

# FORMAL ALGEBRAIC SPACES

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## 1. Introduction

Formal schemes were introduced in [DG67]. A more general version of formal schemes was introduced in [McQ02] and another in [Yas09]. Formal algebraic spaces were introduced in [Knu71]. Related material and much besides can be found in [Abb10] and [FK]. This chapter introduces the notion of formal algebraic spaces we will work with. Our definition is general enough to allow most classes of formal schemes/spaces in the literature as full subcategories.

Although we do discuss the comparison of some of these alternative theories with ours, we do not always give full details when it is not necessary for the logical development of the theory.

Besides introducing formal algebraic spaces, we also prove a few very basic properties and we discuss a few types of morphisms.

## 2. Formal schemes à la EGA

In this section we review the construction of formal schemes in [DG67]. This notion, although very useful in algebraic geometry, may not always be the correct one to consider. Perhaps it is better to say that in the setup of the theory a number of choices are made, where for different purposes others might work better. And indeed in the literature one can find many different closely related theories adapted to the problem the authors may want to consider. Still, one of the major advantages of the theory as sketched here is that one gets to work with definite geometric objects.

Before we start we should point out an issue with the sheaf condition for sheaves of topological rings or more generally sheaves of topological spaces. Namely, the big categories

- (1) category of topological spaces,
- (2) category of topological abelian groups,
- (3) category of topological rings,
- (4) category of topological modules over a given topological ring,

endowed with their natural forgetful functors to *Sets* are not examples of types of algebraic structures as defined in Sheaves, Section 15. Thus we cannot blithely apply to them the machinery developed in that chapter. On the other hand, each of the categories listed above has products, so we can define the notion of a sheaf as in Sheaves, Definition 9.1. Moreover, products still commute with the forgetful functor to *Sets* so the underlying presheaf of sets of a sheaf of topological spaces (for example) is a sheaf of sets. The key difference is that for an open covering  $U = \bigcup_{i \in I} U_i$  the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

has to be an equalizer diagram in the category of topological spaces, groups, rings, modules, i.e., that the first map identifies  $\mathcal{F}(U)$  with a subspace of  $\prod_{i \in I} \mathcal{F}(U_i)$  which is endowed with the product topology.

The stalk  $\mathcal{F}_x$  of a sheaf  $\mathcal{F}$  of topological spaces, abelian groups, rings, or modules at a point  $x \in X$  is defined as the colimit over open neighbourhoods

$$\mathcal{F}_x = \operatorname{colim}_{x \in U} \mathcal{F}(U)$$

in the corresponding category, which is the same as taking the colimit on the level of underlying sheaves of sets, groups, rings, or modules but comes equipped with a topology (the colimit topology). One can sheafify presheaves of topological spaces, abelian groups, rings, or modules and taking stalks commutes with this operation, see Remark 2.4.

Let  $f : X \rightarrow Y$  be a continuous map of topological spaces. There is a functor  $f_*$  from the category of sheaves of topological spaces, abelian groups, rings, modules, to the corresponding category of sheaves on  $Y$  which is defined by setting  $f_*\mathcal{F}(V) =$

$\mathcal{F}(f^{-1}V)$  as usual. (We delay discussing the pullback in this setting till later.) We define the notion of an  $f$ -map  $\xi : \mathcal{G} \rightarrow \mathcal{F}$  between a sheaf of topological spaces  $\mathcal{G}$  on  $Y$  and a sheaf of topological spaces  $\mathcal{F}$  on  $X$  in exactly the same manner as in Sheaves, Definition 21.7 with the additional constraint that  $\xi_V : \mathcal{G}(V) \rightarrow \mathcal{F}(f^{-1}V)$  be continuous for every open  $V \subset Y$ . We have

$$\{f\text{-maps from } \mathcal{G} \text{ to } \mathcal{F}\} = \text{Mor}_{Sh(Y, Top)}(\mathcal{G}, f_*\mathcal{F})$$

as in Sheaves, Lemma 21.8. Similarly for sheaves of topological abelian groups, rings, modules. Finally, let  $\xi : \mathcal{G} \rightarrow \mathcal{F}$  be an  $f$ -map as above. Then given  $x \in X$  with image  $y = f(x)$  there is a continuous map

$$\xi_x : \mathcal{G}_y \longrightarrow \mathcal{F}_x$$

of stalks defined in exactly the same manner as in the discussion following Sheaves, Definition 21.9.

Using the discussion above, we can define a category  $LTRS$  of “locally topologically ringed spaces”. An object is a pair  $(X, \mathcal{O}_X)$  consisting of a topological space  $X$  and a sheaf of topological rings  $\mathcal{O}_X$  whose stalks  $\mathcal{O}_{X,x}$  are local rings (if one forgets about the topology). A morphism  $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  of  $LTRS$  is a pair  $(f, f^\#)$  where  $f : X \rightarrow Y$  is a continuous map of topological spaces and  $f^\# : \mathcal{O}_Y \rightarrow \mathcal{O}_X$  is an  $f$ -map such that for every  $x \in X$  the induced map

$$f_x^\# : \mathcal{O}_{Y, f(x)} \longrightarrow \mathcal{O}_{X, x}$$

is a local homomorphism of local rings (forgetting about the topologies). The composition works in exactly the same manner as composition of morphisms of locally ringed spaces.

Assume now that the topological space  $X$  has a basis consisting of quasi-compact opens. Given a sheaf  $\mathcal{G}$  of sets, abelian groups, rings, modules over a ring, one can endow  $\mathcal{G}$  with the structure of a sheaf of topological spaces, abelian groups, rings, modules. Namely, if  $U \subset X$  is quasi-compact open, we endow  $\mathcal{F}(U)$  with the discrete topology. If  $U \subset X$  is arbitrary, then we choose an open covering  $U = \bigcup_{i \in I} U_i$  by quasi-compact opens and we endow  $\mathcal{F}(U)$  with the induced topology from  $\prod_{i \in I} \mathcal{F}(U_i)$  (as we should do according to our discussion above). The reader may verify (omitted) that we obtain a sheaf of topological spaces, groups, rings, modules in this fashion. Let us say that a sheaf of topological spaces, groups, rings, modules is *pseudo-discrete* if the topology on  $\mathcal{F}(U)$  is discrete for every quasi-compact open  $U \subset X$ . Then the construction given above is an adjoint to the forgetful functor and induces an equivalence between the category of sheaves of sets and the category of pseudo-discrete sheaves of topological space (similarly for groups, rings, modules).

Grothendieck and Dieudonné first define formal affine schemes. These correspond to admissible topological rings  $A$ , see More on Algebra, Definition 26.1. Namely, given  $A$  one considers a fundamental system  $I_\lambda$  of ideals of definition for the ring  $A$ . (In any admissible topological ring the family of all ideals of definition forms a fundamental system.) For each  $\lambda$  we can consider the scheme  $\text{Spec}(A/I_\lambda)$ . For  $I_\lambda \subset I_\mu$  the induced morphism

$$\text{Spec}(A/I_\mu) \rightarrow \text{Spec}(A/I_\lambda)$$

is a thickening because  $I_\mu^n \subset I_\lambda$  for some  $n$ . Another way to see this, is to notice that the image of each of the maps

$$\mathrm{Spec}(A/I_\lambda) \rightarrow \mathrm{Spec}(A)$$

is a homeomorphism onto the set of open prime ideals of  $A$ . This motivates the definition

$$\mathrm{Spf}(A) = \{\text{open prime ideals } \mathfrak{p} \subset A\}$$

endowed with the topology coming from  $\mathrm{Spec}(A)$ . For each  $\lambda$  we can consider the structure sheaf  $\mathcal{O}_{\mathrm{Spec}(A/I_\lambda)}$  as a sheaf on  $\mathrm{Spf}(A)$ . Let  $\mathcal{O}_\lambda$  be the corresponding pseudo-discrete sheaf of topological rings, see above. Then we set

$$\mathcal{O}_{\mathrm{Spf}(A)} = \lim \mathcal{O}_\lambda$$

where the limit is taken in the category of sheaves of topological rings. The pair  $(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})$  is called the *formal spectrum* of  $A$ .

At this point one should check several things. The first is that the stalks  $\mathcal{O}_{\mathrm{Spf}(A),x}$  are local rings (forgetting about the topology). The second is that given  $f \in A$ , for the corresponding open  $D(f) \cap \mathrm{Spf}(A)$  we have

$$\Gamma(D(f) \cap \mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}) = A_{\{f\}} = \lim(A/I_\lambda)_f$$

as topological rings where  $I_\lambda$  is a fundamental system of ideals of definition as above. Moreover, the ring  $A_{\{f\}}$  is admissible too and  $(\mathrm{Spf}(A_f), \mathcal{O}_{\mathrm{Spf}(A_{\{f\}})})$  is isomorphic to  $(D(f) \cap \mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}|_{D(f) \cap \mathrm{Spf}(A)})$ . Finally, given a pair of admissible topological rings  $A, B$  we have

$$(2.0.1) \quad \mathrm{Mor}_{LTRS}((\mathrm{Spf}(B), \mathcal{O}_{\mathrm{Spf}(B)}), (\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})) = \mathrm{Hom}_{\mathrm{cont}}(A, B)$$

where  $LTRS$  is the category of “locally topologically ringed spaces” as defined above.

Having said this, in [DG67] a *formal scheme* is defined as a pair  $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$  where  $\mathfrak{X}$  is a topological space and  $\mathcal{O}_\mathfrak{X}$  is a sheaf of topological rings such that every point has an open neighbourhood isomorphic (in  $LTRS$ ) to an affine formal scheme. A *morphism of formal schemes*  $f : (\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \rightarrow (\mathfrak{Y}, \mathcal{O}_\mathfrak{Y})$  is a morphism in the category  $LTRS$ .

Let  $A$  be a ring endowed with the discrete topology. Then  $A$  is admissible and the formal scheme  $\mathrm{Spf}(A)$  is equal to  $\mathrm{Spec}(A)$ . The structure sheaf  $\mathcal{O}_{\mathrm{Spf}(A)}$  is the pseudo-discrete sheaf of topological rings associated to  $\mathcal{O}_{\mathrm{Spec}(A)}$ , in other words, its underlying sheaf of rings is equal to  $\mathcal{O}_{\mathrm{Spec}(A)}$  and the ring  $\mathcal{O}_{\mathrm{Spf}(A)}(U) = \mathcal{O}_{\mathrm{Spec}(A)}(U)$  over a quasi-compact open  $U$  has the discrete topology, but not in general. Thus we can associate to every affine scheme a formal affine scheme. In exactly the same manner we can start with a general scheme  $(X, \mathcal{O}_X)$  and associate to it  $(X, \mathcal{O}'_X)$  where  $\mathcal{O}'_X$  is the pseudo-discrete sheaf of topological rings whose underlying sheaf of rings is  $\mathcal{O}_X$ . This construction is compatible with morphisms and defines a functor

$$(2.0.2) \quad \text{Schemes} \longrightarrow \text{Formal Schemes}$$

It follows in a straightforward manner from (2.0.1) that this functor is fully faithful.

Let  $\mathfrak{X}$  be a formal scheme. Let us define the *size* of the formal scheme by the formula  $\mathrm{size}(\mathfrak{X}) = \max(\aleph_0, \kappa_1, \kappa_2)$  where  $\kappa_1$  is the cardinality of the formal affine opens of  $\mathfrak{X}$  and  $\kappa_2$  is the supremum of the cardinalities of  $\mathcal{O}_\mathfrak{X}(\mathfrak{U})$  where  $\mathfrak{U} \subset \mathfrak{X}$  is such a formal affine open.

**Lemma 2.1.** *Choose a category of schemes  $Sch_\alpha$  as in Sets, Lemma 9.2. Given a formal scheme  $\mathfrak{X}$  let*

$$h_{\mathfrak{X}} : (Sch_\alpha)^{opp} \longrightarrow Sets, \quad h_{\mathfrak{X}}(S) = \text{Mor}_{\text{Formal Schemes}}(S, \mathfrak{X})$$

*be its functor of points. Then we have*

$$\text{Mor}_{\text{Formal Schemes}}(\mathfrak{X}, \mathfrak{Y}) = \text{Mor}_{PSh(Sch_\alpha)}(h_{\mathfrak{X}}, h_{\mathfrak{Y}})$$

*provided the size of  $\mathfrak{X}$  is not too large.*

**Proof.** First we observe that  $h_{\mathfrak{X}}$  satisfies the sheaf property for the Zariski topology for any formal scheme  $\mathfrak{X}$  (see Schemes, Definition 15.3). This follows from the local nature of morphisms in the category of formal schemes. Also, for an open immersion  $\mathfrak{V} \rightarrow \mathfrak{W}$  of formal schemes, the corresponding transformation of functors  $h_{\mathfrak{V}} \rightarrow h_{\mathfrak{W}}$  is injective and representable by open immersions (see Schemes, Definition 15.3). Choose an open covering  $\mathfrak{X} = \bigcup \mathfrak{U}_i$  of a formal scheme by affine formal schemes  $\mathfrak{U}_i$ . Then the collection of functors  $h_{\mathfrak{U}_i}$  covers  $h_{\mathfrak{X}}$  (see Schemes, Definition 15.3). Finally, note that

$$h_{\mathfrak{U}_i} \times_{h_{\mathfrak{X}}} h_{\mathfrak{U}_j} = h_{\mathfrak{U}_i \cap \mathfrak{U}_j}$$

Hence in order to give a map  $h_{\mathfrak{X}} \rightarrow h_{\mathfrak{Y}}$  is equivalent to giving a family of maps  $h_{\mathfrak{U}_i} \rightarrow h_{\mathfrak{Y}}$  which agree on overlaps. Thus we can reduce the bijectivity (resp. injectivity) of the map of the lemma to bijectivity (resp. injectivity) for the pairs  $(\mathfrak{U}_i, \mathfrak{Y})$  and injectivity (resp. nothing) for  $(\mathfrak{U}_i \cap \mathfrak{U}_j, \mathfrak{Y})$ . In this way we reduce to the case where  $\mathfrak{X}$  is an affine formal scheme. Say  $\mathfrak{X} = \text{Spf}(A)$  for some admissible topological ring  $A$ . Also, choose a fundamental system of ideals of definition  $I_\lambda \subset A$ .

We can also localize on  $\mathfrak{Y}$ . Namely, suppose that  $\mathfrak{V} \subset \mathfrak{Y}$  is an open formal subscheme and  $\varphi : h_{\mathfrak{X}} \rightarrow h_{\mathfrak{Y}}$ . Then

$$h_{\mathfrak{V}} \times_{h_{\mathfrak{Y}}, \varphi} h_{\mathfrak{X}} \rightarrow h_{\mathfrak{X}}$$

is representable by open immersions. Pulling back to  $\text{Spec}(A/I_\lambda)$  for all  $\lambda$  we find an open subscheme  $U_\lambda \subset \text{Spec}(A/I_\lambda)$ . However, for  $I_\lambda \subset I_\mu$  the morphism  $\text{Spec}(A/I_\lambda) \rightarrow \text{Spec}(A/I_\mu)$  pulls back  $U_\mu$  to  $U_\lambda$ . Thus these glue to give an open formal subscheme  $\mathfrak{U} \subset \mathfrak{X}$ . A straightforward argument (omitted) shows that

$$h_{\mathfrak{U}} = h_{\mathfrak{V}} \times_{h_{\mathfrak{Y}}, \varphi} h_{\mathfrak{X}}$$

In this way we see that given an open covering  $\mathfrak{Y} = \bigcup \mathfrak{V}_j$  and a transformation of functors  $\varphi : h_{\mathfrak{X}} \rightarrow h_{\mathfrak{Y}}$  we obtain a corresponding open covering of  $\mathfrak{X}$ . Since  $\mathfrak{X}$  is affine, we can refine this covering by a finite open covering  $\mathfrak{X} = \mathfrak{U}_1 \cup \dots \cup \mathfrak{U}_n$  by affine formal subschemes. In other words, for each  $i$  there is a  $j$  and a map  $\varphi_i : h_{\mathfrak{U}_i} \rightarrow h_{\mathfrak{V}_j}$  such that

$$\begin{array}{ccc} h_{\mathfrak{U}_i} & \xrightarrow{\varphi_i} & h_{\mathfrak{V}_j} \\ \downarrow & & \downarrow \\ h_{\mathfrak{X}} & \xrightarrow{\varphi} & h_{\mathfrak{Y}} \end{array}$$

commutes. With a few additional arguments (which we omit) this implies that it suffices to prove the bijectivity of the lemma in case both  $\mathfrak{X}$  and  $\mathfrak{Y}$  are affine formal schemes.

Assume  $\mathfrak{X}$  and  $\mathfrak{Y}$  are affine formal schemes. Say  $\mathfrak{X} = \text{Spf}(A)$  and  $\mathfrak{Y} = \text{Spf}(B)$ . Let  $\varphi : h_{\mathfrak{X}} \rightarrow h_{\mathfrak{Y}}$  be a transformation of functors. Let  $I_\lambda \subset A$  be a fundamental system

of ideals of definition. The canonical inclusion morphism  $i_\lambda : \operatorname{Spec}(A/I_\lambda) \rightarrow \mathfrak{X}$  maps to a morphism  $\varphi(i_\lambda) : \operatorname{Spec}(A/I_\lambda) \rightarrow \mathfrak{Y}$ . By (2.0.1) this corresponds to a continuous map  $\chi_\lambda : B \rightarrow A/I_\lambda$ . Since  $\varphi$  is a transformation of functors it follows that for  $I_\lambda \subset I_\mu$  the composition  $B \rightarrow A/I_\lambda \rightarrow A/I_\mu$  is equal to  $\chi_\mu$ . In other words we obtain a ring map

$$\chi = \lim \chi_\lambda : B \longrightarrow \lim A/I_\lambda = A$$

This is a continuous homomorphism because the inverse image of  $I_\lambda$  is open for all  $\lambda$  (as  $A/I_\lambda$  has the discrete topology and  $\chi_\lambda$  is continuous). Thus we obtain a morphism  $\operatorname{Spf}(\chi) : \mathfrak{X} \rightarrow \mathfrak{Y}$  by (2.0.1). We omit the verification that this construction is the inverse to the map of the lemma in this case.

Set theoretic remarks. To make this work on the given category of schemes  $Sch_\alpha$  we just have to make sure all the schemes used in the proof above are isomorphic to objects of  $Sch_\alpha$ . In fact, a careful analysis shows that it suffices if the schemes  $\operatorname{Spec}(A/I_\lambda)$  occurring above are isomorphic to objects of  $Sch_\alpha$ . For this it certainly suffices to assume the size of  $\mathfrak{X}$  is at most the size of a scheme contained in  $Sch_\alpha$ .  $\square$

**Lemma 2.2.** *Let  $\mathfrak{X}$  be a formal scheme. The functor of points  $h_{\mathfrak{X}}$  (see Lemma 2.1) satisfies the sheaf condition for fpqc coverings.*

**Proof.** Topologies, Lemma 8.13 reduces us to the case of a Zariski covering and a covering  $\{\operatorname{Spec}(S) \rightarrow \operatorname{Spec}(R)\}$  with  $R \rightarrow S$  faithfully flat. We observed in the proof of Lemma 2.1 that  $h_{\mathfrak{X}}$  satisfies the sheaf condition for Zariski coverings.

Suppose that  $R \rightarrow S$  is a faithfully flat ring map. Denote  $\pi : \operatorname{Spec}(S) \rightarrow \operatorname{Spec}(R)$  the corresponding morphism of schemes. It is surjective and flat. Let  $f : \operatorname{Spec}(S) \rightarrow \mathfrak{X}$  be a morphism such that  $f \circ \operatorname{pr}_1 = f \circ \operatorname{pr}_2$  as maps  $\operatorname{Spec}(S \otimes_R S) \rightarrow \mathfrak{X}$ . By Descent, Lemma 9.1 we see that as a map on the underlying sets  $f$  is of the form  $f = g \circ \pi$  for some (set theoretic) map  $g : \operatorname{Spec}(R) \rightarrow \mathfrak{X}$ . By Morphisms, Lemma 26.10 and the fact that  $f$  is continuous we see that  $g$  is continuous.

Pick  $y \in \operatorname{Spec}(R)$ . Choose  $\mathfrak{U} \subset \mathfrak{X}$  an affine formal open subscheme containing  $g(y)$ . Say  $\mathfrak{U} = \operatorname{Spf}(A)$  for some admissible topological ring  $A$ . By the above we may choose an  $r \in R$  such that  $x \in D(r) \subset g^{-1}(\mathfrak{U})$ . The restriction of  $f$  to  $\pi^{-1}(D(r))$  into  $\mathfrak{U}$  corresponds to a continuous ring map  $A \rightarrow S_r$  by (2.0.1). The two induced ring maps  $A \rightarrow S_r \otimes_{R_r} S_r = (S \otimes_R S)_r$  are equal by assumption on  $f$ . Note that  $R_r \rightarrow S_r$  is faithfully flat. By Descent, Lemma 3.6 the equalizer of the two arrows  $S_r \rightarrow S_r \otimes_{R_r} S_r$  is  $R_r$ . We conclude that  $A \rightarrow S_r$  factors uniquely through a map  $A \rightarrow R_r$  which is also continuous as it has the same (open) kernel as the map  $A \rightarrow S_r$ . This map in turn gives a morphism  $D(r) \rightarrow \mathfrak{U}$  by (2.0.1).

What have we proved so far? We have shown that for any  $y \in \operatorname{Spec}(R)$  there exists a standard affine open  $y \in D(r) \subset \operatorname{Spec}(R)$  such that the morphism  $f|_{\pi^{-1}(D(r))} : \pi^{-1}(D(r)) \rightarrow \mathfrak{X}$  factors uniquely through some morphism  $D(r) \rightarrow \mathfrak{X}$ . We omit the verification that these morphisms glue to the desired morphism  $\operatorname{Spec}(R) \rightarrow \mathfrak{X}$ .  $\square$

**Remark 2.3** (McQuillan's variant). There is a variant of the construction of formal schemes due to McQuillan, see [McQ02]. He suggests a slight weakening of the condition of admissibility. Namely, recall that an admissible topological ring is a complete (and separated by our conventions) topological ring  $A$  which is linearly topologized such that there exists an ideal of definition: an open ideal  $I$  such that any neighbourhood of 0 contains  $I^n$ . McQuillan works with what we will call *weakly*

*admissible* topological rings. A weakly admissible topological ring  $A$  is a complete (and separated by our conventions) topological ring which is linearly topologized such that there exists an *weak ideal of definition*: an open ideal  $I$  such that for all  $f \in I$  we have  $f^n \rightarrow 0$  for  $n \rightarrow \infty$ . Similarly to the admissible case, if  $I$  is a weak ideal of definition and  $J \subset A$  is an open ideal, then  $I \cap J$  is a weak ideal of definition. Thus the weak ideals of definition form a fundamental system of open neighbourhoods of 0 and one can proceed along much the same route as above to define a larger category of formal schemes based on this notion. The analogues of Lemmas 2.1 and 2.2 still hold in this setting (with the same proof).

**Remark 2.4** (Sheafification of presheaves of topological spaces). In this remark we briefly discuss sheafification of presheaves of topological spaces. The exact same arguments work for presheaves of topological abelian groups, topological rings, and topological modules (over a given topological ring). In order to do this in the correct generality let us work over a site  $\mathcal{C}$ . The reader who is interested in the case of (pre)sheaves over a topological space  $X$  should think of objects of  $\mathcal{C}$  as the opens of  $X$ , of morphisms of  $\mathcal{C}$  as inclusions of opens, and of coverings in  $\mathcal{C}$  as coverings in  $X$ , see Sites, Example 6.4. Denote  $Sh(\mathcal{C}, Top)$  the category of sheaves of topological spaces on  $\mathcal{C}$  and denote  $PSh(\mathcal{C}, Top)$  the category of presheaves of topological spaces on  $\mathcal{C}$ . Let  $\mathcal{F}$  be a presheaf of topological spaces on  $\mathcal{C}$ . The sheafification  $\mathcal{F}^\#$  should satisfy the formula

$$\text{Mor}_{PSh(\mathcal{C}, Top)}(\mathcal{F}, \mathcal{G}) = \text{Mor}_{Sh(\mathcal{C}, Top)}(\mathcal{F}^\#, \mathcal{G})$$

functorially in  $\mathcal{G}$  from  $Sh(\mathcal{C}, Top)$ . In other words, we are trying to construct the left adjoint to the inclusion functor  $Sh(\mathcal{C}, Top) \rightarrow PSh(\mathcal{C}, Top)$ . We first claim that  $Sh(\mathcal{C}, Top)$  has limits and that the inclusion functor commutes with them. Namely, given a category  $\mathcal{I}$  and a functor  $i \mapsto \mathcal{G}_i$  into  $Sh(\mathcal{C}, Top)$  we simply define

$$(\lim \mathcal{G}_i)(U) = \lim \mathcal{G}_i(U)$$

where we take the limit in the category of topological spaces (Topology, Lemma 13.1). This defines a sheaf because limits commute with limits (Categories, Lemma 14.9) and in particular products and equalizers (which are the operations used in the sheaf axiom). Finally, a morphism of presheaves from  $\mathcal{F} \rightarrow \lim \mathcal{G}_i$  is clearly the same thing as a compatible system of morphisms  $\mathcal{F} \rightarrow \mathcal{G}_i$ . In other words, the object  $\lim \mathcal{G}_i$  is the limit in the category of presheaves of topological spaces and a fortiori in the category of sheaves of topological spaces. Our second claim is that any morphism of presheaves  $\mathcal{F} \rightarrow \mathcal{G}$  with  $\mathcal{G}$  an object of  $Sh(\mathcal{C}, Top)$  factors through a subsheaf  $\mathcal{G}' \subset \mathcal{G}$  whose size is bounded. Here we define the *size*  $|\mathcal{H}|$  of a sheaf of topological spaces  $\mathcal{H}$  to be the cardinal  $\sup_{U \in \text{Ob}(\mathcal{C})} |\mathcal{H}(U)|$ . To prove our claim we let

$$\mathcal{G}'(U) = \left\{ s \in \mathcal{G}(U) \mid \begin{array}{l} \text{there exists a covering } \{U_i \rightarrow U\}_{i \in I} \\ \text{such that } s|_{U_i} \in \text{Im}(\mathcal{F}(U_i) \rightarrow \mathcal{G}(U_i)) \end{array} \right\}$$

We endow  $\mathcal{G}'(U)$  with the induced topology. Then  $\mathcal{G}'$  is a sheaf of topological spaces (details omitted) and  $\mathcal{G}' \rightarrow \mathcal{G}$  is a morphism through which the given map  $\mathcal{F} \rightarrow \mathcal{G}$  factors. Moreover, the size of  $\mathcal{G}'$  is bounded by some cardinal  $\kappa$  depending only on  $\mathcal{C}$  and the presheaf  $\mathcal{F}$  (hint: use that coverings in  $\mathcal{C}$  form a set by our conventions). Putting everything together we see that the assumptions of Categories, Theorem 40.3 are satisfied and we obtain sheafification as the left adjoint of the inclusion functor from sheaves to presheaves. Finally, let  $p$  be a point of the site  $\mathcal{C}$  given by

a functor  $u : \mathcal{C} \rightarrow \mathbf{Sets}$ , see Sites, Definition 31.2. For a topological space  $M$  the presheaf defined by the rule

$$U \mapsto \mathrm{Map}(u(U), M) = \prod_{x \in u(U)} M$$

endowed with the product topology is a sheaf of topological spaces. Hence the exact same argument as given in the proof of Sites, Lemma 31.5 shows that  $\mathcal{F}_p = \mathcal{F}_p^\#$ , in other words, sheafification commutes with taking stalks at a point.

### 3. Conventions and notation

The conventions from now on will be similar to the conventions in Properties of Spaces, Section 2. Thus from now on the standing assumption is that all schemes are contained in a big fppf site  $Sch_{fppf}$ . And all rings  $A$  considered have the property that  $\mathrm{Spec}(A)$  is (isomorphic) to an object of this big site. For topological rings  $A$  we assume only that all discrete quotients have this property (but usually we assume more, compare with Remark 7.6).

Let  $S$  be a scheme and let  $X$  be a “space” over  $S$ , i.e., a sheaf on  $(Sch/S)_{fppf}$ . In this chapter we will write  $X \times_S X$  for the product of  $X$  with itself in the category of sheaves on  $(Sch/S)_{fppf}$  instead of  $X \times X$ . Moreover, if  $X$  and  $Y$  are “spaces” then we say “let  $f : X \rightarrow Y$  be a morphism” to indicate that  $f$  is a natural transformation of functors, i.e., a map of sheaves on  $(Sch/S)_{fppf}$ . Similarly, if  $U$  is a scheme over  $S$  and  $X$  is a “space” over  $S$ , then we say “let  $f : U \rightarrow X$  be a morphism” or “let  $g : X \rightarrow U$  be a morphism” to indicate that  $f$  or  $g$  is a map of sheaves  $h_U \rightarrow X$  or  $X \rightarrow h_U$  where  $h_U$  is as in Categories, Example 3.4.

### 4. Topological rings and modules

This section is a continuation of More on Algebra, Section 26. Let  $R$  be a topological ring and let  $M$  be a linearly topologized  $R$ -module. When we say “let  $M_\lambda$  be a fundamental system of open submodules” we will mean that each  $M_\lambda$  is an open submodule and that any neighbourhood of 0 contains one of the  $M_\lambda$ . In other words, this means that  $M_\lambda$  is a fundamental system of neighbourhoods of 0 in  $M$  consisting of submodules. Similarly, if  $R$  is a linearly topologized ring, then we say “let  $I_\lambda$  be a fundamental system of open ideals” to mean that  $I_\lambda$  is a fundamental system of neighbourhoods of 0 in  $R$  consisting of ideals.

**Example 4.1.** Let  $R$  be a linearly topologized ring and let  $M$  be a linearly topologized  $A$ -module. Let  $I_\lambda$  be a fundamental system of open ideals in  $R$  and let  $M_\mu$  be a fundamental system of open submodules of  $M$ . The continuity of  $+$  :  $M \times M \rightarrow M$  is automatic and the continuity of  $R \times M \rightarrow M$  signifies

$$\forall f, x, \mu \exists \lambda, \nu, (f + I_\lambda)(x + M_\nu) \subset fx + M_\mu$$

Since  $fM_\nu + I_\lambda M_\nu \subset M_\mu$  if  $M_\nu \subset M_\mu$  we see that the condition is equivalent to

$$\forall x, \mu \exists \lambda I_\lambda x \subset M_\mu$$

However, it need not be the case that given  $\mu$  there is a  $\lambda$  such that  $I_\lambda M \subset M_\mu$ . For example, consider  $R = k[[t]]$  with the  $t$ -adic topology and  $M = \bigoplus_{n \in \mathbf{N}} R$  with fundamental system of open submodules given by

$$M_m = \bigoplus_{n \in \mathbf{N}} t^{nm} R$$



Since every  $x \in M$  has finitely many nonzero coordinates we see that, given  $m$  and  $x$  there exists a  $k$  such that  $t^k x \in M_m$ . Thus  $M$  is a linearly topologized  $R$ -module, but it isn't true that given  $m$  there is a  $k$  such that  $t^k M \subset M_m$ . On the other hand, if  $R \rightarrow S$  is a continuous map of linearly topologized rings, then the corresponding statement does hold, i.e., for every open ideal  $J \subset S$  there exists an open ideal  $I \subset R$  such that  $IS \subset J$  (as the reader can easily deduce from continuity of the map  $R \rightarrow S$ ).

**Lemma 4.2.** *Let  $R$  be a topological ring. Let  $M$  be a linearly topologized  $R$ -module and let  $M_\lambda$ ,  $\lambda \in \Lambda$  be a fundamental system of open submodules. Let  $N \subset M$  be a submodule. The closure of  $N$  is  $\bigcap_{\lambda \in \Lambda} (N + M_\lambda)$ .*

**Proof.** Since each  $N + M_\lambda$  is open, it is also closed. Hence the intersection is closed. If  $x \in M$  is not in the closure of  $N$ , then  $(x + M_\lambda) \cap N = 0$  for some  $\lambda$ . Hence  $x \notin N + M_\lambda$ . This proves the lemma.  $\square$

Unless otherwise mentioned we endow submodules and quotient modules with the induced topology. Let  $M$  be a linearly topologized module over a topological ring  $R$ , and let  $0 \rightarrow N \rightarrow M \rightarrow Q \rightarrow 0$  is a short exact sequence of  $R$ -modules. If  $M_\lambda$  is a fundamental system of open submodules of  $M$ , then  $N \cap M_\lambda$  is a fundamental system of open submodules of  $N$ . If  $\pi : M \rightarrow Q$  is the quotient map, then  $\pi(M_\lambda)$  is a fundamental system of open submodules of  $Q$ . In particular these induced topologies are linear topologies.

**Lemma 4.3.** *Let  $R$  be a topological ring. Let  $M$  be a linearly topologized  $R$ -module. Let  $N \subset M$  be a submodule. Then*

- (1)  $0 \rightarrow N^\wedge \rightarrow M^\wedge \rightarrow (M/N)^\wedge$  is exact, and
- (2)  $N^\wedge$  is the closure of the image of  $N \rightarrow M^\wedge$ .

**Proof.** Let  $M_\lambda$ ,  $\lambda \in \Lambda$  be a fundamental system of open submodules. Then  $N \cap M_\lambda$  is a fundamental system of open submodules of  $N$  and  $M_\lambda + N/N$  is a fundamental system of open submodules of  $M/N$ . Thus we see that (1) follows from the exactness of the sequences

$$0 \rightarrow N/N \cap M_\lambda \rightarrow M/M_\lambda \rightarrow M/(M_\lambda + N) \rightarrow 0$$

and the fact that taking limits commutes with limits. The second statement follows from this and the fact that  $N \rightarrow N^\wedge$  has dense image and that the kernel of  $M^\wedge \rightarrow (M/N)^\wedge$  is closed.  $\square$

**Lemma 4.4.** *Let  $R$  be a topological ring. Let  $M$  be a complete, linearly topologized  $R$ -module. Let  $N \subset M$  be a closed submodule. If  $M$  has a countable fundamental system of neighbourhoods of 0, then  $M/N$  is complete and the map  $M \rightarrow M/N$  is open.*

**Proof.** Let  $M_n$ ,  $n \in \mathbf{N}$  be a fundamental system of open submodules of  $M$ . We may assume  $M_{n+1} \subset M_n$  for all  $n$ . The  $(M_n + N)/N$  is a fundamental system in  $M/N$ . Hence we have to show that  $M/N = \lim M/(M_n + N)$ . Consider the short exact sequences

$$0 \rightarrow N/N \cap M_n \rightarrow M/M_n \rightarrow M/(M_n + N) \rightarrow 0$$

Since the transition maps of the system  $\{N/N \cap M_n\}$  are surjective we see that  $M = \lim M/M_n$  (by completeness of  $M$ ) surjects onto  $\lim M/(M_n + N)$  by Algebra, Lemma 83.4. As  $N$  is closed we see that the kernel of  $M \rightarrow \lim M/(M_n + N)$  is

$N$  (see Lemma 4.2). Finally,  $M \rightarrow M/N$  is open by definition of the quotient topology.  $\square$

**Lemma 4.5.** *Let  $R$  be a topological ring. Let  $M$  be a linearly topologized  $R$ -module. Let  $N \subset M$  be a submodule. Assume  $M$  has a countable fundamental system of neighbourhoods of 0. Then*

- (1)  $0 \rightarrow N^\wedge \rightarrow M^\wedge \rightarrow (M/N)^\wedge \rightarrow 0$  is exact,
- (2)  $N^\wedge$  is the closure of the image of  $N \rightarrow M^\wedge$ ,
- (3)  $M^\wedge \rightarrow (M/N)^\wedge$  is open.

**Proof.** We have  $0 \rightarrow N^\wedge \rightarrow M^\wedge \rightarrow (M/N)^\wedge$  is exact and statement (2) by Lemma 4.3. This produces a canonical map  $c : M^\wedge/N^\wedge \rightarrow (M/N)^\wedge$ . The module  $M^\wedge/N^\wedge$  is complete and  $M^\wedge \rightarrow M^\wedge/N^\wedge$  is open by Lemma 4.4. By the universal property of completion we obtain a canonical map  $b : (M/N)^\wedge \rightarrow M^\wedge/N^\wedge$ . Then  $b$  and  $c$  are mutually inverse as they are on a dense subset.  $\square$

**Definition 4.6.** Let  $R$  be a topological ring. Let  $M$  and  $N$  be linearly topologized  $R$ -modules. The *tensor product* of  $M$  and  $N$  is the (usual) tensor product  $M \otimes_R N$  endowed with the linear topology defined by declaring

$$\text{Im}(M_\mu \otimes_R N + M \otimes_R N_\nu \longrightarrow M \otimes_R N)$$

to be a fundamental system of open submodules, where  $M_\mu \subset M$  and  $N_\nu \subset N$  run through fundamental systems of open submodules in  $M$  and  $N$ . The *completed tensor product*

$$M \hat{\otimes}_R N = \lim M \otimes_R N / (M_\mu \otimes_R N + M \otimes_R N_\nu) = \lim M/M_\mu \otimes_R N/N_\nu$$

is the completion of the tensor product.

Observe that the topology on  $R$  is immaterial for the construction of the tensor product or the completed tensor product. If  $R \rightarrow A$  and  $R \rightarrow B$  are continuous maps of linearly topologized rings, then the construction above gives a tensor product  $A \otimes_R B$  and a completed tensor product  $A \hat{\otimes}_R B$ .

We record here the notions introduced in Remark 2.3.

**Definition 4.7.** Let  $A$  be a linearly topologized ring.

- (1) An element  $f \in A$  is called *topologically nilpotent* if  $f^n \rightarrow 0$  as  $n \rightarrow \infty$ .
- (2) A *weak ideal of definition* for  $A$  is an open ideal  $I \subset A$  consisting entirely of topologically nilpotent elements.
- (3) We say  $A$  is *weakly pre-admissible* if  $A$  has a weak ideal of definition.
- (4) We say  $A$  is *weakly admissible* if  $A$  is weakly pre-admissible and complete<sup>1</sup>.

Given a weak ideal of definition  $I$  in a linearly topologized ring  $A$  and an open ideal  $J$  the intersection  $I \cap J$  is a weak ideal of definition. Hence if there is one weak ideal of definition, then there is a fundamental system of open ideals consisting of weak ideals of definition. In particular, given a weakly admissible topological ring  $A$  then  $A = \lim A/I_\lambda$  where  $\{I_\lambda\}$  is a fundamental system of weak ideals of definition.

**Lemma 4.8.** *Let  $\varphi : A \rightarrow B$  be a continuous map of linearly topologized rings.*

- (1) *If  $f \in A$  is topologically nilpotent, then  $\varphi(f)$  is topologically nilpotent.*

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<sup>1</sup>By our conventions this includes separated.

- (2) If  $I \subset A$  consists of topologically nilpotent elements, then the closure of  $\varphi(I)B$  consists of topologically nilpotent elements.

**Proof.** Part (1) is clear. Let  $g$  be an element of the closure of  $\varphi(I)B$ . Let  $J \subset B$  be an open ideal. We have to show  $g^e \in J$  for some  $e$ . We have  $g \in \varphi(I)B + J$  by Lemma 4.2. Hence  $g = \sum_{i=1, \dots, n} f_i b_i + h$  for some  $f_i \in I$ ,  $b_i \in B$  and  $h \in J$ . Pick  $e_i$  such that  $\varphi(f_i^{e_i}) \in J$ . Then  $g^{e_1 + \dots + e_n + 1} \in J$ .  $\square$

**Definition 4.9.** Let  $\varphi : A \rightarrow B$  be a continuous map of linearly topologized rings. We say  $\varphi$  is *taut*<sup>2</sup> if for every open ideal  $I \subset A$  the closure of the ideal  $\varphi(I)B$  is open and these closures form a fundamental system of open ideals.

If  $\varphi : A \rightarrow B$  is a continuous map of linearly topologized rings and  $I_\lambda$  a fundamental system of open ideals of  $A$ , then  $\varphi$  is taut if and only if the closures of  $I_\lambda B$  are open and form a fundamental system of open ideals in  $A$ .

**Lemma 4.10.** Let  $\varphi : A \rightarrow B$  be a continuous map of weakly admissible topological rings. The following are equivalent

- (1)  $\varphi$  is taut,
- (2) for every weak ideal of definition  $I \subset A$  the closure of  $\varphi(I)B$  is a weak ideal of definition of  $B$  and these form a fundamental system of weak ideals of definition of  $B$ .

**Proof.** It is clear that (2) implies (1). The other implication follows from Lemma 4.8.  $\square$

**Lemma 4.11.** Let  $A \rightarrow B$  be a continuous map of linearly topologized rings. Let  $I \subset A$  be an ideal. The closure of  $IB$  is the kernel of  $B \rightarrow B \widehat{\otimes}_A A/I$ .

**Proof.** Let  $J_\mu$  be a fundamental system of open ideals of  $B$ . The closure of  $IB$  is  $\bigcap (IB + J_\lambda)$  by Lemma 4.2. Let  $I_\mu$  be a fundamental system of open ideals in  $A$ . Then

$$B \widehat{\otimes}_A A/I = \lim (B/J_\lambda \otimes_A A/(I_\mu + I)) = \lim B/(J_\lambda + I_\mu B + IB)$$

Since  $A \rightarrow B$  is continuous, for every  $\lambda$  there is a  $\mu$  such that  $I_\mu B \subset J_\lambda$ , see discussion in Example 4.1. Hence the limit can be written as  $\lim B/(J_\lambda + IB)$  and the result is clear.  $\square$

**Lemma 4.12.** Let  $\varphi : A \rightarrow B$  be a continuous homomorphism of linearly topologized rings. If

- (1)  $\varphi$  is taut,
- (2)  $\varphi$  has dense image,
- (3)  $A$  is complete,
- (4)  $B$  is separated, and
- (5)  $A$  has a countable fundamental system of neighbourhoods of 0.

Then  $\varphi$  is surjective and open,  $B$  is complete, and  $B = A/K$  for some closed ideal  $K \subset A$ .

<sup>2</sup>This is nonstandard notation. The definition generalizes to modules, by saying a linearly topologized  $A$ -module  $M$  is  $A$ -taut if for every open ideal  $I \subset A$  the closure of  $IM$  in  $M$  is open and these closures form a fundamental system of neighbourhoods of 0 in  $M$ .

**Proof.** We may choose a sequence of open ideals  $A \supset I_1 \supset I_2 \supset I_3 \supset \dots$  which form a fundamental system of neighbourhoods of 0. For each  $i$  let  $J_i \subset B$  be the closure of  $\varphi(I_i)B$ . As  $\varphi$  is taut we see that these form a fundamental system of open ideals of  $B$ . Set  $I_0 = A$  and  $J_0 = B$ . Let  $n \geq 0$  and let  $y_n \in J_n$ . Since  $J_{n+1}$  is the closure of  $\varphi(I_n)B$  we can write

$$y_n = \sum_t \varphi(f_t)b_t + y'_{n+1}$$

for some  $f_t \in I_n$ ,  $b_t \in B$ , and  $y'_{n+1} \in J_{n+1}$ . Since  $\varphi$  has dense image we can choose  $a_t \in A$  with  $\varphi(a_t) = b_t \bmod J_{n+1}$ . Thus

$$y_n = \varphi(f_n) + y_{n+1}$$

with  $f_n = \sum f_t a_t \in I_n$  and  $y_{n+1} = y'_{n+1} + \sum f_t(b_t - \varphi(a_t)) \in J_{n+1}$ . Thus, starting with any  $y = y_0 \in B$ , we can find by induction a sequence  $f_m \in I_m$ ,  $m \geq 0$  such that

$$y = y_0 = \varphi(f_0 + f_1 + \dots + f_n) + y_{n+1}$$

with  $y_{n+1} \in J_{n+1}$ . Since  $A$  is complete we see that

$$x = x_0 = f_0 + f_1 + f_2 + \dots$$

exists. Since the partial sums approximate  $x$  in  $A$ , since  $\varphi$  is continuous, and since  $B$  is separated we find that  $\varphi(x) = y$  because above we've shown that the images of the partial sums approximate  $y$  in  $B$ . Thus  $\varphi$  is surjective. In exactly the same manner we find that  $\varphi(I_n) = J_n$  for all  $n \geq 1$ . This proves the lemma.  $\square$

The next lemma says “ $\varphi$  is taut” if and only if “ $\varphi$  is adic” for continuous maps  $\varphi : A \rightarrow B$  between adic rings if  $A$  has a finitely generated ideal of definition. In some sense the previously introduced notion of tautness for continuous ring maps supersedes the notion of an adic map between adic rings. See also Section 17.

**Lemma 4.13.** *Let  $\varphi : A \rightarrow B$  be a continuous map of linearly topologized rings. Let  $I \subset A$  be an ideal. Assume*

- (1)  *$I$  is finitely generated,*
- (2)  *$A$  has the  $I$ -adic topology,*
- (3)  *$B$  is complete, and*
- (4)  *$\varphi$  is taut.*

*Then the topology on  $B$  is the  $I$ -adic topology.*

**Proof.** Let  $J_n$  be the closure of  $\varphi(I^n)B$  in  $B$ . Since  $B$  is complete we have  $B = \lim B/J_n$ . Let  $B' = \lim B/I^n B$  be the  $I$ -adic completion of  $B$ . By Algebra, Lemmas 93.6 and 93.7, the  $I$ -adic topology on  $B'$  is complete and  $B'/I^n B' = B/I^n B$ . Thus the ring map  $B' \rightarrow B$  is continuous and has dense image as  $B' \rightarrow B/I^n B \rightarrow B/J_n$  is surjective for all  $n$ . Finally, the map  $B' \rightarrow B$  is taut because  $(I^n B')B = I^n B$  and  $A \rightarrow B$  is taut. By Lemma 4.12 we see that  $B' \rightarrow B$  is open and surjective which implies the lemma.  $\square$

## 5. Affine formal algebraic spaces

In this section we introduce affine formal algebraic spaces. These will in fact be the same as what are called affine formal schemes in [BD]. However, we will call them affine formal algebraic spaces, in order to prevent confusion with the notion of an affine formal scheme as defined in [DG67].

Recall that a thickening of schemes is a closed immersion which induces a surjection on underlying topological spaces, see More on Morphisms, Definition 2.1.

**Definition 5.1.** Let  $S$  be a scheme. We say a sheaf  $X$  on  $(Sch/S)_{fppf}$  is an *affine formal algebraic space* if there exist

- (1) a directed partially ordered set  $\Lambda$ ,
- (2) a system  $(X_\lambda, f_{\lambda\mu})$  over  $\Lambda$  in  $(Sch/S)_{fppf}$  where
  - (a) each  $X_\lambda$  is affine,
  - (b) each  $f_{\lambda\mu} : X_\lambda \rightarrow X_\mu$  is a thickening,

such that

$$X \cong \operatorname{colim}_{\lambda \in \Lambda} X_\lambda$$

as fppf sheaves and  $X$  satisfies a set theoretic condition (see Remark 7.6). A *morphism of affine formal algebraic spaces* over  $S$  is a map of sheaves.

Observe that the system  $(X_\lambda, f_{\lambda\mu})$  is not part of the data. Suppose that  $U$  is a quasi-compact scheme over  $S$ . Since the transition maps are monomorphisms, we see that

$$X(U) = \operatorname{colim} X_\lambda(U)$$

by Sites, Lemma 11.2. Thus the fppf sheafification inherent in the colimit of the definition is a Zariski sheafification which does not do anything for quasi-compact schemes.

**Lemma 5.2.** *Let  $S$  be a scheme. If  $X$  is an affine formal algebraic space over  $S$ , then the diagonal morphism  $\Delta : X \rightarrow X \times_S X$  is representable and a closed immersion.*

**Proof.** Suppose given  $U \rightarrow X$  and  $V \rightarrow X$  where  $U, V$  are schemes over  $S$ . Let us show that  $U \times_X V$  is representable. Write  $X = \operatorname{colim} X_\lambda$  as in Definition 5.1. The discussion above shows that Zariski locally on  $U$  and  $V$  the morphisms factors through some  $X_\lambda$ . In this case  $U \times_X V = U \times_{X_\lambda} V$  which is a scheme. Thus the diagonal is representable, see Spaces, Lemma 5.10. Given  $(a, b) : W \rightarrow X \times_S X$  where  $W$  is a scheme over  $S$  consider the map  $X \times_{\Delta, X \times_S X, (a, b)} W \rightarrow W$ . As before locally on  $W$  the morphisms  $a$  and  $b$  map into the affine scheme  $X_\lambda$  for some  $\lambda$  and then we get the morphism  $X_\lambda \times_{\Delta_\lambda, X_\lambda \times_S X_\lambda, (a, b)} W \rightarrow W$ . This is the base change of  $\Delta_\lambda : X_\lambda \rightarrow X_\lambda \times_S X_\lambda$  which is a closed immersion as  $X_\lambda \rightarrow S$  is separated (because  $X_\lambda$  is affine). Thus  $X \rightarrow X \times_S X$  is a closed immersion.  $\square$

A morphism of schemes  $X \rightarrow X'$  is a thickening if it is a closed immersion and induces a surjection on underlying sets of points, see (More on Morphisms, Definition 2.1). Hence the property of being a thickening is preserved under arbitrary base change and fpqc local on the target, see Spaces, Section 4. Thus Spaces, Definition 5.1 applies to “thickening” and we know what it means for a representable transformation  $F \rightarrow G$  of presheaves on  $(Sch/S)_{fppf}$  to be a thickening. We observe that this does not clash with our definition (More on Morphisms of Spaces, Definition 8.1) of thickenings in case  $F$  and  $G$  are algebraic spaces.

**Lemma 5.3.** *Let  $X_\lambda, \lambda \in \Lambda$  and  $X = \operatorname{colim} X_\lambda$  be as in Definition 5.1. Then  $X_\lambda \rightarrow X$  is representable and a thickening.*

**Proof.** The statement makes sense by the discussion in Spaces, Section 3 and 5. By Lemma 5.2 the morphisms  $X_\lambda \rightarrow X$  are representable. Given  $U \rightarrow X$

where  $U$  is a scheme, then the discussion following Definition 5.1 shows that Zariski locally on  $U$  the morphism factors through some  $X_\mu$  with  $\lambda \leq \mu$ . In this case  $U \times_X X_\lambda = U \times_{X_\mu} X_\lambda$  so that  $U \times_X X_\lambda \rightarrow U$  is a base change of the thickening  $X_\lambda \rightarrow X_\mu$ .  $\square$

**Lemma 5.4.** *Let  $X_\lambda, \lambda \in \Lambda$  and  $X = \operatorname{colim} X_\lambda$  be as in Definition 5.1. If  $Y$  is a quasi-compact algebraic space over  $S$ , then any morphism  $Y \rightarrow X$  factors through an  $X_\lambda$ .*

**Proof.** Choose an affine scheme  $V$  and a surjective étale morphism  $V \rightarrow Y$ . The composition  $V \rightarrow Y \rightarrow X$  factors through  $X_\lambda$  for some  $\lambda$  by the discussion following Definition 5.1. Since  $V \rightarrow Y$  is a surjection of sheaves, we conclude.  $\square$

**Lemma 5.5.** *Let  $S$  be a scheme. Let  $X$  be a sheaf on  $(\operatorname{Sch}/S)_{fppf}$ . Then  $X$  is an affine formal algebraic space if and only if the following hold*

- (1) *any morphism  $U \rightarrow X$  where  $U$  is an affine scheme over  $S$  factors through a morphism  $T \rightarrow X$  which is representable and a thickening with  $T$  an affine scheme over  $S$ , and*
- (2) *a set theoretic condition as in Remark 7.6.*

**Proof.** It follows from Lemmas 5.3 and 5.4 that an affine formal algebraic space satisfies (1) and (2). In order to prove the converse we may assume  $X$  is not empty. Let  $\Lambda$  be the category of representable morphisms  $T \rightarrow X$  which are thickenings where  $T$  is an affine scheme over  $S$ . This category is directed. Since  $X$  is not empty,  $\Lambda$  contains at least one object. If  $T \rightarrow X$  and  $T' \rightarrow X$  are in  $\Lambda$ , then we can factor  $T \amalg T' \rightarrow X$  through  $T'' \rightarrow X$  in  $\Lambda$ . Between any two objects of  $\Lambda$  there is a unique arrow or none. Thus  $\Lambda$  is a directed partially ordered set and by assumption  $X = \operatorname{colim}_{T \rightarrow X \text{ in } \Lambda} T$ .  $\square$

For a general affine formal algebraic space  $X$  there is no guarantee that  $X$  has enough functions to separate points (for example). See Examples, Section 62. To characterize those that do we offer the following lemma.

**Lemma 5.6.** *Let  $S$  be a scheme. Let  $X$  be an fppf sheaf on  $(\operatorname{Sch}/S)_{fppf}$  which satisfies the set theoretic condition of Remark 7.6. The following are equivalent:*

- (1) *there exists a weakly admissible topological ring  $A$  over  $S$  (see Remark 2.3) such that  $X = \operatorname{colim}_{I \subset A \text{ weak ideal of definition}} \operatorname{Spec}(A/I)$ ,*
- (2)  *$X$  is an affine formal algebraic space and there exists an  $S$ -algebra  $A$  and a map  $X \rightarrow \operatorname{Spec}(A)$  such that for a closed immersion  $T \rightarrow X$  with  $T$  an affine scheme the composition  $T \rightarrow \operatorname{Spec}(A)$  is a closed immersion,*
- (3)  *$X$  is an affine formal algebraic space and there exists an  $S$ -algebra  $A$  and a map  $X \rightarrow \operatorname{Spec}(A)$  such that for a closed immersion  $T \rightarrow X$  with  $T$  a scheme the composition  $T \rightarrow \operatorname{Spec}(A)$  is a closed immersion,*
- (4)  *$X$  is an affine formal algebraic space and for some choice of  $X = \operatorname{colim} X_\lambda$  as in Definition 5.1 the projections  $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) \rightarrow \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$  are surjective,*
- (5)  *$X$  is an affine formal algebraic space and for any choice of  $X = \operatorname{colim} X_\lambda$  as in Definition 5.1 the projections  $\lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) \rightarrow \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$  are surjective.*

Moreover, the weakly admissible topological ring is  $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$  endowed with its limit topology and the weak ideals of definition classify exactly the morphisms  $T \rightarrow X$  which are representable and thickenings.

**Proof.** It is clear that (5) implies (4).

Assume (4) for  $X = \operatorname{colim} X_\lambda$  as in Definition 5.1. Set  $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ . Let  $T \rightarrow X$  be a closed immersion with  $T$  a scheme (note that  $T \rightarrow X$  is representable by Lemma 5.2). Since  $X_\lambda \rightarrow X$  is a thickening, so is  $X_\lambda \times_X T \rightarrow T$ . On the other hand,  $X_\lambda \times_X T \rightarrow X_\lambda$  is a closed immersion, hence  $X_\lambda \times_X T$  is affine. Hence  $T$  is affine by Limits, Proposition 10.2. Then  $T \rightarrow X$  factors through  $X_\lambda$  for some  $\lambda$  by Lemma 5.4. Thus  $A \rightarrow \Gamma(X_\lambda, \mathcal{O}) \rightarrow \Gamma(T, \mathcal{O})$  is surjective. In this way we see that (3) holds.

It is clear that (3) implies (2).

Assume (2) for  $A$  and  $X \rightarrow \operatorname{Spec}(A)$ . Write  $X = \operatorname{colim} X_\lambda$  as in Definition 5.1. Then  $A_\lambda = \Gamma(X_\lambda, \mathcal{O})$  is a quotient of  $A$  by assumption (2). Hence  $A^\wedge = \lim A_\lambda$  is a complete topological ring, see discussion in More on Algebra, Section 26. The maps  $A^\wedge \rightarrow A_\lambda$  are surjective as  $A \rightarrow A_\lambda$  is. We claim that for any  $\lambda$  the kernel  $I_\lambda \subset A^\wedge$  of  $A^\wedge \rightarrow A_\lambda$  is a weak ideal of definition. Namely, it is open by definition of the limit topology. If  $f \in I_\lambda$ , then for any  $\mu \in \Lambda$  the image of  $f$  in  $A_\mu$  is zero in all the residue fields of the points of  $X_\mu$ . Hence it is a nilpotent element of  $A_\mu$ . Hence some power  $f^n \in I_\mu$ . Thus  $f^n \rightarrow 0$  as  $n \rightarrow 0$ . Thus  $A^\wedge$  is weakly admissible. Finally, suppose that  $I \subset A^\wedge$  is a weak ideal of definition. Then  $I \subset A^\wedge$  is open and hence there exists some  $\lambda$  such that  $I \supset I_\lambda$ . Thus we obtain a morphism  $\operatorname{Spec}(A^\wedge/I) \rightarrow \operatorname{Spec}(A_\lambda) \rightarrow X$ . Then it follows that  $X = \operatorname{colim} \operatorname{Spec}(A^\wedge/I)$  where now the colimit is over all weak ideals of definition. Thus (1) holds.

Assume (1). In this case it is clear that  $X$  is an affine formal algebraic space. Let  $X = \operatorname{colim} X_\lambda$  be any presentation as in Definition 5.1. For each  $\lambda$  we can find a weak ideal of definition  $I \subset A$  such that  $X_\lambda \rightarrow X$  factors through  $\operatorname{Spec}(A/I) \rightarrow X$ , see Lemma 5.4. Then  $X_\lambda = \operatorname{Spec}(A/I_\lambda)$  with  $I \subset I_\lambda$ . Conversely, for any weak ideal of definition  $I \subset A$  the morphism  $\operatorname{Spec}(A/I) \rightarrow X$  factors through  $X_\lambda$  for some  $\lambda$ , i.e.,  $I_\lambda \subset I$ . It follows that each  $I_\lambda$  is a weak ideal of definition and that they form a cofinal subset of the set of weak ideals of definition. Hence  $A = \lim A/I = \lim A/I_\lambda$  and we see that (5) is true and moreover that  $A = \lim \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ .  $\square$

With this lemma in hand we can make the following definition.

**Definition 5.7.** Let  $S$  be a scheme. Let  $X$  be an affine formal algebraic space over  $S$ . We say  $X$  is *McQuillan* if  $X$  satisfies the equivalent conditions of Lemma 5.6. Let  $A$  be the weakly admissible topological ring associated to  $X$ . We say

- (1)  $X$  is *classical* if  $X$  is McQuillan and  $A$  is admissible,
- (2)  $X$  is *adic* if  $X$  is McQuillan and  $A$  is adic,
- (3)  $X$  is *adic\** if  $X$  is McQuillan,  $A$  is adic, and  $A$  has a finitely generated ideal of definition, and
- (4)  $X$  is *Noetherian* if  $X$  is McQuillan and  $A$  is both Noetherian and adic.

In [FK] they use the terminology “of finite ideal type” for the property that an adic topological ring  $A$  contains a finitely generated ideal of definition.

**Remark 5.8.** The classical affine formal algebraic spaces correspond to the affine formal schemes considered in EGA ([DG67]). To explain this we assume our base

scheme is  $\mathrm{Spec}(\mathbf{Z})$ . Let  $\mathfrak{X} = \mathrm{Spf}(A)$  be an affine formal scheme. Let  $h_{\mathfrak{X}}$  be its functor of points as in Lemma 2.1. Then  $h_{\mathfrak{X}} = \mathrm{colim} h_{\mathrm{Spec}(A/I)}$  where the colimit is over the collection of ideals of definition of the admissible topological ring  $A$ . This follows from (2.0.1) when evaluating on affine schemes and it suffices to check on affine schemes as both sides are fppf sheaves, see Lemma 2.2. Thus  $h_{\mathfrak{X}}$  is an affine formal algebraic space. In fact, it is a classical affine formal algebraic space by Definition 5.7. Thus Lemma 2.1 tells us the category of affine formal schemes is equivalent to the category of classical affine formal algebraic spaces.

Having made the connection with affine formal schemes above, it seems natural to make the following definition.

**Definition 5.9.** Let  $S$  be a scheme. Let  $A$  be a weakly admissible topological ring over  $S$ , see Definition 4.7<sup>3</sup>. The *formal spectrum* of  $A$  is the affine formal algebraic space

$$\mathrm{Spf}(A) = \mathrm{colim} \mathrm{Spec}(A/I)$$

where the colimit is over the set of weak ideals of definition of  $A$  and taken in the category  $\mathrm{Sh}((\mathrm{Sch}/S)_{\mathrm{fppf}})$ .

Such a formal spectrum is McQuillan by construction and conversely every McQuillan affine formal algebraic space is isomorphic to a formal spectrum. To be sure, in our theory there exist affine formal algebraic spaces which are not the formal spectrum of any weakly admissible topological ring. Following [Yas09] we could introduce  $S$ -pro-rings to be pro-objects in the category of  $S$ -algebras, see Categories, Remark 22.4. Then every affine formal algebraic space over  $S$  would be the formal spectrum of such an  $S$ -pro-ring. We will not do this and instead we will work directly with the corresponding affine formal algebraic spaces.

The construction of the formal spectrum is functorial. To explain this let  $\varphi : B \rightarrow A$  be a continuous map of weakly admissible topological rings over  $S$ . Then

$$\mathrm{Spf}(\varphi) : \mathrm{Spf}(B) \rightarrow \mathrm{Spf}(A)$$

is the unique morphism of affine formal algebraic spaces such that the diagrams

$$\begin{array}{ccc} \mathrm{Spec}(B/J) & \longrightarrow & \mathrm{Spec}(A/I) \\ \downarrow & & \downarrow \\ \mathrm{Spf}(B) & \longrightarrow & \mathrm{Spf}(A) \end{array}$$

commute for all weak ideals of definition  $I \subset A$  and  $J \subset B$  with  $\varphi(I) \subset J$ . Since continuity of  $\varphi$  implies that for every weak ideal of definition  $J \subset B$  there is a weak ideal of definition  $I \subset A$  with the required property, we see that the required commutativities uniquely determine and define  $\mathrm{Spf}(\varphi)$ .

**Lemma 5.10.** *Let  $S$  be a scheme. Let  $A, B$  be weakly admissible topological rings over  $S$ . Any morphism  $f : \mathrm{Spf}(B) \rightarrow \mathrm{Spf}(A)$  of affine formal algebraic spaces over  $S$  is equal to  $\mathrm{Spf}(f^{\sharp})$  for a unique continuous  $S$ -algebra map  $f^{\sharp} : A \rightarrow B$ .*

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<sup>3</sup>See More on Algebra, Definition 26.1 for the classical case and see Remark 2.3 for a discussion of differences.



**Proof.** Let  $f : \mathrm{Spf}(B) \rightarrow \mathrm{Spf}(A)$  be as in the lemma. Let  $J \subset B$  be a weak ideal of definition. By Lemma 5.4 there exists a weak ideal of definition  $I \subset A$  such that  $\mathrm{Spec}(B/J) \rightarrow \mathrm{Spf}(B) \rightarrow \mathrm{Spf}(A)$  factors through  $\mathrm{Spec}(A/I)$ . By Schemes, Lemma 6.4 we obtain an  $S$ -algebra map  $A/I \rightarrow B/J$ . These maps are compatible for varying  $J$  and define the map  $f^\# : A \rightarrow B$ . This map is continuous because for every weak ideal of definition  $J \subset B$  there is a weak ideal of definition  $I \subset A$  such that  $f^\#(I) \subset J$ . The equality  $f = \mathrm{Spf}(f^\#)$  holds by our choice of the ring maps  $A/I \rightarrow B/J$  which make up  $f^\#$ .  $\square$

**Lemma 5.11.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a map of presheaves on  $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ . If  $X$  is an affine formal algebraic space and  $f$  is representable by algebraic spaces and locally quasi-finite, then  $f$  is representable (by schemes).*

**Proof.** Let  $T$  be a scheme over  $S$  and  $T \rightarrow Y$  a map. We have to show that the algebraic space  $X \times_Y T$  is a scheme. Write  $X = \mathrm{colim} X_\lambda$  as in Definition 5.1. Let  $W \subset X \times_Y T$  be a quasi-compact open subspace. The restriction of the projection  $X \times_Y T \rightarrow X$  to  $W$  factors through  $X_\lambda$  for some  $\lambda$ . Then

$$W \rightarrow X_\lambda \times_S T$$

is a monomorphism (hence separated) and locally quasi-finite (because  $W \rightarrow X \times_Y T \rightarrow T$  is locally quasi-finite by our assumption on  $X \rightarrow Y$ , see Morphisms of Spaces, Lemma 26.8). Hence  $W$  is a scheme by Morphisms of Spaces, Proposition 44.2. Thus  $X \times_Y T$  is a scheme by Properties of Spaces, Lemma 10.1.  $\square$

## 6. Countably indexed affine formal algebraic spaces

These are the affine formal algebraic spaces as in the following lemma.

**Lemma 6.1.** *Let  $S$  be a scheme. Let  $X$  be an affine formal algebraic space over  $S$ . The following are equivalent*

- (1) *there exists a system  $X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \dots$  of thickenings of affine schemes over  $S$  such that  $X = \mathrm{colim} X_n$ ,*
- (2) *there exists a choice  $X = \mathrm{colim} X_\lambda$  as in Definition 5.1 such that  $\Lambda$  is countable.*

**Proof.** This follows from the observation that a countable directed partially ordered set has a cofinal subset isomorphic to  $(\mathbb{N}, \geq)$ . See proof of Algebra, Lemma 83.3.  $\square$

**Definition 6.2.** Let  $S$  be a scheme. Let  $X$  be an affine formal algebraic space over  $S$ . We say  $X$  is *countably indexed* if the equivalent conditions of Lemma 6.1 are satisfied.

In the language of [BD] this is expressed by saying that  $X$  is an  $\aleph_0$ -ind scheme.

**Lemma 6.3.** *Let  $X$  be an affine formal algebraic space over a scheme  $S$ .*

- (1) *If  $X$  is Noetherian, then  $X$  is  $\mathrm{adic}^*$ .*
- (2) *If  $X$  is  $\mathrm{adic}^*$ , then  $X$  is  $\mathrm{adic}$ .*
- (3) *If  $X$  is  $\mathrm{adic}$ , then  $X$  is countably indexed.*
- (4) *If  $X$  is countably indexed, then  $X$  is McQuillan.*

**Proof.** Parts (1) and (2) are immediate from the definitions.

Proof of (3). By definition there exists an adic topological ring  $A$  such that  $X = \text{colim Spec}(A/I)$  where the colimit is over the ideals of definition of  $A$ . As  $A$  is adic, there exists an ideal  $I$  such that  $\{I^n\}$  forms a fundamental system of neighbourhoods of 0. Then each  $I^n$  is an ideal of definition and  $X = \text{colim Spec}(A/I^n)$ . Thus  $X$  is countably indexed.

Proof of (4). Write  $X = \lim X_n$  for some system  $X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \dots$  of thickenings of affine schemes over  $S$ . Then

$$A = \lim \Gamma(X_n, \mathcal{O}_{X_n})$$

surjects onto each  $\Gamma(X_n, \mathcal{O}_{X_n})$  because the transition maps are surjections as the morphisms  $X_n \rightarrow X_{n+1}$  are closed immersions.  $\square$

**Lemma 6.4.** *Let  $S$  be a scheme. Let  $X$  be a presheaf on  $(\text{Sch}/S)_{fppf}$ . The following are equivalent*

- (1)  $X$  is a countably indexed affine formal algebraic space,
- (2)  $X = \text{Spf}(A)$  where  $A$  is a weakly admissible topological  $S$ -algebra which has a countable fundamental system of neighbourhoods of 0,
- (3)  $X = \text{Spf}(A)$  where  $A$  is a weakly admissible topological  $S$ -algebra which has a fundamental system  $A \supset I_1 \supset I_2 \supset I_3 \supset \dots$  of weak ideals of definition,
- (4)  $X = \text{Spf}(A)$  where  $A$  is a complete topological  $S$ -algebra with a fundamental system of open neighbourhoods of 0 given by a countable sequence  $A \supset I_1 \supset I_2 \supset I_3 \supset \dots$  of ideals such that  $I_n/I_{n+1}$  is locally nilpotent, and
- (5)  $X = \text{Spf}(A)$  where  $A = \lim B/J_n$  with the limit topology where  $B \supset J_1 \supset J_2 \supset J_3 \supset \dots$  is a sequence of ideals in an  $S$ -algebra  $B$  with  $J_n/J_{n+1}$  locally nilpotent.

**Proof.** Assume (1). By Lemma 6.3 we can write  $X = \text{Spf}(A)$  where  $A$  is a weakly admissible topological  $S$ -algebra. For any presentation  $X = \text{colim } X_n$  as in Lemma 6.1 part (1) we see that  $A = \lim A_n$  with  $X_n = \text{Spec}(A_n)$  and  $A_n = A/I_n$  for some weak ideal of definition  $I_n \subset A$ . This follows from the final statement of Lemma 5.6 which moreover implies that  $\{I_n\}$  is a fundamental system of open neighbourhoods of 0. Thus we have a sequence

$$A \supset I_1 \supset I_2 \supset I_3 \supset \dots$$

of weak ideals of definition with  $A = \lim A/I_n$ . In this way we see that condition (1) implies each of the conditions (2) – (5).

Assume (5). First note that the limit topology on  $A = \lim B/J_n$  is a linearly topologized, complete topology, see More on Algebra, Section 26. If  $f \in A$  maps to zero in  $B/J_1$ , then some power maps to zero in  $B/J_2$  as its image in  $J_1/J_2$  is nilpotent, then a further power maps to zero in  $J_2/J_3$ , etc, etc. In this way we see the open ideal  $\text{Ker}(A \rightarrow B/J_1)$  is a weak ideal of definition. Thus  $A$  is weakly admissible. In this way we see that (5) implies (2).

It is clear that (4) is a special case of (5) by taking  $B = A$ . It is clear that (3) is a special case of (2).

Assume  $A$  is as in (2). Let  $E_n$  be a countable fundamental system of neighbourhoods of 0 in  $A$ . Since  $A$  is a weakly admissible topological ring we can find open ideals  $I_n \subset E_n$ . We can also choose a weak ideal of definition  $J \subset A$ . Then  $J \cap I_n$  is

a fundamental system of weak ideals of definition of  $A$  and we get  $X = \mathrm{Spf}(A) = \mathrm{colim} \mathrm{Spec}(A/(J \cap I_n))$  which shows that  $X$  is a countably indexed affine formal algebraic space.  $\square$

**Lemma 6.5.** *Let  $S$  be a scheme. Let  $X$  be an affine formal algebraic space. The following are equivalent*

- (1)  $X$  is Noetherian,
- (2)  $X$  is  $\mathrm{adic}^*$  and for some choice of  $X = \mathrm{colim} X_\lambda$  as in Definition 5.1 the schemes  $X_\lambda$  are Noetherian,
- (3)  $X$  is  $\mathrm{adic}^*$  and for a closed immersion  $T \rightarrow X$  with  $T$  a scheme,  $T$  is Noetherian.

**Proof.** This follows from the fact that if  $A$  is a ring complete with respect to a finitely generated ideal  $I$ , then  $A$  is Noetherian if and only if  $A/I$  is Noetherian, see Algebra, Lemma 93.9. Details omitted.  $\square$

## 7. Formal algebraic spaces

We take a break from our habit of introducing new concepts first for rings, then for schemes, and then for algebraic spaces, by introducing formal algebraic spaces without first introducing formal schemes. The general idea will be that a formal algebraic space is a sheaf in the fppf topology which étale locally is an affine formal scheme in the sense of [BD]. Related material can be found in [Yas09].

In the definition of a formal algebraic space we are going to borrow some terminology from Bootstrap, Sections 3 and 4.

**Definition 7.1.** Let  $S$  be a scheme. We say a sheaf  $X$  on  $(\mathrm{Sch}/S)_{\mathrm{fppf}}$  is a *formal algebraic space* if there exist a family of maps  $\{X_i \rightarrow X\}_{i \in I}$  of sheaves such that

- (1)  $X_i$  is an affine formal algebraic space,
- (2)  $X_i \rightarrow X$  is representable by algebraic spaces and étale,
- (3)  $\coprod X_i \rightarrow X$  is surjective as a map of sheaves

and  $X$  satisfies a set theoretic condition (see Remark 7.6). A *morphism of formal algebraic spaces* over  $S$  is a map of sheaves.

Discussion. Sanity check: an affine formal algebraic space is a formal algebraic space. In the situation of the definition the morphisms  $X_i \rightarrow X$  are representable (by schemes), see Lemma 5.11. By Bootstrap, Lemma 4.6 we could instead of asking  $\coprod X_i \rightarrow X$  to be surjective as a map of sheaves, require that it be surjective (which makes sense because it is representable).

Our notion of a formal algebraic space is **very general**. In fact, even affine formal algebraic spaces as defined above are very nasty objects. However, they do have an underlying reduced algebraic space as the following lemma demonstrates.

**Lemma 7.2.** *Let  $S$  be a scheme. Let  $X$  be a formal algebraic space over  $S$ . There exists a reduced algebraic space  $X_{\mathrm{red}}$  and a representable morphism  $X_{\mathrm{red}} \rightarrow X$  which is a thickening. A morphism  $U \rightarrow X$  with  $U$  a reduced algebraic space factors uniquely through  $X_{\mathrm{red}}$ .*

**Proof.** First assume that  $X$  is an affine formal algebraic space. Say  $X = \mathrm{colim} X_\lambda$  as in Definition 5.1. Since the transition morphisms are thickenings, the affine schemes  $X_\lambda$  all have isomorphic reductions  $X_{\mathrm{red}}$ . The morphism  $X_{\mathrm{red}} \rightarrow X$  is

representable and a thickening by Lemma 5.3 and the fact that compositions of thickenings are thickenings. We omit the verification of the universal property (use Schemes, Definition 12.5, Schemes, Lemma 12.6, Properties of Spaces, Definition 9.5, and Properties of Spaces, Lemma 9.4).

Let  $X$  and  $\{X_i \rightarrow X\}_{i \in I}$  be as in Definition 7.1. For each  $i$  let  $X_{i,red} \rightarrow X_i$  be the reduction as constructed above. For  $i, j \in I$  the projection  $X_{i,red} \times_X X_j \rightarrow X_{i,red}$  is an étale (by assumption) morphism of schemes (by Lemma 5.11). Hence  $X_{i,red} \times_X X_j$  is reduced (see Descent, Lemma 14.1). Thus the projection  $X_{i,red} \times_X X_j \rightarrow X_j$  factors through  $X_{j,red}$  by the universal property. We conclude that

$$R_{ij} = X_{i,red} \times_X X_j = X_{i,red} \times_X X_{j,red} = X_i \times_X X_{j,red}$$

because the morphisms  $X_{i,red} \rightarrow X_i$  are injections of sheaves. Set  $U = \coprod X_{i,red}$ , set  $R = \coprod R_{ij}$ , and denote  $s, t : R \rightarrow U$  the two projections. As a sheaf  $R = U \times_X U$  and  $s$  and  $t$  are étale. Then  $(t, s) : R \rightarrow U$  defines an étale equivalence relation by our observations above. Thus  $X_{red} = U/R$  is an algebraic space by Spaces, Theorem 10.5. By construction the diagram

$$\begin{array}{ccc} \coprod X_{i,red} & \longrightarrow & \coprod X_i \\ \downarrow & & \downarrow \\ X_{red} & \longrightarrow & X \end{array}$$

is cartesian. Since the right vertical arrow is étale surjective and the top horizontal arrow is representable and a thickening we conclude that  $X_{red} \rightarrow X$  is representable by Bootstrap, Lemma 5.2 (to verify the assumptions of the lemma use that a surjective étale morphism is surjective, flat, and locally of finite presentation and use that thickenings are separated and locally quasi-finite). Then we can use Spaces, Lemma 5.6 to conclude that  $X_{red} \rightarrow X$  is a thickening (use that being a thickening is equivalent to being a surjective closed immersion).

Finally, suppose that  $U \rightarrow X$  is a morphism with  $U$  a reduced algebraic space over  $S$ . Then each  $X_i \times_X U$  is étale over  $U$  and therefore reduced (by our definition of reduced algebraic spaces in Properties of Spaces, Section 7). Then  $X_i \times_X U \rightarrow X_i$  factors through  $X_{i,red}$ . Hence  $U \rightarrow X$  factors through  $X_{red}$  because  $\{X_i \times_X U \rightarrow U\}$  is an étale covering.  $\square$

**Lemma 7.3.** *Let  $S$  be a scheme. If  $X$  is a formal algebraic space over  $S$ , then the diagonal morphism  $\Delta : X \rightarrow X \times_S X$  is representable, a monomorphism, locally quasi-finite, locally of finite type, and separated.*

**Proof.** Suppose given  $U \rightarrow X$  and  $V \rightarrow X$  with  $U, V$  schemes over  $S$ . Then  $U \times_X V$  is a sheaf. Choose  $\{X_i \rightarrow X\}$  as in Definition 7.1. For every  $i$  the morphism

$$(U \times_X X_i) \times_{X_i} (V \times_X X_i) = (U \times_X V) \times_X X_i \rightarrow U \times_X V$$

is representable and étale as a base change of  $X_i \rightarrow X$  and its source is a scheme (use Lemmas 5.2 and 5.11). These maps are jointly surjective hence  $U \times_X V$  is an algebraic space by Bootstrap, Theorem 10.1. The morphism  $U \times_X V \rightarrow U \times_S V$  is a monomorphism. It is also locally quasi-finite, because on precomposing with the morphism displayed above we obtain the composition

$$(U \times_X X_i) \times_{X_i} (V \times_X X_i) \rightarrow (U \times_X X_i) \times_S (V \times_X X_i) \rightarrow U \times_S V$$

which is locally quasi-finite as a composition of a closed immersion (Lemma 5.2) and an étale morphism, see Descent on Spaces, Lemma 17.2. Hence we conclude that  $U \times_X V$  is a scheme by Morphisms of Spaces, Proposition 44.2. Thus  $\Delta$  is representable, see Spaces, Lemma 5.10.

Let  $W \rightarrow X \times_S X$  be a morphism where  $W$  is a scheme over  $S$ . For each  $i$  consider the diagram

$$\begin{array}{ccc} W \times_{(X \times_S X)} X_i & \longrightarrow & W \times_{(X \times_S X)} (X_i \times_S X_i) \\ \downarrow & & \downarrow \\ W \times_{(X \times_S X)} X & \longrightarrow & W \end{array}$$

The vertical arrows are étale because  $X_i \rightarrow X$  is étale. The top horizontal arrow is a closed immersion by Lemma 5.2. Because the maps  $X_i \rightarrow X$  are jointly surjective, this certainly implies that the lower horizontal arrow is locally of finite type, see Morphisms of Spaces, Lemma 23.4. Since it is clearly a monomorphism, we conclude that it is locally quasi-finite (Morphisms of Spaces, Lemma 26.10). It is separated because any monomorphism is separated (Morphisms of Spaces, Lemma 10.3). This finishes the proof.  $\square$

**Lemma 7.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism from an algebraic space over  $S$  to a formal algebraic space over  $S$ . Then  $f$  is representable by algebraic spaces.*

**Proof.** Let  $Z \rightarrow Y$  be a morphism where  $Z$  is a scheme over  $S$ . We have to show that  $X \times_Y Z$  is an algebraic space. Choose a scheme  $U$  and a surjective étale morphism  $U \rightarrow X$ . Then  $U \times_Y Z \rightarrow X \times_Y Z$  is representable surjective étale (Spaces, Lemma 5.5) and  $U \times_Y Z$  is a scheme by Lemma 7.3. Hence the result by Bootstrap, Theorem 10.1.  $\square$

**Remark 7.5.** Modulo set theoretic issues the category of formal schemes à la EGA (see Section 2) is equivalent to a full subcategory of the category of formal algebraic spaces. To explain this we assume our base scheme is  $\text{Spec}(\mathbf{Z})$ . By Lemma 2.2 the functor of points  $h_{\mathfrak{X}}$  associated to a formal scheme  $\mathfrak{X}$  is a sheaf in the fppf topology. By Lemma 2.1 the assignment  $\mathfrak{X} \mapsto h_{\mathfrak{X}}$  is a fully faithful embedding of the category of formal schemes into the category of fppf sheaves. Given a formal scheme  $\mathfrak{X}$  we choose an open covering  $\mathfrak{X} = \bigcup \mathfrak{X}_i$  with  $\mathfrak{X}_i$  affine formal schemes. Then  $h_{\mathfrak{X}_i}$  is an affine formal algebraic space by Remark 5.8. The morphisms  $h_{\mathfrak{X}_i} \rightarrow h_{\mathfrak{X}}$  are representable and open immersions. Thus  $\{h_{\mathfrak{X}_i} \rightarrow h_{\mathfrak{X}}\}$  is a family as in Definition 7.1 and we see that  $h_{\mathfrak{X}}$  is a formal algebraic space.

**Remark 7.6.** Let  $S$  be a scheme and let  $(Sch/S)_{fppf}$  be a big fppf site as in Topologies, Definition 7.8. As our set theoretic condition on  $X$  in Definitions 5.1 and 7.1 we take: there exist objects  $U, R$  of  $(Sch/S)_{fppf}$ , a morphism  $U \rightarrow X$  which is a surjection of fppf sheaves, and a morphism  $R \rightarrow U \times_X U$  which is a surjection of fppf sheaves. In other words, we require our sheaf to be a coequalizer of two maps between representable sheaves. Here are some observations which imply this notion behaves reasonably well:

- (1) Suppose  $X = \text{colim}_{\lambda \in \Lambda} X_{\lambda}$  and the system satisfies conditions (1) and (2) of Definition 5.1. Then  $U = \coprod_{\lambda \in \Lambda} X_{\lambda} \rightarrow X$  is a surjection of fppf sheaves. Moreover,  $U \times_X U$  is a closed subscheme of  $U \times_S U$  by Lemma 5.2. Hence

- if  $U$  is representable by an object of  $(Sch/S)_{fppf}$  then  $U \times_S U$  is too (see Sets, Lemma 9.9) and the set theoretic condition is satisfied. This is always the case if  $\Lambda$  is countable, see Sets, Lemma 9.9.
- (2) Sanity check. Let  $\{X_i \rightarrow X\}_{i \in I}$  be as in Definition 7.1 (with the set theoretic condition as formulated above) and assume that each  $X_i$  is actually an affine scheme. Then  $X$  is an algebraic space. Namely, if we choose a larger big fppf site  $(Sch'/S)_{fppf}$  such that  $U' = \coprod X_i$  and  $R' = \coprod X_i \times_X X_j$  are representable by objects in it, then  $X' = U'/R'$  will be an object of the category of algebraic spaces for this choice. Then an application of Spaces, Lemma 15.2 shows that  $X$  is an algebraic space for  $(Sch/S)_{fppf}$ .
- (3) Let  $\{X_i \rightarrow X\}_{i \in I}$  be a family of maps of sheaves satisfying conditions (1), (2), (3) of Definition 7.1. For each  $i$  we can pick  $U_i \in \text{Ob}((Sch/S)_{fppf})$  and  $U_i \rightarrow X_i$  which is a surjection of sheaves. Thus if  $I$  is not too large (for example countable) then  $U = \coprod U_i \rightarrow X$  is a surjection of sheaves and  $U$  is representable by an object of  $(Sch/S)_{fppf}$ . To get  $R \in \text{Ob}((Sch/S)_{fppf})$  surjecting onto  $U \times_X U$  it suffices to assume the diagonal  $\Delta : X \rightarrow X \times_S X$  is not too wild, for example this always works if the diagonal of  $X$  is quasi-compact, i.e.,  $X$  is quasi-separated.

### 8. Colimits of algebraic spaces along thickenings

A special type of formal algebraic space is one which can globally be written as a cofiltered colimit of algebraic spaces along thickenings as in the following lemma. We will see later (in Section 13) that any quasi-compact and quasi-separated formal algebraic space is such a global colimit.

**Lemma 8.1.** *Let  $S$  be a scheme. Suppose given a directed partially ordered set  $\Lambda$  and a system of algebraic spaces  $(X_\lambda, f_{\lambda\mu})$  over  $\Lambda$  where each  $f_{\lambda\mu} : X_\lambda \rightarrow X_\mu$  is a thickening. Then  $X = \text{colim}_{\lambda \in \Lambda} X_\lambda$  is a formal algebraic space over  $S$ .*

**Proof.** Since we take the colimit in the category of fppf sheaves, we see that  $X$  is a sheaf. Choose and fix  $\lambda \in \Lambda$ . Choose an étale covering  $\{X_{i,\lambda} \rightarrow X_\lambda\}$  where  $X_i$  is an affine scheme over  $S$ , see Properties of Spaces, Lemma 6.1. For each  $\mu \geq \lambda$  there exists a cartesian diagram

$$\begin{array}{ccc} X_{i,\lambda} & \longrightarrow & X_{i,\mu} \\ \downarrow & & \downarrow \\ X_\lambda & \longrightarrow & X_\mu \end{array}$$

with étale vertical arrows, see More on Morphisms of Spaces, Theorem 7.1 (this also uses that a thickening is a surjective closed immersion which satisfies the conditions of the theorem). Moreover, these diagrams are unique up to unique isomorphism and hence  $X_{i,\mu} = X_\mu \times_{X_\lambda} X_{i,\lambda}$  for  $\mu \geq \lambda$ . The morphisms  $X_{i,\mu} \rightarrow X_{i,\mu'}$  is a thickening as a base change of a thickening. Each  $X_{i,\mu}$  is an affine scheme by Limits of Spaces, Proposition 15.2 and the fact that  $X_{i,\lambda}$  is affine. Set  $X_i = \text{colim}_{\mu \geq \lambda} X_{i,\mu}$ . Then  $X_i$  is an affine formal algebraic space. The morphism  $X_i \rightarrow X$  is étale because given an affine scheme  $U$  any  $U \rightarrow X$  factors through  $X_\mu$  for some  $\mu \geq \lambda$  (details omitted). In this way we see that  $X$  is a formal algebraic space.  $\square$

Let  $S$  be a scheme. Let  $X$  be a formal algebraic space over  $S$ . How does one prove or check that  $X$  is a global colimit as in Lemma 8.1? To do this we look for maps

$i : Z \rightarrow X$  where  $Z$  is an algebraic space over  $S$  and  $i$  is surjective and a closed immersion, in other words,  $i$  is a thickening. This makes sense as  $i$  is representable by algebraic spaces (Lemma 7.4) and we can use Bootstrap, Definition 4.1 as before.

**Remark 8.2** (Weak ideals of definition). Let  $\mathfrak{X}$  be a formal scheme in the sense of McQuillan, see Remark 2.3. An *weak ideal of definition* for  $\mathfrak{X}$  is an ideal sheaf  $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$  such that for all  $\mathfrak{U} \subset \mathfrak{X}$  affine formal open subscheme the ideal  $\mathcal{I}(\mathfrak{U}) \subset \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$  is a weak ideal of definition of the weakly admissible topological ring  $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ . It suffices to check the condition on the members of an affine open covering. There is a one-to-one correspondence

$$\{\text{weak ideals of definition for } \mathfrak{X}\} \leftrightarrow \{\text{thickenings } i : Z \rightarrow h_{\mathfrak{X}} \text{ as above}\}$$

This correspondence associates to  $\mathcal{I}$  the scheme  $Z = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I})$  together with the obvious morphism to  $\mathfrak{X}$ . A *fundamental system of weak ideals of definition* is a collection of weak ideals of definition  $\mathcal{I}_{\lambda}$  such that on every affine open formal subscheme  $\mathfrak{U} \subset \mathfrak{X}$  the ideals

$$I_{\lambda} = \mathcal{I}_{\lambda}(\mathfrak{U}) \subset A = \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$$

form a fundamental system of weak ideals of definition of the weakly admissible topological ring  $A$ . It suffices to check on the members of an affine open covering. We conclude that the formal algebraic space  $h_{\mathfrak{X}}$  associated to the McQuillan formal scheme  $\mathfrak{X}$  is a colimit of schemes as in Lemma 8.1 if and only if there exists a fundamental system of weak ideals of definition for  $\mathfrak{X}$ .

**Remark 8.3** (Ideals of definition). Let  $\mathfrak{X}$  be a formal scheme à la EGA. An *ideal of definition* for  $\mathfrak{X}$  is an ideal sheaf  $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$  such that for all  $\mathfrak{U} \subset \mathfrak{X}$  affine formal open subscheme the ideal  $\mathcal{I}(\mathfrak{U}) \subset \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$  is an ideal of definition of the admissible topological ring  $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ . It suffices to check the condition on the members of an affine open covering. We do **not** get the same correspondence between ideals of definition and thickenings  $Z \rightarrow \mathfrak{X}$  as in Remark 8.2. A *fundamental system of ideals of definition* is a collection of ideals of definition  $\mathcal{I}_{\lambda}$  such that on every affine open formal subscheme  $\mathfrak{U} \subset \mathfrak{X}$  the ideals

$$I_{\lambda} = \mathcal{I}_{\lambda}(\mathfrak{U}) \subset A = \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$$

form a fundamental system of ideals of definition of the admissible topological ring  $A$ . It suffices to check on the members of an affine open covering. Suppose that  $\mathfrak{X}$  is quasi-compact and that  $\{\mathcal{I}_{\lambda}\}_{\lambda \in \Lambda}$  is a fundamental system of weak ideals of definition. If  $A$  is an admissible topological ring  $A$  then all sufficiently small open ideals are ideals of definition (namely any open ideal contained in an ideal of definition is an ideal of definition). Thus since we only need to check on the finitely many members of an affine open covering we see that  $\mathcal{I}_{\lambda}$  is an ideal of definition for  $\lambda$  sufficiently large. Using the discussion in Remark 8.2 we conclude that the formal algebraic space  $h_{\mathfrak{X}}$  associated to the quasi-compact formal scheme  $\mathfrak{X}$  à la EGA is a colimit of schemes as in Lemma 8.1 if and only if there exists a fundamental system of ideals of definition for  $\mathfrak{X}$ .

## 9. Completion along a closed subset

Our notion of a formal algebraic space is well adapted to taking the completion along a closed.

**Lemma 9.1.** *Let  $S$  be a scheme. Let  $X$  be an affine scheme over  $S$ . Let  $T \subset |X|$  be a closed subset. Then the functor*

$$(Sch/S)_{fppf} \longrightarrow Sets, \quad U \longmapsto \{f : U \rightarrow X \mid f(|U|) \subset T\}$$

*is a McQuillan affine formal algebraic space.*

**Proof.** Say  $X = \text{Spec}(A)$  and  $T$  corresponds to the radical ideal  $I \subset A$ . Let  $U = \text{Spec}(B)$  be an affine scheme over  $S$  and let  $f : U \rightarrow X$  be an element of  $F(U)$ . Then  $f$  corresponds to a ring map  $\varphi : A \rightarrow B$  such that every prime of  $B$  contains  $\varphi(I)B$ . Thus every element of  $\varphi(I)$  is nilpotent in  $B$ , see Algebra, Lemma 16.2. Setting  $J = \text{Ker}(\varphi)$  we conclude that  $I/J$  is a locally nilpotent ideal in  $A/J$ . Equivalently,  $V(J) = V(I) = T$ . In other words, the functor of the lemma equals  $\text{colim Spec}(A/J)$  where the colimit is over the collection of ideals  $J$  with  $V(J) = T$ . Thus our functor is an affine formal algebraic space. It is McQuillan (Definition 5.7) because the maps  $A \rightarrow A/J$  are surjective and hence  $A^\wedge = \lim A/J \rightarrow A/J$  is surjective, see Lemma 5.6.  $\square$

**Lemma 9.2.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $T \subset |X|$  be a closed subset. Then the functor*

$$(Sch/S)_{fppf} \longrightarrow Sets, \quad U \longmapsto \{f : U \rightarrow X \mid f(|U|) \subset T\}$$

*is a formal algebraic space.*

**Proof.** Denote  $F$  the functor. Let  $\{U_i \rightarrow U\}$  be an fppf covering. Then  $\coprod |U_i| \rightarrow |U|$  is surjective. Since  $X$  is an fppf sheaf, it follows that  $F$  is an fppf sheaf.

Let  $\{g_i : X_i \rightarrow X\}$  be an étale covering such that  $X_i$  is affine for all  $i$ , see Properties of Spaces, Lemma 6.1. The morphisms  $F \times_X X_i \rightarrow F$  are étale (see Spaces, Lemma 5.5) and the map  $\coprod F \times_X X_i \rightarrow F$  is a surjection of sheaves. Thus it suffices to prove that  $F \times_X X_i$  is an affine formal algebraic space. A  $U$ -valued point of  $F \times_X X_i$  is a morphism  $U \rightarrow X_i$  whose image is contained in the closed subset  $g_i^{-1}(T) \subset |X_i|$ . Thus this follows from Lemma 9.1.  $\square$

**Definition 9.3.** Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $T \subset |X|$  be a closed subset. The formal algebraic space of Lemma 9.2 is called the *completion of  $X$  along  $T$* .

In [DG67, Chapter I, Section 10.8] the notation  $X_{/T}$  is used to denote the completion and we will occasionally use this notation as well. Let  $f : X \rightarrow X'$  be a morphism of algebraic spaces over a scheme  $S$ . Suppose that  $T \subset |X|$  and  $T' \subset |X'|$  are closed subsets such that  $|f|(T) \subset T'$ . Then it is clear that  $f$  defines a morphism of formal algebraic spaces

$$X_{/T} \longrightarrow X'_{/T'}$$

between the completions.

**Lemma 9.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$ . Let  $T \subset |Y|$  be a closed subset and let  $T' = |f|^{-1}(T) \subset |X|$ . Then  $X_{/T'} \rightarrow Y_{/T}$  is representable by algebraic spaces.*

**Proof.** Namely, suppose that  $V \rightarrow Y$  is a morphism from a scheme into  $Y$  such that  $|V|$  maps into  $T$ . Then  $V \times_Y X \rightarrow X$  is a morphism of algebraic spaces such that  $|V \times_Y X|$  maps into  $T'$ . Hence the functor  $V \times_{Y_{/T}} X_{/T}$  is represented by  $V \times_Y X$  and we see that the lemma holds.  $\square$



The following lemma is due to Ofer Gabber.

**Lemma 9.5.** *Let  $S$  be a scheme. Let  $X = \operatorname{Spec}(A)$  be an affine scheme over  $S$ . Let  $T \subset X$  be a closed subscheme.*

- (1) *If the formal completion  $X_{/T}$  is countably indexed and there exist countably many  $f_1, f_2, f_3, \dots \in A$  such that  $T = V(f_1, f_2, f_3, \dots)$ , then  $X_{/T}$  is adic\*.*
- (2) *The conclusion of (1) is wrong if we omit the assumption that  $T$  can be cut out by countably many functions in  $X$ .*

**Proof.** The assumption that  $X_{/T}$  is countably indexed means that there exists a sequence of ideals

$$A \supset J_1 \supset J_2 \supset J_3 \supset \dots$$

with  $V(J_n) = T$  such that every ideal  $J \subset A$  with  $V(J) = T$  there exists an  $n$  such that  $J \supset J_n$ .

To construct an example for (2) let  $\omega_1$  be the first uncountable ordinal. Let  $k$  be a field and let  $A$  be the  $k$ -algebra generated by  $x_\alpha$ ,  $\alpha \in \omega_1$  and  $y_{\alpha\beta}$  with  $\alpha \in \beta \in \omega_1$  subject to the relations  $x_\alpha = y_{\alpha\beta}x_\beta$ . Let  $T = V(x_\alpha)$ . Let  $J_n = (x_\alpha^n)$ . If  $J \subset A$  is an ideal such that  $V(J) = T$ , then  $x_\alpha^{n_\alpha} \in J$  for some  $n_\alpha \geq 1$ . One of the sets  $\{\alpha \mid n_\alpha = n\}$  must be unbounded in  $\omega_1$ . Then the relations imply that  $J_n \subset J$ .

To see that (2) holds it now suffices to show that  $A^\wedge = \lim A/J_n$  is not a ring complete with respect to a finitely generated ideal. For  $\gamma \in \omega_1$  let  $A_\gamma$  be the quotient of  $A$  by the ideal generated by  $x_\alpha$ ,  $\alpha \in \gamma$  and  $y_{\alpha\beta}$ ,  $\alpha \in \gamma$ . As  $A/J_1$  is reduced, every topologically nilpotent element  $f$  of  $\lim A/J_n$  is in  $J_1^\wedge = \lim J_1/J_n$ . This means  $f$  is an infinite series involving only a countable number of generators. Hence  $f$  dies in  $A_\gamma^\wedge = \lim A_\gamma/J_n A_\gamma$  for some  $\gamma$ . Note that  $A^\wedge \rightarrow A_\gamma^\wedge$  is continuous and open by Lemma 4.5. If the topology on  $A^\wedge$  was  $I$ -adic for some finitely generated ideal  $I \subset A^\wedge$ , then  $I$  would go to zero in some  $A_\gamma^\wedge$ . This would mean that  $A_\gamma^\wedge$  is discrete, which is not the case as there is a surjective continuous and open (by Lemma 4.5) map  $A_\gamma^\wedge \rightarrow k[[t]]$  given by  $x_\alpha \mapsto t$ ,  $y_{\alpha\beta} \mapsto 1$  for  $\gamma = \alpha$  or  $\gamma \in \alpha$ .

Before we prove (1) we first prove the following: If  $I \subset A^\wedge$  is a finitely generated ideal whose closure  $\bar{I}$  is open, then  $I = \bar{I}$ . Since  $V(J_n^2) = T$  there exists an  $m$  such that  $J_n^2 \supset J_m$ . Thus, we may assume that  $J_n^2 \supset J_{n+1}$  for all  $n$  by passing to a subsequence. Set  $J_n^\wedge = \lim_{k \geq n} J_n/J_k \subset A^\wedge$ . Since the closure  $\bar{I} = \bigcap (I + J_n^\wedge)$  (Lemma 4.2) is open we see that there exists an  $m$  such that  $I + J_n^\wedge \supset J_m^\wedge$  for all  $n \geq m$ . Fix such an  $m$ . We have

$$J_{n-1}^\wedge I + J_{n+1}^\wedge \supset J_{n-1}^\wedge (I + J_{n+1}^\wedge) \supset J_{n-1}^\wedge J_m^\wedge \supset J_n^\wedge$$

for all  $n \geq m+1$ . Namely, the first inclusion is trivial. The second was shown above. The third as  $J_{n-1} J_m \supset J_{n-1}^2 \supset J_n$ , hence  $J_{n-1}^\wedge J_m^\wedge \supset J_n^\wedge$ . Say  $I = (g_1, \dots, g_t)$ . Pick  $f \in J_{m+1}^\wedge$ . Using the displayed inclusions above, valid for all  $n \geq m+1$ , we can write by induction on  $c \geq 0$

$$f = \sum f_{i,c} g_i \mod J_{m+1+c}^\wedge$$

with  $f_{i,c} \in J_m^\wedge$  and  $f_{i,c} \equiv f_{i,c-1} \mod J_{m+c}^\wedge$ . It follows that  $I J_m^\wedge \supset J_{m+1}^\wedge$ . Combined with  $I + J_{m+1}^\wedge \supset J_m^\wedge$  we conclude that  $I$  is open.

Proof of (1). Assume  $T = V(f_1, f_2, f_3, \dots)$ . Let  $I_m \subset A^\wedge$  be the ideal generated by  $f_1, \dots, f_m$ . Case I: For some  $m$  the closure of  $I_m$  is open. Then  $I_m$  is open by the result of the previous paragraph. Since in  $A^\wedge$  the product of open ideals is open,

we see that  $I_m^k$  is open for all  $k$ . As each element of  $I_m$  is topologically nilpotent, we conclude that  $I_m$  is an ideal of definition which proves that  $A^\wedge$  is adic with a finitely generated ideal of definition, i.e.,  $X$  is adic\*.

Case II. For all  $m$  the closure  $\bar{I}_m$  of  $I_m$  is not open. Then the topology on  $A^\wedge/\bar{I}_m$  is not discrete. This means we can pick  $\phi(m) \geq m$  such that

$$\text{Im}(J_{\phi(m)} \rightarrow A/(f_1, \dots, f_m)) \neq \text{Im}(J_{\phi(m)+1} \rightarrow A/(f_1, \dots, f_m))$$

To see this we have used that  $A^\wedge/(\bar{I}_m + J_n^\wedge) = A/((f_1, \dots, f_m) + J_n)$ . Choose exponents  $e_i > 0$  such that  $f_i^{e_i} \in J_{\phi(m)+1}$  for  $0 < m < i$ . Let  $J = (f_1^{e_1}, f_2^{e_2}, f_3^{e_3}, \dots)$ . Then  $V(J) = T$ . We claim that  $J \not\supset J_n$  for all  $n$  which is a contradiction proving Case II does not occur. Namely, the image of  $J$  in  $A/(f_1, \dots, f_m)$  is contained in the image of  $J_{\phi(m)+1}$  which is properly contained in the image of  $J_m$ .  $\square$

## 10. Fibre products

Obligatory section about fibre products of formal algebraic spaces.

**Lemma 10.1.** *Let  $S$  be a scheme. Let  $\{X_i \rightarrow X\}_{i \in I}$  be a family of maps of sheaves on  $(\text{Sch}/S)_{fppf}$ . Assume (a)  $X_i$  is a formal algebraic space over  $S$ , (b)  $X_i \rightarrow X$  is representable by algebraic spaces and étale, and (c)  $\coprod X_i \rightarrow X$  is a surjection of sheaves. Then  $X$  is a formal algebraic space over  $S$ .*

**Proof.** For each  $i$  pick  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$  as in Definition 7.1. Then  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is a family as in Definition 7.1 for  $X$ .  $\square$

**Lemma 10.2.** *Let  $S$  be a scheme. Let  $X, Y$  be formal algebraic spaces over  $S$  and let  $Z$  be a sheaf whose diagonal is representable by algebraic spaces. Then  $X \times_Z Y$  is a formal algebraic space.*

**Proof.** Choose  $\{X_i \rightarrow X\}$  and  $\{Y_j \rightarrow Y\}$  as in Definition 7.1. Then  $\{X_i \times_Z Y_j \rightarrow X \times_Z Y\}$  is a family of maps which are representable by algebraic spaces and étale. Thus Lemma 10.1 tells us it suffices to show that  $X \times_Z Y$  is a formal algebraic space when  $X$  and  $Y$  are affine formal algebraic spaces.

Assume  $X$  and  $Y$  are affine formal algebraic spaces. Write  $X = \text{colim } X_\lambda$  and  $Y = \text{colim } Y_\mu$  as in Definition 5.1. Then  $X \times_Z Y = \text{colim } X_\lambda \times_Z Y_\mu$ . Each  $X_\lambda \times_Z Y_\mu$  is an algebraic space. For  $\lambda \leq \lambda'$  and  $\mu \leq \mu'$  the morphism

$$X_\lambda \times_Z Y_\mu \rightarrow X_\lambda \times_Z Y_{\mu'} \rightarrow X_{\lambda'} \times_Z Y_{\mu'}$$

is a thickening as a composition of base changes of thickenings. Thus we conclude by applying Lemma 8.1.  $\square$

**Lemma 10.3.** *Let  $S$  be a scheme. The category of formal algebraic spaces over  $S$  has fibre products.*

**Proof.** Special case of Lemma 10.2 because formal algebraic spaces have representable diagonals, see Lemma 7.3.  $\square$

We have already proved the following lemma (without knowing that fibre products exist).

**Lemma 10.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . The diagonal morphism  $\Delta : X \rightarrow X \times_Y X$  is representable (by schemes), a monomorphism, locally quasi-finite, locally of finite type, and separated.*

**Proof.** Let  $T$  be a scheme and let  $T \rightarrow X \times_Y X$  be a morphism. Then

$$T \times_{(X \times_Y X)} X = T \times_{(X \times_S X)} X$$

Hence the result follows immediately from Lemma 7.3.  $\square$

### 11. Separation axioms for formal algebraic spaces

This section is about “absolute” separation conditions on formal algebraic spaces. We will discuss separation conditions for morphisms of formal algebraic spaces later.

**Lemma 11.1.** *Let  $S$  be a scheme. Let  $X$  be a formal algebraic space over  $S$ . The following are equivalent*

- (1) *the reduction of  $X$  (Lemma 7.2) is a quasi-separated algebraic space,*
- (2) *for  $U \rightarrow X$ ,  $V \rightarrow X$  with  $U$ ,  $V$  quasi-compact schemes the fibre product  $U \times_X V$  is quasi-compact,*
- (3) *for  $U \rightarrow X$ ,  $V \rightarrow X$  with  $U$ ,  $V$  affine the fibre product  $U \times_X V$  is quasi-compact.*

**Proof.** Observe that  $U \times_X V$  is a scheme by Lemma 7.3. Let  $U_{red}, V_{red}, X_{red}$  be the reduction of  $U, V, X$ . Then

$$U_{red} \times_{X_{red}} V_{red} = U_{red} \times_X V_{red} \rightarrow U \times_X V$$

is a thickening of schemes. From this the equivalence of (1) and (2) is clear, keeping in mind the analogous lemma for algebraic spaces, see Properties of Spaces, Lemma 3.3. We omit the proof of the equivalence of (2) and (3).  $\square$

**Lemma 11.2.** *Let  $S$  be a scheme. Let  $X$  be a formal algebraic space over  $S$ . The following are equivalent*

- (1) *the reduction of  $X$  (Lemma 7.2) is a separated algebraic space,*
- (2) *for  $U \rightarrow X$ ,  $V \rightarrow X$  with  $U$ ,  $V$  affine the fibre product  $U \times_X V$  is affine and*

$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$

*is surjective.*

**Proof.** If (2) holds, then  $X_{red}$  is a separated algebraic space by applying Properties of Spaces, Lemma 3.3 to morphisms  $U \rightarrow X_{red}$  and  $V \rightarrow X_{red}$  with  $U, V$  affine and using that  $U \times_{X_{red}} V = U \times_X V$ .

Assume (1). Let  $U \rightarrow X$  and  $V \rightarrow X$  be as in (2). Observe that  $U \times_X V$  is a scheme by Lemma 7.3. Let  $U_{red}, V_{red}, X_{red}$  be the reduction of  $U, V, X$ . Then

$$U_{red} \times_{X_{red}} V_{red} = U_{red} \times_X V_{red} \rightarrow U \times_X V$$

is a thickening of schemes. It follows that  $(U \times_X V)_{red} = (U_{red} \times_{X_{red}} V_{red})_{red}$ . In particular, we see that  $(U \times_X V)_{red}$  is an affine scheme and that

$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}((U \times_X V)_{red})$$

is surjective, see Properties of Spaces, Lemma 3.3. Then  $U \times_X V$  is affine by Limits of Spaces, Proposition 15.2. On the other hand, the morphism  $U \times_X V \rightarrow U \times V$  of affine schemes is the composition

$$U \times_X V = X \times_{(X \times_S X)} (U \times_S V) \rightarrow U \times_S V \rightarrow U \times V$$

The first morphism is a monomorphism and locally of finite type (Lemma 7.3). The second morphism is an immersion (Schemes, Lemma 21.10). Hence the composition is a monomorphism which is locally of finite type. On the other hand, the composition is integral as the map on underlying reduced affine schemes is a closed immersion by the above and hence universally closed (use Morphisms, Lemma 44.7). Thus the ring map

$$\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \longrightarrow \mathcal{O}(U \times_X V)$$

is an epimorphism which is integral of finite type hence finite hence surjective (use Morphisms, Lemma 44.4 and Algebra, Lemma 103.6).  $\square$

**Definition 11.3.** Let  $S$  be a scheme. Let  $X$  be a formal algebraic space over  $S$ . We say

- (1)  $X$  is *quasi-separated* if the equivalent conditions of Lemma 11.1 are satisfied.
- (2)  $X$  is *separated* if the equivalent conditions of Lemma 11.2 are satisfied.

The following lemma implies in particular that the completed tensor product of weakly admissible topological rings is a weakly admissible topological ring.

**Lemma 11.4.** *Let  $S$  be a scheme. Let  $X \rightarrow Z$  and  $Y \rightarrow Z$  be morphisms of formal algebraic spaces over  $S$ . Assume  $Z$  separated.*

- (1) *If  $X$  and  $Y$  are affine formal algebraic spaces, then so is  $X \times_Z Y$ .*
- (2) *If  $X$  and  $Y$  are McQuillan affine formal algebraic spaces, then so is  $X \times_Z Y$ .*
- (3) *If  $X$ ,  $Y$ , and  $Z$  are McQuillan formal algebraic spaces corresponding to the weakly admissible topological  $S$ -algebras  $A$ ,  $B$ , and  $C$ , then  $X \times_Z Y$  corresponds to  $A \hat{\otimes}_C B$ .*

**Proof.** Write  $X = \text{colim } X_\lambda$  and  $Y = \text{colim } Y_\mu$  as in Definition 5.1. Then  $X \times_Z Y = \text{colim } X_\lambda \times_Z Y_\mu$ . Since  $Z$  is separated the fibre products are affine, hence we see that (1) holds. Assume  $X$  and  $Y$  corresponds to the weakly admissible topological  $S$ -algebras  $A$  and  $B$  and  $X_\lambda = \text{Spec}(A/I_\lambda)$  and  $Y_\mu = \text{Spec}(B/J_\mu)$ . Then

$$X_\lambda \times_Z Y_\mu \rightarrow X_\lambda \times Y_\mu \rightarrow \text{Spec}(A \otimes B)$$

is a closed immersion. Thus one of the conditions of Lemma 5.6 holds and we conclude that  $X \times_Z Y$  is McQuillan. If also  $Z$  is McQuillan corresponding to  $C$ , then

$$X_\lambda \times_Z Y_\mu = \text{Spec}(A/I_\lambda \otimes_C B/J_\mu)$$

hence we see that the weakly admissible topological ring corresponding to  $X \times_Z Y$  is the completed tensor product (see Definition 4.6).  $\square$

**Lemma 11.5.** *Let  $S$  be a scheme. Let  $X$  be a formal algebraic space over  $S$ . Let  $U \rightarrow X$  be a morphism where  $U$  is a separated algebraic space over  $S$ . Then  $U \rightarrow X$  is separated.*

**Proof.** The statement makes sense because  $U \rightarrow X$  is representable by algebraic spaces (Lemma 7.4). Let  $T$  be a scheme and  $T \rightarrow X$  a morphism. We have to show that  $U \times_X T \rightarrow T$  is separated. Since  $U \times_X T \rightarrow U \times_S T$  is a monomorphism, it suffices to show that  $U \times_S T \rightarrow T$  is separated. As this is the base change of  $U \rightarrow S$  this follows. We used in the argument above: Morphisms of Spaces, Lemmas 4.4, 4.8, 10.3, and 4.11.  $\square$

## 12. Quasi-compact formal algebraic spaces

Here is the characterization of quasi-compact formal algebraic spaces.

**Lemma 12.1.** *Let  $S$  be a scheme. Let  $X$  be a formal algebraic space over  $S$ . The following are equivalent*

- (1) *the reduction of  $X$  (Lemma 7.2) is a quasi-compact algebraic space,*
- (2) *we can find  $\{X_i \rightarrow X\}_{i \in I}$  as in Definition 7.1 with  $I$  finite,*
- (3) *there exists a morphism  $Y \rightarrow X$  representable by algebraic spaces which is étale and surjective and where  $Y$  is an affine formal algebraic space.*

**Proof.** Omitted. □

**Definition 12.2.** Let  $S$  be a scheme. Let  $X$  be a formal algebraic space over  $S$ . We say  $X$  is *quasi-compact* if the equivalent conditions of Lemma 12.1 are satisfied.

**Lemma 12.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . The following are equivalent*

- (1) *the induced map  $f_{\text{red}} : X_{\text{red}} \rightarrow Y_{\text{red}}$  between reductions (Lemma 7.2) is a quasi-compact morphism of algebraic spaces,*
- (2) *for every quasi-compact scheme  $T$  and morphism  $T \rightarrow Y$  the fibre product  $X \times_Y T$  is a quasi-compact formal algebraic space,*
- (3) *for every affine scheme  $T$  and morphism  $T \rightarrow Y$  the fibre product  $X \times_Y T$  is a quasi-compact formal algebraic space, and*
- (4) *there exists a covering  $\{Y_j \rightarrow Y\}$  as in Definition 7.1 such that each  $X \times_Y Y_j$  is a quasi-compact formal algebraic space.*

**Proof.** Omitted. □

**Definition 12.4.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . We say  $f$  is *quasi-compact* if the equivalent conditions of Lemma 12.3 are satisfied.

This agrees with the already existing notion when the morphism is representable by algebraic spaces (and in particular when it is representable).

**Lemma 12.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$  which is representable by algebraic spaces. Then  $f$  is quasi-compact in the sense of Definition 12.4 if and only if  $f$  is quasi-compact in the sense of Bootstrap, Definition 4.1.*

**Proof.** This is immediate from the definitions and Lemma 12.3. □

## 13. Quasi-compact and quasi-separated formal algebraic spaces

The following result is due to Yasuda, see [Yas09, Proposition 3.32].

**Lemma 13.1.** *Let  $S$  be a scheme. Let  $X$  be a quasi-compact and quasi-separated formal algebraic space over  $S$ . Then  $X = \text{colim } X_\lambda$  for a system of algebraic spaces  $(X_\lambda, f_{\lambda\mu})$  over a directed partially ordered set  $\Lambda$  where each  $f_{\lambda\mu} : X_\lambda \rightarrow X_\mu$  is a thickening.*

**Proof.** By Lemma 12.1 we may choose an affine formal algebraic space  $Y$  and a representable surjective étale morphism  $Y \rightarrow X$ . Write  $Y = \text{colim } Y_\lambda$  as in Definition 5.1.

Pick  $\lambda \in \Lambda$ . Then  $Y_\lambda \times_X Y$  is a scheme by Lemma 5.11. The reduction (Lemma 7.2) of  $Y_\lambda \times_X Y$  is equal to the reduction of  $Y_{red} \times_{X_{red}} Y_{red}$  which is quasi-compact as  $X$  is quasi-separated and  $Y_{red}$  is affine. Therefore  $Y_\lambda \times_X Y$  is a quasi-compact scheme. Hence there exists a  $\mu \geq \lambda$  such that  $\text{pr}_2 : Y_\lambda \times_X Y \rightarrow Y$  factors through  $Y_\mu$ , see Lemma 5.4. Let  $Z_\lambda$  be the scheme theoretic image of the morphism  $\text{pr}_2 : Y_\lambda \times_X Y \rightarrow Y_\mu$ . This is independent of the choice of  $\mu$  and we can and will think of  $Z_\lambda \subset Y$  as the scheme theoretic image of the morphism  $\text{pr}_2 : Y_\lambda \times_X Y \rightarrow Y$ . Observe that  $Z_\lambda$  is also equal to the scheme theoretic image of the morphism  $\text{pr}_1 : Y \times_X Y_\lambda \rightarrow Y$  since this is isomorphic to the morphism used to define  $Z_\lambda$ . We claim that  $Z_\lambda \times_X Y = Y \times_X Z_\lambda$  as subfunctors of  $Y \times_X Y$ . Namely, since  $Y \rightarrow X$  is étale we see that  $Z_\lambda \times_X Y$  is the scheme theoretic image of the morphism

$$\text{pr}_{13} = \text{pr}_1 \times \text{id}_Y : Y \times_X Y_\lambda \times_X Y \longrightarrow Y \times_X Y$$

by Morphisms of Spaces, Lemma 16.3. By the same token,  $Y \times_X Z_\lambda$  is the scheme theoretic image of the morphism

$$\text{pr}_{13} = \text{id}_Y \times \text{pr}_2 : Y \times_X Y_\lambda \times_X Y \longrightarrow Y \times_X Y$$

The claim follows. Then  $R_\lambda = Z_\lambda \times_X Y = Y \times_X Z_\lambda$  together with the morphism  $R_\lambda \rightarrow Z_\lambda \times_S Z_\lambda$  defines an étale equivalence relation. In this way we obtain an algebraic space  $X_\lambda = Z_\lambda / R_\lambda$ . By construction the diagram

$$\begin{array}{ccc} Z_\lambda & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_\lambda & \longrightarrow & X \end{array}$$

is cartesian (because  $X$  is the coequalizer of the two projections  $R = Y \times_X Y \rightarrow Y$ , because  $Z_\lambda \subset Y$  is  $R$ -invariant, and because  $R_\lambda$  is the restriction of  $R$  to  $Z_\lambda$ ). Hence  $X_\lambda \rightarrow X$  is representable and a closed immersion, see Spaces, Lemma 11.3. On the other hand, since  $Y_\lambda \subset Z_\lambda$  we see that  $(X_\lambda)_{red} = X_{red}$ , in other words,  $X_\lambda \rightarrow X$  is a thickening. Finally, we claim that

$$X = \text{colim } X_\lambda$$

We have  $Y \times_X X_\lambda = Z_\lambda \supset Y_\lambda$ . Every morphism  $T \rightarrow X$  where  $T$  is a scheme over  $S$  lifts étale locally to a morphism into  $Y$  which lifts étale locally into a morphism into some  $Y_\lambda$ . Hence  $T \rightarrow X$  lifts étale locally on  $T$  to a morphism into  $X_\lambda$ . This finishes the proof.  $\square$

**Remark 13.2.** In this remark we translate the statement and proof of Lemma 13.1 into the language of formal schemes à la EGA. Looking at Remark 8.3 we see that the lemma can be translated as follows

- (\*) Every quasi-compact and quasi-separated formal scheme has a fundamental system of ideals of definition.

To prove this we first use the induction principle (reformulated for quasi-compact and quasi-separated formal schemes) of Cohomology of Schemes, Lemma 4.1 to reduce to the following situation:  $\mathfrak{X} = \mathfrak{U} \cup \mathfrak{V}$  with  $\mathfrak{U}, \mathfrak{V}$  open formal subschemes, with  $\mathfrak{V}$  affine, and the result is true for  $\mathfrak{U}, \mathfrak{V}$ , and  $\mathfrak{U} \cap \mathfrak{V}$ . Pick any ideals of definition  $\mathcal{I} \subset \mathcal{O}_{\mathfrak{U}}$  and  $\mathcal{J} \subset \mathcal{O}_{\mathfrak{V}}$ . By our assumption that we have a fundamental system of

ideals of definition on  $\mathfrak{U}$  and  $\mathfrak{V}$  and because  $\mathfrak{U} \cap \mathfrak{V}$  is quasi-compact, we can find ideals of definition  $\mathcal{I}' \subset \mathcal{I}$  and  $\mathcal{J}' \subset \mathcal{J}$  such that

$$\mathcal{I}'|_{\mathfrak{U} \cap \mathfrak{V}} \subset \mathcal{J}|_{\mathfrak{U} \cap \mathfrak{V}} \quad \text{and} \quad \mathcal{J}'|_{\mathfrak{U} \cap \mathfrak{V}} \subset \mathcal{I}|_{\mathfrak{U} \cap \mathfrak{V}}$$

Let  $U \rightarrow U' \rightarrow \mathfrak{U}$  and  $V \rightarrow V' \rightarrow \mathfrak{V}$  be the closed immersions determined by the ideals of definition  $\mathcal{I}' \subset \mathcal{I} \subset \mathcal{O}_{\mathfrak{U}}$  and  $\mathcal{J}' \subset \mathcal{J} \subset \mathcal{O}_{\mathfrak{V}}$ . Let  $\mathfrak{U} \cap V$  denote the open subscheme of  $V$  whose underlying topological space is that of  $\mathfrak{U} \cap \mathfrak{V}$ . Similarly for  $U \cap \mathfrak{V}$ . Then we consider

$$Z_U = \text{scheme theoretic image of } U \amalg (\mathfrak{U} \cap V) \longrightarrow U'$$

and

$$Z_V = \text{scheme theoretic image of } (U \cap \mathfrak{V}) \amalg V \longrightarrow V'$$

Since taking scheme theoretic images of quasi-compact morphisms commutes with restriction to opens (Morphisms, Lemma 6.3) we see that  $Z_U \cap \mathfrak{V} = \mathfrak{U} \cap Z_V$ . Thus  $Z_U$  and  $Z_V$  glue to a scheme  $Z$  which comes equipped with a morphism  $Z \rightarrow \mathfrak{X}$ . Analogous to the discussion in Remark 8.2 we see that  $Z$  corresponds to a weak ideal of definition  $\mathcal{I}_Z \subset \mathcal{O}_{\mathfrak{X}}$ . Note that  $U \subset Z_U \subset Z$  and that  $V \subset Z_V \subset Z$ . Thus the collection of all  $\mathcal{I}_Z$  constructed in this manner forms a fundamental system of weak ideals of definition. Hence a subfamily gives a fundamental system of ideals of definition, see Remark 8.3.

#### 14. Morphisms representable by algebraic spaces

Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces which is representable by algebraic spaces. For these types of morphisms we have a lot of theory at our disposal, thanks to the work done in the chapters on algebraic spaces.

**Lemma 14.1.** *The composition of morphisms representable by algebraic spaces is representable by algebraic spaces. The same holds for representable (by schemes).*

**Proof.** See Bootstrap, Lemma 3.8.  $\square$

**Lemma 14.2.** *A base change of a morphism representable by algebraic spaces is representable by algebraic spaces. The same holds for representable (by schemes).*

**Proof.** See Bootstrap, Lemma 3.3.  $\square$

**Lemma 14.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be morphisms of formal algebraic spaces over  $S$ . If  $g \circ f : X \rightarrow Z$  is representable by algebraic spaces, then  $f : X \rightarrow Y$  is representable by algebraic spaces.*

**Proof.** Note that the diagonal of  $Y \rightarrow Z$  is representable by Lemma 10.4. Thus  $U \rightarrow V$  is representable by algebraic spaces by Bootstrap, Lemma 3.10.  $\square$

The property of being representable by algebraic spaces is local on the source and the target.

**Lemma 14.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . The following are equivalent:*

- (1) *the morphism  $f$  is representable by algebraic spaces,*

- (2) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where  $U, V$  are formal algebraic spaces, the vertical arrows are representable by algebraic spaces,  $U \rightarrow X$  is surjective étale, and  $U \rightarrow V$  is representable by algebraic spaces,

- (3) *for any commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where  $U, V$  are formal algebraic spaces and the vertical arrows are representable by algebraic spaces, the morphism  $U \rightarrow V$  is representable by algebraic spaces,

- (4) *there exists a covering  $\{Y_j \rightarrow Y\}$  as in Definition 7.1 and for each  $j$  a covering  $\{X_{ji} \rightarrow Y_j \times_Y X\}$  as in Definition 7.1 such that  $X_{ji} \rightarrow Y_j$  is representable by algebraic spaces for each  $j$  and  $i$ ,*
- (5) *there exist a covering  $\{X_i \rightarrow X\}$  as in Definition 7.1 and for each  $i$  a factorization  $X_i \rightarrow Y_i \rightarrow Y$  where  $Y_i$  is an affine formal algebraic space,  $Y_i \rightarrow Y$  is representable by algebraic spaces, such that  $X_i \rightarrow Y_i$  is representable by algebraic spaces, and*
- (6) *add more here.*

**Proof.** It is clear that (1) implies (2) because we can take  $U = X$  and  $V = Y$ . Conversely, (2) implies (1) by Bootstrap, Lemma 11.3 applied to  $U \rightarrow X \rightarrow Y$ .

Assume (1) is true and consider a diagram as in (3). Then  $U \rightarrow Y$  is representable by algebraic spaces (as the composition  $U \rightarrow X \rightarrow Y$ , see Bootstrap, Lemma 3.8) and factors through  $V$ . Thus  $U \rightarrow V$  is representable by algebraic spaces by Lemma 14.3.

It is clear that (3) implies (2). Thus now (1) – (3) are equivalent.

Observe that the condition in (4) makes sense as the fibre product  $Y_j \times_Y X$  is a formal algebraic space by Lemma 10.3. It is clear that (4) implies (5).

Assume  $X_i \rightarrow Y_i \rightarrow Y$  as in (5). Then we set  $V = \coprod Y_i$  and  $U = \coprod X_i$  to see that (5) implies (2).

Finally, assume (1) – (3) are true. Thus we can choose any covering  $\{Y_j \rightarrow Y\}$  as in Definition 7.1 and for each  $j$  any covering  $\{X_{ji} \rightarrow Y_j \times_Y X\}$  as in Definition 7.1. Then  $X_{ij} \rightarrow Y_j$  is representable by algebraic spaces by (3) and we see that (4) is true. This concludes the proof.  $\square$

**Lemma 14.5.** *Let  $S$  be a scheme. Let  $Y$  be an affine formal algebraic space over  $S$ . Let  $f : X \rightarrow Y$  be a map of sheaves on  $(\text{Sch}/S)_{\text{fppf}}$  which is representable by algebraic spaces. Then  $X$  is a formal algebraic space.*



**Proof.** Write  $Y = \operatorname{colim} Y_\lambda$  as in Definition 5.1. For each  $\lambda$  the fibre product  $X \times_Y Y_\lambda$  is an algebraic space. Hence  $X = \operatorname{colim} X \times_Y Y_\lambda$  is a formal algebraic space by Lemma 8.1.  $\square$

**Lemma 14.6.** *Let  $S$  be a scheme. Let  $Y$  be a formal algebraic space over  $S$ . Let  $f : X \rightarrow Y$  be a map of sheaves on  $(\operatorname{Sch}/S)_{fppf}$  which is representable by algebraic spaces. Then  $X$  is a formal algebraic space.*

**Proof.** Let  $\{Y_i \rightarrow Y\}$  be as in Definition 7.1. Then  $X \times_Y Y_i \rightarrow X$  is a family of morphisms representable by algebraic spaces, étale, and jointly surjective. Thus it suffices to show that  $X \times_Y Y_i$  is a formal algebraic space, see Lemma 10.1. This follows from Lemma 14.5.  $\square$

**Lemma 14.7.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of affine formal algebraic spaces which is representable by algebraic spaces. Then  $f$  is representable (by schemes) and affine.*

**Proof.** The first assertion follows from Lemma 5.11. Write  $Y = \operatorname{colim} Y_\mu$  and  $X = \operatorname{colim} X_\lambda$  as in Definition 5.1. For the second, let  $T \rightarrow Y$  be a morphism where  $T$  is a scheme over  $S$ . We have to show that  $X \times_Y T \rightarrow T$  is affine, see Spaces, Definition 5.1. To do this we may assume that  $T$  is affine and we have to prove that  $X \times_Y T$  is affine. In this case  $T \rightarrow Y$  factors through  $Y_\mu \rightarrow Y$  for some  $\mu$ , see Lemma 5.4. Since  $f$  is quasi-compact we see that  $X \times_Y T$  is quasi-compact (Lemma 12.3). Hence  $X \times_Y T \rightarrow X$  factors through  $X_\lambda$  for some  $\lambda$ . Similarly  $X_\lambda \rightarrow Y$  factors through  $Y_\mu$  after increasing  $\mu$ . Then  $X \times_Y T = X_\lambda \times_{Y_\mu} T$ . We conclude as fibre products of affine schemes are affine.  $\square$

**Lemma 14.8.** *Let  $S$  be a scheme. Let  $Y$  be an affine formal algebraic space. Let  $f : X \rightarrow Y$  be a map of sheaves on  $(\operatorname{Sch}/S)_{fppf}$  which is representable and affine. Then*

- (1)  $X$  is an affine formal algebraic space.
- (2) if  $Y$  is countably indexed, then  $X$  is countably indexed.
- (3) if  $Y$  is  $\text{adic}^*$ , then  $X$  is  $\text{adic}^*$ ,
- (4) if  $Y$  is Noetherian and  $f$  is (locally) of finite type, then  $X$  is Noetherian.

**Proof.** Proof of (1). Write  $Y = \operatorname{colim}_{\lambda \in \Lambda} Y_\lambda$  as in Definition 5.1. Since  $f$  is representable and affine, the fibre products  $X_\lambda = Y_\lambda \times_Y X$  are affine. And  $X = \operatorname{colim} Y_\lambda \times_Y X$ . Thus  $X$  is an affine formal algebraic space.

Proof of (2). If  $Y$  is countably indexed, then in the argument above we may assume  $\Lambda$  is countable. Then we immediately see that  $X$  is countably indexed too.

Proof of (3). Assume  $Y$  is  $\text{adic}^*$ . Then  $Y = \operatorname{Spf}(B)$  for some adic topological ring  $B$  which has a finitely generated ideal  $J$  such that  $\{J^n\}$  is a fundamental system of open ideals. Of course, then  $Y = \operatorname{colim} \operatorname{Spec}(B/J^n)$ . The schemes  $X \times_Y \operatorname{Spec}(B/J^n)$  are affine and we can write  $X \times_Y \operatorname{Spec}(B/J^n) = \operatorname{Spec}(A_n)$ . Then  $X = \operatorname{colim} \operatorname{Spec}(A_n)$ . The  $B$ -algebra maps  $A_{n+1} \rightarrow A_n$  are surjective and induce isomorphisms  $A_{n+1}/J^n A_{n+1} \rightarrow A_n$ . By Algebra, Lemma 94.1 the ring  $A = \lim A_n$  is  $J$ -adically complete and  $A/J^n A = A_n$ . Hence  $X = \operatorname{Spf}(A^\wedge)$  is  $\text{adic}^*$ .

Proof of (4). Combining (3) with Lemma 6.3 we see that  $X$  is  $\text{adic}^*$ . Thus we can use the criterion of Lemma 6.5. First, it tells us the affine schemes  $Y_\lambda$  are

Noetherian. Then  $X_\lambda \rightarrow Y_\lambda$  is of finite type, hence  $X_\lambda$  is Noetherian too (Morphisms, Lemma 16.6). Then the criterion tells us  $X$  is Noetherian and the proof is complete.  $\square$

**Lemma 14.9.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of affine formal algebraic spaces which is representable by algebraic spaces. Then*

- (1) *if  $Y$  is countably indexed, then  $X$  is countably indexed.*
- (2) *if  $Y$  is  $\text{adic}^*$ , then  $X$  is  $\text{adic}^*$ ,*
- (3) *if  $Y$  is Noetherian and  $f$  is (locally) of finite type, then  $X$  is Noetherian.*

**Proof.** Combine Lemmas 14.7 and 14.8.  $\square$

**Lemma 14.10.** *Let  $S$  be a scheme. Let  $\varphi : A \rightarrow B$  be a continuous map of weakly admissible topological rings over  $S$ . The following are equivalent*

- (1)  *$\text{Spf}(\varphi) : \text{Spf}(B) \rightarrow \text{Spf}(A)$  is representable by algebraic spaces,*
- (2)  *$\text{Spf}(\varphi) : \text{Spf}(B) \rightarrow \text{Spf}(A)$  is representable (by schemes),*
- (3)  *$\varphi$  is taut, see Definition 4.9.*

**Proof.** Parts (1) and (2) are equivalent by Lemma 14.7.

Assume the equivalent conditions (1) and (2) hold. If  $I \subset A$  is a weak ideal of definition, then  $\text{Spec}(A/I) \rightarrow \text{Spf}(A)$  is representable and a thickening (this is clear from the construction of the formal spectrum but it also follows from Lemma 5.6). Then  $\text{Spec}(A/I) \times_{\text{Spf}(A)} \text{Spf}(B) \rightarrow \text{Spf}(B)$  is representable and a thickening as a base change. Hence by Lemma 5.6 there is a weak ideal of definition  $J(I) \subset B$  such that  $\text{Spec}(A/I) \times_{\text{Spf}(A)} \text{Spf}(B) = \text{Spec}(B/J(I))$  as subfunctors of  $\text{Spf}(B)$ . We obtain a cartesian diagram

$$\begin{array}{ccc} \text{Spec}(B/J(I)) & \longrightarrow & \text{Spec}(A/I) \\ \downarrow & & \downarrow \\ \text{Spf}(B) & \longrightarrow & \text{Spf}(A) \end{array}$$

By Lemma 11.4 we see that  $B/J(I) = \widehat{B \otimes_A A/I}$ . It follows that  $J(I)$  is the closure of the ideal  $\varphi(I)B$ , see Lemma 4.11. Since  $\text{Spf}(A) = \text{colim Spec}(A/I)$  with  $I$  as above, we find that  $\text{Spf}(B) = \text{colim Spec}(B/J(I))$ . Thus the ideals  $J(I)$  form a fundamental system of weak ideals of definition (see Lemma 5.6). Hence (3) holds.

Assume (3) holds. For a weak ideal of definition  $I \subset A$  denote  $J(I)$  the closure of  $\varphi(I)B$ . By Lemma 4.8 the ideal  $J(I)$  is a weak ideal of definition of  $B$ . Using Lemmas 11.4 and 4.11 we see that the diagram displayed above is cartesian. Since every morphism  $T \rightarrow \text{Spf}(A)$  with  $T$  quasi-compact factors through  $\text{Spec}(A/I)$  for some weak ideal of definition  $I$  (Lemma 5.4) we conclude that  $\text{Spf}(\varphi)$  is representable, i.e., (2) holds. This finishes the proof.  $\square$

**Example 14.11.** Let  $B$  be a weakly admissible topological ring. Let  $B \rightarrow A$  be a ring map (no topology). Then we can consider

$$A^\wedge = \lim A/JA$$

where the limit is over all weak ideals of definition  $J$  of  $B$ . Then  $A^\wedge$  (endowed with the limit topology) is a complete linearly topologized ring. The (open) kernel  $I$  of the surjection  $A^\wedge \rightarrow A/JA$  is the closure of  $JA^\wedge$ , see Lemma 4.2. By Lemma 4.8 we see that  $I$  consists of topologically nilpotent elements. Thus  $I$  is a weak ideal of

definition of  $A^\wedge$  and we conclude  $A^\wedge$  is a weakly admissible topological ring. Thus  $\varphi : B \rightarrow A^\wedge$  is taut map of weakly admissible topological rings and

$$\mathrm{Spf}(A^\wedge) \longrightarrow \mathrm{Spf}(B)$$

is a special case of the phenomenon studied in Lemma 14.10.

**Remark 14.12** (Warning). Lemma 14.10 does **not** mean that given a morphism  $f : X \rightarrow Y$  of affine formal algebraic spaces with  $f$  representable and  $Y$  McQuillan we have that  $X$  is McQuillan.

The warning above notwithstanding, we do have the following result.

**Lemma 14.13.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of affine formal algebraic spaces over  $S$ . Assume*

- (1)  *$Y$  is McQuillan, i.e., equal to  $\mathrm{Spf}(B)$  for some weakly admissible topological  $S$ -algebra  $B$ , and*
- (2)  *$f : Y \rightarrow X$  is representable by algebraic spaces and étale.*

*Then there exists an étale ring map  $B \rightarrow A$  such that*

$$Y = \mathrm{Spf}(A^\wedge) \quad \text{where} \quad A^\wedge = \lim A/JA$$

*with  $J \subset B$  running over the weak ideals of definition of  $B$ . In particular,  $Y$  is McQuillan.*

**Proof.** Choose a weak ideal of definition  $J_0 \subset B$ . Set  $Y_0 = \mathrm{Spec}(B/J_0)$  and  $X_0 = Y_0 \times_Y X$ . Then  $X_0 \rightarrow Y_0$  is an étale morphism of affine schemes (see Lemma 14.7). Say  $X_0 = \mathrm{Spec}(A_0)$ . By Algebra, Lemma 138.11 we can find an étale algebra map  $B \rightarrow A$  such that  $A_0 \cong A/J_0A$ . Consider an ideal of definition  $J \subset J_0$ . As above we may write  $\mathrm{Spec}(B/J) \times_Y X = \mathrm{Spec}(\bar{A})$  for some étale ring map  $B/J \rightarrow \bar{A}$ . Then both  $B/J \rightarrow \bar{A}$  and  $B/J \rightarrow A/JA$  are étale ring maps lifting the étale ring map  $B/J_0 \rightarrow A_0$ . By More on Algebra, Lemma 7.2 there is a unique  $B/J$ -algebra isomorphism  $\varphi_J : A/JA \rightarrow \bar{A}$  lifting the identification modulo  $J_0$ . Since the maps  $\varphi_J$  are unique they are compatible for varying  $J$ . Thus

$$X = \mathrm{colim} \mathrm{Spec}(B/J) \times_Y X = \mathrm{colim} \mathrm{Spec}(A/JA)$$

and we see that the lemma holds.  $\square$

**Lemma 14.14.** *With notation and assumptions as in Lemma 14.13. The following are equivalent*

- (1)  *$f : X \rightarrow Y$  is surjective,*
- (2)  *$B \rightarrow A$  is faithfully flat,*
- (3) *for every weak ideal of definition  $J \subset B$  the ring map  $B \rightarrow A/JA$  is faithfully flat, and*
- (4) *for some weak ideal of definition  $J \subset B$  the ring map  $B \rightarrow A/JA$  is faithfully flat.*

**Proof.** Let  $J \subset B$  be an ideal of definition. As every element of  $J$  is topologically nilpotent, we see that every element of  $1+J$  is a unit. It follows that  $J$  is contained in the Jacobson radical of  $B$  (Algebra, Lemma 18.1). Hence a flat ring map  $B \rightarrow A$  is faithfully flat if and only if  $B/J \rightarrow A/JA$  is faithfully flat (Algebra, Lemma 38.15). In this way we see that (2) – (4) are equivalent. If (1) holds, then for every weak ideal of definition  $J \subset B$  the morphism  $\mathrm{Spec}(A/JA) = \mathrm{Spec}(B/J) \times_Y X \rightarrow \mathrm{Spec}(B/J)$  is surjective which implies (3). Conversely, assume (3). A morphism  $T \rightarrow Y$  with

$T$  quasi-compact factors through  $\mathrm{Spec}(B/J)$  for some ideal of definition  $J$  of  $B$  (Lemma 5.4). Hence  $X \times_Y T = \mathrm{Spec}(A/JA) \times_{\mathrm{Spec}(B/J)} T \rightarrow T$  is surjective as a base change of the surjective morphism  $\mathrm{Spec}(A/JA) \rightarrow \mathrm{Spec}(B/J)$ . Thus (1) holds.  $\square$

### 15. Types of formal algebraic spaces

In this section we define Noetherian, adic\*, and countably indexed formal algebraic spaces. The types adic, classical, and McQuillan are missing as we do not know how to prove the analogue of the following lemmas for those cases.

**Lemma 15.1.** *Let  $S$  be a scheme. Let  $X \rightarrow Y$  be a morphism of affine formal schemes which is representable by algebraic spaces, surjective, and flat. Then  $X$  is countably indexed if and only if  $Y$  is countably indexed.*

**Proof.** Assume  $X$  is countably indexed. We write  $X = \mathrm{colim} X_n$  as in Lemma 6.1. Write  $Y = \mathrm{colim} Y_\lambda$  as in Definition 5.1. For every  $n$  we can pick a  $\lambda_n$  such that  $X_n \rightarrow Y$  factors through  $Y_{\lambda_n}$ , see Lemma 5.4. On the other hand, for every  $\lambda$  the scheme  $Y_\lambda \times_Y X$  is affine (Lemma 14.7) and hence  $Y_\lambda \times_Y X \rightarrow X$  factors through  $X_n$  for some  $n$  (Lemma 5.4). Picture

$$\begin{array}{ccccccc}
 & & & \curvearrowright & & & \\
 Y_\mu \times_Y X & \longrightarrow & Y_\lambda \times_Y X & \longrightarrow & X_n & \longrightarrow & X \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 Y_\mu & \longrightarrow & Y_\lambda & \cdots \longrightarrow & Y_{\lambda_n} & \longrightarrow & Y \\
 & & & \curvearrowleft & & & 
 \end{array}$$

If we can show the dotted arrow exists, then we conclude that  $Y = \mathrm{colim} Y_{\lambda_n}$  and  $Y$  is countably indexed. To do this we pick a  $\mu$  such that we have the solid arrows in the diagram. Say  $Y_\mu = \mathrm{Spec}(B_\mu)$ , the closed subscheme  $Y_\lambda$  corresponds to  $J \subset B_\mu$ , and the closed subscheme  $Y_{\lambda_n}$  corresponds to  $J' \subset B_\mu$ . We are trying to show that  $J \subset J'$ . By the diagram above we know  $JA_\mu \subset J'A_\mu$  where  $Y_\mu \times_Y X = \mathrm{Spec}(A_\mu)$ . Since  $X \rightarrow Y$  is surjective and flat the morphism  $Y_\lambda \times_Y X \rightarrow Y_\lambda$  is a faithfully flat morphism of affine schemes, hence  $B_\mu \rightarrow A_\mu$  is faithfully flat. Thus  $J \subset J'$  as desired.

Assume  $Y$  is countably indexed. Then  $X$  is countably indexed by Lemma 14.9.  $\square$

**Lemma 15.2.** *Let  $S$  be a scheme. Let  $X \rightarrow Y$  be a morphism of affine formal schemes which is representable by algebraic spaces, surjective, and flat. Then  $X$  is adic\* if and only if  $Y$  is adic\*.*

**Proof.** Assume  $Y$  is adic\*. Then  $X$  is adic\* by Lemma 14.9.

Assume  $X$  is adic\*. Write  $X = \mathrm{Spf}(A)$  for some adic ring  $A$  which has a finitely generated ideal  $I$  such that  $\{I^n\}$  is a fundamental system of open ideals. By Lemmas 15.1 we see that  $Y$  is countably indexed. Thus, by Lemma 6.4, we can write  $Y = \mathrm{Spf}(B)$  where  $B$  is a weakly admissible topological ring with a countable fundamental system  $\{J_m\}$  of weak ideals of definition. Set  $Y_m = \mathrm{Spec}(B/J_m)$  so that  $Y = \mathrm{colim} Y_m$ . The scheme  $Y_m \times_Y X$  is affine (Lemma 14.7) and we have

$X = \text{colim } Y_m \times_Y X$ . Say  $Y_m \times_Y X = \text{Spec}(A_m)$  so that  $B/J_m \rightarrow A_m$  is a faithfully flat ring map. It follows from Lemma 11.4 that  $\text{Ker}(A \rightarrow A_m)$  is the closure of  $J_m A$ .

Choose  $n \geq 1$ . There exists an  $m$  such that  $\text{Spec}(A/I^n) \rightarrow Y$  factors through  $Y_m$ . In terms of ideals

$$(15.2.1) \quad \forall n \exists m, J_m A \subset I^n.$$

Choose  $m \geq 1$ . We can find an  $n$  such that the morphism  $\text{Spec}(A_m) \rightarrow X$  factors through  $\text{Spec}(A/I^n)$ . In terms of ideals

$$(15.2.2) \quad \forall m \exists n, I^n \subset \text{Ker}(A \rightarrow A_m).$$

Fix an  $m$ . Pick  $n$  such that  $I^n \subset \text{Ker}(A \rightarrow A_m)$  (15.2.2). Choose generators  $f_1, \dots, f_r$  of  $I$ . For any  $E = (e_1, \dots, e_r)$  with  $|E| = \sum e_i = n$  write

$$f_1^{e_1} \dots f_r^{e_r} = \sum g_{E,j} a_{E,j} + \delta_E$$

with  $g_{E,j} \in J_m$ ,  $a_{E,j} \in A$ , and  $\delta_E \in I^{n+1}$  (possible by the above). Let  $J = (g_{E,j}) \subset B$ . Then we see that

$$I^n \subset JA + I^{n+1}$$

As  $I$  is contained in the radical of  $A$  and  $I^n$  is finitely generated we see that  $I^n \subset JA$  by Algebra, Lemma 19.1.

We first apply what we just proved as follows: since  $I^n \subset J_m A$  we see that  $J_m A$  is open in  $A$ , hence closed, hence  $\text{Ker}(A \rightarrow A_m) = J_m A$ , in other words,  $A_m = A/J_m A$ . This holds for every  $m$ .

Next, we pick  $m$  with  $J_m A \subset I$  (15.2.1). Then choose  $J \subset J_m$  with  $I^n \subset JA \subset I$  as above. For every  $k \geq 1$  we define  $\mathfrak{b}_k = \text{Ker}(B \rightarrow A/J^k A)$ . For every  $k$  there exists an  $m'$  with  $J_{m'} \subset \mathfrak{b}_k$  as we have  $I^{nk} \subset J^k A$  and we can apply (15.2.1). On the other hand, for every  $m'$  there exists a  $k$  such that  $I^k \subset J_{m'} A$  because  $J_{m'} A$  is open. Then  $\mathfrak{b}_k$  maps to zero in  $A/J_{m'} A$  which is faithfully flat over  $B/J_{m'}$ . Hence  $\mathfrak{b}_k \subset J_{m'}$ . In other words, we see that the topology on  $B$  is defined by the sequence of ideals  $\mathfrak{b}_k$ . Note that  $J^k \subset \mathfrak{b}_k$  which implies that  $\mathfrak{b}_k A = J^k A$ . In other words, we have reduced the problem to the situation discussed in the following paragraph.

We are given a ring map  $B \rightarrow A$  where

- (1)  $B$  is a weakly admissible topological ring with a fundamental system  $J_1 \supset J_2 \supset J_3 \supset \dots$  of ideals of definition,
- (2)  $A$  is a ring complete with respect to a finitely generated ideal  $I$ ,
- (3) we have  $J_k A = I^k$  for all  $k$ , and
- (4)  $B/J_k \rightarrow A/I^k$  is faithfully flat.

Pick  $g_1, \dots, g_r \in J_1$  whose images in  $A/I^2$  generate  $I/I^2$ ; this is possible because  $J_1 A/J_2 A = I/I^2$ . Then for all  $k \geq 1$  we see that the elements  $g^E = g_1^{e_1} \dots g_r^{e_r}$  with  $|E| = k$  are in  $J_k$  and their classes in  $J_k/J_{k+1}$  map to generators of  $I^k/I^{k+1}$ . Since  $B/J_{k+1} \rightarrow A/I^{k+1}$  and  $B/J_k \rightarrow A/I^k$  are flat we see that

$$J_k/J_{k+1} \otimes_{B/J_1} A/I = J_k/J_{k+1} \otimes_{B/J_{k+1}} A/I^{k+1} \rightarrow I^k/I^{k+1}$$

is an isomorphism (see More on Morphisms, Lemma 8.1). Since  $B/J_1 \rightarrow A/I$  is faithfully flat, we conclude that the classes of the elements  $g^E$ ,  $|E| = k$  generate

$J_k/J_{k+1}$ . We claim that  $J_k = (g^E, |E| = k)$ . Namely, suppose that  $x_k \in J_k$ . By the above we can write

$$x_k = \sum_{|E|=k} b_{E,0} g^E + x_{k+1}$$

with  $x_{k+1} \in J_{k+1}$  and some  $b_{E,0} \in B$ . Now we can write  $x_{k+1}$  as follows

$$x_{k+1} = \sum_{|E|=k} \left( \sum_{|E'|=1} b_{E,E'} g^{E'} \right) g^E + x_{k+2}$$

because every multi-index of degree  $k+1$  is a sum of a multi-index of degree  $k$  and a multi-index of degree 1. Continuing in this manner we can find  $b_{E,E'} \in B$  such that for every  $l > 1$  we have

$$x_k = \sum_{|E|=k} \left( \sum_{0 \leq |E'| < l} b_{E,E'} g^{E'} \right) g^E + x_{k+l}$$

with some  $x_{k+l} \in J_{k+l}$ . Then we can finally define

$$b_E = \sum_{E'} b_{E,E'} g^{E'}$$

as an element in  $B$  and we see that  $x_k = \sum b_E g^E$  as desired. This finishes the proof as now  $J_1$  is finitely generated and  $J_k = J_1^k$  for all  $k \geq 1$ .  $\square$

**Lemma 15.3.** *Let  $S$  be a scheme. Let  $X \rightarrow Y$  be a morphism of affine formal schemes which is representable by algebraic spaces, surjective, flat, and (locally) of finite type. Then  $X$  is Noetherian if and only if  $Y$  is Noetherian.*

**Proof.** Observe that a Noetherian affine formal algebraic space is  $\text{adic}^*$ , see Lemma 6.3. Thus by Lemma 15.2 we may assume that both  $X$  and  $Y$  are  $\text{adic}^*$ . We will use the criterion of Lemma 6.5 to see that the lemma holds. Namely, write  $Y = \text{colim } Y_n$  as in Lemma 6.1. For each  $n$  set  $X_n = Y_n \times_Y X$ . Then  $X_n$  is an affine scheme (Lemma 14.7) and  $X = \text{colim } X_n$ . Each of the morphisms  $X_n \rightarrow Y_n$  is faithfully flat and of finite type. Thus the lemma follows from the fact that in this situation  $X_n$  is Noetherian if and only if  $Y_n$  is Noetherian, see Algebra, Lemma 152.1 (to go down) and Algebra, Lemma 30.1 (to go up).  $\square$

**Lemma 15.4.** *Let  $S$  be a scheme. Let  $P \in \{\text{countably indexed, } \text{adic}^*, \text{Noetherian}\}$ . Let  $X$  be a formal algebraic space over  $S$ . The following are equivalent*

- (1) *if  $Y$  is an affine formal algebraic space and  $f : Y \rightarrow X$  is representable by algebraic spaces and étale, then  $Y$  has property  $P$ ,*
- (2) *for some  $\{X_i \rightarrow X\}_{i \in I}$  as in Definition 7.1 each  $X_i$  has property  $P$ .*

**Proof.** It is clear that (1) implies (2). Assume (2) and let  $Y \rightarrow X$  be as in (1). Since the fibre products  $X_i \times_Y X$  are formal algebraic spaces (Lemma 10.2) we can pick coverings  $\{X_{ij} \rightarrow X_i \times_X Y\}$  as in Definition 7.1. Since  $Y$  is quasi-compact, there exist  $(i_1, j_1), \dots, (i_n, j_n)$  such that

$$X_{i_1 j_1} \amalg \dots \amalg X_{i_n j_n} \longrightarrow Y$$

is surjective. Then  $X_{i_k j_k} \rightarrow X_{i_k}$  is representable by algebraic spaces and étale hence  $X_{i_k j_k}$  has property  $P$  by Lemma 14.9. Then  $X_{i_1 j_1} \amalg \dots \amalg X_{i_n j_n}$  is an affine formal algebraic space with property  $P$  (small detail omitted on finite disjoint unions of affine formal algebraic spaces). Hence we conclude by applying one of Lemmas 15.1, 15.2, and 15.3.  $\square$

The previous lemma clears the way for the following definition.

**Definition 15.5.** Let  $S$  be a scheme. Let  $X$  be a formal algebraic space over  $S$ . We say  $X$  is *locally countably indexed*, *locally adic\**, or *locally Noetherian* if the equivalent conditions of Lemma 15.4 hold for the corresponding property.

The formal completion of a locally Noetherian algebraic space along a closed is a locally Noetherian formal algebraic space.

**Lemma 15.6.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $T \subset |X|$  be a closed subset. Let  $X_{/T}$  be the formal completion of  $X$  along  $T$ .*

- (1) *If  $X \setminus T \rightarrow X$  is quasi-compact, then  $X_{/T}$  is locally adic\*.*
- (2) *If  $X$  is locally Noetherian, then  $X_{/T}$  is locally Noetherian.*

**Proof.** Choose a surjective étale morphism  $U \rightarrow X$  with  $U = \coprod U_i$  a disjoint union of affine schemes, see Properties of Spaces, Lemma 6.1. Let  $T_i \subset U_i$  be the inverse image of  $T$ . We have  $X_{/T} \times_X U_i = (U_i)_{/T_i}$  (small detail omitted). Hence  $\{(U_i)_{/T_i} \rightarrow X_{/T}\}$  is a covering as in Definition 7.1. Moreover, if  $X \setminus T \rightarrow X$  is quasi-compact, so is  $U_i \setminus T_i \rightarrow U_i$  and if  $X$  is locally Noetherian, so is  $U_i$ . Thus it suffices to prove the lemma in case  $X$  is affine.

Assume  $X = \operatorname{Spec}(A)$  is affine and  $X \setminus T \rightarrow X$  is quasi-compact. Then there exists a finitely generated ideal  $I = (f_1, \dots, f_r) \subset A$  cutting out  $T$  (Algebra, Lemma 28.1). If  $Z = \operatorname{Spec}(B)$  is an affine scheme and  $g : Z \rightarrow X$  is a morphism with  $g(Z) \subset T$  (set theoretically), then  $g^\sharp(f_i)$  is nilpotent in  $B$  for each  $i$ . Thus  $I^n$  maps to zero in  $B$  for some  $n$ . Hence we see that  $X_{/T} = \operatorname{colim} \operatorname{Spec}(A/I^n)$  and  $X$  is adic\*.

Assume  $X = \operatorname{Spec}(A)$  is affine with  $A$  Noetherian. By the above we see that  $X_{/T} = \operatorname{Spf}(A^\wedge)$  where  $A^\wedge$  is the  $I$ -adic completion of  $A$  with respect to some ideal  $I \subset A$ . Then  $X_{/T}$  is Noetherian because  $A^\wedge$  is so, see Algebra, Lemma 93.10.  $\square$

## 16. Morphisms and continuous ring maps

In this section we denote  $WAdm$  the category of weakly admissible topological rings and continuous ring homomorphisms. We define full subcategories

$$WAdm \supset WAdm^{count} \supset WAdm^{adic*} \supset WAdm^{Noeth}$$

whose objects are

- (1)  $WAdm^{count}$ : those weakly admissible topological rings  $A$  which have a countable fundamental system of neighbourhoods of 0,
- (2)  $WAdm^{adic*}$ : the adic topological rings which have a finitely generated ideal of definition, and
- (3)  $WAdm^{Noeth}$ : the adic topological rings which are Noetherian.

Clearly, the formal spectra of these types of rings are the basic building blocks of locally countably indexed, locally adic\*, and locally Noetherian formal algebraic spaces.

We briefly review the relationship between morphisms of countably indexed, affine formal algebraic spaces and morphisms of  $WAdm^{count}$ . Let  $S$  be a scheme. Let  $X$  and  $Y$  be countably indexed, affine formal algebraic spaces. Write  $X = \operatorname{Spf}(A)$  and  $Y = \operatorname{Spf}(B)$  topological  $S$ -algebras  $A$  and  $B$  in  $WAdm^{count}$ , see Lemma 6.4. By Lemma 5.10 there is a 1-to-1 correspondence between morphisms  $f : X \rightarrow Y$  and continuous maps

$$\varphi : B \rightarrow A$$

of topological  $S$ -algebras. The relationship is given by  $f \mapsto f^\#$  and  $\varphi \mapsto \mathrm{Spf}(\varphi)$ .

Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of locally countably indexed formal algebraic spaces. Consider a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with  $U$  and  $V$  affine formal algebraic spaces and  $U \rightarrow X$  and  $V \rightarrow Y$  representable by algebraic spaces and étale. By Definition 15.5 (and hence via Lemma 15.4) we see that  $U$  and  $V$  are countably indexed affine formal algebraic spaces. By the discussion in the previous paragraph we see that  $U \rightarrow V$  is isomorphic to  $\mathrm{Spf}(\varphi)$  for some continuous map

$$\varphi : B \longrightarrow A$$

of topological  $S$ -algebras in  $WAdm^{count}$ .

**Lemma 16.1.** *Let  $A \in \mathrm{Ob}(WAdm)$ . Let  $A \rightarrow A'$  be a ring map (no topology). Let  $(A')^\wedge = \lim_{I \subset A \text{ w.i.d.}} A'/IA'$  be the object of  $WAdm$  constructed in Example 14.11.*

- (1) *If  $A$  is in  $WAdm^{count}$ , so is  $(A')^\wedge$ .*
- (2) *If  $A$  is in  $WAdm^{adic*}$ , so is  $(A')^\wedge$ .*
- (3) *If  $A$  is in  $WAdm^{Noeth}$  and  $A'$  is Noetherian, then  $(A')^\wedge$  is in  $WAdm^{Noeth}$ .*

**Proof.** Part (1) is clear from the construction. Assume  $A$  has a finitely generated ideal of definition  $I \subset A$ . Then  $I^n(A')^\wedge = \mathrm{Ker}((A')^\wedge \rightarrow A'/I^n A')$  by Algebra, Lemmas 93.6 and 93.7. Thus  $I(A')^\wedge$  is a finitely generated ideal of definition and we see that (2) holds. Finally, assume that  $A$  is Noetherian and adic. By (2) we know that  $(A')^\wedge$  is adic. By Algebra, Lemma 93.10 we see that  $(A')^\wedge$  is Noetherian. Hence (3) holds.  $\square$

Let  $P$  be a property of morphisms of  $WAdm^{count}$ . Consider commutative diagrams

$$(16.1.1) \quad \begin{array}{ccc} A & \longrightarrow & (A')^\wedge \\ \uparrow \varphi & & \uparrow \varphi' \\ B & \longrightarrow & (B')^\wedge \end{array}$$

satisfying the following conditions

- (1)  $A$  and  $B$  are objects of  $WAdm^{count}$ ,
- (2)  $A \rightarrow A'$  and  $B \rightarrow B'$  are étale ring maps,
- (3)  $(A')^\wedge = \lim A'/IA'$ , resp.  $(B')^\wedge = \lim B'/JB'$  where  $I \subset A$ , resp.  $J \subset B$  runs through the weakly admissible ideals of definition of  $A$ , resp.  $B$ ,
- (4)  $\varphi : A \rightarrow B$  and  $\varphi' : (A')^\wedge \rightarrow (B')^\wedge$  are continuous.

By Lemma 16.1 the topological rings  $(A')^\wedge$  and  $(B')^\wedge$  are objects of  $WAdm^{count}$ . We say  $P$  is a *local property* if the following axioms hold:

- (1) for any diagram (16.1.1) we have  $P(\varphi) \Rightarrow P(\varphi')$ ,
- (2) for any diagram (16.1.1) with  $A \rightarrow A'$  faithfully flat we have  $P(\varphi') \Rightarrow P(\varphi)$ ,
- (3) if  $P(A \rightarrow B_i)$  for  $i = 1, \dots, n$ , then  $P(A \rightarrow \prod_{i=1, \dots, n} B_i)$ .

Axiom (3) makes sense as  $WAdm^{count}$  has finite products.



**Lemma 16.2.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of locally countably indexed formal algebraic spaces over  $S$ . Let  $P$  be a local property of morphisms of  $WAdm^{count}$ . The following are equivalent*

- (1) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

*with  $U$  and  $V$  affine formal algebraic spaces,  $U \rightarrow X$  and  $V \rightarrow Y$  representable by algebraic spaces and étale, the morphism  $U \rightarrow V$  corresponds to a morphism of  $WAdm^{count}$  with property  $P$ ,*

- (2) *there exists a covering  $\{Y_j \rightarrow Y\}$  as in Definition 7.1 and for each  $j$  a covering  $\{X_{ji} \rightarrow Y_j \times_Y X\}$  as in Definition 7.1 such that each  $X_{ji} \rightarrow Y_j$  corresponds to a morphism of  $WAdm^{count}$  with property  $P$ , and*
- (3) *there exist a covering  $\{X_i \rightarrow X\}$  as in Definition 7.1 and for each  $i$  a factorization  $X_i \rightarrow Y_i \rightarrow Y$  where  $Y_i$  is an affine formal algebraic space,  $Y_i \rightarrow Y$  is representable by algebraic spaces and étale, and  $X_i \rightarrow Y_i$  corresponds to a morphism of  $WAdm^{count}$  with property  $P$ .*

**Proof.** It is clear that (1) implies (2) and that (2) implies (3). Assume  $\{X_i \rightarrow X\}$  and  $X_i \rightarrow Y_i \rightarrow Y$  as in (3) and let a diagram as in (1) be given. Since  $Y_i \times_Y V$  is a formal algebraic space (Lemma 10.2) we may pick coverings  $\{Y_{ij} \rightarrow Y_i \times_Y V\}$  as in Definition 7.1. For each  $(i, j)$  we may similarly choose coverings we can pick coverings  $\{X_{ijk} \rightarrow Y_{ij} \times_{Y_i} X_i \times_X U\}$  as in Definition 7.1. Since  $U$  is quasi-compact we can choose  $(i_1, j_1, k_1), \dots, (i_n, j_n, k_n)$  such that

$$X_{i_1 j_1 k_1} \amalg \dots \amalg X_{i_n j_n k_n} \longrightarrow U$$

is surjective. For  $s = 1, \dots, n$  consider the commutative diagram

$$\begin{array}{ccccccc} & & & X_{i_s j_s k_s} & & & \\ & & \swarrow & \downarrow & \searrow & & \\ X & \longleftarrow & X_{i_s} & \longleftarrow & X_{i_s} \times_X U & \longrightarrow & U \longrightarrow X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & Y_{i_s} & \longleftarrow & Y_{i_s} \times_Y V & \longrightarrow & V \longrightarrow Y \end{array}$$

Let us say that  $P$  holds for a morphism of countably indexed affine formal algebraic spaces if it holds for the corresponding morphism of  $WAdm^{count}$ . Observe that the maps  $X_{i_s j_s k_s} \rightarrow X_{i_s}$ ,  $Y_{i_s j_s} \rightarrow Y_{i_s}$  are given by completions of étale ring maps, see Lemma 14.13. Hence we see that  $P(X_{i_s} \rightarrow Y_{i_s})$  implies  $P(X_{i_s j_s k_s} \rightarrow Y_{i_s j_s})$  by axiom (1). By axiom (2) (and the fact that identities are faithfully flat ring maps) we conclude that  $P(X_{i_s j_s k_s} \rightarrow V)$  holds. By axiom (3) we find that  $P(\coprod_{s=1, \dots, n} X_{i_s j_s k_s} \rightarrow V)$  holds. Since the morphism  $\coprod X_{i_s j_s k_s} \rightarrow U$  is surjective by construction, the corresponding morphism of  $WAdm^{count}$  is the completion of a faithfully flat étale ring map, see Lemma 14.14. One more application of axiom (2) implies that  $P(U \rightarrow V)$  is true as desired.  $\square$

**Remark 16.3** (Variant for adic-star). Let  $P$  be a property of morphisms of  $WAdm^{adic*}$ . We say  $P$  is a *local property* if axioms (1), (2), (3), hold for diagrams (16.1.1) with  $\varphi$  a morphism of  $WAdm^{adic*}$ . In exactly the same way we obtain a variant of Lemma 16.2 for morphisms between locally adic\* formal algebraic spaces over  $S$ .

**Remark 16.4** (Variant for Noetherian). Let  $P$  be a property of morphisms of  $WAdm^{Noeth}$ . We say  $P$  is a *local property* if axioms (1), (2), (3), hold for diagrams (16.1.1) with  $\varphi$  a morphism of  $WAdm^{Noeth}$ . In exactly the same way we obtain a variant of Lemma 16.2 for morphisms between locally Noetherian formal algebraic spaces over  $S$ .

**Lemma 16.5.** *Let  $B \rightarrow A$  be an arrow of  $WAdm^{count}$ . The following are equivalent*

- (a)  $B \rightarrow A$  is taut (Definition 4.9),
- (b) for  $B \supset J_1 \supset J_2 \supset J_3 \supset \dots$  a fundamental system of weak ideals of definition there exist a commutative diagram

$$\begin{array}{ccccccc}
 A & \longrightarrow & \dots & \longrightarrow & A_3 & \longrightarrow & A_2 \longrightarrow A_1 \\
 \uparrow & & & & \uparrow & & \uparrow \\
 B & \longrightarrow & \dots & \longrightarrow & B/J_3 & \longrightarrow & B/J_2 \longrightarrow B/J_1
 \end{array}$$

such that  $A_{n+1}/J_n A_{n+1} = A_n$  and  $A = \lim A_n$  as topological ring.

Moreover, these equivalent conditions define a local property, i.e., they satisfy axioms (1), (2), (3).

**Proof.** The equivalence of (a) and (b) is immediate. Below we will give an algebraic proof of the axioms, but it turns out we've already proven them. Namely, using Lemma 14.10 (a) and (b) translate to saying the corresponding morphism of affine formal schemes is representable, and this condition is “étale local on the source and target” by Lemma 14.4.

Let a diagram (16.1.1) be given. By Example 14.11 the maps  $A \rightarrow (A')^\wedge$  and  $B \rightarrow (B')^\wedge$  satisfy (a) and (b).

Assume (a) and (b) hold for  $\varphi$ . Let  $J \subset B$  be a weak ideal of definition. Then the closure of  $JA$ , resp.  $J(B')^\wedge$  is a weak ideal of definition  $I \subset A$ , resp.  $J' \subset (B')^\wedge$ . Then the closure of  $I(A')^\wedge$  is a weak ideal of definition  $I' \subset (A')^\wedge$ . A topological argument shows that  $I'$  is also the closure of  $J(A')^\wedge$  and of  $J'(A')^\wedge$ . Finally, as  $J$  runs over a fundamental system of weak ideals of definition of  $B$  so do the ideals  $I$  and  $I'$  in  $A$  and  $(A')^\wedge$ . It follows that (a) holds for  $\varphi'$ . This proves (1).

Assume  $A \rightarrow A'$  is faithfully flat and that (a) and (b) hold for  $\varphi'$ . Let  $J \subset B$  be a weak ideal of definition. Using (a) and (b) for the maps  $B \rightarrow (B')^\wedge \rightarrow (A')^\wedge$  we find that the closure  $I'$  of  $J(A')^\wedge$  is a weak ideal of definition. In particular,  $I'$  is open and hence the inverse image of  $I'$  in  $A$  is open. Now we have (explanation below)

$$\begin{aligned}
 A \cap I' &= A \cap \bigcap (J(A')^\wedge + \text{Ker}((A')^\wedge \rightarrow A'/I_0 A')) \\
 &= A \cap \bigcap \text{Ker}((A')^\wedge \rightarrow A'/JA' + I_0 A') \\
 &= \bigcap (JA + I_0)
 \end{aligned}$$

which is the closure of  $JA$  by Lemma 4.2. The intersections are over weak ideals of definition  $I_0 \subset A$ . The first equality because a fundamental system of neighbourhoods of 0 in  $(A')^\wedge$  are the kernels of the maps  $(A')^\wedge \rightarrow A'/I_0A'$ . The second equality is trivial. The third equality because  $A \rightarrow A'$  is faithfully flat, see Algebra, Lemma 79.11. Thus the closure of  $JA$  is open. By Lemma 4.8 the closure of  $JA$  is a weak ideal of definition of  $A$ . Finally, given a weak ideal of definition  $I \subset A$  we can find  $J$  such that  $J(A')^\wedge$  is contained in the closure of  $I(A')^\wedge$  by property (a) for  $B \rightarrow (B')^\wedge$  and  $\varphi'$ . Thus we see that (a) holds for  $\varphi$ . This proves (2).

We omit the proof of (3).  $\square$

**Lemma 16.6.** *Let  $P = \text{"taut"}$  viewed as a property of morphisms of  $WAdm^{count}$ . Then under the assumptions of Lemma 16.2 the equivalent conditions (1), (2), and (3) are also equivalent to the condition*

(4)  *$f$  is representable by algebraic spaces.*

**Proof.** Property  $P$  is a local property by Lemma 16.5. By Lemma 14.10 condition  $P$  on morphisms of  $WAdm^{count}$  corresponds to “representable by algebraic spaces” for the corresponding morphisms of countably indexed affine formal algebraic spaces. Thus the lemma follows from Lemma 14.4.  $\square$

## 17. Adic morphisms

Suppose that  $\varphi : A \rightarrow B$  is a continuous map between adic topological rings. One says  $\varphi$  is *adic* if there exists an ideal of definition  $I \subset A$  such that the topology on  $B$  is  $I$ -adic. However, this is not a good notion unless we assume  $A$  has a finitely generated ideal of definition. In this case, the condition is equivalent to  $\varphi$  being taut, see Lemma 4.13.

Let  $P$  be the property of morphisms  $\varphi : A \rightarrow B$  of  $WAdm^{adic*}$  defined by

$$P(\varphi) = \text{“}\varphi \text{ is adic”} = \text{“}\varphi \text{ is taut”}$$

(see above for the equivalence). Since  $WAdm^{adic*}$  is a full subcategory of  $WAdm^{count}$  it follows trivially from Lemma 16.5 that  $P$  is a local property on morphisms of  $WAdm^{adic*}$ , see Remark 16.3. Combining Lemmas 16.2 and 16.6 we obtain the result stated in the next paragraph.

Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of locally adic\* formal algebraic spaces over  $S$ . Then the following are equivalent

- (1)  $f$  is representable by algebraic spaces (in other words, the equivalent conditions of Lemma 14.4 hold),
- (2) for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with  $U$  and  $V$  affine formal algebraic spaces,  $U \rightarrow X$  and  $V \rightarrow Y$  representable by algebraic spaces and étale, the morphism  $U \rightarrow V$  corresponds to an adic map in  $WAdm^{adic*}$  (in other words, the equivalent conditions of Lemma 16.2 hold with  $P$  as above).

In this situation we will sometimes say that  $f$  is an *adic morphism*. Here it is understood that this notion is only defined for morphisms between formal algebraic spaces which are locally adic\*.

**Definition 17.1.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . Assume  $X$  and  $Y$  are locally adic\*. We say  $f$  is an *adic morphism* if  $f$  is representable by algebraic spaces. See discussion above.

## 18. Morphisms of finite type

Due to how things are setup in the Stacks project, the following is really the correct thing to do and stronger notions should have a different name.

**Definition 18.1.** Let  $S$  be a scheme. Let  $f : Y \rightarrow X$  be a morphism of formal algebraic spaces over  $S$ .

- (1) We say  $f$  is *locally of finite type* if  $f$  is representable by algebraic spaces and is locally of finite type in the sense of Bootstrap, Definition 4.1.
- (2) We say  $f$  is of *finite type* if  $f$  is locally of finite type and quasi-compact (Definition 12.4).

We will discuss the relationship between finite type morphisms of certain formal algebraic spaces and continuous ring maps  $A \rightarrow B$  which are topologically of finite type in Restricted Power Series, Section 3.

**Lemma 18.2.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . The following are equivalent

- (1)  $f$  is of finite type,
- (2)  $f$  is representable by algebraic spaces and is of finite type in the sense of Bootstrap, Definition 4.1.

**Proof.** This follows from Bootstrap, Lemma 4.5, the implication “quasi-compact + locally of finite type  $\Rightarrow$  finite type” for morphisms of algebraic spaces, and Lemma 12.5.  $\square$

**Lemma 18.3.** The composition of finite type morphisms is of finite type. The same holds for locally of finite type.

**Proof.** See Bootstrap, Lemma 4.3 and use Morphisms of Spaces, Lemma 23.2.  $\square$

**Lemma 18.4.** A base change of a finite type morphism is finite type. The same holds for locally of finite type.

**Proof.** See Bootstrap, Lemma 4.2 and use Morphisms of Spaces, Lemma 23.3.  $\square$

**Lemma 18.5.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be morphisms of formal algebraic spaces over  $S$ . If  $g \circ f : X \rightarrow Z$  is locally of finite type, then  $f : X \rightarrow Y$  is locally of finite type.

**Proof.** By Lemma 14.3 we see that  $f$  is representable by algebraic spaces. Let  $T$  be a scheme and let  $T \rightarrow Z$  be a morphism. Then we can apply Morphisms of Spaces, Lemma 23.6 to the morphisms  $T \times_Z X \rightarrow T \times_Z Y \rightarrow T$  of algebraic spaces to conclude.  $\square$

Being locally of finite type is local on the source and the target.

**Lemma 18.6.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . The following are equivalent:*

- (1) *the morphism  $f$  is locally of finite type,*
- (2) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

*where  $U, V$  are formal algebraic spaces, the vertical arrows are representable by algebraic spaces and étale,  $U \rightarrow X$  is surjective, and  $U \rightarrow V$  is locally of finite type,*

- (3) *for any commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

*where  $U, V$  are formal algebraic spaces and vertical arrows representable by algebraic spaces and étale, the morphism  $U \rightarrow V$  is locally of finite type,*

- (4) *there exists a covering  $\{Y_j \rightarrow Y\}$  as in Definition 7.1 and for each  $j$  a covering  $\{X_{ji} \rightarrow Y_j \times_Y X\}$  as in Definition 7.1 such that  $X_{ji} \rightarrow Y_j$  is locally of finite type for each  $j$  and  $i$ ,*
- (5) *there exist a covering  $\{X_i \rightarrow X\}$  as in Definition 7.1 and for each  $i$  a factorization  $X_i \rightarrow Y_i \rightarrow Y$  where  $Y_i$  is an affine formal algebraic space,  $Y_i \rightarrow Y$  is representable by algebraic spaces and étale, such that  $X_i \rightarrow Y_i$  is locally of finite type, and*
- (6) *add more here.*

**Proof.** In each of the 5 cases the morphism  $f : X \rightarrow Y$  is representable by algebraic spaces, see Lemma 14.4. We will use this below without further mention.

It is clear that (1) implies (2) because we can take  $U = X$  and  $V = Y$ . Conversely, assume given a diagram as in (2). Let  $T$  be a scheme and let  $T \rightarrow Y$  be a morphism. Then we can consider

$$\begin{array}{ccc} U \times_Y T & \longrightarrow & V \times_Y T \\ \downarrow & & \downarrow \\ X \times_Y T & \longrightarrow & T \end{array}$$

The vertical arrows are étale and the top horizontal arrow is locally of finite type as base changes of such morphisms. Hence by Morphisms of Spaces, Lemma 23.4 we conclude that  $X \times_Y T \rightarrow T$  is locally of finite type. In other words (1) holds.

Assume (1) is true and consider a diagram as in (3). Then  $U \rightarrow Y$  is locally of finite type (as the composition  $U \rightarrow X \rightarrow Y$ , see Bootstrap, Lemma 4.3). Let  $T$  be a scheme and let  $T \rightarrow V$  be a morphism. Then the projection  $T \times_V U \rightarrow U$  factors as

$$T \times_V U = (T \times_Y U) \times_{(V \times_Y V)} V \rightarrow T \times_Y U \rightarrow U$$

The second arrow is locally of finite type (see above) and the first is the base change of the diagonal  $V \rightarrow V \times_Y V$  which is locally of finite type by Lemma 10.4.

It is clear that (3) implies (2). Thus now (1) – (3) are equivalent.

Observe that the condition in (4) makes sense as the fibre product  $Y_j \times_Y X$  is a formal algebraic space by Lemma 10.3. It is clear that (4) implies (5).

Assume  $X_i \rightarrow Y_i \rightarrow Y$  as in (5). Then we set  $V = \coprod Y_i$  and  $U = \coprod X_i$  to see that (5) implies (2).

Finally, assume (1) – (3) are true. Thus we can choose any covering  $\{Y_j \rightarrow Y\}$  as in Definition 7.1 and for each  $j$  any covering  $\{X_{ji} \rightarrow Y_j \times_Y X\}$  as in Definition 7.1. Then  $X_{ij} \rightarrow Y_j$  is locally of finite type by (3) and we see that (4) is true. This concludes the proof.  $\square$

**Example 18.7.** Let  $S$  be a scheme. Let  $A$  be a weakly admissible topological ring over  $S$ . Let  $A \rightarrow A'$  be a finite type ring map. Then

$$(A')^\wedge = \lim_{I \subset A \text{ w.i.d.}} A'/IA'$$

is a weakly admissible ring and the corresponding morphism  $\mathrm{Spf}((A')^\wedge) \rightarrow \mathrm{Spf}(A)$  is representable, see Example 14.11. If  $T \rightarrow \mathrm{Spf}(A)$  is a morphism where  $T$  is a quasi-compact scheme, then this factors through  $\mathrm{Spec}(A/I)$  for some weak ideal of definition  $I \subset A$  (Lemma 5.4). Then  $T \times_{\mathrm{Spf}(A)} \mathrm{Spf}((A')^\wedge)$  is equal to  $T \times_{\mathrm{Spec}(A/I)} \mathrm{Spec}(A'/IA')$  and we see that  $\mathrm{Spf}((A')^\wedge) \rightarrow \mathrm{Spf}(A)$  is of finite type.

**Lemma 18.8.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . If  $Y$  is locally Noetherian and  $f$  locally of finite type, then  $X$  is locally Noetherian.*

**Proof.** Pick  $\{Y_j \rightarrow Y\}$  and  $\{X_{ij} \rightarrow Y_j \times_Y X\}$  as in Lemma 18.6. Then it follows from Lemma 14.8 that each  $X_{ij}$  is Noetherian. This proves the lemma.  $\square$

**Lemma 18.9.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  and  $Z \rightarrow Y$  be morphisms of formal algebraic spaces over  $S$ . If  $Z$  is locally Noetherian and  $f$  locally of finite type, then  $Z \times_Y X$  is locally Noetherian.*

**Proof.** The morphism  $Z \times_Y X \rightarrow Z$  is locally of finite type by Lemma 18.4. Hence this follows from Lemma 18.8.  $\square$

**Lemma 18.10.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces over  $S$  which is locally of finite type. Let  $T \subset |Y|$  be a closed subset and let  $T' = |f|^{-1}(T) \subset |X|$ . Then  $X_{/T'} \rightarrow Y_{/T}$  is locally of finite type.*

**Proof.** Namely, suppose that  $V \rightarrow Y$  is a morphism from a scheme into  $Y$  such that  $|V|$  maps into  $T$ . In the proof of Lemma 9.4 we have seen that  $V \times_Y X \rightarrow X$  is an algebraic space representing  $V \times_{Y_{/T'}} X_{/T}$ . Since  $V \times_Y X \rightarrow V$  is locally of finite type (by Morphisms of Spaces, Lemma 23.3) we conclude.  $\square$

## 19. Monomorphisms

Here is the definition.

**Definition 19.1.** Let  $S$  be a scheme. A morphism of formal algebraic spaces over  $S$  is called a *monomorphism* if it is an injective map of sheaves.

An example is the following. Let  $X$  be an algebraic space and let  $T \subset |X|$  be a closed subset. Then the morphism  $X_{/T} \rightarrow X$  from the formal completion of  $X$  along  $T$  to  $X$  is a monomorphism. In particular, monomorphisms of formal algebraic spaces are in general not representable.

## 20. Closed immersions

Here is the definition.

**Definition 20.1.** Let  $S$  be a scheme. Let  $f : Y \rightarrow X$  be a morphism of formal algebraic spaces over  $S$ . We say  $f$  is a *closed immersion* if  $f$  is representable by algebraic spaces and a closed immersion in the sense of Bootstrap, Definition 4.1.

**Lemma 20.2.** *Let  $S$  be a scheme. Let  $X$  be a McQuillan affine formal algebraic space over  $S$ . Let  $f : Y \rightarrow X$  be a closed immersion of formal algebraic spaces over  $S$ . Then  $Y$  is a McQuillan affine formal algebraic space and  $f$  corresponds to a continuous homomorphism  $A \rightarrow B$  of weakly admissible topological  $S$ -algebras which is taut, has closed kernel, and has dense image.*

**Proof.** Write  $X = \mathrm{Spf}(A)$  where  $A$  is a weakly admissible topological ring. Let  $I_\lambda$  be a fundamental system of weakly admissible ideals of definition in  $A$ . Then  $Y \times_X \mathrm{Spec}(A/I_\lambda)$  is a closed subscheme of  $\mathrm{Spec}(A/I_\lambda)$  and hence affine (Definition 20.1). Say  $Y \times_X \mathrm{Spec}(A/I_\lambda) = \mathrm{Spec}(B_\lambda)$ . The ring map  $A/I_\lambda \rightarrow B_\lambda$  is surjective. Hence the projections

$$B = \lim B_\lambda \longrightarrow B_\lambda$$

are surjective as the compositions  $A \rightarrow B \rightarrow B_\lambda$  are surjective. It follows that  $Y$  is McQuillan by Lemma 5.6. The ring map  $A \rightarrow B$  is taut by Lemma 14.10. The kernel is closed because  $B$  is complete and  $A \rightarrow B$  is continuous. Finally, as  $A \rightarrow B_\lambda$  is surjective for all  $\lambda$  we see that the image of  $A$  in  $B$  is dense.  $\square$

Even though we have the result above, in general we do not know how closed immersions behave when the target is a McQuillan affine formal algebraic space, see Restricted Power Series, Remark 3.4.

**Example 20.3.** Let  $S$  be a scheme. Let  $A$  be a weakly admissible topological ring over  $S$ . Let  $K \subset A$  be a closed ideal. Setting

$$B = (A/K)^\wedge = \lim_{I \subset A \text{ w.i.d.}} A/(I + K)$$

the morphism  $\mathrm{Spf}(B) \rightarrow \mathrm{Spf}(A)$  is representable, see Example 14.11. If  $T \rightarrow \mathrm{Spf}(A)$  is a morphism where  $T$  is a quasi-compact scheme, then this factors through  $\mathrm{Spec}(A/I)$  for some weak ideal of definition  $I \subset A$  (Lemma 5.4). Then  $T \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B)$  is equal to  $T \times_{\mathrm{Spec}(A/I)} \mathrm{Spec}(A/(K + I))$  and we see that  $\mathrm{Spf}(B) \rightarrow \mathrm{Spf}(A)$  is a closed immersion. The kernel of  $A \rightarrow B$  is  $K$  as  $K$  is closed, but beware that in general the ring map  $A \rightarrow B = (A/K)^\wedge$  need not be surjective.

## 21. Separation axioms for morphisms

This section is the analogue of Morphisms of Spaces, Section 4 for morphisms of formal algebraic spaces.

**Definition 21.1.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . Let  $\Delta_{X/Y} : X \rightarrow X \times_Y X$  be the diagonal morphism.

- (1) We say  $f$  is *separated* if  $\Delta_{X/Y}$  is a closed immersion.
- (2) We say  $f$  is *quasi-separated* if  $\Delta_{X/Y}$  is quasi-compact.

Since  $\Delta_{X/Y}$  is representable (by schemes) by Lemma 10.4 we can test this by considering morphisms  $T \rightarrow X \times_Y X$  from affine schemes  $T$  and checking whether

$$E = T \times_{X \times_Y X} X \longrightarrow T$$

is quasi-compact or a closed immersion, see Lemma 12.5 or Definition 20.1. Note that the scheme  $E$  is the equalizer of two morphisms  $a, b : T \rightarrow X$  which agree as morphisms into  $Y$  and that  $E \rightarrow T$  is a monomorphism and locally of finite type.

**Lemma 21.2.** *All of the separation axioms listed in Definition 21.1 are stable under base change.*

**Proof.** Let  $f : X \rightarrow Y$  and  $Y' \rightarrow Y$  be morphisms of formal algebraic spaces. Let  $f' : X' \rightarrow Y'$  be the base change of  $f$  by  $Y' \rightarrow Y$ . Then  $\Delta_{X'/Y'}$  is the base change of  $\Delta_{X/Y}$  by the morphism  $X' \times_{Y'} X' \rightarrow X \times_Y X$ . Each of the properties of the diagonal used in Definition 21.1 is stable under base change. Hence the lemma is true.  $\square$

**Lemma 21.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Z$ ,  $g : Y \rightarrow Z$  and  $Z \rightarrow T$  be morphisms of formal algebraic spaces over  $S$ . Consider the induced morphism  $i : X \times_Z Y \rightarrow X \times_T Y$ . Then*

- (1)  *$i$  is representable (by schemes), locally of finite type, locally quasi-finite, separated, and a monomorphism,*
- (2) *if  $Z \rightarrow T$  is separated, then  $i$  is a closed immersion, and*
- (3) *if  $Z \rightarrow T$  is quasi-separated, then  $i$  is quasi-compact.*

**Proof.** By general category theory the following diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{i} & X \times_T Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\Delta_{Z/T}} & Z \times_T Z \end{array}$$

is a fibre product diagram. Hence  $i$  is the base change of the diagonal morphism  $\Delta_{Z/T}$ . Thus the lemma follows from Lemma 10.4.  $\square$

**Lemma 21.4.** *All of the separation axioms listed in Definition 21.1 are stable under composition of morphisms.*

**Proof.** Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be morphisms of formal algebraic spaces to which the axiom in question applies. The diagonal  $\Delta_{X/Z}$  is the composition

$$X \longrightarrow X \times_Y X \longrightarrow X \times_Z X.$$

Our separation axiom is defined by requiring the diagonal to have some property  $\mathcal{P}$ . By Lemma 21.3 above we see that the second arrow also has this property. Hence the lemma follows since the composition of (representable) morphisms with property  $\mathcal{P}$  also is a morphism with property  $\mathcal{P}$ .  $\square$

**Lemma 21.5.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . Let  $\mathcal{P}$  be any of the separation axioms of Definition 21.1. The following are equivalent*

- (1)  *$f$  is  $\mathcal{P}$ ,*
- (2) *for every scheme  $Z$  and morphism  $Z \rightarrow Y$  the base change  $Z \times_Y X \rightarrow Z$  of  $f$  is  $\mathcal{P}$ ,*
- (3) *for every affine scheme  $Z$  and every morphism  $Z \rightarrow Y$  the base change  $Z \times_Y X \rightarrow Z$  of  $f$  is  $\mathcal{P}$ ,*
- (4) *for every affine scheme  $Z$  and every morphism  $Z \rightarrow Y$  the formal algebraic space  $Z \times_Y X$  is  $\mathcal{P}$  (see Definition 11.3),*



- (5) *there exists a covering  $\{Y_j \rightarrow Y\}$  as in Definition 7.1 such that the base change  $Y_j \times_Y X \rightarrow Y_j$  has  $\mathcal{P}$  for all  $j$ .*

**Proof.** We will repeatedly use Lemma 21.2 without further mention. In particular, it is clear that (1) implies (2) and (2) implies (3).

Assume (3) and let  $Z \rightarrow Y$  be a morphism where  $Z$  is an affine scheme. Let  $U, V$  be affine schemes and let  $a : U \rightarrow Z \times_Y X$  and  $b : V \rightarrow Z \times_Y X$  be morphisms. Then

$$U \times_{Z \times_Y X} V = (Z \times_Y X) \times_{\Delta, (Z \times_Y X) \times_Z (Z \times_Y X)} (U \times_Z V)$$

and we see that this is quasi-compact if  $\mathcal{P}$  = “quasi-separated” or an affine scheme equipped with a closed immersion into  $U \times_Z V$  if  $\mathcal{P}$  = “separated”. Thus (4) holds.

Assume (4) and let  $Z \rightarrow Y$  be a morphism where  $Z$  is an affine scheme. Let  $U, V$  be affine schemes and let  $a : U \rightarrow Z \times_Y X$  and  $b : V \rightarrow Z \times_Y X$  be morphisms. Reading the argument above backwards, we see that  $U \times_{Z \times_Y X} V \rightarrow U \times_Z V$  is quasi-compact if  $\mathcal{P}$  = “quasi-separated” or a closed immersion if  $\mathcal{P}$  = “separated”. Since we can find an étale covering of  $Z \times_Y X$  by  $U$  and  $V$  as above, we find that the corresponding morphisms

$$U \times_Z V \rightarrow (Z \times_Y X) \times_Z (Z \times_Y X)$$

form an étale covering by affines. Hence we conclude that  $\Delta : (Z \times_Y X) \rightarrow (Z \times_Y X) \times_Z (Z \times_Y X)$  is quasi-compact, resp. a closed immersion. Thus (3) holds.

Let us prove that (3) implies (5). Assume (3) and let  $\{Y_j \rightarrow Y\}$  be as in Definition 7.1. We have to show that the morphisms

$$\Delta_j : Y_j \times_Y X \rightarrow (Y_j \times_Y X) \times_{Y_j} (Y_j \times_Y X) = Y_j \times_Y X \times_Y X$$

has the corresponding property (i.e., is quasi-compact or a closed immersion). Write  $Y_j = \operatorname{colim} Y_{j,\lambda}$  as in Definition 5.1. Replacing  $Y_j$  by  $Y_{j,\lambda}$  in the formula above, we have the property by our assumption that (3) holds. Since the displayed arrow is the colimit of the arrows  $\Delta_{j,\lambda}$  and since we can test whether  $\Delta_j$  has the corresponding property by testing after base change by affine schemes mapping into  $Y_j \times_Y X \times_Y X$ , we conclude by Lemma 5.4.

Let us prove that (5) implies (1). Let  $\{Y_j \rightarrow Y\}$  be as in (5). Then we have the fibre product diagram

$$\begin{array}{ccc} \coprod Y_j \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod Y_j \times_Y X \times_Y X & \longrightarrow & X \times_Y X \end{array}$$

By assumption the left vertical arrow is quasi-compact or a closed immersion. It follows from Spaces, Lemma 5.6 that also the right vertical arrow is quasi-compact or a closed immersion.  $\square$

## 22. Proper morphisms

Here is the definition we will use.

**Definition 22.1.** Let  $S$  be a scheme. Let  $f : Y \rightarrow X$  be a morphism of formal algebraic spaces over  $S$ . We say  $f$  is *proper* if  $f$  is representable by algebraic spaces and is proper in the sense of Bootstrap, Definition 4.1.

It follows from the definitions that a proper morphism is of finite type.

**Lemma 22.2.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of formal algebraic spaces over  $S$ . The following are equivalent*

- (1)  $f$  is proper,
- (2) for every scheme  $Z$  and morphism  $Z \rightarrow Y$  the base change  $Z \times_Y X \rightarrow Z$  of  $f$  is proper,
- (3) for every affine scheme  $Z$  and every morphism  $Z \rightarrow Y$  the base change  $Z \times_Y X \rightarrow Z$  of  $f$  is proper,
- (4) for every affine scheme  $Z$  and every morphism  $Z \rightarrow Y$  the formal algebraic space  $Z \times_Y X$  is an algebraic space proper over  $Z$ ,
- (5) there exists a covering  $\{Y_j \rightarrow Y\}$  as in Definition 7.1 such that the base change  $Y_j \times_Y X \rightarrow Y_j$  is proper for all  $j$ .

**Proof.** Omitted. □

### 23. Formal algebraic spaces and fpqc coverings

This section is the analogue of Properties of Spaces, Section 14. Please read that section first.

**Lemma 23.1.** *Let  $S$  be a scheme. Let  $X$  be a formal algebraic space over  $S$ . Then  $X$  satisfies the sheaf property for the fpqc topology.*

**Proof.** The proof is **identical** to the proof of Properties of Spaces, Proposition 14.1. Since  $X$  is a sheaf for the Zariski topology it suffices to show the following. Given a surjective flat morphism of affines  $f : T' \rightarrow T$  we have:  $X(T)$  is the equalizer of the two maps  $X(T') \rightarrow X(T' \times_T T')$ . See Topologies, Lemma 8.13.

Let  $a, b : T \rightarrow X$  be two morphisms such that  $a \circ f = b \circ f$ . We have to show  $a = b$ . Consider the fibre product

$$E = X \times_{\Delta_{X/S}, X \times_S X, (a,b)} T.$$

By Lemma 7.3 the morphism  $\Delta_{X/S}$  is a representable monomorphism. Hence  $E \rightarrow T$  is a monomorphism of schemes. Our assumption that  $a \circ f = b \circ f$  implies that  $T' \rightarrow T$  factors (uniquely) through  $E$ . Consider the commutative diagram

$$\begin{array}{ccc} T' \times_T E & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ T' & \longrightarrow & T \end{array}$$

Since the projection  $T' \times_T E \rightarrow T'$  is a monomorphism with a section we conclude it is an isomorphism. Hence we conclude that  $E \rightarrow T$  is an isomorphism by Descent, Lemma 19.15. This means  $a = b$  as desired.

Next, let  $c : T' \rightarrow X$  be a morphism such that the two compositions  $T' \times_T T' \rightarrow T' \rightarrow X$  are the same. We have to find a morphism  $a : T \rightarrow X$  whose composition with  $T' \rightarrow T$  is  $c$ . Choose a formal affine scheme  $U$  and an étale morphism  $U \rightarrow X$  such that the image of  $|U| \rightarrow |X_{red}|$  contains the image of  $|c| : |T'| \rightarrow |X_{red}|$ . This is possible by Definition 7.1, Properties of Spaces, Lemma 4.6, the fact that a finite union of formal affine algebraic spaces is a formal affine algebraic space, and the

fact that  $|T'|$  is quasi-compact (small argument omitted). The morphism  $U \rightarrow X$  is representable by schemes (Lemma 5.11) and separated (Lemma 11.5). Thus

$$V = U \times_{X,c} T' \longrightarrow T'$$

is an étale and separated morphism of schemes. It is also surjective by our choice of  $U \rightarrow X$  (if you do not want to argue this you can replace  $U$  by a disjoint union of formal affine algebraic spaces so that  $U \rightarrow X$  is surjective everything else still works as well). The fact that  $c \circ \text{pr}_0 = c \circ \text{pr}_1$  means that we obtain a descent datum on  $V/T'/T$  (Descent, Definition 30.1) because

$$\begin{aligned} V \times_{T'} (T' \times_T T') &= U \times_{X, c \circ \text{pr}_0} (T' \times_T T') \\ &= (T' \times_T T') \times_{c \circ \text{pr}_1, X} U \\ &= (T' \times_T T') \times_{T'} V \end{aligned}$$

The morphism  $V \rightarrow T'$  is ind-quasi-affine by More on Morphisms, Lemma 46.4 (because étale morphisms are locally quasi-finite, see Morphisms, Lemma 37.6). By More on Groupoids, Lemma 14.3 the descent datum is effective. Say  $W \rightarrow T$  is a morphism such that there is an isomorphism  $\alpha : T' \times_T W \rightarrow V$  compatible with the given descent datum on  $V$  and the canonical descent datum on  $T' \times_T W$ . Then  $W \rightarrow T$  is surjective and étale (Descent, Lemmas 19.6 and 19.27). Consider the composition

$$b' : T' \times_T W \longrightarrow V = U \times_{X,c} T' \longrightarrow U$$

The two compositions  $c' \circ (\text{pr}_0, 1), c' \circ (\text{pr}_1, 1) : (T' \times_T T') \times_T W \rightarrow T' \times_T W \rightarrow U$  agree by our choice of  $\alpha$  and the corresponding property of  $c$  (computation omitted). Hence  $b'$  descends to a morphism  $b : W \rightarrow U$  by Descent, Lemma 9.3. The diagram

$$\begin{array}{ccccc} T' \times_T W & \longrightarrow & W & \xrightarrow{b} & U \\ \downarrow & & & & \downarrow \\ T' & \xrightarrow{c} & & & X \end{array}$$

is commutative. What this means is that we have proved the existence of a étale locally on  $T$ , i.e., we have an  $a' : W \rightarrow X$ . However, since we have proved uniqueness in the first paragraph, we find that this étale local solutions satisfy the glueing condition, i.e., we have  $\text{pr}_0^* a' = \text{pr}_1^* a'$  as elements of  $X(W \times_T W)$ . Since  $X$  is an étale sheaf we find an unique  $a \in X(T)$  restricting to  $a'$  on  $W$ .  $\square$

## 24. Maps out of affine formal schemes

We prove a few results that will be useful later. In the paper [Bha14] the reader can find very general results of a similar nature.

**Lemma 24.1.** *Let  $S$  be a scheme. Let  $A$  be a weakly admissible topological  $S$ -algebra. Let  $X$  be an affine scheme over  $S$ . Then the natural map*

$$\text{Mor}_S(\text{Spec}(A), X) \longrightarrow \text{Mor}_S(\text{Spf}(A), X)$$

*is bijective.*

**Proof.** If  $X$  is affine, say  $X = \text{Spec}(B)$ , then we see from Lemma 5.10 that morphisms  $\text{Spf}(A) \rightarrow \text{Spec}(B)$  correspond to continuous  $S$ -algebra maps  $B \rightarrow A$  where  $B$  has the discrete topology. These are just  $S$ -algebra maps, which correspond to morphisms  $\text{Spec}(A) \rightarrow \text{Spec}(B)$ .  $\square$

**Lemma 24.2.** *Let  $S$  be a scheme. Let  $A$  be a weakly admissible topological  $S$ -algebra such that  $A/I$  is a local ring for some weak ideal of definition  $I \subset A$ . Let  $X$  be a scheme over  $S$ . Then the natural map*

$$\mathrm{Mor}_S(\mathrm{Spec}(A), X) \longrightarrow \mathrm{Mor}_S(\mathrm{Spf}(A), X)$$

*is bijective.*

**Proof.** Let  $\varphi : \mathrm{Spf}(A) \rightarrow X$  be a morphism. Since  $\mathrm{Spec}(A/I)$  is local we see that  $\varphi$  maps  $\mathrm{Spec}(A/I)$  into an affine open  $U \subset X$ . However, this then implies that  $\mathrm{Spec}(A/J)$  maps into  $U$  for every ideal of definition  $J$ . Hence we may apply Lemma 24.1 to see that  $\varphi$  comes from a morphism  $\mathrm{Spec}(A) \rightarrow X$ . This proves surjectivity of the map. We omit the proof of injectivity.  $\square$

**Lemma 24.3.** *Let  $S$  be a scheme. Let  $R$  be a complete local Noetherian  $S$ -algebra. Let  $X$  be an algebraic space over  $S$ . Then the natural map*

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \mathrm{Mor}_S(\mathrm{Spf}(R), X)$$

*is bijective.*

**Proof.** Let  $\mathfrak{m}$  be the maximal ideal of  $R$ . We have to show that

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n), X)$$

is bijective for  $R$  as above.

**Injectivity:** Let  $x, x' : \mathrm{Spec}(R) \rightarrow X$  be two morphisms mapping to the same element in the right hand side. Consider the fibre product

$$T = \mathrm{Spec}(R) \times_{(x, x'), X \times_S X, \Delta} X$$

Then  $T$  is a scheme and  $T \rightarrow \mathrm{Spec}(R)$  is locally of finite type, monomorphism, separated, and locally quasi-finite, see Morphisms of Spaces, Lemma 4.1. In particular  $T$  is locally Noetherian, see Morphisms, Lemma 16.6. Let  $t \in T$  be the unique point mapping to the closed point of  $\mathrm{Spec}(R)$  which exists as  $x$  and  $x'$  agree over  $R/\mathfrak{m}$ . Then  $R \rightarrow \mathcal{O}_{T,t}$  is a local ring map of Noetherian rings such that  $R/\mathfrak{m}^n \rightarrow \mathcal{O}_{T,t}/\mathfrak{m}^n \mathcal{O}_{T,t}$  is an isomorphism for all  $n$  (because  $x$  and  $x'$  agree over  $\mathrm{Spec}(R/\mathfrak{m}^n)$  for all  $n$ ). Since  $\mathcal{O}_{T,t}$  maps injectively into its completion (see Algebra, Lemma 49.4) we conclude that  $R = \mathcal{O}_{T,t}$ . Hence  $x$  and  $x'$  agree over  $R$ .

**Surjectivity:** Let  $(x_n)$  be an element of the right hand side. Choose a scheme  $U$  and a surjective étale morphism  $U \rightarrow X$ . Denote  $x_0 : \mathrm{Spec}(k) \rightarrow X$  the morphism induced on the residue field  $k = R/\mathfrak{m}$ . The morphism of schemes  $U \times_{X, x_0} \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k)$  is surjective étale. Thus  $U \times_{X, x_0} \mathrm{Spec}(k)$  is a nonempty disjoint union of spectra of finite separable field extensions of  $k$ , see Morphisms, Lemma 37.7. Hence we can find a finite separable field extension  $k \subset k'$  and a  $k'$ -point  $u_0 : \mathrm{Spec}(k') \rightarrow U$  such that

$$\begin{array}{ccc} \mathrm{Spec}(k') & \xrightarrow{u_0} & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{x_0} & X \end{array}$$

commutes. Let  $R \subset R'$  be the finite étale extension of Noetherian complete local rings which induces  $k \subset k'$  on residue fields (see Algebra, Lemmas 145.8 and 145.10). Denote  $x'_n$  the restriction of  $x_n$  to  $\mathrm{Spec}(R'/\mathfrak{m}^n R')$ . By More on Morphisms of Spaces, Lemma 13.8 we can find an element  $(u'_n) \in \lim \mathrm{Mor}_S(\mathrm{Spec}(R'/\mathfrak{m}^n R'), U)$

mapping to  $(x'_n)$ . By Lemma 24.2 the family  $(u'_n)$  comes from a unique morphism  $u' : \mathrm{Spec}(R') \rightarrow U$ . Denote  $x' : \mathrm{Spec}(R') \rightarrow X$  the composition. Note that  $R' \otimes_R R'$  is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Hence the diagram

$$\begin{array}{ccc} \mathrm{Spec}(R' \otimes_R R') & \longrightarrow & \mathrm{Spec}(R') \\ \downarrow & & \downarrow x' \\ \mathrm{Spec}(R') & \xrightarrow{x'} & X \end{array}$$

is commutative by the injectivity shown above and the fact that  $x'_n$  is the restriction of  $x_n$  which is defined over  $R/\mathfrak{m}^n$ . Since  $\{\mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)\}$  is an fppf covering we conclude that  $x'$  descends to a morphism  $x : \mathrm{Spec}(R) \rightarrow X$ . We omit the proof that  $x_n$  is the restriction of  $x$  to  $\mathrm{Spec}(R/\mathfrak{m}^n)$ .  $\square$

## 25. Other chapters

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