COHOMOLOGY OF SHEAVES

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1. Introduction

In this document we work out some topics on cohomology of sheaves on topological spaces. We mostly work in the generality of modules over a sheaf of rings and we work with morphisms of ringed spaces. To see what happens for sheaves on sites take a look at the chapter Cohomology on Sites, Section 1. Basic references are [God73] and [Ive86].

2. Topics

Here are some topics that should be discussed in this chapter, and have not yet been written.

(1) Ext-groups.
(2) Ext sheaves.
(3) Tor functors.
(4) Derived pullback for morphisms between ringed spaces.
(5) Cup-product.
(6) Etc, etc, etc.

3. Cohomology of sheaves

Let X be a topological space. Let $\mathcal{F}$ be a abelian sheaf. We know that the category of abelian sheaves on X has enough injectives, see Injectives, Lemma 4.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet)) \tag{3.0.1}$$

to be the *ith cohomology group of the abelian sheaf* $\mathcal{F}$. The family of functors $H^i((X, -)$ forms a universal δ-functor from $Ab(X) \to Ab$.

Let $f : X \to Y$ be a continuous map of topological spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^if_*\mathcal{F} = H^i(f_*\mathcal{I}^\bullet) \tag{3.0.2}$$

to be the *ith higher direct image of* $\mathcal{F}$. The family of functors R^if_* forms a universal δ-functor from $Ab(X) \to Ab(Y)$.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We know that the category of $\mathcal{O}_X$-modules on X has enough injectives, see Injectives, Lemma 5.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet)) \tag{3.0.3}$$

to be the *ith cohomology group of* $\mathcal{F}$. The family of functors $H^i((X, -)$ forms a universal δ-functor from $Mod(\mathcal{O}_X) \to \text{Mod}_{\mathcal{O}_X(X)}$.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^if_*\mathcal{F} = H^i(f_*\mathcal{I}^\bullet) \tag{3.0.4}$$

to be the *ith higher direct image of* $\mathcal{F}$. The family of functors R^if_* forms a universal δ-functor from $Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$.

4. Derived functors

We briefly explain an approach to right derived functors using resolution functors. Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is abelian, see Modules, Lemma 3.1. In this chapter we will write

$$K(X) = K(\mathcal{O}_X) = K(Mod(\mathcal{O}_X)) \quad \text{and} \quad D(X) = D(\mathcal{O}_X) = D(Mod(\mathcal{O}_X)).$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 8.1 and Definition 11.3. By Derived Categories, Remark 24.3 there exists a resolution functor

$$j = j_X : K^+(Mod(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $Mod(\mathcal{O}_X)$ consisting of injective sheaves. For any left exact functor $F : Mod(\mathcal{O}_X) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote RF the right derived functor described in Derived Categories, Section 20 and constructed using the resolution functor j_X just described:

$$RF = F \circ j'_X : D^+(X) \longrightarrow D^+(\mathcal{B}) \tag{4.0.5}$$

see Derived Categories, Lemma 25.1 for notation. Note that we may think of RF as defined on $Mod(\mathcal{O}_X)$, $\text{Comp}^+(Mod(\mathcal{O}_X))$, $K^+(X)$, or $D^+(X)$ depending on the situation. According to Derived Categories, Definition 17.2 we obtain the ith right derived functor

$$R^iF = H^i \circ RF : Mod(\mathcal{O}_X) \longrightarrow \mathcal{B} \tag{4.0.6}$$

so that $R^0F = F$ and $\{R^iF, \delta\}_{i \geq 0}$ is universal δ-functor, see Derived Categories, Lemma 20.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\text{Mod}_R)$ and $D(R) = D(\text{Mod}_R)$ and similarly for bounded versions. For any open $U \subset X$ we have a left exact functor $\Gamma(U, -) : Mod(\mathcal{O}_X) \longrightarrow \text{Mod}_{\mathcal{O}_X(U)}$ which gives rise to

$$R\Gamma(U, -) : D^+(X) \longrightarrow D^+(\mathcal{O}_X(U)) \tag{4.0.7}$$

by the discussion above. We set $H^i(U, -) = R^i\Gamma(U, -)$. If $U = X$ we recover (3.0.3). If $f : X \to Y$ is a morphism of ringed spaces, then we have the left exact functor $f_* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Y)$ which gives rise to the *derived pushforward*

$$Rf_* : D^+(X) \longrightarrow D^+(Y) \tag{4.0.8}$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^if_*\mathcal{F}^\bullet$ and called the ith *higher direct image* in accordance with (3.0.4). The two displayed functors above are exact functor of derived categories.

Abuse of notation: When the functor Rf_*, or any other derived functor, is applied to a sheaf $\mathcal{F}$ on X or a complex of sheaves it is understood that $\mathcal{F}$ has been replaced by a suitable resolution of $\mathcal{F}$. To facilitate this kind of operation we will say, given an object $\mathcal{F}^\bullet \in D(X)$, that a bounded below complex $\mathcal{I}^\bullet$ of injectives of $Mod(\mathcal{O}_X)$ *represents* $\mathcal{F}^\bullet$ *in the derived category* if there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. In the same vein the phrase "let $\alpha : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a morphism of $D(X)$"

does not mean that α is represented by a morphism of complexes. If we have an actual morphism of complexes we will say so.

5. First cohomology and torsors

Definition 5.1. Let X be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on X. A *torsor*, or more precisely a *$\mathcal{G}$-torsor*, is a sheaf of sets $\mathcal{F}$ on X endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive, and
(2) for every $x \in X$ the stalk $\mathcal{F}_x$ is nonempty.

A *morphism of $\mathcal{G}$-torsors* $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

Lemma 5.2. *Let X be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on X. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\mathcal{F}(X) \neq \emptyset$.*

Proof. Omitted. □

Lemma 5.3. *Let X be a topological space. Let $\mathcal{H}$ be an abelian sheaf on X. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(X, \mathcal{H})$.*

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \subset X$ open the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by the local section of the form $[s] - [s']$. There is a canonical map $a : \mathrm{Ker}(\sigma) \to \mathcal{H}$ which maps $[s] - [s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s = s'$. Consider the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\sigma) \arrow[r] \arrow[d, "a"] & \mathbf{Z}[\mathcal{F}] \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{E} \arrow[r] & \underline{\mathbf{Z}} \arrow[r] & 0 \end{tikzcd}$$

Here $\mathcal{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_{\mathcal{F}} \in H^1(X, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(X, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(X, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(X, \mathcal{Q})$ because $H^1(X, \mathcal{I}) = 0$ (see Derived Categories, Lemma 20.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a torsor.

We omit the verification that the two constructions given above are mutually inverse. $\square$

6. First cohomology and invertible sheaves

The Picard group of a ringed space is defined in Modules, Section 21.

Lemma 6.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. There is a canonical isomorphism*

$$H^1(X, \mathcal{O}_X^*) = Pic(X).$$

of abelian groups.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s \cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}_X^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}_X^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally by the very definition of an invertible sheaf. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}_X^*$-torsor. Thus we get a map

$$\begin{matrix} \text{invertible sheaves on } (X, \mathcal{O}_X) \\ \text{up to isomorphism} \end{matrix} \longrightarrow \begin{matrix} \mathcal{O}_X^*\text{-torsors} \\ \text{up to isomorphism} \end{matrix}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 5.3 the right hand side is canonically bijective to $H^1(X, \mathcal{O}_X^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 5.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}_X$ is the neutral element in $\text{Pic}(X)$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}_X^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}_X(U))/\mathcal{O}_X^*(U)$$

where the action of $f \in \mathcal{O}_X^*(U)$ on (s, g) is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}_X$-modules by setting $(s, g) + (s', g') = (s, g + (s'/s)g')$ where s'/s is the local section f of $\mathcal{O}_X^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for h a local section of $\mathcal{O}_X$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}_X$-module whose associated $\mathcal{O}_X^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. $\square$

7. Locality of cohomology

The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an open.

Lemma 7.1. *Let X be a ringed space. Let $U \subset X$ be an open subspace.*

(1) *If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*
(2) *For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}|_U)$.*

Proof. Denote $j : U \to X$ the open immersion. Recall that the functor j^{-1} of restriction to U is a right adjoint to the functor $j_!$ of extension by 0, see Sheaves, Lemma 31.8. Moreover, $j_!$ is exact. Hence (1) follows from Homology, Lemma 25.1.

By definition $H^p(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Mod(\mathcal{O}_X)$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in

$Mod(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\Gamma(U, \mathcal{I}^\bullet|_U))$. Of course $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on X. Hence the equality in (2). $\square$

Let X be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset V \subset X$ be open subsets. Then there is a canonical *restriction mapping*

$$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U \tag{7.1.1}$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}_X$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 12.3.

Lemma 7.2. *Let X be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subspace. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\mathrm{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \mathrm{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. $\square$

Lemma 7.3. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. The sheaves $R^i f_* \mathcal{F}$ are the sheaves associated to the presheaves*

$$V \longmapsto H^i(f^{-1}(V), \mathcal{F})$$

with restriction mappings as in Equation (7.1.1). There is a similar statement for $R^i f_*$ applied to a bounded below complex $\mathcal{F}^\bullet$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^i f_* \mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_Y$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(f^{-1}(V)) \to \mathcal{I}^{i+1}(f^{-1}(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(f^{-1}(V)) \to \mathcal{I}^i(f^{-1}(V)))}$$

which is equal to $H^i(f^{-1}(V), \mathcal{F})$ and we win. □

Lemma 7.4. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Denote $g : f^{-1}(V) \to V$ the restriction of f. Then we have*

$$R^p g_*(\mathcal{F}|_{f^{-1}(V)}) = (R^p f_* \mathcal{F})|_V$$

There is a similar statement for the derived image $Rf_*\mathcal{F}^\bullet$ where $\mathcal{F}^\bullet$ is a bounded below complex of $\mathcal{O}_X$-modules.

Proof. First proof. Apply Lemmas 7.3 and 7.1 to see the displayed equality. Second proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and use that $\mathcal{F}|_{f^{-1}(V)} \to \mathcal{I}^\bullet|_{f^{-1}(V)}$ is an injective resolution also. □

Remark 7.5. Here is a different approach to the proofs of Lemmas 7.2 and 7.3 above. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i_X : \mathit{Mod}(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$ be the inclusion functor and let # be the sheafification functor. Recall that i_X is left exact and # is exact.

(1) First prove Lemma 12.3 below which says that the right derived functors of i_X are given by $R^p i_X \mathcal{F} = \underline{H}^p(\mathcal{F})$. Here is another proof: The equality is clear for $p = 0$. Both $(R^p i_X)_{p \geq 0}$ and $(\underline{H}^p)_{p \geq 0}$ are delta functors vanishing on injectives, hence both are universal, hence they are isomorphic. See Homology, Section 11.
(2) A restatement of Lemma 7.2 is that $(\underline{H}^p(\mathcal{F}))^\# = 0$, $p > 0$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. To see this is true, use that $^\#$ is exact so
$$(\underline{H}^p(\mathcal{F}))^\# = (R^p i_X \mathcal{F})^\# = R^p(\# \circ i_X)(\mathcal{F}) = 0$$
because $\# \circ i_X$ is the identity functor.
(3) Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The presheaf $V \mapsto H^p(f^{-1}V, \mathcal{F})$ is equal to $R^p(i_Y \circ f_*)\mathcal{F}$. You can prove this by noticing that both give universal delta functors as in the argument of (1) above. Hence Lemma 7.3 says that $R^p f_* \mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$. Again using that # is exact a that $\# \circ i_Y$ is the identity functor we see that
$$R^p f_* \mathcal{F} = R^p(\# \circ i_Y \circ f_*)\mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$$
as desired.

8. Projection formula

In this section we collect variants of the projection formula. The most basic version is Lemma 8.2.

Lemma 8.1. *Let X be a ringed space. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{E}$ is finite locally free on X, see Modules, Definition 14.1. Then $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}$ is an injective $\mathcal{O}_X$-module.*

Proof. This is true because under the assumptions of the lemma we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\wedge, \mathcal{I})$$

where $\mathcal{E}^\wedge = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual of $\mathcal{E}$ which is finite locally free also. Since tensoring with a finite locally free sheaf is an exact functor we win by Homology, Lemma 23.2. □

Lemma 8.2. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Assume $\mathcal{E}$ is finite locally free on Y, see Modules, Definition 14.1. Then there exist isomorphisms*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} R^q f_* \mathcal{F} \longrightarrow R^q f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

for all $q \geq 0$. In fact there exists an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_Y} Rf_* \mathcal{F} \longrightarrow Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

in $D^+(Y)$ functorial in $\mathcal{F}$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on X. Note that $f^*\mathcal{E}$ is finite locally free also, hence we get a resolution

$$f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet$$

which is an injective resolution by Lemma 8.1. Apply f_* to see that

$$Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet).$$

Hence the lemma follows if we can show that $f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{O}_Y} f_*(\mathcal{F})$ functorially in the $\mathcal{O}_X$-module $\mathcal{F}$. This is clear when $\mathcal{E} = \mathcal{O}_Y^{\oplus n}$, and follows in general by working locally on Y. Details omitted. □

9. Mayer-Vietoris

Below will construct the Čech-to-cohomology spectral sequence, see Lemma 12.4. A special case of that spectral sequence is the Mayer-Vietoris long exact sequence. Since it is such a basic, useful and easy to understand variant of the spectral sequence we treat it here separately.

Lemma 9.1. *Let X be a ringed space. Let $U' \subset U \subset X$ be open subspaces. For any injective $\mathcal{O}_X$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.*

Proof. Let $j : U \to X$ and $j' : U' \to X$ be the open immersions. Recall that $j_!\mathcal{O}_U$ is the extension by zero of $\mathcal{O}_U = \mathcal{O}_X|_U$, see Sheaves, Section 31. Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

see Sheaves, Lemma 31.8. Similarly, the sheaf $j'_!\mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover there is an obvious canonical map of $\mathcal{O}_X$-modules

$$j'_!\mathcal{O}_{U'} \longrightarrow j_!\mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 3.5). By the description of the stalks of the sheaves $j'_!\mathcal{O}_{U'}$, $j_!\mathcal{O}_U$ we see that the displayed map above is injective (see lemma cited above). Hence if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then the map

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{I}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(j'_!\mathcal{O}_{U'}, \mathcal{I})$$

is surjective, see Homology, Lemma 23.2. Putting everything together we obtain the lemma. □

Lemma 9.2 (Mayer-Vietoris)**.** *Let X be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact cohomology sequence*

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. The sheaf condition says that the kernel of $(1, -1) : \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$ is equal to the image of $\mathcal{F}(X)$ by the first map for any abelian sheaf $\mathcal{F}$. Lemma 9.1 above implies that the map $(1, -1) : \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V)$ is surjective whenever $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. Hence if $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$, then we get a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

Taking cohomology gives the result (use Homology, Lemma 12.12). We omit the proof of the functoriality of the sequence. □

Lemma 9.3 (Relative Mayer-Vietoris)**.** *Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. We claim that we get a short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \cap V} \to 0.$$

Namely, for any open $W \subset Y$, and for any $n \geq 0$ the corresponding sequence of groups of sections over W

$$0 \to \mathcal{I}^n(f^{-1}(W)) \to \mathcal{I}^n(U \cap f^{-1}(W)) \oplus \mathcal{I}^n(V \cap f^{-1}(W)) \to \mathcal{I}^n(U \cap V \cap f^{-1}(W)) \to 0$$

was shown to be short exact in the proof of Lemma 9.2. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^\bullet|_V$, $\mathcal{I}^\bullet|_{U \cap V}$ see Lemma 7.1. □

10. The Čech complex and Čech cohomology

Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering, see Topology, Basic notion (10). As is customary we denote $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ for the $(p+1)$-fold intersection of members of $\mathcal{U}$. Let $\mathcal{F}$ be an abelian presheaf on X. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0 \ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in $\mathcal{F}(U_{i_0 \ldots i_p})$. Note that if $s \in \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{F})$ and $i, j \in I$ then s_{ij} and s_{ji} are both elements of $\mathcal{F}(U_i \cap U_j)$ but there is no imposed relation between s_{ij} and s_{ji}. In other words, we are *not* working with alternating cochains (these will be defined in Section 24). We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}|_{U_{i_0 \ldots i_{p+1}}} \tag{10.0.1}$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 10.1. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Čech cohomology groups* associated to $\mathcal{F}$ and the covering $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

Lemma 10.2. *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. The following are equivalent*

(1) *$\mathcal{F}$ is an abelian sheaf and*
(2) *for every open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ the natural map*

$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$

is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every open covering. $\square$

11. Čech cohomology as a functor on presheaves

Warning: In this section we work almost exclusively with presheaves and categories of presheaves and the results are completely wrong in the setting of sheaves and categories of sheaves!

Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules. We have the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ just by thinking of $\mathcal{F}$ as a presheaf of abelian groups. However, each term $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ has a natural structure of a $\mathcal{O}_X(U)$-module and the differential is given by $\mathcal{O}_X(U)$-module maps. Moreover, it is clear that the construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : \textit{PMod}(\mathcal{O}_X) \longrightarrow \text{Comp}^+(\text{Mod}_{\mathcal{O}_X(U)}) \tag{11.0.1}$$

see Derived Categories, Definition 8.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 12.9.

Lemma 11.1. *The functor given by Equation (11.0.1) is an exact functor (see Homology, Lemma 7.1).*

Proof. For any open $W \subset U$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $\textit{PMod}(\mathcal{O}_X)$ to $\text{Mod}_{\mathcal{O}_X(U)}$. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. $\square$

Lemma 11.2. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules (see Homology, Definition 11.1).*

Proof. By Lemma 11.1 a short exact sequence of presheaves of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathcal{O}_X(U)$-modules. Hence we can use Homology, Lemma 12.12 to get the boundary maps $\delta_{\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. □

In the formulation of the following lemma we use the functor $j_{p!}$ of extension by 0 for presheaves of modules relative to an open immersion $j : U \to X$. See Sheaves, Section 31. For any open $W \subset X$ and any presheaf $\mathcal{G}$ of $\mathcal{O}_X|_U$-modules we have

$$(j_{p!}\mathcal{G})(W) = \begin{cases} \mathcal{G}(W) & \text{if } W \subset U \\ 0 & \text{else.} \end{cases}$$

Moreover, the functor $j_{p!}$ is a left adjoint to the restriction functor see Sheaves, Lemma 31.8. In particular we have the following formula

$$\mathrm{Hom}_{\mathcal{O}_X}(j_{p!}\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U).$$

Since the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor on the category of presheaves we conclude that the presheaf $j_{p!}\mathcal{O}_U$ is a projective object in the category *PMod*($\mathcal{O}_X$), see Homology, Lemma 24.2.

Note that if we are given open subsets $U \subset V \subset X$ with associated open immersions j_U, j_V, then we have a canonical map $(j_U)_{p!}\mathcal{O}_U \to (j_V)_{p!}\mathcal{O}_V$. It is the identity on sections over any open $W \subset U$ and 0 else. In terms of the identification $\mathrm{Hom}_{\mathcal{O}_X}((j_U)_{p!}\mathcal{O}_U, (j_V)_{p!}\mathcal{O}_V) = (j_V)_{p!}\mathcal{O}_V(U) = \mathcal{O}_V(U)$ it corresponds to the element $1 \in \mathcal{O}_V(U)$.

Lemma 11.3. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Denote $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to X$ the open immersion. Consider the chain complex $K(\mathcal{U})_\bullet$ of presheaves of $\mathcal{O}_X$-modules*

$$\ldots \to \bigoplus_{i_0 i_1 i_2} (j_{i_0 i_1 i_2})_{p!}\mathcal{O}_{U_{i_0 i_1 i_2}} \to \bigoplus_{i_0 i_1} (j_{i_0 i_1})_{p!}\mathcal{O}_{U_{i_0 i_1}} \to \bigoplus_{i_0} (j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$(j_{i_0 \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0 \ldots i_{p+1}}} \longrightarrow (j_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in \mathrm{Ob}(PMod(\mathcal{O}_X))$.

Proof. We saw in the discussion just above the lemma that

$$\mathrm{Hom}_{\mathcal{O}_X}((j_{i_0 \ldots i_p})_{p!}\mathcal{O}_{U_{i_0 \ldots i_p}}, \mathcal{F}) = \mathcal{F}(U_{i_0 \ldots i_p}).$$

Hence we see that it is indeed the case that the direct sum

$$\bigoplus\nolimits_{i_0 \ldots i_p} (j_{i_0 \ldots i_p})_{p!}\mathcal{O}_{U_{i_0 \ldots i_p}}$$

represents the functor

$$\mathcal{F} \longmapsto \prod\nolimits_{i_0 \ldots i_p} \mathcal{F}(U_{i_0 \ldots i_p}).$$

Hence by Categories, Yoneda Lemma 3.5 we see that there is a complex $K(\mathcal{U})_\bullet$ with terms as given. It is a simple matter to see that the maps are as given in the lemma. □

Lemma 11.4. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be a covering. Let $\mathcal{O}_\mathcal{U} \subset \mathcal{O}_X$ be the image presheaf of the map $\bigoplus j_{p!}\mathcal{O}_{U_i} \to \mathcal{O}_X$. The chain complex $K(\mathcal{U})_\bullet$ of presheaves of Lemma 11.3 above has homology presheaves*

$$H_i(K(\mathcal{U})_\bullet) = \begin{cases} 0 & if \quad i \neq 0 \\ \mathcal{O}_\mathcal{U} & if \quad i = 0 \end{cases}$$

Proof. Consider the extended complex $K^{ext}_\bullet$ one gets by putting $\mathcal{O}_\mathcal{U}$ in degree -1 with the obvious map $K(\mathcal{U})_0 = \bigoplus_{i_0} (j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to \mathcal{O}_\mathcal{U}$. It suffices to show that taking sections of this extended complex over any open $W \subset X$ leads to an acyclic complex. In fact, we claim that for every $W \subset X$ the complex $K^{ext}_\bullet(W)$ is homotopy equivalent to the zero complex. Write $I = I_1 \amalg I_2$ where $W \subset U_i$ if and only if $i \in I_1$.

If $I_1 = \emptyset$, then the complex $K^{ext}_\bullet(W) = 0$ so there is nothing to prove.

If $I_1 \neq \emptyset$, then $\mathcal{O}_\mathcal{U}(W) = \mathcal{O}_X(W)$ and

$$K^{ext}_p(W) = \bigoplus\nolimits_{i_0\ldots i_p\in I_1} \mathcal{O}_X(W).$$

This is true because of the simple description of the presheaves $(j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$. Moreover, the differential of the complex $K^{ext}_\bullet(W)$ is given by

$$d(s)_{i_0\ldots i_p} = \sum\nolimits_{j=0,\ldots,p+1} \sum\nolimits_{i\in I_1} (-1)^j s_{i_0\ldots i_{j-1} i i_j\ldots i_p}.$$

The sum is finite as the element s has finite support. Fix an element $i_{\text{fix}} \in I_1$. Define a map

$$h : K^{ext}_p(W) \longrightarrow K^{ext}_{p+1}(W)$$

by the rule

$$h(s)_{i_0\ldots i_{p+1}} = \begin{cases} 0 & \text{if} \quad i_0 \neq i \\ s_{i_1\ldots i_{p+1}} & \text{if} \quad i_0 = i_{\text{fix}} \end{cases}$$

We will use the shorthand $h(s)_{i_0\ldots i_{p+1}} = (i_0 = i_{\text{fix}}) s_{i_1\ldots i_p}$ for this. Then we compute

$$\begin{aligned} & (dh + hd)(s)_{i_0\ldots i_p} \\ = & \sum_j \sum_{i\in I_1} (-1)^j h(s)_{i_0\ldots i_{j-1} i i_j\ldots i_p} + (i = i_0) d(s)_{i_1\ldots i_p} \\ = & s_{i_0\ldots i_p} + \sum_{j\geq 1} \sum_{i\in I_1} (-1)^j (i_0 = i_{\text{fix}}) s_{i_1\ldots i_{j-1} i i_j\ldots i_p} + (i_0 = i_{\text{fix}}) d(s)_{i_1\ldots i_p} \end{aligned}$$

which is equal to $s_{i_0\ldots i_p}$ as desired. □

Lemma 11.5. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering of $U \subset X$. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : PMod(\mathcal{O}_X) \longrightarrow Mod_{\mathcal{O}_X(U)}.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the right derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(PMod(\mathcal{O}_X)) \longrightarrow D^+(\mathcal{O}_X(U))$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of presheaves of $\mathcal{O}_X$-modules has enough injectives, see Injectives, Proposition 8.5. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 20.

Let $\mathcal{I}$ be a injective presheaf of $\mathcal{O}_X$-modules. In this case the functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact on $PMod(\mathcal{O}_X)$. By Lemma 11.3 we have

$$\mathrm{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 11.4 we have that $K(\mathcal{U})_\bullet$ is quasi-isomorphic to $\mathcal{O}_\mathcal{U}[0]$. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 11.2) satisfies the assumptions of Homology, Lemma 11.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 20.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 11.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any presheaf of $\mathcal{O}_X$-modules. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $PMod(\mathcal{O}_X)$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q).$$

Consider the simple complex $sA^\bullet$ associated to this double complex. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to A^{p,0} = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to A^{0,q} = \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 22.7. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 11.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathcal{O}_X(U))$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. □

12. Čech cohomology and cohomology

Lemma 12.1. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. An injective $\mathcal{O}_X$-module is also injective as an object in the category $PMod(\mathcal{O}_X)$ (for example since sheafification is an exact left adjoint to the inclusion functor, using Homology, Lemma 25.1). Hence we can apply Lemma 11.5 (or its proof) to see the result. □

Lemma 12.2. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $Mod(\mathcal{O}_X) \to D^+(\mathcal{O}_X(U))$. In particular this provides canonical maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $Mod(\mathcal{O}_X)$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 22.7 to see that α is a quasi-isomorphism. Namely, Lemma 12.1 implies that the qth row of the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathcal{O}_X(U))$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

Lemma 12.3. *Let X be a ringed space. Consider the functor $i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X)$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open U are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

Lemma 12.4. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 22.2) for the functors

$$i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PMod(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(U)}.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 10.2. We have that $i(\mathcal{I})$ is Čech acyclic by Lemma 12.1. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $PMod(\mathcal{O}_X)$ by Lemma 11.5. Putting everything together gives the lemma. □

Lemma 12.5. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that $H^i(U_{i_0 \ldots i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$ as $\mathcal{O}_X(U)$-modules.*

Proof. We will use the spectral sequence of Lemma 12.4. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. $\square$

Lemma 12.6. *Let X be a ringed space. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subset. If there exists a cofinal system of open coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose an open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} - s_{i_0}|_{U_{i_0 i_1}}.$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0 i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0 i_1}} - t_{i_0}|_{U_{i_0 i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. $\square$

Lemma 12.7. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$$

for all $p > 0$ and any open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ of an open of X. Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any open $U \subset X$.

Proof. Let $\mathcal{F}$ be a sheaf satisfying the assumption of the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any open covering". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 12.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any open covering. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 12.6 and our assumptions this sequence is actually exact as a sequence of presheaves! In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U}$, see Lemma 11.2 for example. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all open coverings.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\
& H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\
& \ldots & \ldots & \ldots
\end{tikzcd}$$

for any open $U \subset X$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

Lemma 12.8. *(Variant of Lemma 12.7.) Let X be a ringed space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there exists a set of open coverings Cov with the following properties:*

1. *For every $\mathcal{U} \in Cov$ with $\mathcal{U} : U = \bigcup_{i \in I} U_i$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0 \ldots i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the open coverings of U occurring in Cov is a cofinal system of open coverings of U.*
3. *For every $\mathcal{U} \in Cov$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 12.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 12.6 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in$ Cov we get a short exact sequence of Čech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U} \in$ Cov. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in$ Cov.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\
& H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\
& \ldots & \ldots & \ldots
\end{tikzcd}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in$ Cov we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

Lemma 12.9. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

(1) $\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ *for all* $p > 0$ *and any open covering* $\mathcal{V} : V = \bigcup_{j\in J} V_j$ *of* Y.
(2) $H^p(V, f_*\mathcal{I}) = 0$ *for all* $p > 0$ *and every open* $V \subset Y$.

In other words, $f_*\mathcal{I}$ is right acyclic for $\Gamma(U, -)$ (see Derived Categories, Definition 16.3) for any $U \subset X$ open.

Proof. Set $\mathcal{U} : f^{-1}(V) = \bigcup_{j\in J} f^{-1}(V_j)$. It is an open covering of X and

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

This is true because

$$f_*\mathcal{I}(V_{j_0\ldots j_p}) = \mathcal{I}(f^{-1}(V_{j_0\ldots j_p})) = \mathcal{I}(f^{-1}(V_{j_0}) \cap \ldots \cap f^{-1}(V_{j_p})) = \mathcal{I}(U_{j_0\ldots j_p}).$$

Thus the first statement of the lemma follows from Lemma 12.1. The second statement follows from the first and Lemma 12.7. □

The following lemma implies in particular that $f_* : Ab(X) \to Ab(Y)$ transforms injective abelian sheaves into injective abelian sheaves.

Lemma 12.10. *Let $f : X \to Y$ be a morphism of ringed spaces. Assume f is flat. Then $f_*\mathcal{I}$ is an injective $\mathcal{O}_Y$-module for any injective $\mathcal{O}_X$-module $\mathcal{I}$.*

Proof. In this case the functor f^* transforms injections into injections (Modules, Lemma 17.2). Hence the result follows from Homology, Lemma 25.1. □

13. Flasque sheaves

Here is the definition.

Definition 13.1. Let X be a topological space. We say a presheaf of sets $\mathcal{F}$ is *flasque* or *flabby* if for every $U \subset V$ open in X the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

We will use this terminology also for abelian sheaves and sheaves of modules if X is a ringed space. Clearly it suffices to assume the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for every open $U \subset X$.

Lemma 13.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then any injective $\mathcal{O}_X$-module is flasque.*

Proof. This is a reformulation of Lemma 9.1. □

Lemma 13.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any flasque $\mathcal{O}_X$-module is acyclic for $R\Gamma(X, -)$ as well as $R\Gamma(U, -)$ for any open U of X.*

Proof. We will prove this using Derived Categories, Lemma 16.6. Since every injective module is flasque we see that we can embed every $\mathcal{O}_X$-module into a flasque module, see Injectives, Lemma 4.1. Thus it suffices to show that given a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

with $\mathcal{F}$, $\mathcal{G}$ flasque, then $\mathcal{H}$ is flasque and the sequence remains short exact after taking sections on any open of X. In fact, the second statement implies the first. Thus, let $U \subset X$ be an open subspace. Let $s \in \mathcal{H}(U)$. We will show that we can lift s to a sequence of $\mathcal{G}$ over U. To do this consider the set T of pairs (V, t) where

$V \subset U$ is open and $t \in \mathcal{G}(V)$ is a section mapping to $s|_V$ in $\mathcal{H}$. We put a partial ordering on T by setting $(V, t) \leq (V', t')$ if and only if $V \subset V'$ and $t'|_V = t$. If (V_α, t_α), $\alpha \in A$ is a totally ordered subset of T, then $V = \bigcup V_\alpha$ is open and there is a unique section $t \in \mathcal{G}(V)$ restricting to t_α over V_α by the sheaf condition on $\mathcal{G}$. Thus by Zorn's lemma there exists a maximal element (V, t) in T. We will show that $V = U$ thereby finishing the proof. Namely, pick any $x \in U$. We can find a small open neighbourhood $W \subset U$ of x and $t' \in \mathcal{H}(W)$ mapping to $s|_W$ in $\mathcal{H}$. Then $t'|_{W\cap V} - t|_{W\cap V}$ maps to zero in $\mathcal{H}$, hence comes from some section $r' \in \mathcal{F}(W \cap V)$. Using that $\mathcal{F}$ is flasque we find a section $r \in \mathcal{F}(W)$ restricting to r' over $W' \cap V$. Modifying t' by the image of r we may assume that t and t' restrict to the same section over $W \cap V$. By the sheaf condition of $\mathcal{G}$ we can find a section $\tilde{t}$ of $\mathcal{G}$ over $W \cup V$ restricting to t and t'. By maximality of (V, t) we see that $V \cap W = V$. Thus $x \in V$ and we are done. □

The following lemma does not hold for flasque presheaves.

Lemma 13.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{U} : U = \bigcup U_i$ be an open covering. If $\mathcal{F}$ is flasque, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.*

Proof. The presheaves $\underline{H}^q(\mathcal{F})$ used in the statement of Lemma 12.4 are zero by Lemma 13.3. Hence $\check{H}^p(U, \mathcal{F}) = H^p(U, \mathcal{F}) = 0$ by Lemma 13.3 again. □

Lemma 13.5. *Let $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is flasque, then $R^p f_* \mathcal{F} = 0$ for $p > 0$.*

Proof. Immediate from Lemma 7.3 and Lemma 13.3. □

The following lemma can be proved by an elementary induction argument for finite coverings, compare with the discussion of Čech cohomology in [Vak].

Lemma 13.6. *Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume the restriction mappings $\mathcal{F}(U) \to \mathcal{F}(U')$ are surjective for U' an arbirtrary union of opens of the form $U_{i_0 \ldots i_p}$. Then $\check{H}^p(\mathcal{U}, \mathcal{F})$ vanishes for $p > 0$.*

Proof. Let Y be the set of nonempty subsets of I. We will use the letters $A, B, C, \ldots$ to denote elements of Y, i.e., nonempty subsets of I. For a finite nonempty subset $J \subset I$ let

$$V_J = \{A \in Y \mid J \subset A\}$$

This means that $V_{\{i\}} = \{A \in Y \mid i \in A\}$ and $V_J = \bigcap_{j \in J} V_{\{j\}}$. Then $V_J \subset V_K$ if and only if $J \supset K$. There is a unique topology on Y such that the collection of subsets V_J is a basis for the topology on Y. Any open is of the form

$$V = \bigcup\nolimits_{t \in T} V_{J_t}$$

for some family of finite subsets J_t. If $J_t \subset J_{t'}$ then we may remove $J_{t'}$ from the family without changing V. Thus we may assume there are no inclusions among the J_t. In this case the minimal elements of V are the sets $A = J_t$. Hence we can read off the family $(J_t)_{t \in T}$ from the open V.

We can completely understand open coverings in Y. First, because the elements $A \in Y$ are nonempty subsets of I we have

$$Y = \bigcup\nolimits_{i \in I} V_{\{i\}}$$

To understand other coverings, let V be as above and let $V_s \subset Y$ be an open corresponding to the family $(J_{s,t})_{t \in T_s}$. Then

$$V = \bigcup\nolimits_{s \in S} V_s$$

if and only if for each $t \in T$ there exists an $s \in S$ and $t_s \in T_s$ such that $J_t = J_{s,t_s}$. Namely, as the family $(J_t)_{t \in T}$ is minimal, the minimal element $A = J_t$ has to be in V_s for some s, hence $A \in V_{J_{t_s}}$ for some $t_s \in T_s$. But since A is also minimal in V_s we conclude that $J_{t_s} = J_t$.

Next we map the set of opens of Y to opens of X. Namely, we send Y to U, we use the rule

$$V_J \mapsto U_J = \bigcap\nolimits_{i \in J} U_i$$

on the opens V_J, and we extend it to arbitrary opens V by the rule

$$V = \bigcup\nolimits_{t \in T} V_{J_t} \mapsto \bigcup\nolimits_{t \in T} U_{J_t}$$

The classification of open coverings of Y given above shows that this rule transforms open coverings into open coverings. Thus we obtain an abelian sheaf $\mathcal{G}$ on Y by setting $\mathcal{G}(Y) = \mathcal{F}(U)$ and for $V = \bigcup_{t \in T} V_{J_t}$ setting

$$\mathcal{G}(V) = \mathcal{F}\left(\bigcup\nolimits_{t \in T} U_{J_t}\right)$$

and using the restriction maps of $\mathcal{F}$.

With these preliminaries out of the way we can prove our lemma as follows. We have an open covering $\mathcal{V} : Y = \bigcup_{i \in I} V_{\{i\}}$ of Y. By construction we have an equality

$$\check{C}^\bullet(\mathcal{V}, \mathcal{G}) = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$$

of Čech complexes. Since the sheaf $\mathcal{G}$ is flasque on Y (by our assumption on $\mathcal{F}$ in the statement of the lemma) the vanishing follows from Lemma 13.4. $\square$

14. The Leray spectral sequence

Lemma 14.1. *Let $f : X \to Y$ be a morphism of ringed spaces. There is a commutative diagram*

$$\begin{tikzcd} D^+(X) \arrow[r, "R\Gamma(X,-)"'] \arrow[d, "Rf_*"'] & D^+(\mathcal{O}_X(X)) \arrow[d, "restriction"] \\ D^+(Y) \arrow[r, "R\Gamma(Y,-)"] & D^+(\mathcal{O}_Y(Y)) \end{tikzcd}$$

More generally for any $V \subset Y$ open and $U = f^{-1}(V)$ there is a commutative diagram

$$\begin{tikzcd} D^+(X) \arrow[r, "R\Gamma(U,-)"'] \arrow[d, "Rf_*"'] & D^+(\mathcal{O}_X(U)) \arrow[d, "restriction"] \\ D^+(Y) \arrow[r, "R\Gamma(V,-)"] & D^+(\mathcal{O}_Y(V)) \end{tikzcd}$$

See also Remark 14.2 for more explanation.

Proof. Let $\Gamma_{res} : \mathit{Mod}(\mathcal{O}_X) \to \text{Mod}_{\mathcal{O}_Y(Y)}$ be the functor which associates to an $\mathcal{O}_X$-module $\mathcal{F}$ the global sections of $\mathcal{F}$ viewed as a $\mathcal{O}_Y(Y)$-module via the map $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Let *restriction* : $\text{Mod}_{\mathcal{O}_X(X)} \to \text{Mod}_{\mathcal{O}_Y(Y)}$ be the restriction functor induced by $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Note that *restriction* is exact so that its right derived functor is computed by simply applying the restriction functor, see Derived Categories, Lemma 17.8. It is clear that

$$\Gamma_{res} = \mathit{restriction} \circ \Gamma(X, -) = \Gamma(Y, -) \circ f_*$$

We claim that Derived Categories, Lemma 22.1 applies to both compositions. For the first this is clear by our remarks above. For the second, it follows from Lemma 12.9 which implies that injective $\mathcal{O}_X$-modules are mapped to $\Gamma(Y, -)$-acyclic sheaves on Y. $\square$

Remark 14.2. Here is a down-to-earth explanation of the meaning of Lemma 14.1. It says that given $f : X \to Y$ and $\mathcal{F} \in \mathit{Mod}(\mathcal{O}_X)$ and given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ we have

$$\begin{matrix} R\Gamma(X, \mathcal{F}) & \text{is represented by} & \Gamma(X, \mathcal{I}^\bullet) \\ Rf_*\mathcal{F} & \text{is represented by} & f_*\mathcal{I}^\bullet \\ R\Gamma(Y, Rf_*\mathcal{F}) & \text{is represented by} & \Gamma(Y, f_*\mathcal{I}^\bullet) \end{matrix}$$

the last fact coming from Leray's acyclicity lemma (Derived Categories, Lemma 17.7) and Lemma 12.9. Finally, it combines this with the trivial observation that

$$\Gamma(X, \mathcal{I}^\bullet) = \Gamma(Y, f_*\mathcal{I}^\bullet).$$

to arrive at the commutativity of the diagram of the lemma.

Lemma 14.3. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

(1) *The cohomology groups $H^i(U, \mathcal{F})$ for $U \subset X$ open of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.*
(2) *Let $f : X \to Y$ be a morphism of ringed spaces. The higher direct images $R^if_*\mathcal{F}$ of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.*

There are similar statements in the case of bounded below complexes of $\mathcal{O}_X$-modules.

Proof. Consider the morphism of ringed spaces $(X, \mathcal{O}_X) \to (X, \underline{\mathbf{Z}}_X)$ given by the identity on the underlying topological space and by the unique map of sheaves of rings $\underline{\mathbf{Z}}_X \to \mathcal{O}_X$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{ab}$ the same sheaf seen as an $\underline{\mathbf{Z}}_X$-module, i.e., seen as a sheaf of abelian groups. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. By Remark 14.2 we see that $\Gamma(X, \mathcal{I}^\bullet)$ computes both $R\Gamma(X, \mathcal{F})$ and $R\Gamma(X, \mathcal{F}_{ab})$. This proves (1).

To prove (2) we use (1) and Lemma 7.3. The result follows immediately. $\square$

Lemma 14.4 (Leray spectral sequence)**.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^qf_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(X, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 22.2 coming from the composition of functors $\Gamma_{res} = \Gamma(Y, -) \circ f_*$ where Γ_{res} is as in the proof of Lemma 14.1. To see that the assumptions of Derived Categories, Lemma 22.2 are satisfied, see the proof of Lemma 14.1 or Remark 14.2. □

Remark 14.5. The Leray spectral sequence, the way we proved it in Lemma 14.4 is a spectral sequence of $\Gamma(Y, \mathcal{O}_Y)$-modules. However, it is quite easy to see that it is in fact a spectral sequence of $\Gamma(X, \mathcal{O}_X)$-modules. For example f gives rise to a morphism of ringed spaces $f' : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$. By Lemma 14.3 the terms $E_r^{p,q}$ of the Leray spectral sequence for an $\mathcal{O}_X$-module $\mathcal{F}$ and f are identical with those for $\mathcal{F}$ and f' at least for $r \geq 2$. Namely, they both agree with the terms of the Leray spectral sequence for $\mathcal{F}$ as an abelian sheaf. And since $(f_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$ we see the result. It is often the case that the Leray spectral sequence carries additional structure.

Lemma 14.6. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

1. *If $R^q f_*\mathcal{F} = 0$ for $q > 0$, then $H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F})$ for all p.*
2. *If $H^p(Y, R^q f_*\mathcal{F}) = 0$ for all q and $p > 0$, then $H^q(X, \mathcal{F}) = H^0(Y, R^q f_*\mathcal{F})$ for all q.*

Proof. These are two simple conditions that force the Leray spectral sequence to converge. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. □

Lemma 14.7. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. In this case $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors from $D^+(X) \to D^+(Z)$.*

Proof. We are going to apply Derived Categories, Lemma 22.1. It is clear that $g_* \circ f_* = (g \circ f)_*$, see Sheaves, Lemma 21.2. It remains to show that $f_*\mathcal{I}$ is g_*-acyclic. This follows from Lemma 12.9 and the description of the higher direct images $R^i g_*$ in Lemma 7.3. □

Lemma 14.8 (Relative Leray spectral sequence)**.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. There is a spectral sequence with*

$$E_2^{p,q} = R^p g_*(R^q f_*\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_*\mathcal{F}$. This spectral sequence is functorial in $\mathcal{F}$, and there is a version for bounded below complexes of $\mathcal{O}_X$-modules.

Proof. This is a Grothendieck spectral sequence for composition of functors and follows from Lemma 14.7 and Derived Categories, Lemma 22.2. □

15. Functoriality of cohomology

Lemma 15.1. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}^\bullet$, resp. $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_Y$-modules, resp. $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet$ be a morphism of complexes. There is a canonical morphism*

$$\mathcal{G}^\bullet \longrightarrow Rf_*(\mathcal{F}^\bullet)$$

in $D^+(Y)$. Moreover this construction is functorial in the triple $(\mathcal{G}^\bullet, \mathcal{F}^\bullet, \varphi)$.

Proof. Choose an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. By definition $Rf_*(\mathcal{F}^\bullet)$ is represented by $f_*\mathcal{I}^\bullet$ in $K^+(\mathcal{O}_Y)$. The composition

$$\mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet \to f_*\mathcal{I}^\bullet$$

is a morphism in $K^+(Y)$ which turns into the morphism of the lemma upon applying the localization functor $j_Y : K^+(Y) \to D^+(Y)$. $\square$

Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Recall that an f-map φ from $\mathcal{G}$ to $\mathcal{F}$ is a map $\varphi : \mathcal{G} \to f_*\mathcal{F}$, or what is the same thing, a map $\varphi : f^*\mathcal{G} \to \mathcal{F}$. See Sheaves, Definition 21.7. Such an f-map gives rise to a morphism of complexes

$$\varphi : R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X, \mathcal{F}) \tag{15.1.1}$$

in $D^+(\mathcal{O}_Y(Y))$. Namely, we use the morphism $\mathcal{G} \to Rf_*\mathcal{F}$ in $D^+(Y)$ of Lemma 15.1, and we apply $R\Gamma(Y, -)$. By Lemma 14.1 we see that $R\Gamma(X, \mathcal{F}) = R\Gamma(Y, Rf_*\mathcal{F})$ and we get the displayed arrow. We spell this out completely in Remark 15.2 below. In particular it gives rise to maps on cohomology

$$\varphi : H^i(Y, \mathcal{G}) \longrightarrow H^i(X, \mathcal{F}). \tag{15.1.2}$$

Remark 15.2. Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let φ be an f-map from $\mathcal{G}$ to $\mathcal{F}$. Choose a resolution $\mathcal{F} \to \mathcal{I}^\bullet$ by a complex of injective $\mathcal{O}_X$-modules. Choose resolutions $\mathcal{G} \to J^\bullet$ and $f_*\mathcal{I} \to (\mathcal{J}')^\bullet$ by complexes of injective $\mathcal{O}_Y$-modules. By Derived Categories, Lemma 18.6 there exists a map of complexes β such that the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & f_*\mathcal{F} \arrow[r] & f_*\mathcal{I}^\bullet \arrow[d] \\ \mathcal{J}^\bullet \arrow[rr, "\beta"] & & (\mathcal{J}')^\bullet \end{tikzcd} \tag{15.2.1}$$

commutes. Applying global section functors we see that we get a diagram

$$\begin{tikzcd} & \Gamma(Y, f_*\mathcal{I}^\bullet) \arrow[r, equal] \arrow[d, "qis"'] & \Gamma(X, \mathcal{I}^\bullet) \\ \Gamma(Y, \mathcal{J}^\bullet) \arrow[r, "\beta"] & \Gamma(Y, (\mathcal{J}')^\bullet) & \end{tikzcd}$$

The complex on the bottom left represents $R\Gamma(Y, \mathcal{G})$ and the complex on the top right represents $R\Gamma(X, \mathcal{F})$. The vertical arrow is a quasi-isomorphism by Lemma 14.1 which becomes invertible after applying the localization functor $K^+(\mathcal{O}_Y(Y)) \to D^+(\mathcal{O}_Y(Y))$. The arrow (15.1.1) is given by the composition of the horizontal map by the inverse of the vertical map.

16. Refinements and Čech cohomology

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : X = \bigcup_{j \in J} V_j$ be open coverings. Assume that $\mathcal{U}$ is a refinement of $\mathcal{V}$. Choose a map $c : I \to J$ such that $U_i \subset V_{c(i)}$ for all $i \in I$. This induces a map of Čech complexes

$$\gamma : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}), \quad (\xi_{j_0 \ldots j_p}) \longmapsto (\xi_{c(i_0) \ldots c(i_p)}|_{U_{i_0 \ldots i_p}})$$

functorial in the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Suppose that $c' : I \to J$ is a second map such that $U_i \subset V_{c'(i)}$ for all $i \in I$. Then the corresponding maps γ and γ' are homotopic. Namely, $\gamma - \gamma' = \mathrm{d} \circ h + h \circ \mathrm{d}$ with $h : \check{\mathcal{C}}^{p+1}(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ given by the rule

$$h(\xi)_{i_0 \ldots i_p} = \sum\nolimits_{a=0}^{p} (-1)^a \alpha_{c(i_0) \ldots c(i_a) c'(i_a) \ldots c'(i_p)}$$

We omit the computation showing this works; please see the discussion following (26.0.2) for the proof in a more general case. In particular, the map on Čech cohomology groups is independent of the choice of c. Moreover, it is clear that if $\mathcal{W} : X = \bigcup_{k \in K} W_k$ is a third open covering and $\mathcal{V}$ is a refinement of $\mathcal{W}$, then the composition of the maps

$$\check{\mathcal{C}}^\bullet(\mathcal{W}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

associated to maps $I \to J$ and $J \to K$ is the map associated to the composition $I \to K$. In particular, we can define the Čech cohomology groups

$$\check{H}^p(X, \mathcal{F}) = \mathrm{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F})$$

where the colimit is over all open coverings of X partially ordered by refinement.

It turns out that the maps γ defined above are compatible with the map to cohomology, in other words, the composition

$$\check{H}^p(\mathcal{V}, \mathcal{F}) \to \check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{Lemma 12.2}} H^p(X, \mathcal{F})$$

is the canonical map from the first group to cohomology of Lemma 12.2. In the lemma below we will prove this in a slightly more general setting. A consequence is that we obtain a well defined map

$$\check{H}^p(X, \mathcal{F}) = \mathrm{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}) \tag{16.0.2}$$

from Čech cohomology to cohomology.

Lemma 16.1. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\varphi : f^*\mathcal{G} \to \mathcal{F}$ be an f-map from an $\mathcal{O}_Y$-module $\mathcal{G}$ to an $\mathcal{O}_X$-module $\mathcal{F}$. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j \in J} V_j$ be open coverings. Assume that $\mathcal{U}$ is a refinement of $f^{-1}\mathcal{V} : X = \bigcup_{j \in J} f^{-1}(V_j)$. In this case there exists a commutative diagram*

$$\begin{tikzcd}
\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] & R\Gamma(X, \mathcal{F}) \\
\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \arrow[r] \arrow[u, "\gamma"] & R\Gamma(Y, \mathcal{G}) \arrow[u]
\end{tikzcd}$$

in $D^+(\mathcal{O}_X(X))$ with horizontal arrows given by Lemma 12.2 and right vertical arrow by (15.1.1). In particular we get commutative diagrams of cohomology groups

$$\begin{tikzcd}
\check{H}^p(\mathcal{U}, \mathcal{F}) \arrow[r] & H^p(X, \mathcal{F}) \\
\check{H}^p(\mathcal{V}, \mathcal{G}) \arrow[r] \arrow[u, "\gamma"] & H^p(Y, \mathcal{G}) \arrow[u]
\end{tikzcd}$$

where the right vertical arrow is (15.1.2)

Proof. We first define the left vertical arrow. Namely, choose a map $c : I \to J$ such that $U_i \subset f^{-1}(V_{c(i)})$ for all $i \in I$. In degree p we define the map by the rule

$$\gamma(s)_{i_0 \ldots i_p} = \varphi(s)_{c(i_0) \ldots c(i_p)}$$

This makes sense because φ does indeed induce maps $\mathcal{G}(V_{c(i_0)\ldots c(i_p)}) \to \mathcal{F}(U_{i_0\ldots i_p})$ by assumption. It is also clear that this defines a morphism of complexes. Choose injective resolutions $\mathcal{F} \to \mathcal{I}^\bullet$ on X and $\mathcal{G} \to J^\bullet$ on Y. According to the proof of Lemma 12.2 we introduce the double complexes $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$ with terms

$$B^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, \mathcal{J}^q) \quad \text{and} \quad A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q).$$

As in Remark 15.2 above we also choose an injective resolution $f_*\mathcal{I} \to (\mathcal{J}')^\bullet$ on Y and a morphism of complexes $\beta : \mathcal{J} \to (\mathcal{J}')^\bullet$ making (15.2.1) commutes. We introduce some more double complexes, namely $(B')^{\bullet,\bullet}$ and $(B'')\bullet, \bullet$ with

$$(B')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, (\mathcal{J}')^q) \quad \text{and} \quad (B'')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, f_*\mathcal{I}^q).$$

Note that there is an f-map of complexes from $f_*\mathcal{I}^\bullet$ to $\mathcal{I}^\bullet$. Hence it is clear that the same rule as above defines a morphism of double complexes

$$\gamma : (B'')^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet}.$$

Consider the diagram of complexes

$$\begin{tikzcd}
\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] & sA^\bullet & & & \Gamma(X, \mathcal{I}^\bullet) \arrow[llll, "qis"'] \\
\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \arrow[r] \arrow[u, "\gamma"] & sB^\bullet \arrow[r, "\beta"] & s(B')^\bullet & s(B'')^\bullet \arrow[l] \arrow[llu, "s\gamma"'] & \\
& \Gamma(Y, \mathcal{J}^\bullet) \arrow[r, "\beta"] \arrow[u, "qis"] & \Gamma(Y, (\mathcal{J}')^\bullet) \arrow[u] & \Gamma(Y, f_*\mathcal{I}^\bullet) \arrow[l, "qis"'] \arrow[u] \arrow[ruu, equal] &
\end{tikzcd}$$

The two horizontal arrows with targets $sA^\bullet$ and $sB^\bullet$ are the ones explained in Lemma 12.2. The left upper shape (a pentagon) is commutative simply because (15.2.1) is commutative. The two lower squares are trivially commutative. It is also immediate from the definitions that the right upper shape (a square) is commutative. The result of the lemma now follows from the definitions and the fact that going around the diagram on the outer sides from $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})$ to $\Gamma(X, \mathcal{I}^\bullet)$ either on top or on bottom is the same (where you have to invert any quasi-isomorphisms along the way). $\square$

17. Cohomology on Hausdorff quasi-compact spaces

For such a space Čech cohomology agrees with cohomology.

Lemma 17.1. *Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf. Then the map $\check{H}^1(X, \mathcal{F}) \to H^1(X, \mathcal{F})$ defined in (16.0.2) is an isomorphism.*

Proof. Let $\mathcal{U}$ be an open covering of X. By Lemma 12.4 there is an exact sequence

$$0 \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$$

Thus the map is injective. To show surjectivity it suffices to show that any element of $\check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$ maps to zero after replacing $\mathcal{U}$ by a refinement. This is immediate from the definitions and the fact that $\underline{H}^1(\mathcal{F})$ is a presheaf of abelian groups whose sheafification is zero by locality of cohomology, see Lemma 7.2. $\square$

Lemma 17.2. *Let X be a Hausdorff and quasi-compact topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Then the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ defined in (16.0.2) is an isomorphism for all p.*

Proof. We argue by induction on p that the map $c^p_\mathcal{F} : \check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ is an isomorphism. For $p = 0$ the result is clear and for $p = 1$ the result holds by Lemma 17.1. Thus we may assume $p > 1$.

Choose an injective map $a : \mathcal{F} \to \mathcal{I}$, where $\mathcal{I}$ is an injective abelian sheaf. Let $b : \mathcal{I} \to \mathcal{G}$ be the quotient by $\mathcal{F}$. Let $\xi = (\xi_{i_0\ldots i_p})$ be a cocycle of the Čech complex, giving rise to an element $\overline{\xi}$ of $\check{H}^p(\mathcal{U}, \mathcal{F})$. Then $a(\xi) = \mathrm{d}(\eta)$ for some cochain η for $\mathcal{I}$ by Lemma 12.1. The image $\theta = b(\eta)$ of η in the Čech complex for $\mathcal{G}$ is a cocyle, hence gives rise to an element $\overline{\theta}$ in $\check{H}^{p-1}(\mathcal{U}, \mathcal{G})$. A straightforward argument (using $p \geq 2$ and hence the Čech complex of $\mathcal{I}$ is acyclic in degree $p-1$) shows that the rule which assigns the element $\overline{\theta} \in \check{H}^{p-1}(\mathcal{U}, \mathcal{G})$ of θ to the class is well defined. It follows from the construction that $c^p_\mathcal{F}(\overline{\xi}) = \partial(c^{p-1}_\mathcal{G}(\overline{\theta}))$ where $\partial : H^{p-1}(X, \mathcal{G}) \to H^p(X, \mathcal{F})$ is the boundary coming from the short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ (details omitted).

Conversely, let $\theta = (\theta_{i_0\ldots i_{p-1}})$ a cocycle of the Čech complex of $\mathcal{G}$ for some open covering $\mathcal{U}$. We would like to lift θ to a cochain for $\mathcal{I}$. The problem is that the sequence of complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) \to 0$$

may not be exact on the right. Howeover, we know that for all p-tuples $i_0 \ldots i_{p-1}$ of I there exists an open covering

$$U_{i_0} \cap \ldots \cap U_{i_{p-1}} = \bigcup W_{i_0\ldots i_{p-1},k}$$

such that $\theta_{i_0\ldots i_{p-1}}|_{W_{i_0\ldots i_{p-1},k}}$ does lift to a section of $\mathcal{I}$ over $W_{i_0\ldots i_{p-1},k}$. Thus, by Topology, Lemma 12.4 after refining $\mathcal{U}$, we can lift θ to a $(p-1)$-cochain η in the Čech complex of $\mathcal{I}$. Then $\mathrm{d}(\eta) = a(\xi)$ for some p-cocycle ξ for $\mathcal{F}$. In other words, every element of $\operatorname{colim} \check{H}^{p-1}(\mathcal{U}, \mathcal{G})$ comes about by the construction of the previous paragraph from an element of $\operatorname{colim} \check{H}^p(\mathcal{U}, \mathcal{F})$.

By the compatibility of the construction with the boundary map $\partial : H^{p-1}(X, \mathcal{G}) \to H^p(X, \mathcal{F})$, the surjectivity of the map, the induction hypothesis saying $\gamma^{p-1}_\mathcal{G}$ is an isomorphism, and the fact that $H^{p-1}(X, \mathcal{I}) = H^p(X, \mathcal{I}) = 0$, it follows formally that $c^p_\mathcal{F}$ is surjective. To show injectivity one has to show that, given ξ, η, θ linked as above, if θ is a boundary, then ξ becomes a boundary after replacing $\mathcal{U}$ by a refinement. To do this argue as above, once more appealing to Topology, Lemma 12.4. Some details omitted. $\square$

Lemma 17.3. *Let X be a Hausdorff and locally quasi-compact space. Let $Z \subset X$ be a quasi-compact (hence closed) subset. For every abelian sheaf $\mathcal{F}$ on X we have*

$$\operatorname{colim} H^p(U, \mathcal{F}) \longrightarrow H^p(Z, \mathcal{F}|_Z)$$

where the colimit is over open neighbourhoods U of Z in X.

Proof. We first prove this for $p = 0$. Injectivity follows from the definition of $\mathcal{F}|_Z$ and holds in general (for any subset of any topological space X). Next, suppose that $s \in H^0(Z, \mathcal{F}|_Z)$. Then we can find opens $U_i \subset X$ such that $Z \subset \bigcup U_i$ and such that $s|_{Z \cap U_i}$ comes from $s_i \in \mathcal{F}(U_i)$. It follows that there exist opens $W_{ij} \subset U_i \cap U_j$ with $W_{ij} \cap Z = U_i \cap U_j \cap Z$ such that $s_i|_{W_{ij}} = s_j|_{W_{ij}}$. Applying Topology, Lemma 12.5 we find opens V_i of X such that $V_i \subset U_i$ and such that $V_i \cap V_j \subset W_{ij}$. Hence we see that $s_i|_{V_i}$ glue to a section of $\mathcal{F}$ over the open neighbourhood $\bigcup V_i$ of Z.

To finish the proof, it suffices to show that if $\mathcal{I}$ is an injective abelian sheaf on X, then $H^p(Z, \mathcal{I}|_Z) = 0$ for $p > 0$. This follows using short exact sequences and dimension shifting; details omitted. Thus, suppose $\overline{\xi}$ is an element of $H^p(Z, \mathcal{I}|_Z)$ for some $p > 0$. By Lemma 17.2 the element $\overline{\xi}$ comes from $\check{H}^p(\mathcal{V}, \mathcal{I}|_Z)$ for some open covering $\mathcal{V} : Z = \bigcup V_i$ of Z. Say $\overline{\xi}$ is the image of the class of a cocycle $\xi = (\xi_{i_0 \ldots i_p})$ in $\check{\mathcal{C}}^p(\mathcal{V}, \mathcal{I}|_Z)$.

Let $\mathcal{I}' \subset \mathcal{I}|_Z$ be the subpresheaf defined by the rule

$$\mathcal{I}'(V) = \{s \in \mathcal{I}|_Z(V) \mid \exists (U, t),\ U \subset X \text{ open},\ t \in \mathcal{I}(U),\ V = Z \cap U,\ s = t|_{Z \cap U}\}$$

Then $\mathcal{I}|_Z$ is the sheafification of $\mathcal{I}'$. Thus for every $(p+1)$-tuple $i_0 \ldots i_p$ we can find an open covering $V_{i_0 \ldots i_p} = \bigcup W_{i_0 \ldots i_p, k}$ such that $\xi_{i_0 \ldots i_p}|_{W_{i_0 \ldots i_p, k}}$ is a section of $\mathcal{I}'$. Applying Topology, Lemma 12.4 we may after refining $\mathcal{V}$ assume that each $\xi_{i_0 \ldots i_p}$ is a section of the presheaf $\mathcal{I}'$.

Write $V_i = Z \cap U_i$ for some opens $U_i \subset X$. Since $\mathcal{I}$ is flasque (Lemma 13.2) and since $\xi_{i_0 \ldots i_p}$ is a section of $\mathcal{I}'$ for every $(p+1)$-tuple $i_0 \ldots i_p$ we can choose a section $s_{i_0 \ldots i_p} \in \mathcal{I}(U_{i_0 \ldots i_p})$ which restricts to $\xi_{i_0 \ldots i_p}$ on $V_{i_0 \ldots i_p} = Z \cap U_{i_0 \ldots i_p}$. (This appeal to injectives being flasque can be avoided by an additional application of Topology, Lemma 12.5.) Let $s = (s_{i_0 \ldots i_p})$ be the corresponding cochain for the open covering $U = \bigcup U_i$. Since $\mathrm{d}(\xi) = 0$ we see that the sections $\mathrm{d}(s)_{i_0 \ldots i_{p+1}}$ restrict to zero on $Z \cap U_{i_0 \ldots i_{p+1}}$. Hence, by the initial remarks of the proof, there exists open subsets $W_{i_0 \ldots i_{p+1}} \subset U_{i_0 \ldots i_{p+1}}$ with $Z \cap W_{i_0 \ldots i_{p+1}} = Z \cap U_{i_0 \ldots i_{p+1}}$ such that $\mathrm{d}(s)_{i_0 \ldots i_{p+1}}|_{W_{i_0 \ldots i_{p+1}}} = 0$. By Topology, Lemma 12.5 we can find $U'_i \subset U_i$ such that $Z \subset \bigcup U'_i$ and such that $U'_{i_0 \ldots i_{p+1}} \subset W_{i_0 \ldots i_{p+1}}$. Then $s' = (s'_{i_0 \ldots i_p})$ with $s'_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}|_{U'_{i_0 \ldots i_p}}$ is a cocycle for $\mathcal{I}$ for the open covering $U' = \bigcup U'_i$ of an open neighbourhood of Z. Since $\mathcal{I}$ has trivial higher Čech cohomology groups (Lemma 12.1) we conclude that s' is a coboundary. It follows that the image of ξ in the Čech complex for the open covering $Z = \bigcup Z \cap U'_i$ is a coboundary and we are done. $\square$

18. The base change map

We will need to know how to construct the base change map in some cases. Since we have not yet discussed derived pullback we only discuss this in the case of a base change by a flat morphism of ringed spaces. Before we state the result, let us discuss flat pullback on the derived category. Namely, suppose that $g : X \to Y$ is a flat morphism of ringed spaces. By Modules, Lemma 17.2 the functor $g^* : \textit{Mod}(\mathcal{O}_Y) \to \textit{Mod}(\mathcal{O}_X)$ is exact. Hence it has a derived functor

$$g^* : D^+(Y) \to D^+(X)$$

which is computed by simply pulling back an representative of a given object in $D^+(Y)$, see Derived Categories, Lemma 17.8. Hence as indicated we indicate this functor by g^* rather than Lg^*.

Lemma 18.1. *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a commutative diagram of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. Assume both g and g' are flat. Then there exists a canonical base change map

$$g^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*(g')^*\mathcal{F}^\bullet$$

in $D^+(S')$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^*\mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 12.10 we see that $(g')_*\mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^*\mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 18.6 and 18.7 the arrow β in the diagram

$$\begin{tikzcd}
(g')_*(g')^*\mathcal{F}^\bullet \arrow[r] & (g')_*\mathcal{J}^\bullet \\
\mathcal{F}^\bullet \arrow[r] \arrow[u, "adjunction"] & \mathcal{I}^\bullet \arrow[u, "\beta"']
\end{tikzcd}$$

exists and is unique up to homotopy. Pushing down to S we get

$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$

By adjunction of g^* and g_* we get a map of complexes $g^*f_*\mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. $\square$

Remark 18.2. The "correct" version of the base change map is map

$$Lg^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*L(g')^*\mathcal{F}^\bullet.$$

The construction of this map involves unbounded complexes, see Remark 29.2.

19. Proper base change in topology

In this section we prove a very general version of the proper base change theorem in topology. It tells us that the stalks of the higher direct images R^pf_* can be computed on the fibre.

Lemma 19.1. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $y \in Y$. Assume that*

1. *X is Hausdorff and locally quasi-compact,*
2. *$f^{-1}(y)$ is quasi-compact, and*
3. *f is closed.*

Then for E in $D^+(\mathcal{O}_X)$ we have $(Rf_*E)_y = R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$ in $D^+(\mathcal{O}_{Y,y})$.

Proof. The base change map of Lemma 18.1 gives a canonical map $(Rf_*E)_y \to R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$. To prove this map is an isomorphism, we represent E by a bounded below complex of injectives $\mathcal{I}^\bullet$. By Lemma 17.3 the restrictions $\mathcal{I}^n|_{f^{-1}(y)}$ are acyclic for $\Gamma(f^{-1}(y), -)$. Thus $R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$ is represented by the complex $\Gamma(f^{-1}(y), \mathcal{I}^\bullet|_{f^{-1}(y)})$, see Derived Categories, Lemma 17.7. In other words, we have to show the map

$$\mathrm{colim}_V\, \mathcal{I}^\bullet(f^{-1}(V)) \longrightarrow \Gamma(f^{-1}(y), \mathcal{I}^\bullet|_{f^{-1}(y)})$$

is an isomorphism. Using Lemma 17.3 we see that it suffices to show that the collection of open neighbourhoods $f^{-1}(V)$ of $f^{-1}(y)$ is cofinal in the system of all open neighbourhoods. If $f^{-1}(y) \subset U$ is an open neighbourhood, then as f is closed the set $V = Y \setminus f(X \setminus U)$ is an open neighbourhood of y with $f^{-1}(V) \subset U$. This proves the lemma. $\square$

Theorem 19.2 (Proper base change)**.** *Consider a cartesian square of Hausdorff, locally quasi-compact topological spaces*

$$\begin{tikzcd} X' = Y' \times_Y X \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

and assume that f is proper. Let E be an object of $D^+(X)$. Then the base change map

$$g^{-1}Rf_*E \longrightarrow Rf'_*(g')^{-1}E$$

of Lemma 18.1 is an isomorphism in $D^+(Y')$.

Proof. Let $y' \in Y'$ be a point with image $y \in Y$. It suffices to show that the base change map induces an isomorphism on stalks at y'. As f is proper it follows that f' is proper, the fibres of f and f' are quasi-compact and f and f' are closed, see Topology, Theorem 16.5. Thus we can apply Lemma 19.1 twice to see that

$$(Rf'_*(g')^{-1}E)_{y'} = R\Gamma((f')^{-1}(y'), (g')^{-1}E|_{(f')^{-1}(y')})$$

and

$$(Rf_*E)_y = R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$$

The induced map of fibres $(f')^{-1}(y') \to f^{-1}(y)$ is a homeomorphism of topological spaces and the pull back of $E|_{f^{-1}(y)}$ is $(g')^{-1}E|_{(f')^{-1}(y')}$. The desired result follows. $\square$

20. Cohomology and colimits

Let X be a ringed space. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a directed system of sheaves of $\mathcal{O}_X$-modules over the partially ordered set I, see Categories, Section 21. Since for each i there is a canonical map $\mathcal{F}_i \to \operatorname{colim}_i \mathcal{F}_i$ we get a canonical map

$$\operatorname{colim}_i H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, \operatorname{colim}_i \mathcal{F}_i)$$

for every $p \geq 0$. Of course there is a similar map for every open $U \subset X$. These maps are in general not isomorphisms, even for $p = 0$. In this section we generalize the results of Sheaves, Lemma 29.1. See also Modules, Lemma 11.6 (in the special case $\mathcal{G} = \mathcal{O}_X$).

Lemma 20.1. *Let X be a ringed space. Assume that the underlying topological space of X has the following properties:*

1. *there exists a basis of quasi-compact open subsets, and*
2. *the intersection of any two quasi-compact opens is quasi-compact.*

Then for any directed system $(\mathcal{F}_i, \varphi_{ii'})$ of sheaves of $\mathcal{O}_X$-modules and for any quasi-compact open $U \subset X$ the canonical map

$$\operatorname{colim}_i H^q(U, \mathcal{F}_i) \longrightarrow H^q(U, \operatorname{colim}_i \mathcal{F}_i)$$

is an isomorphism for every $q \geq 0$.

Proof. It is important in this proof to argue for all quasi-compact opens $U \subset X$ at the same time. The result is true for $i = 0$ and any quasi-compact open $U \subset X$ by Sheaves, Lemma 29.1 (combined with Topology, Lemma 26.1). Assume that we have proved the result for all $q \leq q_0$ and let us prove the result for $q = q_0 + 1$.

By our conventions on directed systems the index set I is directed, and any system of $\mathcal{O}_X$-modules $(\mathcal{F}_i, \varphi_{ii'})$ over I is directed. By Injectives, Lemma 5.1 the category of $\mathcal{O}_X$-modules has functorial injective embeddings. Thus for any system $(\mathcal{F}_i, \varphi_{ii'})$ there exists a system $(\mathcal{I}_i, \varphi_{ii'})$ with each $\mathcal{I}_i$ an injective $\mathcal{O}_X$-module and a morphism of systems given by injective $\mathcal{O}_X$-module maps $\mathcal{F}_i \to \mathcal{I}_i$. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

We claim that the sequence

$$0 \to \mathrm{colim}_i\, \mathcal{F}_i \to \mathrm{colim}_i\, \mathcal{I}_i \to \mathrm{colim}_i\, \mathcal{Q}_i \to 0.$$

is also a short exact sequence of $\mathcal{O}_X$-modules. We may check this on stalks. By Sheaves, Sections 28 and 29 taking stalks commutes with colimits. Since a directed colimit of short exact sequences of abelian groups is short exact (see Algebra, Lemma 8.9) we deduce the result. We claim that $H^q(U, \mathrm{colim}_i\, \mathcal{I}_i) = 0$ for all quasi-compact open $U \subset X$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{tikzcd}
\mathrm{colim}_i\, H^{q_0}(U, \mathcal{I}_i) \arrow[r] \arrow[d] & \mathrm{colim}_i\, H^{q_0}(U, \mathcal{Q}_i) \arrow[r] \arrow[d] & \mathrm{colim}_i\, H^{q_0+1}(U, \mathcal{F}_i) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^{q_0}(U, \mathrm{colim}_i\, \mathcal{I}_i) \arrow[r] & H^{q_0}(U, \mathrm{colim}_i\, \mathcal{Q}_i) \arrow[r] & H^{q_0+1}(U, \mathrm{colim}_i\, \mathcal{F}_i) \arrow[r] & 0
\end{tikzcd}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 8.9. Hence by the snake lemma we deduce the result for $q = q_0 + 1$.

It remains to show that the claim is true. We will use Lemma 12.8. Let $\mathcal{B}$ be the collection of all quasi-compact open subsets of X. This is a basis for the topology on X by assumption. Let Cov be the collection of finite open coverings $\mathcal{U} : U = \bigcup_{j=1,\ldots,m} U_j$ with each of U, U_j quasi-compact open in X. By the result for $q = 0$ we see that for $\mathcal{U} \in \mathrm{Cov}$ we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{colim}_i\, \mathcal{I}_i) = \mathrm{colim}_i\, \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the multiple intersections $U_{j_0\ldots j_p}$ are quasi-compact. By Lemma 12.1 each of the complexes in the colimit of Čech complexes is acyclic in degree ≥ 1. Hence by Algebra, Lemma 8.9 we see that also the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{colim}_i\, \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, \mathrm{colim}_i\, \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 12.8 are satisfied and the claim follows. $\square$

Next we formulate the analogy of Sheaves, Lemma 29.4 for cohomology. Let X be a spectral space which is written as a cofiltered limit of spectral spaces X_i for a diagram with spectral transition morphisms as in Topology, Lemma 23.5. Assume given

(1) an abelian sheaf $\mathcal{F}_i$ on X_i for all $i \in \mathrm{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ an f_a-map $\varphi_a : \mathcal{F}_i \to \mathcal{F}_j$ of abelian sheaves (see Sheaves, Definition 21.7)

such that $\varphi_c = \varphi_b \circ \varphi_a$ whenever $c = a \circ b$. Set $\mathcal{F} = \mathrm{colim}\, p_i^{-1}\mathcal{F}_i$ on X.

Lemma 20.2. *In the situation discussed above. Let $i \in \mathrm{Ob}(\mathcal{I})$ and let $U_i \subset X_i$ be quasi-compact open. Then*

$$\mathrm{colim}_{a:j\to i} H^p(f_a^{-1}(U_i), \mathcal{F}_j) = H^p(p_i^{-1}(U_i), \mathcal{F})$$

for all $p \geq 0$. In particular we have $H^p(X, \mathcal{F}) = \mathrm{colim}\, H^p(X_i, \mathcal{F}_i)$.

Proof. The case $p = 0$ is Sheaves, Lemma 29.4.

In this paragraph we show that we can find a map of systems $(\gamma_i) : (\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ with $\mathcal{G}_i$ an injective abelian sheaf and γ_i injective. For each i we pick an injection $\mathcal{F}_i \to \mathcal{I}_i$ where $\mathcal{I}_i$ is an injective abelian sheaf on X_i. Then we can consider the family of maps

$$\gamma_i : \mathcal{F}_i \longrightarrow \prod\nolimits_{b:k\to i} f_{b,*}\mathcal{I}_k = \mathcal{G}_i$$

where the component maps are the maps adjoint to the maps $f_b^{-1}\mathcal{F}_i \to \mathcal{F}_k \to \mathcal{I}_k$. For $a : j \to i$ in $\mathcal{I}$ there is a canonical map

$$\psi_a : f_a^{-1}\mathcal{G}_i \to \mathcal{G}_j$$

whose components are the canonical maps $f_b^{-1} f_{a\circ b,*}\mathcal{I}_k \to f_{b,*}\mathcal{I}_k$ for $b : k \to j$. Thus we find an injection $\{\gamma_i\} : \{\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ of systems of abelian sheaves. Note that $\mathcal{G}_i$ is an injective sheaf of abelian groups on $\mathcal{C}_i$, see Lemma 12.10 and Homology, Lemma 23.3. This finishes the construction.

Arguing exactly as in the proof of Lemma 20.1 we see that it suffices to prove that $H^p(X, \mathrm{colim}\, f_i^{-1}\mathcal{G}_i) = 0$ for $p > 0$.

Set $\mathcal{G} = \mathrm{colim}\, f_i^{-1}\mathcal{G}_i$. To show vanishing of cohomology of $\mathcal{G}$ on every quasi-compact open of X, it suffices to show that the Čech cohomology of $\mathcal{G}$ for any covering $\mathcal{U}$ of a quasi-compact open of X by finitely many quasi-compact opens is zero, see Lemma 12.8. Such a covering is the inverse by p_i of such a covering $\mathcal{U}_i$ on the space X_i for some i by Topology, Lemma 23.6. We have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) = \mathrm{colim}_{a:j\to i}\, \check{\mathcal{C}}^\bullet(f_a^{-1}(\mathcal{U}_i), \mathcal{G}_j)$$

by the case $p = 0$. The right hand side is a filtered colimit of complexes each of which is acyclic in positive degrees by Lemma 12.1. Thus we conclude by Algebra, Lemma 8.9. $\square$

21. Vanishing on Noetherian topological spaces

The aim is to prove a theorem of Grothendieck namely Proposition 21.6. See [Gro57].

Lemma 21.1. *Let $i : Z \to X$ be a closed immersion of topological spaces. For any abelian sheaf $\mathcal{F}$ on Z we have $H^p(Z, \mathcal{F}) = H^p(X, i_*\mathcal{F})$.*

Proof. This is true because i_* is exact (see Modules, Lemma 6.1), and hence $R^p i_* = 0$ as a functor (Derived Categories, Lemma 17.8). Thus we may apply Lemma 14.6. $\square$

Lemma 21.2. *Let X be an irreducible topological space. Then $H^p(X, \underline{A}) = 0$ for all $p > 0$ and any abelian group A.*

Proof. Recall that $\underline{A}$ is the constant sheaf as defined in Sheaves, Definition 7.4. It is clear that for any nonempty open $U \subset X$ we have $\underline{A}(U) = A$ as X is irreducible (and hence U is connected). We will show that the higher Čech cohomology groups $\check{H}^p(\mathcal{U}, \underline{A})$ are zero for any open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ of an open $U \subset X$. Then the lemma will follow from Lemma 12.7.

Recall that the value of an abelian sheaf on the empty open set is 0. Hence we may clearly assume $U_i \neq \emptyset$ for all $i \in I$. In this case we see that $U_i \cap U_{i'} \neq \emptyset$ for all $i, i' \in I$. Hence we see that the Čech complex is simply the complex

$$\prod_{i_0 \in I} A \to \prod_{(i_0, i_1) \in I^2} A \to \prod_{(i_0, i_1, i_2) \in I^3} A \to \ldots$$

We have to see this has trivial higher cohomology groups. We can see this for example because this is the cech complex for the covering of a 1-point space and Čech cohomology agrees with cohomology on such a space. (You can also directly verify it by writing an explicit homotopy.) $\square$

Lemma 21.3. *Let X be a topological space such that the intersection of any two quasi-compact opens is quasi-compact. Let $\mathcal{F} \subset \underline{\mathbf{Z}}$ be a subsheaf generated by finitely many sections over quasi-compact opens. Then there exists a finite filtration*

$$(0) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

by abelian subsheaves such that for each $0 < i \leq n$ there exists a short exact sequence

$$0 \to j'_!\underline{\mathbf{Z}}_V \to j_!\underline{\mathbf{Z}}_U \to \mathcal{F}_i/\mathcal{F}_{i-1} \to 0$$

with $j : U \to X$ and $j' : V \to X$ the inclusion of quasi-compact opens into X.

Proof. Say $\mathcal{F}$ is generated by the sections $s_1, \ldots, s_t$ over the quasi-compact opens $U_1, \ldots, U_t$. Since U_i is quasi-compact and s_i a locally constant function to $\mathbf{Z}$ we may assume, after possibly replacing U_i by the parts of a finite decomposition into open and closed subsets, that s_i is a constant section. Say $s_i = n_i$ with $n_i \in \mathbf{Z}$. Of course we can remove (U_i, n_i) from the list if $n_i = 0$. Flipping signs if necessary we may also assume $n_i > 0$. Next, for any subset $I \subset \{1, \ldots, t\}$ we may add $\bigcup_{i \in I} U_i$ and $\gcd(n_i, i \in I)$ to the list. After doing this we see that our list $(U_1, n_1), \ldots, (U_t, n_t)$ satisfies the following property: For $x \in X$ set $I_x = \{i \in \{1, \ldots, t\} \mid x \in U_i\}$. Then $\gcd(n_i, i \in I_x)$ is attained by n_i for some $i \in I_x$.

As our filtration we take $\mathcal{F}_0 = (0)$ and $\mathcal{F}_n$ generated by the sections n_i over U_i for those i such that $n_i \leq n$. It is clear that $\mathcal{F}_n = \mathcal{F}$ for $n \gg 0$. Moreover, the quotient $\mathcal{F}_n/\mathcal{F}_{n-1}$ is generated by the section n over $U = \bigcup_{n_i \leq n} U_i$ and the kernel of the map $j_!\underline{\mathbf{Z}}_U \to \mathcal{F}_n/\mathcal{F}_{n-1}$ is generated by the section n over $V = \bigcup_{n_i \leq n-1} U_i$. Thus a short exact sequence as in the statment of the lemma. $\square$

Lemma 21.4. *Let X be a topological space. Let $d \geq 0$ be an integer. Assume*

(1) *X is quasi-compact,*
(2) *the quasi-compact opens form a basis for X, and*
(3) *the intersection of two quasi-compact opens is quasi-compact.*
(4) *$H^p(X, j_!\underline{\mathbf{Z}}_U) = 0$ for all $p > d$ and any quasi-compact open $j : U \to X$.*

Then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and any abelian sheaf $\mathcal{F}$ on X.

Proof. Let $S = \coprod_{U \subset X} \mathcal{F}(U)$ where U runs over the quasi-compact opens of X. For any finite subset $A = \{s_1, \ldots, s_n\} \subset S$, let $\mathcal{F}_A$ be the subsheaf of $\mathcal{F}$ generated by all s_i (see Modules, Definition 4.5). Note that if $A \subset A'$, then $\mathcal{F}_A \subset \mathcal{F}_{A'}$. Hence $\{\mathcal{F}_A\}$ forms a system over the directed partially ordered set of finite subsets of S. By Modules, Lemma 4.6 it is clear that

$$\operatorname{colim}_A \mathcal{F}_A = \mathcal{F}$$

by looking at stalks. By Lemma 20.1 we have

$$H^p(X, \mathcal{F}) = \operatorname{colim}_A H^p(X, \mathcal{F}_A)$$

Hence it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}_A$. In other words, it suffices to prove the result when $\mathcal{F}$ is generated by finitely many local sections over quasi-compact opens of X.

Suppose that $\mathcal{F}$ is generated by the local sections $s_1, \ldots, s_n$. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf generated by $s_1, \ldots, s_{n-1}$. Then we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$$

From the long exact sequence of cohomology we see that it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}'$ and $\mathcal{F}/\mathcal{F}'$ which are generated by fewer than n local sections. Hence it suffices to prove the vanishing for sheaves generated by at most one local section. These sheaves are exactly the quotients of the sheaves $j_!\underline{\mathbf{Z}}_U$ where U is a quasi-compact open of X.

Assume now that we have a short exact sequence

$$0 \to \mathcal{K} \to j_!\underline{\mathbf{Z}}_U \to \mathcal{F} \to 0$$

with U quasi-compact open in X. It suffices to show that $H^q(X, \mathcal{K})$ is zero for $q \geq d+1$. As above we can write $\mathcal{K}$ as the filtered colimit of subsheaves $\mathcal{K}'$ generated by finitely many sections over quasi-compact opens. Then $\mathcal{F}$ is the filtered colimit of the sheaves $j_!\underline{\mathbf{Z}}_U/\mathcal{K}'$. In this way we reduce to the case that $\mathcal{K}$ is generated by finitely many sections over quasi-compact opens. Note that $\mathcal{K}$ is a subsheaf of $\underline{\mathbf{Z}}_X$. Thus by Lemma 21.3 there exists a finite filtration of $\mathcal{K}$ whose successive quotients $\mathcal{Q}$ fit into a short exact sequence

$$0 \to j''_!\underline{\mathbf{Z}}_W \to j'_!\underline{\mathbf{Z}}_V \to \mathcal{Q} \to 0$$

with $j'' : W \to X$ and $j' : V \to X$ the inclusions of quasi-compact opens. Hence the vanishing of $H^p(X, \mathcal{Q})$ for $p > d$ follows from our assumption (in the lemma) on the vanishing of the cohomology groups of $j''_!\underline{\mathbf{Z}}_W$ and $j'_!\underline{\mathbf{Z}}_V$. Returning to $\mathcal{K}$ this, via an induction argument using the long exact cohomology sequence, implies the desired vanishing for it as well. $\square$

Lemma 21.5. *Let X be an irreducible topological space. Let $\mathcal{H} \subset \underline{\mathbf{Z}}$ be an abelian subsheaf of the constant sheaf. Then there exists a nonempty open $U \subset X$ such that $\mathcal{H}|_U = \underline{d\mathbf{Z}_U}$ for some $d \in \mathbf{Z}$.*

Proof. Recall that $\underline{\mathbf{Z}}(V) = \mathbf{Z}$ for any nonempty open V of X (see proof of Lemma 21.2). If $\mathcal{H} = 0$, then the lemma holds with $d = 0$. If $\mathcal{H} \neq 0$, then there exists a nonempty open $U \subset X$ such that $\mathcal{H}(U) \neq 0$. Say $\mathcal{H}(U) = n\mathbf{Z}$ for some $n \geq 1$. Hence we see that $\underline{n\mathbf{Z}_U} \subset \mathcal{H}|_U \subset \underline{\mathbf{Z}}_U$. If the first inclusion is strict we can find a nonempty $U' \subset U$ and an integer $1 \leq n' < n$ such that $\underline{n'\mathbf{Z}_{U'}} \subset \mathcal{H}|_{U'} \subset \underline{\mathbf{Z}}_{U'}$. This process has to stop after a finite number of steps, and hence we get the lemma. $\square$

Proposition 21.6 (Grothendieck)**.** *Let X be a Noetherian topological space. If $\dim(X) \leq d$, then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and any abelian sheaf $\mathcal{F}$ on X.*

Proof. We prove this lemma by induction on d. So fix d and assume the lemma holds for all Noetherian topological spaces of dimension $< d$.

Let $\mathcal{F}$ be an abelian sheaf on X. Suppose $U \subset X$ is an open. Let $Z \subset X$ denote the closed complement. Denote $j : U \to X$ and $i : Z \to X$ the inclusion maps. Then there is a short exact sequence

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

see Modules, Lemma 7.1. Note that $j_!j^*\mathcal{F}$ is supported on the topological closure Z' of U, i.e., it is of the form $i'_*\mathcal{F}'$ for some abelian sheaf $\mathcal{F}'$ on Z', where $i' : Z' \to X$ is the inclusion.

We can use this to reduce to the case where X is irreducible. Namely, according to Topology, Lemma 8.2 X has finitely many irreducible components. If X has more than one irreducible component, then let $Z \subset X$ be an irreducible component of X and set $U = X \setminus Z$. By the above, and the long exact sequence of cohomology, it suffices to prove the vanishing of $H^p(X, i_*i^*\mathcal{F})$ and $H^p(X, i'_*\mathcal{F}')$ for $p > d$. By Lemma 21.1 it suffices to prove $H^p(Z, i^*\mathcal{F})$ and $H^p(Z', \mathcal{F}')$ vanish for $p > d$. Since Z' and Z have fewer irreducible components we indeed reduce to the case of an irreducible X.

If $d = 0$ and $X = \{*\}$, then every sheaf is constant and higher cohomology groups vanish (for example by Lemma 21.2).

Suppose X is irreducible of dimension d. By Lemma 21.4 we reduce to the case where $\mathcal{F} = j_!\underline{\mathbf{Z}}_U$ for some open $U \subset X$. In this case we look at the short exact sequence

$$0 \to j_!(\underline{\mathbf{Z}}_U) \to \underline{\mathbf{Z}}_X \to i_*\underline{\mathbf{Z}}_Z \to 0$$

where $Z = X \setminus U$. By Lemma 21.2 we have the vanishing of $H^p(X, \underline{\mathbf{Z}}_X)$ for all $p \geq 1$. By induction we have $H^p(X, i_*\underline{\mathbf{Z}}_Z) = H^p(Z, \underline{\mathbf{Z}}_Z) = 0$ for $p \geq d$. Hence we win by the long exact cohomology sequence. $\square$

22. Cohomology with support in a closed

Let X be a topological space and let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be an abelian sheaf on X. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \mathrm{Supp}(s) \subset Z\}$$

be the sections with support in Z (Modules, Definition 5.1). This is a left exact functor which is not exact in general. Hence we obtain a derived functor

$$R\Gamma_Z(X, -) : D(X) \longrightarrow D(Ab)$$

and cohomology groups with support in Z defined by $H^q_Z(X, \mathcal{F}) = R^q\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on X. Let $U = X \setminus Z$. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Lemma 9.1) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D(X)$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(Ab)$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \ldots$$

for any K in $D(X)$.

For an abelian sheaf $\mathcal{F}$ on X we can consider the *subsheaf of sections with support in Z*, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \cap Z\}$$

Using the equivalence of Modules, Lemma 6.1 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on Z (see also Modules, Lemmas 6.2 and 6.3). Thus we obtain a functor

$$Ab(X) \longrightarrow Ab(Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as a sheaf on } Z$$

which is left exact, but in general not exact.

Lemma 22.1. *Let $i : Z \to X$ be the inclusion of a closed subset. Let $\mathcal{I}$ be an injective abelian sheaf on X. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on Z.*

Proof. Observe that for any abelian sheaf $\mathcal{G}$ on Z we have

$$\text{Hom}_Z(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \text{Hom}_X(i_*\mathcal{G}, \mathcal{F})$$

because after all any section of $i_*\mathcal{G}$ has support in Z. Since i_* is exact (Modules, Lemma 6.1) and $\mathcal{I}$ injective on X we conclude that $\mathcal{H}_Z(\mathcal{I})$ is injective on Z. ☐

Denote

$$R\mathcal{H}_Z : D(X) \longrightarrow D(Z)$$

the derived functor. We set $\mathcal{H}^q_Z(\mathcal{F}) = R^q\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$. By the lemma above we have a Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow H^{p+q}_Z(X, \mathcal{F})$$

Lemma 22.2. *Let $i : Z \to X$ be the inclusion of a closed subset. Let $\mathcal{G}$ be an injective abelian sheaf on Z. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.*

Proof. This is true because the functor i_* is exact and transforms injective abelian sheaves into injective abelian sheaves by Lemma 12.10. ☐

Let X be a topological space and let $Z \subset X$ be a closed subset. We denote $D_Z(X)$ the strictly full saturated triangulated subcategory of $D(X)$ consisting of complexes whose cohomology sheaves are supported on Z.

Lemma 22.3. *Let $i : Z \to X$ be the inclusion of a closed subset of a topological space X. The map $Ri_* = i_* : D(Z) \to D(X)$ induces an equivalence $D(Z) \to D_Z(X)$ with quasi-inverse*

$$i^{-1}|_{D_Z(X)} = R\mathcal{H}_Z|_{D_Z(X)}$$

Proof. Recall that i^{-1} and i_* is an adjoint pair of exact functors such that $i^{-1}i_*$ is isomorphic to the identify functor on abelian sheaves. See Modules, Lemmas 3.3 and 6.1. Thus $i_* : D(Z) \to D_Z(X)$ is fully faithfull and i^{-1} determines a left inverse. On the other hand, suppose that K is an object of $D_Z(X)$ and consider the adjunction map $K \to i_*i^{-1}K$. Using exactness of i_* and i^{-1} this induces the adjunction maps $H^n(K) \to i_*i^{-1}H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on Z we see these adjunction maps are isomorphisms and we conclude that $D(Z) \to D_Z(X)$ is an equivalence.

To finish the proof we have to show that $R\mathcal{H}_Z(K) = i^{-1}K$ if K is an object of $D_Z(X)$. To do this we can use that $K = i_*i^{-1}K$ as we've just proved this is the case. Then we can choose a K-injective representative $\mathcal{I}^\bullet$ for $i^{-1}K$. Since i_* is the right adjoint to the exact functor i^{-1}, the complex $i_*\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 29.10). We see that $R\mathcal{H}_Z(K)$ is computed by $\mathcal{H}_Z(i_*\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ as desired. $\square$

23. Cohomology on spectral spaces

A key result on the cohomology of spectral spaces is Lemma 20.2 which loosely speaking says that cohomology commutes with cofiltered limits in the category of spectral spaces as defined in Topology, Definition 22.1. This can be applied to give analogues of Lemmas 17.3 and 19.1 as follows.

Lemma 23.1. *Let X be a spectral space. Let $\mathcal{F}$ be an abelian sheaf on X. Let $E \subset X$ be a quasi-compact subset. Let $W \subset X$ be the set of points of X which specialize to a point of E.*

1. *$H^p(W, \mathcal{F}|_W) = \operatorname{colim} H^p(U, \mathcal{F})$ where the colimit is over quasi-compact open neighbourhoods of E,*
2. *$H^p(W \setminus E, \mathcal{F}|_{W\setminus E}) = \operatorname{colim} H^p(U \setminus E, \mathcal{F}|_{U\setminus E})$ if E is a constructible subset.*

Proof. From Topology, Lemma 23.7 we see that $W = \lim U$ where the limit is over the quasi-compact opens containing E. Each U is a spectral space by Topology, Lemma 22.4. Thus we may apply Lemma 20.2 to conclude that (1) holds. The same proof works for part (2) except we use Topology, Lemma 23.8. $\square$

Lemma 23.2. *Let $f : X \to Y$ be a spectral map of spectral spaces. Let $y \in Y$. Let $E \subset Y$ be the set of points specializing to y. Let $\mathcal{F}$ be an abelian sheaf on X. Then $(R^pf_*\mathcal{F})_y = H^p(f^{-1}(E), \mathcal{F}|_{f^{-1}(E)})$.*

Proof. Observe that $E = \bigcap V$ where V runs over the quasi-compact open neighbrouhoods of y in Y. Hence $f^{-1}(E) = \bigcap f^{-1}(V)$. This implies that $f^{-1}(E) = \lim f^{-1}(V)$ as topological spaces. Since f is spectral, each $f^{-1}(V)$ is a spectral space too (Topology, Lemma 22.4). We conclude that $f^{-1}(E)$ is a spectral space and that

$$H^p(f^{-1}(E), \mathcal{F}|_{f^{-1}(E)}) = \operatorname{colim} H^p(f^{-1}(V), \mathcal{F})$$

by Lemma 20.2. On the other hand, the stalk of $R^pf_*\mathcal{F}$ at y is given by the colimit on the right. $\square$

Lemma 23.3. *Let X be a profinite topological space. Then $H^q(X, \mathcal{F}) = 0$ for all $q > 0$ and all abelian sheaves $\mathcal{F}$.*

Proof. Any open covering of X can be refined by a finite disjoint union decomposition with open parts, see Topology, Lemma 21.3. Hence if $\mathcal{F} \to \mathcal{G}$ is a surjection of abelian sheaves on X, then $\mathcal{F}(X) \to \mathcal{G}(X)$ is surjective. In other words, the global sections functor is an exact functor. Therefore its higher derived functors are zero, see Derived Categories, Lemma 17.8. $\square$

The following result on cohomological vanishing improves Grothendieck's result (Proposition 21.6) and can be found in [Sch92].

Proposition 23.4. *Let X be a spectral space of Krull dimension d. Let $\mathcal{F}$ be an abelian sheaf on X.*

1. *$H^q(X, \mathcal{F}) = 0$ for $q > d$,*
2. *$H^d(X, \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective for every quasi-compact open $U \subset X$,*
3. *$H^q_Z(X, \mathcal{F}) = 0$ for $q > d$ and any constructible closed subset $Z \subset X$.*

Proof. We prove this result by induction on d.

If $d = 0$, then X is a profinite space, see Topology, Lemma 22.7. Thus (1) holds by Lemma 23.3. If $U \subset X$ is quasi-compact open, then U is also closed as a quasi-compact subset of a Hausdorff space. Hence $X = U \coprod (X \setminus U)$ as a topological space and we see that (2) holds. Given Z as in (3) we consider the long exact sequence

$$H^{q-1}(X, \mathcal{F}) \to H^{q-1}(X \setminus Z, \mathcal{F}) \to H^q_Z(X, \mathcal{F}) \to H^q(X, \mathcal{F})$$

Since X and $U = X \setminus Z$ are profinite (namely U is quasi-compact because Z is constructible) and since we have (2) and (1) we obtain the desired vanishing of the cohomology groups with support in Z.

Induction step. Assume $d \geq 1$ and assume the proposition is valid for all spectral spaces of dimension $< d$. We first prove part (2) for X. Let U be a quasi-compact open. Let $\xi \in H^d(U, \mathcal{F})$. Set $Z = X \setminus U$. Let $W \subset X$ be the set of points specializing to Z. By Lemma 23.1 we have

$$H^d(W \setminus Z, \mathcal{F}|_{W \setminus Z}) = \mathrm{colim}_{Z \subset V} H^d(V \setminus Z, \mathcal{F})$$

where the colimit is over the quasi-compact open neighbourhoods V of Z in X. By Topology, Lemma 23.7 we see that $W \setminus Z$ is a spectral space. Since every point of W specializes to a point of Z, we see that $W \setminus Z$ is a spectral space of Krull dimension $< d$. By induction hypothesis we see that the image of ξ in $H^d(W \setminus Z, \mathcal{F}|_{W \setminus Z})$ is zero. By the displayed formula, there exists a $Z \subset V \subset X$ quasi-compact open such that $\xi|_{V \setminus Z} = 0$. Since $V \setminus Z = V \cap U$ we conclude by the Mayer-Vietoris (Lemma 9.2) for the covering $X = U \cap V$ that there exists a $\tilde{\xi} \in H^d(X, \mathcal{F})$ which restricts to ξ on U and to zero on V. In other words, part (2) is true.

Proof of part (1) assuming (2). Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Set

$$\mathcal{G} = \mathrm{Im}(\mathcal{I}^{d-1} \to \mathcal{I}^d) = \mathrm{Ker}(\mathcal{I}^d \to \mathcal{I}^{d+1})$$

For $U \subset X$ quasi-compact open we have a map of exact sequences as follows

$$\begin{tikzcd}
\mathcal{I}^{d-1}(X) \arrow[r] \arrow[d] & \mathcal{G}(X) \arrow[r] \arrow[d] & H^d(X, \mathcal{F}) \arrow[r] \arrow[d] & 0 \\
\mathcal{I}^{d-1}(U) \arrow[r] & \mathcal{G}(U) \arrow[r] & H^d(U, \mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

The sheaf $\mathcal{I}^{d-1}$ is flasque by Lemma 13.2 and the fact that $d \geq 1$. By part (2) we see that the right vertical arrow is surjective. We conclude by a diagram chase that the map $\mathcal{G}(X) \to \mathcal{G}(U)$ is surjective. By Lemma 13.6 we conclude that $\check{H}^q(\mathcal{U}, \mathcal{G}) = 0$ for $q > 0$ and any finite covering $\mathcal{U} : U = U_1 \cup \ldots \cup U_n$ of a quasi-compact open by quasi-compact opens. Applying Lemma 12.8 we find that $H^q(U, \mathcal{G}) = 0$ for all $q > 0$ and all quasi-compact opens U of X. By Leray's acyclicity lemma (Derived Categories, Lemma 17.7) we conclude that

$$H^q(X, \mathcal{F}) = H^q\left(\Gamma(X, \mathcal{I}^0) \to \ldots \to \Gamma(X, \mathcal{I}^{d-1}) \to \Gamma(X, \mathcal{G})\right)$$

In particular the cohomology group vanishes if $q > d$.

Proof of (3). Given Z as in (3) we consider the long exact sequence

$$H^{q-1}(X, \mathcal{F}) \to H^{q-1}(X \setminus Z, \mathcal{F}) \to H^q_Z(X, \mathcal{F}) \to H^q(X, \mathcal{F})$$

Since X and $U = X \setminus Z$ are spectral spaces (Topology, Lemma 22.4) of dimension $\leq d$ and since we have (2) and (1) we obtain the desired vanishing. $\square$

24. The alternating Čech complex

This section compares the Čech complex with the alternating Čech complex and some related complexes.

Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. For $p \geq 0$ set

$$\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = \left\{ \begin{matrix} s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \text{ such that } s_{i_0 \ldots i_p} = 0 \text{ if } i_n = i_m \text{ for some } n \neq m \\ \text{and } s_{i_0 \ldots i_n \ldots i_m \ldots i_p} = -s_{i_0 \ldots i_m \ldots i_n \ldots i_p} \text{ in any case.} \end{matrix} \right\}$$

We omit the verification that the differential d of Equation (10.0.1) maps $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F})$ into $\check{\mathcal{C}}^{p+1}_{alt}(\mathcal{U}, \mathcal{F})$.

Definition 24.1. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ is the *alternating Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$.

Hence there is a canonical morphism of complexes

$$\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

namely the inclusion of the alternating Čech complex into the usual Čech complex.

Suppose our covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ comes equipped with a total ordering $<$ on I. In this case, set

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}, i_0 < \ldots < i_p} \mathcal{F}(U_{i_0 \ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in $\mathcal{F}(U_{i_0 \ldots i_p})$. We define

$$d : \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}_{ord}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_p}|_{U_{i_0 \ldots i_{p+1}}}$$

for any $i_0 < \ldots < i_{p+1}$. Note that this formula is identical to Equation (10.0.1). It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 24.2. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume given a total ordering on I. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is the *ordered Čech complex* associated to $\mathcal{F}$, the open covering $\mathcal{U}$ and the given total ordering on I.

This complex is sometimes called the alternating Čech complex. The reason is that there is an obvious comparison map between the ordered Čech complex and the alternating Čech complex. Namely, consider the map

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

given by the rule

$$c(s)_{i_0 \ldots i_p} = \begin{cases} 0 & \text{if} \quad i_n = i_m \text{ for some } n \neq m \\ \operatorname{sgn}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}} & \text{if} \quad i_{\sigma(0)} < i_{\sigma(1)} < \ldots < i_{\sigma(p)} \end{cases}$$

Here σ denotes a permutation of $\{0, \ldots, p\}$ and $\operatorname{sgn}(\sigma)$ denotes its sign. The alternating and ordered Čech complexes are often identified in the literature via the map c. Namely we have the following easy lemma.

Lemma 24.3. *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map c is a morphism of complexes. In fact it induces an isomorphism*

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

of complexes.

Proof. Omitted. □

There is also a map

$$\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

which is described by the rule

$$\pi(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}$$

whenever $i_0 < i_1 < \ldots < i_p$.

Lemma 24.4. *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map $\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a morphism of complexes. It induces an isomorphism*

$$\pi : \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

of complexes which is a left inverse to the morphism c.

Proof. Omitted. □

Remark 24.5. This means that if we have two total orderings $<_1$ and $<_2$ on the index set I, then we get an isomorphism of complexes $\tau = \pi_2 \circ c_1 : \check{\mathcal{C}}_{ord\text{-}1}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}_{ord\text{-}2}(\mathcal{U}, \mathcal{F})$. It is clear that

$$\tau(s)_{i_0 \ldots i_p} = \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

where $i_0 <_1 i_1 <_1 \ldots <_1 i_p$ and $i_{\sigma(0)} <_2 i_{\sigma(1)} <_2 \ldots <_2 i_{\sigma(p)}$. This is the sense in which the ordered Čech complex is independent of the chosen total ordering.

Lemma 24.6. *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map $c \circ \pi$ is homotopic to the identity on $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$. In particular the inclusion map $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a homotopy equivalence.*

Proof. For any multi-index $(i_0, \ldots, i_p) \in I^{p+1}$ there exists a unique permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ such that

$$i_{\sigma(0)} \leq i_{\sigma(1)} \leq \ldots \leq i_{\sigma(p)} \quad \text{and} \quad \sigma(j) < \sigma(j+1) \quad \text{if} \quad i_{\sigma(j)} = i_{\sigma(j+1)}.$$

We denote this permutation $\sigma = \sigma^{i_0 \ldots i_p}$.

For any permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ and any a, $0 \leq a \leq p$ we denote σ_a the permutation of $\{0, \ldots, p\}$ such that

$$\sigma_a(j) = \begin{cases} \sigma(j) & \text{if} \quad 0 \leq j < a, \\ \min\{j' \mid j' > \sigma_a(j-1), j' \neq \sigma(k), \forall k < a\} & \text{if} \quad a \leq j \end{cases}$$

So if $p = 3$ and σ, τ are given by

$$\begin{matrix} \text{id} & 0 & 1 & 2 & 3 \\ \sigma & 3 & 2 & 1 & 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \text{id} & 0 & 1 & 2 & 3 \\ \tau & 3 & 0 & 2 & 1 \end{matrix}$$

then we have

$$\begin{array}{ccccc} \text{id} & 0 & 1 & 2 & 3 \\ \sigma_0 & 0 & 1 & 2 & 3 \\ \sigma_1 & 3 & 0 & 1 & 2 \\ \sigma_2 & 3 & 2 & 0 & 1 \\ \sigma_3 & 3 & 2 & 1 & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccc} \text{id} & 0 & 1 & 2 & 3 \\ \tau_0 & 0 & 1 & 2 & 3 \\ \tau_1 & 3 & 0 & 1 & 2 \\ \tau_2 & 3 & 0 & 1 & 2 \\ \tau_3 & 3 & 0 & 2 & 1 \end{array}$$

It is clear that always $\sigma_0 = \text{id}$ and $\sigma_p = \sigma$.

Having introduced this notation we define for $s \in \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$ the element $h(s) \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ to be the element with components

$$h(s)_{i_0 \ldots i_p} = \sum\nolimits_{0 \leq a \leq p} (-1)^a \text{sign}(\sigma_a) s_{i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}} \tag{24.6.1}$$

where $\sigma = \sigma^{i_0 \ldots i_p}$. The index $i_{\sigma(a)}$ occurs twice in $i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}$ once in the first group of $a+1$ indices and once in the second group of $p-a+1$ indices since $\sigma_a(j) = \sigma(a)$ for some $j \geq a$ by definition of σ_a. Hence the sum makes sense since each of the elements $s_{i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}}$ is defined over the open $U_{i_0 \ldots i_p}$. Note also that for $a = 0$ we get $s_{i_0 \ldots i_p}$ and for $a = p$ we get $(-1)^p \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$.

We claim that

$$(dh + hd)(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p} - \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

where $\sigma = \sigma^{i_0 \ldots i_p}$. We omit the verification of this claim. (There is a PARI/gp script called first-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) Write

$$\kappa : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

for the operator given by the rule

$$\kappa(s)_{i_0 \ldots i_p} = \text{sign}(\sigma^{i_0 \ldots i_p}) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}.$$

The claim above implies that κ is a morphism of complexes and that κ is homotopic to the identity map of the Čech complex. This does not immediately imply the lemma since the image of the operator κ is not the alternating subcomplex. Namely, the image of κ is the "semi-alternating" complex $\check{\mathcal{C}}^p_{\textit{semi-alt}}(\mathcal{U}, \mathcal{F})$ where s is a p-cochain of this complex if and only if

$$s_{i_0 \ldots i_p} = \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

for any $(i_0, \ldots, i_p) \in I^{p+1}$ with $\sigma = \sigma^{i_0 \ldots i_p}$. We introduce yet another variant Čech complex, namely the semi-ordered Čech complex defined by

$$\check{\mathcal{C}}^p_{\textit{semi-ord}}(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{i_0 \leq i_1 \leq \ldots \leq i_p} \mathcal{F}(U_{i_0 \ldots i_p})$$

It is easy to see that Equation (10.0.1) also defines a differential and hence that we get a complex. It is also clear (analogous to Lemma 24.4) that the projection map

$$\check{\mathcal{C}}^\bullet_{\textit{semi-alt}}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{\textit{semi-ord}}(\mathcal{U}, \mathcal{F})$$

is an isomorphism of complexes.

Hence the Lemma follows if we can show that the obvious inclusion map

$$\check{\mathcal{C}}^p_{\textit{ord}}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^p_{\textit{semi-ord}}(\mathcal{U}, \mathcal{F})$$

is a homotopy equivalence. To see this we use the homotopy

(24.6.2)
$$h(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} \quad i_0 < i_1 < \ldots < i_p \\ (-1)^a s_{i_0\ldots i_{a-1}i_ai_ai_{a+1}\ldots i_p} & \text{if} \quad i_0 < i_1 < \ldots < i_{a-1} < i_a = i_{a+1} \end{cases}$$

We claim that

$$(dh + hd)(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} \quad i_0 < i_1 < \ldots < i_p \\ s_{i_0\ldots i_p} & \text{else} \end{cases}$$

We omit the verification. (There is a PARI/gp script called second-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) The claim clearly shows that the composition

$$\check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

of the projection with the natural inclusion is homotopic to the identity map as desired. □

25. Alternative view of the Čech complex

In this section we discuss an alternative way to establish the relationship between the Čech complex and cohomology.

Lemma 25.1. *Let X be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{i_0\ldots i_p}$ the restriction of $\mathcal{F}$ to $U_{i_0\ldots i_p}$. There exists a complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{O}_X$-modules with*

$$\mathfrak{C}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{i_0\ldots i_p} (j_{i_0\ldots i_p})_* \mathcal{F}_{i_0\ldots i_p}$$

and differential $d : \mathfrak{C}^p(\mathcal{U}, \mathcal{F}) \to \mathfrak{C}^{p+1}(\mathcal{U}, \mathcal{F})$ as in Equation (10.0.1). Moreover, there exists a canonical map

$$\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$$

which is a quasi-isomorphism, i.e., $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$.

Proof. We check

$$0 \to \mathcal{F} \to \mathfrak{C}^0(\mathcal{U}, \mathcal{F}) \to \mathfrak{C}^1(\mathcal{U}, \mathcal{F}) \to \ldots$$

is exact on stalks. Let $x \in X$ and choose $i_{\text{fix}} \in I$ such that $x \in U_{i_{\text{fix}}}$. Then define

$$h : \mathfrak{C}^p(\mathcal{U}, \mathcal{F})_x \to \mathfrak{C}^{p-1}(\mathcal{U}, \mathcal{F})_x$$

as follows: If $s \in \mathfrak{C}^p(\mathcal{U}, \mathcal{F})_x$, take a representative

$$\widetilde{s} \in \mathfrak{C}^p(\mathcal{U}, \mathcal{F})(V) = \prod\nolimits_{i_0\ldots i_p} \mathcal{F}(V \cap U_{i_0} \cap \ldots \cap U_{i_p})$$

defined on some neighborhood V of x, and set

$$h(s)_{i_0\ldots i_{p-1}} = \widetilde{s}_{i_{\text{fix}}i_0\ldots i_{p-1},x}.$$

By the same formula (for $p = 0$) we get a map $\mathfrak{C}^0(\mathcal{U}, \mathcal{F})_x \to \mathcal{F}_x$. We compute formally as follows:

$$\begin{aligned} (dh + hd)(s)_{i_0\ldots i_p} &= \sum\nolimits_{j=0}^p (-1)^j h(s)_{i_0\ldots \hat{i}_j \ldots i_p} + d(s)_{i_{\text{fix}}i_0\ldots i_p} \\ &= \sum\nolimits_{j=0}^p (-1)^j s_{i_{\text{fix}}i_0\ldots \hat{i}_j\ldots i_p} + s_{i_0\ldots i_p} + \sum\nolimits_{j=0}^p (-1)^{j+1} s_{i_{\text{fix}}i_0\ldots \hat{i}_j\ldots i_p} \\ &= s_{i_0\ldots i_p} \end{aligned}$$

This shows h is a homotopy from the identity map of the extended complex

$$0 \to \mathcal{F}_x \to \mathfrak{C}^0(\mathcal{U}, \mathcal{F})_x \to \mathfrak{C}^1(\mathcal{U}, \mathcal{F})_x \to \ldots$$

to zero and we conclude. $\square$

With this lemma it is easy to reprove the Čech to cohomology spectral sequence of Lemma 12.4. Namely, let X, $\mathcal{U}$, $\mathcal{F}$ as in Lemma 25.1 and let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then we may consider the double complex

$$A^{\bullet,\bullet} = \Gamma(X, \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

By construction we have

$$A^{p,q} = \prod\nolimits_{i_0\ldots i_p} \mathcal{I}^q(U_{i_0\ldots i_p})$$

Consider the two spectral sequences of Homology, Section 22 associated to this double complex, see especially Homology, Lemma 22.4. For the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ we get ${}'E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ because taking products is exact (Homology, Lemma 28.1). For the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$ we get ${}''E_2^{p,q} = 0$ if $p > 0$ and ${}''E_2^{0,q} = H^q(X, \mathcal{F})$. Namely, for fixed q the complex of sheaves $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^q)$ is a resolution (Lemma 25.1) of the injective sheaf $\mathcal{I}^q$ by injective sheaves (by Lemmas 7.1 and 12.10 and Homology, Lemma 23.3). Hence the cohomology of $\Gamma(X, \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^q))$ is zero in positive degrees and equal to $\Gamma(X, \mathcal{I}^q)$ in degree 0. Taking cohomology of the next differential we get our claim about the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$. Whence the result since both spectral sequences converge to the cohomology of the associated total complex of $A^{\bullet,\bullet}$.

Definition 25.2. Let X be a topological space. An open covering $X = \bigcup_{i\in I} U_i$ is said to be *locally finite* if for every $x \in X$ there exists an open neighbourhood W of x such that $\{i \in I \mid W \cap U_i \neq \emptyset\}$ is finite.

Remark 25.3. Let $X = \bigcup_{i\in I} U_i$ be a locally finite open covering. Denote $j_i : U_i \to X$ the inclusion map. Suppose that for each i we are given an abelian sheaf $\mathcal{F}_i$ on U_i. Consider the abelian sheaf $\mathcal{G} = \bigoplus_{i\in I} (j_i)_*\mathcal{F}_i$. Then for $V \subset X$ open we actually have

$$\Gamma(V, \mathcal{G}) = \prod\nolimits_{i\in I} \mathcal{F}_i(V \cap U_i).$$

In other words we have

$$\bigoplus\nolimits_{i\in I} (j_i)_*\mathcal{F}_i = \prod\nolimits_{i\in I} (j_i)_*\mathcal{F}_i$$

This seems strange until you realize that the direct sum of a collection of sheaves is the sheafification of what you think it should be. See discussion in Modules, Section 3. Thus we conclude that in this case the complex of Lemma 25.1 has terms

$$\mathfrak{C}^p(\mathcal{U}, \mathcal{F}) = \bigoplus\nolimits_{i_0\ldots i_p} (j_{i_0\ldots i_p})_*\mathcal{F}_{i_0\ldots i_p}$$

which is sometimes useful.

26. Čech cohomology of complexes

In general for sheaves of abelian groups $\mathcal{F}$ and $\mathcal{G}$ on X there is a cupproduct map

$$H^i(X, \mathcal{F}) \times H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes_{\mathbf{Z}} \mathcal{G}).$$

In this section we define it using Čech cocycles by an explicit formula for the cup product. If you are worried about the fact that cohomology may not equal Čech cohomology, then you can use hypercoverings and still use the cocycle notation. This also has the advantage that it works to define the cup product for hypercohomology on any topos (insert future reference here).

Let $\mathcal{F}^\bullet$ be a bounded below complex of presheaves of abelian groups on X. We can often compute $H^n(X, \mathcal{F}^\bullet)$ using Čech cocycles. Namely, let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Since the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ (Definition 10.1) is functorial in the presheaf $\mathcal{F}$ we obtain a double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$. The associated total complex to $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ is the complex with degree n term

$$\text{Tot}^n(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) = \bigoplus\nolimits_{p+q=n} \prod\nolimits_{i_0 \ldots i_p} \mathcal{F}^q(U_{i_0 \ldots i_p})$$

see Homology, Definition 22.3. A typical element in Tot^n will be denoted $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ where $\alpha_{i_0 \ldots i_p} \in \mathcal{F}^q(U_{i_0 \ldots i_p})$. In other words the $\mathcal{F}$-degree of $\alpha_{i_0 \ldots i_p}$ is $q = n - p$. This notation requires us to be aware of the degree α lives in at all times. We indicate this situation by the formula $\deg_{\mathcal{F}}(\alpha_{i_0 \ldots i_p}) = q$. According to our conventions in Homology, Definition 22.3 the differential of an element α of degree n is given by

$$d(\alpha)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0 \ldots i_{p+1}})$$

where $d_{\mathcal{F}}$ denotes the differential on the complex $\mathcal{F}^\bullet$. The expression $\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$ means the restriction of $\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \in \mathcal{F}(U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})$ to $U_{i_0 \ldots i_{p+1}}$.

The construction of $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

$$\Gamma(X, \mathcal{F}^\bullet) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \tag{26.0.1}$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0 \ldots i_p} = 0$ for $p > 0$.

Refinements. Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a refinement of $\mathcal{U}$. This means there is a map $t : J \to I$ such that $V_j \subset U_{t(j)}$ for all $j \in J$. This gives rise to a functorial transformation

$$T_t : \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)). \tag{26.0.2}$$

defined by the rule

$$T_t(\alpha)_{j_0 \ldots j_p} = \alpha_{t(j_0) \ldots t(j_p)}|_{V_{j_0 \ldots j_p}}.$$

Given two maps $t, t' : J \to I$ as above the maps T_t and $T_{t'}$ constructed above are homotopic. The homotopy is given by

$$h(\alpha)_{j_0 \ldots j_p} = \sum\nolimits_{a=0}^{p} (-1)^a \alpha_{t(j_0) \ldots t(j_a) t'(j_a) \ldots t'(j_p)}$$

for an element α of degree n. This works because of the following computation, again with α an element of degree n (so $d(\alpha)$ has degree $n+1$ and $h(\alpha)$ has degree $n-1$):

$$\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{j_0\ldots j_p} = & \sum\nolimits_{k=0}^{p}(-1)^k h(\alpha)_{j_0\ldots\hat{j}_k\ldots j_p} + \\
& (-1)^p d_{\mathcal{F}}(h(\alpha)_{j_0\ldots j_p}) + \\
& \sum\nolimits_{a=0}^{p}(-1)^a d(\alpha)_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)} \\
= & \sum\nolimits_{k=0}^{p}\sum\nolimits_{a=0}^{k-1}(-1)^{k+a}\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(\hat{j}_k)\ldots t'(j_p)} + \\
& \sum\nolimits_{k=0}^{p}\sum\nolimits_{a=k+1}^{p}(-1)^{k+a-1}\alpha_{t(j_0)\ldots t(\hat{j}_k)\ldots t(j_a)t'(j_a)\ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p}(-1)^{p+a}d_{\mathcal{F}}(\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)}) + \\
& \sum\nolimits_{a=0}^{p}\sum\nolimits_{k=0}^{a}(-1)^{a+k}\alpha_{t(j_0)\ldots t(\hat{j}_k)\ldots t(j_a)t'(j_a)\ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p}\sum\nolimits_{k=a}^{p}(-1)^{a+k+1}\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(\hat{j}_k)\ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p}(-1)^{a+p+1}d_{\mathcal{F}}(\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)}) \\
& = \alpha_{t'(j_0)\ldots t'(j_p)} + (-1)^{2p+1}\alpha_{t(j_0)\ldots t(j_p)} \\
& = T_{t'}(\alpha)_{j_0\ldots j_p} - T_t(\alpha)_{j_0\ldots j_p}
\end{aligned}$$

We leave it to the reader to verify the cancellations. (Note that the terms having both k and a in the 1st, 2nd and 4th, 5th summands cancel, except the ones where $a = k$ which only occur in the 4th and 5th and these cancel against each other except for the two desired terms.) It follows that the induced map

$$H^n(T_t) : H^n(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to H^n(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)))$$

is independent of the choice of t. We define *Čech hypercohomology* as the limit of the Čech cohomology groups over all refinements via the maps $H^\bullet(T_t)$.

In the limit (over all open coverings of X) the following lemma provides a map of Čech hypercohomology into cohomology, which is often an isomorphism and is always an isomorphism if we use hypercoverings.

Lemma 26.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ be an open covering. For a bounded below complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical map*

$$\mathit{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

functorial in $\mathcal{F}^\bullet$ and compatible with (26.0.1) and (26.0.2). There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with

$$E_2^{p,q} = H^p(\mathit{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \underline{H}^q(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(X, \mathcal{F}^\bullet)$.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives. The map (26.0.1) for $\mathcal{I}^\bullet$ is a map $\Gamma(X, \mathcal{I}^\bullet) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from Homology, Lemma 22.7 applied to the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemma 12.1. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of

$\mathcal{F}^\bullet$ into a bounded below complex of injectives. Since $R\Gamma(X, \mathcal{F}^\bullet)$ is represented by the complex $\Gamma(X, \mathcal{I}^\bullet)$ we obtain the map of the lemma using

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

We omit the verification of functoriality and compatibilities. To construct the spectral sequence of the lemma, choose a Cartan-Eilenberg resolution $\mathcal{F}^\bullet \to \mathcal{I}^{\bullet,\bullet}$, see Derived Categories, Lemma 21.2. In this case $\mathcal{F}^\bullet \to \mathrm{Tot}(\mathcal{I}^{\bullet,\bullet})$ is an injective resolution and hence

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Tot}(\mathcal{I}^{\bullet,\bullet})))$$

computes $R\Gamma(X, \mathcal{F}^\bullet)$ as we've seen above. By Homology, Remark 22.9 we can view this as the total complex associated to the triple complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^{\bullet,\bullet})$ hence, using the same remark we can view it as the total complex associate to the double complex $A^{\bullet,\bullet}$ with terms

$$A^{n,m} = \bigoplus\nolimits_{p+q=n} \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^{q,m})$$

Since $\mathcal{I}^{q,\bullet}$ is an injective resolution of $\mathcal{F}^q$ we can apply the first spectral sequence associated to $A^{\bullet,\bullet}$ (Homology, Lemma 22.4) to get a spectral sequence with

$$E_1^{n,m} = \bigoplus\nolimits_{p+q=n} \check{\mathcal{C}}^p(\mathcal{U}, \underline{H}^m(\mathcal{F}^q))$$

which is the nth term of the complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \underline{H}^m(\mathcal{F}^\bullet)))$. Hence we obtain E_2 terms as described in the lemma. Convergence by Homology, Lemma 22.6. $\square$

Let X be a topological space, let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering, and let $\mathcal{F}^\bullet$ be a bounded below complex of presheaves of abelian groups. Consider the map $\tau : \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ defined by

$$\tau(\alpha)_{i_0 \ldots i_p} = (-1)^{p(p+1)/2} \alpha_{i_p \ldots i_0}.$$

Then we have for an element α of degree n that

$$\begin{aligned} & d(\tau(\alpha))_{i_0 \ldots i_{p+1}} \\ & = \sum\nolimits_{j=0}^{p+1} (-1)^j \tau(\alpha)_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}}(\tau(\alpha)_{i_0 \ldots i_{p+1}}) \\ & = \sum\nolimits_{j=0}^{p+1} (-1)^{j + \frac{p(p+1)}{2}} \alpha_{i_{p+1} \ldots \hat{i}_j \ldots i_0} + (-1)^{p+1+\frac{(p+1)(p+2)}{2}} d_{\mathcal{F}}(\alpha_{i_{p+1} \ldots i_0}) \end{aligned}$$

On the other hand we have

$$\begin{aligned} & \tau(d(\alpha))_{i_0 \ldots i_{p+1}} \\ & = (-1)^{\frac{(p+1)(p+2)}{2}} d(\alpha)_{i_{p+1} \ldots i_0} \\ & = (-1)^{\frac{(p+1)(p+2)}{2}} \left(\sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_{p+1} \ldots \hat{i}_{p+1-j} \ldots i_0} + (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_{p+1} \ldots i_0}) \right) \end{aligned}$$

Thus we conclude that $d(\tau(\alpha)) = \tau(d(\alpha))$ because $p(p+1)/2 \equiv (p+1)(p+2)/2 + p + 1 \bmod 2$. In other words τ is an endomorphism of the complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$. Note that the diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}^\bullet) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \\ \downarrow \mathrm{id} & & \downarrow \tau \\ \Gamma(X, \mathcal{F}^\bullet) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \end{array}$$

commutes. In addition τ is clearly compatible with refinements. This suggests that τ acts as the identity on Čech cohomology (i.e., in the limit – provided Čech hypercohomology agrees with hypercohomology, which is always the case if we use hypercoverings). We claim that τ actually is homotopic to the identity on the total Čech complex $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$. To prove this, we use as homotopy

$$h(\alpha)_{i_0 \ldots i_p} = \sum\nolimits_{a=0}^{p} \epsilon_p(a) \alpha_{i_0 \ldots i_a i_p \ldots i_a} \quad \text{with} \quad \epsilon_p(a) = (-1)^{\frac{(p-a)(p-a-1)}{2} + p}$$

for α of degree n. As usual we omit writing $|_{U_{i_0 \ldots i_p}}$. This works because of the following computation, again with α an element of degree n:

$$\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{i_0 \ldots i_p} = & \sum\nolimits_{k=0}^{p} (-1)^k h(\alpha)_{i_0 \ldots \hat{i}_k \ldots i_p} + \\
& (-1)^p d_{\mathcal{F}}(h(\alpha)_{i_0 \ldots i_p}) + \\
& \sum\nolimits_{a=0}^{p} \epsilon_p(a) d(\alpha)_{i_0 \ldots i_a i_p \ldots i_a} \\
= & \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=0}^{k-1} (-1)^k \epsilon_{p-1}(a) \alpha_{i_0 \ldots i_a i_p \ldots \hat{i_k} \ldots i_a} + \\
& \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=k+1}^{p} (-1)^k \epsilon_{p-1}(a-1) \alpha_{i_0 \ldots \hat{i_k} \ldots i_a i_p \ldots i_a} + \\
& \sum\nolimits_{a=0}^{p} (-1)^p \epsilon_p(a) d_{\mathcal{F}}(\alpha_{i_0 \ldots i_a i_p \ldots i_a}) + \\
& \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=0}^{a} \epsilon_p(a) (-1)^k \alpha_{i_0 \ldots \hat{i_k} \ldots i_a i_p \ldots i_a} + \\
& \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=a}^{p} \epsilon_p(a) (-1)^{p+a+1-k} \alpha_{i_0 \ldots i_a i_p \ldots \hat{i_k} \ldots i_a} + \\
& \sum\nolimits_{a=0}^{p} \epsilon_p(a) (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0 \ldots i_a i_p \ldots i_a}) \\
= & \epsilon_p(0) \alpha_{i_p \ldots i_0} + \epsilon_p(p)(-1)^{p+1} \alpha_{i_0 \ldots i_p} \\
= & (-1)^{\frac{p(p+1)}{2}} \alpha_{i_p \ldots i_0} - \alpha_{i_0 \ldots i_p}
\end{aligned}$$

The cancellations follow because

$$(-1)^k \epsilon_{p-1}(a) + \epsilon_p(a)(-1)^{p+a+1-k} = 0 \quad \text{and} \quad (-1)^k \epsilon_{p-1}(a-1) + \epsilon_p(a)(-1)^k = 0$$

We leave it to the reader to verify the cancellations.

Suppose we have two bounded below complexes complexes of abelian sheaves $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. We define the complex $\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet)$ to be to complex with terms $\bigoplus_{p+q=n} \mathcal{F}^p \otimes \mathcal{G}^q$ and differential according to the rule

$$d(\alpha \otimes \beta) = d(\alpha) \otimes \beta + (-1)^{\deg(\alpha)} \alpha \otimes d(\beta) \tag{26.1.1}$$

when α and β are homogeneous, see Homology, Definition 22.3.

Suppose that $M^\bullet$ and $N^\bullet$ are two bounded below complexes of abelian groups. Then if m, resp. n is a cocycle for $M^\bullet$, resp. $N^\bullet$, it is immediate that $m \otimes n$ is a cocycle for $\text{Tot}(M^\bullet \otimes N^\bullet)$. Hence a cupproduct

$$H^i(M^\bullet) \times H^j(N^\bullet) \longrightarrow H^{i+j}(Tot(M^\bullet \otimes N^\bullet)).$$

This is discussed also in More on Algebra, Section 49.

So the construction of the cup product in hypercohomology of complexes rests on a construction of a map of complexes

$$\text{Tot}\left(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}^\bullet))\right) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{Tot}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet))) \tag{26.1.2}$$

This map is denoted $\cup$ and is given by the rule

$$(\alpha \cup \beta)_{i_0 \ldots i_p} = \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_p}.$$

where α has degree n and β has degree m and with

$$\epsilon(n, m, p, r) = (-1)^{(p+r)n + rp + r}.$$

Note that $\epsilon(n, m, p, n) = 1$. Hence if $\mathcal{F}^\bullet = \mathcal{F}[0]$ is the complex consisting in a single abelian sheaf $\mathcal{F}$ placed in degree 0, then there no signs in the formula for $\cup$ (as in that case $\alpha_{i_0 \ldots i_r} = 0$ unless $r = n$). For an explanation of why there has to be a sign and how to compute it see [AGV71, Exposee XVII] by Deligne. To check (26.1.2) is a map of complexes we have to show that

$$d(\alpha \cup \beta) = d(\alpha) \cup \beta + (-1)^{\deg(\alpha)} \alpha \cup d(\beta)$$

by the definition of the differential on $\text{Tot}(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}^\bullet)))$ as given in Homology, Definition 22.3. We compute first

$$\begin{aligned} d(\alpha \cup \beta)_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{j=0}^{p+1} (-1)^j (\alpha \cup \beta)_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F} \otimes \mathcal{G}}((\alpha \cup \beta)_{i_0 \ldots i_{p+1}}) \\ &= \sum\nolimits_{j=0}^{p+1} \sum\nolimits_{r=0}^{j-1} (-1)^j \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots \hat{i}_j \ldots i_{p+1}} + \\ &\quad \sum\nolimits_{j=0}^{p+1} \sum\nolimits_{r=j+1}^{p+1} (-1)^j \epsilon(n, m, p, r-1) \alpha_{i_0 \ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} + \\ &\quad \sum\nolimits_{r=0}^{p+1} (-1)^{p+1} \epsilon(n, m, p+1, r) d_{\mathcal{F} \otimes \mathcal{G}}(\alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}}) \end{aligned}$$

and note that the summands in the last term equal

$$(-1)^{p+1} \epsilon(n, m, p+1, r) \left(d_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) \otimes \beta_{i_r \ldots i_{p+1}} + (-1)^{n-r} \alpha_{i_0 \ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r \ldots i_{p+1}}) \right).$$

because $\deg_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) = n - r$. On the other hand

$$\begin{aligned} (d(\alpha) \cup \beta)_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{r=0}^{p+1} \epsilon(n+1, m, p+1, r) d(\alpha)_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} \\ &= \sum\nolimits_{r=0}^{p+1} \sum\nolimits_{j=0}^{r} \epsilon(n+1, m, p+1, r)(-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} + \\ &\quad \sum\nolimits_{r=0}^{p+1} \epsilon(n+1, m, p+1, r)(-1)^r d_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) \otimes \beta_{i_r \ldots i_{p+1}} \end{aligned}$$

and

$$\begin{aligned} (\alpha \cup d(\beta))_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{r=0}^{p+1} \epsilon(n, m+1, p+1, r) \alpha_{i_0 \ldots i_r} \otimes d(\beta)_{i_r \ldots i_{p+1}} \\ &= \sum\nolimits_{r=0}^{p+1} \sum\nolimits_{j=r}^{p+1} \epsilon(n, m+1, p+1, r)(-1)^{j-r} \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots \hat{i}_j \ldots i_{p+1}} + \\ &\quad \sum\nolimits_{r=0}^{p+1} \epsilon(n, m+1, p+1, r)(-1)^{p+1-r} \alpha_{i_0 \ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r \ldots i_{p+1}}) \end{aligned}$$

The desired equality holds if we have

$$\begin{aligned} (-1)^{p+1} \epsilon(n, m, p+1, r) &= \epsilon(n+1, m, p+1, r)(-1)^r \\ (-1)^{p+1} \epsilon(n, m, p+1, r)(-1)^{n-r} &= (-1)^n \epsilon(n, m+1, p+1, r)(-1)^{p+1-r} \\ \epsilon(n+1, m, p+1, r)(-1)^r &= (-1)^{1+n} \epsilon(n, m+1, p+1, r-1) \\ (-1)^j \epsilon(n, m, p, r) &= (-1)^n \epsilon(n, m+1, p+1, r)(-1)^{j-r} \\ (-1)^j \epsilon(n, m, p, r-1) &= \epsilon(n+1, m, p+1, r)(-1)^j \end{aligned}$$

(The third equality is necessary to get the terms with $r = j$ from $d(\alpha) \cup \beta$ and $(-1)^n \alpha \cup d(\beta)$ to cancel each other.) We leave the verifications to the reader. (Alternatively, check the script signs.gp in the scripts subdirectory of the stacks project.)

Associativity of the cupproduct. Suppose that $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ and $\mathcal{H}^\bullet$ are bounded below complexes of abelian groups on X. The obvious map (without the intervention of signs) is an isomorphism of complexes

$$\text{Tot}(\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet) \otimes_{\mathbf{Z}} \mathcal{H}^\bullet) \longrightarrow \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \text{Tot}(\mathcal{G}^\bullet \otimes_{\mathbf{Z}} \mathcal{H}^\bullet)).$$

Another way to say this is that the triple complex $\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet \otimes_{\mathbf{Z}} \mathcal{H}^\bullet$ gives rise to a well defined total complex with differential satisfying

$$d(\alpha\otimes\beta\otimes\gamma) = d(\alpha)\otimes\beta\otimes\gamma + (-1)^{\deg(\alpha)}\alpha\otimes d(\beta)\otimes\gamma + (-1)^{\deg(\alpha)+\deg(\beta)}\alpha\otimes\beta\otimes d(\gamma)$$

for homogeneous elements. Using this map it is easy to verify that

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$$

namely, if α has degree a, β has degree b and γ has degree c, then

$$\begin{aligned} ((\alpha \cup \beta) \cup \gamma)_{i_0 \ldots i_p} &= \sum\nolimits_{r=0}^{p} \epsilon(a+b, c, p, r)(\alpha \cup \beta)_{i_0 \ldots i_r} \otimes \gamma_{i_r \ldots i_p} \\ &= \sum\nolimits_{r=0}^{p} \sum\nolimits_{s=0}^{r} \epsilon(a+b, c, p, r)\epsilon(a, b, r, s)\alpha_{i_0 \ldots i_s} \otimes \beta_{i_s \ldots i_r} \otimes \gamma_{i_r \ldots i_p} \end{aligned}$$

and

$$\begin{aligned} (\alpha \cup (\beta \cup \gamma)_{i_0 \ldots i_p} &= \sum\nolimits_{s=0}^{p} \epsilon(a, b+c, p, s)\alpha_{i_0 \ldots i_s} \otimes (\beta \cup \gamma)_{i_s \ldots i_p} \\ &= \sum\nolimits_{s=0}^{p} \sum\nolimits_{r=s}^{p} \epsilon(a, b+c, p, s)\epsilon(b, c, p-s, r-s)\alpha_{i_0 \ldots i_s} \otimes \beta_{i_s \ldots i_r} \otimes \gamma_{i_r \ldots i_p} \end{aligned}$$

and a trivial mod 2 calculation shows the signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the stacks project.)

Finally, we indicate why the cup product preserves a graded commutative structure, at least on a cohomological level. For this we use the operator τ introduced above. Let $\mathcal{F}^\bullet$ be a bounded below complexes of abelian groups, and assume we are given a graded commutative multiplication

$$\wedge^\bullet : \text{Tot}(\mathcal{F}^\bullet \otimes \mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet.$$

This means the following: For s a local section of $\mathcal{F}^a$, and t a local section of $\mathcal{F}^b$ we have $s \wedge t$ a local section of $\mathcal{F}^{a+b}$. Graded commutative means we have $s \wedge t = (-1)^{ab} t \wedge s$. Since $\wedge$ is a map of complexes we have $d(s \wedge t) = d(s) \wedge t + (-1)^a s \wedge t$. The composition

$$\text{Tot}(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{F}^\bullet))) \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

induces a cup product on cohomology

$$H^n(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \times H^m(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \longrightarrow H^{n+m}(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)))$$

and so in the limit also a product on Čech cohomology and therefore (using hypercoverings if needed) a product in cohomology of $\mathcal{F}^\bullet$. We claim this product (on cohomology) is graded commutative as well. To prove this we first consider an element α of degree n in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and an element β of degree m in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and we compute

$$\begin{aligned} \wedge^\bullet(\alpha \cup \beta)_{i_0 \ldots i_p} &= \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \wedge \beta_{i_r \ldots i_p} \\ &= \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r)(-1)^{\deg(\alpha_{i_0 \ldots i_r}) \deg(\beta_{i_r \ldots i_p})} \beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r} \end{aligned}$$

because $\wedge$ is graded commutative. On the other hand we have

$$\begin{aligned} \tau(\wedge^\bullet(\tau(\beta) \cup \tau(\alpha)))_{i_0 \ldots i_p} =& \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, r) \tau(\beta)_{i_p \ldots i_{p-r}} \wedge \tau(\alpha)_{i_{p-r} \ldots i_0} \\ =& \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, r) \chi(r) \chi(p-r) \beta_{i_{p-r} \ldots i_p} \wedge \alpha_{i_0 \ldots i_{p-r}} \\ =& \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, p-r) \chi(r) \chi(p-r) \beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r} \end{aligned}$$

where $\chi(t) = (-1)^{\frac{t(t+1)}{2}}$. Since we proved earlier that τ acts as the identity on cohomology we have to verify that

$$\epsilon(n, m, p, r)(-1)^{(n-r)(m-(p-r))} = (-1)^{nm} \chi(p) \epsilon(m, n, p, p-r) \chi(r) \chi(p-r)$$

A trivial mod 2 calculation shows these signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the stacks project.)

Finally, we study the compatibility of cup product with boundary maps. Suppose that

$$0 \to \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to 0 \quad \text{and} \quad 0 \leftarrow \mathcal{G}_1^\bullet \leftarrow \mathcal{G}_2^\bullet \leftarrow \mathcal{G}_3^\bullet \leftarrow 0$$

are short exact sequences of bounded below complexes of abelian sheaves on X. Let $\mathcal{H}^\bullet$ be another bounded below complex of abelian sheaves, and suppose we have maps of complexes

$$\gamma_i : \text{Tot}(\mathcal{F}_i^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_i^\bullet) \longrightarrow \mathcal{H}^\bullet$$

which are compatible with the maps between the complexes, namely such that the diagrams

$$\begin{tikzcd} \text{Tot}(\mathcal{F}_1^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_1^\bullet) \arrow[d, "\gamma_1"'] & \text{Tot}(\mathcal{F}_1^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) \arrow[l] \arrow[d] \\ \mathcal{H}^\bullet & \text{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) \arrow[l, "\gamma_2"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \text{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) \arrow[d, "\gamma_2"'] & \text{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_3^\bullet) \arrow[l] \arrow[d] \\ \mathcal{H}^\bullet & \text{Tot}(\mathcal{F}_3^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_3^\bullet) \arrow[l, "\gamma_3"'] \end{tikzcd}$$

are commutative.

Lemma 26.2. *In the situation above, assume Čech cohomology agrees with cohomology for the sheaves $\mathcal{F}_i^p$ and $\mathcal{G}_j^q$. Let $a_3 \in H^n(X, \mathcal{F}_3^\bullet)$ and $b_1 \in H^m(X, \mathcal{G}_1^\bullet)$. Then we have*

$$\gamma_1(\partial a_3 \cup b_1) = (-1)^{n+1} \gamma_3(a_3 \cup \partial b_1)$$

in $H^{n+m}(X, \mathcal{H}^\bullet)$ where ∂ indicates the boundary map on cohomology associated to the short exact sequences of complexes above.

Proof. We will use the following conventions and notation. We think of $\mathcal{F}_1^p$ as a subsheaf of $\mathcal{F}_2^p$ and we think of $\mathcal{G}_3^q$ as a subsheaf of $\mathcal{G}_2^q$. Hence if s is a local section of $\mathcal{F}_1^p$ we use s to denote the corresponding section of $\mathcal{F}_2^p$ as well. Similarly for local sections of $\mathcal{G}_3^q$. Furthermore, if s is a local section of $\mathcal{F}_2^p$ then we denote $\bar{s}$ its image in $\mathcal{F}_3^p$. Similarly for the map $\mathcal{G}_2^q \to \mathcal{G}_1^q$. In particular if s is a local section of $\mathcal{F}_2^p$ and $\bar{s} = 0$ then s is a local section of $\mathcal{F}_1^p$. The commutativity of the diagrams above implies, for local sections s of $\mathcal{F}_2^p$ and t of $\mathcal{G}_3^q$ that $\gamma_2(s \otimes t) = \gamma_3(\bar{s} \otimes t)$ as sections of $\mathcal{H}^{p+q}$.

Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Suppose that α_3, resp. β_1 is a degree n, resp. m cocycle of $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3^\bullet))$, resp. $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_1^\bullet))$ representing a_3, resp. b_1. After refining $\mathcal{U}$ if necessary, we can find cochains α_2, resp. β_2 of degree n, resp. m in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2^\bullet))$, resp. $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_2^\bullet))$ mapping to α_3, resp. β_1. Then we see that

$$\overline{d(\alpha_2)} = d(\bar{\alpha}_2) = 0 \quad \text{and} \quad \overline{d(\beta_2)} = d(\bar{\beta}_2) = 0.$$

This means that $\alpha_1 = d(\alpha_2)$ is a degree $n+1$ cocycle in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1^\bullet))$ representing ∂a_3. Similarly, $\beta_3 = d(\beta_2)$ is a degree $m+1$ cocycle in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_3^\bullet))$ representing ∂b_1. Thus we may compute

$$\begin{aligned} d(\gamma_2(\alpha_2 \cup \beta_2)) &= \gamma_2(d(\alpha_2 \cup \beta_2)) \\ &= \gamma_2(d(\alpha_2) \cup \beta_2 + (-1)^n \alpha_2 \cup d(\beta_2)) \\ &= \gamma_2(\alpha_1 \cup \beta_2) + (-1)^n \gamma_2(\alpha_2 \cup \beta_3) \\ &= \gamma_1(\alpha_1 \cup \beta_1) + (-1)^n \gamma_3(\alpha_3 \cup \beta_3) \end{aligned}$$

So this even tells us that the sign is $(-1)^{n+1}$ as indicated in the lemma[1]. □

27. Flat resolutions

A reference for the material in this section is [Spa88]. Let $(X, \mathcal{O}_X)$ be a ringed space. By Modules, Lemma 16.6 any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module. By Derived Categories, Lemma 16.5 any bounded above complex of $\mathcal{O}_X$-modules has a left resolution by a bounded above complex of flat $\mathcal{O}_X$-modules. However, for unbounded complexes, it turns out that flat resolutions aren't good enough.

Lemma 27.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The functor*

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is an exact functor of triangulated categories.

Proof. Omitted. Hint: See More on Algebra, Lemmas 45.1 and 45.2. □

Definition 27.2. Let $(X, \mathcal{O}_X)$ be a ringed space. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the complex

$$\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

is acyclic.

---

[1]The sign depends on the convention for the signs in the long exact sequence in cohomology associated to a triangle in $D(X)$. The conventions in the stacks project are (a) distinguished triangles correspond to termwise split exact sequences and (b) the boundary maps in the long exact sequence are given by the maps in the snake lemma without the intervention of signs. See Derived Categories, Section 10.

Lemma 27.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 27.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. $\square$

Lemma 27.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then $\mathcal{K}^\bullet$ is K-flat if and only if for all $x \in X$ the complex $\mathcal{K}^\bullet_x$ of $\mathcal{O}_{X,x}$ is K-flat (More on Algebra, Definition 45.3).*

Proof. If $\mathcal{K}^\bullet_x$ is K-flat for all $x \in X$ then we see that $\mathcal{K}^\bullet$ is K-flat because $\otimes$ and direct sums commute with taking stalks and because we can check exactness at stalks, see Modules, Lemma 3.1. Conversely, assume $\mathcal{K}^\bullet$ is K-flat. Pick $x \in X$ $M^\bullet$ be an acyclic complex of $\mathcal{O}_{X,x}$-modules. Then $i_{x,*}M^\bullet$ is an acyclic complex of $\mathcal{O}_X$-modules. Thus $\text{Tot}(i_{x,*}M^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$ is acyclic. Taking stalks at x shows that $\text{Tot}(M^\bullet \otimes_{\mathcal{O}_{X,x}} \mathcal{K}^\bullet_x)$ is acyclic. $\square$

Lemma 27.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}_X$-modules, then $Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. Follows from the isomorphism

$$\text{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)) = \text{Tot}(\text{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$$

and the definition. $\square$

Lemma 27.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(Mod(\mathcal{O}_X))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 27.1 and the fact that in a distinguished triangle in $K(Mod(\mathcal{O}_X))$ if two out of three are acyclic, so is the third. $\square$

Lemma 27.7. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback of a K-flat complex of $\mathcal{O}_Y$-modules is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. We can check this on stalks, see Lemma 27.4. Hence this follows from Sheaves, Lemma 26.4 and More on Algebra, Lemma 45.5. $\square$

Lemma 27.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. A bounded above complex of flat $\mathcal{O}_X$-modules is K-flat.*

Proof. We can check this on stalks, see Lemma 27.4. Thus this lemma follows from Modules, Lemma 16.2 and More on Algebra, Lemma 45.8. $\square$

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

Lemma 27.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\text{colim}_i \mathcal{K}_i^\bullet$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$\text{colim}_i \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_i^\bullet) = \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \text{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. $\square$

Lemma 27.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules there exists a commutative diagram of complexes of $\mathcal{O}_X$-modules*

$$\begin{tikzcd} \mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}\mathcal{G}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{G}^\bullet \arrow[r] & \ldots \end{tikzcd}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules, Lemma 16.6 and Derived Categories, Lemma 28.1. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. $\square$

Lemma 27.11. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ there exists a K-flat complex $\mathcal{K}^\bullet$ and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$.*

Proof. Choose a diagram as in Lemma 27.10. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules, Lemma 16.5. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 27.8. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism by construction. Since $\operatorname{colim} \mathcal{K}_n^\bullet$ is K-flat by Lemma 27.9 we win. $\square$

Lemma 27.12. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}_X$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the induced map*

$$Tot(id_{\mathcal{F}^\bullet} \otimes \alpha) : Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \longrightarrow Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 27.11. Consider the commutative diagram

$$\begin{tikzcd} \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \arrow[r] \arrow[d] & \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \arrow[d] \\ \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \arrow[r] & \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \end{tikzcd}$$

The result follows as by Lemma 27.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. $\square$

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 27.11. By Lemma 27.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}_X) \longrightarrow K(\mathcal{O}_X), \quad \mathcal{G}^\bullet \longmapsto \operatorname{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

By Lemma 27.3 this functor induces a functor $D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ simply because $D(\mathcal{O}_X)$ is the localization of $K(\mathcal{O}_X)$ at quasi-isomorphisms. By Lemma 27.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

Definition 27.13. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. The *derived tensor product*

$$- \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}^\bullet : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O}_X)$. Hence when we write $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

Definition 27.14. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. The *Tor*'s of $\mathcal{F}$ and $\mathcal{G}$ are define by the formula

$$\text{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we have a long exact cohomology sequence

$$\begin{tikzcd}
\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r] & \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r] & \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r] & 0 \\
\text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) \arrow[r] & \text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G}) \arrow[r] & \text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{G}) \arrow[llu]
\end{tikzcd}$$

for every $\mathcal{O}_X$-module $\mathcal{G}$. This will be called the long exact sequence of Tor associated to the situation.

Lemma 27.15. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is a flat $\mathcal{O}_X$-module, and*
2. *$Tor_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{O}_X$-module $\mathcal{G}$.*

Proof. If $\mathcal{F}$ is flat, then $\mathcal{F} \otimes_{\mathcal{O}_X} -$ is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if $\mathcal{G} \to \mathcal{H}$ is injective with cokernel $\mathcal{Q}$, the long exact sequence of Tor shows that the kernel of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}$ is a quotient of $\text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})$ which is zero by assumption. Hence $\mathcal{F}$ is flat. $\square$

28. Derived pullback

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$$

Namely, for every complex of $\mathcal{O}_Y$-modules $\mathcal{G}^\bullet$ we can choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ and set $Lf^*\mathcal{G}^\bullet = f^*\mathcal{K}^\bullet$. You can use Lemmas 27.7, 27.11, and 27.12 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

Lemma 28.1. *The construction above is independent of choices and defines an exact functor of triangulated categories $Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$.*

Proof. To see this we use the general theory developed in Derived Categories, Section 15. Set $\mathcal{D} = K(\mathcal{O}_Y)$ and $\mathcal{D}' = D(\mathcal{O}_X)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}_Y)$. This gives a situation as in Derived Categories, Situation 15.1 so that Derived Categories, Definition 15.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 15.15 with $\mathcal{P} \subset \text{Ob}(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 27.11 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between K-flat complexes of $\mathcal{O}_Y$-modules the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. By Lemma 27.7 applied to the morphism $(X, f^{-1}\mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ the complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}_Y$-modules. Hence Lemma 27.12 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}_Y) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O}_X)$$

see Derived Categories, Equation (15.9.1). Finally, Derived Categories, Lemma 15.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is K-flat, i.e., $Lf^* = LF$ is indeed computed in the way described above. □

Lemma 28.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \text{Ob}(D(X))$.

Proof. We may assume that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat complexes. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet$. By Lemma 27.5 $\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$ is K-flat also. Hence the isomorphism of the lemma comes from the isomorphism

$$\text{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{G}^\bullet) \longrightarrow f^*\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}_X} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}_Y} \mathcal{G}^q)$ of Modules, Lemma 15.4. □

Lemma 28.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism*

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet$ in $D(X)$ and $\mathcal{G}^\bullet$ in $D(Y)$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$ because $f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$. The lemma follows from this and the definitions. □

29. Cohomology of unbounded complexes

Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \subset X \text{ open}} j_{U!}\mathcal{O}_U,$$

see Modules, Section 3 and Lemmas 16.5 and 16.6. By Injectives, Theorem 12.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}_X$-modules. Hence we can define

$$R\Gamma(X, \mathcal{F}^\bullet) = \Gamma(X, \mathcal{I}^\bullet)$$

and similarly for any left exact functor, see Derived Categories, Lemma 29.6. For any morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ we obtain

$$Rf_* : D(X) \longrightarrow D(Y)$$

on the unbounded derived categories.

Lemma 29.1. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The functor Rf_* defined above and the functor Lf^* defined in Lemma 28.1 are adjoint:*

$$\mathrm{Hom}_{D(X)}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(Y)}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in \mathrm{Ob}(D(X))$ and $\mathcal{G}^\bullet \in \mathrm{Ob}(D(Y))$.

Proof. This follows formally from the fact that Rf_* and Lf^* exist, see Derived Categories, Lemma 28.4. □

Remark 29.2. The construction of unbounded derived functor Lf^* and Rf_* allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

is a commutative diagram of ringed spaces. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then there exists a canonical base change map

$$Lg^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*L(g')^*\mathcal{F}^\bullet$$

in $D(\mathcal{O}_{S'})$. Namely, this map is adjoint to a map $L(f')^*Lg^*Rf_*\mathcal{F}^\bullet \to L(g')^*\mathcal{F}^\bullet$ Since $L(f')^*Lg^* = L(g')^*Lf^*$ we see this is the same as a map $L(g')^*Lf^*Rf_*\mathcal{F}^\bullet \to L(g')^*\mathcal{F}^\bullet$ which we can take to be $L(g')^*$ of the adjunction map $Lf^*Rf_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet$.

30. Unbounded Mayer-Vietoris

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be an open subset. Denote $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ the corresponding open immersion. The pullback functor j^* is exact as it is just the restriction functor. Thus derived pullback Lj^* is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}_X) \to D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero $j_! : Mod(\mathcal{O}_U) \to Mod(\mathcal{O}_X)$ (see Sheaves, Section 31) is an exact functor (Modules, Lemma 3.4). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \to D(\mathcal{O}_X), \quad F \mapsto j_!F$$

by simply applying $j_!$ to any complex representing the object F.

Lemma 30.1. *Let X be a ringed space. Let $U \subset X$ be an open subspace. The restriction of a K-injective complex of $\mathcal{O}_X$-modules to U is a K-injective complex of $\mathcal{O}_U$-modules.*

Proof. Follows immediately from Derived Categories, Lemma 29.10 and the fact that the restriction functor has the exact adjoint $j_!$. See discussion above. $\square$

Lemma 30.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Given an open subspace $V \subset Y$, set $U = f^{-1}(V)$ and denote $g : U \to V$ the induced morphism. Then $(Rf_*E)|_V = Rg_*(E|_U)$ for E in $D(\mathcal{O}_X)$.*

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_U) = g_*(\mathcal{I}^\bullet|_U)$ by Lemma 30.1. Hence the result follows from Lemma 7.4 (with $p = 0$). $\square$

Lemma 30.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be an open subset. Denote $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ the corresponding open immersion. The restriction functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is a right adjoint to extension by zero $j_! : D(\mathcal{O}_U) \to D(\mathcal{O}_X)$.*

Proof. We have to show that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(j_!E, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E, F|_U)$$

Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_U$-modules representing E and choose a K-injective complex $\mathcal{I}^\bullet$ representing F. By Lemma 30.1 the complex $\mathcal{I}^\bullet|_U$ is K-injective as well. Hence we see that the formula above becomes

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(j_!\mathcal{E}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{I}^\bullet|_U)$$

which holds as $|_U$ and $j_!$ are adjoint functors (Sheaves, Lemma 31.8) and Derived Categories, Lemma 29.2. $\square$

Lemma 30.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces. For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$j_{U\cap V!}E|_{U\cap V} \to j_{U!}E|_U \oplus j_{V!}E|_V \to E \to j_{U\cap V!}E|_{U\cap V}[1]$$

in $D(\mathcal{O}_X)$.

Proof. We have seen above that the restriction functors and the extension by zero functors are computed by just applying the functors to any complex. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules representing E. The distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes of $\mathcal{O}_X$-modules

$$0 \to j_{U\cap V!}\mathcal{E}^\bullet|_{U\cap V} \to j_{U!}\mathcal{E}^\bullet|_U \oplus j_{V!}\mathcal{E}^\bullet|_V \to \mathcal{E}^\bullet \to 0$$

To see this sequence is exact one checks on stalks using Sheaves, Lemma 31.8 (computation omitted). $\square$

Lemma 30.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces. For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U\cap V,*}E|_{U\cap V} \to E[1]$$

in $D(\mathcal{O}_X)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E whose terms $\mathcal{I}^n$ are injective objects of $Mod(\mathcal{O}_X)$, see Injectives, Theorem 12.6. We have seen that $\mathcal{I}^\bullet|U$ is a K-injective complex as well (Lemma 30.1). Hence $Rj_{U,*}E|_U$ is represented by $j_{U,*}\mathcal{I}^\bullet|_U$. Similarly for V and $U \cap V$. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U\cap V,*}\mathcal{I}^\bullet|_{U\cap V} \to 0.$$

This sequence is exact because for any $W \subset X$ open and any n the sequence

$$0 \to \mathcal{I}^n(W) \to \mathcal{I}^n(W \cap U) \oplus \mathcal{I}^n(W \cap V) \to \mathcal{I}^n(W \cap U \cap V) \to 0$$

is exact (see proof of Lemma 9.2). $\square$

Lemma 30.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces of X. For objects E, F of $D(\mathcal{O}_X)$ we have a Mayer-Vietoris sequence*

$$\begin{tikzcd} & \ldots \arrow[r] & Ext^{-1}(E_{U\cap V}, F_{U\cap V}) \arrow[dll] \\ \mathrm{Hom}(E, F) \arrow[r] & \mathrm{Hom}(E_U, F_U) \oplus \mathrm{Hom}(E_V, F_V) \arrow[r] & \mathrm{Hom}(E_{U\cap V}, F_{U\cap V}) \end{tikzcd}$$

where the subscripts denote restrictions to the relevant opens and the Hom*'s are taken in the relevant derived categories.*

Proof. Use the distinguished triangle of Lemma 30.4 to obtain a long exact sequence of Hom's (from Derived Categories, Lemma 4.2) and use that $\mathrm{Hom}(j_{U!}E|_U, F) = \mathrm{Hom}(E|_U, F|_U)$ by Lemma 30.3. $\square$

Lemma 30.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For an object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$R\Gamma(X, E) \to R\Gamma(U, E) \oplus R\Gamma(V, E) \to R\Gamma(U \cap V, E) \to R\Gamma(X, E)[1]$$

and in particular a long exact cohomology sequence

$$\ldots \to H^n(X, E) \to H^n(U, E) \oplus H^0(V, E) \to H^n(U \cap V, E) \to H^{n+1}(X, E) \to \ldots$$

The construction of the distinguished triangle and the long exact sequence is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $Mod(\mathcal{O}_X)$ for all n, see Injectives, Theorem 12.6. Then $R\Gamma(X, E)$ is computed by $\Gamma(X, \mathcal{I}^\bullet)$. Similarly for U, V, and $U \cap V$ by Lemma 30.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

We have seen this is a short exact sequence in the proof of Lemma 9.2. The final statement follows from the functoriality of the construction in Injectives, Theorem 12.6. $\square$

Lemma 30.8. *Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U\cap V} : U \cap V \to Y$. For every object E of $D(\mathcal{O}_X)$ there exists a distinguished triangle*

$$Rf_*E \to Ra_*(E|_U) \oplus Rb_*(E|_V) \to Rc_*(E|_{U\cap V}) \to Rf_*E[1]$$

This triangle is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $\mathit{Mod}(\mathcal{O}_X)$ for all n, see Injectives, Theorem 12.6. Then Rf_*E is computed by $f_*\mathcal{I}^\bullet$. Similarly for U, V, and $U \cap V$ by Lemma 30.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U\cap V} \to 0.$$

This is a short exact sequence of complexes by Lemma 9.3 and the fact that $R^1f_*\mathcal{I} = 0$ for an injective object $\mathcal{I}$ of $\mathit{Mod}(\mathcal{O}_X)$. The final statement follows from the functoriality of the construction in Injectives, Theorem 12.6. ☐

Lemma 30.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $j : U \to X$ be an open subspace. Let $T \subset X$ be a closed subset contained in U.*

1. *If E is an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T, then $E \to Rj_*(E|_U)$ is an isomorphism.*
2. *If F is an object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on T, then $j_!F \to Rj_*F$ is an isomorphism.*

Proof. Let $V = X \setminus T$ and $W = U \cap V$. Note that $X = U \cup V$ is an open covering of X. Denote $j_W : W \to V$ the open immersion. Let E be an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T. By Lemma 30.2 we have $(Rj_*E|_U)|_V = Rj_{W,*}(E|_W) = 0$ because $E|_W = 0$ by our assumption. On the other hand, $Rj_*(E|_U)|_U = E|_U$. Thus (1) is clear. Let F be an object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on T. By Lemma 30.2 we have $(Rj_*F)|_V = Rj_{W,*}(F|_W) = 0$ because $F|_W = 0$ by our assumption. We also have $(j_!F)|_V = j_{W!}(F|_W) = 0$ (the first equality is immediate from the definition of extension by zero). Since both $(Rj_*F)|_U = F$ and $(j_!F)|_U = F$ we see that (2) holds. ☐

We can glue complexes!

Lemma 30.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces of X. Suppose given*

1. *an object E of $D(\mathcal{O}_X)$,*
2. *a morphism $a : A \to E|_U$ of $D(\mathcal{O}_U)$,*
3. *a morphism $b : B \to E|_V$ of $D(\mathcal{O}_V)$,*
4. *an isomorphism $c : A|_{U\cap V} \to B|_{U\cap V}$*

such that

$$a|_{U\cap V} = b|_{U\cap V} \circ c.$$

Then there exists a morphism $F \to E$ in $D(\mathcal{O}_X)$ whose restriction to U is isomorphic to a and whose restriction to V is isomorphic to b.

Proof. Denote j_U, j_V, $j_{U\cap V}$ the corresponding open immersions. Choose a distinguished triangle

$$F \to Rj_{U,*}A \oplus Rj_{V,*}B \to Rj_{U\cap V,*}(B|_{U\cap V}) \to F[1]$$

where the map $Rj_{V,*}B \to Rj_{U\cap V,*}(B|_{U\cap V})$ is the obvious one and where $Rj_{U,*}A \to Rj_{U\cap V,*}(B|_{U\cap V})$ is the composition of $Rj_{U,*}A \to Rj_{U\cap V,*}(A|_{U\cap V})$ with $Rj_{U\cap V,*}c$. Restricting to U we obtain

$$F|_U \to A \oplus (Rj_{V,*}B)|_U \to (Rj_{U\cap V,*}(B|_{U\cap V}))|_U \to F|_U[1]$$

Denote $j : U \cap V \to U$. Compatibility of restriction to opens and cohomology shows that both $(Rj_{V,*}B)|_U$ and $(Rj_{U\cap V,*}(B|_{U\cap V}))|_U$ are canonically isomorphic to $Rj_*(B|_{U\cap V})$. Hence the second arrow of the last displayed diagram has a section, and we conclude that the morphism $F|_U \to A$ is an isomorphism. Similarly, the morphism $F|_V \to B$ is an isomorphism. The existence of the morphism $F \to E$ follows from the Mayer-Vietoris sequence for Hom, see Lemma 30.6. $\square$

31. Producing K-injective resolutions

First a technical lemma about the cohomology sheaves of the inverse limit of a system of complexes of sheaves. In some sense this lemma is the wrong thing to try to prove as one should take derived limits and not actual inverse limits. This will be discussed in Cohomology on Sites, Section 22.

Lemma 31.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{F}_n^\bullet)$ be an inverse system of complexes of $\mathcal{O}_X$-modules. Let $m \in \mathbf{Z}$. Assume there exist a set $\mathcal{B}$ of open subsets of X and an integer n_0 such that*

1. *every open in X has a covering whose members are elements of $\mathcal{B}$,*
2. *for every $U \in \mathcal{B}$*
   - (a) *the systems of abelian groups $\mathcal{F}_n^{m-2}(U)$ and $\mathcal{F}_n^{m-1}(U)$ have vanishing $R^1\lim$ (for example these have the Mittag-Leffler condition),*
   - (b) *the system of abelian groups $H^{m-1}(\mathcal{F}_n^\bullet(U))$ has vanishing $R^1\lim$ (for example it has the Mittag-Leffler condition), and*
   - (c) *we have $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$ for all $n \geq n_0$.*

Then the maps $H^m(\mathcal{F}^\bullet) \to \lim H^m(\mathcal{F}_n^\bullet) \to H^m(\mathcal{F}_{n_0}^\bullet)$ are isomorphisms of sheaves where $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$ is the termwise inverse limit.

Proof. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{F}^\bullet(U))$ is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \to \lim_n \mathcal{F}_n^{m-1}(U) \to \lim_n \mathcal{F}_n^{m}(U) \to \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply More on Algebra, Lemma 61.2 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all $n \geq n_0$. By assumption (1) we conclude that the sheafification of $U \mapsto H^m(\mathcal{F}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$ for all $n \geq n_0$. Thus the inverse system of sheaves $H^m(\mathcal{F}_n^\bullet)$ is constant for $n \geq n_0$ with value $H^m(\mathcal{F}^\bullet)$ which proves the lemma. $\square$

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The category $Mod(\mathcal{O}_X)$ has enough injectives, hence we can use Derived Categories, Lemma 28.3 produce a diagram

$$\begin{tikzcd} \ldots \arrow[r] & \tau_{\geq -2}\mathcal{F}^\bullet \arrow[r] \arrow[d] & \tau_{\geq -1}\mathcal{F}^\bullet \arrow[d] \\ \ldots \arrow[r] & \mathcal{I}_2^\bullet \arrow[r] & \mathcal{I}_1^\bullet \end{tikzcd}$$

in the category of complexes of $\mathcal{O}_X$-modules such that

(1) the vertical arrows are quasi-isomorphisms,
(2) $\mathcal{I}_n^\bullet$ is a bounded above complex of injectives,
(3) the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}_X$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 29.4 and 29.7 this is a K-injective complex. In general the canonical map

$$\mathcal{F}^\bullet \to \mathcal{I}^\bullet \tag{31.1.1}$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

Lemma 31.2. *In the situation described above. Denote $\mathcal{H}^m = H^m(\mathcal{F}^\bullet)$ the mth cohomology sheaf. Let $\mathcal{B}$ be a set of open subsets of X. Let $d \in \mathbf{N}$. Assume*

(1) *every open in X has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$[2].*

Then (31.1.1) is a quasi-isomorphism.

Proof. Let $m \in \mathbf{Z}$. We have to show that the map $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism $\mathcal{H}^m \to H^m(\mathcal{I}^\bullet)$. Since $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}\mathcal{F}^\bullet$ it suffices to show that $H^m(\mathcal{I}^\bullet) \to H^m(\mathcal{I}_n^\bullet)$ is an isomorphism for n large enough. To do this we will verify the hypotheses (1), (2)(a), (2)(b), (2)(c) of Lemma 31.1.

Hypothesis (1) is assumption (1) above. Hypothesis (2)(a) follows from the fact that the maps $\mathcal{I}_{n+1}^k \to \mathcal{I}_n^k$ are split surjections. We will prove hypothesis (2)(b) and (2)(c) simultaneously by proving that for $U \in \mathcal{B}$ the system $H^m(\mathcal{I}_n^\bullet(U))$ becomes constant for $n \geq -m + d$. Namely, recalling that $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}\mathcal{F}^\bullet$ we obtain for all n a distinguished triangle

$$\mathcal{H}^{-n}[n] \to \mathcal{I}_n^\bullet \to \mathcal{I}_{n-1}^\bullet \to \mathcal{H}^{-n}[n+1]$$

(Derived Categories, Remark 12.4) in $D(\mathcal{O}_X)$. By assumption (2) we see that if $m > d - n$ then

$$H^m(U, \mathcal{H}^{-n}[n]) = 0 \quad \text{and} \quad H^m(U, \mathcal{H}^{-n}[n+1]) = 0.$$

Observe that $H^m(\mathcal{I}_n^\bullet(U)) = H^m(U, \mathcal{I}_n^\bullet)$ as $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives. Unwinding the long exact sequence of cohomology associated to the distinguished triangle above this implies that

$$H^m(\mathcal{I}_n^\bullet(U)) \to H^m(\mathcal{I}_{n-1}^\bullet(U))$$

is an isomorphism for $m > d - n$, i.e., $n > d - m$ and we win. □

[2] In fact, analyzing the proof we see that it suffices if there exists a function $d : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ such that $H^p(U, \mathcal{H}^q) = 0$ for $p > d(q)$ where $q + d(q) \to -\infty$ as $q \to -\infty$

Lemma 31.3. *With assumptions and notation as in Lemma 31.2. Let K denote the object of $D(\mathcal{O}_X)$ represented by the complex $\mathcal{F}^\bullet$. Then there exists a distinguished triangle*

$$K \to \prod\nolimits_{n \geq 0} \tau_{\geq -n} K \to \prod\nolimits_{n \geq 0} \tau_{\geq -n} K \to K[1]$$

in $D(\mathcal{O}_X)$. In other words, K is the derived limit of its canonical truncations.

Proof. The proof of Injectives, Lemma 13.4 shows that $\prod \tau_{\geq -n} K$ is represented by the complex $\prod \mathcal{I}_n^\bullet$. Because the transition maps $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections, we have a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to \prod \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to 0$$

Since $\mathcal{I}^\bullet$ represents K by Lemma 31.2 the distinguished triangle of the lemma is the distinguished triangle associated to the short exact sequence above (Derived Categories, Lemma 12.1). $\square$

32. Čech cohomology of unbounded complexes

The construction of Section 26 isn't the "correct" one for unbounded complexes. The problem is that in the Stacks project we use direct sums in the totalization of a double complex and we would have to replace this by a product. Instead of doing so in this section we assume the covering is finite and we use the alternating Čech complex.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of presheaves of $\mathcal{O}_X$-modules. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a **finite** open covering of X. Since the alternating Čech complex $\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F})$ (Section 24) is functorial in the presheaf $\mathcal{F}$ we obtain a double complex $\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$. In this section we work with the associated total complex. The construction of $\text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

$$\Gamma(X, \mathcal{F}^\bullet) \longrightarrow \text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \tag{32.0.1}$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0 \ldots i_p} = 0$ for $p > 0$.

Lemma 32.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite open covering. For a complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical map*

$$Tot(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

functorial in $\mathcal{F}^\bullet$ and compatible with (32.0.1).

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex whose terms are injective $\mathcal{O}_X$-modules. The map (32.0.1) for $\mathcal{I}^\bullet$ is a map $\Gamma(X, \mathcal{I}^\bullet) \to \text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from Homology, Lemma 22.7 applied to the double complex $\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemmas 12.1 and 24.6. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of $\mathcal{F}^\bullet$ into a K-injective complex whose terms are injectives (Injectives, Theorem 12.6). Since $R\Gamma(X, \mathcal{F}^\bullet)$ is represented by the complex $\Gamma(X, \mathcal{I}^\bullet)$ we obtain the map of the lemma using

$$\text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

We omit the verification of functoriality and compatibilities. $\square$

Lemma 32.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite open covering. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Let $\mathcal{B}$ be a set of open subsets of X. Assume*

1. *every open in X has a covering whose members are elements of $\mathcal{B}$,*
2. *we have $U_{i_0 \ldots i_p} \in \mathcal{B}$ for all $i_0, \ldots, i_p \in I$,*
3. *for every $U \in \mathcal{B}$ and $p > 0$ we have*
   (a) $H^p(U, \mathcal{F}^q) = 0$,
   (b) $H^p(U, Coker(\mathcal{F}^{q-1} \to \mathcal{F}^q)) = 0$, *and*
   (c) $H^p(U, H^q(\mathcal{F})) = 0$.

Then the map

$$Tot(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

of Lemma 32.1 is an isomorphism in $D(Ab)$.

Proof. If $\mathcal{F}^\bullet$ is bounded below, this follows from assumption (3)(a) and the spectral sequence of Lemma 26.1 and the fact that

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

is a quasi-isomorphism by Lemma 24.6 (some details omitted). In general, by assumption (3)(c) we may choose a resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet = \lim \mathcal{I}_n^\bullet$ as in Lemma 31.2. Then the map of the lemma becomes

$$\lim_n \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \tau_{\geq -n}\mathcal{F}^\bullet)) \longrightarrow \lim_n \Gamma(X, \mathcal{I}_n^\bullet)$$

Note that (3)(b) shows that $\tau_{\geq -n}\mathcal{F}^\bullet$ is a bounded below complex satisfying the hypothesis of the lemma. Thus the case of bounded below complexes shows each of the maps

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \tau_{\geq -n}\mathcal{F}^\bullet)) \longrightarrow \Gamma(X, \mathcal{I}_n^\bullet)$$

is a quasi-isomorphism. The cohomologies of the complexes on the left hand side in given degree are eventually constant (as the alternating Čech complex is finite). Hence the same is true on the right hand side. Thus the cohomology of the limit on the right hand side is this constant value by Homology, Lemma 27.7 and we win. □

33. Hom complexes

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ be two complexes of $\mathcal{O}_X$-modules. We construct a complex of $\mathcal{O}_X$-modules $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. Namely, for each n we set

$$\mathcal{H}om^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

It is a good idea to think of $\mathcal{H}om^n$ as the sheaf of $\mathcal{O}_X$-modules of all $\mathcal{O}_X$-linear maps from $\mathcal{L}^\bullet$ to $\mathcal{M}^\bullet$ (viewed as graded $\mathcal{O}_X$-modules) which are homogenous of degree n. In this terminology, we define the differential by the rule

$$\mathrm{d}(f) = \mathrm{d}_\mathcal{M} \circ f - (-1)^n f \circ \mathrm{d}_\mathcal{L}$$

for $f \in \mathcal{H}om^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. We omit the verification that $\mathrm{d}^2 = 0$. This construction is a special case of Differential Graded Algebra, Example 19.6. It follows immediately from the construction that we have

$$H^n(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n]) \quad (33.0.1)$$

for all $n \in \mathbf{Z}$ and every open $U \subset X$.

Lemma 33.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is an isomorphism*

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)) = \mathcal{H}om^\bullet(\mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}_X$-modules functorial in $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 54.1. $\square$

Lemma 33.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$\mathit{Tot}(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}_X$-modules.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 54.2. $\square$

Lemma 33.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$\mathit{Tot}(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}_X$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 54.3. $\square$

Lemma 33.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism*

$$\mathcal{K}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}_X$-modules functorial in both complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 54.5. $\square$

Lemma 33.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules. Let $\mathcal{L}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then*

$$H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

for all $U \subset X$ open.

Proof. We have

$$\begin{aligned} H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) &= \mathrm{Hom}_{K(\mathcal{O}_U)}(L|_U, M|_U) \\ &= \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U) \end{aligned}$$

The first equality is (33.0.1). The second equality is true because $\mathcal{I}^\bullet|_U$ is K-injective by Lemma 30.1. $\square$

Lemma 33.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{I}')^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism of K-injective complexes of $\mathcal{O}_X$-modules. Let $(\mathcal{L}')^\bullet \to \mathcal{L}^\bullet$ be a quasi-isomorphism of complexes of $\mathcal{O}_X$-modules. Then*

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet) \longrightarrow \mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet)$$

is a quasi-isomorphism.

Proof. Let M be the object of $D(\mathcal{O}_X)$ represented by $\mathcal{I}^\bullet$ and $(\mathcal{I}')^\bullet$. Let L be the object of $D(\mathcal{O}_X)$ represented by $\mathcal{L}^\bullet$ and $(\mathcal{L}')^\bullet$. By Lemma 33.5 we see that the sheaves

$$H^0(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet)) \quad \text{and} \quad H^0(\mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet))$$

are both equal to the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

Thus the map is a quasi-isomorphism. $\square$

Lemma 33.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules. Let $\mathcal{L}^\bullet$ be a K-flat complex of $\mathcal{O}_X$-modules. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}_X$-modules.*

Proof. Namely, if $\mathcal{K}^\bullet$ is an acyclic complex of $\mathcal{O}_X$-modules, then

$$\begin{aligned}
\mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) &= H^0(\Gamma(X, \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)))) \\
&= H^0(\Gamma(X, \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet))) \\
&= \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet) \\
&= 0
\end{aligned}$$

The first equality by (33.0.1). The second equality by Lemma 33.1. The third equality by (33.0.1). The final equality because $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ is acyclic because $\mathcal{L}^\bullet$ is K-flat (Definition 27.2) and because $\mathcal{I}^\bullet$ is K-injective. $\square$

34. Internal hom in the derived category

Let $(X, \mathcal{O}_X)$ be a ringed space. Let L, M be objects of $D(\mathcal{O}_X)$. We would like to construct an object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O}_X)$ such that for every third object K of $D(\mathcal{O}_X)$ there exists a canonical bijection

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, M) \tag{34.0.1}$$

Observe that this formula defines $R\mathcal{H}om(L, M)$ up to unique isomorphism by the Yoneda lemma (Categories, Lemma 3.5).

To construct such an object, choose a K-injective complex $\mathcal{I}^\bullet$ representing M and any complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing L. Then we set

$$R\mathcal{H}om(L, M) = \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where the right hand side is the complex of $\mathcal{O}_X$-modules constructed in Section 33. This is well defined by Lemma 33.6. We get a functor

$$D(\mathcal{O}_X)^{opp} \times D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X), \quad (K, L) \longmapsto R\mathcal{H}om(K, L)$$

As a prelude to proving (34.0.1) we compute the cohomology groups of $R\mathcal{H}om(K, L)$.

Lemma 34.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let L, M be objects of $D(\mathcal{O}_X)$. For every open U we have*

$$H^0(U, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

and in particular $H^0(X, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(L, M)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules representing M and a K-flat complex $\mathcal{L}^\bullet$ representing L. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is K-injective by Lemma 33.7. Hence we can compute cohomology over U by simply taking sections over U and the result follows from Lemma 33.5. $\square$

Lemma 34.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L, M be objects of $D(\mathcal{O}_X)$. With the construction as described above there is a canonical isomorphism*

$$R\mathcal{H}om(K, R\mathcal{H}om(L, M)) = R\mathcal{H}om(K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L, M)$$

in $D(\mathcal{O}_X)$ functorial in K, L, M which recovers (34.0.1) by taking $H^0(X, -)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M and a K-flat complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing L. Let $\mathcal{H}^\bullet$ be the complex described above. For any complex of $\mathcal{O}_X$-modules $\mathcal{K}^\bullet$ we have

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) = \mathcal{H}om^\bullet(\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet)$$

by Lemma 33.1. Note that the left hand side represents $R\mathcal{H}om(K, R\mathcal{H}om(L, M))$ (use Lemma 33.7) and that the right hand side represents $R\mathcal{H}om(K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L, M)$. This proves the displayed formula of the lemma. Taking global sections and using Lemma 34.1 we obtain (34.0.1). $\square$

Lemma 34.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$. The construction of $R\mathcal{H}om(K, L)$ commutes with restrictions to opens, i.e., for every open U we have $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$.*

Proof. This is clear from the construction and Lemma 30.1. $\square$

Lemma 34.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. The bifunctor $R\mathcal{H}om(-, -)$ transforms distinguished triangles into distinguished triangles in both variables.*

Proof. This follows from the observation that the assignment

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \longmapsto \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$$

transforms a termwise split short exact sequences of complexes in either variable into a termwise split short exact sequence. Details omitted. $\square$

Lemma 34.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L, M be objects of $D(\mathcal{O}_X)$. There is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes_{\mathcal{O}_X}^{\mathbf{L}} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

in $D(\mathcal{O}_X)$ functorial in K, L, M.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and a K-flat complex $\mathcal{K}^\bullet$ representing K. The map is defined using the map

$$\text{Tot}(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet), \mathcal{I}^\bullet)$$

of Lemma 33.3. By our particular choice of complexes the left hand side represents $R\mathcal{H}om(L, M) \otimes_{\mathcal{O}_X}^{\mathbf{L}} K$ and the right hand side represents $R\mathcal{H}om(R\mathcal{H}om(K, L), M)$. We omit the proof that this is functorial in all three objects of $D(\mathcal{O}_X)$. $\square$

Lemma 34.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Given K, L, M in $D(\mathcal{O}_X)$ there is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes_{\mathcal{O}_X}^{\mathbf{L}} R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, M)$$

in $D(\mathcal{O}_X)$.

Proof. In general (without suitable finiteness conditions) we do not see how to get this map from Lemma 33.2. Instead, we use the maps

$$\begin{tikzcd} R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(K, L) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \arrow[d] \\ R\mathcal{H}om(R\mathcal{H}om(K, L), M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(K, L) \arrow[d] \\ M \end{tikzcd}$$

gotten by applying Lemma 34.5 twice. Finally, we use Lemma 34.2 to translate the composition

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(K, L) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow M$$

into a map as in the statement of the lemma. □

Lemma 34.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Given K, L in $D(\mathcal{O}_X)$ there is a canonical morphism*

$$K \longrightarrow R\mathcal{H}om(L, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

in $D(\mathcal{O}_X)$ functorial in both K and L.

Proof. Choose K-flat complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ represeting K and L. Choose a K-injective complex $\mathcal{I}^\bullet$ and a quasi-isomorphism $\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet) \to \mathcal{I}^\bullet$. Then we use

$$\mathcal{K}^\bullet \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where the first map comes from Lemma 33.4. □

Lemma 34.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let L be an object of $D(\mathcal{O}_X)$. Set $L^\wedge = R\mathcal{H}om(L, \mathcal{O}_X)$. For M in $D(\mathcal{O}_X)$ there is a canonical map*

$$L^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \longrightarrow R\mathcal{H}om(L, M) \tag{34.8.1}$$

which induces a canonical map

$$H^0(X, L^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \longrightarrow \text{Hom}_{D(\mathcal{O}_X)}(L, M)$$

functorial in M in $D(\mathcal{O}_X)$.

Proof. The map (34.8.1) is a special case of Lemma 34.6 using the identification $M = R\mathcal{H}om(\mathcal{O}_X, M)$. □

Remark 34.9. Let $h : X \to Y$ be a morphism of ringed spaces. Let K, L be objects of $D(\mathcal{O}_Y)$. We claim there is a canonical map

$$Lh^*R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_X)$. Namely, by (34.0.1) proved in Lemma 34.2 such a map is the same thing as a map

$$Lh^*R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} Lh^*K \longrightarrow Lh^*L$$

The source of this arrow is $Lh^*(\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K)$ by Lemma 28.2 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\mathrm{id} : R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, L)$$

via (34.0.1).

Remark 34.10. Suppose that

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

is a commutative diagram of ringed spaces. Let K, L be objects of $D(\mathcal{O}_X)$. We claim there exists a canonical base change map

$$Lg^*Rf_*R\mathcal{H}om(K, L) \longrightarrow R(f')_*R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_{S'})$. Namely, we take the map adjoint to the composition

$$\begin{aligned} L(f')^*Lg^*Rf_*R\mathcal{H}om(K, L) &= Lh^*Lf^*Rf_*R\mathcal{H}om(K, L) \\ &\to Lh^*R\mathcal{H}om(K, L) \\ &\to R\mathcal{H}om(Lh^*K, Lh^*L) \end{aligned}$$

where the first arrow uses the adjunction mapping $Lf^*Rf_* \to \mathrm{id}$ and the second arrow is the canonical map constructed in Remark 34.9.

35. Strictly perfect complexes

Strictly perfect complexes of modules are used to define the notions of pseudo-coherent and perfect complexes later on. They are defined as follows.

Definition 35.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $\mathcal{E}^\bullet$ is *strictly perfect* if $\mathcal{E}^i$ is zero for all but finitely many i and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}_X$-module for all i.

Warning: Since we do not assume that X is a locally ringed space, it may not be true that a direct summand of a finite free $\mathcal{O}_X$-module is finite locally free.

Lemma 35.2. *The cone on a morphism of strictly perfect complexes is strictly perfect.*

Proof. This is immediate from the definitions. □

Lemma 35.3. *The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.*

Proof. Omitted. □

Lemma 35.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_Y$-modules, then $f^*\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_X$-modules.*

Proof. The pullback of a finite free module is finite free. The functor f^* is additive functor hence preserves direct summands. The lemma follows. □

Lemma 35.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Given a solid diagram of $\mathcal{O}_X$-modules*

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[dr, dotted] & \mathcal{F} \\ & \mathcal{G} \arrow[u, "p"'] \end{tikzcd}$$

with $\mathcal{E}$ a direct summand of a finite free $\mathcal{O}_X$-module and p surjective, then a dotted arrow making the diagram commute exists locally on X.

Proof. We may assume $\mathcal{E} = \mathcal{O}_X^{\oplus n}$ for some n. In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in $\Gamma(X, \mathcal{F})$. This is locally possible by the characterization of surjective maps of sheaves (Sheaves, Section 16). $\square$

Lemma 35.6. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

1. *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{F}^\bullet$ acyclic. Then α is locally on X homotopic to zero.*
2. *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^i = 0$ for $i < a$, and $H^i(\mathcal{F}^\bullet) = 0$ for $i \geq a$. Then α is locally on X homotopic to zero.*

Proof. The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex $\mathcal{E}^\bullet$. If $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$ for some direct summand $\mathcal{E}$ of a finite free $\mathcal{O}_X$-module and integer $n \geq a$, then the result follows from Lemma 35.5 and the fact that $\mathcal{F}^{n-1} \to \mathrm{Ker}(\mathcal{F}^n \to \mathcal{F}^{n+1})$ is surjective by the assumed vanishing of $H^n(\mathcal{F}^\bullet)$. If $\mathcal{E}^i$ is zero except for $i \in [a, b]$, then we have a split exact sequence of complexes

$$0 \to \mathcal{E}^b[-b] \to \mathcal{E}^\bullet \to \sigma_{\leq b-1}\mathcal{E}^\bullet \to 0$$

which determines a distinguished triangle in $K(\mathcal{O}_X)$. Hence an exact sequence

$$\mathrm{Hom}_{K(\mathcal{O}_X)}(\sigma_{\leq b-1}\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition $\mathcal{E}^b[-b] \to \mathcal{F}^\bullet$ is locally homotopic to zero, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is locally zero by induction hypothesis. $\square$

Lemma 35.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Given a solid diagram of complexes of $\mathcal{O}_X$-modules*

$$\begin{tikzcd} \mathcal{E}^\bullet \arrow[r, "\alpha"'] \arrow[dr, dotted] & \mathcal{F}^\bullet \\ & \mathcal{G}^\bullet \arrow[u, "f"'] \end{tikzcd}$$

with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^j = 0$ for $j < a$ and $H^j(f)$ an isomorphism for $j > a$ and surjective for $j = a$, then a dotted arrow making the diagram commute up to homotopy exists locally on X.

Proof. Our assumptions on f imply the cone $C(f)^\bullet$ has vanishing cohomology sheaves in degrees $\geq a$. Hence Lemma 35.6 guarantees there is an open covering

$X = \bigcup U_i$ such that the composition $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet$ is homotopic to zero over U_i. Since

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet \to \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in $K(\mathcal{O}_{U_i})$ we see that we can lift $\alpha|_{U_i}$ up to homotopy to a map $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{G}^\bullet|_{U_i}$ as desired. $\square$

Lemma 35.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect.*

1. *For any element $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ there exists an open covering $X = \bigcup U_i$ such that $\alpha|_{U_i}$ is given by a morphism of complexes $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.*
2. *Given a morphism of complexes $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ whose image in the group $\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is zero, there exists an open covering $X = \bigcup U_i$ such that $\alpha|_{U_i}$ is homotopic to zero.*

Proof. Proof of (1). By the construction of the derived category we can find a quasi-isomorphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and a map of complexes $\beta : \mathcal{E}^\bullet \to \mathcal{G}^\bullet$ such that $\alpha = f^{-1}\beta$. Thus the result follows from Lemma 35.7. We omit the proof of (2). $\square$

Lemma 35.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect. Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms*

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 34.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex. Let $(\mathcal{H}')^\bullet$ be the complex with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^{-q}, \mathcal{I}^p)$$

which represents $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ by the construction in Section 34. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an open $U \subset X$ we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U) \to H^0((\mathcal{H}')^\bullet(U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U)$$

By Lemma 35.8 the sheafification of $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^0((\mathcal{H}')^\bullet(U))$. A similar argument can be given for the other cohomology sheaves. Thus $\mathcal{H}^\bullet$ is quasi-isomorphic to $(\mathcal{H}')^\bullet$ which proves the lemma. $\square$

Lemma 35.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with*

1. *$\mathcal{F}^n = 0$ for $n \ll 0$,*
2. *$\mathcal{E}^n = 0$ for $n \gg 0$, and*
3. *$\mathcal{E}^n$ isomorphic to a direct summand of a finite free $\mathcal{O}_X$-module.*

Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 34.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives. Note that $\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 29.4). Hence the construction in Section 34 shows that $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $(\mathcal{H}')^\bullet$ with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p) = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

(equality because there are only finitely many nonzero terms). Note that $\mathcal{H}^\bullet$ is the total complex associated to the double complex with terms $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$ and similarly for $(\mathcal{H}')^\bullet$. The natural map $(\mathcal{H}')^\bullet \to \mathcal{H}^\bullet$ comes from a map of double complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 22.6)

$${}'E_1^{p,q} = H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{H}^\bullet)$ and similarly for $(\mathcal{H}')^\bullet$. To finish the proof of the lemma it suffices to show that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism

$$H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}^\bullet)) \longrightarrow H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{I}^\bullet))$$

on cohomology sheaves whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module. Since this is clear when $\mathcal{E}$ is finite free the result follows. □

36. Pseudo-coherent modules

In this section we discuss pseudo-coherent complexes.

Definition 36.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Let $m \in \mathbf{Z}$.

1. We say $\mathcal{E}^\bullet$ is *m-pseudo-coherent* if there exists an open covering $X = \bigcup U_i$ and for each i a morphism of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ where $\mathcal{E}_i$ is strictly perfect on U_i and $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective.
2. We say $\mathcal{E}^\bullet$ is *pseudo-coherent* if it is m-pseudo-coherent for all m.
3. We say an object E of $D(\mathcal{O}_X)$ is *m-pseudo-coherent* (resp. *pseudo-coherent*) if and only if it can be represented by a m-pseudo-coherent (resp. pseudo-coherent) complex of $\mathcal{O}_X$-modules.

If X is quasi-compact, then an m-pseudo-coherent object of $D(\mathcal{O}_X)$ is in $D^-(\mathcal{O}_X)$. But this need not be the case if X is not quasi-compact.

Lemma 36.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$.*

1. *If there exists an open covering $X = \bigcup U_i$, strictly perfect complexes $\mathcal{E}_i^\bullet$ on U_i, and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $H^j(\alpha_i)$ an isomorphism for $j > m$ and $H^m(\alpha_i)$ surjective, then E is m-pseudo-coherent.*
2. *If E is m-pseudo-coherent, then any complex representing E is m-pseudo-coherent.*

Proof. Let $\mathcal{F}^\bullet$ be any complex representing E and let $X = \bigcup U_i$ and $\alpha_i : \mathcal{E}_i \to E|_{U_i}$ be as in (1). We will show that $\mathcal{F}^\bullet$ is m-pseudo-coherent as a complex, which will prove (1) and (2) simultaneously. By Lemma 35.8 we can after refining the open covering $X = \bigcup U_i$ represent the maps α_i by maps of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{F}^\bullet|_{U_i}$. By assumption $H^j(\alpha_i)$ are isomorphisms for $j > m$, and $H^m(\alpha_i)$ is surjective whence $\mathcal{F}^\bullet$ is m-pseudo-coherent. □

Lemma 36.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E is m-pseudo-coherent, then Lf^*E is m-pseudo-coherent.*

Proof. Represent E by a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Y$-modules and choose an open covering $Y = \bigcup V_i$ and $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{V_i}$ as in Definition 36.1. Set $U_i = f^{-1}(V_i)$. By Lemma 36.2 it suffices to show that $Lf^*\mathcal{E}^\bullet|_{U_i}$ is m-pseudo-coherent. Choose a distinguished triangle

$$\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{V_i} \to C \to \mathcal{E}_i^\bullet[1]$$

The assumption on α_i means exactly that the cohomology sheaves $H^j(C)$ are zero for all $j \geq m$. Denote $f_i : U_i \to V_i$ the restriction of f. Note that $Lf^*\mathcal{E}^\bullet|_{U_i} = Lf_i^*(\mathcal{E}|_{V_i})$. Applying Lf_i^* we obtain the distinguished triangle

$$Lf_i^*\mathcal{E}_i^\bullet \to Lf_i^*\mathcal{E}|_{V_i} \to Lf_i^*C \to Lf_i^*\mathcal{E}_i^\bullet[1]$$

By the construction of Lf_i^* as a left derived functor we see that $H^j(Lf_i^*C) = 0$ for $j \geq m$ (by the dual of Derived Categories, Lemma 17.1). Hence $H^j(Lf_i^*\alpha_i)$ is an isomorphism for $j > m$ and $H^m(Lf^*\alpha_i)$ is surjective. On the other hand, $Lf_i^*\mathcal{E}_i^\bullet = f_i^*\mathcal{E}_i^\bullet$. is strictly perfect by Lemma 35.4. Thus we conclude. □

Lemma 36.4. *Let $(X, \mathcal{O}_X)$ be a ringed space and $m \in \mathbf{Z}$. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$.*

1. *If K is $(m+1)$-pseudo-coherent and L is m-pseudo-coherent then M is m-pseudo-coherent.*
2. *If K anf M are m-pseudo-coherent, then L is m-pseudo-coherent.*
3. *If L is $(m+1)$-pseudo-coherent and M is m-pseudo-coherent, then K is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Choose an open covering $X = \bigcup U_i$ and maps $\alpha_i : \mathcal{K}_i^\bullet \to K|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{K}_i^\bullet$ strictly perfect and $H^j(\alpha_i)$ isomorphisms for $j > m+1$ and surjective for $j = m+1$. We may replace $\mathcal{K}_i^\bullet$ by $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$ and hence we may assume that $\mathcal{K}_i^j = 0$ for $j < m+1$. After refining the open covering we may choose maps $\beta_i : \mathcal{L}_i^\bullet \to L|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{L}_i^\bullet$ strictly perfect such that $H^j(\beta)$ is an isomorphism for $j > m$ and surjective for $j = m$. By Lemma 35.7 we can, after refining the covering, find maps of complexes $\gamma_i : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ such that the diagrams

$$\begin{tikzcd}
K|_{U_i} \arrow[r] & L|_{U_i} \\
\mathcal{K}_i^\bullet \arrow[u, "\alpha_i"] \arrow[r, "\gamma_i"] & \mathcal{L}_i^\bullet \arrow[u, "\beta_i"']
\end{tikzcd}$$

are commutative in $D(\mathcal{O}_{U_i})$ (this requires representing the maps α_i, β_i and $K|_{U_i} \to L|_{U_i}$ by actual maps of complexes; some details omitted). The cone $C(\gamma_i)^\bullet$ is strictly perfect (Lemma 35.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 5.19 and 5.20 that $C(\gamma_i)^\bullet \to M|_{U_i}$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence M is m-pseudo-coherent by Lemma 36.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. □

Lemma 36.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$.*

(1) *If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.*
(2) *If K and L are pseudo-coherent, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is pseudo-coherent.*

Proof. Proof of (1). By replacing X by the members of an open covering we may assume there exist strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ and maps $\alpha : \mathcal{K}^\bullet \to K$ and $\beta : \mathcal{L}^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet) \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$$

induces isomorphisms on cohomology sheaves in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted).

Proof of (2). We may first replace X by the members of an open covering to reduce to the case that K and L are bounded above. Then the statement follows immediately from case (1). □

Lemma 36.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. If $K \oplus L$ is m-pseudo-coherent (resp. pseudo-coherent) in $D(\mathcal{O}_X)$ so are K and L.*

Proof. Assume that $K \oplus L$ is m-pseudo-coherent. After replacing X by the members of an open covering we may assume $K \oplus L \in D^-(\mathcal{O}_X)$, hence $L \in D^-(\mathcal{O}_X)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 4.9. By Lemma 36.4 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. Since L is bounded above we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n-1], L[n-2], \ldots, L$ are m-pseudo-coherent as desired. □

Lemma 36.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. Let $\mathcal{F}^\bullet$ be a (locally) bounded above complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $(m-i)$-pseudo-coherent for all i. Then $\mathcal{F}^\bullet$ is m-pseudo-coherent.*

Proof. Omitted. Hint: use Lemma 36.4 and truncations as in the proof of More on Algebra, Lemma 50.9. □

Lemma 36.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. If E is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent for all i, then E is m-pseudo-coherent.*

Proof. Omitted. Hint: use Lemma 36.4 and truncations as in the proof of More on Algebra, Lemma 50.10. □

Lemma 36.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$.*

(1) *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m$, then $H^m(K)$ is a finite type $\mathcal{O}_X$-module.*

(2) *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m + 1$, then $H^{m+1}(K)$ is a finitely presented $\mathcal{O}_X$-module.*

Proof. Proof of (1). We may work locally on X. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It suffices to prove the result for $\mathcal{E}^\bullet$. Let n be the largest integer such that $\mathcal{E}^n \neq 0$. If $n = m$, then $H^m(\mathcal{E}^\bullet)$ is a quotient of $\mathcal{E}^n$ and the result is clear. If $n > m$, then $\mathcal{E}^{n-1} \to \mathcal{E}^n$ is surjective as $H^n(E^\bullet) = 0$. By Lemma 35.5 we can locally find a section of this surjection and write $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$. Hence it suffices to prove the result for the complex $(\mathcal{E}')^\bullet$ which is the same as $\mathcal{E}^\bullet$ except has $\mathcal{E}'$ in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). We may work locally on X. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $\mathcal{E}^i = 0$ for $i > m + 1$. Then we see that $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \text{Coker}(\mathcal{E}^m \to \mathcal{E}^{m+1})$ which is of finite presentation. □

Lemma 36.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

(1) *$\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is 0-pseudo-coherent if and only if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and*
(2) *$\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is (-1)-pseudo-coherent if and only if $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Use Lemma 36.9 to prove the implications in one direction and Lemma 36.8 for the other. □

37. Tor dimension

In this section we take a closer look at resolutions by flat modules.

Definition 37.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$.

(1) We say E has *tor-amplitude in* $[a, b]$ if $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$ and all $i \notin [a, b]$.
(2) We say E has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.
(3) We say E *locally has finite tor dimension* if there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ has finite tor dimension for all i.

Note that if E has finite tor dimension, then E is an object of $D^b(\mathcal{O}_X)$ as can be seen by taking $\mathcal{F} = \mathcal{O}_X$ in the definition above.

Lemma 37.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a bounded above complex of flat $\mathcal{O}_X$-modules with tor-amplitude in $[a, b]$. Then $\text{Coker}(d^{a-1}_{\mathcal{E}^\bullet})$ is a flat $\mathcal{O}_X$-module.*

Proof. As $\mathcal{E}^\bullet$ is a bounded above complex of flat modules we see that $\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F} = \mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ for any $\mathcal{O}_X$-module $\mathcal{F}$. Hence for every $\mathcal{O}_X$-module $\mathcal{F}$ the sequence

$$\mathcal{E}^{a-2} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact in the middle. Since $\mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to \mathcal{E}^a \to \text{Coker}(d^{a-1}) \to 0$ is a flat resolution this implies that $\text{Tor}_1^{\mathcal{O}_X}(\text{Coker}(d^{a-1}), \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$. This means that $\text{Coker}(d^{a-1})$ is flat, see Lemma 27.15. □

Lemma 37.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent*

(1) *E has tor-amplitude in $[a, b]$.*
(2) *E is represented by a complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}_X$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$.*

Proof. If (2) holds, then we may compute $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}$ and it is clear that (1) holds.

Assume that (1) holds. We may represent E by a bounded above complex of flat $\mathcal{O}_X$-modules $\mathcal{K}^\bullet$, see Section 27. Let n be the largest integer such that $\mathcal{K}^n \neq 0$. If $n > b$, then $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is surjective as $H^n(\mathcal{K}^\bullet) = 0$. As $\mathcal{K}^n$ is flat we see that $\text{Ker}(\mathcal{K}^{n-1} \to \mathcal{K}^n)$ is flat (Modules, Lemma 16.8). Hence we may replace $\mathcal{K}^\bullet$ by $\tau_{\leq n-1}\mathcal{K}^\bullet$. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of flat $\mathcal{O}_X$-modules with $\mathcal{K}^i = 0$ for $i > b$.

Set $\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet$. Everything is clear except that $\mathcal{E}^a$ is flat which follows immediately from Lemma 37.2 and the definitions. □

Lemma 37.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E has tor amplitude in $[a, b]$, then Lf^*E has tor amplitude in $[a, b]$.*

Proof. Assume E has tor amplitude in $[a, b]$. By Lemma 37.3 we can represent E by a complex of $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$. Then Lf^*E is represented by $f^*\mathcal{E}^\bullet$. By Modules, Lemma 17.2 the modules $f^*\mathcal{E}^i$ are flat. Thus by Lemma 37.3 we conclude that Lf^*E has tor amplitude in $[a, b]$. □

Lemma 37.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent*

(1) *E has tor-amplitude in $[a, b]$.*
(2) *for every $x \in X$ the object E_x of $D(\mathcal{O}_{X,x})$ has tor-amplitude in $[a, b]$.*

Proof. Taking stalks at x is the same thing as pulling back by the morphism of ringed spaces $(x, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$. Hence the implication (1) $\Rightarrow$ (2) follows from Lemma 37.4. For the converse, note that taking stalks commutes with tensor products (Modules, Lemma 15.1). Hence

$$(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x = E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{F}_x$$

On the other hand, taking stalks is exact, so

$$H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x = H^i((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x) = H^i(E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{F}_x)$$

and we can check whether $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})$ is zero by checking whether all of its stalks are zero (Modules, Lemma 3.1). Thus (2) implies (1). □

Lemma 37.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$.*

(1) *If K has tor-amplitude in $[a + 1, b + 1]$ and L has tor-amplitude in $[a, b]$ then M has tor-amplitude in $[a, b]$.*

(2) *If K and M have tor-amplitude in $[a, b]$, then L has tor-amplitude in $[a, b]$.*
(3) *If L has tor-amplitude in $[a+1, b+1]$ and M has tor-amplitude in $[a, b]$, then K has tor-amplitude in $[a+1, b+1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

Lemma 37.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$. If K has tor-amplitude in $[a, b]$ and L has tor-amplitude in $[c, d]$ then $K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L$ has tor amplitude in $[a+c, b+d]$.*

Proof. Omitted. Hint: use the spectral sequence for tors. □

Lemma 37.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $a, b \in \mathbf{Z}$. For K, L objects of $D(\mathcal{O}_X)$ if $K \oplus L$ has tor amplitude in $[a, b]$ so do K and L.*

Proof. Clear from the fact that the Tor functors are additive. □

38. Perfect complexes

In this section we discuss properties of perfect complexes on ringed spaces.

Definition 38.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $\mathcal{E}^\bullet$ is *perfect* if there exists an open covering $X = \bigcup U_i$ such that for each i there exists a morphism of complexes $\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ which is a quasi-isomorphism with $\mathcal{E}_i^\bullet$ strictly perfect. An object E of of $D(\mathcal{O}_X)$ is *perfect* if it can be represented by a perfect complex of $\mathcal{O}_X$-modules.

Lemma 38.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$.*
(1) *If there exists an open covering $X = \bigcup U_i$, strictly perfect complexes $\mathcal{E}_i^\bullet$ on U_i, and isomorphisms $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$, then E is perfect.*
(2) *If E is perfect, then any complex representing E is perfect.*

Proof. Identical to the proof of Lemma 36.2. □

Lemma 38.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a \leq b$ be integers. If E has tor amplitude in $[a, b]$ and is $(a-1)$-pseudo-coherent, then E is perfect.*

Proof. After replacing X by the members of an open covering we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to E$ such that $H^i(\alpha)$ is an isomorphism for $i \geq a$. We may and do replace $\mathcal{E}^\bullet$ by $\sigma_{\geq a-1}\mathcal{E}^\bullet$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to E \to C \to \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of E and $\mathcal{E}^\bullet$ and the assumption on α we obtain $C \cong \mathcal{K}[a-2]$ with $\mathcal{K} = \mathrm{Ker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Applying $- \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}$ the assumption that E has tor amplitude in $[a, b]$ implies $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F}$ has image $\mathrm{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}_X} \mathcal{F})$. It follows that $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{E}', \mathcal{F}) = 0$ where $\mathcal{E}' = \mathrm{Coker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Hence $\mathcal{E}'$ is flat (Lemma 27.15). Thus $\mathcal{E}'$ is locally a direct summand of a finite free module by Modules, Lemma 16.11. Thus locally the complex

$$\mathcal{E}' \to \mathcal{E}^{a-1} \to \ldots \to \mathcal{E}^b$$

is quasi-isomorphic to E and E is perfect. $\square$

Lemma 38.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent*

1. *E is perfect, and*
2. *E is pseudo-coherent and locally has finite tor dimension.*

Proof. Assume (1). By definition this means there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ is represented by a strictly perfect complex. Thus E is pseudo-coherent (i.e., m-pseudo-coherent for all m) by Lemma 36.2. Moreover, a direct summand of a finite free module is flat, hence $E|_{U_i}$ has finite Tor dimension by Lemma 37.3. Thus (2) holds.

Assume (2). After replacing X by the members of an open covering we may assume there exist integers $a \leq b$ such that E has tor amplitude in $[a, b]$. Since E is m-pseudo-coherent for all m we conclude using Lemma 38.3. $\square$

Lemma 38.5. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E is perfect in $D(\mathcal{O}_Y)$, then Lf^*E is perfect in $D(\mathcal{O}_X)$.*

Proof. This follows from Lemma 38.4, 37.4, and 36.3. (An alternative proof is to copy the proof of Lemma 36.3.) $\square$

Lemma 38.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$. If two out of three of K, L, M are perfect then the third is also perfect.*

Proof. First proof: Combine Lemmas 38.4, 36.4, and 37.6. Second proof (sketch): Say K and L are perfect. After replacing X by the members of an open covering we may assume that K and L are represented by strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$. After replacing X by the members of an open covering we may assume the map $K \to L$ is given by a map of complexes $\alpha : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, see Lemma 35.8. Then M is isomorphic to the cone of α which is strictly perfect by Lemma 35.2. $\square$

Lemma 38.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. If K, L are perfect objects of $D(\mathcal{O}_X)$, then so is $K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L$.*

Proof. Follows from Lemmas 38.4, 36.5, and 37.7. $\square$

Lemma 38.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $K \oplus L$ is a perfect object of $D(\mathcal{O}_X)$, then so are K and L.*

Proof. Follows from Lemmas 38.4, 36.6, and 37.8. $\square$

Lemma 38.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $j : U \to X$ be an open subspace. Let E be a perfect object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on a closed subset $T \subset U$ with $j(T)$ closed in X. Then Rj_*E is a perfect object of $D(\mathcal{O}_X)$.*

Proof. Being a perfect complex is local on X. Thus it suffices to check that Rj_*E is perfect when restricted to U and $V = X \setminus j(T)$. We have $Rj_*E|_U = E$ which is perfect. We have $Rj_*E|_V = 0$ because $E|_{U \setminus T} = 0$. $\square$

Lemma 38.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be a perfect object of $D(\mathcal{O}_X)$. Then $K^\wedge = R\mathcal{H}om(K, \mathcal{O}_X)$ is a perfect object too and $(K^\wedge)^\wedge = K$. There are functorial isomorphisms*

$$H^0(X, K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, M)$$

for M in $D(\mathcal{O}_X)$.

Proof. We will use without further mention that formation of internal hom commutes with restriction to opens (Lemma 34.3). In particular we may check the first two statements locally on X. By Lemma 34.8 to see the final statement it suffices to check that the map (34.8.1)

$$K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \longrightarrow R\mathcal{H}om(K, M)$$

is an isomorphism. This is local on X as well. Hence it suffices to prove the lemma when K is represented by a strictly perfect complex.

Assume K is represented by the strictly perfect complex $\mathcal{E}^\bullet$. Then it follows from Lemma 35.9 that $K^\wedge$ is represented by the complex whose terms are $(\mathcal{E}^{-n})^\wedge = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-n}, \mathcal{O}_X)$ in degree n. Since $\mathcal{E}^{-n}$ is a direct summand of a finite free $\mathcal{O}_X$-module, so is $(\mathcal{E}^{-n})^\wedge$. Hence $K^\wedge$ is represented by a strictly perfect complex too. It is also clear that $(K^\wedge)^\wedge = K$ as we have $((\mathcal{E}^{-n})^\wedge)^\wedge = \mathcal{E}^{-n}$. To see that (34.8.1) is an isomorphism, represent M by a K-flat complex $\mathcal{F}^\bullet$. By Lemma 35.9 the complex $R\mathcal{H}om(K, M)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

On the other hand, then object $K^\wedge \otimes^{\mathbf{L}} M$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{F}^p \otimes_{\mathcal{O}_X} (\mathcal{E}^{-q})^\wedge$$

Thus the assertion that (34.8.1) is an isomorphism reduces to the assertion that the canonical map

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$$

is an isomorphism when $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module and $\mathcal{F}$ is any $\mathcal{O}_X$-module. This follows immediately from the corresponding statement when $\mathcal{E}$ is finite free. □

39. Compact objects

n this section we study compact objects in the derived category of modules on a ringed space. We recall that compact objects are defined in Derived Categories, Definition 34.1. On suitable ringed spaces the perfect objects are compact.

Lemma 39.1. *Let X be a ringed space. Assume that the underlying topological space of X has the following properties:*

1. *X is quasi-compact,*
2. *there exists a basis of quasi-compact open subsets, and*
3. *the intersection of any two quasi-compact opens is quasi-compact.*

Then any perfect object of $D(\mathcal{O}_X)$ is compact.

Proof. Let K be a perfect object and let $K^\wedge$ be its dual, see Lemma 38.10. Then we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M in $D(\mathcal{O}_X)$. Since $K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums (by construction) and H^0 does by Lemma 20.1 and the construction of direct sums in Injectives, Lemma 13.4 we obtain the result of the lemma. $\square$

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