: A is a ring, $I, J \subset A$ ideals and $f \in A$ a nonzerodivisor such that $IJ = (f)$. Thus the result follows from Algebra, Lemma 116.12. $\square$

Recall that we have defined the inverse image of a closed subscheme under any morphism of schemes in Schemes, Definition 17.7.

Lemma 9.10. *Let $f : S' \to S$ be a morphism of schemes. Let $Z \subset S$ be a locally principal closed subscheme. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subscheme of S'.*

Proof. Omitted. $\square$

Definition 9.11. Let $f : S' \to S$ be a morphism of schemes. Let $D \subset S$ be an effective Cartier divisor. We say the *pullback of D by f is defined* if the closed subscheme $f^{-1}(D) \subset S'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the *pullback of the effective Cartier divisor*.

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice. Here is an example lemma.

Lemma 9.12. *Let $f : X \to Y$ be a morphism of schemes. Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:*

(1) *X, Y integral and f dominant,*
(2) *X reduced, and for any generic point ξ of any irreducible component of X we have $f(\xi) \notin D$,*
(3) *X is locally Noetherian and for any associated point x of X we have $f(x) \notin D$,*
(4) *X is locally Noetherian, has no embedded points, and for any generic point ξ of any irreducible component of X we have $f(\xi) \notin D$,*
(5) *f is flat, and*
(6) *add more here as needed.*

Proof. The question is local on X, and hence we reduce to the case where $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$, f is given by $\varphi : R \to A$ and $D = \mathrm{Spec}(R/(t))$ where $t \in R$ is not a zerodivisor. The goal in each case is to show that $\varphi(t) \in A$ is not a zerodivisor.

In case (2) this follows as the intersection of all minimal primes of a ring is the nilradical of the ring, see Algebra, Lemma 16.2.

Let us prove (3). By Lemma 2.2 the associated points of X correspond to the primes $\mathfrak{p} \in \text{Ass}(A)$. By Algebra, Lemma 62.9 we have $\bigcup_{\mathfrak{p} \in \text{Ass}(A)} \mathfrak{p}$ is the set of zerodivisors of A. The hypothesis of (3) is that $\varphi(t) \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}(A)$. Hence $\varphi(t)$ is a nonzerodivisor as desired.

Part (4) follows from (3) and the definitions. $\square$

Lemma 9.13. *Let $f : S' \to S$ be a morphism of schemes. Let D_1, D_2 be effective Cartier divisors on S. If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.*

Proof. Omitted. $\square$

Definition 9.14. Let S be a scheme and let D be an effective Cartier divisor. The *invertible sheaf $\mathcal{O}_S(D)$ associated to D* is given by

$$\mathcal{O}_S(D) := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{I}_D, \mathcal{O}_S) = \mathcal{I}_D^{\otimes -1}.$$

The canonical section, usually denoted 1 or 1_D, is the global section of $\mathcal{O}_S(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \to \mathcal{O}_S$.

Lemma 9.15. *Let S be a scheme. Let D_1, D_2 be effective Cartier divisors on S. Let $D = D_1 + D_2$. Then there is a unique isomorphism*

$$\mathcal{O}_S(D_1) \otimes_{\mathcal{O}_S} \mathcal{O}_S(D_2) \longrightarrow \mathcal{O}_S(D)$$

which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D.

Proof. Omitted. $\square$

Definition 9.16. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{L}$ be an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a *regular section* if the map $\mathcal{O}_X \to \mathcal{L}$, $f \mapsto fs$ is injective.

Lemma 9.17. *Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:*

(1) *f is a regular section, and*
(2) *for any $x \in X$ the image $f \in \mathcal{O}_{X,x}$ is not a zerodivisor.*

If X is a scheme these are also equivalent to

(3) *for any affine open $\text{Spec}(A) = U \subset X$ the image $f \in A$ is not a zerodivisor, and*
(4) *there exists an affine open covering $X = \bigcup \text{Spec}(A_i)$ such that the image of f in A_i is not a zerodivisor for all i.*

Proof. Omitted. $\square$

Note that a global section s of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ may be seen as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$. (See Modules, Definition 21.3 for the definition of the dual invertible sheaf.)

Definition 9.18. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$. The *zero scheme* of s is the closed subscheme $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

Lemma 9.19. *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$.*

(1) *Consider closed immersions $i : Z \to X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L}))$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this ordered set.*
(2) *For any morphism of schemes $f : Y \to X$ we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.*
(3) *The zero scheme $Z(s)$ is a locally principal closed subscheme.*
(4) *The zero scheme $Z(s)$ is an effective Cartier divisor if and only if s is a regular section of $\mathcal{L}$.*

Proof. Omitted. $\square$

Lemma 9.20. *Let X be a scheme.*

(1) *If $D \subset X$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.*
(2) *Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$, then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ which maps 1_D to s.*

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\left\{\textit{effective Cartier divisors on } X\right\} \leftrightarrow \left\{\begin{matrix} \textit{pairs } (\mathcal{L}, s) \textit{ consisting of an invertible} \\ \mathcal{O}_X\textit{-module and a regular global section} \end{matrix}\right\}$$

Proof. Omitted. $\square$

Lemma 9.21. *Let X be a Noetherian scheme. Let $D \subset X$ be a closed subscheme corresponding to the quasi-coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$.*

(1) *If for every $x \in D$ the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ can be generated by one element, then D is locally principal.*
(2) *If for every $x \in D$ the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ can be generated by a single nonzerodivisor, then D is an effective Cartier divisor.*

Proof. Let $\text{Spec}(A)$ be an affine neighbourhood of a point $x \in D$. Let $\mathfrak{p} \subset A$ be the prime corresponding to x. Let $I \subset A$ be the ideal defining the trace of D on $\text{Spec}(A)$. Since A is Noetherian (as X is Noetherian) the ideal I is generated by finitely many elements, say $I = (f_1, \ldots, f_r)$. Under the assumption of (1) we have $I_\mathfrak{p} = (f)$ for some $f \in A_\mathfrak{p}$. Then $f_i = g_i f$ for some $g_i \in A_\mathfrak{p}$. Write $g_i = a_i/h_i$ and $f = f'/h$ for some $h_i, h \in A$, $h_i, h \notin \mathfrak{p}$. Then $I_{h_1 \ldots h_r h} \subset A_{h_1 \ldots h_r h}$ is principal, because it is generated by f'. This proves (1). For (2) we may assume $I = (f)$. The assumption implies that the image of f in $A_\mathfrak{p}$ is a nonzerodivisor. Then f is a nonzero divisor on a neighbourhood of x by Algebra, Lemma 67.8. This proves (2). $\square$

Lemma 9.22. *Let X be a Noetherian scheme. Let $D \subset X$ be an integral closed subscheme which is also an effective Cartier divisor. Then the local ring of X at the generic point of D is a discrete valuation ring.*

Proof. By Lemma 9.2 we may assume $X = \text{Spec}(A)$ and $D = \text{Spec}(A/(f))$ where A is a Noetherian ring and $f \in A$ is a nonzerodivisor. The assumption that D is integral signifies that (f) is prime. Hence the local ring of X at the generic point is $A_{(f)}$ which is a Noetherian local ring whose maximal ideal is generated by a nonzerodivisor. Thus it is a discrete valuation ring by Algebra, Lemma 115.6. $\square$

Lemma 9.23. *Let X be a Noetherian scheme. Let $D \subset X$ be a integral closed subscheme. Assume that*

1. *D has codimension 1 in X, and*
2. *$\mathcal{O}_{X,x}$ is a UFD for all $x \in D$.*

Then D is an effective Cartier divisor.

Proof. Let $x \in D$ and set $A = \mathcal{O}_{X,x}$. Let $\mathfrak{p} \subset A$ correspond to the generic point of D. Then $A_\mathfrak{p}$ has dimension 1 by assumption (1). Thus $\mathfrak{p}$ is a prime ideal of height 1. Since A is a UFD this implies that $\mathfrak{p} = (f)$ for some $f \in A$. Of course f is a nonzerodivisor and we conclude by Lemma 9.21. $\square$

Lemma 9.24. *Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Assume at least one of the following conditions holds*

1. *There exist reduced and irreducible effective Cartier divisors $D_i \subset X$, $i = 1, \ldots, n$ and a closed subset $Z \subset X$ of codimension ≥ 2, such that $\mathcal{O}_X/\mathcal{I}$ is supported on $Z \cup \bigcup D_i$.*
2. *The local ring $\mathcal{O}_{X,x}$ is a UFD for every point x of the support of $\mathcal{O}_X/\mathcal{I}$.*
3. *The scheme X is regular.*

Then there exists an invertible ideal sheaf $\mathcal{I} \subset \mathcal{J} \subset \mathcal{O}_X$ such that the support of $\mathcal{J}/\mathcal{I}$ has codimension ≥ 2. Moreover, in case (1) we have $\mathcal{J}^{\otimes -1} = \mathcal{O}_X(\sum a_i D_i)$ for some $a_i \geq 0$.

Proof. Case (1). Let $\xi_i \in D_i$ be the generic point and let $\mathcal{O}_i = \mathcal{O}_{X,\xi_i}$ be the local ring which is a discrete valuation ring by Lemma 9.22. Let $a_i \geq 0$ be the minimal valuation of an element of $\mathcal{I}_{x_i} \subset \mathcal{O}_i$. We claim that the ideal sheaf $\mathcal{J}$ of the effective Cartier divisor $D = \sum a_i D_i$ works.

Namely, suppose that $x \in D$. Let $A = \mathcal{O}_{X,x}$. Let $f_i \in A$ be a local equation for D_i; we only consider those i such that $x \in D_i$. Then f_i is a nonzerodivisor and $A/(f_i)$ is a domain and $\mathcal{O}_i = A_{(f_i)}$. Let $I = \mathcal{I}_x \subset A$. We chose a_i such that $IA_{(f_i)} = f_i^{a_i} A_{(f_i)}$. It follows that $I \subset (\prod f_i^{a_i})$ because $(\prod f_i^{a_i})$ is the kernel of $A \to \prod A_{(f_i)}/f_i^{a_i} A_{(f_i)}$. This proves that $\mathcal{I} \subset \mathcal{J}$. Moreover, we also see that $\mathcal{I}_{x_i} = \mathcal{J}_{x_i}$ which proves that x_i is not in the support of $\mathcal{J}/\mathcal{I}$. Hence the support of $\mathcal{I}/\mathcal{J}$ has codimension at least 2. This finishes the proof in case (1).

Observe that (3) is a special case of (2) because a regular local ring is a UFD (More on Algebra, Lemma 70.4). In case (2) let D_i be the irreducible components of the support of $\mathcal{O}_X/\mathcal{I}$ which have codimension 1. By Lemma 9.23 each D_i is an effective Cartier divisor. In this way we reduce to case (1). $\square$

10. Relative effective Cartier divisors

The following lemma shows that an effective Cartier divisor which is flat over the base is really a "family of effective Cartier divisors" over the base. For example the restriction to any fibre is an effective Cartier divisor.

Lemma 10.1. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a closed subscheme. Assume*

1. *D is an effective Cartier divisor, and*
2. *$D \to S$ is a flat morphism.*

Then for every morphism of schemes $g : S' \to S$ the pullback $(g')^{-1}D$ is an effective Cartier divisor on $X' = S' \times_S X$.

Proof. Using Lemma 9.2 we translate this as follows into algebra. Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzerodivisor and that B/hB is flat over A. Then

$$0 \to B \xrightarrow{h} B \to B/hB \to 0$$

is a short exact sequence of A-modules with B/hB flat over A. By Algebra, Lemma 38.11 this sequence remains exact on tensoring over A with any module, in particular with any A-algebra A'. □

This lemma is the motivation for the following definition.

Definition 10.2. Let $f : X \to S$ be a morphism of schemes. A *relative effective Cartier divisor* on X/S is an effective Cartier divisor $D \subset X$ such that $D \to S$ is a flat morphism of schemes.

We warn the reader that this may be nonstandard notation. In particular, in [DG67, IV, Section 21.15] the notion of a relative divisor is discussed only when $X \to S$ is flat and locally of finite presentation. Our definition is a bit more general. However, it turns out that if $x \in D$ then $X \to S$ is flat at x in many cases (but not always).

Lemma 10.3. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If $x \in D$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.*

Proof. Set $A = \mathcal{O}_{S,f(x)}$ and $B = \mathcal{O}_{X,x}$. Let $h \in B$ be an element which generates the ideal of D. Then h is a nonzerodivisor in B such that B/hB is a flat local A-algebra. Let $I \subset A$ be a finitely generated ideal. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "h"'] & B \arrow[r] & B/hB \arrow[r] & 0 \\
0 \arrow[r] & B \otimes_A I \arrow[r, "h"] \arrow[u] & B \otimes_A I \arrow[r] \arrow[u] & B/hB \otimes_A I \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The lower sequence is short exact as B/hB is flat over A, see Algebra, Lemma 38.11. The right vertical arrow is injective as B/hB is flat over A, see Algebra, Lemma 38.4. Hence multiplication by h is surjective on the kernel K of the middle vertical arrow. By Nakayama's lemma, see Algebra, Lemma 19.1 we conclude that $K = 0$. Hence B is flat over A, see Algebra, Lemma 38.4. □

The following lemma relies on the algebraic version of openness of the flat locus. The scheme theoretic version can be found in More on Morphisms, Section 12.

Lemma 10.4. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor. If f is locally of finite presentation, then there exists an open subscheme $U \subset X$ such that $D \subset U$ and such that $f|_U : U \to S$ is flat.*

Proof. Pick $x \in D$. It suffices to find an open neighbourhood $U \subset X$ of x such that $f|_U$ is flat. Hence the lemma reduces to the case that $X = \text{Spec}(B)$ and $S = \text{Spec}(A)$ are affine and that D is given by a nonzerodivisor $h \in B$. By assumption B is a finitely presented A-algebra and B/hB is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume h is the image of $h' \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $h', g_1, \ldots, g_m$ are in A_0. Then we can set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and h_0 the image of h' in B_0. Then $B = B_0 \otimes_{A_0} A$ and $B/hB = B_0/h_0B_0 \otimes_{A_0} A$. By Algebra, Lemma 156.1 we may, after enlarging A_0, assume that B_0/h_0B_0 is flat over A_0. Let $K_0 = \text{Ker}(h_0 : B_0 \to B_0)$. As B_0 is of finite type over $\mathbf{Z}$ we see that K_0 is a finitely generated ideal. Let $A_1 \subset A$ be a finite type $\mathbf{Z}$-subalgebra containing A_0 and denote B_1, h_1, K_1 the corresponding objects over A_1. By More on Algebra, Lemma 21.15 the map $K_0 \otimes_{A_0} A_1 \to K_1$ is surjective. On the other hand, the kernel of $h : B \to B$ is zero by assumption. Hence every element of K_0 maps to zero in K_1 for sufficiently large subrings $A_1 \subset A$. Since K_0 is finitely generated, we conclude that $K_1 = 0$ for a suitable choice of A_1.

Set $f_1 : X_1 \to S_1$ equal to Spec of the ring map $A_1 \to B_1$. Set $D_1 = \text{Spec}(B_1/h_1B_1)$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X = X_1 \times_{S_1} S$, it now suffices to prove the lemma for $X_1 \to S_1$ and the relative effective Cartier divisor D_1, see Morphisms, Lemma 26.6. Hence we have reduced to the case where A is a Noetherian ring. In this case we know that the ring map $A \to B$ is flat at every prime $\mathfrak{q}$ of $V(h)$ by Lemma 10.3. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 125.4 we win. □

There is also the following lemma (whose idea is apparently due to Michael Artin, see [Nob77]) which needs no finiteness assumptions at all.

Lemma 10.5. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If f is flat at all points of $X \setminus D$, then f is flat.*

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzerodivisor, that B/hB is flat over A, and that the localization B_h is flat over A. Then B is flat over A. The reason is that we have a short exact sequence

$$0 \to B \to B_h \to \text{colim}_n (1/h^n)B/B \to 0$$

and that the second and third terms are flat over A, which implies that B is flat over A (see Algebra, Lemma 38.12). Note that a filtered colimit of flat modules is flat (see Algebra, Lemma 38.2) and that by induction on n each $(1/h^n)B/B \cong B/h^nB$ is flat over A since it fits into the short exact sequence

$$0 \to B/h^{n-1}B \xrightarrow{h} B/h^nB \to B/hB \to 0$$

Some details omitted. □

Example 10.6. Here is an example of a relative effective Cartier divisor D where the ambient scheme is not flat in a neighbourhood of D. Namely, let $A = k[t]$ and

$$B = k[t, x, y, x^{-1}y, x^{-2}y, \ldots]/(ty, tx^{-1}y, tx^{-2}y, \ldots)$$

Then B is not flat over A but $B/xB \cong A$ is flat over A. Moreover x is a nonzerodivisor and hence defines a relative effective Cartier divisor in $\text{Spec}(B)$ over $\text{Spec}(A)$.

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative effective Cartier divisor in terms of its fibres. See also More on Morphisms, Lemma 18.1 for a slightly different take on this lemma.

Lemma 10.7. *Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $Z \subset X$ be a closed subscheme. Let $x \in Z$ with image $s \in S$.*

1. *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, then there exists an open $U \subset X$ and a relative effective Cartier divisor $D \subset U$ such that $Z \cap U \subset D$.*
2. *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, the morphism $Z \to X$ is of finite presentation, and $Z \to S$ is flat at x, then we can choose U and D such that $Z \cap U = D$.*
3. *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x and Z is a locally principal closed subscheme of X in a neighbourhood of x, then we can choose U and D such that $Z \cap U = D$.*

In particular, if $Z \to S$ is locally of finite presentation and flat and all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor. Similarly, if Z is a locally principal closed subscheme of X such that all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor.

Proof. Choose affine open neighbourhoods $\mathrm{Spec}(A)$ of s and $\mathrm{Spec}(B)$ of x such that $\varphi(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to Z. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $\mathrm{Spec}(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a single element which is a nonzerodivisor in $B \otimes_A \kappa(\mathfrak{p})$. Say $f \in I$ maps to this generator. We claim that after inverting an element $g \in B$, $g \notin \mathfrak{q}$ the closed subscheme $D = V(f) \subset \mathrm{Spec}(B_g)$ is a relative effective Cartier divisor.

By Algebra, Lemma 156.1 we can find a flat finite type ring map $A_0 \to B_0$ of Noetherian rings, an element $f_0 \in B_0$, a ring map $A_0 \to A$ and an isomorphism $A \otimes_{A_0} B_0 \cong B$. If $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$ then we see that

$$B \otimes_A \kappa(\mathfrak{p}) = (B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)) \otimes_{\kappa(\mathfrak{p}_0))} \kappa(\mathfrak{p})$$

hence f_0 is a nonzerodivisor in $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$. By Algebra, Lemma 95.2 we see that f_0 is a nonzerodivisor in $(B_0)_{\mathfrak{q}_0}$ where $\mathfrak{q}_0 = B_0 \cap \mathfrak{q}$ and that $(B_0/f_0B_0)_{\mathfrak{q}_0}$ is flat over A_0. Hence by Algebra, Lemma 67.8 and Algebra, Theorem 125.4 there exists a $g_0 \in B_0$, $g_0 \notin \mathfrak{q}_0$ such that f_0 is a nonzerodivisor in $(B_0)_{g_0}$ and such that $(B_0/f_0B_0)_{g_0}$ is flat over A_0. Hence we see that $D_0 = V(f_0) \subset \mathrm{Spec}((B_0)_{g_0})$ is a relative effective Cartier divisor. Since we know that this property is preserved under base change, see Lemma 10.1, we obtain the claim mentioned above with g equal to the image of g_0 in B.

At this point we have proved (1). To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 124.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 122.6. To see (3), after possibly shrinking U we may assume that the ideal of D is generated by a single nonzerodivisor f and the ideal of Z is generated by an element g. Then $f = gh$. But $g|_{U_s}$ and $f|_{U_s}$ cut out the same effective Cartier divisor in a neighbourhood of x. Hence $h|_{X_s}$ is a unit in $\mathcal{O}_{X_s,x}$, hence h is a unit in $\mathcal{O}_{X,x}$ hence h is a unit in an open neighbourhood of x. I.e., $Z \cap U = D$ after shrinking U.

The final statements of the lemma follow immediately from parts (2) and (3), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 22.3 and 22.11. $\square$

11. The normal cone of an immersion

Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by $\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms, Lemma 4.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the *conormal algebra of Z in X* and is often simply denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms, Section 4.

Let $f : Z \to X$ be an immersion. We define the conormal algebra of f as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 11.1. Let $f : Z \to X$ be an immersion. The *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X* or the *conormal algebra of f* is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

$$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1} \qquad (11.1.1)$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

$$\mathrm{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*} \qquad (11.1.2)$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

Lemma 11.2. *Let $i : Z \to X$ be an immersion. The conormal algebra of i has the following properties:*

(1) *Let $U \subset X$ be any open such that $i(Z)$ is a closed subset of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals corresponding to the closed subscheme $i(Z) \subset U$. Then*
$$\mathcal{C}_{Z/X,*} = i^*\left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n\right) = i^{-1}\left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$
(2) *For any affine open $\mathrm{Spec}(R) = U \subset X$ such that $Z \cap U = \mathrm{Spec}(R/I)$ there is a canonical isomorphism $\Gamma(Z \cap U, \mathcal{C}_{Z/X,*}) = \bigoplus_{n \geq 0} I^n/I^{n+1}$.*

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I^n/I^{n+1} = I^n \otimes_R R/I$. Details omitted. $\square$

Lemma 11.3. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of graded $\mathcal{O}_Z$-algebras

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

characterized by the following property: For every pair of affine opens $(\mathrm{Spec}(R) = U \subset X, \mathrm{Spec}(R') = U' \subset X')$ *with* $f(U) \subset U'$ *such that* $Z \cap U = \mathrm{Spec}(R/I)$ *and* $Z' \cap U' = \mathrm{Spec}(R'/I')$ *the induced map*

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X',*}) = \bigoplus (I')^n/(I')^{n+1} \longrightarrow \bigoplus\nolimits_{n \geq 0} I^n/I^{n+1} = \Gamma(Z \cap U, \mathcal{C}_{Z/X,*})$$

is the one induced by the ring map $f^\sharp : R' \to R$ which has the property $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \big(g^{-1}(\partial Z') \cup \partial Z\big)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 4.6 and 4.7. Hence we get an induced map of quasi-coherent sheaves $g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back by i gives $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to i^*(\mathcal{I}^n/\mathcal{I}^{n+1})$. Note that $i^*(\mathcal{I}^n/\mathcal{I}^{n+1}) = \mathcal{C}_{Z/X,n}$. On the other hand, $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*(i')^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*\mathcal{C}_{Z'/X',n}$. This gives the desired map.

Checking that the map is locally described as the given map $(I')^n/(I')^{n+1} \to I^n/I^{n+1}$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). $\square$

Lemma 11.4. *Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram in the category of schemes with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ of Lemma 11.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $(I')^n/(I')^{n+1} \otimes_{R'} R \to I^n/I^{n+1}$ is surjective. If $R' \to R$ is flat, then $I^n = (I')^n \otimes_{R'} R$ and we see the map is an isomorphism. $\square$

Definition 11.5. Let $i : Z \to X$ be an immersion of schemes. The *normal cone* C_ZX of Z in X is

$$C_ZX = \underline{\mathrm{Spec}}_Z(\mathcal{C}_{Z/X,*})$$

see Constructions, Definitions 7.1 and 7.2. The *normal bundle* of Z in X is the vector bundle

$$N_ZX = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{C}_{Z/X}))$$

see Constructions, Definitions 6.1 and 6.2.

Thus $C_ZX \to Z$ is a cone over Z and $N_ZX \to Z$ is a vector bundle over Z (recall that in our terminology this does not imply that the conormal sheaf is a finite locally free sheaf). Moreover, the canonical surjection (11.1.2) of graded algebras defines a canonical closed immersion

$$C_ZX \longrightarrow N_ZX \tag{11.5.1}$$

of cones over Z.

12. Regular ideal sheaves

In this section we generalize the notion of an effective Cartier divisor to higher codimension. Recall that a sequence of elements $f_1, \ldots, f_r$ of a ring R is a *regular sequence* if for each $i = 1, \ldots, r$ the element f_i is a nonzerodivisor on $R/(f_1, \ldots, f_{i-1})$ and $R/(f_1, \ldots, f_r) \neq 0$, see Algebra, Definition 67.1. There are three closely related weaker conditions that we can impose. The first is to assume that $f_1, \ldots, f_r$ is a *Koszul-regular sequence*, i.e., that $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for $i > 0$, see More on Algebra, Definition 21.1. The sequence is called an *H_1-regular sequence* if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$. Another condition we can impose is that with $J = (f_1, \ldots, f_r)$, the map

$$R/J[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

which maps T_i to f_i mod J^2 is an isomorphism. In this case we say that $f_1, \ldots, f_r$ is a *quasi-regular sequence*, see Algebra, Definition 68.1. Given an R-module M there is also a notion of M-regular and M-quasi-regular sequence.

We can generalize this to the case of ringed spaces as follows. Let X be a ringed space and let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We say that $f_1, \ldots, f_r$ is a *regular sequence* if for each $i = 1, \ldots, r$ the map

$$f_i : \mathcal{O}_X/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_X/(f_1, \ldots, f_{i-1}) \tag{12.0.2}$$

is an injective map of sheaves. We say that $f_1, \ldots, f_r$ is a *Koszul-regular sequence* if the Koszul complex

$$K_\bullet(\mathcal{O}_X, f_\bullet), \tag{12.0.3}$$

see Modules, Definition 20.2, is acyclic in degrees > 0. We say that $f_1, \ldots, f_r$ is a *H_1-regular sequence* if the Koszul complex $K_\bullet(\mathcal{O}_X, f_\bullet)$ is exact in degree 1. Finally, we say that $f_1, \ldots, f_r$ is a *quasi-regular* sequence if the map

$$\mathcal{O}_X/\mathcal{J}[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{J}^d/\mathcal{J}^{d+1} \tag{12.0.4}$$

is an isomorphism of sheaves where $\mathcal{J} \subset \mathcal{O}_X$ is the sheaf of ideals generated by $f_1, \ldots, f_r$. (There is also a notion of $\mathcal{F}$-regular and $\mathcal{F}$-quasi-regular sequence for a given $\mathcal{O}_X$-module $\mathcal{F}$ which we will introduce here if we ever need it.)

Lemma 12.1. *Let X be a ringed space. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We have the following implications $f_1, \ldots, f_r$ is a regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is a Koszul-regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is an H_1-regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is a quasi-regular sequence.*

Proof. Since we may check exactness at stalks, a sequence $f_1, \ldots, f_r$ is a regular sequence if and only if the maps

$$f_i : \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1})$$

are injective for all $x \in X$. In other words, the image of the sequence $f_1, \ldots, f_r$ in the ring $\mathcal{O}_{X,x}$ is a regular sequence for all $x \in X$. The other types of regularity can be checked stalkwise as well (details omitted). Hence the implications follow from More on Algebra, Lemmas 21.2 and 21.5. □

Definition 12.2. Let X be a ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals.

(1) We say $\mathcal{J}$ is *regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(2) We say $\mathcal{J}$ is *Koszul-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a Koszul-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(3) We say $\mathcal{J}$ is *H_1-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a H_1-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(4) We say $\mathcal{J}$ is *quasi-regular* if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.

Many properties of this notion immediately follow from the corresponding notions for regular and quasi-regular sequences in rings.

Lemma 12.3. *Let X be a ringed space. Let $\mathcal{J}$ be a sheaf of ideals. We have the following implications: $\mathcal{J}$ is regular $\Rightarrow$ $\mathcal{J}$ is Koszul-regular $\Rightarrow$ $\mathcal{J}$ is H_1-regular $\Rightarrow$ $\mathcal{J}$ is quasi-regular.*

Proof. The lemma immediately reduces to Lemma 12.1. □

Lemma 12.4. *Let X be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Then $\mathcal{J}$ is quasi-regular if and only if the following conditions are satisfied:*

(1) *$\mathcal{J}$ is an $\mathcal{O}_X$-module of finite type,*
(2) *$\mathcal{J}/\mathcal{J}^2$ is a finite locally free $\mathcal{O}_X/\mathcal{J}$-module, and*
(3) *the canonical maps*
$$Sym^n_{\mathcal{O}_X/\mathcal{J}}(\mathcal{J}/\mathcal{J}^2) \longrightarrow \mathcal{J}^n/\mathcal{J}^{n+1}$$
are isomorphisms for all $n \geq 0$.

Proof. It is clear that if $U \subset X$ is an open such that $\mathcal{J}|_U$ is generated by a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ then $\mathcal{J}|_U$ is of finite type, $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free with basis $f_1, \ldots, f_r$, and the maps in (3) are isomorphisms because they are coordinate free formulation of the degree n part of (12.0.4). Hence it is clear that being quasi-regular implies conditions (1), (2), and (3).

Conversely, suppose that (1), (2), and (3) hold. Pick a point $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$. Then there exists a neighbourhood $U \subset X$ of x such that $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free of rank r over $\mathcal{O}_U/\mathcal{J}|_U$. After possibly shrinking U we may assume there exist $f_1, \ldots, f_r \in \mathcal{J}(U)$ which map to a basis of $\mathcal{J}|_U/\mathcal{J}^2|_U$ as an $\mathcal{O}_U/\mathcal{J}|_U$-module. In particular we see that the images of $f_1, \ldots, f_r$ in $\mathcal{J}_x/\mathcal{J}_x^2$ generate. Hence by Nakayama's lemma (Algebra, Lemma 19.1) we see that $f_1, \ldots, f_r$ generate the stalk $\mathcal{J}_x$. Hence, since $\mathcal{J}$ is of finite type, by Modules, Lemma 9.4 after shrinking U we may assume that $f_1, \ldots, f_r$ generate $\mathcal{J}$. Finally, from (3) and the isomorphism $\mathcal{J}|_U/\mathcal{J}^2|_U = \bigoplus \mathcal{O}_U/\mathcal{J}|_U f_i$ it is clear that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ is a quasi-regular sequence. □

Lemma 12.5. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Let $x \in X$ and $f_1, \ldots, f_r \in \mathcal{J}_x$ whose images give a basis for the $\kappa(x)$-vector space $\mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$.*

(1) *If $\mathcal{J}$ is quasi-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form a quasi-regular sequence generating $\mathcal{J}|_U$.*
(2) *If $\mathcal{J}$ is H_1-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an H_1-regular sequence generating $\mathcal{J}|_U$.*
(3) *If $\mathcal{J}$ is Koszul-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an Koszul-regular sequence generating $\mathcal{J}|_U$.*

Proof. First assume that $\mathcal{J}$ is quasi-regular. We may choose an open neighbourhood $U \subset X$ of x and a quasi-regular sequence $g_1, \ldots, g_s \in \mathcal{O}_X(U)$ which generates $\mathcal{J}|_U$. Note that this implies that $\mathcal{J}/\mathcal{J}^2$ is free of rank s over $\mathcal{O}_U/\mathcal{J}|_U$ (see Lemma 12.4 and its proof) and hence $r = s$. We may shrink U and assume $f_1, \ldots, f_r \in \mathcal{J}(U)$. Thus we may write

$$f_i = \sum a_{ij} g_j$$

for some $a_{ij} \in \mathcal{O}_X(U)$. By assumption the matrix $A = (a_{ij})$ maps to an invertible matrix over $\kappa(x)$. Hence, after shrinking U once more, we may assume that (a_{ij}) is invertible. Thus we see that $f_1, \ldots, f_r$ give a basis for $(\mathcal{J}/\mathcal{J}^2)|_U$ which proves that $f_1, \ldots, f_r$ is a quasi-regular sequence over U.

Note that in order to prove (2) and (3) we may, because the assumptions of (2) and (3) are stronger than the assumption in (1), already assume that $f_1, \ldots, f_r \in \mathcal{J}(U)$ and $f_i = \sum a_{ij} g_j$ with (a_{ij}) invertible as above, where now $g_1, \ldots, g_r$ is a H_1-regular or Koszul-regular sequence. Since the Koszul complex on $f_1, \ldots, f_r$ is isomorphic to the Koszul complex on $g_1, \ldots, g_r$ via the matrix (a_{ij}) (see More on Algebra, Lemma 20.4) we conclude that $f_1, \ldots, f_r$ is H_1-regular or Koszul-regular as desired. $\square$

Lemma 12.6. *Any regular, Koszul-regular, H_1-regular, or quasi-regular sheaf of ideals on a scheme is a finite type quasi-coherent sheaf of ideals.*

Proof. This follows as such a sheaf of ideals is locally generated by finitely many sections. And any sheaf of ideals locally generated by sections on a scheme is quasi-coherent, see Schemes, Lemma 10.1. $\square$

Lemma 12.7. *Let X be a scheme. Let $\mathcal{J}$ be a sheaf of ideals. Then $\mathcal{J}$ is regular (resp. Koszul-regular, H_1-regular, quasi-regular) if and only if for every $x \in Supp(\mathcal{O}_X/\mathcal{J})$ there exists an affine open neighbourhood $x \in U \subset X$, $U = \mathrm{Spec}(A)$ such that $\mathcal{J}|_U = \widetilde{I}$ and such that I is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in A$.*

Proof. By assumption we can find an open neighbourhood U of x over which $\mathcal{J}$ is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$. After shrinking U we may assume that U is affine, say $U = \mathrm{Spec}(A)$. Since $\mathcal{J}$ is quasi-coherent by Lemma 12.6 we see that $\mathcal{J}|_U = \widetilde{I}$ for some ideal $I \subset A$. Now we can use the fact that

$$\widetilde{\ }: \mathrm{Mod}_A \longrightarrow QCoh(\mathcal{O}_U)$$

is an equivalence of categories which preserves exactness. For example the fact that the functions f_i generate $\mathcal{J}$ means that the f_i, seen as elements of A generate I. The fact that (12.0.2) is injective (resp. (12.0.3) is exact, (12.0.3) is exact in degree 1, (12.0.4) is an isomorphism) implies the corresponding property of the map $A/(f_1, \ldots, f_{i-1}) \to A/(f_1, \ldots, f_{i-1})$ (resp. the complex $K_\bullet(A, f_1, \ldots, f_r)$, the map $A/I[T_1, \ldots, T_r] \to \bigoplus I^n/I^{n+1}$). Thus $f_1, \ldots, f_r \in A$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence of the ring A. □

Lemma 12.8. *Let X be a locally Noetherian scheme. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let x be a point of the support of $\mathcal{O}_X/\mathcal{J}$. The following are equivalent*

1. *$\mathcal{J}_x$ is generated by a regular sequence in $\mathcal{O}_{X,x}$,*
2. *$\mathcal{J}_x$ is generated by a Koszul-regular sequence in $\mathcal{O}_{X,x}$,*
3. *$\mathcal{J}_x$ is generated by an H_1-regular sequence in $\mathcal{O}_{X,x}$,*
4. *$\mathcal{J}_x$ is generated by a quasi-regular sequence in $\mathcal{O}_{X,x}$,*
5. *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a regular sequence in A, and*
6. *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a Koszul-regular sequence in A, and*
7. *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by an H_1-regular sequence in A, and*
8. *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a quasi-regular sequence in A,*
9. *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is regular, and*
10. *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is Koszul-regular, and*
11. *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is H_1-regular, and*
12. *there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is quasi-regular.*

In particular, on a locally Noetherian scheme the notions of regular, Koszul-regular, H_1-regular, or quasi-regular ideal sheaf all agree.

Proof. It follows from Lemma 12.7 that (5) $\Leftrightarrow$ (9), (6) $\Leftrightarrow$ (10), (7) $\Leftrightarrow$ (11), and (8) $\Leftrightarrow$ (12). It is clear that (5) $\Rightarrow$ (1), (6) $\Rightarrow$ (2), (7) $\Rightarrow$ (3), and (8) $\Rightarrow$ (4). We have (1) $\Rightarrow$ (5) by Algebra, Lemma 67.8. We have (9) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (12) by Lemma 12.3. Finally, (4) $\Rightarrow$ (1) by Algebra, Lemma 68.6. Now all 12 statements are equivalent. □

13. Regular immersions

Let $i : Z \to X$ be an immersion of schemes. By definition this means there exists an open subscheme $U \subset X$ such that Z is identified with a closed subscheme of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the corresponding quasi-coherent sheaf of ideals. Suppose $U' \subset X$ is a second such open subscheme, and denote $\mathcal{I}' \subset \mathcal{O}_{U'}$ the corresponding quasi-coherent sheaf of ideals. Then $\mathcal{I}|_{U \cap U'} = \mathcal{I}'|_{U \cap U'}$. Moreover, the support of $\mathcal{O}_U/\mathcal{I}$ is Z which is contained in $U \cap U'$ and is also the support of $\mathcal{O}_{U'}/\mathcal{I}'$. Hence it follows from Definition 12.2 that $\mathcal{I}$ is a regular ideal if and only if $\mathcal{I}'$ is a regular ideal. Similarly for being Koszul-regular, H_1-regular, or quasi-regular.

Definition 13.1. Let $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U and denote $\mathcal{I} \subset \mathcal{O}_U$ the corresponding quasi-coherent sheaf of ideals.

1. We say i is a *regular immersion* if $\mathcal{I}$ is regular.
2. We say i is a *Koszul-regular immersion* if $\mathcal{I}$ is Koszul-regular.
3. We say i is a *H_1-regular immersion* if $\mathcal{I}$ is H_1-regular.

(4) We say i is a *quasi-regular immersion* if $\mathcal{I}$ is quasi-regular.

The discussion above shows that this is independent of the choice of U. The conditions are listed in decreasing order of strength, see Lemma 13.2. A Koszul-regular closed immersion is smooth locally a regular immersion, see Lemma 13.11. In the locally Noetherian case all four notions agree, see Lemma 12.8.

Lemma 13.2. *Let $i : Z \to X$ be an immersion of schemes. We have the following implications: i is regular $\Rightarrow$ i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.*

Proof. The lemma immediately reduces to Lemma 12.3. □

Lemma 13.3. *Let $i : Z \to X$ be an immersion of schemes. Assume X is locally Noetherian. Then i is regular $\Leftrightarrow$ i is Koszul-regular $\Leftrightarrow$ i is H_1-regular $\Leftrightarrow$ i is quasi-regular.*

Proof. Follows immediately from Lemma 13.2 and Lemma 12.8. □

Lemma 13.4. *Let $i : Z \to X$ be a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion. Let $X' \to X$ be a flat morphism. Then the base change $i' : Z \times_X X' \to X'$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion.*

Proof. Via Lemma 12.7 this translates into the algebraic statements in Algebra, Lemmas 67.7 and 68.3 and More on Algebra, Lemma 21.4. □

Lemma 13.5. *Let $i : Z \to X$ be an immersion of schemes. Then i is a quasi-regular immersion if and only if the following conditions are satisfied*

(1) *i is locally of finite presentation,*
(2) *the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and*
(3) *the map (11.1.2) is an isomorphism.*

Proof. An open immersion is locally of finite presentation. Hence we may replace X by an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U, i.e., we may assume that i is a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Recall, see Morphisms, Lemma 22.7 that $\mathcal{I}$ is of finite type if and only if i is locally of finite presentation. Hence the equivalence follows from Lemma 12.4 and unwinding the definitions. □

Lemma 13.6. *Let $Z \to Y \to X$ be immersions of schemes. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Morphisms, Lemma 33.5*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 13.5 and Lemma 12.3) it suffices to prove that the sequence is exact. By what was proven in Morphisms, Lemma 33.5 it suffices to show that the first map is injective. Working affine locally this reduces to the following question: Suppose that we have a ring A and ideals $I \subset J \subset A$. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Does this imply that $I/I^2 \otimes_A A/J \to J/J^2$ is injective? Note that $I/I^2 \otimes_A A/J = I/IJ$. Hence we are trying to prove that $I \cap J^2 = IJ$. This is the result of More on Algebra, Lemma 21.8. □

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 68.8. The other types of regular immersions are preserved under composition.

Lemma 13.7. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes.*

(1) *If i and j are regular immersions, so is $j \circ i$.*
(2) *If i and j are Koszul-regular immersions, so is $j \circ i$.*
(3) *If i and j are H_1-regular immersions, so is $j \circ i$.*
(4) *If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.*

Proof. The algebraic version of (1) is Algebra, Lemma 67.9. The algebraic version of (2) is More on Algebra, Lemma 21.12. The algebraic version of (3) is More on Algebra, Lemma 21.10. The algebraic version of (4) is More on Algebra, Lemma 21.9. $\square$

Lemma 13.8. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume that the sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Morphisms, Lemma 33.5 is exact and locally split.

(1) *If $j \circ i$ is a quasi-regular immersion, so is i.*
(2) *If $j \circ i$ is a H_1-regular immersion, so is i.*
(3) *If both j and $j \circ i$ are Koszul-regular immersions, so is i.*

Proof. After shrinking Y and X we may assume that i and j are closed immersions. Denote $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of Y and $\mathcal{J} \subset \mathcal{O}_X$ the ideal sheaf of Z. The conormal sequence is $0 \to \mathcal{I}/\mathcal{I}\mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{J}/(\mathcal{I} + \mathcal{J}^2) \to 0$. Let $z \in Z$ and set $y = i(z)$, $x = j(y) = j(i(z))$. Choose $f_1, \ldots, f_n \in \mathcal{I}_x$ which map to a basis of $\mathcal{I}_x/\mathfrak{m}_z\mathcal{I}_x$. Extend this to $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathcal{J}_x$ which map to a basis of $\mathcal{J}_x/\mathfrak{m}_z\mathcal{J}_x$. This is possible as we have assumed that the sequence of conormal sheaves is split in a neighbourhood of z, hence $\mathcal{I}_x/\mathfrak{m}_x\mathcal{I}_x \to \mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$ is injective.

Proof of (1). By Lemma 12.5 we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a quasi-regular sequence generating $\mathcal{J}$. Hence by Algebra, Lemma 68.5 we see that $g_1, \ldots, g_m$ induces a quasi-regular sequence on $Y \cap U$ cutting out Z.

Proof of (2). Exactly the same as the proof of (1) except using More on Algebra, Lemma 21.11.

Proof of (3). By Lemma 12.5 (applied twice) we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n$ forms a Koszul-regular sequence generating $\mathcal{I}$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a Koszul-regular sequence generating $\mathcal{J}$. Hence by More on Algebra, Lemma 21.13 we see that $g_1, \ldots, g_m$ induces a Koszul-regular sequence on $Y \cap U$ cutting out Z. $\square$

Lemma 13.9. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Pick $z \in Z$ and denote $y \in Y$, $x \in X$ the corresponding points. Assume X is locally Noetherian. The following are equivalent*

(1) *i is a regular immersion in a neighbourhood of z and j is a regular immersion in a neighbourhood of y,*
(2) *i and $j \circ i$ are regular immersions in a neighbourhood of z,*

(3) *$j \circ i$ is a regular immersion in a neighbourhood of z and the conormal sequence*
$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$
is split exact in a neighbourhood of z.

Proof. Since X (and hence Y) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Lemma 12.8. The implication (1) $\Rightarrow$ (2) is Lemma 13.7. The implication (2) $\Rightarrow$ (3) is Lemma 13.6. Thus it suffices to prove that (3) implies (1).

Assume (3). Set $A = \mathcal{O}_{X,x}$. Denote $I \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ and denote $J \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$. Note that any minimal sequence of elements generating J in A is a quasi-regular hence regular sequence, see Lemma 12.5. By assumption the conormal sequence
$$0 \to I/IJ \to J/J^2 \to J/(I + J^2) \to 0$$
is split exact as a sequence of A/J-modules. Hence we can pick a minimal system of generators $f_1, \ldots, f_n, g_1, \ldots, g_m$ of J with $f_1, \ldots, f_n \in I$ a minimal system of generators of I. As pointed out above $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A. It follows directly from the definition of a regular sequence that $f_1, \ldots, f_n$ is a regular sequence in A and $\overline{g}_1, \ldots, \overline{g}_m$ is a regular sequence in A/I. Thus j is a regular immersion at y and i is a regular immersion at z. $\square$

Remark 13.10. In the situation of Lemma 13.9 parts (1), (2), (3) are **not** equivalent to "$j \circ i$ and j are regular immersions at z and y". An example is $X = \mathbf{A}^1_k = \mathrm{Spec}(k[x])$, $Y = \mathrm{Spec}(k[x]/(x^2))$ and $Z = \mathrm{Spec}(k[x]/(x))$.

Lemma 13.11. *Let $i : Z \to X$ be a Koszul regular closed immersion. Then there exists a surjective smooth morphism $X' \to X$ such that the base change $i' : Z \times_X X' \to X'$ of i is a regular immersion.*

Proof. We may assume that X is affine and the ideal of Z generated by a Koszul-regular sequence by replacing X by the members of a suitable affine open covering (affine opens as in Lemma 12.7). The affine case is More on Algebra, Lemma 21.17. $\square$

14. Relative regular immersions

In this section we consider the base change property for regular immersions. The following lemma does not hold for regular immersions or for Koszul immersions, see Examples, Lemma 13.2.

Lemma 14.1. *Let $f : X \to S$ be a morphism of schemes. Let $i : Z \subset X$ be an immersion. Assume*

(1) *i is an H_1-regular (resp. quasi-regular) immersion, and*
(2) *$Z \to S$ is a flat morphism.*

Then for every morphism of schemes $g : S' \to S$ the base change $Z' = S' \times_S Z \to X' = S' \times_S X$ is an H_1-regular (resp. quasi-regular) immersion.

Proof. Unwinding the definitions and using Lemma 12.7 we translate this into algebra as follows. Let $A \to B$ be a ring map and $f_1, \ldots, f_r \in B$. Assume $B/(f_1, \ldots, f_r)B$ is flat over A. Consider a ring map $A \to A'$. Set $B' = B \otimes_A A'$ and $J' = JB'$.

Case I: $f_1, \ldots, f_r$ is quasi-regular. Set $J = (f_1, \ldots, f_r)$. By assumption J^n/J^{n+1} is isomorphic to a direct sum of copies of B/J hence flat over A. By induction and Algebra, Lemma 38.12 we conclude that B/J^n is flat over A. The ideal $(J')^n$ is equal to $J^n \otimes_A A'$, see Algebra, Lemma 38.11. Hence $(J')^n/(J')^{n+1} = J^n/J^{n+1} \otimes_A A'$ which clearly implies that $f_1, \ldots, f_r$ is a quasi-regular sequence in B'.

Case II: $f_1, \ldots, f_r$ is H_1-regular. By More on Algebra, Lemma 21.15 the vanishing of the Koszul homology group $H_1(K_\bullet(B, f_1, \ldots, f_r))$ implies the vanishing of $H_1(K_\bullet(B', f_1', \ldots, f_r'))$ and we win. □

This lemma is the motivation for the following definition.

Definition 14.2. Let $f : X \to S$ be a morphism of schemes. Let $i : Z \to X$ be an immersion.

(1) We say i is a *relative quasi-regular immersion* if $Z \to S$ is flat and i is a quasi-regular immersion.
(2) We say i is a *relative H_1-regular immersion* if $Z \to S$ is flat and i is an H_1-regular immersion.

We warn the reader that this may be nonstandard notation. Lemma 14.1 guarantees that relative quasi-regular (resp. H_1-regular) immersions are preserved under any base change. A relative H_1-regular immersion is a relative quasi-regular immersion, see Lemma 13.2. Please take a look at Lemma 14.5 (or Lemma 14.4) which shows that if $Z \to X$ is a relative H_1-regular (or quasi-regular) immersion and the ambient scheme is (flat and) locally of finite presentation over S, then $Z \to X$ is actually a regular immersion and the same remains true after any base change.

Lemma 14.3. *Let $f : X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative quasi-regular immersion. If $x \in Z$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.*

Proof. Let $f_1, \ldots, f_r \in \mathcal{O}_{X,x}$ be a quasi-regular sequence cutting out the ideal of Z at x. By Algebra, Lemma 68.6 we know that $f_1, \ldots, f_r$ is a regular sequence. Hence f_r is a nonzerodivisor on $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ such that the quotient is a flat $\mathcal{O}_{S,f(x)}$-module. By Lemma 10.3 we conclude that $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ is a flat $\mathcal{O}_{S,f(x)}$-module. Continuing by induction we find that $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module. □

Lemma 14.4. *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be an immersion. Assume*

(1) *$X \to S$ is flat and locally of finite presentation,*
(2) *$Z \to X$ is a relative quasi-regular immersion.*

Then $Z \to X$ is a regular immersion and the same remains true after any base change.

Proof. Pick $x \in Z$ with image $s \in S$. To prove this it suffices to find an affine neighbourhood of x contained in U such that the result holds on that affine open. Hence we may assume that X is affine and there exist a quasi-regular sequence $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ such that $Z = V(f_1, \ldots, f_r)$. By Lemma 14.1 and its proof the sequence $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a quasi-regular sequence in $\Gamma(X_s, \mathcal{O}_{X_s})$. Since X_s is Noetherian, this implies, possibly after shrinking X a bit, that $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a regular sequence, see Algebra, Lemmas 68.6 and 67.8. By Lemma 10.7 it follows that $Z_1 = V(f_1) \subset X$ is a relative effective Cartier divisor, again after possibly shrinking X a bit. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until on reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Since being a relative effective Cartier divisor is preserved under arbitrary base change, see Lemma 10.1, we also see that the final statement of the lemma holds. ☐

Lemma 14.5. *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative H_1-regular immersion. Assume $X \to S$ is locally of finite presentation. Then*

1. *there exists an open subscheme $U \subset X$ such that $Z \subset U$ and such that $U \to S$ is flat, and*
2. *$Z \to X$ is a regular immersion and the same remains true after any base change.*

Proof. Pick $x \in Z$. To prove (1) suffices to find an open neighbourhood $U \subset X$ of x such that $U \to S$ is flat. Hence the lemma reduces to the case that $X = \text{Spec}(B)$ and $S = \text{Spec}(A)$ are affine and that Z is given by an H_1-regular sequence $f_1, \ldots, f_r \in B$. By assumption B is a finitely presented A-algebra and $B/(f_1, \ldots, f_r)B$ is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume f_i is the image of $f'_i \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $f'_1, \ldots, f'_r, g_1, \ldots, g_m$ are in A_0. We set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and we denote $f_{i,0}$ the image of f'_i in B_0. Then $B = B_0 \otimes_{A_0} A$ and

$$B/(f_1, \ldots, f_r) = B_0/(f_{0,1}, \ldots, f_{0,r}) \otimes_{A_0} A.$$

By Algebra, Lemma 156.1 we may, after enlarging A_0, assume that $B_0/(f_{0,1}, \ldots, f_{0,r})$ is flat over A_0. It may not be the case at this point that the Koszul cohomology group $H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ is zero. On the other hand, as B_0 is Noetherian, it is a finitely generated B_0-module. Let $\xi_1, \ldots, \xi_n \in H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ be generators. Let $A_0 \subset A_1 \subset A$ be a larger finite type $\mathbf{Z}$-subalgebra of A. Denote $f_{1,i}$ the image of $f_{0,i}$ in $B_1 = B_0 \otimes_{A_0} A_1$. By More on Algebra, Lemma 21.15 the map

$$H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r})) \otimes_{A_0} A_1 \longrightarrow H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$$

is surjective. Furthermore, it is clear that the colimit (over all choices of A_1 as above) of the complexes $K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})$ is the complex $K_\bullet(B, f_1, \ldots, f_r)$ which is acyclic in degree 1. Hence

$$\text{colim}_{A_0 \subset A_1 \subset A} H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})) = 0$$

by Algebra, Lemma 8.9. Thus we can find a choice of A_1 such that $\xi_1, \ldots, \xi_n$ all map to zero in $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$. In other words, the Koszul cohomology group $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$ is zero.

Consider the morphism of affine schemes $X_1 \to S_1$ equal to Spec of the ring map $A_1 \to B_1$ and $Z_1 = \text{Spec}(B_1/(f_{1,1}, \ldots, f_{1,r}))$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X = X_1 \times_{S_1} S$, and similarly $Z = Z_1 \times_S S_1$, it now suffices to prove (1) for $X_1 \to S_1$ and the relative H_1-regular immersion $Z_1 \to X_1$, see Morphisms, Lemma 26.6. Hence we have reduced to the case where $X \to S$ is a finite type morphism of Noetherian schemes. In this case we know that $X \to S$ is flat at every point of Z by Lemma 14.3. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 125.4 we see that (1) holds. Part (2) then follows from an application of Lemma 14.4. □

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative quasi-regular immersion in terms of its fibres.

Lemma 14.6. *Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $T \subset X$ be a closed subscheme. Let $x \in T$ with image $s \in S$.*

1. *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, then there exists an open $U \subset X$ and a relative quasi-regular immersion $Z \subset U$ such that $Z_s = T_s \cap U_s$ and $T \cap U \subset Z$.*
2. *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, the morphism $T \to X$ is of finite presentation, and $T \to S$ is flat at x, then we can choose U and Z as in (1) such that $T \cap U = Z$.*
3. *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, and T is cut out by c equations in a neighbourhood of x, where $c = \dim_x(X_s) - \dim_x(T_s)$, then we can choose U and Z as in (1) such that $T \cap U = Z$.*

In each case $Z \to U$ is a regular immersion by Lemma 14.4. In particular, if $T \to S$ is locally of finite presentation and flat and all fibres $T_s \subset X_s$ are quasi-regular immersions, then $T \to X$ is a relative quasi-regular immersion.

Proof. Choose affine open neighbourhoods $\mathrm{Spec}(A)$ of s and $\mathrm{Spec}(B)$ of x such that $\varphi(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to T. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $\mathrm{Spec}(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a quasi-regular sequence of elements. After possibly localizing B at some $g \in B$, $g \notin \mathfrak{q}$ we may assume there exist $f_1, \ldots, f_r \in I$ which map to a quasi-regular sequence in $B \otimes_A \kappa(\mathfrak{p})$ which generates $I(B \otimes_A \kappa(\mathfrak{p}))$. By Algebra, Lemmas 68.6 and 67.8 we may assume after another localization that $f_1, \ldots, f_r \in I$ form a regular sequence in $B \otimes_A \kappa(\mathfrak{p})$. By Lemma 10.7 it follows that $Z_1 = V(f_1) \subset \mathrm{Spec}(B)$ is a relative effective Cartier divisor, again after possibly localizing B. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until one reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Then $Z \to \mathrm{Spec}(B)$ is a regular immersion and Z is flat over S, in particular $Z \to \mathrm{Spec}(B)$ is a relative quasi-regular immersion over $\mathrm{Spec}(A)$. This proves (1).

To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 124.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 122.6.

To see (3), after possibly shrinking U we may assume that the ideal of Z is generated by a regular sequence $f_1, \ldots, f_r$ (see our construction of Z above) and the ideal of

T is generated by $g_1, \ldots, g_c$. We claim that $c = r$. Namely,

$$\begin{aligned}
\dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)), \\
\dim_x(T_s) &= \dim(\mathcal{O}_{T_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)), \\
\dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{T_s,x}) + r
\end{aligned}$$

the first two equalities by Algebra, Lemma 112.3 and the second by r times applying Algebra, Lemma 59.11. As $T \subset Z$ we see that $f_i = \sum b_{ij} g_j$. But the ideals of Z and T cut out the same quasi-regular closed subscheme of X_s in a neighbourhood of x. Hence the matrix $(b_{ij}) \bmod \mathfrak{m}_x$ is invertible (some details omitted). Hence (b_{ij}) is invertible in an open neighbourhood of x. In other words, $T \cap U = Z$ after shrinking U.

The final statements of the lemma follow immediately from part (2), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 22.3 and 22.11. □

The following lemma is an enhancement of Morphisms, Lemma 35.20.

Lemma 14.7. *Let $f : X \to S$ be a smooth morphism of schemes. Let $\sigma : S \to X$ be a section of f. Then σ is a regular immersion.*

Proof. By Schemes, Lemma 21.11 the morphism σ is an immersion. After replacing X by an open neighbourhood of $\sigma(S)$ we may assume that σ is a closed immersion. Let $T = \sigma(S)$ be the corresponding closed subscheme of X. Since $T \to S$ is an isomorphism it is flat and of finite presentation. Also a smooth morphism is flat and locally of finite presentation, see Morphisms, Lemmas 35.9 and 35.8. Thus, according to Lemma 14.6, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. This follows immediately from Morphisms, Lemma 35.20 but we can also see it directly as follows. Let k be a field and let A be a smooth k-algebra. Let $\mathfrak{m} \subset A$ be a maximal ideal whose residue field is k. Then $\mathfrak{m}$ is generated by a quasi-regular sequence, possibly after replacing A by A_g for some $g \in A$, $g \notin \mathfrak{m}$. In Algebra, Lemma 135.3 we proved that $A_\mathfrak{m}$ is a regular local ring, hence $\mathfrak{m}A_\mathfrak{m}$ is generated by a regular sequence. This does indeed imply that $\mathfrak{m}$ is generated by a regular sequence (after replacing A by A_g for some $g \in A$, $g \notin \mathfrak{m}$), see Algebra, Lemma 67.8. □

The following lemma has a kind of converse, see Lemma 14.11.

Lemma 14.8. *Let*

$$\begin{tikzcd}
Y \arrow[rr, "i"'] \arrow[rd, "j"'] & & X \arrow[ld] \\
& S &
\end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth, and i, j immersions. If j is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, then so is i.

Proof. We can write i as the composition

$$Y \to Y \times_S X \to X$$

By Lemma 14.7 the first arrow is a regular immersion. The second arrow is a flat base change of $Y \to S$, hence is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, see Lemma 13.4. We conclude by an application of Lemma 13.7. □

Lemma 14.9. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"'] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is syntomic, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. After replacing X by an open neighbourhood of $i(Y)$ we may assume that i is a closed immersion. Let $T = i(Y)$ be the corresponding closed subscheme of X. Since $T \cong Y$ the morphism $T \to S$ is flat and of finite presentation (Morphisms, Lemmas 32.6 and 32.7). Also a smooth morphism is flat and locally of finite presentation (Morphisms, Lemmas 35.9 and 35.8). Thus, according to Lemma 14.6, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. As X_s is locally of finite type over a field, it is Noetherian (Morphisms, Lemma 16.6). Thus we can check that $T_s \subset X_s$ is a quasi-regular immersion at points, see Lemma 12.8. Take $t \in T_s$. By Morphisms, Lemma 32.9 the local ring $\mathcal{O}_{T_s,t}$ is a local complete intersection over $\kappa(s)$. The local ring $\mathcal{O}_{X_s,t}$ is regular, see Algebra, Lemma 135.3. By Algebra, Lemma 130.7 we see that the kernel of the surjection $\mathcal{O}_{X_s,t} \to \mathcal{O}_{T_s,t}$ is generated by a regular sequence, which is what we had to show. □

Lemma 14.10. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"'] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is smooth, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. This is a special case of Lemma 14.9 because a smooth morphism is syntomic, see Morphisms, Lemma 35.7. □

Lemma 14.11. *Let*

$$\begin{tikzcd} Y \arrow[rr, "i"'] \arrow[dr, "j"'] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth, and i, j immersions. If i is a Koszul-regular (resp. H_1-regular, quasi-regular) immersion, then so is j.

Proof. Let $y \in Y$ be any point. Set $x = i(y)$ and set $s = j(y)$. It suffices to prove the result after replacing X, S by open neighbourhoods U, V of x, s and Y by an open neighbourhood of y in $i^{-1}(U) \cap j^{-1}(V)$. Hence we may assume that Y, X and S are affine. In this case we can choose a closed immersion $h : X \to \mathbf{A}^n_S$ over S for some n. Note that h is a regular immersion by Lemma 14.10. Hence $h \circ i$ is a

Koszul-regular (resp. H_1-regular, quasi-regular) immersion, see Lemmas 13.7 and 13.2. In this way we reduce to the case $X = \mathbf{A}^n_S$ and S affine.

After replacing S by an affine open V and replacing Y by $j^{-1}(V)$ we may assume that i is a closed immersion and S affine. Write $S = \mathrm{Spec}(A)$. Then $j : Y \to S$ defines an isomorphism of Y to the closed subscheme $\mathrm{Spec}(A/I)$ for some ideal $I \subset A$. The map $i : Y = \mathrm{Spec}(A/I) \to \mathbf{A}^n_S = \mathrm{Spec}(A[x_1, \ldots, x_n])$ corresponds to an A-algebra homomorphism $i^\sharp : A[x_1, \ldots, x_n] \to A/I$. Choose $a_i \in A$ which map to $i^\sharp(x_i)$ in A/I. Observe that the ideal of the closed immersion i is

$$J = (x_1 - a_1, \ldots, x_n - a_n) + IA[x_1, \ldots, x_n].$$

Set $K = (x_1 - a_1, \ldots, x_n - a_n)$. We claim the sequence

$$0 \to K/KJ \to J/J^2 \to J/(K + J^2) \to 0$$

is split exact. To see this note that K/K^2 is free with basis $x_i - a_i$ over the ring $A[x_1, \ldots, x_n]/K \cong A$. Hence K/KJ is free with the same basis over the ring $A[x_1, \ldots, x_n]/J \cong A/I$. On the other hand, taking derivatives gives a map

$$\mathrm{d}_{A[x_1,\ldots,x_n]/A} : J/J^2 \longrightarrow \Omega_{A[x_1,\ldots,x_n]/A} \otimes_{A[x_1,\ldots,x_n]} A[x_1, \ldots, x_n]/J$$

which maps the generators $x_i - a_i$ to the basis elements $\mathrm{d}x_i$ of the free module on the right. The claim follows. Moreover, note that $x_1 - a_1, \ldots, x_n - a_n$ is a regular sequence in $A[x_1, \ldots, x_n]$ with quotient ring $A[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n) \cong A$. Thus we have a factorization

$$Y \to V(x_1 - a_1, \ldots, x_n - a_n) \to \mathbf{A}^n_S$$

of our closed immersion i where the composition is Koszul-regular (resp. H_1-regular, quasi-regular), the second arrow is a regular immersion, and the associated conormal sequence is split. Now the result follows from Lemma 13.8. $\square$

15. Meromorphic functions and sections

See [Kle79] for some possible pitfalls[1].

Let $(X, \mathcal{O}_X)$ be a locally ringed space. For any open $U \subset X$ we have defined the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ of regular sections of $\mathcal{O}_X$ over U, see Definition 9.16. The restriction of a regular section to a smaller open is regular. Hence $\mathcal{S} : U \mapsto \mathcal{S}(U)$ is a subsheaf (of sets) of $\mathcal{O}_X$. We sometimes denote $\mathcal{S} = \mathcal{S}_X$ if we want to indicate the dependence on X. Moreover, $\mathcal{S}(U)$ is a multiplicative subset of the ring $\mathcal{O}_X(U)$ for each U. Hence we may consider the presheaf of rings

$$U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U),$$

see Modules, Lemma 22.1.

Definition 15.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space. The *sheaf of meromorphic functions on* X is the sheaf $\mathcal{K}_X$ associated to the presheaf displayed above. A *meromorphic function* on X is a global section of $\mathcal{K}_X$.

Since each element of each $\mathcal{S}(U)$ is a nonzerodivisor on $\mathcal{O}_X(U)$ we see that the natural map of sheaves of rings $\mathcal{O}_X \to \mathcal{K}_X$ is injective.

[1] Danger, Will Robinson!

Example 15.2. Let $A = \mathbf{C}[x, \{y_\alpha\}_{\alpha \in \mathbf{C}}]/((x-\alpha)y_\alpha, y_\alpha y_\beta)$. Any element of A can be written uniquely as $f(x) + \sum \lambda_\alpha y_\alpha$ with $f(x) \in \mathbf{C}[x]$ and $\lambda_\alpha \in \mathbf{C}$. Let $X = \mathrm{Spec}(A)$. In this case $\mathcal{O}_X = \mathcal{K}_X$, since on any affine open $D(f)$ the ring A_f any nonzerodivisor is a unit (proof omitted).

Definition 15.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. We say that *pullbacks of meromorphic functions are defined for f* if for every pair of open $U \subset X$, $V \subset Y$ such that $f(U) \subset V$, and any section $s \in \Gamma(V, \mathcal{S}_Y)$ the pullback $f^\sharp(s) \in \Gamma(U, \mathcal{O}_X)$ is an element of $\Gamma(U, \mathcal{S}_X)$.

In this case there is an induced map $f^\sharp : f^{-1}\mathcal{K}_Y \to \mathcal{K}_X$, in other words we obtain a commutative diagram of morphisms of ringed spaces

$$\begin{tikzcd}
(X, \mathcal{K}_X) \arrow[r] \arrow[d, "f"] & (X, \mathcal{O}_X) \arrow[d, "f"] \\
(Y, \mathcal{K}_Y) \arrow[r] & (Y, \mathcal{O}_X)
\end{tikzcd}$$

We sometimes denote $f^*(s) = f^\sharp(s)$ for a section $s \in \Gamma(Y, \mathcal{K}_Y)$.

Lemma 15.4. *Let $f : X \to Y$ be a morphism of schemes. In each of the following cases pullbacks of meromorphic sections are defined.*

1. *X, Y are integral and f is dominant,*
2. *X is integral and the generic point of X maps to a generic point of an irreducible component of Y,*
3. *X is reduced and every generic point of every irreducible component of X maps to the generic point of an irreducible component of Y,*
4. *X is locally Noetherian, and any associated point of X maps to a generic point of an irreducible component of Y, and*
5. *X is locally Noetherian, has no embedded points and any generic point of an irreducible component of X maps to the generic point of an irreducible component of Y.*

Proof. Omitted. Hint: Similar to the proof of Lemma 9.12, using the following fact (on Y): if an element $x \in R$ maps to a nonzerodivisor in $R_\mathfrak{p}$ for a minimal prime $\mathfrak{p}$ of R, then $x \notin \mathfrak{p}$. See Algebra, Lemma 24.1. $\square$

Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Consider the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Its sheafification is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, see Modules, Lemma 22.2.

Definition 15.5. Let X be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

1. We denote $\mathcal{K}_X(\mathcal{F})$ the sheaf of $\mathcal{K}_X$-modules which is the sheafification of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Equivalently $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ (see above).
2. A *meromorphic section of $\mathcal{F}$* is a global section of $\mathcal{K}_X(\mathcal{F})$.

In particular we have

$$\mathcal{K}_X(\mathcal{F})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_{X,x} = \mathcal{S}_x^{-1}\mathcal{F}_x$$

for any point $x \in X$. However, one has to be careful since it may not be the case that $\mathcal{S}_x$ is the set of nonzerodivisors in the local ring $\mathcal{O}_{X,x}$. Namely, there is always an injective map

$$\mathcal{K}_{X,x} \longrightarrow Q(\mathcal{O}_{X,x})$$

to the total quotient ring. It is also surjective if and only if $\mathcal{S}_x$ is the set of nonzerodivisors in $\mathcal{O}_{X,x}$. The sheaves of meromorphic sections aren't quasi-coherent modules in general, but they do have some properties in common with quasi-coherent modules.

Lemma 15.6. *Let X be a quasi-compact scheme. Let $h \in \Gamma(X, \mathcal{O}_X)$ and $f \in \Gamma(X, \mathcal{K}_X)$ such that f restricts to zero on X_h. Then $h^n f = 0$ for some $n \gg 0$.*

Proof. We can find a covering of X by affine opens U such that $f|_U = s^{-1}a$ with $a \in \mathcal{O}_X(U)$ and $s \in \mathcal{S}(U)$. Since X is quasi-compact we can cover it by finitely many affine opens of this form. Thus it suffices to prove the lemma when $X = \operatorname{Spec}(A)$ and $f = s^{-1}a$. Note that $s \in A$ is a nonzerodivisor hence it suffices to prove the result when $f = a$. The condition $f|_{X_h} = 0$ implies that a maps to zero in $A_h = \mathcal{O}_X(X_h)$ as $\mathcal{O}_X \subset \mathcal{K}_X$. Thus $h^n a = 0$ for some $n > 0$ as desired. $\square$

Lemma 15.7. *Let X be a locally Noetherian scheme.*

(1) *For any $x \in X$ we have $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors, and hence $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$.*
(2) *For any affine open $U \subset X$ the ring $\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$.*

Proof. To prove this lemma we may assume X is the spectrum of a Noetherian ring A. Say $x \in X$ corresponds to $\mathfrak{p} \subset A$.

Proof of (1). It is clear that $\mathcal{S}_x$ is contained in the set of nonzero divisors of $\mathcal{O}_{X,x} = A_\mathfrak{p}$. For the converse, let $f, g \in A$, $g \notin \mathfrak{p}$ and assume f/g is a nonzerodivisor in $A_\mathfrak{p}$. Let $I = \{a \in A \mid af = 0\}$. Then we see that $I_\mathfrak{p} = 0$ by exactness of localization. Since A is Noetherian we see that I is finitely generated and hence that $g'I = 0$ for some $g' \in A$, $g' \notin \mathfrak{p}$. Hence f is a nonzerodivisor in $A_{g'}$, i.e., in a Zariski open neighbourhood of $\mathfrak{p}$. Thus f/g is an element of $\mathcal{S}_x$.

Proof of (2). Let $f \in \Gamma(X, \mathcal{K}_X)$ be a meromorphic function. Set $I = \{a \in A \mid af \in A\}$. Fix a prime $\mathfrak{p} \subset A$ corresponding to the point $x \in X$. By (1) we can write the image of f in the stalk at $\mathfrak{p}$ as a/b, $a, b \in A_\mathfrak{p}$ with $b \in A_\mathfrak{p}$ not a zerodivisor. Write $b = c/d$ with $c, d \in A$, $d \notin \mathfrak{p}$. Then $ad - cf$ is a section of $\mathcal{K}_X$ which vanishes in an open neighbourhood of x. Say it vanishes on $D(e)$ with $e \in A$, $e \notin \mathfrak{p}$. Then $e^n(ad - cf) = 0$ for some $n \gg 0$ by Lemma 15.6. Thus $e^n c \in I$ and $e^n c$ maps to a nonzerodivisor in $A_\mathfrak{p}$. Let $\text{Ass}(A) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ be the associated primes of A. By looking at $IA_{\mathfrak{q}_i}$ and using Algebra, Lemma 62.14 the above says that $I \not\subset \mathfrak{q}_i$ for each i. By Algebra, Lemma 14.2 there exists an element $x \in I$, $x \notin \bigcup \mathfrak{q}_i$. By Algebra, Lemma 62.9 we see that x is not a zerodivisor on A. Hence $f = (xf)/x$ is an element of the total ring of fractions of A. This proves (2). $\square$

Lemma 15.8. *Let X be a scheme. Assume X is reduced and any quasi-compact open $U \subset X$ has a finite number of irreducible components.*

(1) *The sheaf $\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras.*
(2) *For any $x \in X$ we have $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors. In particular $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$.*
(3) *For any affine open $\operatorname{Spec}(A) = U \subset X$ we have that $\mathcal{K}_X(U)$ equals the total quotient ring of A.*

Proof. Let X be as in the lemma. Let $X^{(0)} \subset X$ be the set of generic points of irreducible components of X. Let

$$f : Y = \coprod\nolimits_{\eta \in X^{(0)}} \mathrm{Spec}(\kappa(\eta)) \longrightarrow X$$

be the inclusion of the generic points into X using the canonical maps of Schemes, Section 13. (This morphism was used in Morphisms, Definition 48.12 to define the normalization of X.) We claim that $\mathcal{K}_X = f_*\mathcal{O}_Y$. First note that $\mathcal{K}_Y = \mathcal{O}_Y$ as Y is a disjoint union of spectra of field. Next, note that pullbacks of meromorphic functions are defined for f, by Lemma 15.4. This gives a map

$$\mathcal{K}_X \longrightarrow f_*\mathcal{O}_Y.$$

Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. Then A is a reduced ring with finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$. Then we have $Q(A) = \prod A_{\mathfrak{q}_i} = \prod \kappa(\mathfrak{q}_i)$ by Algebra, Lemmas 24.4 and 24.1. In other words, already the value of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U)$ agrees with $f_*\mathcal{O}_Y(U)$ on our affine open U. Hence the displayed map is an isomorphism.

Now we are ready to prove (1), (2) and (3). The morphism f is quasi-compact by our assumption that the set of irreducible components of X is locally finite. Hence f is quasi-compact and quasi-separated (as Y is separated). By Schemes, Lemma 24.1 $f_*\mathcal{O}_Y$ is quasi-coherent. This proves (1). Let $x \in X$. Then

$$(f_*\mathcal{O}_Y)_x = \prod\nolimits_{\eta \in X^{(0)},\ x \in \overline{\{\eta\}}} \kappa(\eta)$$

On the other hand, $\mathcal{O}_{X,x}$ is reduced and has finitely minimal primes $\mathfrak{q}_i$ corresponding exactly to those $\eta \in X^{(0)}$ such that $x \in \overline{\{\eta\}}\kappa(\eta)$. Hence by Algebra, Lemmas 24.4 and 24.1 again we see that $Q(\mathcal{O}_{X,x}) = \prod \kappa(\mathfrak{q}_i)$ is the same as $(f_*\mathcal{O}_Y)_x$. This proves (2). Part (3) we saw during the course of the proof that $\mathcal{K}_X = f_*\mathcal{O}_Y$. $\square$

Lemma 15.9. *Let X be a scheme. Assume X is reduced and any quasi-compact open $U \subset X$ has a finite number of irreducible components. Then the normalization morphism $\nu : X^\nu \to X$ is the morphism*

$$\underline{\mathrm{Spec}}_X(\mathcal{O}') \longrightarrow X$$

where $\mathcal{O}' \subset \mathcal{K}_X$ is the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions.

Proof. Compare the definition of the normalization morphism $\nu : X^\nu \to X$ (see Morphisms, Definition 48.12) with the result $\mathcal{K}_X = f_*\mathcal{O}_Y$ obtained in the proof of Lemma 15.8 above. $\square$

Lemma 15.10. *Let X be an integral scheme with generic point η. We have*

1. *the sheaf of meromorphic functions is isomorphic to the constant sheaf with value the function field (see Morphisms, Definition 10.5) of X.*
2. *for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaf $\mathcal{K}_X(\mathcal{F})$ is isomorphic to the constant sheaf with value $\mathcal{F}_\eta$.*

Proof. Omitted. $\square$

Definition 15.11. Let X be a locally ringed space. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A meromorphic section s of $\mathcal{L}$ is said to be *regular* if the induced map $\mathcal{K}_X \to \mathcal{K}_X(\mathcal{L})$ is injective. (In other words, this means that s is a regular section of the invertible $\mathcal{K}_X$-module $\mathcal{K}_X(\mathcal{L})$. See Definition 9.16.)

First we spell out when (regular) meromorphic sections can be pulled back. After that we discuss the existence of regular meromorphic sections and consequences.

Lemma 15.12. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Assume that pullbacks of meromorphic functions are defined for f (see Definition 15.3).*

1. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical pullback map $f^* : \Gamma(Y, \mathcal{K}_Y(\mathcal{F})) \to \Gamma(X, \mathcal{K}_X(f^*\mathcal{F}))$ for meromorphic sections of $\mathcal{F}$.*
2. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A regular meromorphic section s of $\mathcal{L}$ pulls back to a regular meromorphic section f^*s of $f^*\mathcal{L}$.*

Proof. Omitted. $\square$

In some cases we can show regular meromorphic sections exist.

Lemma 15.13. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. In each of the following cases $\mathcal{L}$ has a regular meromorphic section:*

1. *X is integral,*
2. *X is reduced and any quasi-compact open has a finite number of irreducible components, and*
3. *X is locally Noetherian and has no embedded points.*

Proof. In case (1) we have seen in Lemma 15.10 that $\mathcal{K}_X(\mathcal{L})$ is a constant sheaf with value $\mathcal{L}_\eta$, and hence the result is clear.

Suppose X is a scheme. Let $G \subset X$ be the set of generic points of irreducible components of X. For each $\eta \in G$ denote $j_\eta : \eta \to X$ the canonical morphism of $\eta = \text{Spec}(\kappa(\eta))$ into X (see Schemes, Lemma 13.3). Consider the sheaf

$$\mathcal{G}_X(\mathcal{L}) = \prod\nolimits_{\eta \in G} j_{\eta,*}(\mathcal{L}_\eta).$$

There is a canonical map

$$\varphi : \mathcal{K}_X(\mathcal{L}) \longrightarrow \mathcal{G}_X(\mathcal{L})$$

coming from the maps $\mathcal{K}_X(\mathcal{L})_\eta \to \mathcal{L}_\eta$ and adjunction (see Sheaves, Lemma 27.3).

We claim that in cases (2) and (3) the map φ is an isomorphism for any invertible sheaf $\mathcal{L}$. Before proving this let us show that cases (2) and (3) follow from this. Namely, we can choose $s_\eta \in \mathcal{L}_\eta$ which generate $\mathcal{L}_\eta$, i.e., such that $\mathcal{L}_\eta = \mathcal{O}_{X,\eta}s_\eta$. Since the claim applied to $\mathcal{O}_X$ gives $\mathcal{K}_X = \mathcal{G}_X(\mathcal{O}_X)$ it is clear that the global section $s = \prod_{\eta \in G} s_\eta$ is regular as desired.

To prove that φ is an isomorphism we may work locally on X. For example it suffices to show that sections of $\mathcal{K}_X(\mathcal{L})$ and $\mathcal{G}_X(\mathcal{L})$ agree over small affine opens U. Say $U = \text{Spec}(A)$ and $\mathcal{L}|_U \cong \mathcal{O}_U$. By Lemmas 15.7 and 15.8 we see that $\Gamma(U, \mathcal{K}_X) = Q(A)$ is the total ring of fractions of A. On the other hand, $\Gamma(U, \mathcal{G}_X(\mathcal{O}_X)) = \prod_{\mathfrak{q} \subset A \text{ minimal}} A_\mathfrak{q}$. In both cases we see that the set of minimal primes of A is finite, say $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and that the set of zerodivisors of A is equal to $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ (see Algebra, Lemma 62.9). Hence the result follows from Algebra, Lemma 24.4. $\square$

Lemma 15.14. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. Let us denote $\mathcal{I} \subset \mathcal{O}_X$ the sheaf of ideals defined by the rule*

$$\mathcal{I}(V) = \{f \in \mathcal{O}_Z(V) \mid fs \in \mathcal{L}(V)\}.$$

The formula makes sense since $\mathcal{L}(V) \subset \mathcal{K}_X(\mathcal{L})(V)$. Then $\mathcal{I}$ is a quasi-coherent sheaf of ideals and we have injective maps

$$1 : \mathcal{I} \longrightarrow \mathcal{O}_X, \quad s : \mathcal{I} \longrightarrow \mathcal{L}$$

whose cokernels are supported on closed nowhere dense subsets of X.

Proof. The question is local on X. Hence we may assume that $X = \mathrm{Spec}(A)$, and $\mathcal{L} = \mathcal{O}_X$. After shrinking further we may assume that $s = x/y$ with $a, b \in A$ *both* nonzerodivisors in A. Set $I = \{x \in A \mid x(a/b) \in A\}$.

To show that $\mathcal{I}$ is quasi-coherent we have to show that $I_f = \{x \in A_f \mid x(a/b) \in A_f\}$ for every $f \in A$. If $c/f^n \in A_f$, $(c/f^n)(a/b) \in A_f$, then we see that $f^m c(a/b) \in A$ for some m, hence $c/f^n \in I_f$. Conversely it is easy to see that I_f is contained in $\{x \in A_f \mid x(a/b) \in A_f\}$. This proves quasi-coherence.

Let us prove the final statement. It is clear that $(b) \subset I$. Hence $V(I) \subset V(b)$ is a nowhere dense subset as b is a nonzerodivisor. Thus the cokernel of 1 is supported in a nowhere dense closed set. The same argument works for the cokernel of s since $s(b) = (a) \subset sI \subset A$. □

Definition 15.15. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. The sheaf of ideals $\mathcal{I}$ constructed in Lemma 15.14 is called the *ideal sheaf of denominators of s*.

Here is a lemma which will be used later.

Lemma 15.16. *Suppose given*

1. *X a locally Noetherian scheme,*
2. *$\mathcal{L}$ an invertible $\mathcal{O}_X$-module,*
3. *s a regular meromorphic section of $\mathcal{L}$, and*
4. *$\mathcal{F}$ coherent on X without embedded associated points and Supp$(\mathcal{F}) = X$.*

Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s. Let $T \subset X$ be the union of the supports of $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{L}/s(\mathcal{I})$ which is a nowhere dense closed subset $T \subset X$ according to Lemma 15.14. Then there are canonical injective maps

$$1 : \mathcal{I}\mathcal{F} \to \mathcal{F}, \quad s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$$

whose cokernels are supported on T.

Proof. Reduce to the affine case with $\mathcal{L} \cong \mathcal{O}_X$, and $s = a/b$ with $a, b \in A$ both nonzerodivisors. Proof of reduction step omitted. Write $\mathcal{F} = \widetilde{M}$. Let $I = \{x \in A \mid x(a/b) \in A\}$ so that $\mathcal{I} = \widetilde{I}$ (see proof of Lemma 15.14). Note that $T = V(I) \cup V((a/b)I)$. For any A-module M consider the map $1 : IM \to M$; this is the map that gives rise to the map 1 of the lemma. Consider on the other hand the map $\sigma : IM \to M_b, x \mapsto ax/b$. Since b is not a zerodivisor in A, and since M has support $\mathrm{Spec}(A)$ and no embedded primes we see that b is a nonzerodivisor on M also. Hence $M \subset M_b$. By definition of I we have $\sigma(IM) \subset M$ as submodules of M_b. Hence we get an A-module map $s : IM \to M$ (namely the unique map such that $s(z)/1 = \sigma(z)$ in M_b for all $z \in IM$). It is injective because a is a nonzerodivisor also (on both A and M). It is clear that M/IM is annihilated by I and that $M/s(IM)$ is annihilated by $(a/b)I$. Thus the lemma follows. □

16. Relative Proj

Some results on relative Proj. First some very basic results. Recall that a relative Proj is always separated over the base, see Constructions, Lemma 16.9.

Lemma 16.1. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If one of the following holds*

1. *$\mathcal{A}$ is of finite type as a sheaf of $\mathcal{A}_0$-algebras,*
2. *$\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra and $\mathcal{A}_1$ is a finite type $\mathcal{A}_0$-module,*
3. *there exists a finite type quasi-coherent $\mathcal{A}_0$-submodule $\mathcal{F} \subset \mathcal{A}_+$ such that $\mathcal{A}_+/\mathcal{F}\mathcal{A}$ is a locally nilpotent sheaf of ideals of $\mathcal{A}/\mathcal{F}\mathcal{A}$,*

then p is quasi-compact.

Proof. The question is local on the base, see Schemes, Lemma 19.2. Thus we may assume S is affine. Say $S = \mathrm{Spec}(R)$ and $\mathcal{A}$ corresponds to the graded R-algebra A. Then $X = \mathrm{Proj}(A)$, see Constructions, Section 15. In case (1) we may after possibly localizing more assume that A is generated by homogeneous elements $f_1, \ldots, f_n \in A_+$ over A_0. Then $A_+ = (f_1, \ldots, f_n)$ by Algebra, Lemma 57.1. In case (3) we see that $\mathcal{F} = \widetilde{M}$ for some finite type A_0-module $M \subset A_+$. Say $M = \sum A_0 f_i$. Say $f_i = \sum f_{i,j}$ is the decomposition into homogeneous pieces. The condition in (2) signifies that $A_+ \subset \sqrt{(f_{i,j})}$. Thus in both cases we conclude that $\mathrm{Proj}(A)$ is quasi-compact by Constructions, Lemma 8.9. Finally, (2) follows from (1). $\square$

Lemma 16.2. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{O}_S$-algebras, then p is of finite type.*

Proof. The assumption implies that p is quasi-compact, see Lemma 16.1. Hence it suffices to show that p is locally of finite type. Thus the question is local on the base and target, see Morphisms, Lemma 16.2. Say $S = \mathrm{Spec}(R)$ and $\mathcal{A}$ corresponds to the graded R-algebra A. After further localizing on S we may assume that A is a finite type R-algebra. The scheme X is constructed out of glueing the spectra of the rings $A_{(f)}$ for $f \in A_+$ homogeneous. Each of these is of finite type over R by Algebra, Lemma 55.9. Thus $\mathrm{Proj}(A)$ is of finite type over R. $\square$

Lemma 16.3. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{O}_S \to \mathcal{A}_0$ is an integral algebra map*[2] *and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra, then p is universally closed.*

Proof. The question is local on the base. Thus we may assume that $X = \mathrm{Spec}(R)$ is affine. Let $\mathcal{A}$ be the quasi-coherent $\mathcal{O}_X$-algebra associated to the graded R-algebra A. The assumption is that $R \to A_0$ is integral and A is of finite type over A_0. Write $X \to \mathrm{Spec}(R)$ as the composition $X \to \mathrm{Spec}(A_0) \to \mathrm{Spec}(R)$. Since $R \to A_0$ is an integral ring map, we see that $\mathrm{Spec}(A_0) \to \mathrm{Spec}(R)$ is universally closed, see Morphisms, Lemma 44.7. The quasi-compact (see Constructions, Lemma 8.9) morphism

$$\mathrm{Proj}(A) \to \mathrm{Proj}(A_0)$$

satisfies the existence part of the valuative criterion by Constructions, Lemma 8.11 and hence it is universally closed by Schemes, Proposition 20.6. Thus $X \to \mathrm{Spec}(R)$ is universally closed as a composition of universally closed morphisms. $\square$

[2]In other words, the integral closure of $\mathcal{O}_S$ in $\mathcal{A}_0$, see Morphisms, Definition 48.2, equals $\mathcal{A}_0$.

Lemma 16.4. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. The following conditions are equivalent*

(1) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra,*
(2) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module and $\mathcal{A}$ is of finite type as an $\mathcal{O}_S$-algebra*

If these conditions hold, then p is locally projective and in particular proper.

Proof. Assume that $\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module. Choose an affine open $U = \mathrm{Spec}(R) \subset X$ such that $\mathcal{A}$ corresponds to a graded R-algebra A with A_0 a finite R-module. Condition (1) means that (after possibly localizing further on S) that A is a finite type A_0-algebra and condition (2) means that (after possibly localizing further on S) that A is a finite type R-algebra. Thus these conditions imply each other by Algebra, Lemma 6.2.

A locally projective morphism is proper, see Morphisms, Lemma 43.5. Thus we may now assume that $S = \mathrm{Spec}(R)$ and $X = \mathrm{Proj}(A)$ and that A_0 is finite over R and A of finite type over R. We will show that $X = \mathrm{Proj}(A) \to \mathrm{Spec}(R)$ is projective. We urge the reader to prove this for themselves, by directly constructing a closed immersion of X into a projective space over R, instead of reading the argument we give below.

By Lemma 16.2 we see that X is of finite type over $\mathrm{Spec}(R)$. Constructions, Lemma 10.6 tells us that $\mathcal{O}_X(d)$ is ample on X for some $d \geq 1$ (see Properties, Section 24). Hence $X \to \mathrm{Spec}(R)$ is quasi-projective (by Morphisms, Definition 41.1). By Morphisms, Lemma 43.12 we conclude that X is isomorphic to an open subscheme of a scheme projective over $\mathrm{Spec}(R)$. Therefore, to finish the proof, it suffices to show that $X \to \mathrm{Spec}(R)$ is universally closed (use Morphisms, Lemma 42.7). This follows from Lemma 16.3. $\square$

17. Closed subschemes of relative proj

Some auxiliary lemmas about closed subschemes of relative proj.

Lemma 17.1. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. Denote $\mathcal{I} \subset \mathcal{A}$ the kernel of the canonical map*

$$\mathcal{A} \longrightarrow \bigoplus\nolimits_{d \geq 0} p_*\left((i_*\mathcal{O}_Z)(d)\right)$$

If p is quasi-compact, then there is an isomorphism $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{I})$.

Proof. The morphism p is separated by Constructions, Lemma 16.9. As p is quasi-compact, p_* transforms quasi-coherent modules into quasi-coherent modules, see Schemes, Lemma 24.1. Hence $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_S$-module. In particular, $\mathcal{B} = \mathcal{A}/\mathcal{I}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. The functoriality morphism $Z' = \underline{\mathrm{Proj}}_S(\mathcal{B}) \to \underline{\mathrm{Proj}}_S(\mathcal{A})$ is everywhere defined and a closed immersion, see Constructions, Lemma 18.3. Hence it suffices to prove $Z = Z'$ as closed subschemes of X.

Having said this, the question is local on the base and we may assume that $S = \mathrm{Spec}(R)$ and that $X = \mathrm{Proj}(A)$ for some graded R-algebra A. Assume $\mathcal{I} = \widetilde{I}$ for $I \subset A$ a graded ideal. By Constructions, Lemma 8.9 there exist $f_0, \ldots, f_n \in A_+$ such that $A_+ \subset \sqrt{(f_0, \ldots, f_n)}$ in other words $X = \bigcup D_+(f_i)$. Therefore, it suffices to check that $Z \cap D_+(f_i) = Z' \cap D_+(f_i)$ for each i. By renumbering we may assume $i = 0$. Say $Z \cap D_+(f_0)$, resp. $Z' \cap D_+(f_0)$ is cut out by the ideal J, resp. J' of $A_{(f_0)}$.

The inclusion $J' \subset J$. Let d be the least common multiple of $\deg(f_0), \ldots, \deg(f_n)$. Note that each of the twists $\mathcal{O}_X(nd)$ is invertible, trivialized by $f_i^{nd/\deg(f_i)}$ over $D_+(f_i)$, and that for any quasi-coherent module $\mathcal{F}$ on X the multiplication maps $\mathcal{O}_X(nd) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \to \mathcal{F}(nd + m)$ are isomorphisms, see Constructions, Lemma 10.2. Observe that J' is the ideal generated by the elements g/f_0^e where $g \in I$ is homogeneous of degree $e\deg(f_0)$ (see proof of Constructions, Lemma 11.3). Of course, by replacing g by $f_0^l g$ for suitable l we may always assume that $d|e$. Then, since g vanishes as a section of $\mathcal{O}_X(e\deg(f_0))$ restricted to Z we see that g/f_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $g/f_0^e \in J$. Again we may assume $d|e$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(f_i)$ is cut out by some ideal $J_i \subset A_{(f_i)}$. Moreover,

$$J \cdot A_{(f_0 f_i)} = J_i \cdot A_{(f_0 f_i)}$$

The right hand side is the localization of J_i with respect to $f_0^{\deg(f_i)}/f_i^{\deg(f_0)}$. It follows that

$$f_0^{e_i} g / f_i^{(e_i + e)\deg(f_0)/\deg(f_i)} \in J_i$$

for some $e_i \gg 0$ sufficiently divisible. This proves that $f_0^{\max(e_i)} g$ is an element of I, because its restriction to each affine open $D_+(f_i)$ vanishes on the closed subscheme $Z \cap D_+(f_i)$. Hence $g \in J'$ and we conclude $J \subset J'$ as desired. □

In case the closed subscheme is locally cut out by finitely many equations we can define it by a finite type ideal sheaf of $\mathcal{A}$.

Lemma 17.2. *Let S be a quasi-compact and quasi-separated scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. If p is quasi-compact and i of finite presentation, then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{A}_d$ such that $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{F}\mathcal{A})$.*

Proof. By Lemma 17.1 we know there exists a quasi-coherent graded sheaf of ideals $\mathcal{I} \subset \mathcal{A}$ such that $Z = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{I})$. Since S is quasi-compact we can choose a finite affine open covering $S = U_1 \cup \ldots \cup U_n$. Say $U_i = \mathrm{Spec}(R_i)$. Let $\mathcal{A}|_{U_i}$ correspond to the graded R_i-algebra A_i and $\mathcal{I}|_{U_i}$ to the graded ideal $I_i \subset A_i$. Note that $p^{-1}(U_i) = \mathrm{Proj}(A_i)$ as schemes over R_i. Since p is quasi-compact we can choose finitely many homogeneous elements $f_{i,j} \in A_{i,+}$ such that $p^{-1}(U_i) = D_+(f_{i,j})$. The condition on $Z \to X$ means that the ideal sheaf of Z in $\mathcal{O}_X$ is of finite type, see Morphisms, Lemma 22.7. Hence we can find finitely many homogeneous elements $h_{i,j,k} \in I_i \cap A_{i,+}$ such that the ideal of $Z \cap D_+(f_{i,j})$ is generated by the elements $h_{i,j,k}/f_{i,j}^{e_{i,j,k}}$. Choose $d > 0$ to be a common multiple of all the integers $\deg(f_{i,j})$ and $\deg(h_{i,j,k})$. By Properties, Lemma 20.7 there exists a finite type $\mathcal{F} \subset \mathcal{I}_d$ such that all the local sections

$$h_{i,j,k} f_{i,j}^{(d - \deg(h_{i,j,k}))/\deg(f_{i,j})}$$

are sections of $\mathcal{F}$. By construction $\mathcal{F}$ is a solution. □

The following version of Lemma 17.2 will be used in the proof of Lemma 20.2.

Lemma 17.3. *Let S be a quasi-compact and quasi-separated scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. Let $U \subset X$ be an open. Assume that*

(1) *p is quasi-compact,*
(2) *i of finite presentation,*
(3) *$U \cap p(i(Z)) = \emptyset$,*
(4) *U is quasi-compact,*
(5) *$\mathcal{A}_n$ is a finite type $\mathcal{O}_S$-module for all n.*

Then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{A}_d$ with (a) $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{F}\mathcal{A})$ and (b) the support of $\mathcal{A}_d/\mathcal{F}$ is disjoint from U.

Proof. Let $\mathcal{I} \subset \mathcal{A}$ be the sheaf of quasi-coherent graded ideals constructed in Lemma 17.1. Let U_i, R_i, A_i, I_i, $f_{i,j}$, $h_{i,j,k}$, and d be as constructed in the proof of Lemma 17.2. Since $U \cap p(i(Z)) = \emptyset$ we see that $\mathcal{I}_d|_U = \mathcal{A}_d|_U$ (by our construction of $\mathcal{I}$ as a kernel). Since U is quasi-compact we can choose a finite affine open covering $U = W_1 \cup \ldots \cup W_m$. Since $\mathcal{A}_d$ is of finite type we can find finitely many sections $g_{t,s} \in \mathcal{A}_d(W_t)$ which generate $\mathcal{A}_d|_{W_t} = \mathcal{I}_d|_{W_t}$ as an $\mathcal{O}_{W_t}$-module. To finish the proof, note that by Properties, Lemma 20.7 there exists a finite type $\mathcal{F} \subset \mathcal{I}_d$ such that all the local sections

$$h_{i,j,k} f_{i,j}^{(d - \deg(h_{i,j,k}))/\deg(f_{i,j})} \quad \text{and} \quad g_{t,s}$$

are sections of $\mathcal{F}$. By construction $\mathcal{F}$ is a solution. □

18. Blowing up

Blowing up is an important tool in algebraic geometry.

Definition 18.1. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Schemes, Definition 10.2. The *blowing up of X along Z*, or the *blowing up of X in the ideal sheaf $\mathcal{I}$* is the morphism

$$b : \underline{\text{Proj}}_X\left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n\right) \longrightarrow X$$

The *exceptional divisor* of the blow up is the inverse image $b^{-1}(Z)$. Sometimes Z is called the *center* of the blowup.

We will see later that the exceptional divisor is an effective Cartier divisor. Moreover, the blowing up is characterized as the "smallest" scheme over X such that the inverse image of Z is an effective Cartier divisor.

If $b : X' \to X$ is the blow up of X in Z, then we often denote $\mathcal{O}_{X'}(n)$ the twists of the structure sheaf. Note that these are invertible $\mathcal{O}_{X'}$-modules and that $\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(1)^{\otimes n}$ because X' is the relative Proj of a quasi-coherent graded $\mathcal{O}_X$-algebra which is generated in degree 1, see Constructions, Lemma 16.11. Note that $\mathcal{O}_{X'}(1)$ is b-relatively very ample, even though b need not be of finite type or even quasi-compact, because X' comes equipped with a closed immersion into $\mathbf{P}(\mathcal{I})$, see Morphisms, Example 39.3.

Lemma 18.2. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $U = \text{Spec}(A)$ be an affine open subscheme of X and let $I \subset A$ be the ideal* *corresponding to $\mathcal{I}|_U$. If $b : X' \to X$ is the blow up of X in $\mathcal{I}$, then there is a canonical isomorphism*

$$b^{-1}(U) = Proj(\bigoplus\nolimits_{d \geq 0} I^d)$$

of $b^{-1}(U)$ with the homogeneous spectrum of the Rees algebra of I in A. Moreover, $b^{-1}(U)$ has an affine open covering by spectra of the affine blowup algebras $A[\frac{I}{a}]$.

Proof. The first statement is clear from the construction of the relative Proj via glueing, see Constructions, Section 15. For $a \in I$ denote $a^{(1)}$ the element a seen as an element of degree 1 in the Rees algebra $\bigoplus_{n \geq 0} I^n$. Since these elements generate the Rees algebra over A we see that $\text{Proj}(\bigoplus_{d \geq 0} I^d)$ is covered by the affine opens $D_+(a^{(1)})$. The affine scheme $D_+(a^{(1)})$ is the spectrum of the affine blowup algebra $A' = A[\frac{I}{a}]$, see Algebra, Definition 56.1. This finishes the proof. □

Lemma 18.3. *Let $X_1 \to X_2$ be a flat morphism of schemes. Let $Z_2 \subset X_2$ be a closed subscheme. Let Z_1 be the inverse image of Z_2 in X_1. Let X'_i be the blow up of Z_i in X_i. Then there exists a cartesian diagram*

$$\begin{tikzcd} X'_1 \arrow[r] \arrow[d] & X'_2 \arrow[d] \\ X_1 \arrow[r] & X_2 \end{tikzcd}$$

of schemes.

Proof. Let $\mathcal{I}_2$ be the ideal sheaf of Z_2 in X_2. Denote $g : X_1 \to X_2$ the given morphism. Then the ideal sheaf $\mathcal{I}_1$ of Z_1 is the image of $g^*\mathcal{I}_2 \to \mathcal{O}_{X_1}$ (by definition of the inverse image, see Schemes, Definition 17.7). By Constructions, Lemma 16.10 we see that $X_1 \times_{X_2} X'_2$ is the relative Proj of $\bigoplus_{n \geq 0} g^*\mathcal{I}_2^n$. Because g is flat the map $g^*\mathcal{I}_2^n \to \mathcal{O}_{X_1}$ is injective with image $\mathcal{I}_1^n$. Thus we see that $X_1 \times_{X_2} X'_2 = X'_1$. □

Lemma 18.4. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. The blowing up $b : X' \to X$ of Z in X has the following properties:*

1. *$b|_{b^{-1}(X \setminus Z)} : b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism,*
2. *the exceptional divisor $E = b^{-1}(Z)$ is an effective Cartier divisor on X',*
3. *there is a canonical isomorphism $\mathcal{O}_{X'}(-1) = \mathcal{O}_{X'}(E)$*

Proof. As blowing up commutes with restrictions to open subschemes (Lemma 18.3) the first statement just means that $X' = X$ if $Z = \emptyset$. In this case we are blowing up in the ideal sheaf $\mathcal{I} = \mathcal{O}_X$ and the result follows from Constructions, Example 8.14.

The second statement is local on X, hence we may assume X affine. Say $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$. By Lemma 18.2 we see that X' is covered by the spectra of the affine blowup algebras $A' = A[\frac{I}{a}]$. Then $IA' = aA'$ and a maps to a nonzerodivisor in A' according to Algebra, Lemma 56.2. This proves the lemma as the inverse image of Z in $\text{Spec}(A')$ corresponds to $\text{Spec}(A'/IA') \subset \text{Spec}(A')$.

Consider the canonical map $\psi_{univ,1} : b^*\mathcal{I} \to \mathcal{O}_{X'}(1)$, see discussion following Constructions, Definition 16.7. We claim that this factors through an isomorphism $\mathcal{I}_E \to \mathcal{O}_{X'}(1)$ (which proves the final assertion). Namely, on the affine open corresponding to the blowup algebra $A' = A[\frac{I}{a}]$ mentioned above $\psi_{univ,1}$ corresponds to the A'-module map

$$I \otimes_A A' \longrightarrow \left(\left(\bigoplus\nolimits_{d \geq 0} I^d\right)_{a^{(1)}}\right)_1$$

where $a^{(1)}$ is as in Algebra, Definition 56.1. We omit the verification that this is the map $I \otimes_A A' \to IA' = aA'$. $\square$

Lemma 18.5 (Universal property blowing up)**.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{C}$ be the full subcategory of (Sch/X) consisting of $Y \to X$ such that the inverse image of Z is an effective Cartier divisor on Y. Then the blowing up $b : X' \to X$ of Z in X is a final object of $\mathcal{C}$.*

Proof. We see that $b : X' \to X$ is an object of $\mathcal{C}$ according to Lemma 18.4. Let $f : Y \to X$ be an object of $\mathcal{C}$. We have to show there exists a unique morphism $Y \to X'$ over X. Let $D = f^{-1}(Z)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z and let $\mathcal{I}_D$ be the ideal sheaf of D. Then $f^*\mathcal{I} \to \mathcal{I}_D$ is a surjection to an invertible $\mathcal{O}_Y$-module. This extends to a map $\psi : \bigoplus f^*\mathcal{I}^d \to \bigoplus \mathcal{I}_D^d$ of graded $\mathcal{O}_Y$-algebras. (We observe that $\mathcal{I}_D^d = \mathcal{I}_D^{\otimes d}$ as D is an effective Cartier divisor.) By the material in Constructions, Section 16 the triple $(1, f : Y \to X, \psi)$ defines a morphism $Y \to X'$ over X. The restriction

$$Y \setminus D \longrightarrow X' \setminus b^{-1}(Z) = X \setminus Z$$

is unique. The open $Y \setminus D$ is scheme theoretically dense in Y according to Lemma 9.4. Thus the morphism $Y \to X'$ is unique by Morphisms, Lemma 7.10 (also b is separated by Constructions, Lemma 16.9). $\square$

Lemma 18.6. *Let X be a scheme. Let $Z \subset X$ be an effective Cartier divisor. The blowup of X in Z is the identity morphism of X.*

Proof. Immediate from the universal property of blowups (Lemma 18.5). $\square$

Lemma 18.7. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is integral, then the blow up X' of X in $\mathcal{I}$ is integral.*

Proof. Combine Lemma 18.2 with Algebra, Lemma 56.4. $\square$

Lemma 18.8. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blow up X' of X in $\mathcal{I}$ is reduced.*

Proof. Combine Lemma 18.2 with Algebra, Lemma 56.5. $\square$

Lemma 18.9. *Let X be a scheme. Let $b : X' \to X$ be a blow up of X in a closed subscheme. For any effective Cartier divisor D on X the pullback $b^{-1}D$ is defined (see Definition 9.11).*

Proof. By Lemmas 18.2 and 9.2 this reduces to the following algebra fact: Let A be a ring, $I \subset A$ an ideal, $a \in I$, and $x \in A$ a nonzerodivisor. Then the image of x in $A[\frac{I}{a}]$ is a nonzerodivisor. Namely, suppose that $x(y/a^n) = 0$ in $A[\frac{I}{a}]$. Then $a^m xy = 0$ in A for some m. Hence $a^m y = 0$ as x is a nonzerodivisor. Whence y/a^n is zero in $A[\frac{I}{a}]$ as desired. $\square$

Lemma 18.10. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ and $\mathcal{J}$ be quasi-coherent sheaves of ideals. Let $b : X' \to X$ be the blowing up of X in $\mathcal{I}$. Let $b' : X'' \to X'$ be the blowing up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$. Then $X'' \to X$ is canonically isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$.*

Proof. Let $E \subset X'$ be the exceptional divisor of b which is an effective Cartier divisor by Lemma 18.4. Then $(b')^{-1}E$ is an effective Cartier divisor on X'' by Lemma 18.9. Let $E' \subset X''$ be the exceptional divisor of b' (also an effective Cartier divisor). Consider the effective Cartier divisor $E'' = E' + (b')^{-1}E$. By construction the ideal of E'' is $(b \circ b')^{-1}\mathcal{I}(b \circ b')^{-1}\mathcal{J}\mathcal{O}_{X''}$. Hence according to Lemma 18.5 there is a canonical morphism from X'' to the blowup $c : Y \to X$ of X in $\mathcal{I}\mathcal{J}$. Conversely, as $\mathcal{I}\mathcal{J}$ pulls back to an invertible ideal we see that $c^{-1}\mathcal{I}\mathcal{O}_Y$ defines an effective Cartier divisor, see Lemma 9.9. Thus a morphism $c' : Y \to X'$ over X by Lemma 18.5. Then $(c')^{-1}b^{-1}\mathcal{J}\mathcal{O}_Y = c^{-1}\mathcal{J}\mathcal{O}_Y$ which also defines an effective Cartier divisor. Thus a morphism $c'' : Y \to X''$ over X'. We omit the verification that this morphism is inverse to the morphism $X'' \to Y$ constructed earlier. □

Lemma 18.11. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $b : X' \to X$ be the blowing up of X in the ideal sheaf $\mathcal{I}$ If $\mathcal{I}$ is of finite type, then*

1. *$b : X' \to X$ is a projective morphism, and*
2. *$\mathcal{O}_{X'}(1)$ is a b-relatively ample invertible sheaf.*

Proof. The surjection of graded $\mathcal{O}_X$-algebras

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{I}) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{I}^d$$

defines via Constructions, Lemma 18.5 a closed immersion

$$X' = \underline{\mathrm{Proj}}_X(\bigoplus\nolimits_{d \geq 0} \mathcal{I}^d) \longrightarrow \mathbf{P}(\mathcal{I}).$$

Hence b is projective, see Morphisms, Definition 43.1. The second statement follows for example from the characterization of relatively ample invertible sheaves in Morphisms, Lemma 38.4. Some details omitted. □

Lemma 18.12. *Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subscheme of finite presentation. Let $b : X' \to X$ be the blowing up with center Z. Let $Z' \subset X'$ be a closed subscheme of finite presentation. Let $X'' \to X'$ be the blowing up with center Z'. There exists a closed subscheme $Y \subset X$ of finite presentation, such that*

1. *$Y = Z \cup b(Z')$ set theoretically, and*
2. *the composition $X'' \to X$ is isomorphic to the blowing up of X in Y.*

Proof. The condition that $Z \to X$ is of finite presentation means that Z is cut out by a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, see Morphisms, Lemma 22.7. Write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ so that $X' = \underline{\mathrm{Proj}}(\mathcal{A})$. Note that $X \setminus Z$ is a quasi-compact open of X by Properties, Lemma 22.1. Since $b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism (Lemma 18.4) the same result shows that $b^{-1}(X \setminus Z) \setminus Z'$ is quasi-compact open in X'. Hence $U = X \setminus (Z \cup b(Z'))$ is quasi-compact open in X. By Lemma 17.3 there exist a $d > 0$ and a finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{I}^d$ such that $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ and such that the support of $\mathcal{I}^d/\mathcal{F}$ is contained in $X \setminus U$.

Since $\mathcal{F} \subset \mathcal{I}^d$ is an $\mathcal{O}_X$-submodule we may think of $\mathcal{F} \subset \mathcal{I}^d \subset \mathcal{O}_X$ as a finite type quasi-coherent sheaf of ideals on X. Let's denote this $\mathcal{J} \subset \mathcal{O}_X$ to prevent confusion. Since $\mathcal{I}^d/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}^d$ are supported on $X \setminus U$ we see that $V(\mathcal{J})$ is contained in $X \setminus U$. Conversely, as $\mathcal{J} \subset \mathcal{I}^d$ we see that $Z \subset V(\mathcal{J})$. Over $X \setminus Z \cong X' \setminus b^{-1}(Z)$ the sheaf of ideals $\mathcal{J}$ cuts out Z' (see displayed formula below). Hence $V(\mathcal{J})$ equals

$Z \cup b(Z')$. It follows that also $V(\mathcal{I}\mathcal{J}) = Z \cup b(Z')$ set theoretically. Moreover, $\mathcal{I}\mathcal{J}$ is an ideal of finite type as a product of two such. We claim that $X'' \to X$ is isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$ which finishes the proof of the lemma by setting $Y = V(\mathcal{I}\mathcal{J})$.

First, recall that the blow up of X in $\mathcal{I}\mathcal{J}$ is the same as the blow up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$, see Lemma 18.10. Hence it suffices to show that the blow up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$ agrees with the blow up of X' in Z'. We will show that

$$b^{-1}\mathcal{J}\mathcal{O}_{X'} = \mathcal{I}_E^d\mathcal{I}_{Z'}$$

as ideal sheaves on X''. This will prove what we want as $\mathcal{I}_E^d$ cuts out the effective Cartier divisor dE and we can use Lemmas 18.6 and 18.10.

To see the displayed equality of the ideals we may work locally. With notation A, I, $a \in I$ as in Lemma 18.2 we see that $\mathcal{F}$ corresponds to an R-submodule $M \subset I^d$ mapping isomorphically to an ideal $J \subset R$. The condition $Z' = \underline{\text{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ means that $Z' \cap \text{Spec}(A[\frac{I}{a}])$ is cut out by the ideal generated by the elements m/a^d, $m \in M$. Say the element $m \in M$ corresponds to the function $f \in J$. Then in the affine blowup algebra $A' = A[\frac{I}{a}]$ we see that $f = (a^d m)/a^d = a^d(m/a^d)$. Thus the equality holds. $\square$

19. Strict transform

In this section we briefly discuss strict transform under blowing up. Let S be a scheme and let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up of S in Z and denote $E \subset S'$ the exceptional divisor $E = b^{-1}Z$. In the following we will often consider a scheme X over S and form the cartesian diagram

$$\begin{tikzcd}
\text{pr}_{S'}^{-1}E \arrow[r] \arrow[d] & X \times_S S' \arrow[r, "\text{pr}_X"'] \arrow[d, "\text{pr}_{S'}"'] & X \arrow[d, "f"] \\
E \arrow[r] & S' \arrow[r] & S
\end{tikzcd}$$

Since E is an effective Cartier divisor (Lemma 18.4) we see that $\text{pr}_{S'}^{-1}E \subset X \times_S S'$ is locally principal (Lemma 9.10). Thus the complement of $\text{pr}_{S'}^{-1}E$ in $X \times_S S'$ is retrocompact (Lemma 9.3). Consequently, for a quasi-coherent $\mathcal{O}_{X \times_S S'}$-module $\mathcal{G}$ the subsheaf of sections supported on $\text{pr}_{S'}^{-1}E$ is a quasi-coherent submodule, see Properties, Lemma 22.5. If $\mathcal{G}$ is a quasi-coherent sheaf of algebras, e.g., $\mathcal{G} = \mathcal{O}_{X \times_S S'}$, then this subsheaf is an ideal of $\mathcal{G}$.

Definition 19.1. With $Z \subset S$ and $f : X \to S$ as above.

1. Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the *strict transform* of $\mathcal{F}$ with respect to the blowup of S in Z is the quotient $\mathcal{F}'$ of $\text{pr}_X^*\mathcal{F}$ by the submodule of sections supported on $\text{pr}_{S'}^{-1}E$.
2. The *strict transform* of X is the closed subscheme $X' \subset X \times_S S'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X \times_S S'}$ supported on $\text{pr}_{S'}^{-1}E$.

Note that taking the strict transform along a blowup depends on the closed subscheme used for the blowup (and not just on the morphism $S' \to S$). This notion is often used for closed subschemes of S. It turns out that the strict transform of X is a blowup of X.

Lemma 19.2. *In the situation of Definition 19.1.*

(1) *The strict transform X' of X is the blowup of X in the closed subscheme $f^{-1}Z$ of X.*
(2) *For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform $\mathcal{F}'$ is canonically isomorphic to the pushforward along $X' \to X \times_S S'$ of the strict transform of $\mathcal{F}$ relative to the blowing up $X' \to X$.*

Proof. Let $X'' \to X$ be the blowup of X in $f^{-1}Z$. By the universal property of blowing up (Lemma 18.5) there exists a commutative diagram

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

whence a morphism $X'' \to X \times_S S'$. Thus the first assertion is that this morphism is a closed immersion with image X'. The question is local on X. Thus we may assume X and S are affine. Say that $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and Z is cut out by the ideal $I \subset A$. Set $J = IB$. The map $B \otimes_A \bigoplus_{n \geq 0} I^n \to \bigoplus_{n \geq 0} J^n$ defines a closed immersion $X'' \to X \times_S S'$, see Constructions, Lemmas 11.6 and 11.5. We omit the verification that this morphism is the same as the one constructed above from the universal property. Pick $a \in I$ corresponding to the affine open $\text{Spec}(A[\frac{I}{a}]) \subset S'$, see Lemma 18.2. The inverse image of $\text{Spec}(A[\frac{I}{a}])$ in the strict transform X' of X is the spectrum of

$$B' = (B \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

On the other hand, letting $b \in J$ be the image of a we see that $\text{Spec}(B[\frac{J}{b}])$ is the inverse image of $\text{Spec}(A[\frac{I}{a}])$ in X''. The ring map

$$B \otimes_A A[\tfrac{I}{a}] \longrightarrow B[\tfrac{J}{b}]$$

see Properties, Lemma 22.5. is surjective and annihilates a-power torsion as b is a nonzerodivisor in $B[\frac{J}{b}]$. Hence we obtain a surjective map $B' \to B[\frac{J}{b}]$. To see that the kernel is trivial, we construct an inverse map. Namely, let $z = y/b^n$ be an element of $B[\frac{J}{b}]$, i.e., $y \in J^n$. Write $y = \sum x_i b_i$ with $x_i \in I^n$ and $b_i \in B$. We map z to the class of $\sum b_i \otimes x_i/a^n$ in B'. This is well defined because an element of the kernel of the map $B \otimes_A I^n \to J^n$ is annihilated by a^n, hence maps to zero in B'. This shows that the open $\text{Spec}(B[\frac{J}{b}])$ maps isomorphically to the open subscheme $\text{pr}_{S'}^{-1}(\text{Spec}(A[\frac{I}{a}]))$ of X'. Thus $X'' \to X'$ is an isomorphism.

In the notation above, let $\mathcal{F}$ correspond to the B-module N. The strict transform of $\mathcal{F}$ corresponds to the $B \otimes_A A[\frac{I}{a}]$-module

$$N' = (N \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 22.5. The strict transform of $\mathcal{F}$ relative to the blowup of X in $f^{-1}Z$ corresponds to the $B[\frac{J}{b}]$-module $N \otimes_B B[\frac{J}{b}]/b$-power-torsion. In exactly the same way as above one proves that these two modules are isomorphic. Details omitted. $\square$

Lemma 19.3. *In the situation of Definition 19.1.*

(1) *If X is flat over S at all points lying over Z, then the strict transform of X is equal to the base change $X \times_S S'$.*

(2) *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over S at all points lying over Z, then the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is equal to the pullback $pr_X^*\mathcal{F}$.*

Proof. We will prove part (2) as it implies part (1) by the definition of the strict transform of a scheme over S. The question is local on X. Thus we may assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and that $\mathcal{F}$ corresponds to the B-module N. Then $\mathcal{F}'$ over the open $\mathrm{Spec}(B \otimes_A A[\frac{I}{a}])$ of $X \times_S S'$ corresponds to the module

$$N' = (N \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 22.5. Thus we have to show that the a-power-torsion of $N \otimes_A A[\frac{I}{a}]$ is zero. Let $y \in N \otimes_A A[\frac{I}{a}]$ with $a^n y = 0$. If $\mathfrak{q} \subset B$ is a prime and $a \not\in \mathfrak{q}$, then y maps to zero in $(N \otimes_A A[\frac{I}{a}])_\mathfrak{q}$. on the other hand, if $a \in \mathfrak{q}$, then $N_\mathfrak{q}$ is a flat A-module and we see that $N_\mathfrak{q} \otimes_A A[\frac{I}{a}] = (N \otimes_A A[\frac{I}{a}])_\mathfrak{q}$ has no a-power torsion (as $A[\frac{I}{a}]$ doesn't). Hence y maps to zero in this localization as well. We conclude that y is zero by Algebra, Lemma 23.1. □

Lemma 19.4. *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up of Z in S. Let $g : X \to Y$ be an affine morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g' : X \times_S S' \to Y \times_S S'$ be the base change of g. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ relative to b. Then $g'_*\mathcal{F}'$ is the strict transform of $g_*\mathcal{F}$.*

Proof. Observe that $g'_*\mathrm{pr}_X^*\mathcal{F} = \mathrm{pr}_Y^*g_*\mathcal{F}$ by Cohomology of Schemes, Lemma 5.1. Let $\mathcal{K} \subset \mathrm{pr}_X^*\mathcal{F}$ be the subsheaf of sections supported in the inverse image of Z in $X \times_S S'$. By Properties, Lemma 22.7 the pushforward $g'_*\mathcal{K}$ is the subsheaf of sections of $\mathrm{pr}_Y^*g_*\mathcal{F}$ supported in the inverse image of Z in $Y \times_S S'$. As g' is affine (Morphisms, Lemma 13.8) we see that g'_* is exact, hence we conclude. □

Lemma 19.5. *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $D \subset S$ be an effective Cartier divisor. Let $Z' \subset S$ be the closed subscheme cut out by the product of the ideal sheaves of Z and D. Let $S' \to S$ be the blowup of S in Z.*

(1) *The blowup of S in Z' is isomorphic to $S' \to S$.*
(2) *Let $f : X \to S$ be a morphism of schemes and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ has no nonzero local sections supported in $f^{-1}D$, then the strict transform of $\mathcal{F}$ relative to the blowing up in Z agrees with the strict transform of $\mathcal{F}$ relative to the blowing up of S in Z'.*

Proof. The first statement follows on combining Lemmas 18.10 and 18.6. Using Lemma 18.2 the second statement translates into the following algebra problem. Let A be a ring, $I \subset A$ an ideal, $x \in A$ a nonzerodivisor, and $a \in I$. Let M be an A-module whose x-torsion is zero. To show: the a-power torsion in $M \otimes_A A[\frac{I}{a}]$ is equal to the xa-power torsion. The reason for this is that the kernel and cokernel of the map $A \to A[\frac{I}{a}]$ is a-power torsion, so this map becomes an isomorphism after inverting a. Hence the kernel and cokernel of $M \to M \otimes_A A[\frac{I}{a}]$ are a-power torsion too. This implies the result. □

Lemma 19.6. *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up with center Z. Let $Z' \subset S'$ be a closed subscheme. Let $S'' \to S'$ be the blowing up with center Z'. Let $Y \subset S$ be a closed subscheme such that $Y = Z \cup b(Z')$ set theoretically and the composition $S'' \to S$ is isomorphic to* *the blowing up of S in Y. In this situation, given any scheme X over S and $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have*

1. *the strict transform of $\mathcal{F}$ with respect to the blowing up of S in Y is equal to the strict transform with respect to the blowup $S'' \to S'$ in Z' of the strict transform of $\mathcal{F}$ with respect to the blowup $S' \to S$ of S in Z, and*
2. *the strict transform of X with respect to the blowing up of S in Y is equal to the strict transform with respect to the blowup $S'' \to S'$ in Z' of the strict transform of X with respect to the blowup $S' \to S$ of S in Z.*

Proof. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ with respect to the blowup $S' \to S$ of S in Z. Let $\mathcal{F}''$ be the strict transform of $\mathcal{F}'$ with respect to the blowup $S'' \to S'$ of S' in Z'. Let $\mathcal{G}$ be the strict transform of $\mathcal{F}$ with respect to the blowup $S'' \to S$ of S in Y. We also label the morphisms

$$\begin{tikzcd} X \times_S S'' \arrow[r, "q"'] \arrow[d, "f''"] & X \times_S S' \arrow[r, "p"'] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

By definition there is a surjection $p^*\mathcal{F} \to \mathcal{F}'$ and a surjection $q^*\mathcal{F}' \to \mathcal{F}''$ which combine by right exactness of q^* to a surjection $(p \circ q)^*\mathcal{F} \to \mathcal{F}''$. Also we have the surjection $(p \circ q)^*\mathcal{F} \to \mathcal{G}$. Thus it suffices to prove that these two surjections have the same kernel.

The kernel of the surjection $p^*\mathcal{F} \to \mathcal{F}'$ is supported on $(f \circ p)^{-1}Z$, so this map is an isomorphism at points in the complement. Hence the kernel of $q^*p^*\mathcal{F} \to q^*\mathcal{F}'$ is supported on $(f \circ p \circ q)^{-1}Z$. The kernel of $q^*\mathcal{F}' \to \mathcal{F}''$ is supported on $(f' \circ q)^{-1}Z'$. Combined we see that the kernel of $(p \circ q)^*\mathcal{F} \to \mathcal{F}''$ is supported on $(f \circ p \circ q)^{-1}Z \cup (f' \circ q)^{-1}Z' = (f \circ p \circ q)^{-1}Y$. By construction of $\mathcal{G}$ we see that we obtain a factorization $(p \circ q)^*\mathcal{F} \to \mathcal{F}'' \to \mathcal{G}$. To finish the proof it suffices to show that $\mathcal{F}''$ has no nonzero (local) sections supported on $(f \circ p \circ q)^{-1}(Y) = (f \circ p \circ q)^{-1}Z \cup (f' \circ q)^{-1}Z'$. This follows from Lemma 19.5 applied to $\mathcal{F}'$ on $X \times_S S'$ over S', the closed subscheme Z' and the effective Cartier divisor $b^{-1}Z$. $\square$

Lemma 19.7. *In the situation of Definition 19.1. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of quasi-coherent sheaves on X which remains exact after any base change $T \to S$. Then the strict transforms of $\mathcal{F}'_i$ relative to any blowup $S' \to S$ form a short exact sequence $0 \to \mathcal{F}'_1 \to \mathcal{F}'_2 \to \mathcal{F}'_3 \to 0$ too.

Proof. We may localize on S and X and assume both are affine. Then we may push $\mathcal{F}_i$ to S, see Lemma 19.4. We may assume that our blowup is the morphism $1 : S \to S$ associated to an effective Cartier divisor $D \subset S$. Then the translation into algebra is the following: Suppose that A is a ring and $0 \to M_1 \to M_2 \to M_3 \to 0$ is a universally exact sequence of A-modules. Let $a \in A$. Then the sequence

$$0 \to M_1/a\text{-power torsion} \to M_2/a\text{-power torsion} \to M_3/a\text{-power torsion} \to 0$$

is exact too. Namely, surjectivity of the last map and injectivity of the first map are immediate. The problem is exactness in the middle. Suppose that $x \in M_2$ maps to zero in M_3/a-power torsion. Then $y = a^n x \in M_1$ for some n. Then y maps to zero in M_2/a^nM_2. Since $M_1 \to M_2$ is universally injective we see that y maps to zero in M_1/a^nM_1. Thus $y = a^n z$ for some $z \in M_1$. Thus $a^n(x - y) = 0$. Hence y maps to the class of x in M_2/a-power torsion as desired. $\square$

20. Admissible blowups

To have a bit more control over our blowups we introduce the following standard terminology.

Definition 20.1. Let X be a scheme. Let $U \subset X$ be an open subscheme. A morphism $X' \to X$ is called a *U-admissible blowup* if there exists a closed immersion $Z \to X$ of finite presentation with Z disjoint from U such that X' is isomorphic to the blow up of X in Z.

We recall that $Z \to X$ is of finite presentation if and only if the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is of finite type, see Morphisms, Lemma 22.7. In particular, a U-admissible blowup is a projective morphism, see Lemma 18.11. Note that there can be multiple centers which give rise to the same morphism. Hence the requirement is just the existence of some center disjoint from U which produces X'. Finally, as the morphism $b : X' \to X$ is an isomorphism over U (see Lemma 18.4) we will often abuse notation and think of U as an open subscheme of X' as well.

Lemma 20.2. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open subscheme. Let $b : X' \to X$ be a U-admissible blowup. Let $X'' \to X'$ be a U-admissible blowup. Then the composition $X'' \to X$ is a U-admissible blowup.*

Proof. Immediate from the more precise Lemma 18.12. $\square$

Lemma 20.3. *Let X be a quasi-compact and quasi-separated scheme. Let $U, V \subset X$ be quasi-compact open subschemes. Let $b : V' \to V$ be a $U \cap V$-admissible blowup. Then there exists a U-admissible blowup $X' \to X$ whose restriction to V is V'.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_V$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I})$ is disjoint from $U \cap V$ and such that V' is isomorphic to the blow up of V in $\mathcal{I}$. Let $\mathcal{I}' \subset \mathcal{O}_{U \cup V}$ be the quasi-coherent sheaf of ideals whose restriction to U is $\mathcal{O}_U$ and whose restriction to V is $\mathcal{I}$ (see Sheaves, Section 33). By Properties, Lemma 20.2 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ whose restriction to $U \cup V$ is $\mathcal{I}'$. The lemma follows. $\square$

Lemma 20.4. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open subscheme. Let $b_i : X_i \to X$, $i = 1, \ldots, n$ be U-admissible blowups. There exists a U-admissible blowup $b : X' \to X$ such that (a) b factors as $X' \to X_i \to X$ for $i = 1, \ldots, n$ and (b) each of the morphisms $X' \to X_i$ is a U-admissible blowup.*

Proof. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_i)$ is disjoint from U and such that X_i is isomorphic to the blow up of X in $\mathcal{I}_i$. Set $\mathcal{I} = \mathcal{I}_1 \cdot \ldots \cdot \mathcal{I}_n$ and let X' be the blowup of X in $\mathcal{I}$. Then $X' \to X$ factors through b_i by Lemma 18.10. $\square$

Lemma 20.5. *Let X be a quasi-compact and quasi-separated scheme. Let U, V be quasi-compact disjoint open subschemes of X. Then there exist a $U \cup V$-admissible blowup $b : X' \to X$ such that X' is a disjoint union of open subschemes $X' = X'_1 \amalg X'_2$ with $b^{-1}(U) \subset X'_1$ and $b^{-1}(V) \subset X'_2$.*

Proof. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}$, resp. $\mathcal{J}$ such that $X \setminus U = V(\mathcal{I})$, resp. $X \setminus V = V(\mathcal{J})$, see Properties, Lemma 22.1. Then $V(\mathcal{I}\mathcal{J}) = X$ set theoretically, hence $\mathcal{I}\mathcal{J}$ is a locally nilpotent sheaf of ideals. Since $\mathcal{I}$ and $\mathcal{J}$ are of finite type and X is quasi-compact there exists an $n > 0$ such that $\mathcal{I}^n\mathcal{J}^n = 0$. We may and do replace $\mathcal{I}$ by $\mathcal{I}^n$ and $\mathcal{J}$ by $\mathcal{J}^n$. Whence $\mathcal{I}\mathcal{J} = 0$. Let $b : X' \to X$ be the blowing up in $\mathcal{I} + \mathcal{J}$. This is $U \cup V$-admissible as $V(\mathcal{I} + \mathcal{J}) = X \setminus U \cup V$. We will show that X' is a disjoint union of open subschemes $X' = X'_1 \amalg X'_2$ such that $b^{-1}\mathcal{I}|_{X'_2} = 0$ and $b^{-1}\mathcal{J}|_{X'_1} = 0$ which will prove the lemma.

We will use the description of the blowing up in Lemma 18.2. Suppose that $U = \mathrm{Spec}(A) \subset X$ is an affine open such that $\mathcal{I}|_U$, resp. $\mathcal{J}|_U$ corresponds to the finitely generated ideal $I \subset A$, resp. $J \subset A$. Then

$$b^{-1}(U) = \mathrm{Proj}(A \oplus (I + J) \oplus (I + J)^2 \oplus \ldots)$$

This is covered by the affine open subsets $A[\frac{I+J}{x}]$ and $A[\frac{I+J}{y}]$ with $x \in I$ and $y \in J$. Since $x \in I$ is a nonzerodivisor in $A[\frac{I+J}{x}]$ and $IJ = 0$ we see that $JA[\frac{I+J}{x}] = 0$. Since $y \in J$ is a nonzerodivisor in $A[\frac{I+J}{y}]$ and $IJ = 0$ we see that $IA[\frac{I+J}{y}] = 0$. Moreover,

$$\mathrm{Spec}(A[\tfrac{I+J}{x}]) \cap \mathrm{Spec}(A[\tfrac{I+J}{y}]) = \mathrm{Spec}(A[\tfrac{I+J}{xy}]) = \emptyset$$

because xy is both a nonzerodivisor and zero. Thus $b^{-1}(U)$ is the disjoint union of the open subscheme U_1 defined as the union of the standard opens $\mathrm{Spec}(A[\frac{I+J}{x}])$ for $x \in I$ and the open subscheme U_2 which is the union of the affine opens $\mathrm{Spec}(A[\frac{I+J}{y}])$ for $y \in J$. We have seen that $b^{-1}\mathcal{I}\mathcal{O}_{X'}$ restricts to zero on U_2 and $b^{-1}\mathcal{I}\mathcal{O}_{X'}$ restricts to zero on U_1. We omit the verification that these open subschemes glue to global open subschemes X'_1 and X'_2. $\square$

21. Other chapters

References

[DG67] Jean Dieudonné and Alexander Grothendieck, *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1961–1967).

[Kle79] Steven Lawrence Kleiman, *Misconceptions about* K_X, Enseign. Math. (2) **25** (1979), no. 3-4, 203–206.

[Nob77] Augusto Nobile, *A note on flat algebras*, Proc. Amer. Math. Soc. **64** (1977), no. 2, 206–208.