DIFFERENTIAL GRADED ALGEBRA

Contents

1. Introduction

In this chapter we talk about differential graded algebras, modules, categories, etc. A basic reference is [Kel94]. A survey paper is [Kel06].

Since we do not worry about length of exposition in the Stacks project we first develop the material in the setting of categories of differential graded modules.

After that we redo the constructions in the setting of differential graded modules over differential graded categories.

2. Conventions

In this chapter we hold on to the convention that *ring* means commutative ring with 1. If R is a ring, then an *R-algebra A* will be an R-module A endowed with an R-bilinear map $A \times A \to A$ (multiplication) such that multiplication is associative and has a unit. In other words, these are unital associative R-algebras such that the structure map $R \to A$ maps into the center of A.

3. Differential graded algebras

Just the definitions.

Definition 3.1. Let R be a commutative ring. A *differential graded algebra over R* is either

(1) a chain complex $A_\bullet$ of R-modules endowed with R-bilinear maps $A_n \times A_m \to A_{n+m}$, $(a, b) \mapsto ab$ such that

$$\mathrm{d}_{n+m}(ab) = \mathrm{d}_n(a)b + (-1)^n a\mathrm{d}_m(b)$$

and such that $\bigoplus A_n$ becomes an associative and unital R-algebra, or

(2) a cochain complex $A^\bullet$ of R-modules endowed with R-bilinear maps $A^n \times A^m \to A^{n+m}$, $(a, b) \mapsto ab$ such that

$$\mathrm{d}^{n+m}(ab) = \mathrm{d}^n(a)b + (-1)^n a\mathrm{d}^m(b)$$

and such that $\bigoplus A^n$ becomes an associative and unital R-algebra.

We often just write $A = \bigoplus A_n$ or $A = \bigoplus A^n$ and think of this as an associative unital R-algebra endowed with a $\mathbf{Z}$-grading and an R-linear operator d whose square is zero and which satisfies the Leibniz rule as explained above. In this case we often say "Let (A, d) be a differential graded algebra".

Definition 3.2. A *homomorphism of differential graded algebras* $f : (A, \mathrm{d}) \to (B, \mathrm{d})$ is an algebra map $f : A \to B$ compatible with the gradings and d.

Definition 3.3. Let R be a ring. Let (A, d) be a differential graded algebra over R. The *opposite differential graded algebra* is the differential graded algebra (A^{opp}, d) over R where $A^{opp} = A$ as an R-module, $\mathrm{d} = \mathrm{d}$, and multiplication is given by

$$a \cdot_{opp} b = (-1)^{\deg(a)\deg(b)} ba$$

for homogeneous elements $a, b \in A$.

This makes sense because

$$
\begin{aligned}
\mathrm{d}(a \cdot_{opp} b) &= (-1)^{\deg(a)\deg(b)}\mathrm{d}(ba) \\
&= (-1)^{\deg(a)\deg(b)}\mathrm{d}(b)a + (-1)^{\deg(a)\deg(b)+\deg(b)}b\mathrm{d}(a) \\
&= (-1)^{\deg(a)}a \cdot_{opp} \mathrm{d}(b) + \mathrm{d}(a) \cdot_{opp} b
\end{aligned}
$$

as desired.

Definition 3.4. A differential graded algebra (A, d) is *commutative* if $ab = (-1)^{nm}ba$ for a in degree n and b in degree m. We say A is *strictly commutative* if in addition $a^2 = 0$ for $\deg(a)$ odd.

The following definition makes sense in general but is perhaps "correct" only when tensoring commutative differential graded algebras.

Definition 3.5. Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. The *tensor product differential graded algebra* of A and B is the algebra $A \otimes_R B$ with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'$$

endowed with differential d defined by the rule $\mathrm{d}(a \otimes b) = \mathrm{d}(a) \otimes b + (-1)^m a \otimes \mathrm{d}(b)$ where $m = \deg(b)$.

Lemma 3.6. *Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. Denote $A^\bullet$, $B^\bullet$ the underlying cochain complexes. As cochain complexes of R-modules we have*

$$(A \otimes_R B)^\bullet = Tot(A^\bullet \otimes_R B^\bullet).$$

Proof. Recall that the differential of the total complex is given by $\mathrm{d}_1^{p,q} + (-1)^p \mathrm{d}_2^{p,q}$ on $A^p \otimes_R B^q$. And this is exactly the same as the rule for the differential on $A \otimes_R B$ in Definition 3.5. $\square$

4. Differential graded modules

Just the definitions.

Definition 4.1. Let R be a ring. Let (A, d) be a differential graded algebra over R. A (right) *differential graded module* M over A is a right A-module M which has a grading $M = \bigoplus M^n$ and a differential d such that $M^n A^m \subset M^{n+m}$, such that $\mathrm{d}(M^n) \subset M^{n+1}$, and such that

$$\mathrm{d}(ma) = \mathrm{d}(m)a + (-1)^n m \mathrm{d}(a)$$

for $a \in A$ and $m \in M^n$. A *homomorphism of differential graded modules* $f : M \to N$ is an A-module map compatible with gradings and differentials. The category of (right) differential graded A-modules is denoted $\mathrm{Mod}_{(A,\mathrm{d})}$.

Note that we can think of M as a cochain complex $M^\bullet$ of (right) R-modules. Namely, for $r \in R$ we have $\mathrm{d}(r) = 0$ and r maps to a degree 0 element of A, hence $\mathrm{d}(mr) = \mathrm{d}(m)r$.

We can define *left differential graded A-modules* in exactly the same manner. If M is a left A-module, then we can think of M as a right A^{opp}-module with mulitplication $\cdot_{opp}$ defined by the rule

$$m \cdot_{opp} a = (-1)^{\deg(a)\deg(m)} am$$

for a and m homogeneous. The category of left differential graded A-modules is equivalent to the category of right differential graded A^{opp}-modules. We prefer to work with right modules (essentially because of what happens in Example 19.8), but the reader is free to switch to left modules if (s)he so desires.

Lemma 4.2. *Let (A, d) be a differential graded algebra. The category $\mathrm{Mod}_{(A,d)}$ is abelian and has arbitrary limits and colimits.*

Proof. Kernels and cokernels commute with taking underlying A-modules. Similarly for direct sums and colimits. In other words, these operations in $\mathrm{Mod}_{(A,\mathrm{d})}$ commute with the forgetful functor to the category of A-modules. This is not the case for products and limits. Namely, if N_i, $i \in I$ is a family of differential graded A-modules, then the product $\prod N_i$ in $\mathrm{Mod}_{(A,\mathrm{d})}$ is given by setting $(\prod N_i)^n = \prod N_i^n$ and $\prod N_i = \bigoplus_n (\prod N_i)^n$. Thus we see that the product does commute with the forgetful functor to the category of graded A-modules. A category with products and equalizers has limits, see Categories, Lemma 14.10. $\qquad\square$

Thus, if (A, d) is a differential graded algebra over R, then there is an exact functor

$$\mathrm{Mod}_{(A,\mathrm{d})} \longrightarrow \mathrm{Comp}(R)$$

of abelian categories. For a differential graded module M the cohomology groups $H^n(M)$ are defined as the cohomology of the corresponding complex of R-modules. Therefore, a short exact sequence $0 \to K \to L \to M \to 0$ of differential graded modules gives rise to a long exact sequence

(4.2.1) $$H^n(K) \to H^n(L) \to H^n(M) \to H^{n+1}(K)$$

of cohomology modules, see Homology, Lemma 12.12.

Moreover, from now on we borrow all the terminology used for complexes of modules. For example, we say that a differential graded A-module M is *acyclic* if $H^k(M) = 0$ for all $k \in \mathbf{Z}$. We say that a homomorphism $M \to N$ of differential graded A-modules is a *quasi-isomorphism* if it induces isomorphisms $H^k(M) \to H^k(N)$ for all $k \in \mathbf{Z}$. And so on and so forth.

Definition 4.3. Let (A, d) be a differential graded algebra. Let M be a differential graded module. For any $k \in \mathbf{Z}$ we define the *k-shifted module $M[k]$* as follows

    (1) as A-module $M[k] = M$,
    (2) $M[k]^n = M^{n+k}$,
    (3) $\mathrm{d}_{M[k]} = (-1)^k \mathrm{d}_M$.

For a morphism $f : M \to N$ of differential graded A-modules we let $f[k] : M[k] \to N[k]$ be the map equal to f on underlying A-modules. This defines a functor $[k] : \mathrm{Mod}_{(A,\mathrm{d})} \to \mathrm{Mod}_{(A,\mathrm{d})}$.

The remarks in Homology, Section 14 apply. In particular, we will identify the cohomology groups of all shifts $M[k]$ without the intervention of signs.

At this point we have enough structure to talk about *triangles*, see Derived Categories, Definition 3.1. In fact, our next goal is to develop enough theory to be able to state and prove that the homotopy category of differential graded modules is a triangulated category. First we define the homotopy category.

5. The homotopy category

Our homotopies take into account the A-module structure and the grading, but not the differential (of course).

Definition 5.1. Let (A, d) be a differential graded algebra. Let $f, g : M \to N$ be homomorphisms of differential graded A-modules. A *homotopy between f and g* is an A-module map $h : M \to N$ such that

    (1) $h(M^n) \subset N^{n-1}$ for all n, and
    (2) $f(x) - g(x) = \mathrm{d}_N(h(x)) + h(\mathrm{d}_M(x))$ for all $x \in M$.

If a homotopy exists, then we say f and g are *homotopic*.

Thus h is compatible with the A-module structure and the grading but not with the differential. If $f = g$ and h is a homotopy as in the definition, then h defines a morphism $h : M \to N[-1]$ in $\mathrm{Mod}_{(A,\mathrm{d})}$.

Lemma 5.2. *Let (A, d) be a differential graded algebra. Let $f, g : L \to M$ be homomorphisms of differential graded A-modules. Suppose given further homomorphisms $a : K \to L$, and $c : M \to N$. If $h : L \to M$ is an A-module map which defines a homotopy between f and g, then $c \circ h \circ a$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Immediate from Homology, Lemma 12.7. $\qquad\square$

This lemma allows us to define the homotopy category as follows.

Definition 5.3. Let (A, d) be a differential graded algebra. The *homotopy category*, denoted $K(\mathrm{Mod}_{(A,\mathrm{d})})$, is the category whose objects are the objects of $\mathrm{Mod}_{(A,\mathrm{d})}$ and whose morphisms are homotopy classes of homomorphisms of differential graded A-modules.

The notation $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is not standard but at least is consistent with the use of $K(-)$ in other places of the Stacks project.

Lemma 5.4. *Let (A, d) be a differential graded algebra. The homotopy category $K(Mod_{(A,d)})$ has direct sums and products.*

Proof. Omitted. Hint: Just use the direct sums and products as in Lemma 4.2. This works because we saw that these functors commute with the forgetful functor to the category of graded A-modules and because $\prod$ is an exact functor on the category of families of abelian groups. $\qquad\square$

6. Cones

We introduce cones for the category of differential graded modules.

Definition 6.1. Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded A-modules. The *cone* of f is the differential graded A-module $C(f)$ given by $C(f) = L \oplus K$ with grading $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d_{C(f)} = \begin{pmatrix} \mathrm{d}_L & f \\ 0 & -\mathrm{d}_K \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L \to C(f)$ and $p : C(f) \to K[1]$ induced by the obvious maps $L \to C(f)$ and $C(f) \to K$.

The formation of the cone triangle is functorial in the following sense.

Lemma 6.2. *Let (A, d) be a differential graded algebra. Suppose that*

$$
\begin{array}{ccc}
K_1 & \xrightarrow{\ f_1\ } & L_1 \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\
K_2 & \xrightarrow{\ f_2\ } & L_2
\end{array}
$$

is a diagram of homomorphisms of diferential graded A-modules which is commutative up to homotopy. Then there exists a morphism $c : C(f_1) \to C(f_2)$ which gives rise to a morphism of triangles

$$(a, b, c) : (K_1, L_1, C(f_1), f_1, i_1, p_1) \to (K_1, L_1, C(f_1), f_2, i_2, p_2)$$

in $K(Mod_{(A,d)})$.

Proof. Let $h : K_1 \to L_2$ be a homotopy between $f_2 \circ a$ and $b \circ f_1$. Define c by the matrix

$$c = \begin{pmatrix} b & h \\ 0 & a \end{pmatrix} : L_1 \oplus K_1 \to L_2 \oplus K_2$$

A matrix computation show that c is a morphism of differential graded modules. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. $\square$

7. Admissible short exact sequences

An admissible short exact sequence is the analogue of termwise split exact sequences in the setting of differential graded modules.

Definition 7.1. Let (A, d) be a differential graded algebra.
   (1) A homomorphism $K \to L$ of differential graded A-modules is an *admissible monomorphism* if there exists a graded A-module map $L \to K$ which is left inverse to $K \to L$.
   (2) A homomorphism $L \to M$ of differential graded A-modules is an *admissible epimorphism* if there exists a graded A-module map $M \to L$ which is right inverse to $L \to M$.
   (3) A short exact sequence $0 \to K \to L \to M \to 0$ of differential graded A-modules is an *admissible short exact sequence* if it is split as a sequence of graded A-modules.

Thus the splittings are compatible with all the data except for the differentials. Given an admissible short exact sequence we obtain a triangle; this is the reason that we require our splittings to be compatible with the A-module structure.

Lemma 7.2. *Let (A, d) be a differential graded algebra. Let $0 \to K \to L \to M \to 0$ be an admissible short exact sequence of differential graded A-modules. Let $s : M \to L$ and $\pi : L \to K$ be splittings such that $Ker(\pi) = Im(s)$. Then we obtain a morphism*

$$\delta = \pi \circ d_L \circ s : M \to K[1]$$

of $Mod_{(A,d)}$ which induces the boundary maps in the long exact sequence of cohomology (4.2.1).

Proof. The map $\pi \circ \mathrm{d}_L \circ s$ is compatible with the A-module structure and the gradings by construction. It is compatible with differentials by Homology, Lemmas 14.10. Let R be the ring that A is a differential graded algebra over. The equality of maps is a statement about R-modules. Hence this follows from Homology, Lemmas 14.10 and 14.11. $\square$

Lemma 7.3. *Let (A, d) be a differential graded algebra. Let*

$$\begin{array}{ccc} K & \xrightarrow{\ f\ } & L \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ M & \xrightarrow{\ g\ } & N \end{array}$$

be a diagram of homomorphisms of differential graded A-modules commuting up to homotopy.

(1) *If f is an admissible monomorphism, then b is homotopic to a homomorphism which makes the diagram commute.*
(2) *If g is an admissible epimorphism, then a is homotopic to a morphism which makes the diagram commute.*

Proof. Let $h : K \to N$ be a homotopy between bf and ga, i.e., $bf - ga = \mathrm{d}h + h\mathrm{d}$. Suppose that $\pi : L \to K$ is a graded A-module map left inverse to f. Take $b' = b - \mathrm{d}h\pi - h\pi\mathrm{d}$. Suppose $s : N \to M$ is a graded A-module map right inverse to g. Take $a' = a + \mathrm{d}sh + sh\mathrm{d}$. Computations omitted. $\square$

Lemma 7.4. *Let (A, d) be a differential graded algebra. Let $\alpha : K \to L$ be a homomorphism of differential graded A-modules. There exists a factorization*

$$K \xrightarrow{\ \tilde{\alpha}\ } \tilde{L} \xrightarrow{\ \pi\ } L$$
$$\alpha$$

in $Mod_{(A,d)}$ such that

(1) *$\tilde{\alpha}$ is an admissible monomorphism (see Definition 7.1),*
(2) *there is a morphism $s : L \to \tilde{L}$ such that $\pi \circ s = id_L$ and such that $s \circ \pi$ is homotopic to $id_{\tilde{L}}$.*

Proof. The proof is identical to the proof of Derived Categories, Lemma 9.6. Namely, we set $\tilde{L} = L \oplus C(1_K)$ and we use elementary properties of the cone construction. $\square$

Lemma 7.5. *Let (A, d) be a differential graded algebra. Let $L_1 \to L_2 \to \ldots \to L_n$ be a sequence of composable homomorphisms of differential graded A-modules. There exists a commutative diagram*

$$
\begin{array}{ccccccc}
L_1 & \longrightarrow & L_2 & \longrightarrow & \ldots & \longrightarrow & L_n \\
\uparrow & & \uparrow & & & & \uparrow \\
M_1 & \longrightarrow & M_2 & \longrightarrow & \ldots & \longrightarrow & M_n
\end{array}
$$

in $Mod_{(A,d)}$ such that each $M_i \to M_{i+1}$ is an admissible monomorphism and each $M_i \to L_i$ is a homotopy equivalence.

Proof. The case $n = 1$ is without content. Lemma 7.4 is the case $n = 2$. Suppose we have constructed the diagram except for M_n. Apply Lemma 7.4 to the composition $M_{n-1} \to L_{n-1} \to L_n$. The result is a factorization $M_{n-1} \to M_n \to L_n$ as desired. $\square$

Lemma 7.6. *Let (A, d) be a differential graded algebra. Let $0 \to K_i \to L_i \to M_i \to 0$, $i = 1, 2, 3$ be admissible short exact sequence of differential graded A-modules. Let $b : L_1 \to L_2$ and $b' : L_2 \to L_3$ be homomorphisms of differential graded modules such that*

$$
\begin{array}{ccccc}
K_1 & \longrightarrow & L_1 & \longrightarrow & M_1 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle 0} \\
K_2 & \longrightarrow & L_2 & \longrightarrow & M_2
\end{array}
\quad and \quad
\begin{array}{ccccc}
K_2 & \longrightarrow & L_2 & \longrightarrow & M_2 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle 0} \\
K_3 & \longrightarrow & L_3 & \longrightarrow & M_3
\end{array}
$$

commute up to homotopy. Then $b' \circ b$ is homotopic to 0.

Proof. By Lemma 7.3 we can replace b and b' by homotopic maps such that the right square of the left diagram commutes and the left square of the right diagram commutes. In other words, we have $\mathrm{Im}(b) \subset \mathrm{Im}(K_2 \to L_2)$ and $\mathrm{Ker}((b')^n) \supset \mathrm{Im}(K_2 \to L_2)$. Then $b \circ b' = 0$ as a map of modules. $\square$

8. Distinguished triangles

The following lemma produces our distinguished triangles.

Lemma 8.1. *Let (A, d) be a differential graded algebra. Let $0 \to K \to L \to M \to 0$ be an admissible short exact sequence of differential graded A-modules. The triangle*

$$(8.1.1) \qquad\qquad K \to L \to M \xrightarrow{\delta} K[1]$$

with δ as in Lemma 7.2 is, up to canonical isomorphism in $K(Mod_{(A,d)})$, independent of the choices made in Lemma 7.2.

Proof. Namely, let (s', π') be a second choice of splittings as in Lemma 7.2. Then we claim that δ and δ' are homotopic. Namely, write $s' = s + \alpha \circ h$ and $\pi' = \pi + g \circ \beta$ for some unique homomorphisms of A-modules $h : M \to K$ and $g : M \to K$ of degree -1. Then $g = -h$ and g is a homotopy between δ and δ'. The computations are done in the proof of Homology, Lemma 14.12. $\square$

Definition 8.2. Let (A, d) be a differential graded algebra.
   (1) If $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded A-modules, then the *triangle associated to* $0 \to K \to L \to M \to 0$ is the triangle $(8.1.1)$ of $K(Mod_{(A,d)})$.
   (2) A triangle of $K(Mod_{(A,d)})$ is called a *distinguished triangle* if it is isomorphic to a triangle associated to an admissible short exact sequence of differential graded A-modules.

9. Cones and distinguished triangles

Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded A-modules. Then $(K, L, C(f), f, i, p)$ forms a triangle:

$$K \to L \to C(f) \to K[1]$$

in $Mod_{(A,d)}$ and hence in $K(Mod_{(A,d)})$. Cones are **not** distinguished triangles in general, but the difference is a sign or a rotation (your choice). Here are two precise statements.

Lemma 9.1. *Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded modules. The triangle $(L, C(f), K[1], i, p, f[1])$ is the triangle associated to the admissible short exact sequence*

$$0 \to L \to C(f) \to K[1] \to 0$$

coming from the definition of the cone of f.

Proof. Immediate from the definitions. $\square$

Lemma 9.2. *Let (A, d) be a differential graded algebra. Let $\alpha : K \to L$ and $\beta : L \to M$ define an admissible short exact sequence*

$$0 \to K \to L \to M \to 0$$

of differential graded A-modules. Let $(K, L, M, \alpha, \beta, \delta)$ be the associated triangle. Then the triangles

$$(M[-1], K, L, \delta[-1], \alpha, \beta) \quad and \quad (M[-1], K, C(\delta[-1]), \delta[-1], i, p)$$

are isomorphic.

Proof. Using a choice of splittings we write $L = K \oplus M$ and we identify α and β with the natural inclusion and projection maps. By construction of δ we have

$$d_B = \begin{pmatrix} d_K & \delta \\ 0 & d_M \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : M[-1] \to K$ is given as $C(\delta[-1]) = K \oplus M$ with differential identical with the matrix above! Whence the lemma. $\square$

Lemma 9.3. *Let (A, d) be a differential graded algebra. Let $f_1 : K_1 \to L_1$ and $f_2 : K_2 \to L_2$ be homomorphisms of differential graded A-modules. Let*

$$(a, b, c) : (K_1, L_1, C(f_1), f_1, i_1, p_1) \longrightarrow (K_1, L_1, C(f_1), f_2, i_2, p_2)$$

be any morphism of triangles of $K(\mathrm{Mod}_{(A,d)})$. If a and b are homotopy equivalences then so is c.

Proof. Let $a^{-1} : K_2 \to K_1$ be a homomorphism of differential graded A-modules which is inverse to a in $K(\mathrm{Mod}_{(A,d)})$. Let $b^{-1} : L_2 \to L_1$ be a homomorphism of differential graded A-modules which is inverse to b in $K(\mathrm{Mod}_{(A,d)})$. Let $c' : C(f_2) \to C(f_1)$ be the morphism from Lemma 6.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\mathrm{Mod}_{(A,d)})$ then we win. Hence it suffices to prove the following: Given a morphism of triangles $(1, 1, c) : (K, L, C(f), f, i, p)$ in $K(\mathrm{Mod}_{(A,d)})$ the morphism c is an isomorphism in $K(\mathrm{Mod}_{(A,d)})$. By assumption the two squares in the diagram

$$
\begin{array}{ccccc}
L & \longrightarrow & C(f) & \longrightarrow & K[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 1} \\
L & \longrightarrow & C(f) & \longrightarrow & K[1]
\end{array}
$$

commute up to homotopy. By construction of $C(f)$ the rows form admissible short exact sequences. Thus we see that $(c - 1)^2 = 0$ in $K(\mathrm{Mod}_{(A,d)})$ by Lemma 7.6. Hence c is an isomorphism in $K(\mathrm{Mod}_{(A,d)})$ with inverse $2 - c$. $\square$

The following lemma shows that the collection of triangles of the homotopy category given by cones and the distinguished triangles are the same up to isomorphisms, at least up to sign!

Lemma 9.4. *Let (A, d) be a differential graded algebra.*

(1) *Given an admissible short exact sequence* $0 \to K \xrightarrow{\alpha} L \to M \to 0$ *of differential graded A-modules there exists a homotopy equivalence $C(\alpha) \to M$ such that the diagram*

$$
\begin{array}{ccccccc}
K & \longrightarrow & L & \longrightarrow & C(\alpha) & \xrightarrow{\ -p\ } & K[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K & \xrightarrow{\alpha} & L & \xrightarrow{\beta} & M & \xrightarrow{\delta} & K[1]
\end{array}
$$

defines an isomorphism of triangles in $K(Mod_{(A,d)})$.

(2) *Given a morphism of complexes $f : K \to L$ there exists an isomorphism of triangles*

$$
\begin{array}{ccccccc}
K & \longrightarrow & \tilde{L} & \longrightarrow & M & \xrightarrow{\ \delta\ } & K[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K & \longrightarrow & L & \longrightarrow & C(f) & \xrightarrow{\ -p\ } & K[1]
\end{array}
$$

where the upper triangle is the triangle associated to a admissible short exact sequence $K \to \tilde{L} \to M$.

Proof. Proof of (1). We have $C(\alpha) = L \oplus K$ and we simply define $C(\alpha) \to M$ via the projection onto L followed by β. This defines a morphism of differential graded modules because the compositions $K^{n+1} \to L^{n+1} \to M^{n+1}$ are zero. Choose splittings $s : M \to L$ and $\pi : L \to K$ with $\mathrm{Ker}(\pi) = \mathrm{Im}(s)$ and set $\delta = \pi \circ d_L \circ s$ as usual. To get a homotopy inverse we take $M \to C(\alpha)$ given by $(s, -\delta)$. This is compatible with differentials because δ^n can be characterized as the unique map $M^n \to K^{n+1}$ such that $d \circ s^n - s^{n+1} \circ d = \alpha \circ \delta^n$, see proof of Homology, Lemma 14.10. The composition $M \to C(f) \to M$ is the identity. The composition $C(f) \to M \to C(f)$ is equal to the morphism

$$
\begin{pmatrix} s \circ \beta & 0 \\ -\delta \circ \beta & 0 \end{pmatrix}
$$

To see that this is homotopic to the identity map use the homotopy $h : C(\alpha) \to C(\alpha)$ given by the matrix

$$
\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} : C(\alpha) = L \oplus K \to L \oplus K = C(\alpha)
$$

It is trivial to verify that

$$
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s \\ -\delta \end{pmatrix} \begin{pmatrix} \beta & 0 \end{pmatrix} = \begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} \begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix}
$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha) \to K[1]$ (see Definition 6.1) and $C(\alpha) \to M \to K[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : M \to C(\alpha)$ and check instead that the two maps $M \to K[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K \to \tilde{L}$, $s : L \to \tilde{L}$ and $\pi : L \to L$ be as in Lemma 7.4. By Lemmas 6.2 and 9.3 the triangles $(K, L, C(f), i, p)$ and $(K, \tilde{L}, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may

replace L by $\tilde{L}$ and f by $\tilde{f}$. In other words we may assume that f is an admissible monomorphism. In this case the result follows from part (1). $\square$

10. The homotopy category is triangulated

We first prove that it is pre-triangulated.

Lemma 10.1. *Let (A, d) be a differential graded algebra. The homotopy category $K(Mod_{(A,d)})$ with its natural translation functors and distinguished triangles is a pre-triangulated category.*

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(K, K, 0, 1, 0, 0)$ is distinguished since $0 \to K \to K \to 0 \to 0$ is an admissible short exact sequence. Finally, given any homomorphism $f : K \to L$ of differential graded A-modules the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 9.4.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists an admissible short exact sequence $0 \to K \to L \to M \to 0$ such that the associated triangle $(K, L, M, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(M[-1], K, L, -\delta[-1], \alpha, \beta)$. It follows from Lemma 9.2 that the triangle $(M[-1], K, L, \delta[-1], \alpha, \beta)$ is isomorphic to $(M[-1], K, C(\delta[-1]), \delta[-1], i, p)$. Pre-composing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(M[-1], K, C(\delta[-1]), \delta[-1], i, -p)$. Hence it is distinguished by Lemma 9.4. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 9.4 this means that it is isomorphic to a triangle of the form $(K, L, C(f), f, i, -p)$ for some morphism f of $\text{Mod}_{(A,d)}$. Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L, C(f), K[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L, C(f), K[1], i, p, f[1])$. By Lemma 9.1 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \to X'$ and $b : Y \to Y'$ be morphisms such that $f' \circ a = b \circ f$. By Lemma 9.4 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, -p)$ and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', -p')$. At this point we simply apply Lemma 6.2 to the commutative diagram given by f, f', a, b. $\square$

Before we prove TR4 in general we prove it in a special case.

Lemma 10.2. *Let (A, d) be a differential graded algebra. Suppose that $\alpha : K \to L$ and $\beta : L \to M$ are admissible monomorphisms of differential graded A-modules. Then there exist distinguished triangles $(K, L, Q_1, \alpha, p_1, d_1)$, $(K, M, Q_2, \beta \circ \alpha, p_2, d_2)$ and $(L, M, Q_3, \beta, p_3, d_3)$ for which TR4 holds.*

Proof. Say $\pi_1 : L \to K$ and $\pi_3 : M \to L$ are homomorphisms of graded A-modules which are left inverse to α and β. Then also $K \to M$ is an admissible monomorphism with left inverse $\pi_2 = \pi_1 \circ \pi_3$. Let us write Q_1, Q_2 and Q_3 for the cokernels of $K \to L$, $K \to M$, and $L \to M$. Then we obtain identifications (as graded A-modules) $Q_1 = \text{Ker}(\pi_1)$, $Q_3 = \text{Ker}(\pi_3)$ and $Q_2 = \text{Ker}(\pi_2)$. Then $L = K \oplus Q_1$ and $M = L \oplus Q_3$ as graded A-modules. This implies $M = K \oplus Q_1 \oplus Q_3$.

Note that $\pi_2 = \pi_1 \circ \pi_3$ is zero on both Q_1 and Q_3. Hence $Q_2 = Q_1 \oplus Q_3$. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & K & \to & L & \to & Q_1 & \to & 0 \\
  &   & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & M & \to & Q_2 & \to & 0 \\
  &   & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & M & \to & Q_3 & \to & 0
\end{array}
$$

The rows of this diagram are admissible short exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of triangles

$$(K \to L \to Q_1 \to K[1]) \longrightarrow (K \to M \to Q_2 \to K[1])$$

and

$$(K \to M \to Q_2 \to K[1]) \longrightarrow (L \to M \to Q_3 \to L[1]).$$

Note that the splittings $Q_3 \to M$ of the bottom sequence in the diagram provides a splitting for the split sequence $0 \to Q_1 \to Q_2 \to Q_3 \to 0$ upon composing with $M \to Q_2$. It follows easily from this that the morphism $\delta : Q_3 \to Q_1[1]$ in the corresponding distinguished triangle

$$(Q_1 \to Q_2 \to Q_3 \to Q_1[1])$$

is equal to the composition $Q_3 \to L[1] \to Q_1[1]$. Hence we get a structure as in the conclusion of axiom TR4. $\qquad\square$

Here is the final result.

Proposition 10.3. *Let (A, d) be a differential graded algebra. The homotopy category $K(Mod_{(A,d)})$ of differential graded A-modules with its natural translation functors and distinguished triangles is a triangulated category.*

Proof. We know that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a pre-triangulated category. Hence it suffices to prove TR4 and to prove it we can use Derived Categories, Lemma 4.13. Let $K \to L$ and $L \to M$ be composable morphisms of $K(\mathrm{Mod}_{(A,\mathrm{d})})$. By Lemma 7.5 we may assume that $K \to L$ and $L \to M$ are admissible monomorphisms. In this case the result follows from Lemma 10.2. $\qquad\square$

11. Projective modules over algebras

In this section we discuss projective modules over algebras and over graded algebras. Thus it is the analogue of Algebra, Section 74 in the setting of this chapter.

Algebras and modules. Let R be a ring and let A be an R-algebra, see Section 2 for our conventions. It is clear that A is a projective right A-module since $\mathrm{Hom}_A(A, M) = M$ for any right A-module M (and thus $\mathrm{Hom}_A(A, -)$ is exact). Conversely, let P be a projective right A-module. Then we can choose a surjection $\bigoplus_{i \in I} A \to M$ by choosing a set $\{m_i\}_{i \in I}$ of generators of P over A. Since P is projective there is a left inverse to the surjection, and we find that P is isomorphic to a direct summand of a free module, exactly as in the commutative case (Algebra, Lemma 74.2).

Graded algebras and modules. Let R be a ring. Let A be a graded algebra over R. Let Mod_A denote the category of graded right A-modules. For an integer k let $A[k]$ denote the shift of A. For an graded right A-module we have

$$\mathrm{Hom}_{\mathrm{Mod}_A}(A[k], M) = M^{-k}$$

As the functor $M \mapsto M^{-k}$ is exact on Mod_A we conclude that $A[k]$ is a projective object of Mod_A. Conversely, suppose that P is a projective object of Mod_A. By choosing a set of homogeneous generators of P as an A-module, we can find a surjection

$$\bigoplus_{i \in I} A[k_i] \longrightarrow P$$

Thus we conclude that a projective object of Mod_A is a direct summand of a direct sum of the shifts $A[k]$.

If (A, d) is a differential graded algebra and P is an object of $\mathrm{Mod}_{(A,\mathrm{d})}$ then we say P *is projective as a graded A-module* or sometimes P *is graded projective* to mean that P is a projective object of the abelian category Mod_A of graded A-modules.

Lemma 11.1. *Let (A, d) be a differential graded algebra. Let $M \to P$ be a surjective homomorphism of differential graded A-modules. If P is projective as a graded A-module, then $M \to P$ is an admissible epimorphism.*

Proof. This is immediate from the definitions. $\square$

Lemma 11.2. *Let (A, d) be a differential graded algebra. Then we have*

$$\mathrm{Hom}_{Mod_{(A,d)}}(A[k], M) = Ker(d : M^{-k} \to M^{-k+1})$$

and

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(A[k], M) = H^{-k}(M)$$

for any differential graded A-module M.

Proof. This is clear from the discussion above. $\square$

12. Injective modules over algebras

In this section we discuss injective modules over algebras and over graded algebras. Thus it is the analogue of More on Algebra, Section 42 in the setting of this chapter.

Algebras and modules. Let R be a ring and let A be an R-algebra, see Section 2 for our conventions. For a right A-module M we set

$$M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$$

which we think of as a left A-module by the multiplication $(af)(x) = f(xa)$. Namely, $((ab)f)(x) = f(xab) = (bf)(xa) = (a(bf))(x)$. Conversely, if M is a left A-module, then $M^\vee$ is a right A-module. Since $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group (More on Algebra, Lemma 41.1), the functor $M \mapsto M^\vee$ is exact (More on Algebra, Lemma 42.2). Moreover, the evaluation map $M \to (M^\vee)^\vee$ is injective for all modules M (More on Algebra, Lemma 42.3).

We claim that $A^\vee$ is an injective right A-module. Namely, given a right A-module N we have

$$\mathrm{Hom}_A(N, A^\vee) = \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})) = N^\vee$$

and we conclude because the functor $N \mapsto N^\vee$ is exact. The second equality holds because

$$\mathrm{Hom}_{\mathbf{Z}}(N, \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})) = \mathrm{Hom}_{\mathbf{Z}}(N \otimes_{\mathbf{Z}} A, \mathbf{Q}/\mathbf{Z})$$

by Algebra, Lemma 11.8. Inside this module A-linearity exactly picks out the bilinear maps $\varphi : N \times A \to \mathbf{Q}/\mathbf{Z}$ which have the same value on $x \otimes a$ and $xa \otimes 1$, i.e., come from elements of $N^\vee$.

Finally, for every right A-module M we can choose a surjection $\bigoplus_{i \in I} A \to M^\vee$ to get an injection $M \to (M^\vee)^\vee \to \prod_{i \in I} A^\vee$.

We conclude

  (1) the category of A-modules has enough injectives,
  (2) $A^\vee$ is an injective A-module, and
  (3) every A-module injects into a product of copies of $A^\vee$.

Graded algebras and modules. Let R be a ring. Let A be a graded algebra over R. If M is a graded A-module we set

$$M^\vee = \bigoplus_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(M^{-n}, \mathbf{Q}/\mathbf{Z}) = \bigoplus_{n \in \mathbf{Z}} (M^{-n})^\vee$$

as a graded R-module with the A-module structure defined as above (for homogeneous elements). This again switches left and right modules. On the category of graded A-modules the functor $M \mapsto M^\vee$ is exact (check on graded pieces). Moreover, the evaluation map $M \to (M^\vee)^\vee$ is injective as before (because we can check this on the graded pieces).

We claim that $A^\vee$ is an injective object of the category Mod_A of graded right A-modules. Namely, given a graded right A-module N we have

$$\mathrm{Hom}_{\mathrm{Mod}_A}(N, A^\vee) = \mathrm{Hom}_{\mathrm{Mod}_A}(N, \bigoplus \mathrm{Hom}_{\mathbf{Z}}(A^{-n}, \mathbf{Q}/\mathbf{Z})) = (N^0)^\vee$$

and we conclude because the functor $N \mapsto (N^0)^\vee = (N^\vee)^0$ is exact. To see that the second equality holds we use the equalities

$$\mathrm{Hom}_{\mathbf{Z}}(N^n, \mathrm{Hom}_{\mathbf{Z}}(A^{-n}, \mathbf{Q}/\mathbf{Z})) = \mathrm{Hom}_{\mathbf{Z}}(N^n \otimes_{\mathbf{Z}} A^{-n}, \mathbf{Q}/\mathbf{Z})$$

of Algebra, Lemma 11.8. Thus an element of $\mathrm{Hom}_{\mathrm{Mod}_A}(N, A^\vee)$ corresponds to a family of $\mathbf{Z}$-bilinear maps $\psi_n : N^n \times A^{-n} \to \mathbf{Q}/\mathbf{Z}$ such that $\psi_n(x, a) = \psi_0(xa, 1)$ for all $x \in N^n$ and $a \in A^{-n}$. Moreover, $\psi_0(x, a) = \psi_0(xa, 1)$ for all $x \in N^0$, $a \in A^0$. It follows that the maps ψ_n are determined by ψ_0 and that $\psi_0(x, a) = \varphi(xa)$ for a unique element $\varphi \in (N^0)^\vee$.

Finally, for every graded right A-module M we can choose a surjection (of graded left A-modules)

$$\bigoplus_{i \in I} A[k_i] \to M^\vee$$

where $A[k_i]$ denotes the shift of A by $k_i \in \mathbf{Z}$. (We do this by choosing homogeneous generators for $M^\vee$.) In this way we get an injection

$$M \to (M^\vee)^\vee \to \prod A[k_i]^\vee = \prod A^\vee[-k_i]$$

Observe that the products in the formula above are products in the category of graded modules (in other words, take products in each degree and then take the direct sum of the pieces).

We conclude that

(1) the category of graded A-modules has enough injectives,
(2) for every $k \in \mathbf{Z}$ the module $A^\vee[k]$ is injective, and
(3) every A-module injects into a product in the category of graded modules of copies of shifts $A^\vee[k]$.

If (A, d) is a differential graded algebra and I is an object of $\mathrm{Mod}_{(A,\mathrm{d})}$ then we say I *is injective as a graded A-module* to mean that I is a injective object of the abelian category Mod_A of graded A-modules.

Lemma 12.1. *Let (A, d) be a differential graded algebra. Let $I \to M$ be an injective homomorphism of differential graded A-modules. If I is an injective object of the category of graded A-modules, then $I \to M$ is an admissible monomorphism.*

Proof. This is immediate from the definitions. $\square$

Let (A, d) be a differential graded algebra. If M is a **left** differential graded A-module, then we will endow $M^\vee$ (with its graded module structure as above) with a right differential graded module structure by setting

$$\mathrm{d}_{M^\vee}(f) = -(-1)^n f \circ \mathrm{d}_M^{-n-1} \quad \text{in } (M^\vee)^{n+1}$$

for $f \in (M^\vee)^n = \mathrm{Hom}_{\mathbf{Z}}(M^{-n}, \mathbf{Q}/\mathbf{Z})$ and $\mathrm{d}_M^{-n-1} : M^{-n-1} \to M^{-n}$ the differential of M[1]. We will show by a computation that this works. Namely, if $a \in A^m$, $x \in M^{-n-m-1}$ and $f \in (M^\vee)^n$, then we have

$$\begin{aligned}
\mathrm{d}_{M^\vee}(fa)(x) &= -(-1)^{n+m}(fa)(\mathrm{d}_M(x)) \\
&= -(-1)^{n+m} f(a\mathrm{d}_M(x)) \\
&= -(-1)^n f(\mathrm{d}_M(ax) - \mathrm{d}(a)x) \\
&= -(-1)^n[-(-1)^n \mathrm{d}_{M^\vee}(f)(ax) - (f\mathrm{d}(a))(x)] \\
&= (\mathrm{d}_{M^\vee}(f)a)(x) + (-1)^n(f\mathrm{d}(a))(x)
\end{aligned}$$

the third equality because $\mathrm{d}_M(ax) = \mathrm{d}(a)x + (-1)^m a\mathrm{d}_M(x)$. In other words we have $\mathrm{d}_{M^\vee}(fa) = \mathrm{d}_{M^\vee}(f)a + (-1)^n f\mathrm{d}(a)$ as desired.

If M is a **right** differential graded module, then the sign rule above does not work. The problem seems to be that in defining the left A-module structure on $M^\vee$ our conventions for graded modules above defines af to be the element of $(M^\vee)^{n+m}$ such that $(af)(x) = f(xa)$ for $f \in (M^\vee)^n$, $a \in A^m$ and $x \in M^{-n-m}$ which in some sense is the "wrong" thing to do if m is odd. Anyway, instead of changing the sign rule for the module structure, we fix the problem by using

$$\mathrm{d}_{M^\vee}(f) = (-1)^n f \circ \mathrm{d}_M^{-n-1}$$

---

[1]The sign rule is analogous to the one in Example 19.8, although there we are working with right modules and the same sign rule taken there does not work for left modules. Sigh!

when M is a right differential graded A-module. The computation for $a \in A^m$, $x \in M^{-n-m-1}$ and $f \in (M^\vee)^n$ then becomes

$$
\begin{aligned}
\mathrm{d}_{M^\vee}(af)(x) &= (-1)^{n+m}(fa)(\mathrm{d}_M(x)) \\
&= (-1)^{n+m}f(\mathrm{d}_M(x)a) \\
&= (-1)^{n+m}f(\mathrm{d}_M(ax) - (-1)^{m+n+1}x\mathrm{d}(a)) \\
&= (-1)^m\mathrm{d}_{M^\vee}(f)(ax) + f(x\mathrm{d}(a)) \\
&= (-1)^m(a\mathrm{d}_{M^\vee}(f))(x) + (\mathrm{d}(a)f)(x)
\end{aligned}
$$

the third equality because $\mathrm{d}_M(xa) = \mathrm{d}_M(x)a + (-1)^{n+m+1}x\mathrm{d}(a)$. In other words, we have $\mathrm{d}_{M^\vee}(af) = \mathrm{d}(a)f + (-1)^m a\mathrm{d}_{M^\vee}(f)$ as desired.

We leave it to the reader to show that with the conventions above there is a natural evaluation map $M \to (M^\vee)^\vee$ in the category of differential graded modules if M is either a differential graded left module or a differential graded right module. This works because the sign choices above cancel out and the differentials of $((M^\vee)^\vee$ are the natural maps $((M^n)^\vee)^\vee \to ((M^{n+1})^\vee)^\vee$.

Lemma 12.2. *Let (A, d) be a differential graded algebra. If M is a left differential graded A-module and N is a right differential graded A-module, then*

$$
\mathrm{Hom}_{Mod_{(A,d)}}(N, M^\vee)
$$

is isomorphic to the set of sequences (ψ_n) of $\mathbf{Z}$-bilinear pairings

$$
\psi_n : N^n \times M^{-n} \longrightarrow \mathbf{Q}/\mathbf{Z}
$$

such that $\psi_{n+m}(y, ax) = \psi_{n+m}(ya, x)$ for all $y \in N^n$, $x \in M^{-m}$, and $a \in A^{m-n}$ and such that $\psi_{n+1}(d(y), x) + (-1)^n\psi_n(y, d(x)) = 0$ for all $y \in N^n$ and $x \in M^{-n-1}$.

Proof. If $f \in \mathrm{Hom}_{Mod_{(A,d)}}(N, M^\vee)$, then we map this to the sequence of pairings defined by $\psi_n(y, x) = f(y)(x)$. It is a computation (omitted) to see that these pairings satisfy the conditions as in the lemma. For the converse, use Algebra, Lemma 11.8 to turn a sequence of pairings into a map $f : N \to M^\vee$. $\square$

Lemma 12.3. *Let (A, d) be a differential graded algebra. Then we have*

$$
\mathrm{Hom}_{Mod_{(A,d)}}(M, A^\vee[k]) = Ker(d : (M^\vee)^k \to (M^\vee)^{k+1})
$$

and

$$
\mathrm{Hom}_{K(Mod_{(A,d)})}(M, A^\vee[k]) = H^k(M^\vee)
$$

for any differential graded A-module M.

Proof. This is clear from the discussion above. $\square$

13. P-resolutions

This section is the analogue of Derived Categories, Section 28.

Let (A, d) be a differential graded algebra. Let P be a differential graded A-module. We say P *has property (P)* if it there exists a filtration

$$
0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P
$$

by differential graded submodules such that

(1) $P = \bigcup F_pP$,
(2) the inclusions $F_iP \to F_{i+1}P$ are admissible monomorphisms,

(3) the quotients $F_{i+1}P/F_iP$ are isomorphic as differential graded A-modules to a direct sum of $A[k]$.

In fact, condition (2) is a consequence of condition (3), see Lemma 11.1. Moreover, the reader can verify that as a graded A-module P will be isomorphic to a direct sum of shifts of A.

Lemma 13.1. *Let (A, d) be a differential graded algebra. Let P be a differential graded A-module. If $F_\bullet$ is a filtration as in property (P), then we obtain an admissible short exact sequence*

$$0 \to \bigoplus F_iP \to \bigoplus F_iP \to P \to 0$$

of differential graded A-modules.

Proof. The second map is the direct sum of the inclusion maps. The first map on the summand F_iP of the source is the sum of the identity $F_iP \to F_iP$ and the negative of the inclusion map $F_iP \to P_{i+1}P$. Choose homomorphisms $s_i : F_{i+1}P \to F_iP$ of graded A-modules which are left inverse to the inclusion maps. Composing gives maps $s_{j,i} : F_jP \to F_iP$ for all $j > i$. Then a left inverse of the first arrow maps $x \in F_jP$ to $(s_{j,0}(x), s_{j,1}(x), \ldots, s_{j,j-1}(x), 0, \ldots)$ in $\bigoplus F_iP$. $\square$

The following lemma shows that differential graded modules with property (P) are the dual notion to K-injective modules (i.e., they are K-projective in some sense). See Derived Categories, Definition 29.1.

Lemma 13.2. *Let (A, d) be a differential graded algebra. Let P be a differential graded A-module with property (P). Then*

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(P, N) = 0$$

for all acyclic differential graded A-modules N.

Proof. We will use that $K(\mathrm{Mod}_{(A,d)})$ is a triangulated category (Proposition 10.3). Let $F_\bullet$ be a filtration on P as in property (P). The short exact sequence of Lemma 13.1 produces a distinguished triangle. Hence by Derived Categories, Lemma 4.2 it suffices to show that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,d)})}(F_iP, N) = 0$$

for all acyclic differential graded A-modules N and all i. Each of the differential graded modules F_iP has a finite filtration by admissible monomorphisms, whose graded pieces are direct sums of shifts $A[k]$. Thus it suffices to prove that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,d)})}(A[k], N) = 0$$

for all acyclic differential graded A-modules N and all k. This follows from Lemma 11.2. $\square$

Lemma 13.3. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $P \to M$ of differential graded A-modules with the following properties*

(1) *$P \to M$ is surjective,*
(2) *$Ker(d_P) \to Ker(d_M)$ is surjective, and*
(3) *P sits in an admissible short exact sequence $0 \to P' \to P \to P'' \to 0$ where P', P'' are direct sums of shifts of A.*

Proof. Let P_k be the free A-module with generators x, y in degrees k and $k + 1$. Define the structure of a differential graded A-module on P_k by setting $d(x) = y$ and $d(y) = 0$. For every element $m \in M^k$ there is a homomorphism $P_k \to M$ sending x to m and y to $d(m)$. Thus we see that there is a surjection from a direct sum of copies of P_k to M. This clearly produces $P \to M$ having properties (1) and (3). To obtain property (2) note that if $m \in \mathrm{Ker}(d_M)$ has degree k, then there is a map $A[k] \to M$ mapping 1 to m. Hence we can achieve (2) by adding a direct sum of copies of shifts of A. $\square$

Lemma 13.4. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $P \to M$ of differential graded A-modules such that*

(1) *$P \to M$ is a quasi-isomorphism, and*
(2) *P has property (P).*

Proof. Set $M = M_0$. We inductively choose short exact sequences

$$0 \to M_{i+1} \to P_i \to M_i \to 0$$

where the maps $P_i \to M_i$ are chosen as in Lemma 13.3. This gives a "resolution"

$$\ldots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to M \to 0$$

Then we set

$$P = \bigoplus_{i \geq 0} P_i$$

as an A-module with grading given by $P^n = \bigoplus_{a+b=n} P_{-a}^b$ and differential (as in the construction of the total complex associated to a double complex) by

$$d_P(x) = f_{-a}(x) + (-1)^a d_{P_{-a}}(x)$$

for $x \in P_{-a}^b$. With these conventions P is indeed a differential graded A-module. Recalling that each P_i has a two step filtration $0 \to P_i' \to P_i \to P_i'' \to 0$ we set

$$F_{2i} P = \bigoplus_{i \geq j \geq 0} P_j \subset \bigoplus_{i \geq 0} P_i = P$$

and we add P_{i+1}' to $F_{2i}P$ to get F_{2i+1}. These are differential graded submodules and the successive quotients are direct sums of shifts of A. By Lemma 11.1 we see that the inclusions $F_i P \to F_{i+1} P$ are admissible monomorphisms. Finally, we have to show that the map $P \to M$ (given by the augmentation $P_0 \to M$) is a quasi-isomorphism. This follows from Homology, Lemma 22.10. $\square$

14. I-resolutions

This section is the dual of the section on P-resolutions.

Let (A, d) be a differential graded algebra. Let I be a differential graded A-module. We say *I has property (I)* if it there exists a filtration

$$I = F_0 I \supset F_1 I \supset F_2 I \supset \ldots \supset 0$$

by differential graded submodules such that

(1) $I = \lim I/F_p I$,
(2) the maps $I/F_{i+1}I \to I/F_i I$ are admissible epimorphisms,
(3) the quotients $F_i I/F_{i+1}I$ are isomorphic as differential graded A-modules to products of $A^\vee[k]$.

In fact, condition (2) is a consequence of condition (3), see Lemma 12.1. The reader can verify that as a graded module I will be isomorphic to a product of $A^\vee[k]$.

Lemma 14.1. *Let (A, d) be a differential graded algebra. Let I be a differential graded A-module. If $F_\bullet$ is a filtration as in property (I), then we obtain an admissible short exact sequence*

$$0 \to I \to \prod I/F_i I \to \prod I/F_i I \to 0$$

of differential graded A-modules.

Proof. Omitted. Hint: This is dual to Lemma 13.1. $\square$

The following lemma shows that differential graded modules with property (I) are the analogue of K-injective modules. See Derived Categories, Definition 29.1.

Lemma 14.2. *Let (A, d) be a differential graded algebra. Let I be a differential graded A-module with property (I). Then*

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(N, I) = 0$$

for all acyclic differential graded A-modules N.

Proof. We will use that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a triangulated category (Proposition 10.3). Let $F_\bullet$ be a filtration on I as in property (I). The short exact sequence of Lemma 14.1 produces a distinguished triangle. Hence by Derived Categories, Lemma 4.2 it suffices to show that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(N, I/F_i I) = 0$$

for all acyclic differential graded A-modules N and all i. Each of the differential graded modules $I/F_i I$ has a finite filtration by admissible monomorphisms, whose graded pieces are products of $A^\vee[k]$. Thus it suffices to prove that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(N, A^\vee[k]) = 0$$

for all acyclic differential graded A-modules N and all k. This follows from Lemma 12.3 and the fact that $(-)^\vee$ is an exact functor. $\square$

Lemma 14.3. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $M \to I$ of differential graded A-modules with the following properties*

(1) $M \to I$ *is injective,*
(2) $Coker(d_M) \to Coker(d_I)$ *is injective, and*
(3) I *sits in an admissible short exact sequence $0 \to I' \to I \to I'' \to 0$ where I', I'' are products of shifts of $A^\vee$.*

Proof. For every $k \in \mathbf{Z}$ let Q_k be the free left A-module with generators x, y in degrees k and $k + 1$. Define the structure of a left differential graded A-module on Q_k by setting $\mathrm{d}(x) = y$ and $\mathrm{d}(y) = 0$. Let $I_k = Q^\vee_{-k}$ be the "dual" right differential graded A-module, see Section 12. The next paragraph shows that we can embed M into a product of copies of I_k (for varying k). The dual statement (that any differential graded module is a quotient of a direct sum of of P_k's) is easy to prove (see proof of Lemma 13.3) and using double duals there should be a noncomputational way to deduce what we want. Thus we suggest skipping the next paragraph.

Given a $\mathbf{Z}$-linear map $\lambda : M^k \to \mathbf{Q}/\mathbf{Z}$ we construct pairings

$$\psi_n : M^n \times Q_k^{-n} \longrightarrow \mathbf{Q}/\mathbf{Z}$$

by setting

$$\psi_n(m, ax + by) = \lambda(ma + (-1)^{k+1}\mathrm{d}(mb))$$

for $m \in M^n$, $a \in A^{-n-k}$, and $b \in A^{-n-k-1}$. We compute

$$
\begin{aligned}
\psi_{n+1}(\mathrm{d}(m), ax + by) &= \lambda\left(\mathrm{d}(m)a + (-1)^{k+1}\mathrm{d}(\mathrm{d}(m)b)\right) \\
&= \lambda\left(\mathrm{d}(m)a + (-1)^{k+n}\mathrm{d}(m)\mathrm{d}(b)\right)
\end{aligned}
$$

and because $\mathrm{d}(ax + by) = \mathrm{d}(a)x + (-1)^{-n-k}ay + \mathrm{d}(b)y$ we have

$$
\begin{aligned}
\psi_n(m, \mathrm{d}(ax + by)) &= \lambda\left(m\mathrm{d}(a) + (-1)^{k+1}\mathrm{d}(m((-1)^{-n-k}a + \mathrm{d}(b)))\right) \\
&= \lambda\left(m\mathrm{d}(a) + (-1)^{-n+1}\mathrm{d}(ma) + (-1)^{k+1}\mathrm{d}(m)\mathrm{d}(b))))\right)
\end{aligned}
$$

and we see that

$$\psi_{n+1}(\mathrm{d}(m), ax + by) + (-1)^n \psi_n(m, \mathrm{d}(ax + by)) = 0$$

Thus these pairings define a homomorphism $f_\lambda : M \to I_k$ by Lemma 12.2 such that the composition

$$M^k \xrightarrow{f_\lambda^k} I_k^k = (Q_k^k)^\vee \xrightarrow{\text{evaluation at } x} \mathbf{Q}/\mathbf{Z}$$

is the given map λ. It is clear that we can find an embedding into a product of copies of I_k's by using a map of the form $\prod f_\lambda$ for a suitable choice of the maps λ.

The result of the previous paragraph produces $M \to I$ having properties (1) and (3). To obtain property (2), suppose $\overline{m} \in \mathrm{Coker}(\mathrm{d}_M)$ is a nonzero element of degree k. Pick a map $\lambda : M^k \to \mathbf{Q}/\mathbf{Z}$ which vanishes on $\mathrm{Im}(M^{k-1} \to M^k)$ but not on m. By Lemma 12.3 this corresponds to a homomorphism $M \to A^\vee[k]$ of differential graded A-modules which does not vanish on m. Hence we can achieve (2) by adding a product of copies of shifts of $A^\vee$. $\qquad\square$

Lemma 14.4. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $M \to I$ of differential graded A-modules such that*

(1) *$M \to I$ is a quasi-isomorphism, and*
(2) *I has property (I).*

Proof. Set $M = M_0$. We inductively choose short exact sequences

$$0 \to M_i \to I_i \to M_{i+1} \to 0$$

where the maps $M_i \to I_i$ are chosen as in Lemma 14.3. This gives a "resolution"

$$0 \to M \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} I_1 \to \ldots$$

Then we set

$$I = \prod_{i \geq 0} I_i$$

where we take the product in the category of graded A-modules and differential defined by

$$\mathrm{d}_I(x) = f_a(x) + (-1)^a \mathrm{d}_{I_a}(x)$$

for $x \in I_a^b$. With these conventions I is indeed a differential graded A-module. Recalling that each I_i has a two step filtration $0 \to I_i' \to I_i \to I_i'' \to 0$ we set

$$F_{2i}P = \prod\nolimits_{j \geq i} I_j \subset \prod\nolimits_{i \geq 0} I_i = I$$

and we add a factor I_{i+1}' to $F_{2i}I$ to get $F_{2i+1}I$. These are differential graded submodules and the successive quotients are products of shifts of $A^\vee$. By Lemma 12.1 we see that the inclusions $F_{i+1}I \to F_iI$ are admissible monomorphisms. Finally, we have to show that the map $M \to I$ (given by the augmentation $M \to I_0$) is a quasi-isomorphism. This follows from Homology, Lemma 22.11. $\qquad\square$

15. The derived category

Recall that the notions of acyclic differential graded modules and quasi-isomorphism of differential graded modules make sense (see Section 4).

Lemma 15.1. *Let (A, d) be a differential graded algebra. The full subcategory Ac of $K(Mod_{(A,d)})$ consisting of acyclic modules is a strictly full saturated triangulated subcategory of $K(Mod_{(A,d)})$. The corresponding saturated multiplicative system (see Derived Categories, Lemma 6.10) of $K(Mod_{(A,d)})$ is the class Qis of quasi-isomorphisms. In particular, the kernel of the localization functor*

$$Q : K(Mod_{(A,d)}) \to Qis^{-1}K(Mod_{(A,d)})$$

is Ac. Moreover, the functor H^0 factors through Q.

Proof. We know that H^0 is a homological functor by the long exact sequence of homology (4.2.1). The kernel of H^0 is the subcategory of acyclic objects and the arrows with induce isomorphisms on all H^i are the quasi-isomorphisms. Thus this lemma is a special case of Derived Categories, Lemma 6.11.

Set theoretical remark. The construction of the localization in Derived Categories, Proposition 5.5 assumes the given triangulated category is "small", i.e., that the underlying collection of objects forms a set. Let V_α be a partial universe (as in Sets, Section 5) containing (A, d) and where the cofinality of α is bigger than $\aleph_0$ (see Sets, Proposition 7.2). Then we can consider the category $Mod_{(A,d),\alpha}$ of differential graded A-modules contained in V_α. A straightforward check shows that all the constructions used in the proof of Proposition 10.3 work inside of $Mod_{(A,d),\alpha}$ (because at worst we take finite direct sums of differential graded modules). Thus we obtain a triangulated category $Qis_\alpha^{-1}K(Mod_{(A,d),\alpha})$. We will see below that if $\beta > \alpha$, then the transition functors

$$Qis_\alpha^{-1}K(Mod_{(A,d),\alpha}) \longrightarrow Qis_\beta^{-1}K(Mod_{(A,d),\beta})$$

are fully faithful as the morphism sets in the quotient categories are computed by maps in the homotopy categories from P-resolutions (the construction of a P-resolution in the proof of Lemma 13.4 takes countable direct sums as well as direct sums indexed over subsets of the given module). The reader should therefore think of the category of the lemma as the union of these subcategories. $\qquad\square$

Taking into account the set theoretical remark at the end of the proof of the preceding lemma we define the derived category as follows.

Definition 15.2. Let (A, d) be a differential graded algebra. Let Ac and Qis be as in Lemma 15.1. The *derived category of (A, d)* is the triangulated category

$$D(A, d) = K(\text{Mod}_{(A,d)})/\text{Ac} = \text{Qis}^{-1} K(\text{Mod}_{(A,d)}).$$

We denote $H^0 : D(A, d) \to \text{Mod}_R$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above.

Here is the promised lemma computing morphism sets in the derived category.

Lemma 15.3. *Let (A, d) be a differential graded algebra. Let M and N be differential graded A-modules.*

(1) *Let $P \to M$ be a P-resolution as in Lemma 13.4. Then*

$$\text{Hom}_{D(A,d)}(M, N) = \text{Hom}_{K(Mod_{(A,d)})}(P, N)$$

(2) *Let $N \to I$ be an I-resolution as in Lemma 14.4. Then*

$$\text{Hom}_{D(A,d)}(M, N) = \text{Hom}_{K(Mod_{(A,d)})}(M, I)$$

Proof. Let $P \to M$ be as in (1). Since $P \to M$ is a quasi-isomorphism we see that

$$\text{Hom}_{D(A,d)}(P, N) = \text{Hom}_{D(A,d)}(M, N)$$

by definition of the derived category. A morphism $f : P \to N$ in $D(A, d)$ is equal to $s^{-1} f'$ where $f' : P \to N'$ is a morphism and $s : N \to N'$ is a quasi-isomorphism. Choose a distringuished triangle

$$N \to N' \to Q \to N[1]$$

As s is a quasi-isomorphism, we see that Q is acyclic. Thus $\text{Hom}_{K(\text{Mod}_{(A,d)})}(P, Q[k]) = 0$ for all k by Lemma 13.2. Since $\text{Hom}_{K(\text{Mod}_{(A,d)})}(P, -)$ is cohomological, we conclude that we can lift $f' : P \to N'$ uniquely to a morphism $f : P \to N$. This finishes the proof.

The proof of (2) is dual to that of (1) using Lemma 14.2 in stead of Lemma 13.2. $\square$

Lemma 15.4. *Let (A, d) be a differential graded algebra. Then*

(1) *$D(A, d)$ has both direct sums and products,*
(2) *direct sums are obtained by taking direct sums of differential graded modules,*
(3) *products are obtained by taking products of differential graded modules.*

Proof. We will use that $\text{Mod}_{(A,d)}$ is an abelian category with arbitrary direct sums and products, and that these give rise to direct sums and products in $K(\text{Mod}_{(A,d)})$. See Lemmas 4.2 and 5.4.

Let M_j be a family of differential graded A-modules. Consider the graded direct sum $M = \bigoplus M_j$ which is a differential graded A-module with the obvious. For a differential graded A-module N choose a quasi-isomorphism $N \to I$ where I is a differential graded A-module with property (I). See Lemma 14.4. Using Lemma 15.3 we have

$$\text{Hom}_{D(A,d)}(M, N) = \text{Hom}_{K(A,d)}(M, I)$$
$$= \prod \text{Hom}_{K(A,d)}(M_j, I)$$
$$= \prod \text{Hom}_{D(A,d)}(M_j, N)$$

whence the existence of direct sums in $D(A, d)$ as given in part (2) of the lemma.

Let M_j be a family of differential graded A-modules. Consider the product $M = \prod M_j$ of differential graded A-modules. For a differential graded A-module N choose a quasi-isomorphism $P \to N$ where P is a differential graded A-module with property (P). See Lemma 13.4. Using Lemma 15.3 we have

$$\mathrm{Hom}_{D(A,\mathrm{d})}(N, M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, M)$$
$$= \prod \mathrm{Hom}_{K(A,\mathrm{d})}(P, M_j)$$
$$= \prod \mathrm{Hom}_{D(A,\mathrm{d})}(N, M_j)$$

whence the existence of direct sums in $D(A, \mathrm{d})$ as given in part (3) of the lemma. $\square$

16. The canonical delta-functor

Let (A, d) be a differential graded algebra. Consider the functor $\mathrm{Mod}(\mathcal{A}) \to K(\mathrm{Mod}_{(A,\mathrm{d})})$. This functor is **not** a δ-functor in general. However, it turns out that the functor $\mathrm{Mod}_{(A,\mathrm{d})} \to D(A, \mathrm{d})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to K \xrightarrow{a} L \xrightarrow{b} M \to 0$$

in the abelian category $\mathrm{Mod}_{(A,\mathrm{d})}$. Consider the cone $C(a)$ of the morphism a. We have $C(a) = L \oplus K$ and we define $q : C(a) \to M$ via the projection to L followed by b. Hence a homomorphism of differential graded A-modules

$$q : C(a) \longrightarrow M.$$

It is clear that $q \circ i = b$ where i is as in Definition 6.1. Note that, as a is injective, the kernel of q is identified with the cone of id_K which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 9.4 the triangle

$$(K, L, C(a), a, i, -p)$$

is a distinguished triangle in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. As the localization functor $K(\mathrm{Mod}_{(A,\mathrm{d})}) \to D(A, \mathrm{d})$ is exact we see that $(K, L, C(a), a, i, -p)$ is a distinguished triangle in $D(A, \mathrm{d})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(A, \mathrm{d})$. Hence we deduce that

$$(K, L, M, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(A, \mathrm{d})$. This suggests the following lemma.

Lemma 16.1. *Let (A, d) be a differential graded algebra. The functor $Mod_{(A,d)} \to D(A, d)$ defined has the natural structure of a δ-functor, with*

$$\delta_{K \to L \to M} = -p \circ q^{-1}$$

with p and q as explained above.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show functoriality of this construction, see Derived Categories, Definition 3.6. This follows from Lemma 6.2 with a bit of work. Compare with Derived Categories, Lemma 12.1. $\square$

17. Linear categories

Just the definitions.

Definition 17.1. Let R be a ring. An *R-linear category* $\mathcal{A}$ is a category where every morphism set is given the structure of an R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition law

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \times \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

is R-bilinear.

Thus composition determines an R-linear map

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

of R-modules. Note that we do not assume R-linear categories to be additive.

Definition 17.2. Let R be a ring. A *functor of R-linear categories*, or an *R-linear* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of R-modules.

18. Graded categories

Just some definitions.

Definition 18.1. Let R be a ring. A *graded category* $\mathcal{A}$ *over* R is a category where every morphism set is given the structure of a graded R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition is R-bilinear and induces a homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

of graded R-modules (i.e., preserving degrees).

In this situation we denote $\mathrm{Hom}^i_{\mathcal{A}}(x, y)$ the degree i part of the graded object $\mathrm{Hom}_{\mathcal{A}}(x, y)$, so that

$$\mathrm{Hom}_{\mathcal{A}}(x, y) = \bigoplus\nolimits_{i \in \mathbf{Z}} \mathrm{Hom}^i_{\mathcal{A}}(x, y)$$

is the direct sum decomposition into graded parts.

Definition 18.2. Let R be a ring. A *functor of graded categories over* R, or a *graded functor* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of graded R-modules.

Given a graded category we are often interested in the corresponding "usual" category of maps of degree 0. Here is a formal definition.

Definition 18.3. Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. We let $\mathcal{A}^0$ be the category with the same objects as $\mathcal{A}$ and with

$$\mathrm{Hom}_{\mathcal{A}^0}(x, y) = \mathrm{Hom}^0_{\mathcal{A}}(x, y)$$

the degree 0 graded piece of the graded module of morphisms of $\mathcal{A}$.

Definition 18.4. Let R be a ring. Let $\mathcal{A}$ be a graded category over R. A direct sum (x, y, z, i, j, p, q) in $\mathcal{A}$ (notation as in Homology, Remark 3.6) is a *graded direct sum* if i, j, p, q are homogeneous of degree 0.

Example 18.5 (Graded category of graded objects). Let $\mathcal{B}$ be an additive category. Recall that we have defined the category $\text{Gr}(\mathcal{B})$ of graded objects of $\mathcal{B}$ in Homology, Definition 15.1. In this example, we will construct a graded category $\text{Gr}^{gr}(\mathcal{B})$ over $R = \mathbf{Z}$ whose associated category $\text{Gr}^{gr}(\mathcal{B})^0$ recovers $\text{Gr}(\mathcal{B})$. As objects of $\text{Comp}^{gr}(\mathcal{B})$ we take graded objects of $\mathcal{B}$. Then, given graded objects $A = (A^i)$ and $B = (B^i)$ of $\mathcal{B}$ we set

$$\text{Hom}_{\text{Gr}^{gr}(\mathcal{B})}(A, B) = \bigoplus\nolimits_{n \in \mathbf{Z}} \text{Hom}^n(A, B)$$

where the graded piece of degree n is the abelian group of homogeneous maps of degree n from A to B defined by the rule

$$\text{Hom}^n(A, B) = \text{Hom}_{\text{Gr}(\mathcal{A})}(A, B[n]) = \text{Hom}_{\text{Gr}(\mathcal{A})}(A[-n], B)$$

see Homology, Equation (15.4.1). Explicitly we have

$$\text{Hom}^n(A, B) = \prod\nolimits_{p+q=n} \text{Hom}_{\mathcal{B}}(A^{-q}, B^p)$$

(observe reversal of indices and observe that we have a product here and not a direct sum). In other words, a degree n morphism f from A to B can be seen as a system $f = (f_{p,q})$ where $p, q \in \mathbf{Z}$, $p + q = n$ with $f_{p,q} : A^{-q} \to B^p$ a morphism of $\mathcal{B}$. Given graded objects A, B, C of $\mathcal{B}$ composition of morphisms in $\text{Gr}^{gr}(\mathcal{B})$ is defined via the maps

$$\text{Hom}^m(B, C) \times \text{Hom}^n(A, B) \longrightarrow \text{Hom}^{n+m}(A, C)$$

by simple composition $(g, f) \mapsto g \circ f$ of homogeneous maps of graded objects. In terms of components we have

$$(g \circ f)_{p,r} = g_{p,q} \circ f_{-q,r}$$

where q is such that $p + q = m$ and $-q + r = n$.

Example 18.6 (Graded category of graded modules). Let A be a $\mathbf{Z}$-graded algebra over a ring R. We will construct a graded category Mod_A^{gr} over R whose associated category $(\text{Mod}_A^{gr})^0$ is the category of graded A-modules. As objects of Mod_A^{gr} we take right graded A-modules (see Section 11). Given graded A-modules L and M we set

$$\text{Hom}_{\text{Mod}_A^{gr}}(L, M) = \bigoplus\nolimits_{n \in \mathbf{Z}} \text{Hom}^n(L, M)$$

where $\text{Hom}^n(L, M)$ is the set of right A-module maps $L \to M$ which are homogeneous of degree n, i.e., $f(L^i) \subset M^{i+n}$ for all $i \in \mathbf{Z}$. In terms of components, we have that

$$\text{Hom}^n(L, M) \subset \prod\nolimits_{p+q=n} \text{Hom}_R(L^{-q}, M^p)$$

(observe reversal of indices) is the subset consisting of those $f = (f_{p,q})$ such that

$$f_{p,q}(ma) = f_{p-i,q+i}(m)a$$

for $a \in A^i$ and $m \in L^{-q-i}$. For graded A-modules K, L, M we define composition in Mod_A^{gr} via the maps

$$\text{Hom}^m(L, M) \times \text{Hom}^n(K, L) \longrightarrow \text{Hom}^{n+m}(K, M)$$

by simple composition of right A-module maps: $(g, f) \mapsto g \circ f$.

Remark 18.7. Let R be a ring. Let $\mathcal{D}$ be an R-linear category endowed with a collection of R-linear functors $[n] : \mathcal{D} \to \mathcal{D}$, $x \mapsto x[n]$ indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n+m]$ and $[0] = \mathrm{id}_{\mathcal{D}}$ (equality as functors). This allows us to construct a graded category $\mathcal{D}^{gr}$ over R with the same objects of $\mathcal{D}$ setting

$$\mathrm{Hom}_{\mathcal{D}^{gr}}(x, y) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{D}}(x, y[n])$$

for x, y in $\mathcal{D}$. Observe that $(\mathcal{D}^{gr})^0 = \mathcal{D}$ (see Definition 18.3). Moreover, the graded category $\mathcal{D}^{gr}$ inherits R-linear graded functors $[n]$ satisfying $[n] \circ [m] = [n+m]$ and $[0] = \mathrm{id}_{\mathcal{D}^{gr}}$ with the property that

$$\mathrm{Hom}_{\mathcal{D}^{gr}}(x, y[n]) = \mathrm{Hom}_{\mathcal{D}^{gr}}(x, y)[n]$$

as graded R-modules compatible with composition of morphisms.

Conversely, suppose given a graded category $\mathcal{A}$ over R endowed with a collection of R-linear graded functors $[n]$ satisfying $[n] \circ [m] = [n+m]$ and $[0] = \mathrm{id}_{\mathcal{A}}$ which are moreover equipped with isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(x, y[n]) = \mathrm{Hom}_{\mathcal{A}}(x, y)[n]$$

as graded R-modules compatible with composition of morphisms. Then the reader easily shows that $\mathcal{A} = (\mathcal{A}^0)^{gr}$.

Here are two examples of the relationship $\mathcal{D} \leftrightarrow \mathcal{A}$ we established above:

(1) Let $\mathcal{B}$ be an additive category. If $\mathcal{D} = \mathrm{Gr}(\mathcal{B})$, then $\mathcal{A} = \mathrm{Gr}^{gr}(\mathcal{B})$ as in Example 18.5.
(2) If A is a graded ring and $\mathcal{D} = \mathrm{Mod}_A$ is the category of graded right A-modules, then $\mathcal{A} = \mathrm{Mod}_A^{gr}$, see Example 18.6.

19. Differential graded categories

Note that if R is a ring, then R is a differential graded algebra over itself (with $R = R^0$ of course). In this case a differential graded R-module is the same thing as a complex of R-modules. In particular, given two differential graded R-modules M and N we denote $M \otimes_R N$ the differential graded R-module corresponding to the total complex associated to the double complex obtained by the tensor product of the complexes of R-modules associated to M and N.

Definition 19.1. Let R be a ring. A *differential graded category* $\mathcal{A}$ *over* R is a category where every morphism set is given the structure of a differential graded R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition is R-bilinear and induces a homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

of differential graded R-modules.

The final condition of the definition signifies the following: if $f \in \mathrm{Hom}_{\mathcal{A}}^n(x, y)$ and $g \in \mathrm{Hom}_{\mathcal{A}}^m(y, z)$ are homogeneous of degrees n and m, then

$$\mathrm{d}(g \circ f) = \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f)$$

in $\mathrm{Hom}_{\mathcal{A}}^{n+m+1}(x, z)$. This follows from the sign rule for the differential on the total complex of a double complex, see Homology, Definition 22.3.

Definition 19.2. Let R be a ring. A *functor of differential graded categories over R* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of differential graded R-modules.

Given a diffferential graded category we are often interested in the corresponding categories of complexes and homotopy category. Here is a formal definition.

Definition 19.3. Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. Then we let

(1) the *category of complexes of* $\mathcal{A}^2$ be the category $\mathrm{Comp}(\mathcal{A})$ whose objects are the same as the objects of $\mathcal{A}$ and with

$$\mathrm{Hom}_{\mathrm{Comp}(\mathcal{A})}(x, y) = \mathrm{Ker}(d : \mathrm{Hom}^0_{\mathcal{A}}(x, y) \to \mathrm{Hom}^0_{\mathcal{A}}(x, y))$$

(2) the *homotopy category of* $\mathcal{A}$ be the category $K(\mathcal{A})$ whose objects are the same as the objects of $\mathcal{A}$ and with

$$\mathrm{Hom}_{\mathrm{Comp}(\mathcal{A})}(x, y) = H^0(\mathrm{Hom}_{\mathcal{A}}(x, y))$$

Our use of the symbol $K(\mathcal{A})$ is nonstandard, but at least is compatible with the use of $K(-)$ in other chapters of the Stacks project.

Definition 19.4. Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. A direct sum (x, y, z, i, j, p, q) in $\mathcal{A}$ (notation as in Homology, Remark 3.6) is a *differential graded direct sum* if i, j, p, q are homogeneous of degree 0 and closed, i.e., $\mathrm{d}(i) = 0$, etc.

Lemma 19.5. *Let R be a ring. A functor $F : \mathcal{A} \to \mathcal{B}$ of differential graded categories over R induces functors $\mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{B})$ and $K(\mathcal{A}) \to K(\mathcal{B})$.*

Proof. Omitted. $\square$

Example 19.6 (Differential graded category of complexes)**.** Let $\mathcal{B}$ be an additive category. We will construct a differential graded category $\mathrm{Comp}^{dg}(\mathcal{B})$ over $R = \mathbf{Z}$ whose associated category of complexes is $\mathrm{Comp}(\mathcal{B})$ and whose associated homotopy category is $K(\mathcal{B})$. As objects of $\mathrm{Comp}^{dg}(\mathcal{B})$ we take complexes of $\mathcal{B}$. Given complexes $A^\bullet$ and $B^\bullet$ of $\mathcal{B}$, we sometimes also denote $A^\bullet$ and $B^\bullet$ the corresponding graded objects of $\mathcal{B}$ (i.e., forget about the differential). Using this abuse of notation, we set

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathcal{B})}(A^\bullet, B^\bullet)$$

as a graded $\mathbf{Z}$-module where the right hand side is defined in Example 18.5. In other words, the nth graded piece is the abelian group of homogeneous morphism of degree n of graded objects

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Gr}(\mathcal{B})}(A^\bullet, B^\bullet[n]) = \prod_{p+q=n} \mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^p)$$

(observe reversal of indices and observe we have a direct product and not a direct sum). For an element $f \in \mathrm{Hom}^n(A^\bullet, B^\bullet)$ of degree n we set

$$\mathrm{d}(f) = \mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A$$

To make sense of this we think of d_B and d_A as maps of graded objects of $\mathcal{B}$ homogeneous of degree 1 and we use composition in the category $\mathrm{Gr}^{gr}(\mathcal{B})$ on the

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[2]This may be nonstandard terminology.

right hand side. In terms of components, if $f = (f_{p,q})$ with $f_{p,q} : A^{-q} \to B^p$ we have

(19.6.1) $$\mathrm{d}(f_{p,q}) = \mathrm{d}_B \circ f_{p,q} + (-1)^{p+q+1} f_{p,q} \circ \mathrm{d}_A$$

Note that the first term of this expression is in $\mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^{p+1})$ and the second term is in $\mathrm{Hom}_{\mathcal{B}}(A^{-q-1}, B^p)$. In other words, given $p + q = n + 1$ we have

$$\mathrm{d}(f)_{p,q} = \mathrm{d}_B \circ f_{p-1,q} - (-1)^n f_{p,q-1} \circ \mathrm{d}_A$$

with obvious notation. The reader checks[3] that

(1) d has square zero,
(2) an element f in $\mathrm{Hom}^n(A^\bullet, B^\bullet)$ has $\mathrm{d}(f) = 0$ if and only if the morphism $f : A^\bullet \to B^\bullet[n]$ of graded objects of $\mathcal{B}$ is actually a map of complexes,
(3) in particular, the category of complexes of $\mathrm{Comp}^{dg}(\mathcal{B})$ is equal to $\mathrm{Comp}(\mathcal{B})$,
(4) the morphism of complexes defined by f as in (2) is homotopy equivalent to zero if and only if $f = \mathrm{d}(g)$ for some $g \in \mathrm{Hom}^{n-1}(A^\bullet, B^\bullet)$.
(5) in particular, we obtain a canonical isomorphism

$$\mathrm{Hom}_{K(\mathcal{B})}(A^\bullet, B^\bullet) \longrightarrow H^0(\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet))$$

and the homotopy category of $\mathrm{Comp}^{dg}(\mathcal{B})$ is equal to $K(\mathcal{B})$.

Given complexes $A^\bullet$, $B^\bullet$, $C^\bullet$ we define composition

$$\mathrm{Hom}^m(B^\bullet, C^\bullet) \times \mathrm{Hom}^n(A^\bullet, B^\bullet) \longrightarrow \mathrm{Hom}^{n+m}(A^\bullet, C^\bullet)$$

by composition $(g, f) \mapsto g \circ f$ in the graded category $\mathrm{Gr}^{gr}(\mathcal{B})$, see Example 18.5. This defines a map of differential graded modules as in Definition 19.1 because

$$\begin{aligned}
\mathrm{d}(g \circ f) &= \mathrm{d}_C \circ g \circ f - (-1)^{n+m} g \circ f \circ \mathrm{d}_A \\
&= (\mathrm{d}_C \circ g - (-1)^m g \circ \mathrm{d}_B) \circ f + (-1)^m g \circ (\mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A) \\
&= \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f)
\end{aligned}$$

as desired.

Lemma 19.7. *Let $F : \mathcal{B} \to \mathcal{B}'$ be an additive functor between additive categories. Then F induces a functor of differential graded categories*

$$F : \mathrm{Comp}^{dg}(\mathcal{B}) \to \mathrm{Comp}^{dg}(\mathcal{B}')$$

of Example 19.6 inducing the usual functors on the category of complexes and the homotopy categories.

Proof. Omitted. $\square$

Example 19.8 (Differential graded category of differential graded modules)**.** Let (A, d) be a differential graded algebra over a ring R. We will construct a differential graded category $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ over R whose category of complexes is $\mathrm{Mod}_{(A,\mathrm{d})}$ and whose homotopy category is $K(\mathrm{Mod}_{(A,\mathrm{d})})$. As objects of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ we take the

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[3]What may be useful here is to think of the double complex $H^{\bullet,\bullet}$ with terms $H^{p,q} = \mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^p)$ and differentials d_1 of degree $(1, 0)$ given by d_B and d_2 of degree $(0, 1)$ given by the contragredient of d_A. Up to sign and up to replacing the direct sum by a direct product, the differential graded **Z**-module $\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet)$ is the total complex associated to $H^{\bullet,\bullet}$, see Homology, Definition 22.3. To get the sign correct, change $d_2^{p,q} : H^{p,q} \to H^{p,q+1}$ by $(-1)^{q+1}$ (after this change we still have a double complex).

differential graded A-modules. Given differential graded A-modules L and M we set

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L, M) = \mathrm{Hom}_{\mathrm{Mod}^{gr}_{A}}(L, M) = \bigoplus \mathrm{Hom}^n(L, M)$$

as a graded R-module where the right hand side is defined as in Example 18.6. In other words, the nth graded piece $\mathrm{Hom}^n(L, M)$ is the R-module of right A-module maps homogeneous of degree n. For an element $f \in \mathrm{Hom}^n(L, M)$ we set

$$\mathrm{d}(f) = \mathrm{d}_M \circ f - (-1)^n f \circ \mathrm{d}_L$$

To make sense of this we think of d_M and d_L as graded R-module maps and we use composition of graded R-module maps. It is clear that $\mathrm{d}(f)$ is homogeneous of degree $n + 1$ as a graded R-module map, and it is linear because

$$\mathrm{d}(f)(xa) = \mathrm{d}_M(f(x)a) - (-1)^n f(\mathrm{d}_L(xa))$$
$$= \mathrm{d}_M(f(x))a + (-1)^{\deg(x)+n} f(x)\mathrm{d}(a) - (-1)^n f(\mathrm{d}_L(x))a - (-1)^{n+\deg(x)} f(x)\mathrm{d}(a)$$
$$= \mathrm{d}(f)(x)a$$

as desired (observe that this calculation would not work without the sign in the definition of our differential on Hom). Similar formulae to those of Example 19.6 hold for the differential of f in terms of components. The reader checks (in the same way as in Example 19.6) that

1. d has square zero,
2. an element f in $\mathrm{Hom}^n(L, M)$ has $\mathrm{d}(f) = 0$ if and only if $f : L \to M[n]$ is a homomorphism of differential graded A-modules,
3. in particular, the category of complexes of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ is $\mathrm{Mod}_{(A,\mathrm{d})}$,
4. the homomorphism defined by f as in (2) is homotopy equivalent to zero if and only if $f = \mathrm{d}(g)$ for some $g \in \mathrm{Hom}^{n-1}(L, M)$.
5. in particular, we obtain a canonical isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(L, M) \longrightarrow H^0(\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L, M))$$

and the homotopy category of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ is $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Given differential graded A-modules K, L, M we define composition

$$\mathrm{Hom}^m(L, M) \times \mathrm{Hom}^n(K, L) \longrightarrow \mathrm{Hom}^{n+m}(K, M)$$

by composition of homogeneous right A-module maps $(g, f) \mapsto g \circ f$. This defines a map of differential graded modules as in Definition 19.1 because

$$\mathrm{d}(g \circ f) = \mathrm{d}_M \circ g \circ f - (-1)^{n+m} g \circ f \circ \mathrm{d}_K$$
$$= (\mathrm{d}_M \circ g - (-1)^m g \circ \mathrm{d}_L) \circ f + (-1)^m g \circ (\mathrm{d}_L \circ f - (-1)^n f \circ \mathrm{d}_K)$$
$$= \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f)$$

as desired.

Lemma 19.9. *Let $\varphi : (A, d) \to (E, d)$ be a homomorphism of differential graded algebras. Then φ induces a functor of differential graded categories*

$$F : Mod^{dg}_{(E,d)} \longrightarrow Mod^{dg}_{(A,d)}$$

of Example 19.8 inducing obvious restriction functors on the categories of differential graded modules and homotopy categories.

Proof. Omitted. $\qquad\square$

Lemma 19.10. *Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. Let x be an object of $\mathcal{A}$. Let*

$$(E, d) = \operatorname{Hom}_{\mathcal{A}}(x, x)$$

be the differential graded R-algebra of endomorphisms of x. We obtain a functor

$$\mathcal{A} \longrightarrow Mod^{dg}_{(E,d)}, \quad y \longmapsto \operatorname{Hom}_{\mathcal{A}}(x, y)$$

of differential graded categories by letting E act on $\operatorname{Hom}_{\mathcal{A}}(x, y)$ via composition in $\mathcal{A}$. This functor induces functors

$$\operatorname{Comp}(\mathcal{A}) \to Mod_{(A,d)} \quad and \quad K(\mathcal{A}) \to K(Mod_{(A,d)})$$

by an application of Lemma 19.5.

Proof. This lemma proves itself. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

20. Obtaining triangulated categories

In this section we discuss the most general setup to which the arguments proving Derived Categories, Proposition 10.3 and Proposition 10.3 apply.

Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. To make our argument work, we impose some axioms on $\mathcal{A}$:

- (A) $\mathcal{A}$ has a zero object and differential graded direct sums of two objects (as in Definition 19.4).
- (B) there are functors $[n] : \mathcal{A} \longrightarrow \mathcal{A}$ of differential graded categories such that $[0] = \operatorname{id}_{\mathcal{A}}$ and $[n + m] = [n] \circ [m]$ and given isomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(x, y[n]) = \operatorname{Hom}_{\mathcal{A}}(x, y)[n]$$

  of differential graded R-modules compatible with composition.

Given our differential graded category $\mathcal{A}$ we say

- (1) a sequence $x \to y \to z$ of morphisms of $\operatorname{Comp}(\mathcal{A})$ is an *admissible short exact sequence* if there exists an isomorphism $y \cong x \oplus z$ in the underlying graded category such that $x \to z$ and $y \to z$ are (co)projections.
- (2) a morphism $x \to y$ of $\operatorname{Comp}(\mathcal{A})$ is an *admissible monomorphism* if it extends to an admissible short exact sequence $x \to y \to z$.
- (3) a morphism $y \to z$ of $\operatorname{Comp}(\mathcal{A})$ is an *admissible epimorphism* if it extends to an admissible short exact sequence $x \to y \to z$.

The next lemma tells us an admissible short exact sequence gives a triangle, provided we have axioms (A) and (B).

Lemma 20.1. *Let $\mathcal{A}$ be a differential graded category satisfying axioms (A) and (B). Given an admissible short exact sequence $x \to y \to z$ we obtain (see proof) a triangle*

$$x \to y \to z \to x[1]$$

in $\operatorname{Comp}(\mathcal{A})$ with the property that any two compositions in $z[-1] \to x \to y \to z \to x[1]$ are zero in $K(\mathcal{A})$.

Proof. Choose a diagram

$$
\begin{array}{ccc}
x & \xrightarrow{\quad 1 \quad} & x \\
\end{array}
$$

giving the isomorphism of graded objects $y \cong x \oplus z$ as in the defintion of an admissible short exact sequence. Here are some equations that hold in this situation

(1) $1 = \pi a$ and hence $\mathrm{d}(\pi)a = 0$,
(2) $1 = bs$ and hence $b\mathrm{d}(s) = 0$,
(3) $1 = a\pi + sb$ and hence $a\mathrm{d}(\pi) + \mathrm{d}(s)b = 0$,
(4) $\pi s = 0$ and hence $\mathrm{d}(\pi)s + \pi\mathrm{d}(s) = 0$,
(5) $\mathrm{d}(s) = a\pi\mathrm{d}(s)$ because $\mathrm{d}(s) = (a\pi + sb)\mathrm{d}(s)$ and $b\mathrm{d}(s) = 0$,
(6) $\mathrm{d}(\pi) = \mathrm{d}(\pi)sb$ because $\mathrm{d}(\pi) = \mathrm{d}(\pi)(a\pi + sb)$ and $\mathrm{d}(\pi)a = 0$,
(7) $\mathrm{d}(\pi\mathrm{d}(s)) = 0$ because if we postcompose it with the monomorphism a we get $\mathrm{d}(a\pi\mathrm{d}(s)) = \mathrm{d}(\mathrm{d}(s)) = 0$, and
(8) $\mathrm{d}(\mathrm{d}(\pi)s) = 0$ as by (4) it is the negative of $\mathrm{d}(\pi\mathrm{d}(s))$ which is 0 by (7).

We've used repeatedly that $\mathrm{d}(a) = 0$, $\mathrm{d}(b) = 0$, and that $\mathrm{d}(1) = 0$. By (7) we see that

$$
\delta = \pi\mathrm{d}(s) = -\mathrm{d}(\pi)s : z \to x[1]
$$

is a morphism in $\mathrm{Comp}(\mathcal{A})$. By (5) we see that the composition $a\delta = a\pi\mathrm{d}(s) = \mathrm{d}(s)$ is homotopic to zero. By (6) we see that the composition $\delta b = -\mathrm{d}(\pi)sb = \mathrm{d}(-\pi)$ is homotopic to zero. $\qquad\square$

Besides axioms (A) and (B) we need an axiom concerning the existence of cones. We formalize everything as follows.

Situation 20.2. Here R is a ring and $\mathcal{A}$ is a differential graded category over R having axioms (A), (B), and

(C) given an arrow $f : x \to y$ of degree 0 with $\mathrm{d}(f) = 0$ there exists an admissible short exact sequence $y \to c(f) \to x[1]$ in $\mathrm{Comp}(\mathcal{A})$ such that the map $x[1] \to y[1]$ of Lemma 20.1 is equal to $f[1]$.

We will call $c(f)$ a *cone* of the morphism f. If (A), (B), and (C) hold, then cones are functorial in a weak sense.

Lemma 20.3. *In Situation 20.2 suppose that*

$$
\begin{array}{ccc}
x_1 & \xrightarrow{\ f_1\ } & y_1 \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\
x_2 & \xrightarrow{\ f_2\ } & y_2
\end{array}
$$

is a diagram of $\mathrm{Comp}(\mathcal{A})$ commutative up to homotopy. Then there exists a morphism $c : c(f_1) \to c(f_2)$ which gives rise to a morphism of triangles

$$
(a, b, c) : (x_1, y_1, c(f_1)) \to (x_1, y_1, c(f_1))
$$

in $K(\mathcal{A})$.

Proof. The assumption means there exists a morphism $h : x_1 \to y_2$ of degree -1 such that $d(h) = bf_1 - f_2 a$. Choose isomorphisms $c(f_i) = y_i \oplus x_i[1]$ of graded objects compatible with the morphisms $y_i \to c(f_i) \to x_i[1]$. Let's denote $a_i : y_i \to c(f_i)$, $b_i : c(f_i) \to x_i[1]$, $s_i : x_i[1] \to c(f_i)$, and $\pi_i : c(f_i) \to y_i$ the given morphisms. Recall that $x_i[1] \to y_i[1]$ is given by $\pi_i d(s_i)$. By axiom (C) this means that

$$f_i = \pi_i d(s_i) = -d(\pi_i)s_i$$

(we identify $\mathrm{Hom}(x_i, y_i)$ with $\mathrm{Hom}(x_i[1], y_i[1])$ using the shift functor $[1]$). Set $c = a_2 b \pi_1 + s_2 a b_1 + a_2 h b$. Then, using the equalities found in the proof of Lemma 20.1 we obtain

$$
\begin{aligned}
d(c) &= a_2 b d(\pi_1) + d(s_2) a b_1 + a_2 d(h) b_1 \\
&= -a_2 b f_1 b_1 + a_2 f_2 a b_1 + a_2 (bf_1 - f_2 a) b_1 \\
&= 0
\end{aligned}
$$

(where we have used in particular that $d(\pi_1) = d(\pi_1)s_1 b_1 = f_1 b_1$ and $d(s_2) = a_2 \pi_2 d(s_2) = a_2 f_2$). Thus c is a degree 0 morphism $c : c(f_1) \to c(f_2)$ of $\mathcal{A}$ compatible with the given morphisms $y_i \to c(f_i) \to x_i[1]$. $\square$

In Situation 20.2 we say that a triangle (x, y, z, f, g, h) in $K(\mathcal{A})$ is a *distinguished triangle* if there exists an admissible short exact sequence $x' \to y' \to z'$ such that (x, y, z, f, g, h) is isomorphic as a triangle in $K(\mathcal{A})$ to the triangle $(x', y', z', x' \to y', y' \to z', \delta)$ constructed in Lemma 20.1. We will show below that

$$\boxed{K(\mathcal{A}) \text{ is a triangulated category}}$$

This result, although not as general as one might think, applies to a number of natural generalizations of the cases covered so far in the Stacks project. Here are some examples:

(1) Let $(X, \mathcal{O}_X)$ be a ringed space. Let (A, d) be a sheaf of differential graded $\mathcal{O}_X$-algebras. Let $\mathcal{A}$ be the differential graded category of differential graded A-modules. Then $K(\mathcal{A})$ is a triangulated category.

(2) Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (A, d) be a sheaf of differential graded $\mathcal{O}$-algebras. Let $\mathcal{A}$ be the differential graded category of differential graded A-modules. Then $K(\mathcal{A})$ is a triangulated category.

(3) Two examples with a different flavor may be found in Examples, Section 59.

The following simple lemma is a key to the construction.

Lemma 20.4. *In Situation 20.2 given any object x of $\mathcal{A}$, and the cone $C(1_x)$ of the identity morphism $1_x : x \to x$, the identity morphism on $C(1_x)$ is homotopic to zero.*

Proof. Consider the admissible short exact sequence given by axiom (C).

$$x \underset{\pi}{\overset{a}{\rightleftarrows}} C(1_x) \underset{s}{\overset{b}{\rightleftarrows}} x[1]$$

Then by Lemma 20.1, identifying hom-sets under shifting, we have $1_x = \pi d(s) = -d(\pi)s$ where s is regarded as a morphism in $\mathrm{Hom}_{\mathcal{A}}^{-1}(x, C(1_x))$. Therefore $a = a\pi d(s) = d(s)$ using formula (5) of Lemma 20.1, and $b = -d(\pi)sb = -d(\pi)$ by formula (6) of Lemma 20.1. Hence

$$1_{C(1_x)} = a\pi + sb = d(s)\pi - sd(\pi) = d(s\pi)$$

since s is of degree -1. □

A more general version of the above lemma will appear in Lemma 20.13. The following lemma is the analogue of Lemma 7.3.

Lemma 20.5. *In Situation 20.2 given a diagram*

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ a \downarrow & & \downarrow b \\ z & \xrightarrow{g} & w \end{array}$$

in Comp($\mathcal{A}$) commuting up to homotopy. Then

(1) *If f is an admissible monomorphism, then b is homotopic to a morphism b' which makes the diagram commute.*
(2) *If g is an admissible epimorphism, then a is homotopic to a morphism a' which makes the diagram commute.*

Proof. To prove (1), observe that the hypothesis implies that there is some $h \in \mathrm{Hom}_{\mathcal{A}}(x, w)$ of degree -1 such that $bf - ga = d(h)$. Since f is an admissible monomorphism, there is a morphism $\pi : y \to x$ in the category $\mathcal{A}$ of degree 0. Let $b' = b - d(h\pi)$. Then

$$\begin{aligned} b'f = bf - d(h\pi)f &= bf - d(h\pi f) \quad (\text{since } d(f) = 0) \\ &= bf - d(h) \\ &= ga \end{aligned}$$

as desired. The proof for (2) is omitted. □

The following lemma is the analogue of Lemma 7.4.

Lemma 20.6. *In Situation 20.2 let $\alpha : x \to y$ be a morphism in Comp($\mathcal{A}$). Then there exists a factorization in Comp($\mathcal{A}$):*

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \underset{s}{\overset{\pi}{\rightleftarrows}} y$$

such that

(1) *$\tilde{\alpha}$ is an admissible monomorphism, and $\pi\tilde{\alpha} = \alpha$.*
(2) *There exists a morphism $s : y \to \tilde{y}$ in Comp($\mathcal{A}$) such that $\pi s = 1_y$ and $s\pi$ is homotopic to $1_{\tilde{y}}$.*

Proof. By axiom (B), we may let $\tilde{y}$ be the differential graded direct sum of y and $C(1_x)$, i.e., there exists a diagram

$$y \underset{\pi}{\overset{s}{\rightleftarrows}} y \oplus C(1_x) \underset{t}{\overset{p}{\rightleftarrows}} C(1_x)$$

where all morphisms are of degree zero, and in Comp($\mathcal{A}$). Let $\tilde{y} = y \oplus C(1_x)$. Then $1_{\tilde{y}} = s\pi + tp$. Consider now the diagram

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \underset{s}{\overset{\pi}{\rightleftarrows}} y$$

where $\tilde{\alpha}$ is induced by the morphism $x \xrightarrow{\alpha} y$ and the natural morphism $x \to C(1_x)$ fitting in the admissible short exact sequence

$$x \rightleftarrows C(1_x) \rightleftarrows x[1]$$

So the morphism $C(1_x) \to x$ of degree 0 in this diagram, together with the zero morphism $y \to x$, induces a degree-0 morphism $\beta : \tilde{y} \to x$. Then $\tilde{\alpha}$ is an admissible monomorphism since it fits into the admissible short exact sequence

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \longrightarrow x[1]$$

Furthermore, $\pi\tilde{\alpha} = \alpha$ by the construction of $\tilde{\alpha}$, and $\pi s = 1_y$ by the first diagram. It remains to show that $s\pi$ is homotopic to $1_{\tilde{y}}$. Write 1_x as $d(h)$ for some degree -1 map. Then, our last statement follows from

$$
\begin{aligned}
1_{\tilde{y}} - s\pi &= tp \\
&= t(dh)p \quad \text{(by Lemma 20.4)} \\
&= d(thp)
\end{aligned}
$$

since $dt = dp = 0$, and t is of degree zero. $\qquad\square$

The following lemma is the analogue of Lemma 7.5.

Lemma 20.7. *In Situation 20.2 let $x_1 \to x_2 \to \ldots \to x_n$ be a sequence of composable morphisms in $\mathrm{Comp}(\mathcal{A})$. Then there exists a commutative diagram in $\mathrm{Comp}(\mathcal{A})$:*

$$
\begin{array}{ccccccc}
x_1 & \longrightarrow & x_2 & \longrightarrow & \ldots & \longrightarrow & x_n \\
\uparrow & & \uparrow & & & & \uparrow \\
y_1 & \longrightarrow & y_2 & \longrightarrow & \ldots & \longrightarrow & y_n
\end{array}
$$

such that each $y_i \to y_{i+1}$ is an admissible monomorphism and each $y_i \to x_i$ is a homotopy equivalence.

Proof. The case for $n = 1$ is trivial: one simply takes $y_1 = x_1$ and the identity morphism on x_1 is in particular a homotopy equivalence. The case $n = 2$ is given by Lemma 20.6. Suppose we have constructed the diagram up to x_{n-1}. We apply Lemma 20.6 to the composition $y_{n-1} \to x_{n-1} \to x_n$ to obtain y_n. Then $y_{n-1} \to y_n$ will be an admissible monomorphism, and $y_n \to x_n$ a homotopy equivalence. $\qquad\square$

The following lemma is the analogue of Lemma 7.6.

Lemma 20.8. *In Situation 20.2 let $x_i \to y_i \to z_i$ be morphisms in $\mathcal{A}$ $(i = 1, 2, 3)$ such that $x_2 \to y_2 \to z_2$ is an admissible short exact sequence. Let $b : y_1 \to y_2$ and $b' : y_2 \to y_3$ be morphisms in $\mathrm{Comp}(\mathcal{A})$ such that*

$$
\begin{array}{ccccc}
x_1 & \longrightarrow & y_1 & \longrightarrow & z_1 \\
{\scriptstyle 0}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle 0}\downarrow \\
x_2 & \longrightarrow & y_2 & \longrightarrow & z_2
\end{array}
\quad and \quad
\begin{array}{ccccc}
x_2 & \longrightarrow & y_2 & \longrightarrow & z_2 \\
{\scriptstyle 0}\downarrow & & {\scriptstyle b'}\downarrow & & {\scriptstyle 0}\downarrow \\
x_3 & \longrightarrow & y_3 & \longrightarrow & z_3
\end{array}
$$

commute up to homotopy. Then $b' \circ b$ is homotopic to 0.

Proof. By Lemma 20.5, we can replace b and b' by homotopic maps $\tilde{b}$ and $\tilde{b}'$, such that the right square of the left diagram commutes and the left square of the right diagram commutes. Say $b = \tilde{b} + d(h)$ and $b' = \tilde{b}' + d(h')$ for degree -1 morphisms h and h' in $\mathcal{A}$. Hence

$$b'b = \tilde{b}'\tilde{b} + d(\tilde{b}'h + h'\tilde{b} + h'd(h))$$

since $d(\tilde{b}) = d(\tilde{b}') = 0$, i.e. $b'b$ is homotopic to $\tilde{b}'\tilde{b}$. We now want to show that $\tilde{b}'\tilde{b} = 0$. Because $x_2 \xrightarrow{f} y_2 \xrightarrow{g} z_2$ is an admissible short exact sequence, there exist degree 0 morphisms $\pi : y_2 \to x_2$ and $s : z_2 \to y_2$ such that $\mathrm{id}_{y_2} = f\pi + sg$. Therefore

$$\tilde{b}'\tilde{b} = \tilde{b}'(f\pi + sg)\tilde{b} = 0$$

since $g\tilde{b} = 0$ and $\tilde{b}'f = 0$ as consequences of the two commuting squares. $\square$

The following lemma is the analogue of Lemma 8.1.

Lemma 20.9. *In Situation 20.2 let $0 \to x \to y \to z \to 0$ be an admissible short exact sequence in $\mathrm{Comp}(\mathcal{A})$. The triangle*

$$x \longrightarrow y \longrightarrow z \xrightarrow{\delta} x[1]$$

with $\delta : z \to x[1]$ as defined in Lemma 20.1 is up to canonical isomorphism in $K(\mathcal{A})$, independent of the choices made in Lemma 20.1.

Proof. Suppose δ is defined by the splitting

$$x \underset{\pi}{\overset{a}{\rightleftarrows}} y \underset{s}{\overset{b}{\rightleftarrows}} z$$

and δ' is defined by the splitting with π', s' in place of π, s. Then

$$s' - s = (a\pi + sb)(s' - s) = a\pi s'$$

since $bs' = bs = 1_z$ and $\pi s = 0$. Similarly,

$$\pi' - \pi = (\pi' - \pi)(a\pi + sb) = \pi' sb$$

Since $\delta = \pi d(s)$ and $\delta' = \pi' d(s')$ as constructed in Lemma 20.1, we may compute

$$\delta' = \pi' d(s') = (\pi + \pi' sb)d(s + a\pi s') = \delta + d(\pi s')$$

using $\pi a = 1_x$, $ba = 0$, and $\pi' sbd(s') = \pi' sba\pi d(s') = 0$ by formula (5) in Lemma 20.1. $\square$

The following lemma is the analogue of Lemma 9.1.

Lemma 20.10. *In Situation 20.2 let $f : x \to y$ be a morphism in $\mathrm{Comp}(\mathcal{A})$. The triangle $(y, c(f), x[1], i, p, f[1])$ is the triangle associated to the admissible short exact sequence*

$$y \longrightarrow c(f) \longrightarrow x[1]$$

where the cone $c(f)$ is defined as in Lemma 20.1.

Proof. This follows from axiom (C). $\square$

The following lemma is the analogue of Lemma 9.2.

Lemma 20.11. *In Situation 20.2 let $\alpha : x \to y$ and $\beta : y \to z$ define an admissible short exact sequence*

$$x \longrightarrow y \longrightarrow z$$

in $\mathrm{Comp}(\mathcal{A})$. Let $(x, y, z, \alpha, \beta, \delta)$ be the associated triangle in $K(\mathcal{A})$. Then, the triangles

$$(z[-1], x, y, \delta[-1], \alpha, \beta) \quad \text{and} \quad (z[-1], x, c(\delta[-1]), \delta[-1], i, p)$$

are isomorphic.

Proof. We have a diagram of the form

with splittings to α, β, i, and p given by $\tilde{\alpha}, \tilde{\beta}, \tilde{i}$, and $\tilde{p}$ respectively. Define a morphism $y \to c(\delta[-1])$ by $i\tilde{\alpha} + \tilde{p}\beta$ and a morphism $c(\delta[-1]) \to y$ by $\alpha\tilde{i} + \tilde{\beta}p$. Let us first check that these define morphisms in $\mathrm{Comp}(\mathcal{A})$. We remark that by identities from Lemma 20.1, we have the relation $\delta[-1] = \tilde{\alpha}d(\tilde{\beta}) = -d(\tilde{\alpha})\tilde{\beta}$ and the relation $\delta[-1] = \tilde{i}d(\tilde{p})$. Then

$$d(\tilde{\alpha}) = d(\tilde{\alpha})\tilde{\beta}\beta$$
$$= -\delta[-1]\beta$$

where we have used equation (6) of Lemma 20.1 for the first equality and the preceeding remark for the second. Similarly, we obtain $d(\tilde{p}) = i\delta[-1]$. Hence

$$d(i\tilde{\alpha} + \tilde{p}\beta) = d(i)\tilde{\alpha} + id(\tilde{\alpha}) + d(\tilde{p})\beta + \tilde{p}d(\beta)$$
$$= id(\tilde{\alpha}) + d(\tilde{p})\beta$$
$$= -i\delta[-1]\beta + i\delta[-1]\beta$$
$$= 0$$

so $i\tilde{\alpha} + \tilde{p}\beta$ is indeed a morphism of $\mathrm{Comp}(\mathcal{A})$. By a similar calculation, $\alpha\tilde{i} + \tilde{\beta}p$ is also a morphism of $\mathrm{Comp}(\mathcal{A})$. It is immediate that these morphisms fit in the commutative diagram. We compute:

$$(i\tilde{\alpha} + \tilde{p}\beta)(\alpha\tilde{i} + \tilde{\beta}p) = i\tilde{\alpha}\alpha\tilde{i} + i\tilde{\alpha}\tilde{\beta}p + \tilde{p}\beta\alpha\tilde{i} + \tilde{p}\beta\tilde{\beta}p$$
$$= i\tilde{i} + \tilde{p}p$$
$$= 1_{c(\delta[-1])}$$

where we have freely used the identities of Lemma 20.1. Similarly, we compute $(\alpha\tilde{i} + \tilde{\beta}p)(i\tilde{\alpha} + \tilde{p}\beta) = 1_y$, so we conclude $y \cong c(\delta[-1])$. Hence, the two triangles in question are isomorphic. $\square$

The following lemma is the analogue of Lemma 9.3.

Lemma 20.12. *In Situation 20.2 let $f_1 : x_1 \to y_1$ and $f_2 : x_2 \to y_2$ be morphisms in $\mathrm{Comp}(\mathcal{A})$. Let*

$$(a, b, c) : (x_1, y_1, c(f_1), f_1, i_1, p_1) \to (x_2, y_2, c(f_2), f_2, i_1, p_1)$$

be any morphism of triangles in $K(\mathcal{A})$. If a and b are homotopy equivalences, then so is c.

Proof. Since a and b are homotopy equivalences, they are invertible in $K(\mathcal{A})$ so let a^{-1} and b^{-1} denote their inverses in $K(\mathcal{A})$, giving us a commutative diagram

$$
\begin{array}{ccccc}
x_2 & \xrightarrow{f_2} & y_2 & \xrightarrow{i_2} & c(f_2) \\
\downarrow{\scriptstyle a^{-1}} & & \downarrow{\scriptstyle b^{-1}} & & \downarrow{\scriptstyle c'} \\
x_1 & \xrightarrow{f_1} & y_1 & \xrightarrow{i_1} & c(f_1)
\end{array}
$$

where the map c' is defined via Lemma 20.3 applied to the left commutative box of the above diagram. Since the diagram commutes in $K(\mathcal{A})$, it suffices by Lemma 20.8 to prove the following: given a morphism of triangle $(1, 1, c) : (x, y, c(f), f, i, p) \to (x, y, c(f), f, i, p)$ in $K(\mathcal{A})$, the map c is an isomorphism in $K(\mathcal{A})$. We have the commutative diagrams in $K(\mathcal{A})$:

$$
\begin{array}{ccccc}
y & \longrightarrow & c(f) & \longrightarrow & x[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 1} \\
y & \longrightarrow & c(f) & \longrightarrow & x[1]
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccccc}
y & \longrightarrow & c(f) & \longrightarrow & x[1] \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle c-1} & & \downarrow{\scriptstyle 0} \\
y & \longrightarrow & c(f) & \longrightarrow & x[1]
\end{array}
$$

Since the rows are admissible short exact sequences, we obtain the identity $(c-1)^2 = 0$ by Lemma 20.8, from which we conclude that $2 - c$ is inverse to c in $K(\mathcal{A})$ so that c is an isomorphism. $\square$

The following lemma is the analogue of Lemma 9.4.

Lemma 20.13. *In Situation 20.2.*

(1) *Given an admissible short exact sequence $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$. Then there exists a homotopy equivalence $e : C(\alpha) \to z$ such that the diagram*

(20.13.1)
$$
\begin{array}{ccccccc}
x & \xrightarrow{\alpha} & y & \xrightarrow{b} & C(\alpha) & \xrightarrow{-c} & x[1] \\
\downarrow & & \downarrow & & \vdots{\scriptstyle e} & & \downarrow \\
x & \xrightarrow{\alpha} & y & \xrightarrow{\beta} & z & \xrightarrow{\delta} & x[1]
\end{array}
$$

defines an isomorphism of triangles in $K(\mathcal{A})$. Here $y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]$ is the admissible short exact sequence given as in axiom (C).

(2) *Given a morphism $\alpha : x \to y$ in $Comp(\mathcal{A})$, let $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to y$ be the factorization given as in Lemma 20.6, where the admissible monomorphism $x \xrightarrow{\tilde{\alpha}} y$ extends to the admissible short exact sequence*

$$
x \xrightarrow{\tilde{\alpha}} \tilde{y} \longrightarrow z
$$

Then there exists an isomorphism of triangles

$$
\begin{array}{ccccccc}
x & \xrightarrow{\tilde{\alpha}} & \tilde{y} & \longrightarrow & z & \xrightarrow{\delta} & x[1] \\
\downarrow & & \downarrow & & \vdots{\scriptstyle e} & & \downarrow \\
x & \xrightarrow{\alpha} & y & \longrightarrow & C(\alpha) & \xrightarrow{-c} & x[1]
\end{array}
$$

where the upper triangle is the triangle associated to the sequence $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to z$.

Proof. For (1), we consider the more complete diagram, *without* the sign change on c:

$$x \underset{\pi}{\overset{\alpha}{\rightleftarrows}} y \underset{p}{\overset{b}{\rightleftarrows}} C(\alpha) \underset{\sigma}{\overset{c}{\rightleftarrows}} x[1] \underset{\pi}{\overset{\alpha}{\rightleftarrows}} y[1]$$

$$x \underset{\pi}{\overset{\alpha}{\rightleftarrows}} y \underset{s}{\overset{\beta}{\rightleftarrows}} z \xrightarrow{\delta} x[1]$$

where the admissible short exact sequence $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ is given the splitting π, s, and the admissible short exact sequence $y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]$ is given the splitting p, σ. Note that (identifying hom-sets under shifting)

$$\alpha = pd(\sigma) = -d(p)\sigma, \quad \delta = \pi d(s) = -d(\pi)s$$

by the construction in Lemma 20.1.

We define $e = \beta p$ and $f = bs - \sigma\delta$. We first check that they are morphisms in Comp($\mathcal{A}$). To show that $d(e) = \beta d(p)$ vanishes, it suffices to show that $\beta d(p)b$ and $\beta d(p)\sigma$ both vanish, whereas

$$\beta d(p)b = \beta d(pb) = \beta d(1_y) = 0, \quad \beta d(p)\sigma = -\beta\alpha = 0$$

Similarly, to check that $d(f) = bd(s) - d(\sigma)\delta$ vanishes, it suffices to check the post-compositions by p and c both vanish, whereas

$$pbd(s) - pd(\sigma)\delta = d(s) - \alpha\delta = d(s) - \alpha\pi d(s) = 0$$
$$cbd(s) - cd(\sigma)\delta = -cd(\sigma)\delta = -d(c\sigma)\delta = 0$$

The commutativity of left two squares of the diagram 20.13.1 follows directly from definition. Before we prove the commutativity of the right square (up to homotopy), we first check that e is a homotopy equivalence. Clearly,

$$ef = \beta p(bs - \sigma\delta) = \beta s = 1_z$$

To check that fe is homotopic to $1_{C(\alpha)}$, we first observe

$$b\alpha = bpd(\alpha) = d(\sigma), \quad \alpha c = -d(p)\sigma c = -d(p), \quad d(\pi)p = d(\pi)s\beta p = -\delta\beta p$$

Using these identities, we compute

$$\begin{aligned}
1_{C(\alpha)} =&bp + \sigma c \quad \text{(from } y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]) \\
=&b(\alpha\pi + s\beta)p + \sigma(\pi\alpha)c \quad \text{(from } x \xrightarrow{\alpha} y \xrightarrow{\beta} z) \\
=&d(\sigma)\pi p + bs\beta p - \sigma\pi d(p) \quad \text{(by the first two identities above)} \\
=&d(\sigma)\pi p + bs\beta p - \sigma\delta\beta p + \sigma\delta\beta p - \sigma\pi d(p) \\
=&(bs - \sigma\delta)\beta p + d(\sigma)\pi p - \sigma d(\pi)p - \sigma\pi d(p) \quad \text{(by the third identity above)} \\
=&fe + d(\sigma\pi p)
\end{aligned}$$

since $\sigma \in \text{Hom}^{-1}(x, C(\alpha))$ (cf. proof of Lemma 20.4). Hence e and f are homotopy inverses. Finally, to check that the right square of diagram 20.13.1 commutes up to homotopy, it suffices to check that $-cf = \delta$. This follows from

$$-cf = -c(bs - \sigma\delta) = c\sigma\delta = \delta$$

since $cb = 0$.

For (2), consider the factorization $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to y$ given as in Lemma 20.6, so the second morphism is a homotopy equivalence. By Lemmas 20.3 and 20.12, there exists an isomorphism of triangles between

$$x \xrightarrow{\alpha} y \to C(\alpha) \to x[1] \quad \text{and} \quad x \xrightarrow{\tilde{\alpha}} \tilde{y} \to C(\tilde{\alpha}) \to x[1]$$

Since we can compose isomorphisms of triangles, by replacing α by $\tilde{\alpha}$, y by $\tilde{y}$, and $C(\alpha)$ by $C(\tilde{\alpha})$, we may assume α is an admissible monomorphism. In this case, the result follows from (1). $\qquad\square$

The following lemma is the analogue of Lemma 10.1.

Lemma 20.14. *In Situation 20.2 the homotopy category $K(\mathcal{A})$ with its natural translation functors and distinguished triangles is a pre-triangulated category.*

Proof. We will verify each of TR1, TR2, and TR3.

Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Since

$$x \xrightarrow{\quad 1_x \quad} x \longrightarrow 0$$

is an admissible short exact sequence, $(x, x, 0, 1_x, 0, 0)$ is a distinguished triangle. Moreover, given a morphism $\alpha : x \to y$ in $\mathrm{Comp}(\mathcal{A})$, the triangle given by $(x, y, c(\alpha), \alpha, i, -p)$ is distinguished by Lemma 20.13.

Proof of TR2. Let $(x, y, z, \alpha, \beta, \gamma)$ be a triangle and suppose $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is distinguished. Then there exists an admissible short exact sequence $0 \to x' \to y' \to z' \to 0$ such that the associated triangle $(x', y', z', \alpha', \beta', \gamma')$ is isomorphic to $(y, z, x[1], \beta, \gamma, -\alpha[1])$. After rotating, we conclude that $(x, y, z, \alpha, \beta, \gamma)$ is isomorphic to $(z'[-1], x', y', \gamma'[-1], \alpha', \beta')$. By Lemma 20.11, we deduce that $(z'[-1], x', y', \gamma'[-1], \alpha', \beta')$ is isomorphic to $(z'[-1], x', c(\gamma'[-1]), \gamma'[-1], i, p)$. Composing the two isomorphisms with sign changes as indicated in the following diagram:

$$
\begin{array}{ccccccc}
x & \xrightarrow{\;\alpha\;} & y & \xrightarrow{\;\beta\;} & z & \xrightarrow{\;\gamma\;} & x[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
z'[-1] & \xrightarrow{-\gamma'[-1]} & x & \xrightarrow{\;\alpha'\;} & y' & \xrightarrow{\;\beta'\;} & z' \\
{\scriptstyle -1_{z'[-1]}}\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle -1_{z'}} \\
z'[-1] & \xrightarrow{\;\gamma'[-1]\;} & x & \xrightarrow{\;\alpha'\;} & c(\gamma'[-1]) & \xrightarrow{\;-p\;} & z'
\end{array}
$$

We conclude that $(x, y, z, \alpha, \beta, \gamma)$ is distinguished by Lemma 20.13 (2). Conversely, suppose that $(x, y, z, \alpha, \beta, \gamma)$ is distinguished, so that by Lemma 20.13 (1), it is isomorphic to a triangle of the form $(x', y', c(\alpha'), \alpha', i, -p)$ for some morphism $\alpha' : x' \to y'$ in $\mathrm{Comp}(\mathcal{A})$. The rotated triangle $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is isomorphic to the triangle $(y', c(\alpha'), x'[1], i, -p, -\alpha[1])$ which is isomorphic to $(y', c(\alpha'), x'[1], i, p, \alpha[1])$. By Lemma 20.10, this triangle is distinguished, from which it follows that $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is distinguished.

Proof of TR3: Suppose $(x, y, z, \alpha, \beta, \gamma)$ and $(x', y', z', \alpha', \beta', \gamma')$ are distinguished triangles of $\mathrm{Comp}(\mathcal{A})$ and let $f : x \to x'$ and $g : y \to y'$ be morphisms such that $\alpha' \circ f = g \circ \alpha$. By Lemma 20.13, we may assume that $(x, y, z, \alpha, \beta, \gamma) =$

$(x, y, c(\alpha), \alpha, i, -p)$ and $(x', y', z', \alpha', \beta', \gamma') = (x', y', c(\alpha'), \alpha', i', -p')$. Now apply Lemma 20.3 and we are done. $\qquad\square$

The following lemma is the analogue of Lemma 10.2.

Lemma 20.15. *In Situation 20.2 given admissible monomorphisms $x \xrightarrow{\alpha} y$, $y \xrightarrow{\beta} z$ in $\mathcal{A}$, there exist distinguished triangles $(x, y, q_1, \alpha, p_1, \delta_1)$, $(x, z, q_2, \beta\alpha, p_2, \delta_2)$ and $(y, z, q_3, \beta, p_3, \delta_3)$ for which TR4 holds.*

Proof. Given admissible monomorphisms $x \xrightarrow{\alpha} y$ and $y \xrightarrow{\beta} z$, we can find distinguished triangles, via their extensions to admissible short exact sequences,

$$x \underset{\pi_1}{\overset{\alpha}{\rightleftarrows}} y \underset{s_1}{\overset{p_1}{\rightleftarrows}} q_1 \xrightarrow{\delta_1} x[1]$$

$$x \underset{\pi_1\pi_3}{\overset{\beta\alpha}{\rightleftarrows}} z \underset{s_2}{\overset{p_2}{\rightleftarrows}} q_2 \xrightarrow{\delta_2} x[1]$$

$$y \underset{\pi_3}{\overset{\beta}{\rightleftarrows}} z \underset{s_3}{\overset{p_3}{\rightleftarrows}} q_3 \xrightarrow{\delta_3} x[1]$$

In these diagrams, the maps δ_i are defined as $\delta_i = \pi_i d(s_i)$ analogous to the maps defined in Lemma 20.1. They fit in the following solid commutative diagram

where we have defined the dashed arrows as indicated. Clearly, their composition $p_3 s_2 p_2 \beta s_1 = 0$ since $s_2 p_2 = 0$. We claim that they both are morphisms of $\mathrm{Comp}(\mathcal{A})$. We can check this using equations in Lemma 20.1:

$$d(p_2 \beta s_1) = p_2 \beta d(s_1) = p_2 \beta \alpha \pi_1 d(s_1) = 0$$

since $p_2 \beta \alpha = 0$, and

$$d(p_3 s_2) = p_3 d(s_2) = p_3 \beta \alpha \pi_1 \pi_3 d(s_2) = 0$$

since $p_3 \beta = 0$. To check that $q_1 \to q_2 \to q_3$ is an admissible short exact sequence, it remains to show that in the underlying graded category, $q_2 = q_1 \oplus q_3$ with the above two morphisms as coprojection and projection. To do this, observe that in the underlying graded category $\mathcal{C}$, there hold

$$y = x \oplus q_1, \quad z = y \oplus q_3 = x \oplus q_1 \oplus q_3$$

where $\pi_1\pi_3$ gives the projection morphism onto the first factor: $x \oplus q_1 \oplus q_3 \to z$. By axiom (A) on $\mathcal{A}$, $\mathcal{C}$ is an additive category, hence we may apply Homology, Lemma 3.10 and conclude that

$$\mathrm{Ker}(\pi_1\pi_3) = q_1 \oplus q_3$$

in $\mathcal{C}$. Another application of Homology, Lemma 3.10 to $z = x \oplus q_2$ gives $\mathrm{Ker}(\pi_1\pi_3) = q_2$. Hence $q_2 \cong q_1 \oplus q_3$ in $\mathcal{C}$. It is clear that the dashed morphisms defined above give coprojection and projection.

Finally, we have to check that the morphism $\delta : q_3 \to q_1[1]$ induced by the admissible short exact sequence $q_1 \to q_2 \to q_3$ agrees with $p_1\delta_3$. By the construction in Lemma 20.1, the morphism δ is given by

$$
\begin{aligned}
p_1\pi_3 s_2 d(p_2 s_3) &= p_1\pi_3 s_2 p_2 d(s_3) \\
&= p_1\pi_3(1 - \beta\alpha\pi_1\pi_3)d(s_3) \\
&= p_1\pi_3 d(s_3) \quad (\text{since } \pi_3\beta = 0) \\
&= p_1\delta_3
\end{aligned}
$$

as desired. The proof is complete. $\square$

Putting everything together we finally obtain the analogue of Proposition 10.3.

Proposition 20.16. *In Situation 20.2 the homotopy category $K(\mathcal{A})$ with its natural translation functors and distinguished triangles is a triangulated category.*

Proof. By Lemma 20.14 we know that $K(\mathcal{A})$ is pre-triangulated. Combining Lemmas 20.7 and 20.15 with Derived Categories, Lemma 4.13, we conclude that $K(\mathcal{A})$ is a triangulated category. $\square$

21. Derived Hom

Let R be a ring. Let (B, d) be a differential graded algebra over R. Denote $\mathcal{B} = \mathrm{Mod}^{dg}_{(B,\mathrm{d})}$ the differential graded category of differential graded B-modules, see Example 19.8. Let N be a differential graded B-module. Then the endomorphisms of N in $\mathcal{B}$

$$\mathrm{Hom}_{\mathcal{B}}(N, N)$$

is differential graded algebra over R. Now let N' be a second differential graded B-module. Then

$$\mathrm{Hom}_{\mathcal{B}}(N, N')$$

becomes a right differential graded $\mathrm{Hom}_{\mathcal{B}}(N, N)$-module by the composition

$$\mathrm{Hom}_{\mathcal{B}}(N, N') \times \mathrm{Hom}_{\mathcal{B}}(N, N) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(N, N')$$

We need one more piece of data, in order to be able to formulate the results in the correct generality. Namely, let (A, d) be a differential graded R-algebra and let $A \to \mathrm{Hom}_{\mathcal{B}}(N, N)$ be a homomorphism of differential graded R-algebras[4]. Using this homomorphism we obtain a functor

(21.0.1) $$\mathrm{Mod}_{(B,\mathrm{d})} \longrightarrow \mathrm{Mod}_{(A,\mathrm{d})}, \quad N' \longmapsto \mathrm{Hom}_{\mathcal{B}}(N, N')$$

where A acts on $\mathrm{Hom}_{\mathcal{B}}(N, N')$ via the given homomorphism and the action of $\mathrm{Hom}_{\mathcal{B}}(N, N)$ given above.

---

[4]A very interesting case is when $A = \mathrm{Hom}_{\mathcal{B}}(N, N)$.

Lemma 21.1. *The functor (21.0.1) defines an exact functor of triangulated categories* $K(Mod_{(B,d)}) \to K(Mod_{(A,d)})$.

Proof. Combining Lemmas 19.9, 19.10, and 19.5 we obtain the functor of the statement. We have to show that (21.0.1) transforms distinguished triangles into distinguished triangles. To see this suppose that $0 \to N_1 \to N_2 \to N_3 \to 0$ is an admissible short exact sequence of differential graded B-modules. Let $s : N_3 \to N_2$ be a graded B-module homomorphism which is left inverse to $N_2 \to N_3$. Then s defines a graded A-module homomorphism $\mathrm{Hom}_{\mathcal{B}}(N, N_3) \to \mathrm{Hom}_{\mathcal{B}}(N, N_2)$ which is left inverse to $\mathrm{Hom}_{\mathcal{B}}(N, N_2) \to \mathrm{Hom}_{\mathcal{B}}(N, N_3)$. This finishes the proof. $\square$

At this point we can consider the diagram

$$
\begin{array}{ccc}
K(\mathrm{Mod}_{(B,\mathrm{d})}) & \xrightarrow{\ \mathrm{Hom}_{\mathcal{B}}(N,-)\ } & K(\mathrm{Mod}_{(A,\mathrm{d})}) \\
\downarrow & \ \ \searrow^{F} & \downarrow \\
D(B,\mathrm{d}) & \dashrightarrow & D(A,\mathrm{d})
\end{array}
$$

We would like to construct a dotted arrow as the *right derived functor* of the composition F. (*Warning*: the diagram will not commute.) Namely, in the general setting of Derived Categories, Section 15 we want to compute the right derived functor of F with respect to the multplicative system of quasi-isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Lemma 21.2. *In the situation above, the right derived functor of F exists. We denote it* $R\,\mathrm{Hom}(N, -) : D(B, d) \to D(A, d)$.

Proof. We will use Derived Categories, Lemma 15.15 to prove this. As our collection $\mathcal{I}$ of objects we will use the objects with property (I). Property (1) was shown in Lemma 14.4. Property (2) holds because if $s : I \to I'$ is a quasi-isomorphism of modules with property (I), then s is a homotopy equivalence by Lemma 15.3. $\square$

22. Variant of derived Hom

Let $\mathcal{A}$ be an abelian category. Consider the differential graded category $\mathrm{Comp}^{dg}(\mathcal{A})$ of complexes of $\mathcal{A}$, see Example 19.6. Let $K^\bullet$ be a complex of $\mathcal{A}$. Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, K^\bullet)$$

and consider the functor of differential graded categories

$$\mathrm{Comp}^{dg}(\mathcal{A}) \longrightarrow \mathrm{Mod}^{dg}_{(E,\mathrm{d})}, \quad X^\bullet \longmapsto \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, X^\bullet)$$

of Lemma 19.10.

Lemma 22.1. *In the situation above. If the right derived functor $R\,\mathrm{Hom}(K^\bullet, -)$ of $\mathrm{Hom}(K^\bullet, -) : K(\mathcal{A}) \to D(Ab)$ is everywhere defined on $D(\mathcal{A})$, then we obtain a canonical exact functor*

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{A}) \longrightarrow D(E, d)$$

of triangulated categories which reduces to the usual one on taking associated complexes of abelian groups.

Proof. Note that we have an associated functor $K(\mathcal{A}) \to K(\mathrm{Mod}_{(E,\mathrm{d})})$ by Lemma 19.10. We claim this functor is an exact functor of triangulated categories. Namely, let $f : A^\bullet \to B^\bullet$ be a map of complexes of $\mathcal{A}$. Then a computation shows that

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, C(f)^\bullet) = C\left(\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, A^\bullet) \to \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, B^\bullet)\right)$$

where the right hand side is the cone in $\mathrm{Mod}_{(E,\mathrm{d})}$ defined earlier in this chapter. This shows that our functor is compatible with cones, hence with distinguished triangles. Let $X^\bullet$ be an object of $K(\mathcal{A})$. Consider the category of quasi-isomorphisms $s : X^\bullet \to Y^\bullet$. We are given that the functor $(s : X^\bullet \to Y^\bullet) \mapsto \mathrm{Hom}_{\mathcal{A}}(K^\bullet, Y^\bullet)$ is essentially constant when viewed in $D(Ab)$. But since the forgetful functor $D(E,\mathrm{d}) \to D(Ab)$ is compatible with taking cohomology, the same thing is true in $D(E,\mathrm{d})$. This proves the lemma. $\qquad\square$

Warning: Although the lemma holds as stated and may be useful as stated, the differential algebra E isn't the "correct" one unless $H^n(E) = \mathrm{Ext}^n_{D(\mathcal{A})}(K^\bullet, K^\bullet)$ for all $n \in \mathbf{Z}$.

23. Tensor product

This section should be moved somewhere else. Let R be a ring. Let A be an R-algebra (see Section 2). Given a right A-module M and a left A-module N there is a *tensor product*

$$M \otimes_A N$$

This tensor product is a module over R. In fact, it is the receptacle of the universal A-bilinear map $M \times N \to M \otimes_A N$, $(m, n) \mapsto m \otimes n$.

We list some properties of the tensor product

(1) In each variable the tensor product is right exact, in fact commutes with direct sums and arbitrary colimits.
(2) If A, M, N are graded and the module structures are compatible with gradings then $M \otimes_A N$ is graded as well. Then nth graded piece $(M \otimes_A N)^n$ of $M \otimes_A N$ is the quotient of $\bigoplus_{p+q=n} M^p \otimes_{A^0} N^q$ by the submodule generated by $m \otimes an - ma \otimes n$ where $m \in M^p$, $n \in N^q$, and $a \in A^{n-p-q}$.
(3) If (A, d) is a differential graded algebra, and M and N are (left and right) differential graded A-modules, then $M \otimes_A N$ is a differential graded R-module with differential

$$\mathrm{d}(m \otimes n) = \mathrm{d}(m) \otimes n + (-1)^i m \otimes \mathrm{d}(n)$$

for $m \in M^i$ and $n \in N$.
(4) If N is a (A, B)-bimodule then $M \otimes_A N$ is a right B-module.
(5) If A and B are graded algebras, M is a graded A-module, and N is an (A, B)-bimodule which comes with a grading such that it is both a left graded A-module and a right graded B-module, then $M \otimes_A N$ is a graded B-module.
(6) If (A, d) and (B, d) are differential graded algebras, M is a differential graded A-module, and N is an (A, B)-bimodule which comes with a grading and a differential such that it is both a left differential graded A-module and a right differential graded B-module, then $M \otimes_A N$ is a differential graded B-module.

In the last item, the condition may be more succintly stated by saying that N is a differential graded module over $A^{opp} \otimes_R B$. We state the following as a lemma.

Lemma 23.1. *Let (A, d) and (B, d) be differential graded algebras, and let N be an (A, B)-bimodule which comes with a grading and a differential such that it is both*

a left differential graded A-module and a right differential graded B-module. Then $M \mapsto M \otimes_A N$ defines a functor

$$- \otimes_A N : Mod^{dg}_{(A,d)} \longrightarrow Mod^{dg}_{(B,d)}$$

of differential graded categories. This functor induces functors

$$Mod_{(A,d)} \to Mod_{(B,d)} \quad and \quad K(Mod_{(A,d)}) \to K(Mod_{(B,d)})$$

by an application of Lemma 19.5.

Proof. This follows from the discussion above. $\square$

If A is an algebra and M, M' are right A-modules, then we define

$$\mathrm{Hom}_A(M, M') = \{f : M \to M' \mid f \text{ is } A\text{-linear}\}$$

as usual. If A is graded and M and M' are graded A-modules, then we recall (Example 18.6) that

$$\mathrm{Hom}_{\mathrm{Mod}^{gr}_A}(M, M') = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}^n(M, M')$$

where $\mathrm{Hom}^n(M, M')$ is the collection of all A-module maps $M \to M'$ which are homogeneous of degree n.

Lemma 23.2. *Let A and B be algebras. Let M be a right A-module, N an (A, B)-bimodule, and N' a right B-module. Then we have*

$$\mathrm{Hom}_B(M \otimes_A N, N') = \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, N'))$$

If A, B, M, N, N' are compatibly graded, then we have

$$\mathrm{Hom}_{Mod^{gr}_B}(M \otimes_A N, N') = \mathrm{Hom}_{Mod^{gr}_A}(M, \mathrm{Hom}_{Mod^{gr}_B}(N, N'))$$

for the graded versions.

Proof. This follows by interpreting both sides as A-bilinear maps $\psi : M \times N \to N'$ which are B-linear on the right. $\square$

24. Derived tensor product

This section is analogous to More on Algebra, Section 46.

Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a (A, B)-bimodule equipped with a grading and differential such that N is a left differential graded A-module and a right differential graded B-module. In other words, N is a differential graded $A^{opp} \otimes_R B$-module. Consider the functor

(24.0.1) $$\mathrm{Mod}_{(A,\mathrm{d})} \longrightarrow \mathrm{Mod}_{(B,\mathrm{d})}, \quad M \longmapsto M \otimes_A N$$

defined in Section 23.

Lemma 24.1. *The functor (24.0.1) defines an exact functor of triangulated categories $K(Mod_{(A,d)}) \to K(Mod_{(B,d)})$.*

Proof. The functor was constructed in Lemma 23.1. We have to show that $- \otimes_A N$ transforms distinguished triangles into distinguished triangles. Suppose that $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded A-modules. Let $s : M \to L$ be a graded A-module homomorphism which is left inverse to $L \to M$. Then s defines a graded B-module homomorphism $M \otimes_A N \to L \otimes_A N$ which is left inverse to $L \otimes_A N \to M \otimes_A N$. $\square$

At this point we can consider the diagram

$$\begin{array}{ccc} K(\mathrm{Mod}_{(A,\mathrm{d})}) & \xrightarrow{\quad -\otimes_A N \quad} & K(\mathrm{Mod}_{(B,\mathrm{d})}) \\ \downarrow & \overset{F}{\searrow} & \downarrow \\ D(A,\mathrm{d}) & \dashrightarrow & D(B,\mathrm{d}) \end{array}$$

The dotted arrow that we will construct below will be the *left derived functor* of the composition F. (*Warning*: the diagram will not commute.) Namely, in the general setting of Derived Categories, Section 15 we want to compute the left derived functor of F with respect to the multplicative system of quasi-isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Lemma 24.2. *In the situation above, the left derived functor of F exists. We denote it* $-\otimes_A^{\mathbf{L}} N : D(A,d) \to D(B,d)$.

Proof. We will use Derived Categories, Lemma 15.15 to prove this. As our collection $\mathcal{P}$ of objects we will use the objects with property (P). Property (1) was shown in Lemma 13.4. Property (2) holds because if $s : P \to P'$ is a quasi-isomorphism of modules with property (P), then s is a homotopy equivalence by Lemma 15.3. $\square$

Remark 24.3. Let (A,d) and (B,d) be differential graded algebras. Let $f : N \to N'$ be a homomorphism of differential graded $A^{opp} \otimes_R B$-modules. Then f induces a morphism of functors

$$1 \otimes f : -\otimes_A^{\mathbf{L}} N \longrightarrow -\otimes_A^{\mathbf{L}} N'$$

If f is a quasi-isomorphism, then $1 \otimes f$ is an isomorphism of functors.

Lemma 24.4. *Let (A,d) and (B,d) be differential graded algebras. Let N be an (A,B)-bimodule which comes with a grading and a differential such that it is a differential graded module for both A and B. Then the functors*

$$-\otimes_A^{\mathbf{L}} N : D(A,d) \longrightarrow D(B,d)$$

of Lemma 24.2 and

$$R\mathrm{Hom}(N,-) : D(B,d) \longrightarrow D(A,d)$$

of Lemma 21.2 are adjoint.

Proof. The statement means that we have

$$\mathrm{Hom}_{D(A,\mathrm{d})}(M, R\mathrm{Hom}(N,N')) = \mathrm{Hom}_{D(B,\mathrm{d})}(M \otimes_A^{\mathbf{L}} N, N')$$

bifunctorially in M and N'. To see this we may assume that M is a differential graded A-module with property (P) and that N' is a differential graded B-module with property (I). The computation of the derived functors given in the lemmas referenced in the statement combined with Lemma 15.3 translates the above into

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(M, \mathrm{Hom}_{\mathcal{B}}(N,N')) = \mathrm{Hom}_{K(\mathrm{Mod}_{(B,\mathrm{d})})}(M \otimes_A N, N')$$

where $\mathcal{B} = \mathrm{Mod}_{(B,\mathrm{d})}^{dg}$. Thus it is certainly sufficient to show that

$$\mathrm{Hom}_{\mathcal{A}}(M, \mathrm{Hom}_{\mathcal{B}}(N,N')) = \mathrm{Hom}_{\mathcal{B}}(M \otimes_A N, N')$$

as differential graded $\mathbf{Z}$-modules where $\mathcal{A} = \mathrm{Mod}_{(A,\mathrm{d})}^{dg}$. This follows from the fact that the isomorphism (Lemma 23.2)

$$\mathrm{Hom}_A(M, \mathrm{Hom}_B(N,N')) = \mathrm{Hom}_B(M \otimes_A N, N')$$

of internal homs of graded modules respects the differentials. $\qquad\square$

Lemma 24.5. *Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded algebras over R. Let N be a differential graded $A^{opp} \otimes_R B$-module. Let N' be a differential graded $B^{opp} \otimes_R C$-module. If C is K-flat as a complex of R-modules, then the composition*

$$D(A, d) \xrightarrow{\ -\otimes_A^{\mathbf{L}} N\ } D(B, d) \xrightarrow{\ -\otimes_B^{\mathbf{L}} N'\ } D(C, d)$$

is isomorphic to $- \otimes_A^{\mathbf{L}} N''$ for some differential graded $A^{opp} \otimes_R C$-module N''.

Proof. We will use the construction of the functor $- \otimes^{\mathbf{L}} -$ of the proof of Lemma 24.2 without further mention. By Remark 24.3 we may replace N' by a quasi-isomorphic bimodule. Thus we assume that N' has property (P) as a differential graded $B^{opp} \otimes_R C$-module, see Lemma 13.4. Let $F_\bullet$ be the corresponding filtration on N'. We claim that $N'' = N \otimes_B N'$ works.

Let M be an object of $D(A, \mathrm{d})$. Using the lemma we may and do assume that M has property (P) as a differential graded A-module. Then $M \otimes_A^{\mathbf{L}} N = M \otimes_A N$. Next, we choose a quasi-isomorphism $P \to M \otimes_A N$ where P is a differential graded B-module with property (P). Then

$$(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} N' = P \otimes_B N'$$

The map $P \to M \otimes_A N$ induces a map

$$P \otimes_B N' \to (M \otimes_A N) \otimes_B N' = M \otimes_A N''$$

This construction is functorial in M (details omitted) and hence it suffices to prove this map is a quasi-isomorphism.

Since $N' = \operatorname{colim} F_i N'$ it suffices to prove

$$P \otimes_B F_i N' \to M \otimes_A N \otimes_B F_i N'$$

is a quasi-isomorphism for all i. Using the short exact sequences $0 \to F_{i-1} N' \to F_i N' \to F_i N'/F_{i-1} N' \to 0$ which are graded split, we see that it suffices to prove that the maps

$$P \otimes_B F_i N'/F_{i-1} N' \to M \otimes_A N \otimes_B F_i N'/F_{i-1} N'$$

are quasi-isomorphisms for all i. Since $F_i N'/F_{i-1} N'$ is a direct sum of shifts of $B^{opp} \otimes_R C$ we finally reduce to showing that the map

$$P \otimes_B (B^{opp} \otimes_R C) \to M \otimes_A N \otimes_B (B^{opp} \otimes_R C)$$

is a quasi-isomorphism. In other words, we have to show that

$$P \otimes_R C \to M \otimes_A N \otimes_R C$$

is a quasi-isomorphism. Since $P \to M \otimes_A N$ is a quasi-isomorphism we conclude using More on Algebra, Lemma 45.4. $\qquad\square$

Lemma 24.6. *With notation and assumptions as in Lemma 24.4. Assume*

  (1) *N defines a compact object of $D(B, d)$, and*
  (2) *the map $H^k(A) \to \operatorname{Hom}_{D(B,d)}(N, N[k])$ is an isomorphism for all $k \in \mathbf{Z}$.*

Then the functor $- \otimes_A^{\mathbf{L}} N$ is fully faithful.

Proof. Because our functor has a left adjoint given by $R\operatorname{Hom}(N, -)$ by Lemma 24.4 it suffices to show that for a differential graded A-module M the map

$$H^0(M) \longrightarrow \operatorname{Hom}_{D(B,\mathrm{d})}(N, M \otimes_A^{\mathbf{L}} N)$$

is an isomorphism. We may assume that $M = P$ is a differential graded A-module which has property (P). Since N defines a compact object, we reduce using Lemma 13.1 to the case where P has a finite filtration whose graded pieces are direct sums of $A[k]$. Again using compactness we reduce to the case $P = A[k]$. Assumption (2) on N is that the result holds for these. $\square$

25. Variant of derived tensor product

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Then we have the functors

$$\operatorname{Comp}(\mathcal{O}) \to K(\mathcal{O}) \to D(\mathcal{O})$$

and as we've seen above we have differential graded enhancement $\operatorname{Comp}^{dg}(\mathcal{O})$. Namely, this is the differential graded category of Example 19.6 associated to the abelian category $Mod(\mathcal{O})$. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules in other words, an object of $\operatorname{Comp}^{dg}(\mathcal{O})$. Set

$$(E, \mathrm{d}) = \operatorname{Hom}_{\operatorname{Comp}^{dg}(\mathcal{O})}(K^\bullet, K^\bullet)$$

This is a differential graded $\mathbf{Z}$-algebra. We claim there is an analogue of the derived base change in this situation.

Lemma 25.1. *In the situation above there is a functor*

$$- \otimes_E K^\bullet : Mod^{dg}_{(E,d)} \longrightarrow Comp^{dg}(\mathcal{O})$$

of differential graded categories. This functor sends E to $K^\bullet$ and commutes with direct sums.

Proof. Let M be a differential graded E-module. For every object U of $\mathcal{C}$ the complex $K^\bullet(U)$ is a left differential graded E-module as well as a right $\mathcal{O}(U)$-module. The actions commute, so we have a bimodule. Thus, by the constructions in Section 23 we can form the tensor product

$$M \otimes_E K^\bullet(U)$$

which is a differential graded $\mathcal{O}(U)$-module, i.e., a complex of $\mathcal{O}(U)$-modules. This construction is functorial with respect to U, hence we can sheafify to get a complex of $\mathcal{O}$-modules which we denote

$$M \otimes_E K^\bullet$$

Moreover, for each U the construction determines a functor $Mod^{dg}_{(E,\mathrm{d})} \to \operatorname{Comp}^{dg}(\mathcal{O}(U))$ of differential graded categories by Lemma 23.1. It is therefore clear that we obtain a functor as stated in the lemma. $\square$

Lemma 25.2. *The functor of Lemma 25.1 defines an exact functor of triangulated categories $K(Mod_{(Ed)}) \to K(\mathcal{O})$.*

Proof. The functor induces a functor between homotopy categories by Lemma 19.5. We have to show that $- \otimes_E K^\bullet$ transforms distinguished triangles into distinguished triangles. Suppose that $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded E-modules. Let $s : M \to L$ be a graded E-module homomorphism which is left inverse to $L \to M$. Then s defines a map

$M \otimes_E K^\bullet \to L \otimes_E K^\bullet$ of graded $\mathcal{O}$-modules (i.e., respecting $\mathcal{O}$-module structure and grading, but not differentials) which is left inverse to $L \otimes_E K^\bullet \to M \otimes_E K^\bullet$. Thus we see that

$$0 \to K \otimes_E K^\bullet \to L \otimes_E K^\bullet \to M \otimes_E K^\bullet \to 0$$

is a termwise split short exact sequences of complexes, i.e., a defines a distinguished triangle in $K(\mathcal{O})$. $\square$

Lemma 25.3. *The functor $K(Mod_{(E,d)}) \to K(\mathcal{O})$ of Lemma 25.2 has a left derived version defined on all of $D(E,d)$. We denote it $- \otimes_E^{\mathbf{L}} K^\bullet : D(E,d) \to D(\mathcal{O})$.*

Proof. We will use Derived Categories, Lemma 15.15 to prove this. As our collection $\mathcal{P}$ of objects we will use the objects with property (P). Property (1) was shown in Lemma 13.4. Property (2) holds because if $s : P \to P'$ is a quasi-isomorphism of modules with property (P), then s is a homotopy equivalence by Lemma 15.3. $\square$

Lemma 25.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. Then the functors*

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E,d) \longrightarrow D(\mathcal{O})$$

of Lemma 25.3 and

$$R\operatorname{Hom}(K^\bullet, -) : D(\mathcal{O}) \longrightarrow D(E,d)$$

of Lemma 22.1 are adjoint.

Proof. The statement means that we have

$$\operatorname{Hom}_{D(E,\mathrm{d})}(M, R\operatorname{Hom}(K^\bullet, L^\bullet)) = \operatorname{Hom}_{D(\mathcal{O})}(M \otimes_E^{\mathbf{L}} K^\bullet, L^\bullet)$$

bifunctorially in M and $L^\bullet$. To see this we may replace M by a differential graded E-module P with property (P). We also may replace $L^\bullet$ by a K-injective complex of $\mathcal{O}$-modules $I^\bullet$. The computation of the derived functors given in the lemmas referenced in the statement combined with Lemma 15.3 translates the above into

$$\operatorname{Hom}_{K(\operatorname{Mod}_{(E,\mathrm{d})})}(P, \operatorname{Hom}_{\mathcal{B}}(K^\bullet, I^\bullet)) = \operatorname{Hom}_{K(\mathcal{O})}(P \otimes_E K^\bullet, I^\bullet)$$

where $\mathcal{B} = \operatorname{Comp}^{dg}(\mathcal{O})$. There is an evaluation map from right to left functorial in P and $I^\bullet$ (details omitted). Choose a filtration $F_\bullet$ on P as in the definition of property (P). By Lemma 13.1 and the fact that both sides of the equation are homological functors in P on $K(\operatorname{Mod}_{(E,\mathrm{d})})$ we reduce to the case where P is replaced by the differential graded E-module $\bigoplus F_i P$. Since both sides turn direct sums in the variable P into direct products we reduce to the case where P is one of the differential graded E-modules $F_i P$. Since each $F_i P$ has a finite filtration (given by admissible monomorpisms) whose graded pieces are graded projective E-modules we reduce to the case where P is a graded projective E-module. In this case we clearly have

$$\operatorname{Hom}_{\operatorname{Mod}_{(E,\mathrm{d})}^{dg}}(P, \operatorname{Hom}_{\mathcal{B}}(K^\bullet, I^\bullet)) = \operatorname{Hom}_{\operatorname{Comp}^{dg}(\mathcal{O})}(P \otimes_E K^\bullet, I^\bullet)$$

as graded $\mathbf{Z}$-modules (because this statement reduces to the case $P = E[k]$ where it is obvious). As the isomorphism is compatible with differentials we conclude. $\square$

Lemma 25.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. Assume*

   (1) *$K^\bullet$ represents a compact object of $D(\mathcal{O})$, and*

(2) $E = \mathrm{Hom}_{Comp^{dg}(\mathcal{O})}(K^{\bullet}, K^{\bullet})$ computes the ext groups of $K^{\bullet}$ in $D(\mathcal{O})$.
Then the functor

$$- \otimes_E^{\mathbf{L}} K^{\bullet} : D(E, d) \longrightarrow D(\mathcal{O})$$

of Lemma 25.3 is fully faithful.

Proof. Because our functor has a left adjoint given by $R\,\mathrm{Hom}(K^{\bullet}, -)$ by Lemma 25.4 it suffices to show for a differential graded E-module M that the map

$$H^0(M) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(K^{\bullet}, M \otimes_E^{\mathbf{L}} K^{\bullet})$$

is an isomorphism. We may assume that $M = P$ is a differential graded E-module which has property (P). Since $K^{\bullet}$ defines a compact object, we reduce using Lemma 13.1 to the case where P has a finite filtration whose graded pieces are direct sums of $E[k]$. Again using compactness we reduce to the case $P = E[k]$. The assumption on $K^{\bullet}$ is that the result holds for these. $\qquad\square$

26. Characterizing compact objects

Compact objects of additive categories are defined in Derived Categories, Definition 34.1. In this section we characterize compact objects of the derived category of a differential graded algebra.

Lemma 26.1. *Let (A, d) be a differential graded algebra. Let P, M be differential graded A-modules. Assume P is projective as a graded A-module.*

   (1) *If M is acyclic, then $\mathrm{Hom}_{K(A,d)}(P, M) = 0$.*
   (2) *In general $\mathrm{Hom}_{K(A,d)}(P, M) = \mathrm{Hom}_{D(A,d)}(P, M)$.*

Proof. Consider the functor

$$N \longmapsto \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(P, N)$$

that to a differential graded A-module N associates the differential graded R-module whose degree n summand is the degree n homogeneous A-module maps from P to N, see Example 19.8. Since P is graded projective, this functor transforms any short exact sequence of differential graded A-modules into a short exact sequence of complexes of R-modules. In particular, if $Z \subset M$ is the kernel of d_M, then we get the short exact sequence $0 \to Z \to M \to Z[1] \to 0$ because M is acyclic. This in turn produces the short exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(P, Z) \to \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(P, M) \to \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(P, Z[1]) \to 0$$

The associated long exact sequence of cohomology modules is

$$\mathrm{Hom}_{K(A,\mathrm{d})}(P, Z) \to \mathrm{Hom}_{K(A,\mathrm{d})}(P, Z) \to$$
$$\mathrm{Hom}_{K(A,\mathrm{d})}(P, M) \to \mathrm{Hom}_{K(A,\mathrm{d})}(P, Z[1]) \to \mathrm{Hom}_{K(A,\mathrm{d})}(P, Z[1])$$

A computation, which we omit, shows that the first and the last arrows are identity maps, thereby proving part (1).

Choose a quasi-isomorphism $M \to I$ where I has property (I) which is possible by Lemma 14.4. Choose a distinguished triangle

$$M \to I \to Q \to M[1]$$

in $K(A, \mathrm{d})$. Since $M \to I$ is a quasi-isomorphism, we see that Q is acyclic. By part (1) we have $\mathrm{Hom}_{K(A,\mathrm{d})}(P, Q[n]) = 0$ for all n. As $\mathrm{Hom}_{K(A,\mathrm{d})}(P, -)$ is homological

(Derived Categories, Lemma 4.2) we see that $\mathrm{Hom}_{K(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, I)$. We finish by applying Lemma 15.3. $\qquad\square$

Let us say a differential graded A-module M is *finite* if M is generated, as a right A-module, by finitely many elements.

Lemma 26.2. *Let (A, d) be a differential graded algebra. If P is a differential graded A-module which is finite graded projective, then P is a compact object of $D(A, d)$.*

Proof. Suppose that P is finite and projective as a graded module. Choose a quasi-isomorphism $P' \to P$ where P' has property (P). Let $K = \mathrm{Ker}(P' \to P)$. Then K is an acyclic differential graded A-module, whence the boundary $\delta : P \to K[1]$ (see Lemmas 11.1 and 7.2) is zero by Lemma 26.1. Thus we see that $P' \to P$ is a homotopy equivalence (by general properties of triangulated categories). It follows that $\mathrm{Hom}_{K(A,\mathrm{d})}(P', -) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, -)$ as functors, whence we see that

$$\mathrm{Hom}_{D(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, M)$$

for any differential graded A-module M as this is true for P' by Lemma 15.3. Now, if $M = \bigoplus M_i$, then because P is a finite A-module, we see that

$$\mathrm{Hom}_{K(A,\mathrm{d})}(P, M) = \bigoplus \mathrm{Hom}_{K(A,\mathrm{d})}(P, M_i)$$

(look at the image of the generators of P for any map or homotopy). Moreover, the direct sum is the direct sum in the derived category, see Lemma 15.4. Thus P represents a compact object of $D(A, \mathrm{d})$. $\qquad\square$

Lemma 26.3. *Let (A, d) be a differential graded algebra. Let E be a compact object of $D(A, \mathrm{d})$. Let P be a differential graded A-module which has a finite filtration*

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset F_nP = P$$

by differential graded submodules such that

$$F_{i+1}P/F_iP \cong \bigoplus\nolimits_{j \in J_i} A[k_{i,j}]$$

as differential graded A-modules for some sets J_i and integers $k_{i,j}$. Let $E \to P$ be a morphism of $D(A, \mathrm{d})$. Then there exists a differential graded submodule $P' \subset P$ such that $F_{i+1}P \cap P'/(F_iP \cap P')$ is equal to $\bigoplus_{j \in J'_i} A[k_{i,j}]$ for some finite subsets $J'_i \subset J_i$ and such that $E \to P$ factors through P'.

Proof. We will prove by induction on $-1 \le m \le n$ that there exists a differential graded submodule $P' \subset P$ such that

(1) $F_mP \subset P'$,
(2) for $i \ge m$ the quotient $F_{i+1}P \cap P'/(F_iP \cap P')$ is isomorphic to $\bigoplus_{j \in J'_i} A[k_{i,j}]$ for some finite subsets $J'_i \subset J_i$, and
(3) $E \to P$ factors through P'.

The base case is $m = n$ where we can take $P' = P$.

Induction step. Assume P' works for m. For $i \ge m$ and $j \in J'_i$ let $x_{i,j} \in F_{i+1}P \cap P'$ be a homogeneous element of degree $k_{i,j}$ whose image in $F_{i+1}P \cap P'/(F_iP \cap P')$ is the generator in the summand corresponding to $j \in J_i$. The $x_{i,j}$ generate P'/F_mP as an A-module. Write

$$\mathrm{d}(x_{i,j}) = \sum x_{i',j'} a_{i,j}^{i',j'} + y_{i,j}$$

with $y_{i,j} \in F_m P$ and $a_{i,j}^{i',j'} \in A$. There exists a finite subset $J'_{m-1} \subset J_{m-1}$ such that each $y_{i,j}$ maps to an element of the submodule $\bigoplus_{j \in J'_{m-1}} A[k_{m-1,j}]$ of $F_m P/F_{m-1}P$. Let $P'' \subset F_m P$ be the inverse image of $\bigoplus_{j \in J'_{m-1}} A[k_{m-1,j}]$ under the map $F_m P \to F_m P/F_{m-1}P$. Then we see that the A-submodule

$$P'' + \sum x_{i,j} A$$

is a differential graded submodule of the type we are looking for. Moreover

$$P'/(P'' + \sum x_{i,j} A) = \bigoplus_{j \in J_{m-1} \setminus J'_{m-1}} A[k_{m-1,j}]$$

Since E is compact, the composition of the given map $E \to P'$ with the quotient map, factors through a finite direct subsum of the module displayed above. Hence after enlarging J'_{m-1} we may assume $E \to P'$ factors through $P'' + \sum x_{i,j} A$ as desired. $\square$

It is not true that every compact object of $D(A, d)$ comes from a finite graded projective differential graded A-module, see Examples, Section 58.

Proposition 26.4. *Let (A, d) be a differential graded algebra. Let E be an object of $D(A, d)$. Then the followoing are equivalent*

(1) *E is a compact object,*
(2) *E is a direct summand of an object of $D(A, d)$ which is represented by a finite graded projective differential graded A-module, and*
(3) *E is a direct summand of an object of $D(A, d)$ which is represented by a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_i P/F_{i-1}P$ are finite direct sums of shifts of A.*

Proof. Assume E is compact. By Lemma 13.4 we may assume that E is represented by a differential graded A-module P with propery (P). Consider the distinguished triangle

$$\bigoplus F_i P \to \bigoplus F_i P \to P \xrightarrow{\delta} \bigoplus F_i P[1]$$

coming from the admissible short exact sequence of Lemma 13.1. Since E is compact we have $\delta = \sum_{i=1,\dots,n} \delta_i$ for some $\delta_i : P \to F_i P[1]$. Since the compostion of δ with the map $\bigoplus F_i P[1] \to \bigoplus F_i P[1]$ is zero (Derived Categories, Lemma 4.1) it follows that $\delta = 0$ (follows as $\bigoplus F_i P \to \bigoplus F_i P$ maps the summand $F_i P$ via the difference of id and the inclusion map into $F_{i-1}P$). Thus we see that the identity on E factors through $\bigoplus F_i P$ in $D(A, d)$ (by Derived Categories, Lemma 4.10). Next, we use that P is compact again to see that the map $E \to \bigoplus F_i P$ factors through $\bigoplus_{i=1,\dots,n} F_i P$ for some n. In other words, the identity on E factors through $\bigoplus_{i=1,\dots,n} F_i P$. By Lemma 26.3 we see that the identity of E factors as $E \to P \to E$ where P is as in part (3) of the statement of the lemma. In other words, we have proven that (1) implies (3).

It is clear that (3) implies (2). Finally, (2) implies (1) by Lemma 26.2 and Derived Categories, Lemma 34.2. $\square$

Lemma 26.5. *Let (A, d) be a differential graded algebra. For every compact object E of $D(A, d)$ there exist integers $a \le b$ such that $\operatorname{Hom}_{D(A,d)}(E, M) = 0$ if $H^i(M) = 0$ for $i \in [a, b]$.*

Proof. Observe that the collection of objects of $D(A, \mathrm{d})$ for which such a pair of integers exists is a saturated, strictly full triangulated subcategory of $D(A, \mathrm{d})$. Thus by Proposition 26.4 it suffices to prove this when E is represented by a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_i P / F_{i-1} P$ are finite direct sums of shifts of A. Using the compatibility with triangles, we see that it suffices to prove it for $P = A$. In this case $\mathrm{Hom}_{D(A,\mathrm{d})}(A, M) = H^0(M)$ and the result holds with $a = b = 0$. $\qquad\square$

If (A, d) is just a graded algebra or more generally lives in only a finite number of degrees, then we do obtain the more precise description of compact objects.

Lemma 26.6. *Let (A, d) be a differential graded algebra. Assume that $A^n = 0$ for $|n| \gg 0$. Then an object of $D(A, d)$ is compact if and only if it can be represented by a finite graded projective differential graded A-module.*

Proof. One implication is Lemma 26.2. For the converse, let E be a compact object of $D(A, \mathrm{d})$. Fix $a \leq b$ as in Lemma 26.5. After decreasing a and increasing b if necessary, we may also assume that $H^i(E) = 0$ for $i \notin [a, b]$ (this follows from Proposition 26.4 and our assumption on A). Moreover, fix an integer $c > 0$ such that $A^n = 0$ if $|n| \geq c$.

By Proposition 26.4 we see that E is a direct summand, in $D(A, \mathrm{d})$, of a differential graded A-module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_i P / F_{i-1} P$ are finite direct sums of shifts of A. Note that P is a finite graded free A-module. We will use without further mention that for such a differential graded A-module we have $\mathrm{Hom}_{D(A,\mathrm{d})}(P, P) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, P)$ by Lemma 26.1.

Choose $n > 0$ such that $b + 4c - n < a$. Represent the projector onto E by an endomorphism $\varphi : P \to P$ of differential graded A-modules. Consider the distinguished triangle

$$P \xrightarrow{1-\varphi} P \to C \to P[1]$$

in $K(A, \mathrm{d})$ where C is the cone of the first arrow. Then $C \cong E \oplus E[1]$ in $D(A, \mathrm{d})$ and C is a finite graded free A-module. Next, consider a distinguished triangle

$$C[1] \to C \to C' \to C[2]$$

in $K(A, \mathrm{d})$ where C' is the cone on a morphism $C[1] \to C$ representing the composition

$$C[1] \cong E[1] \oplus E[2] \to E[1] \to E \oplus E[1] \cong C$$

in $D(A, \mathrm{d})$. Then we see that C' represents $E \oplus E[2]$. Continuing in this manner we see that we can find a differential graded A-module P which is a finite graded free A-module representing $E \oplus E[n]$.

Choose a basis $x_i, i \in I$ of homogeneous element for P as an A-module. Let $d_i = \deg(x_i)$. Let P_1 be the A-submodule of P generated by x_i and $\mathrm{d}(x_i)$ for $d_i \leq a - c - 1$. Let P_2 be the A-submodule of P generated by x_i and $\mathrm{d}(x_i)$ for $d_i \geq b - n + c$. We observe

(1) P_1 and P_2 are differential graded submodules of P,
(2) $P_1^t = 0$ for $t \geq a$,
(3) $P_1^t = P^t$ for $t \leq a - 2c$,
(4) $P_2^t = 0$ for $t \leq b - n$,

(5) $P_2^t = P^t$ for $t \geq b - n + 2c$.

As $b - n + 2c \geq a - 2c$ by our choice of n we obtain a short exact sequence of differential graded A-modules

$$0 \to P_1 \cap P_2 \to P_1 \oplus P_2 \xrightarrow{\pi} P \to 0$$

Since P is projective as a graded A-module this is an admissible short exact sequence (Lemma 11.1). Hence we obtain a boundary map $\delta : P \to (P_1 \cap P_2)[1]$ in $K(A, \mathrm{d})$, see Lemma 7.2. Since $P = E \oplus E[n]$ and since $P_1 \cap P_2$ lives in degrees $(b - n, a)$ we find that $\mathrm{Hom}_{D(A,\mathrm{d})}(E \oplus E[n], (P_1 \cap P_2)[1])$ is zero. Therefore $\delta = 0$ as a morphism in $K(A, \mathrm{d})$. By Derived Categories, Lemma 4.10 we can find a map $s : P \to P_1 \oplus P_2$ such that $\pi \circ s = \mathrm{id}_P + \mathrm{d}h + h\mathrm{d}$ for some $h : P \to P$ of degree -1. Since $P_1 \oplus P_2 \to P$ is surjective and since P is projective as a graded A-module we can choose a homogeneous lift $\tilde{h} : P \to P_1 \oplus P_2$ of h. Then we change s into $s + \mathrm{d}\tilde{h} + \tilde{h}\mathrm{d}$ to get $\pi \circ s = \mathrm{id}_P$. This means we obtain a direct sum decomposition $P = s^{-1}(P_1) \oplus s^{-1}(P_2)$. Since $s^{-1}(P_2)$ is equal to P in degrees $\geq b - n + 2c$ we see that $s^{-1}(P_2) \to P \to E$ is a quasi-isomorphism, i.e., an isomorphism in $D(A, \mathrm{d})$. This finishes the proof. $\square$

27. Equivalences of derived categories

Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. A natural question that arises in nature is what it means that $D(A, \mathrm{d})$ is equivalent to $D(B, \mathrm{d})$ as an R-linear triangulated category. This is a rather subtle question and it will turn out it isn't always the correct question to ask. Nonetheless, in this section we collection some conditions that guarantee this is the case.

We strongly urge the reader to take a look at the groundbreaking paper [Ric89] on this topic.

Lemma 27.1. *Let R be a ring. Let $(A, d) \to (B, d)$ be a homomorphism of differential graded algebras over R, which induces an isomorphism on cohomology algebras. Then*

$$- \otimes_A^{\mathbf{L}} B : D(A, d) \to D(B, d)$$

gives an R-linear equivalence of triangulated categories with quasi-inverse the restriction functor $N \mapsto N_A$.

Proof. By Lemma 24.6 the functor $M \longmapsto M \otimes_A^{\mathbf{L}} B$ is fully faithful. By Lemma 24.4 the functor $N \longmapsto R\,\mathrm{Hom}(B, N) = N_A$ is a right adjoint. It is clear that the kernel of $R\,\mathrm{Hom}(B, -)$ is zero. Hence the result follows from Derived Categories, Lemma 7.2. $\square$

When we analyze the proof above we see that we obtain the following generalization for free.

Lemma 27.2. *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be an (A, B)-bimodule which comes with a grading and a differential such that it is a differential graded module for both A and B. Assume that*

    (1) *N defines a compact object of $D(B, d)$,*
    (2) *if $N' \in D(B, d)$ and $\mathrm{Hom}_{D(B,d)}(N, N'[n]) = 0$ for $n \in \mathbf{Z}$, then $N' = 0$, and*
    (3) *the map $H^k(A) \to \mathrm{Hom}_{D(B,d)}(N, N[k])$ is an isomorphism for all $k \in \mathbf{Z}$.*

Then

$$- \otimes_A^{\mathbf{L}} N : D(A, d) \to D(B, d)$$

gives an R-linear equivalence of triangulated categories.

Proof. By Lemma 24.6 the functor $M \longmapsto M \otimes_A^{\mathbf{L}} N$ is fully faithful. By Lemma 24.4 the functor $N' \longmapsto R\operatorname{Hom}(N, N')$ is a right adjoint. By assumption (3) the kernel of $R\operatorname{Hom}(N, -)$ is zero. Hence the result follows from Derived Categories, Lemma 7.2. $\square$

Remark 27.3. In Lemma 27.2 we can replace condition (2) by the condition that N is a classical generator for $D_{compact}(B, d)$, see Derived Categories, Proposition 34.6. Moreover, if we knew that $R\operatorname{Hom}(N, B)$ is a compact object of $D(A, \mathrm{d})$, then it suffices to check that N is a weak generator for $D_{compact}(B, \mathrm{d})$. We omit the proof; we will add it here if we ever need it in the Stacks project.

Sometimes the B-module P in the lemma below is called an "(A, B)-tilting complex".

Lemma 27.4. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Assume that $A = H^0(A)$. The following are equivalent*

   (1) *$D(A, d)$ and $D(B, d)$ are equivalent as R-linear triangulated categories, and*
   (2) *there exists an object P of $D(B, d)$ such that*
      (a) *P is a compact object of $D(B, d)$,*
      (b) *if $N \in D(B, d)$ with $\operatorname{Hom}_{D(B,d)}(P, N[i]) = 0$ for $i \in \mathbf{Z}$, then $N = 0$,*
      (c) *$\operatorname{Hom}_{D(B,d)}(P, P[i]) = 0$ for $i \neq 0$ and equal to A for $i = 0$.*

Proof. Let $F : D(A, \mathrm{d}) \to D(B, \mathrm{d})$ be an equivalence. Then F maps compact objects to compact objects. Hence $P = F(A)$ is compact, i.e., (2)(a) holds. Conditions (2)(b) and (2)(c) are immediate from the fact that F is an equivalence.

Let P be an object as in (2). Represent P by a differential graded module with property (P). Set

$$(E, \mathrm{d}) = \operatorname{Hom}_{\operatorname{Mod}_B^{dg}}(P, P)$$

Then $H^0(E) = A$ and $H^k(E) = 0$ for $k \neq 0$ by Lemma 15.3 and assumption (2)(c). Viewing P as a (E, B)-bimodule and using Lemma 27.2 and assumption (2)(b) we obtain an equivalence

$$D(E, \mathrm{d}) \to D(B, \mathrm{d})$$

Let $E' \subset E$ be the differential graded R-subalgebra with

$$(E')^i = \begin{cases} E^i & \text{if } i < 0 \\ \operatorname{Ker}(E^0 \to E^1) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then there are quasi-isomorphisms of differential graded algebras $(A, \mathrm{d}) \leftarrow (E', \mathrm{d}) \to (E, \mathrm{d})$ Thus we obtain equivalences

$$D(A, \mathrm{d}) \leftarrow D(E', \mathrm{d}) \to D(E, \mathrm{d}) \to D(B, \mathrm{d})$$

by Lemma 27.1. $\square$

Remark 27.5. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Suppose given an R-linear equivalence

$$F : D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$$

of triangulated categories. Set $N = F(A)$. Then N is a differential graded B-module. Since F is an equivalence and A is a compact object of $D(A, \mathrm{d})$, we conclude that N is a compact object of $D(B, \mathrm{d})$. Moreover, since $H^k(A) = \mathrm{Hom}_{D(A,\mathrm{d})}(A, A[k])$ and F an equivalence we see that F induces an isomorphism $H^k(A) = \mathrm{Hom}_{D(B,\mathrm{d})}(N, N[k])$ for all k. In order to conclude that there is an equivalence $D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$ which arises from the construction in Lemma 27.2 all we need is a right A-module structure on N or on any differential graded B-module quasi-isomorphic to B. This module structure can be constructed in certain cases. For example, if we assume that F can be lifted to a differential graded functor

$$F^{dg} : \mathrm{Mod}^{dg}_{(A,\mathrm{d})} \longrightarrow \mathrm{Mod}^{dg}_{(B,\mathrm{d})}$$

(for notation see Example 19.8) between the associated differential graded categories, then this holds. Another case is discussed in the proposition below.

Proposition 27.6. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $F : D(A, d) \to D(B, d)$ be an R-linear equivalence of triangulated cateories. Assume that*

(1) *$A = H^0(A)$, and*
(2) *B is K-flat as a complex of R-modules.*

Then there exists an (A, B)-bimodule N as in Lemma 27.2.

Proof. As in Remark 27.5 above, we set $N = F(A)$ in $D(B, \mathrm{d})$. We may assume that N is a differential graded B-module with property (P). Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, N)$$

Then $H^0(E) = A$ and $H^k(E) = 0$ for $k \neq 0$ by Lemma 15.3. Moreover, by the discussion preceding the proposition and Lemma 27.2 we see that N as a (E, B)-bimodule induces an equivalence $- \otimes^{\mathbf{L}}_E N : D(E, \mathrm{d}) \to D(B, \mathrm{d})$. Let $E' \subset E$ be the differential graded R-subalgebra with

$$(E')^i = \begin{cases} E^i & \text{if } i < 0 \\ \mathrm{Ker}(E^0 \to E^1) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then there are quasi-isomorphisms of differential graded algebras $(A, \mathrm{d}) \leftarrow (E', \mathrm{d}) \to (E, \mathrm{d})$ Thus we obtain equivalences

$$D(A, \mathrm{d}) \leftarrow D(E', \mathrm{d}) \to D(E, \mathrm{d}) \to D(B, \mathrm{d})$$

by Lemma 27.1. Note that the quasi-inverse $D(A, \mathrm{d}) \to D(E', \mathrm{d})$ of the left vertical arrow is given by $M \mapsto M \otimes^{\mathbf{L}}_A A$ where A is viewed as a $A^{opp} \otimes_R E'$-module. On the other hand the functor $D(E', \mathrm{d}) \to D(B, \mathrm{d})$ is given by $M \mapsto M \otimes^{\mathbf{L}}_{E'} N$ where N is as above. We conclude by Lemma 24.5. $\square$

Remark 27.7. Let A, B, F, N be as in Proposition 27.6. It is not clear that F and the functor $G(-) = - \otimes^{\mathbf{L}}_A N$ are isomorphic. By construction there is an isomorphism $N = G(A) \to F(A)$ in $D(B, \mathrm{d})$. It is straigthforward to extend this to a functorial isomorphism $G(M) \to F(M)$ for M is a differential graded A-module which is graded projective (e.g., a sum of shifts of A). Then one can conclude that $G(M) \cong F(M)$ when M is a cone of a map between such modules. We don't know whether more is true in general.

Lemma 27.8. *Let R be a ring. Let A and B be R-algebras. The following are equivalent*

(1) *there is an R-linear equivalence $D(A) \to D(B)$ of triangulated categories,*
(2) *there exists an object P of $D(B)$ such that*
    (a) *P can be represented by a finite complex of finite projective B-modules,*
    (b) *if $K \in D(B)$ with $\mathrm{Ext}_B^i(P, K) = 0$ for $i \in \mathbf{Z}$, then $K = 0$, and*
    (c) *$\mathrm{Ext}_B^i(P, P) = 0$ for $i \neq 0$ and equal to A for $i = 0$.*

Moreover, if B is flat as an R-module, then this is also equivalent to

(3) *there exists an (A, B)-bimodule N such that $- \otimes_A^{\mathbf{L}} N : D(A) \to D(B)$ is an equivalence.*

Proof. The equivalence of (1) and (2) is a special case of Lemma 27.4 combined with the result of Lemma 26.6 characterizing compact objects of $D(B)$. The equivalence with (3) if B is R-flat follows from Proposition 27.6. $\qquad\square$

Remark 27.9. Let R be a ring. Let A and B be R-algebras. If $D(A)$ and $D(B)$ are equivalent as R-linear triangulated categories, then the centers of A and B are isomorphic as R-algebras. In particular, if A and B are commutative, then $A \cong B$. The rather tricky proof can be found in [Ric89, Proposition 9.2] or [KZ98, Proposition 6.3.2]. Another approach might be to use Hochschild cohomology (see remark below).

Remark 27.10. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras which are derived equivalent, i.e., such that there exists an R-linear equivalence $D(A, \mathrm{d}) \to D(B, \mathrm{d})$ of triangulated categories. We would like to show that certain invariants of (A, d) and (B, d) coincide. In many situations one has more control of the situation. For example, it may happen that there is an equivalence of the form

$$- \otimes_A \Omega : D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$$

for some differential graded $A^{opp} \otimes_R B$-module Ω (this happens in the situation of Proposition 27.6 and is often true if the equivalence comes from a geometric construction). If also the quasi-inverse of our functor is given as

$$- \otimes_A^{\mathbf{L}} \Omega' : D(B, \mathrm{d}) \longrightarrow D(A, \mathrm{d})$$

for a differential graded $B^{opp} \otimes_R A$-module Ω' (and as before such a module Ω' often exists in practice) then we can consider the functor

$$D(A^{opp} \otimes_R A, \mathrm{d}) \longrightarrow D(B^{opp} \otimes_R B, \mathrm{d}), \quad M \longmapsto \Omega' \otimes_A^{\mathbf{L}} M \otimes_A^{\mathbf{L}} \Omega$$

Observe that this functor sends the (A, A)-bimodule A to the (B, B)-bimodule B. Under suitable conditions (e.g., flatness of A, B, Ω, etc) this functor will be an equivalence as well. If this is the case, then it follows that we have isomorphisms of Hochschild cohomology groups

$$HH^i(A, \mathrm{d}) = \mathrm{Hom}_{D(A^{opp} \otimes_R A, \mathrm{d})}(A, A[i]) \longrightarrow \mathrm{Hom}_{D(B^{opp} \otimes_R B, \mathrm{d})}(B, B[i]) = HH^i(B, \mathrm{d}).$$

For example, if $A = H^0(A)$, then $HH^0(A, \mathrm{d})$ is equal to the center of A, and this gives a conceptual proof of the result mentioned in Remark 27.9. If we ever need this remark we will provide a precise statement with a detailed proof here.

28. Other chapters

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