LIMITS OF ALGEBRAIC SPACES

Contents

1. Introduction

In this chapter we put material related to limits of algebraic spaces. A first topic is the characterization of algebraic spaces F locally of finite presentation over the base S as limit preserving functors. We continue with a study of limits of inverse systems over directed partially ordered sets with affine transition maps. We discuss absolute Noetherian approximation for quasi-compact and quasi-separated algebraic spaces following [CLO12]. Another approach is due to David Rydh (see [Ryd08]) whose results also cover absolute Noetherian approximation for certain algebraic stacks.

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

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3. Morphisms of finite presentation

In this section we generalize Limits, Proposition 5.1 to morphisms of algebraic spaces. The motivation for the following definition comes from the proposition just cited.

Definition 3.1. Let S be a scheme.

(1) A functor $F : (Sch/S)_{fppf}^{opp} \to Sets$ is said to be *locally of finite presentation* or *limit preserving* if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S, we have
$$F(T) = \mathrm{colim}\, F(T_i).$$
We sometimes say that F is *locally of finite presentation over S*.

(2) Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. A transformation of functors $a : F \to G$ is *locally of finite presentation* if for every scheme T over S and every $y \in G(T)$ the functor
$$F_y : (Sch/T)_{fppf}^{opp} \longrightarrow Sets, \quad T'/T \longmapsto \{x \in F(T') \mid a(x) = y|_{T'}\}$$
is locally of finite presentation over T[1]. We sometimes say that F is *relatively limit preserving* over G.

The functor F_y is in some sense the fiber of $a : F \to G$ over y, except that it is a presheaf on the big fppf site of T. A formula for this functor is:

$$F_y = F|_{(Sch/T)_{fppf}} \times_{G|_{(Sch/T)_{fppf}}} * \tag{3.1.1}$$

Here $*$ is the final object in the category of (pre)sheaves on $(Sch/T)_{fppf}$ (see Sites, Example 10.2) and the map $* \to G|_{(Sch/T)_{fppf}}$ is given by y. Note that if $j : (Sch/T)_{fppf} \to (Sch/S)_{fppf}$ is the localization functor, then the formula above becomes $F_y = j^{-1}F \times_{j^{-1}G} *$ and $j_!F_y$ is just the fiber product $F \times_{G,y} T$. (See Sites, Section 24, for information on localization, and especially Sites, Remark 24.9 for information on $j_!$ for presheaves.)

At this point we temporarily have two definitions of what it means for a morphism $X \to Y$ of algebraic spaces over S to be locally of finite presentation. Namely, one by Morphisms of Spaces, Definition 27.1 and one using that $X \to Y$ is a transformation of functors so that Definition 3.1 applies. We will show in Proposition 3.9 that these two definitions agree.

Lemma 3.2. *Let S be a scheme. Let $a : F \to G$ be a transformation of functors $(Sch/S)_{fppf}^{opp} \to Sets$. The following are equivalent*

(1) *F is relatively limit preserving over G, and*
(2) *for every every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S the diagram of sets*
$$\begin{array}{ccc} \mathrm{colim}_i\, F(T_i) & \longrightarrow & F(T) \\ \downarrow a & & \downarrow a \\ \mathrm{colim}_i\, G(T_i) & \longrightarrow & G(T) \end{array}$$
is a fibre product diagram.

[1] The characterization (2) in Lemma 3.2 may be easier to parse.

Proof. Assume (1). Consider $T = \lim_{i \in I} T_i$ as in (2). Let (y, x_T) be an element of the fibre product $\mathrm{colim}_i\, G(T_i) \times_{G(T)} F(T)$. Then y comes from $y_i \in G(T_i)$ for some i. Consider the functor F_{y_i} on $(Sch/T_i)_{fppf}$ as in Definition 3.1. We see that $x_T \in F_{y_i}(T)$. Moreover $T = \lim_{i' \geq i} T_{i'}$ is a directed system of affine schemes over T_i. Hence (1) implies that x_T the image of a unique element x of $\mathrm{colim}_{i' \geq i} F_{y_i}(T_{i'})$. Thus x is the unique element of $\mathrm{colim}\, F(T_i)$ which maps to the pair (y, x_T). This proves that (2) holds.

Assume (2). Let T be a scheme and $y_T \in G(T)$. We have to show that F_{y_T} is limit preserving. Let $T' = \lim_{i \in I} T'_i$ be an affine scheme over T which is the directed limit of affine scheme T'_i over T. Let $x_{T'} \in F_{y_T}$. Pick $i \in I$ which is possible as I is a directed partially ordered set. Denote $y_i \in F(T'_i)$ the image of $y_{T'}$. Then we see that $(y_i, x_{T'})$ is an element of the fibre product $\mathrm{colim}_i\, G(T'_i) \times_{G(T')} F(T')$. Hence by (2) we get a unique element x of $\mathrm{colim}_i\, F(T'_i)$ mapping to $(y_i, x_{T'})$. It is clear that x defines an element of $\mathrm{colim}_i\, F_y(T'_i)$ mapping to $x_{T'}$ and we win. □

Lemma 3.3. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to$ Sets. Let $a : F \to G$, $b : G \to H$ be transformations of functors. If a and b are locally of finite presentation, then*

$$b \circ a : F \longrightarrow H$$

is locally of finite presentation.

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 3.2. Consider the diagram

$$\begin{tikzcd}
\mathrm{colim}_i\, F(T_i) \arrow[r] \arrow[d, "a"] & F(T) \arrow[d, "a"] \\
\mathrm{colim}_i\, G(T_i) \arrow[r] \arrow[d, "b"] & G(T) \arrow[d, "b"] \\
\mathrm{colim}_i\, H(T_i) \arrow[r] & H(T)
\end{tikzcd}$$

By assumption the two squares are fibre product squares. Hence the outer rectangle is a fibre product diagram too which proves the lemma. □

Lemma 3.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to$ Sets. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram*

$$\begin{tikzcd}
H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"] & F \arrow[d, "a"] \\
H \arrow[r, "b"] & G
\end{tikzcd}$$

If a is locally of finite presentation, then the base change a' is locally of finite presentation.

Proof. Omitted. Hint: This is formal. □

Lemma 3.5. *Let T be an affine scheme which is written as a limit $T = \lim_{i \in I} T_i$ of a directed inverse system of affine schemes.*

(1) *Let $\mathcal{V} = \{V_j \to T\}_{j=1,\ldots,m}$ be a standard fppf covering of T, see Topologies, Definition 7.5. Then there exists an index i and a standard fppf covering $\mathcal{V}_i = \{V_{i,j} \to T_i\}_{j=1,\ldots,m}$ whose base change $T \times_{T_i} \mathcal{V}_i$ to T is isomorphic to $\mathcal{V}$.*
(2) *Let $\mathcal{V}_i$, $\mathcal{V}'_i$ be a pair of standard fppf coverings of T_i. If $f : T \times_{T_i} \mathcal{V} \to T \times_{T_i} \mathcal{V}'_i$ is a morphism of coverings of T, then there exists an index $i' \geq i$ and a morphism $f_{i'} : T_{i'} \times_{T_i} \mathcal{V} \to T_{i'} \times_{T_i} \mathcal{V}'_i$ whose base change to T is f.*
(3) *If $f, g : \mathcal{V} \to \mathcal{V}'_i$ are morphisms of standard fppf coverings of T_i whose base changes f_T, g_T to T are equal then there exists an index $i' \geq i$ such that $f_{T_{i'}} = g_{T_{i'}}$.*

In other words, the category of standard fppf coverings of T is the colimit over I of the categories of standard fppf coverings of T_i

Proof. By Limits, Lemma 9.1 the category of schemes of finite presentation over T is the colimit over I of the categories of finite presentation over T_i. By Limits, Lemmas 7.2 and 7.6 the same is true for category of schemes which are affine, flat and of finite presentation over T. To finish the proof of the lemma it suffices to show that if $\{V_{j,i} \to T_i\}_{j=1,\ldots,m}$ is a finite family of flat finitely presented morphisms with $V_{j,i}$ affine, and the base change $\coprod_j T \times_{T_i} V_{j,i} \to T$ is surjective, then for some $i' \geq i$ the morphism $\coprod T_{i'} \times_{T_i} V_{j,i} \to T_{i'}$ is surjective. Denote $W_{i'} \subset T_{i'}$, resp. $W \subset T$ the image. Of course $W = T$ by assumption. Since the morphisms are flat and of finite presentation we see that W_i is a quasi-compact open of T_i, see Morphisms, Lemma 26.9. Moreover, $W = T \times_{T_i} W_i$ (formation of image commutes with base change). Hence by Limits, Lemma 3.8 we conclude that $W_{i'} = T_{i'}$ for some large enough i' and we win. $\square$

Lemma 3.6. *Let S be a scheme contained in Sch_{fppf}. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. If F is locally of finite presentation over S then its sheafification $F^\#$ is locally of finite presentation over S.*

Proof. Assume F is locally of finite presentation. It suffices to show that F^+ is locally of finite presentation, since $F^\# = (F^+)^+$, see Sites, Theorem 10.10. Let T be an affine scheme over S, and let $T = \lim T_i$ be written as the directed limit of an inverse system of affine S schemes. Recall that $F^+(T)$ is the colimit of $\check{H}^0(\mathcal{V}, F)$ where the limit is over all coverings of T in $(Sch/S)_{fppf}$. Any fppf covering of an affine scheme can be refined by a standard fppf covering, see Topologies, Lemma 7.4. Hence we can write

$$F^+(T) = \text{colim}_{\mathcal{V} \text{ standard covering } T}\, \check{H}^0(\mathcal{V}, F).$$

By Lemma 3.5 we may rewrite this as

$$\text{colim}_{i \in I}\, \text{colim}_{\mathcal{V}_i \text{ standard covering } T_i}\, \check{H}^0(T \times_{T_i} \mathcal{V}_i, F).$$

(The order of the colimits is irrelevant by Categories, Lemma 14.9.) Given a standard fppf covering $\mathcal{V}_i = \{V_j \to T_i\}_{j=1,\ldots,m}$ of T_i we see that

$$T \times_{T_i} V_j = \lim_{i' \geq i} T_{i'} \times_T V_j$$

by Limits, Lemma 2.3, and similarly

$$T \times_{T_i} (V_j \times_{T_i} V_{j'}) = \lim_{i' \geq i} T_{i'} \times_T (V_j \times_{T_i} V_{j'}).$$

As the presheaf F is locally of finite presentation this means that

$$\check{H}^0(T \times_{T_i} \mathcal{V}_i, F) = \text{colim}_{i' \geq i}\, \check{H}^0(T_{i'} \times_{T_i} \mathcal{V}_i, F).$$

Hence the colimit expression for $F^+(T)$ above collapses to

$$\mathrm{colim}_{i \in I}\, \mathrm{colim}_{\mathcal{V} \text{ standard covering } T_i}\, \check{H}^0(\mathcal{V}, F). = \mathrm{colim}_{i \in I}\, F^+(T_i).$$

In other words $F^+(T) = \mathrm{colim}_i F^+(T_i)$ and hence the lemma holds. □

Lemma 3.7. *Let S be a scheme. Let $F : (Sch/S)_{fppf}^{opp} \to Sets$ be a functor. Assume that*

(1) *F is a sheaf, and*
(2) *there exists an fppf covering $\{U_j \to S\}_{j \in J}$ such that $F|_{(Sch/U_j)_{fppf}}$ is locally of finite presentation.*

Then F is locally of finite presentation.

Proof. Let T be an affine scheme over S. Let I be a directed partially ordered set, and let T_i be an inverse system of affine schemes over S such that $T = \lim T_i$. We have to show that the canonical map $\mathrm{colim}\, F(T_i) \to F(T)$ is bijective.

Choose some $0 \in I$ and choose a standard fppf covering $\{V_{0,k} \to T_0\}_{k=1,\ldots,m}$ which refines the pullback $\{U_j \times_S T_0 \to T_0\}$ of the given fppf covering of S. For each $i \geq 0$ we set $V_{i,k} = T_i \times_{T_0} V_{0,k}$, and we set $V_k = T \times_{T_0} V_{0,k}$. Note that $V_k = \lim_{i \geq 0} V_{i,k}$, see Limits, Lemma 2.3.

Suppose that $x, x' \in \mathrm{colim}\, F(T_i)$ map to the same element of $F(T)$. Say x, x' are given by elements $x_i, x'_i \in F(T_i)$ for some $i \in I$ (we may choose the same i for both as I is directed). By assumption (2) and the fact that x_i, x'_i map to the same element of $F(T)$ this implies that

$$x_i|_{V_{i',k}} = x'_i|_{V_{i',k}}$$

for some suitably large $i' \in I$. We can choose the same i' for each k as $k \in \{1, \ldots, m\}$ ranges over a finite set. Since $\{V_{i',k} \to T_{i'}\}$ is an fppf covering and F is a sheaf this implies that $x_i|_{T_{i'}} = x'_i|_{T_{i'}}$ as desired. This proves that the map $\mathrm{colim}\, F(T_i) \to F(T)$ is injective.

To show surjectivity we argue in a similar fashion. Let $x \in F(T)$. By assumption (2) for each k we can choose a i such that $x|_{V_k}$ comes from an element $x_{i,k} \in F(V_{i,k})$. As before we may choose a single i which works for all k. By the injectivity proved above we see that

$$x_{i,k}|_{V_{i',k} \times_{T_{i'}} V_{i',l}} = x_{i,l}|_{V_{i',k} \times_{T_{i'}} V_{i',l}}$$

for some large enough i'. Hence by the sheaf condition of F the elements $x_{i,k}|_{V_{i',k}}$ glue to an element $x_{i'} \in F(T_{i'})$ as desired. □

Lemma 3.8. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$ be functors. If $a : F \to G$ is a transformation which is locally of finite presentation, then the induced transformation of sheaves $F^\# \to G^\#$ is of finite presentation.*

Proof. Suppose that T is a scheme and $y \in G^\#(T)$. We have to show the functor $F_y^\# : (Sch/T)_{fppf}^{opp} \to Sets$ constructed from $F^\# \to G^\#$ and y as in Definition 3.1 is locally of finite presentation. By Equation (3.1.1) we see that $F_y^\#$ is a sheaf. Choose an fppf covering $\{V_j \to T\}_{j \in J}$ such that $y|_{V_j}$ comes from an element $y_j \in F(V_j)$. Note that the restriction of $F^\#$ to $(Sch/V_j)_{fppf}$ is just $F_{y_j}^\#$. If we can show that $F_{y_j}^\#$ is locally of finite presentation then Lemma 3.7 guarantees that $F_y^\#$ is locally of finite presentation and we win. This reduces us to the case $y \in G(T)$.

Let $y \in G(T)$. In this case we claim that $F_y^\# = (F_y)^\#$. This follows from Equation (3.1.1). Thus this case follows from Lemma 3.6. $\square$

Proposition 3.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is a morphism of algebraic spaces which is locally of finite presentation, see Morphisms of Spaces, Definition 27.1.*
(2) *The morphism $f : X \to Y$ is locally of finite presentation as a transformation of functors, see Definition 3.1.*

Proof. Assume (1). Let T be a scheme and let $y \in Y(T)$. We have to show that $T \times_X Y$ is locally of finite presentation over T in the sense of Definition 3.1. Hence we are reduced to proving that if X is an algebraic space which is locally of finite presentation over S as an algebraic space, then it is locally of finite presentation as a functor $X : (Sch/S)_{fppf}^{opp} \to Sets$. To see this choose a presentation $X = U/R$, see Spaces, Definition 9.3. It follows from Morphisms of Spaces, Definition 27.1 that both U and R are schemes which are locally of finite presentation over S. Hence by Limits, Proposition 5.1 we have

$$U(T) = \operatorname{colim} U(T_i), \quad R(T) = \operatorname{colim} R(T_i)$$

whenever $T = \lim_i T_i$ in $(Sch/S)_{fppf}$. It follows that the presheaf

$$(Sch/S)_{fppf}^{opp} \longrightarrow Sets, \quad W \longmapsto U(W)/R(W)$$

is locally of finite presentation. Hence by Lemma 3.6 its sheafification $X = U/R$ is locally of finite presentation too.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to Y$. Next, choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By Lemma 3.4 the transformation of functors $V \times_Y X \to V$ is locally of finite presentation. By Morphisms of Spaces, Lemma 36.8 the morphism of algebraic spaces $U \to V \times_Y X$ is locally of finite presentation, hence locally of finite presentation as a transformation of functors by the first part of the proof. By Lemma 3.3 the composition $U \to V \times_Y X \to V$ is locally of finite presentation as a transformation of functors. Hence the morphism of schemes $U \to V$ is locally of finite presentation by Limits, Proposition 5.1 (modulo a set theoretic remark, see last paragraph of the proof). This means, by definition, that (1) holds.

Set theoretic remark. Let $U \to V$ be a morphism of $(Sch/S)_{fppf}$. In the statement of Limits, Proposition 5.1 we characterize $U \to V$ as being locally of finite presentation if for *all* directed inverse systems $(T_i, f_{ii'})$ of affine schemes over V we have $U(T) = \operatorname{colim} V(T_i)$, but in the current setting we may only consider affine schemes T_i over V which are (isomorphic to) an object of $(Sch/S)_{fppf}$. So we have to make sure that there are enough affines in $(Sch/S)_{fppf}$ to make the proof work. Inspecting the proof of (2) $\Rightarrow$ (1) of Limits, Proposition 5.1 we see that the question reduces to the case that U and V are affine. Say $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$. By construction of $(Sch/S)_{fppf}$ the spectrum of any ring of cardinality $\leq |B|$ is isomorphic to an object of $(Sch/S)_{fppf}$. Hence it suffices to observe that in the "only if" part of the proof of Algebra, Lemma 123.2 only A-algebras of cardinality $\leq |B|$ are used. $\square$

Remark 3.10. Here is an important special case of Proposition 3.9. Let S be a scheme. Let X be an algebraic space over S. Then X is locally of finite presentation over S if and only if X, as a functor $(Sch/S)^{opp} \to Sets$, is limit preserving. Compare with Limits, Remark 5.2.

4. Limits of algebraic spaces

The following lemma explains how we think of limits of algebraic spaces in this chapter. We will use (without further mention) that the base change of an affine morphism of algebraic spaces is affine (see Morphisms of Spaces, Lemma 20.5).

Lemma 4.1. *Let S be a scheme. Let I be a directed partially ordered set. Let $(X_i, f_{ii'})$ be an inverse system over I in the category of algebraic spaces over S. If the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine, then the limit $X = \lim_i X_i$ (as an fppf sheaf) is an algebraic space. Moreover,*

(1) *each of the morphisms $f_i : X \to X_i$ is affine,*
(2) *for any $i \in I$ and any morphism of algebraic spaces $T \to X_i$ we have*

$$X \times_{X_i} T = \lim_{i' \geq i} X_{i'} \times_{X_i} T.$$

as algebraic spaces over S.

Proof. Part (2) is a formal consequence of the existence of the limit $X = \lim X_i$ as an algebraic space over S. Choose an element $0 \in I$ (this is possible as a directed partially ordered set is nonempty). Choose a scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $R_0 = U_0 \times_{X_0} U_0$ so that $X_0 = U_0/R_0$. For $i \geq 0$ set $U_i = X_i \times_{X_0} U_0$ and $R_i = X_i \times_{X_0} R_0 = U_i \times_{X_i} U_i$. By Limits, Lemma 2.2 we see that $U = \lim_{i \geq 0} U_i$ and $R = \lim_{i \geq 0} R_i$ are schemes. Moreover, the two morphisms $s, t : R \to U$ are the base change of the two projections $R_0 \to U_0$ by the morphism $U \to U_0$, in particular étale. The morphism $R \to U \times_S U$ defines an equivalence relation as directed a limit of equivalence relations is an equivalence relation. Hence the morphism $R \to U \times_S U$ is an étale equivalence relation. We claim that the natural map

$$U/R \longrightarrow \lim X_i \tag{4.1.1}$$

is an isomorphism of fppf sheaves on the category of schemes over S. The claim implies $X = \lim X_i$ is an algebraic space by Spaces, Theorem 10.5.

Let Z be a scheme and let $a : Z \to \lim X_i$ be a morphism. Then $a = (a_i)$ where $a_i : Z \to X_i$. Set $W_0 = Z \times_{a_0, X_0} U_0$. Note that $W_0 = Z \times_{a_i, X_i} U_i$ for all $i \geq 0$ by our choice of $U_i \to X_i$ above. Hence we obtain a morphism $W_0 \to \lim_{i \geq 0} U_i = U$. Since $W_0 \to Z$ is surjective and étale, we conclude that (4.1.1) is a surjective map of sheaves. Finally, suppose that Z is a scheme and that $a, b : Z \to U/R$ are two morphisms which are equalized by (4.1.1). We have to show that $a = b$. After replacing Z by the members of an fppf covering we may assume there exist morphisms $a', b' : Z \to U$ which give rise to a and b. The condition that a, b are equalized by (4.1.1) means that for each $i \geq 0$ the compositions $a'_i, b'_i : Z \to U \to U_i$ are equal as morphisms into $U_i/R_i = X_i$. Hence $(a'_i, b'_i) : Z \to U_i \times_S U_i$ factors through R_i, say by some morphism $c_i : Z \to R_i$. Since $R = \lim_{i \geq 0} R_i$ we see that $c = \lim c_i : Z \to R$ is a morphism which shows that a, b are equal as morphisms of Z into U/R.

Part (1) follows as we have seen above that $U_i \times_{X_i} X = U$ and $U \to U_i$ is affine by construction. $\square$

Lemma 4.2. *Let S be a scheme. Let I be a directed partially ordered set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S with affine transition maps. Let $X = \lim_i X_i$. Let $0 \in I$. Suppose that $T \to X_0$ is a morphism of algebraic spaces. Then*

$$T \times_{X_0} X = \lim_{i \geq 0} T \times_{X_0} X_i$$

as algebraic spaces over S.

Proof. The limit X is an algebraic space by Lemma 4.1. The equality is formal, see Categories, Lemma 14.9. $\square$

5. Descending properties

This section is the analogue of Limits, Section 3.

Situation 5.1. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

The following lemma holds a little bit more generally (namely when we just assume each X_i is a decent algebraic space).

Lemma 5.2. *In Situation 5.1 we have $|X| = \lim |X_i|$.*

Proof. There is a canonical map $|X| \to \lim |X_i|$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$. Recall that $U = \lim U_i$ and $R = \lim R_i$, see proof of Lemma 4.1. Recall that $|X| = |U|/|R|$ and $|X_i| = |U_i|/|R_i|$. By Limits, Lemma 3.2 we have $|U| = \lim |U_i|$ and $|R| = \lim |R_i|$.

Surjectivity of $|X| \to \lim |X_i|$. Let $(x_i) \in \lim |X_i|$. Denote $S_i \subset |U_i|$ the inverse image of x_i. This is a finite nonempty set by Properties of Spaces, Lemma 12.3. Hence $\lim S_i$ is nonempty, see Categories, Lemma 21.5. Let $(u_i) \in \lim S_i \subset \lim |U_i|$. By the above this determines a point $u \in |U|$ which maps to an $x \in |X|$ mapping to the given element (x_i) of $\lim |X_i|$.

Injectivity of $|X| \to \lim |X_i|$. Suppose that $x, x' \in |X|$ map to the same point of $\lim |X_i|$. Choose lifts $u, u' \in |U|$ and denote $u_i, u'_i \in |U_i|$ the images. For each i let $T_i \subset |R_i|$ be the set of points mapping to $(u_i, u'_i) \in |U_i| \times |U_i|$. This is a finite set by Properties of Spaces, Lemma 12.3 which is nonempty as we've assumed that x and x' map to the same point of X_i. Hence $\lim T_i$ is nonempty, see Categories, Lemma 21.5. As before let $r \in |R| = \lim |R_i|$ be a point corresponding to an element of $\lim T_i$. Then r maps to (u, u') in $|U| \times |U|$ by construction and we see that $x = x'$ in $|X|$ as desired. $\square$

Lemma 5.3. *In Situation 5.1, if each X_i is nonempty, then $|X|$ is nonempty.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Then each U_i is a nonempty affine scheme. Hence $U = \lim U_i$ is nonempty (Limits, Lemma 3.4) and thus X is nonempty. $\square$

Lemma 5.4. *Notation and assumptions as in Situation 5.1. Suppose that $\mathcal{F}_0$ is a quasi-coherent sheaf on X_0. Set $\mathcal{F}_i = f_{0i}^*\mathcal{F}_0$ for $i \geq 0$ and set $\mathcal{F} = f_0^*\mathcal{F}_0$. Then*

$$\Gamma(X, \mathcal{F}) = \mathrm{colim}_{i \geq 0} \Gamma(X_i, \mathcal{F}_i)$$

Proof. Choose a surjective étale morphism $U_0 \to X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. In the proof of Lemma 4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Hence Limits, Lemma 3.3 implies that

$$\mathcal{F}(U) = \mathrm{colim}\, \mathcal{F}_i(U_i) \quad \text{and} \quad \mathcal{F}(R) = \mathrm{colim}\, \mathcal{F}_i(R_i).$$

The lemma follows as $\Gamma(X, \mathcal{F}) = \mathrm{Ker}(\mathcal{F}(U) \to \mathcal{F}(R))$ and similarly $\Gamma(X_i, \mathcal{F}_i) = \mathrm{Ker}(\mathcal{F}_i(U_i) \to \mathcal{F}_i(R_i))$ □

Lemma 5.5. *Notation and assumptions as in Situation 5.1. For any quasi-compact open subspace $U \subset X$ there exists an i and a quasi-compact open $U_i \subset X_i$ whose inverse image in X is U.*

Proof. Follows formally from the construction of limits in Lemma 4.1 and the corresponding result for schemes: Limits, Lemma 3.8. □

The following lemma will be superseded by the stronger Lemma 6.9.

Lemma 5.6. *Notation and assumptions as in Situation 5.1. Let $f_0 : Y_0 \to Z_0$ be a morphism of algebraic spaces over X_0. Assume (a) $Y_0 \to X_0$ and $Z_0 \to X_0$ are representable, (b) Y_0, Z_0 quasi-compact and quasi-separated, (c) f_0 locally of finite presentation, and (d) $Y_0 \times_{X_0} X \to Z_0 \times_{X_0} X$ an isomorphism. Then there exists an $i \geq 0$ such that $Y_0 \times_{X_0} X_i \to Z_0 \times_{X_0} X_i$ is an isomorphism.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = U_0 \times_{X_0} X_i$ and $U = U_0 \times_{X_0} X$. Apply Limits, Lemma 7.9 to see that $Y_0 \times_{X_0} U_i \to Z_0 \times_{X_0} U_i$ is an isomorphism of schemes for some $i \geq 0$ (details omitted). As $U_i \to X_i$ is surjective étale, it follows that $Y_0 \times_{X_0} X_i \to Z_0 \times_{X_0} X_i$ is an isomorphism (details omitted). □

Lemma 5.7. *Notation and assumptions as in Situation 5.1. If X is separated, then X_i is separated for some $i \in I$.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. For $i \geq 0$ set $U_i = U_0 \times_{X_0} X_i$ and set $U = U_0 \times_{X_0} X$. Note that U_i and U are affine schemes which come equipped with surjective étale morphisms $U_i \to X_i$ and $U \to X$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$ with projections $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$. Note that R_i and R are quasi-compact separated schemes (as the algebraic spaces X_i and X are quasi-separated). The maps $s_i : R_i \to U_i$ and $s : R \to U$ are of finite type. By definition X_i is separated if and only if $(t_i, s_i) : R_i \to U_i \times U_i$ is a closed immersion, and since X is separated by assumption, the morphism $(t, s) : R \to U \times U$ is a closed immersion. Since $R \to U$ is of finite type, there exists an i such that the morphism $R \to U_i \times U$ is a closed immersion (Limits, Lemma 3.13). Fix such an $i \in I$. Apply Limits, Lemma 7.4 to the system of morphisms $R_{i'} \to U_i \times U_{i'}$ for $i' \geq i$ (this is permissible as indeed $R_{i'} = R_i \times_{U_i \times U_i} U_i \times U_{i'}$) to see that $R_{i'} \to U_i \times U_{i'}$ is a closed immersion for

i' sufficiently large. This implies immediately that $R_{i'} \to U_{i'} \times U_{i'}$ is a closed immersion finishing the proof of the lemma. □

Lemma 5.8. *Notation and assumptions as in Situation 5.1. If X is affine, then there exists an i such that X_i is affine.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U = U_0 \times_{X_0} X$ and $U_i = U_0 \times_{X_0} X_i$ for $i \geq 0$. Since the transition morphisms are affine, the algebraic spaces U_i and U are affine. Thus $U \to X$ is an étale morphism of affine schemes. Hence we can write $X = \text{Spec}(A)$, $U = \text{Spec}(B)$ and

$$B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_n)$$

such that $\Delta = \det(\partial g_\lambda/\partial x_\mu)$ is invertible in B, see Algebra, Lemma 138.2. Set $A_i = \mathcal{O}_{X_i}(X_i)$. We have $A = \text{colim}\, A_i$ by Lemma 5.4. After increasing 0 we may assume we have $g_{1,i}, \ldots, g_{n,i} \in A_i[x_1, \ldots, x_n]$ mapping to $g_1, \ldots, g_n$. Set

$$B_i = A_i[x_1, \ldots, x_n]/(g_{1,i}, \ldots, g_{n,i})$$

for all $i \geq 0$. Increasing 0 if necessary we may assume that $\Delta_i = \det(\partial g_{\lambda,i}/\partial x_\mu)$ is invertible in B_i for all $i \geq 0$. Thus $A_i \to B_i$ is an étale ring map. After increasing 0 we may assume also that $\text{Spec}(B_i) \to \text{Spec}(A_i)$ is surjective, see Limits, Lemma 7.11. Increasing 0 yet again we may choose elements $h_{1,i}, \ldots, h_{n,i} \in \mathcal{O}_{U_i}(U_i)$ which map to the classes of $x_1, \ldots, x_n$ in $B = \mathcal{O}_U(U)$ and such that $g_{\lambda,i}(h_{\nu,i}) = 0$ in $\mathcal{O}_{U_i}(U_i)$. Thus we obtain a commutative diagram

$$(5.8.1) \qquad \begin{tikzcd} X_i \arrow[d] & U_i \arrow[l] \arrow[d] \\ \text{Spec}(A_i) & \text{Spec}(B_i) \arrow[l] \end{tikzcd}$$

By construction $B_i = B_0 \otimes_{A_0} A_i$ and $B = B_0 \otimes_{A_0} A$. Consider the morphism

$$f_0 : U_0 \longrightarrow X_0 \times_{\text{Spec}(A_0)} \text{Spec}(B_0)$$

This is a morphism of quasi-compact and quasi-separated algebraic spaces representable, separated and étale over X_0. The base change of f_0 to X is an isomorphism by our choices. Hence Lemma 5.6 guarantees that there exists an i such that the base change of f_0 to X_i is an isomorphism, in other words the diagram (5.8.1) is cartesian. Thus Descent, Lemma 35.1 applied to the fppf covering $\{\text{Spec}(B_i) \to \text{Spec}(A_i)\}$ combined with Descent, Lemma 33.1 give that $X_i \to \text{Spec}(A_i)$ is representable by a scheme affine over $\text{Spec}(A_i)$ as desired. (Of course it then also follows that $X_i = \text{Spec}(A_i)$ but we don't need this.) □

Lemma 5.9. *Notation and assumptions as in Situation 5.1. If X is a scheme, then there exists an i such that X_i is a scheme.*

Proof. Choose a finite affine open covering $X = \bigcup W_j$. By Lemma 5.5 we can find an $i \in I$ and open subspaces $W_{j,i} \subset X_i$ whose base change to X is $W_j \to X$. By Lemma 5.8 we may assume that each $W_{j,i}$ is an affine scheme. This means that X_i is a scheme (see for example Properties of Spaces, Section 10). □

Lemma 5.10. *Let S be a scheme. Let B be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over B with affine transition morphisms. Let $Y \to X$ be a morphism of algebraic spaces over B.*

(1) *If $Y \to X$ is a closed immersion, X_i quasi-compact, and $Y \to B$ locally of finite type, then $Y \to X_i$ is a closed immersion for i large enough.*
(2) *If $Y \to X$ is an immersion, X_i quasi-separated, $Y \to B$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is an immersion for i large enough.*
(3) *If $Y \to X$ is an isomorphism, X_i quasi-compact, $X_i \to B$ locally of finite type, the transition morphisms $X_{i'} \to X_i$ are closed immersions, and $Y \to B$ is locally of finite presentation, then $Y \to X_i$ is an isomorphism for i large enough.*
(4) *If $Y \to X$ is a monomorphism, X_i quasi-separated, $Y \to B$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is a monomorphism for i large enough.*

Proof. Proof of (1). Choose $0 \in I$. As X_0 is quasi-compact, we can choose an affine scheme W and an étale morphism $W \to B$ such that the image of $|X_0| \to |B|$ is contained in $|W| \to |B|$. Choose an affine scheme U_0 and an étale morphism $U_0 \to X_0 \times_B W$ such that $U_0 \to X_0$ is surjective. (This is possible by our choice of W and the fact that X_0 is quasi-compact; details omitted.) Let $V \to Y$, resp. $U \to X$, resp. $U_i \to X_i$ be the base change of $U_0 \to X_0$ (for $i \geq 0$). It suffices to prove that $V \to U_i$ is a closed immersion for i sufficiently large. Thus we reduce to proving the result for $V \to U = \lim U_i$ over W. This follows from the case of schemes, which is Limits, Lemma 3.13.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open subspace $X_0' \subset X_0$ such that $Y \to X_0$ factors through X_0'. After replacing X_i by the inverse image of X_0' for $i \geq 0$ we may assume all X_i' are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ (U exists as Y is quasi-compact). By Lemma 5.5 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some i. Thus (2) holds.

Proof of (3). Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$, $U = X \times_{X_0} U_0 = Y \times_{X_0} U_0$. Then $U = \lim U_i$ is a limit of affine schemes, the transition maps of the system are closed immersions, and $U \to U_0$ is of finite presentation (because $U \to B$ is locally of finite presentation and $U_0 \to B$ is locally of finite type and Morphisms of Spaces, Lemma 27.9). Thus we've reduced to the following algebra fact: If $A = \lim A_i$ is a directed colimit of R-algebras with surjective transition maps and A of finite presentation over A_0, then $A = A_i$ for some i. Namely, write $A = A_0/(f_1, \ldots, f_n)$. Pick i such that $f_1, \ldots, f_n$ map to zero under the surjective map $A_0 \to A_i$.

Proof of (4). Set $Z_i = Y \times_{X_i} Y$. As the transition morphisms $X_{i'} \to X_i$ are affine hence separated, the transition morphisms $Z_{i'} \to Z_i$ are closed immersions, see Morphisms of Spaces, Lemma 4.5. We have $\lim Z_i = Y \times_X Y = Y$ as $Y \to X$ is a monomorphism. Choose $0 \in I$. Since $Y \to X_0$ is locally of finite type (Morphisms of Spaces, Lemma 23.6) the morphism $Y \to Z_0$ is locally of finite presentation (Morphisms of Spaces, Lemma 27.10). The morphisms $Z_i \to Z_0$ are locally of finite type (they are closed immersions). Finally, $Z_i = Y \times_{X_i} Y$ is quasi-compact as X_i is quasi-separated and Y is quasi-compact. Thus part (3) applies to $Y = \lim_{i \geq 0} Z_i$ over Z_0 and we conclude $Y = Z_i$ for some i. This proves (4) and the lemma. $\square$

Lemma 5.11. *Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

(1) *Y is quasi-separated,*
(2) *X_i is quasi-compact and quasi-separated,*
(3) *the morphism $X \to Y$ is separated.*

Then $X_i \to Y$ is separated for all i large enough.

Proof. Let $0 \in I$. Choose an affine scheme W and an étale morphism $W \to Y$ such that the image of $|W| \to |Y|$ contains the image of $|X_0| \to |Y|$. This is possible as X_0 is quasi-compact. It suffices to check that $W \times_Y X_i \to W$ is separated for some $i \geq 0$ because the diagonal of $W \times_Y X_i$ over W is the base change of $X_i \to X_i \times_Y X_i$ by the surjective étale morphism $(X_i \times_Y X_i) \times_Y W \to X_i \times_Y X_i$. Since Y is quasi-separated the algebraic spaces $W \times_Y X_i$ are quasi-compact (as well as quasi-separated). Thus we may base change to W and assume Y is an affine scheme. When Y is an affine scheme, we have to show that X_i is a separated algebraic space for i large enough and we are given that X is a separated algebraic space. Thus this case follows from Lemma 5.7. □

Lemma 5.12. *Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

(1) *Y quasi-compact and quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *$X \to Y$ affine.*

Then $X_i \to Y$ is affine for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is affine and it suffices to check that $X_i \times_Y W$ is affine for some i (Morphisms of Spaces, Lemma 20.3). This follows from Lemma 5.8. □

Lemma 5.13. *Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

(1) *Y quasi-compact and quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *the transition morphisms $X_{i'} \to X_i$ are finite,*
(4) *$X_i \to Y$ locally of finite type*
(5) *$X \to Y$ integral.*

Then $X_i \to Y$ is finite for i large enough.

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is finite over W and it suffices to check that $X_i \times_Y W$ is finite over W for some i (Morphisms of Spaces, Lemma 41.3). By Lemma 5.9 this reduces us to the case of schemes. In the case of schemes it follows from Limits, Lemma 3.16. □

Lemma 5.14. *Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

(1) *Y quasi-compact and quasi-separated,*

(2) X_i *quasi-compact and quasi-separated,*
(3) *the transition morphisms* $X_{i'} \to X_i$ *are closed immersions,*
(4) $X_i \to Y$ *locally of finite type*
(5) $X \to Y$ *is a closed immersion.*

Then $X_i \to Y$ *is a closed immersion for* i *large enough.*

Proof. Choose an affine scheme W and a surjective étale morphism $W \to Y$. Then $X \times_Y W$ is a closed subspace of W and it suffices to check that $X_i \times_Y W$ is a closed subspace W for some i (Morphisms of Spaces, Lemma 12.1). By Lemma 5.9 this reduces us to the case of schemes. In the case of schemes it follows from Limits, Lemma 3.17. □

6. Descending properties of morphisms

This section is the analogue of Section 5 for properties of morphisms. We will work in the following situation.

Situation 6.1. Let S be a scheme. Let $B = \lim B_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms (Lemma 4.1). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of algebraic spaces over B_0. Assume B_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to B_i and let $f : X \to Y$ be the base change of f_0 to B.

Lemma 6.2. *With notation and assumptions as in Situation 6.1. If*
(1) f *is étale,*
(2) f_0 *is locally of finite presentation,*

then f_i *is étale for some* $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $X_i \to Y_i$ is étale if and only if $U_i \to V_i$ is étale and similarly $X \to Y$ is étale if and only if $U \to V$ is étale (Morphisms of Spaces, Definition 36.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 7.8. □

Lemma 6.3. *With notation and assumptions as in Situation 6.1. If*
(1) f *is surjective,*
(2) f_0 *is locally of finite presentation,*

then f_i is surjective for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, the limit of the morphisms $U_i \to V_i$ is $U \to V$, and the morphisms $U_i \to X_i \times_{Y_i} V_i$ and $U \to X \times_Y V$ are surjective (as base changes of $U_0 \to X_0 \times_{Y_0} V_0$). In particular, we see that $X_i \to Y_i$ is surjective if and only if $U_i \to V_i$ is surjective and similarly $X \to Y$ is surjective if and only if $U \to V$ is surjective. Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from the case of schemes (Limits, Lemma 7.11). □

Lemma 6.4. *Notation and assumptions as in Situation 6.1. If*

(1) *f is universally injective,*
(2) *f_0 is locally of finite type,*

then f_i is universally injective for some $i \geq 0$.

Proof. Recall that a morphism $X \to Y$ is universally injective if and only if the diagonal $X \to X \times_Y X$ is surjective (Morphisms of Spaces, Definition 19.3 and Lemma 19.2). Observe that $X_0 \to X_0 \times_{Y_0} X_0$ is of locally of finite presentation (Morphisms of Spaces, Lemma 27.10). Hence the lemma follows from Lemma 6.3 by considering the morphism $X_0 \to X_0 \times_{Y_0} X_0$. □

Lemma 6.5. *Notation and assumptions as in Situation 6.1. If f is affine, then f_i is affine for some $i \geq 0$.*

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is affine we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is affine. By Lemma 5.8 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. For this i the morphism f_i is affine (Morphisms of Spaces, Lemma 20.3). □

Lemma 6.6. *Notation and assumptions as in Situation 6.1. If*

(1) *f is finite,*
(2) *f_0 is locally of finite type,*

then f_i is finite for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is finite we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a scheme finite over V. By Lemma 5.8 we see that $V_i \times_{Y_i} X_i$ is affine for some

$i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \to V_i$ is finite by Limits, Lemma 7.3. For this i the morphism f_i is finite (Morphisms of Spaces, Lemma 41.3). □

Lemma 6.7. *Notation and assumptions as in Situation 6.1. If*

(1) *f is a closed immersion,*
(2) *f_0 is locally of finite type,*

then f_i is a closed immersion for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is a closed immersion we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a closed subscheme of the affine scheme V. By Lemma 5.8 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \to V_i$ is a closed immersion by Limits, Lemma 7.4. For this i the morphism f_i is a closed immersion (Morphisms of Spaces, Lemma 41.3). □

Lemma 6.8. *Notation and assumptions as in Situation 6.1. If f is separated, then f_i is separated for some $i \geq 0$.*

Proof. Apply Lemma 6.7 to the diagonal morphism $\Delta_{X_0/Y_0} : X_0 \to X_0 \times_{Y_0} X_0$. (Diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated. Some details omitted.) □

Lemma 6.9. *Notation and assumptions as in Situation 6.1. If*

(1) *f is a isomorphism,*
(2) *f_0 is locally of finite presentation,*

then f_i is a isomorphism for some $i \geq 0$.

Proof. Being an isomorphism is equivalent to being étale, universally injective, and surjective, see Morphisms of Spaces, Lemma 45.2. Thus the lemma follows from Lemmas 6.2, 6.3, and 6.4. □

Lemma 6.10. *Notation and assumptions as in Situation 6.1. If*

(1) *f is a monomorphism,*
(2) *f_0 is locally of finite type,*

then f_i is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism is a monomorphism if and only if the diagonal is an isomorphism. The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms of Spaces, Lemma 27.10. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated we conclude from Lemma 6.9 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. □

Lemma 6.11. *Notation and assumptions as in Situation 6.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module and denote $\mathcal{F}_i$ the pullback to X_i and $\mathcal{F}$ the pullback to X. If*

(1) *$\mathcal{F}$ is flat over Y,*
(2) *$\mathcal{F}_0$ is of finite presentation, and*
(3) *f_0 is locally of finite presentation,*

then $\mathcal{F}_i$ is flat over Y_i for some $i \geq 0$. In particular, if f_0 is locally of finite presentation and f is flat, then f_i is flat for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[d] \\ X_i \arrow[r] & Y_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $\mathcal{F}_i$ is flat over Y_i if and only if $\mathcal{F}_i|_{U_i}$ is flat over V_i and similarly $\mathcal{F}$ is flat over Y if and only if $\mathcal{F}|_U$ is flat over V (Morphisms of Spaces, Definition 28.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 9.3. □

Lemma 6.12. *Assumptions and notation as in Situation 6.1. If*

(1) *f is proper, and*
(2) *f_0 is locally of finite type,*

then there exists an i such that f_i is proper.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = Y_i \times_{Y_0} V_0$ and $V = Y \times_{Y_0} V_0$. It suffices to prove that the base change of f_i to V_i is proper, see Morphisms of Spaces, Lemma 37.2. Thus we may assume Y_0 is affine.

By Lemma 6.8 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact. Thus f_0 is separated and of finite type. By Cohomology of Spaces, Lemma 17.1 we can choose a diagram

$$\begin{tikzcd} X_0 \arrow[rd] & X'_0 \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{Y_0} \arrow[ld] \\ & Y_0 & \end{tikzcd}$$

where $X'_0 \to \mathbf{P}^n_{Y_0}$ is an immersion, and $\pi : X'_0 \to X_0$ is proper and surjective. Introduce $X' = X'_0 \times_{Y_0} Y$ and $X'_i = X'_0 \times_{Y_0} Y_i$. By Morphisms of Spaces, Lemmas 37.4 and 37.3 we see that $X' \to Y$ is proper. Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms of Spaces, Lemma 37.6). By Morphisms of Spaces, Lemma 37.7 it suffices to prove that $X'_i \to Y_i$ is proper for some i. By Lemma 6.7 we find that $X'_i \to \mathbf{P}^n_{Y_i}$ is a closed immersion for i large enough. Then $X'_i \to Y_i$ is proper and we win. □

7. Descending relative objects

The following lemma is typical of the type of results in this section.

Lemma 7.1. *Let S be a scheme. Let I be a directed partially ordered set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S. Assume*

(1) *the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine,*
(2) *the spaces X_i are quasi-compact and quasi-separated.*

Let $X = \lim_i X_i$. Then the category of algebraic spaces of finite presentation over X is the colimit over I of the categories of algebraic spaces of finite presentation over X_i.

Proof. Pick $0 \in I$. Choose a surjective étale morphism $U_0 \to X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$ the two projections. In the proof of Lemma 4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Let Y be an algebraic space over S and let $Y \to X$ be a morphism of finite presentation. Set $V = U \times_X Y$. This is an algebraic space of finite presentation over U. Choose an affine scheme W and a surjective étale morphism $W \to V$. Then $W \to Y$ is surjective étale as well. Set $R' = W \times_Y W$ so that $Y = W/R'$ (see Spaces, Section 9). Note that W is a scheme of finite presentation over U and that R' is a scheme of finite presentation over R (details omitted). By Limits, Lemma 9.1 we can find an index i and a morphism of schemes $W_i \to U_i$ of finite presentation whose base change to U gives $W \to U$. Similarly we can find, after possibly increasing i, a scheme R'_i of finite presentation over R_i whose base change to R is R'. The projection morphisms $s', t' : R' \to W$ are morphisms over the projection morphisms $s, t : R \to U$. Hence we can view s', resp. t' as a morphism between schemes of finite presentation over U (with structure morphism $R' \to U$ given by $R' \to R$ followed by s, resp. t). Hence we can apply Limits, Lemma 9.1 again to see that, after possibly increasing i, there exist morphisms $s'_i, t'_i : R'_i \to W_i$, whose base change to U is S', t'. By Limits, Lemmas 7.8 and 7.10 we may assume that s'_i, t'_i are étale and that $j'_i : R'_i \to W_i \times_{X_i} W_i$ is a monomorphism (here we view j'_i as a morphism of schemes of finite presentation over U_i via one of the projections – it doesn't matter which one). Setting $Y_i = W_i/R'_i$ (see Spaces, Theorem 10.5) we obtain an algebraic space of finite presentation over X_i whose base change to X is isomorphic to Y.

This shows that every algebraic space of finite presentation over X comes from an algebraic space of finite presentation over some X_i, i.e., it shows that the functor of the lemma is essentially surjective. To show that it is fully faithful, consider an index $0 \in I$ and two algebraic spaces Y_0, Z_0 of finite presentation over X_0. Set $Y_i = X_i \times_{X_0} Y_0$, $Y = X \times_{X_0} Y_0$, $Z_i = X_i \times_{X_0} Z_0$, and $Z = X \times_{X_0} Z_0$. Let $\alpha : Y \to Z$ be a morphism of algebraic spaces over X. Choose a surjective étale morphism $V_0 \to Y_0$ where V_0 is an affine scheme. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$ which are affine schemes endowed with surjective étale morphisms to Y_i and Y. The composition $V \to Y \to Z \to Z_0$ comes from a (essentially unique) morphism $V_i \to Z_0$ for some $i \geq 0$ by Proposition 3.9 (applied to $Z_0 \to X_0$ which is of finite presentation by assumption). After increasing i the two compositions

$$V_i \times_{Y_i} V_i \to V_i \to Z_0$$

are equal as this is true in the limit. Hence we obtain a (essentially unique) morphism $Y_i \to Z_0$. Since this is a morphism over X_0 it induces a morphism into $Z_i = Z_0 \times_{X_0} X_i$ as desired. □

Lemma 7.2. *With notation and assumptions as in Lemma 7.1. The category of $\mathcal{O}_X$-modules of finite presentation is the colimit over I of the categories $\mathcal{O}_{X_i}$-modules of finite presentation.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$ the two projections. In the proof of Lemma 4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Moreover, it is also true that $R \times_{s,U,t} R = \text{colim}\, R_i \times_{s_i,U_i,t_i} R_i$. Thus we know that $QCoh(\mathcal{O}_U) = \text{colim}\, QCoh(\mathcal{O}_{U_i})$, $QCoh(\mathcal{O}_R) = \text{colim}\, QCoh(\mathcal{O}_{R_i})$, and $QCoh(\mathcal{O}_{R \times_{s,U,t} R}) = \text{colim}\, QCoh(\mathcal{O}_{R_i \times_{s_i,U_i,t_i} R_i})$ by Limits, Lemma 9.2. We have $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$ and $QCoh(\mathcal{O}_{X_i}) = QCoh(U_i, R_i, s_i, t_i, c_i)$, see Properties of Spaces, Proposition 30.1. Thus the result follows formally. □

8. Absolute Noetherian approximation

The following result is [CLO12, Theorem 1.2.2]. A key ingredient in the proof is Decent Spaces, Lemma 8.5.

Proposition 8.1. *Let X be a quasi-compact and quasi-separated algebraic space over* $\text{Spec}(\mathbf{Z})$*. There exist a directed partially ordered set I and an inverse system of algebraic spaces $(X_i, f_{ii'})$ over I such that*

1. *the transition morphisms $f_{ii'}$ are affine*
2. *each X_i is quasi-separated and of finite type over* $\mathbf{Z}$*, and*
3. *$X = \lim X_i$.*

Proof. We apply Decent Spaces, Lemma 8.5 to get open subspaces $U_p \subset X$, schemes V_p, and morphisms $f_p : V_p \to U_p$ with properties as stated. Note that $f_n : V_n \to U_n$ is an étale morphism of algebraic spaces whose restriction to the inverse image of $T_n = (V_n)_{red}$ is an isomorphism. Hence f_n is an isomorphism, for example by Morphisms of Spaces, Lemma 45.2. In particular U_n is a quasi-compact and separated scheme. Thus we can write $U_n = \lim U_{n,i}$ as a directed limit of schemes of finite type over $\mathbf{Z}$ with affine transition morphisms, see Limits, Proposition 4.4. Thus, applying descending induction on p, we see that we have reduced to the problem posed in the following paragraph.

Here we have $U \subset X$, $U = \lim U_i$, $Z \subset X$, and $f : V \to X$ with the following properties

1. X is a quasi-compact and quasi-separated algebraic space,
2. V is a quasi-compact and separated scheme,
3. $U \subset X$ is a quasi-compact open subspace,
4. $(U_i, g_{ii'})$ is a directed system of quasi-separated algebraic spaces of finite type over $\mathbf{Z}$ with affine transition morphisms whose limit is U,

(5) $Z \subset X$ is a closed subspace such that $|X| = |U| \amalg |Z|$,
(6) $f : V \to X$ is a surjective étale morphism such that $f^{-1}(Z) \to Z$ is an isomorphism.

Problem: Show that the conclusion of the proposition holds for X.

Note that $W = f^{-1}(U) \subset V$ is a quasi-compact open subscheme étale over U. Hence we may apply Lemmas 7.1 and 6.2 to find an index $0 \in I$ and an étale morphism $W_0 \to U_0$ of finite presentation whose base change to U produces W. Setting $W_i = W_0 \times_{U_0} U_i$ we see that $W = \lim_{i \geq 0} W_i$. After increasing 0 we may assume the W_i are schemes, see Lemma 5.9. Moreover, W_i is of finite type over $\mathbf{Z}$.

Apply Limits, Lemma 4.3 to $W = \lim_{i \geq 0} W_i$ and the inclusion $W \subset V$. Replace I by the directed partially ordered set J found in that lemma. This allows us to write V as a directed limit $V = \lim V_i$ of finite type schemes over $\mathbf{Z}$ with affine transition maps such that each V_i contains W_i as an open subscheme (compatible with transition morphisms). For each i we can form the push out

$$\begin{tikzcd} W_i \arrow[r] \arrow[d, "\Delta"'] & V_i \arrow[d] \\ W_i \times_{U_i} W_i \arrow[r] & R_i \end{tikzcd}$$

in the category of schemes. Namely, the left vertical and upper horizontal arrows are open immersions of schemes. In other words, we can construct R_i as the glueing of V_i and $W_i \times_{U_i} W_i$ along the common open W_i (see Schemes, Section 14). Note that the étale projection maps $W_i \times_{U_i} W_i \to W_i$ extend to étale morphisms $s_i, t_i : R_i \to V_i$. It is clear that the morphism $j_i = (t_i, s_i) : R_i \to V_i \times V_i$ is an étale equivalence relation on V_i. Note that $W_i \times_{U_i} W_i$ is quasi-compact (as U_i is quasi-separated and W_i quasi-compact) and V_i is quasi-compact, hence R_i is quasi-compact. For $i \geq i'$ the diagram

$$\begin{tikzcd} R_i \arrow[r] \arrow[d, "s_i"'] & R_{i'} \arrow[d, "s_{i'}"] \\ V_i \arrow[r] & V_{i'} \end{tikzcd} \tag{8.1.1}$$

is cartesian because

$$(W_{i'} \times_{U_{i'}} W_{i'}) \times_{U_{i'}} U_i = W_{i'} \times_{U_{i'}} U_i \times_{U_i} U_i \times_{U_{i'}} W_{i'} = W_i \times_{U_i} W_i.$$

Consider the algebraic space $X_i = V_i/R_i$ (see Spaces, Theorem 10.5). As V_i is of finite type over $\mathbf{Z}$ and R_i is quasi-compact we see that X_i is quasi-separated and of finite type over $\mathbf{Z}$ (see Properties of Spaces, Lemma 6.5 and Morphisms of Spaces, Lemmas 8.5 and 23.4). As the construction of R_i above is compatible with transition morphisms, we obtain morphisms of algebraic spaces $X_i \to X_{i'}$ for $i \geq i'$. The commutative diagrams

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & V_{i'} \arrow[d] \\ X_i \arrow[r] & X_{i'} \end{tikzcd}$$

are cartesian as (8.1.1) is cartesian, see Groupoids, Lemma 18.7. Since $V_i \to V_{i'}$ is affine, this implies that $X_i \to X_{i'}$ is affine, see Morphisms of Spaces, Lemma 20.3.

Thus we can form the limit $X' = \lim X_i$ by Lemma 4.1. We claim that $X \cong X'$ which finishes the proof of the proposition.

Proof of the claim. Set $R = \lim R_i$. By construction the algebraic space X' comes equipped with a surjective étale morphism $V \to X'$ such that

$$V \times_{X'} V \cong R$$

(use Lemma 4.1). By construction $\lim W_i \times_{U_i} W_i = W \times_U W$ and $V = \lim V_i$ so that R is the union of $W \times_U W$ and V glued along W. Property (6) implies the projections $V \times_X V \to V$ are isomorphisms over $f^{-1}(Z) \subset V$. Hence the scheme $V \times_X V$ is the union of the opens $\Delta_{V/X}(V)$ and $W \times_U W$ which intersect along $\Delta_{W/X}(W)$. We conclude that there exists a unique isomorphism $R \cong V \times_X V$ compatible with the projections to V. Since $V \to X$ and $V \to X'$ are surjective étale we see that

$$X = V/V \times_X V = V/R = V/V \times_{X'} V = X'$$

by Spaces, Lemma 9.1 and we win. $\square$

9. Applications

The following lemma can also be deduced directly from Decent Spaces, Lemma 8.5 without passing through absolute Noetherian approximation.

Lemma 9.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Every quasi-coherent $\mathcal{O}_X$-module is a filtered colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. We may view as an algebraic space over $\text{Spec}(\mathbf{Z})$, see Spaces, Definition 16.2 and Properties of Spaces, Definition 3.1. Thus we may apply Proposition 8.1 and write $X = \lim X_i$ with X_i of finite presentation over $\mathbf{Z}$. Thus X_i is a Noetherian algebraic space, see Morphisms of Spaces, Lemma 27.6. The morphism $X \to X_i$ is affine, see Lemma 4.1. Conclusion by Cohomology of Spaces, Lemma 14.2. $\square$

The rest of this section consists of straightforward applications of Lemma 9.1.

Lemma 9.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is the directed colimit of its finite type quasi-coherent submodules.*

Proof. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are finite type quasi-coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another finite type quasi-coherent $\mathcal{O}_X$-submodule which contains both of them. In this way we see that the system is directed. To show that $\mathcal{F}$ is the colimit of this system, write $\mathcal{F} = \text{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 9.1. Then the images $\mathcal{G}_i = \text{Im}(\mathcal{F}_i \to \mathcal{F})$ are finite type quasi-coherent subsheaves of $\mathcal{F}$. Since $\mathcal{F}$ is the colimit of these the result follows. $\square$

Lemma 9.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \lim \mathcal{F}_i$ where each $\mathcal{F}_i$ is an $\mathcal{O}_X$-module of finite presentation and all transition maps $\mathcal{F}_i \to \mathcal{F}_{i'}$ surjective.*

Proof. Write $\mathcal{F} = \operatorname{colim} \mathcal{G}_i$ as a filtered colimit of finitely presented $\mathcal{O}_X$-modules (Lemma 9.1). We claim that $\mathcal{G}_i \to \mathcal{F}$ is surjective for some i. Namely, choose an étale surjection $U \to X$ where U is an affine scheme. Choose finitely many sections $s_k \in \mathcal{F}(U)$ generating $\mathcal{F}|_U$. Since U is affine we see that s_k is in the image of $\mathcal{G}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{G}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{G}_i$ be the kernel of the map $\mathcal{G}_i \to \mathcal{F}$. Write $\mathcal{K} = \operatorname{colim} \mathcal{K}_a$ as the filtered colimit of its finite type quasi-coherent submodules (Lemma 9.2). Then $\mathcal{F} = \operatorname{colim} \mathcal{G}_i/\mathcal{K}_a$ is a solution to the problem posed by the lemma. □

Let X be an algebraic space. In the following lemma we use the notion of a *finitely presented quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine $U = \operatorname{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite presentation as an R-algebra.

Lemma 9.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is a directed colimit of finitely presented quasi-coherent $\mathcal{O}_X$-algebras.*

Proof. First we write $\mathcal{A} = \operatorname{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 9.1. For each i let $\mathcal{B}_i = \operatorname{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\mathcal{I}_i = \operatorname{Ker}(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = \operatorname{colim}_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 9.2. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \operatorname{colim}_{i,j} \mathcal{A}_{i,j}$. □

Let X be an algebraic space. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type*. This means that for every affine $U = \operatorname{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite type as an R-algebra.

Lemma 9.5. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.*

Proof. Omitted. Hint: Compare with the proof of Lemma 9.2. □

Let X be an algebraic space. In the following lemma we use the notion of a *finite (resp. integral) quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine $U = \operatorname{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is finite (resp. integral) as an R-algebra.

Lemma 9.6. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a finite quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ is a directed colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras with surjective transition maps.*

Proof. By Lemma 9.3 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ and a surjection $\mathcal{F} \to \mathcal{A}$. Using the algebra structure we obtain a surjection

$$\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{A}$$

Denote $\mathcal{J}$ the kernel. Write $\mathcal{J} = \operatorname{colim} \mathcal{E}_i$ as a filtered colimit of finite type $\mathcal{O}_X$-submodules $\mathcal{E}_i$ (Lemma 9.2). Set

$$\mathcal{A}_i = \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F})/(\mathcal{E}_i)$$

where $(\mathcal{E}_i)$ indicates the ideal sheaf generated by the image of $\mathcal{E}_i \to \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F})$. Then each $\mathcal{A}_i$ is a finitely presented $\mathcal{O}_X$-algebra, the transition maps are surjective, and $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$. To finish the proof we still have to show that $\mathcal{A}_i$ is a finite $\mathcal{O}_X$-algebra for i sufficiently large. To do this we choose an étale surjective map $U \to X$ where U is an affine scheme. Take generators $f_1, \ldots, f_m \in \Gamma(U, \mathcal{F})$. As $\mathcal{A}(U)$ is a finite $\mathcal{O}_X(U)$-algebra we see that for each j there exists a monic polynomial $P_j \in \mathcal{O}(U)[T]$ such that $P_j(f_j)$ is zero in $\mathcal{A}(U)$. Since $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ by construction, we have $P_j(f_j) = 0$ in $\mathcal{A}_i(U)$ for all sufficiently large i. For such i the algebras $\mathcal{A}_i$ are finite. $\square$

Lemma 9.7. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be an integral quasi-coherent $\mathcal{O}_X$-algebra. Then*

1. *$\mathcal{A}$ is the directed colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras, and*
2. *$\mathcal{A}$ is a directed colimit of finite and finitely presented $\mathcal{O}_X$-algebras.*

Proof. By Lemma 9.5 we have $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}$ runs through the quasi-coherent $\mathcal{O}_X$-sub algebras of finite type. Any finite type quasi-coherent $\mathcal{O}_X$-subalgebra of $\mathcal{A}$ is finite (use Algebra, Lemma 35.5 on affine schemes étale over X). This proves (1).

To prove (2), write $\mathcal{A} = \operatorname{colim} \mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_X$-modules using Lemma 9.1. For each i, let $\mathcal{J}_i$ be the kernel of the map

$$\text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$ and we have $\mathcal{A} = \operatorname{colim} \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (see above). Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{tikzcd}
\mathcal{J}_i \arrow[r] & \mathcal{J}_{i'} \\
\mathcal{E}_{ik} \arrow[r] \arrow[u] & \mathcal{E}_{i'k'} \arrow[u]
\end{tikzcd}$$

commute. This follows from Cohomology of Spaces, Lemma 4.3. This induces a map

$$\mathcal{A}_{ik} = \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for k large enough (see proof of Lemma 9.6). Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 9.6 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. $\square$

Lemma 9.8. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of* *finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. Denote $j : U \to X$ the inclusion morphism. As X is quasi-separated and U quasi-compact, the morphism j is quasi-compact. Hence $j_*\mathcal{G} \subset j_*\mathcal{F}|_U$ are quasi-coherent modules on X (Morphisms of Spaces, Lemma 11.2). Let $\mathcal{H} = \mathrm{Ker}(j_*\mathcal{G} \oplus \mathcal{F} \to j_*\mathcal{F}|_U)$. Then $\mathcal{H}|_U = \mathcal{G}$. By Lemma 9.2 we can find a finite type quasi-coherent submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{H}|_U = \mathcal{G}$. Set $\mathcal{G}' = \mathrm{Im}(\mathcal{H}' \to \mathcal{F})$ to conclude. □

10. Relative approximation

The title of this section refers to the following result.

Lemma 10.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

1. *X is quasi-compact and quasi-separated, and*
2. *Y is quasi-separated.*

Then $X = \lim X_i$ is a limit of a directed system of algebraic spaces X_i of finite presentation over Y with affine transition morphisms over Y.

Proof. Since $|f|(|X|)$ is quasi-compact we may replace Y by a quasi-compact open subspace whose set of points contains $|f|(|X|)$. Hence we may assume Y is quasi-compact as well. Write $X = \lim X_a$ and $Y = \lim Y_b$ as in Proposition 8.1, i.e., with X_a and Y_b of finite type over $\mathbf{Z}$ and with affine transition morphisms. By Proposition 3.9 we find that for each b there exists an a and a morphism $f_{a,b} : X_a \to Y_b$ making the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_a & \longrightarrow & Y_b \end{array}$$

commute. Moreover the same proposition implies that, given a second triple $(a', b', f_{a',b'})$, there exists an $a'' \geq a'$ such that the compositions $X_{a''} \to X_a \to X_b$ and $X_{a''} \to X_{a'} \to X_{b'} \to X_b$ are equal. Consider the set of triples $(a, b, f_{a,b})$ endowed with the partial ordering

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a',\ b' \geq b, \text{ and } f_{a',b'} \circ h_{a,a'} = g_{b',b} \circ f_{a,b}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b',b} : Y_{b'} \to Y_b$ are the transition morphisms. The remarks above show that this system is directed. It follows formally from the equalities $X = \lim X_a$ and $Y = \lim Y_b$ that

$$X = \lim\nolimits_{(a,b,f_{a,b})} X_a \times_{f_{a,b},Y_b} Y.$$

where the limit is over our directed system above. The transition morphisms $X_a \times_{Y_b} Y \to X_{a'} \times_{Y_{b'}} Y$ are affine as the composition

$$X_a \times_{Y_b} Y \to X_a \times_{Y_{b'}} Y \to X_{a'} \times_{Y_{b'}} Y$$

where the first morphism is a closed immersion (by Morphisms of Spaces, Lemma 4.5) and the second is a base change of an affine morphism (Morphisms of Spaces, Lemma 20.5) and the composition of affine morphisms is affine (Morphisms of Spaces, Lemma 20.4). The morphisms $f_{a,b}$ are of finite presentation (Morphisms of

Spaces, Lemmas 27.7 and 27.9) and hence the base changes $X_a \times_{f_{a,b}, S_b} S \to S$ are of finite presentation (Morphisms of Spaces, Lemma 27.3). $\square$

11. Finite type closed in finite presentation

This section is the analogue of Limits, Section 8.

Lemma 11.1. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. If Y quasi-compact and quasi-separated, then X is a directed limit $X = \lim X_i$ with each X_i affine and of finite presentation over Y.*

Proof. Consider the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 9.4 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$. Set $X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 20.8. By construction $X_i \to Y$ is affine and of finite presentation and $X = \lim X_i$. $\square$

Lemma 11.2. *Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where X_i are finite and of finite presentation over Y.*

Proof. Consider the finite quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 9.7 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$. Set $X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 20.8. By construction $X_i \to Y$ is finite and of finite presentation and $X = \lim X_i$. $\square$

Lemma 11.3. *Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over Y.*

Proof. Consider the finite quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 9.6 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$ with surjective transition maps. Set $X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 20.8. By construction $X_i \to Y$ is finite and of finite presentation, the transition maps are closed immersions, and $X = \lim X_i$. $\square$

Lemma 11.4. *Let S be a scheme. Let $f : X \to Y$ be a closed immersion of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the morphisms $X_i \to Y$ are closed immersions of finite presentation.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subspace of Y. By Lemma 9.2 we can write $\mathcal{I} = \operatorname{colim} \mathcal{I}_i$ as the filtered colimit of its finite type quasi-coherent submodules. Let X_i be the closed subspace of X cut out by $\mathcal{I}_i$. Then $X_i \to Y$ is a closed immersion of finite presentation, and $X = \lim X_i$. Some details omitted. $\square$

Lemma 11.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

1. *f is locally of finite type and quasi-affine, and*
2. *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ over Y.

Proof. By Morphisms of Spaces, Lemma 21.6 we can find a factorization $X \to Z \to Y$ where $X \to Z$ is a quasi-compact open immersion and $Z \to Y$ is affine. Write $Z = \lim Z_i$ with Z_i affine and of finite presentation over Y (Lemma 11.1). For some $0 \in I$ we can find a quasi-compact open $U_0 \subset Z_0$ such that X is isomorphic to the inverse image of U_0 in Z (Lemma 5.5). Let U_i be the inverse image of U_0 in Z_i, so $U = \lim U_i$. By Lemma 5.10 we see that $X \to U_i$ is a closed immersion for some i large enough. Setting $X' = U_i$ finishes the proof. $\square$

Lemma 11.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume:*

(1) *f is of locally of finite type.*
(2) *X is quasi-compact and quasi-separated, and*
(3) *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ of algebraic spaces over Y.

Proof. By Proposition 8.1 we can write $X = \lim_i X_i$ with X_i quasi-separated of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[rd] & X_{i,Y} \arrow[r] \arrow[d] & X_i \arrow[d] \\ & Y \arrow[r] & \mathrm{Spec}(\mathbf{Z}) \end{tikzcd}$$

Note that X_i is of finite presentation over $\mathrm{Spec}(\mathbf{Z})$, see Morphisms of Spaces, Lemma 27.7. Hence the base change $X_{i,Y} \to Y$ is of finite presentation by Morphisms of Spaces, Lemma 27.3. Observe that $\lim X_{i,Y} = X \times Y$ and that $X \to X \times Y$ is a monomorphism. By Lemma 5.10 we see that $X \to X_{i,Y}$ is a monomorphism for i large enough. Fix such an i. Note that $X \to X_{i,Y}$ is locally of finite type (Morphisms of Spaces, Lemma 23.6) and a monomorphism, hence separated and locally quasi-finite (Morphisms of Spaces, Lemma 26.10). Hence $X \to X_{i,Y}$ is representable. Hence $X \to X_{i,Y}$ is quasi-affine because we can use the principle Spaces, Lemma 5.8 and the result for morphisms of schemes More on Morphisms, Lemma 31.2. Thus Lemma 11.5 gives a factorization $X \to X' \to X_{i,Y}$ with $X \to X'$ a closed immersion and $X' \to X_{i,Y}$ of finite presentation. Finally, $X' \to Y$ is of finite presentation as a composition of morphisms of finite presentation (Morphisms of Spaces, Lemma 27.2). $\square$

Proposition 11.7. *Let S be a scheme. $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is of finite type and separated, and*
(2) *Y is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ over Y.

Proof. By Lemma 11.6 there is a closed immersion $X \to Z$ with Z/Y of finite presentation. Let $\mathcal{I} \subset \mathcal{O}_Z$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Lemma 9.2 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = \mathrm{colim}_{a \in A} \mathcal{I}_a$ of its quasi-coherent sheaves of ideals of finite type. Let $X_a \subset Z$ be the closed subspace defined by $\mathcal{I}_a$. These form an inverse system indexed by A. The transition morphisms $X_a \to X_{a'}$ are affine because they are closed immersions. Each X_a is quasi-compact and quasi-separated since it is a closed subspace of Z and Z is quasi-compact and quasi-separated by our assumptions. We have $X = \lim_a X_a$ as follows directly from the fact that $\mathcal{I} = \operatorname{colim}_{a \in A} \mathcal{I}_a$. Each of the morphisms $X_a \to Z$ is of finite presentation, see Morphisms, Lemma 22.7. Hence the morphisms $X_a \to Y$ are of finite presentation. Thus it suffices to show that $X_a \to Y$ is separated for some $a \in A$. This follows from Lemma 5.11 as we have assumed that $X \to Y$ is separated. $\square$

12. Approximating proper morphisms

Lemma 12.1. *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y quasi-compact and quasi-separated. Then $X = \lim X_i$ with $X_i \to Y$ proper and of finite presentation.*

Proof. By Proposition 11.7 we can find a closed immersion $X \to X'$ with X' separated and of finite presentation over Y. By Lemma 11.4 we can write $X = \lim X_i$ with $X_i \to X'$ a closed immersion of finite presentation. We claim that for all i large enough the morphism $X_i \to Y$ is proper which finishes the proof.

To prove this we may assume that Y is an affine scheme, see Morphisms of Spaces, Lemma 37.2. Next, we use the weak version of Chow's lemma, see Cohomology of Spaces, Lemma 17.1, to find a diagram

$$\begin{tikzcd} X' \arrow[dr] & X'' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where $X'' \to \mathbf{P}^n_Y$ is an immersion, and $\pi : X'' \to X'$ is proper and surjective. Denote $X'_i \subset X''$, resp. $\pi^{-1}(X)$ the scheme theoretic inverse image of $X_i \subset X'$, resp. $X \subset X'$. Then $\lim X'_i = \pi^{-1}(X)$. Since $\pi^{-1}(X) \to Y$ is proper (Morphisms of Spaces, Lemmas 37.4), we see that $\pi^{-1}(X) \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms of Spaces, Lemmas 37.6 and 12.3). Hence for i large enough we find that $X'_i \to \mathbf{P}^n_Y$ is a closed immersion by Lemma 5.14. Thus X'_i is proper over Y. For such i the morphism $X_i \to Y$ is proper by Morphisms of Spaces, Lemma 37.7. $\square$

Lemma 12.2. *Let $f : X \to Y$ be a proper morphism of algebraic spaces over $\mathbf{Z}$ with Y quasi-compact and quasi-separated. Then $(X \to Y) = \lim(X_i \to Y_i)$ with Y_i of finite presentation over $\mathbf{Z}$ and $X_i \to Y_i$ proper and of finite presentation.*

Proof. By Lemma 12.1 we can write $X = \lim_{k \in K} X_k$ with $X_k \to Y$ proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 8.1) we can write $Y = \lim_{j \in J} Y_j$ with Y_j of finite presentation over $\mathbf{Z}$. For each k there exists a j and a morphism $X_{k,j} \to Y_j$ of finite presentation with $X_k \cong Y \times_{Y_j} X_{k,j}$ as algebraic spaces over Y, see Lemma 7.1. After increasing j we may assume $X_{k,j} \to Y_j$ is proper, see Lemma 6.12. The set I will be consist of these pairs (k, j) and the corresponding morphism is $X_{k,j} \to Y_j$. For every $k' \geq k$ we can find a $j' \geq j$ and a morphism $X_{j',k'} \to X_{j,k}$ over $Y_{j'} \to Y_j$ whose base change to Y gives the morphism $X_{k'} \to X_k$ (follows again from Lemma 7.1). These morphisms form the transition morphisms of the system. Some details omitted. $\square$

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms of Spaces, Definition 15.4.

Lemma 12.3. *Assumptions and notation as in Situation 6.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module. Denote $\mathcal{F}$ and $\mathcal{F}_i$ the pullbacks of $\mathcal{F}_0$ to X and X_i. Assume*

1. *f_0 is locally of finite type,*
2. *$\mathcal{F}_0$ is of finite type,*
3. *the scheme theoretic support of $\mathcal{F}$ is proper over Y.*

Then the scheme theoretic support of $\mathcal{F}_i$ is proper over Y_i for some i.

Proof. We may replace X_0 by the scheme theoretic support of $\mathcal{F}_0$. By Morphisms of Spaces, Lemma 15.2 this guarantees that X_i is the support of $\mathcal{F}_i$ and X is the support of $\mathcal{F}$. Then, if $Z \subset X$ denotes the scheme theoretic support of $\mathcal{F}$, we see that $Z \to X$ is a universal homeomorphism. We conclude that $X \to Y$ is proper as this is true for $Z \to Y$ by assumption, see Morphisms, Lemma 42.8. By Lemma 6.12 we see that $X_i \to Y$ is proper for some i. Then it follows that the scheme theoretic support Z_i of $\mathcal{F}_i$ is proper over Y by Morphisms of Spaces, Lemmas 37.5 and 37.4. □

13. Embedding into affine space

Some technical lemmas to be used in the proof of Chow's lemma later.

Lemma 13.1. *Let S be a scheme. Let $f : U \to X$ be a morphism of algebraic spaces over S. Assume U is an affine scheme, f is locally of finite type, and X quasi-separated and locally separated. Then there exists an immersion $U \to \mathbf{A}^n_X$ over X.*

Proof. Say $U = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}\, A_i$ as a filtered colimit of finite type $\mathbf{Z}$-subalgebras. For each i the morphism $U \to U_i = \mathrm{Spec}(A_i)$ induces a morphism

$$U \longrightarrow X \times U_i$$

over X. In the limit the morphism $U \to X \times U$ is an immersion as X is locally separated, see Morphisms of Spaces, Lemma 4.6. By Lemma 5.10 we see that $U \to X \times U_i$ is an immersion for some i. Since U_i is isomorphic to a closed subscheme of $\mathbf{A}^n_{\mathbf{Z}}$ the lemma follows. □

Remark 13.2. We have seen in Examples, Section 22 that Lemma 13.1 does not hold if we drop the assumption that X be locally separated. This raises the question: Does Lemma 13.1 hold if we drop the assumption that X be quasi-separated? If you know the answer, please email stacks.project@gmail.com.

Lemma 13.3. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume X Noetherian and f of finite presentation. Then there exists a dense open $V \subset Y$ and an immersion $V \to \mathbf{A}^n_X$.*

Proof. The assumptions imply that Y is Noetherian (Morphisms of Spaces, Lemma 27.6). Then Y is quasi-separated, hence has a dense open subscheme (Properties of Spaces, Proposition 10.3). Thus we may assume that Y is a Noetherian scheme. By removing intersections of irreducible components of Y (use Topology, Lemma 8.2 and Properties, Lemma 5.5) we may assume that Y is a disjoint union of irreducible Noetherian schemes. Since there is an immersion

$$\mathbf{A}^n_X \amalg \mathbf{A}^m_X \longrightarrow \mathbf{A}^{\max(n,m)+1}_X$$

(details omitted) we see that it suffices to prove the result in case Y is irreducible.

Assume Y is an irreducible scheme. Let $T \subset |X|$ be the closure of the image of $f : Y \to X$. Note that since $|Y|$ and $|X|$ are sober topological spaces (Properties of Spaces, Lemma 12.4) T is irreducible with a unique generic point ξ which is the image of the generic point η of Y. Let $\mathcal{I} \subset X$ be a quasi-coherent sheaf of ideals cutting out the reduced induced space structure on T (Properties of Spaces, Definition 9.5). Since $\mathcal{O}_{Y,\eta}$ is an Artinian local ring we see that for some $n > 0$ we have $f^{-1}\mathcal{I}^n\mathcal{O}_{Y,\eta} = 0$. As $f^{-1}\mathcal{I}\mathcal{O}_Y$ is a finite type quasi-coherent ideal we conclude that $f^{-1}\mathcal{I}^n\mathcal{O}_V = 0$ for some nonempty open $V \subset Y$. Let $Z \subset X$ be the closed subspace cut out by $\mathcal{I}^n$. By construction $V \to Y \to X$ factors through Z. Because $\mathbf{A}^n_Z \to \mathbf{A}^n_X$ is an immersion, we may replace X by Z and Y by V. Hence we reach the situation where Y and X are irreducible and $Y \to X$ maps the generic point of Y onto the generic point of X.

Assume Y and X are irreducible, Y is a scheme, and $Y \to X$ maps the generic point of Y onto the generic point of X. By Properties of Spaces, Proposition 10.3 X has a dense open subscheme $U \subset X$. Choose a nonempty affine open $V \subset Y$ whose image in X is contained in U. By Morphisms, Lemma 40.2 we may factor $V \to U$ as $V \to \mathbf{A}^n_U \to U$. Composing with $\mathbf{A}^n_U \to \mathbf{A}^n_X$ we obtain the desired immersion. $\square$

14. Sections with support in a closed subset

This section is the analogue of Properties, Section 22.

Lemma 14.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U \subset X$ be an open subspace. The following are equivalent:*

1. *$U \to X$ is quasi-compact,*
2. *U is quasi-compact, and*
3. *there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $|X| \setminus |U| = |V(\mathcal{I})|$.*

Proof. Let W be an affine scheme and let $\varphi : W \to X$ be a surjective étale morphism, see Properties of Spaces, Lemma 6.3. If (1) holds, then $\varphi^{-1}(U) \to W$ is quasi-compact, hence $\varphi^{-1}(U)$ is quasi-compact, hence U is quasi-compact (as $|\varphi^{-1}(U)| \to |U|$ is surjective). If (2) holds, then $\varphi^{-1}(U)$ is quasi-compact because φ is quasi-compact since X is quasi-separated (Morphisms of Spaces, Lemma 8.9). Hence $\varphi^{-1}(U) \to W$ is a quasi-compact morphism of schemes by Properties, Lemma 22.1. It follows that $U \to X$ is quasi-compact by Morphisms of Spaces, Lemma 8.7. Thus (1) and (2) are equivalent.

Assume (1) and (2). By Properties of Spaces, Lemma 9.3 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subspace structure on $|X| \setminus |U|$. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. As U is quasi-compact it follows from Lemma 9.2 that there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. Then $|X| \setminus |U| = |V(\mathcal{I})|$ and we obtain (3). Conversely, if $\mathcal{I}$ is as in (3), then $\varphi^{-1}(U) \subset W$ is a quasi-compact open by the lemma for schemes (Properties, Lemma 22.1) applied to $\varphi^{-1}\mathcal{I}$ on W. Thus (2) holds. $\square$

Lemma 14.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every object U of $X_{\acute{e}tale}$ the module*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume $\mathcal{I}$ is of finite type. Then

1. *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,*
2. *for affine U in $X_{\acute{e}tale}$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and*
3. *$\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$. Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 22.2. $\square$

Definition 14.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 14.2 above is called the *subsheaf of sections annihilated by $\mathcal{I}$*.

Lemma 14.4. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections annihilated by $f^{-1}\mathcal{I}\mathcal{O}_X$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections annihilated by $\mathcal{I}$.*

Proof. Omitted. Hint: The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent (Morphisms of Spaces, Lemma 11.2) so that Lemma 14.2 applies to $\mathcal{I}$ and $f_*\mathcal{F}$. $\square$

Next we come to the sheaf of sections supported in a closed subset. Again this isn't always a quasi-coherent sheaf, but if the complement of the closed is "retrocompact" in the given algebraic space, then it is.

Lemma 14.5. *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset and let $U \subset X$ be the open subspace such that $T \amalg |U| = |X|$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every object $\varphi : W \to X$ of $X_{\acute{e}tale}$ the module*

$$\mathcal{F}'(W) = \{s \in \mathcal{F}(W) \mid \text{the support of } s \text{ is contained in } |\varphi|^{-1}(T)\}$$

If $U \to X$ is quasi-compact, then

1. *for W affine there exist a finitely generated ideal $I \subset \mathcal{O}_X(W)$ such that $|\varphi|^{-1}(T) = V(I)$,*
2. *for W and I as in (1) we have $\mathcal{F}'(W) = \{x \in \mathcal{F}(W) \mid I^n x = 0 \text{ for some } n\}$,*
3. *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$. Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 22.5. $\square$

Definition 14.6. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset whose complement corresponds to an open subspace $U \subset X$ with quasi-compact inclusion morphism $U \to X$. Let $\mathcal{F}$ be a quasi-coherent

$\mathcal{O}_X$-module. The quasi-coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 14.5 above is called the *subsheaf of sections supported on T*.

Lemma 14.7. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $T \subset |Y|$ be a closed subset. Assume $|Y| \setminus T$ corresponds to an open subspace $V \subset Y$ such that $V \to Y$ is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections supported on $|f|^{-1}T$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections supported on T.*

Proof. Omitted. Hints: $|X| \setminus |f|^{-1}T$ is the support of the open subspace $U = f^{-1}V \subset X$. Since $V \to Y$ is quasi-compact, so is $U \to X$ (by base change). The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent. Hence Lemma 14.5 applies to T and $f_*\mathcal{F}$ as well as to $|f|^{-1}T$ and $\mathcal{F}$. The equality of the given quasi-coherent modules is immediate from the definitions. $\square$

15. Characterizing affine spaces

This section is the analogue of Limits, Section 10.

Lemma 15.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is surjective and finite, and assume that X is affine. Then Y is affine.*

Proof. We may and do view $f : X \to Y$ as a morphism of algebraic space over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 16.2). Note that a finite morphism is affine and universally closed, see Morphisms of Spaces, Lemma 41.7. By Morphisms of Spaces, Lemma 9.8 we see that Y is a separated algebraic space. As f is surjective and X is quasi-compact we see that Y is quasi-compact.

By Lemma 11.3 we can write $X = \lim X_a$ with each $X_a \to Y$ finite and of finite presentation. By Lemma 5.8 we see that X_a is affine for a large enough. Hence we may and do assume that $f : X \to Y$ is finite, surjective, and of finite presentation.

By Proposition 8.1 we may write $Y = \lim Y_i$ as a directed limit of algebraic spaces of finite presentation over $\mathbf{Z}$. By Lemma 7.1 we can find $0 \in I$ and a morphism $X_0 \to Y_0$ of finite presentation such that $X_i = X_0 \times_{Y_0} Y_i$ for $i \geq 0$ and such that $X = \lim_i X_i$. By Lemma 6.6 we see that $X_i \to Y_i$ is finite for i large enough. By Lemma 6.3 we see that $X_i \to Y_i$ is surjective for i large enough. By Lemma 5.8 we see that X_i is affine for i large enough. Hence for i large enough we can apply Cohomology of Spaces, Lemma 16.1 to conclude that Y_i is affine. This implies that Y is affine and we conclude. $\square$

Proposition 15.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is surjective and integral, and assume that X is affine. Then Y is affine.*

Proof. We may and do view $f : X \to Y$ as a morphism of algebraic space over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 16.2). Note that integral morphisms are affine and universally closed, see Morphisms of Spaces, Lemma 41.7. By Morphisms of Spaces, Lemma 9.8 we see that Y is a separated algebraic space. As f is surjective and X is quasi-compact we see that Y is quasi-compact.

Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, see Morphisms of Spaces, Lemma 11.2. By Lemma 9.1 we can write $\mathcal{A} = \mathrm{colim}_i \mathcal{F}_i$ as a filtered colimit of finite type $\mathcal{O}_Y$-modules. Let $\mathcal{A}_i \subset \mathcal{A}$ be the $\mathcal{O}_Y$-subalgebra generated by $\mathcal{F}_i$. Since the map of algebras $\mathcal{O}_Y \to \mathcal{A}$ is integral, we see that each $\mathcal{A}_i$ is a finite quasi-coherent $\mathcal{O}_Y$-algebra. Hence

$$X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i) \longrightarrow Y$$

is a finite morphism of algebraic spaces. (Insert future reference to $\underline{\mathrm{Spec}}$ construction for algebraic spaces here.) It is clear that $X = \lim_i X_i$. Hence by Lemma 5.8 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to Y$ factors through each X_i we see that $X_i \to Y$ is surjective. Hence we conclude that Y is affine by Lemma 15.1. $\square$

The following corollary of the result above can be found in [CLO12].

Lemma 15.3. *Let S be a scheme. Let X be an algebraic space over S. If X_{red} is a scheme, then X is a scheme.*

Proof. Let $U' \subset X_{red}$ be an open affine subscheme. Let $U \subset X$ be the open subspace corresponding to the open $|U'| \subset |X_{red}| = |X|$. Then $U' \to U$ is surjective and integral. Hence U is affine by Proposition 15.2. Thus every point is contained in an open subscheme of X, i.e., X is a scheme. $\square$

Lemma 15.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral and induces a bijection $|X| \to |Y|$. Then X is a scheme if and only if Y is a scheme.*

Proof. An integral morphism is representable by definition, hence if Y is a scheme, so is X. Conversely, assume that X is a scheme. Let $U \subset X$ be an affine open. An integral morphism is closed and $|f|$ is bijective, hence $|f|(|U|) \subset |Y|$ is open as the complement of $|f|(|X| \setminus |U|)$. Let $V \subset Y$ be the open subspace with $|V| = |f|(|U|)$, see Properties of Spaces, Lemma 4.8. Then $U \to V$ is integral and surjective, hence V is an affine scheme by Proposition 15.2. This concludes the proof. $\square$

Lemma 15.5. *Let S be a scheme. Let $f : X \to B$ and $B' \to B$ be morphisms of algebraic spaces over S. Assume*

1. *$B' \to B$ is a closed immersion,*
2. *$|B'| \to |B|$ is bijective,*
3. *$X \times_B B' \to B'$ is a closed immersion, and*
4. *$X \to B$ is of finite type or $B' \to B$ is of finite presentation.*

Then $f : X \to B$ is a closed immersion.

Proof. Assumptions (1) and (2) imply that $B_{red} = B'_{red}$. Set $X' = X \times_B B'$. Then $X' \to X$ is closed immersion and $X'_{red} = X_{red}$. Let $U \to B$ be an étale morphism with U affine. Then $X' \times_B U \to X \times_B U$ is a closed immersion of algebraic spaces inducing an isomorphism on underlying reduced spaces. Since $X' \times_B U$ is a scheme (as $B' \to B$ and $X' \to B'$ are representable) so is $X \times_B U$ by Lemma 15.3. Hence $X \to B$ is representable too. Thus we reduce to the case of schemes, see Morphisms, Lemma 45.5. $\square$

16. Finite cover by a scheme

As an application of Zariski's main theorem and the limit results of this chapter, we prove that given any quasi-compact and quasi-separated algebraic space X, there is a scheme Y and a surjective, finite morphism $Y \to X$. The following lemma will be obsoleted by the full result later on.

Lemma 16.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S.*

(1) *There exists a surjective integral morphism $Y \to X$ where Y is a scheme,*
(2) *given a surjective étale morphism $U \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U.*

Proof. Part (1) is the special case of part (2) where $U = X$. Choose a surjective étale morphism $U' \to U$ where U' is a scheme. It is clear that we may replace U by U' and hence we may assume U is a scheme. Since X is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \coprod U_i \to X$ is surjective. After replacing U by U' again, we see that we may assume U is affine. Since X is quasi-separated, hence reasonable, there exists an integer d bounding the degree of the geometric fibres of $U \to X$ (see Decent Spaces, Lemma 5.1). We will prove the lemma by induction on d for all quasi-compact and separated schemes U mapping surjective and étale onto X. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Morphisms of Spaces, Lemma 46.2 and we obtain a factorization

$$\begin{tikzcd} U \arrow[rr, "j"'] \arrow[dr] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

with π integral and j a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in Y. Note that

$$U \times_X Y = U \amalg W$$

where the first summand is the image of $U \to U \times_X Y$ (which is closed by Morphisms of Spaces, Lemma 4.6 and open because it is étale as a morphism between algebraic spaces étale over Y) and the second summand is the (open and closed) complement. The image $V \subset Y$ of W is an open subspace containing $Y \setminus U$.

The étale morphism $W \to Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $|U| \subset |Y|$ is dense, it holds for all geometric points of Y for example by Decent Spaces, Lemma 8.1 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \to V$ and find a surjective integral morphism $Z \to V$ with Z a scheme, which Zariski locally factors through W. Choose a factorization $Z \to Z' \to Y$ with $Z' \to Y$ integral and $Z \to Z'$ open immersion (Morphisms of Spaces, Lemma 46.2). After replacing Z' by the scheme theoretic closure of Z in Z' we may assume that Z is scheme theoretically dense in Z'. After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced closed subspace structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \longrightarrow X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \to X$ factors through U. On the other hand, let $z \in Z'$ be a point.

If $z \notin Z$, then z maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of z on which the morphism factors through U. If $z \in Z$, then we have a neighbourhood $V \subset Z$ which factors through $W \subset U \times_X Y$ and hence through U. □

Proposition 16.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S.*

(1) *There exists a surjective finite morphism $Y \to X$ of finite presentation where Y is a scheme,*
(2) *given a surjective étale morphism $U \to X$ we may choose $Y \to X$ such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U.*

Proof. Part (1) is the special case of (2) with $U = X$. Let $Y \to X$ be as in Lemma 16.1. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through U. We can write $Y = \lim Y_i$ with $Y_i \to X$ finite and of finite presentation, see Lemma 11.2. For large enough i the algebraic space Y_i is a scheme, see Lemma 5.9. For large enough i we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in Y recovers V_j, see Lemma 5.5. For even larger i the morphisms $V_j \to U$ over X come from morphisms $V_{i,j} \to U$ over X, see Proposition 3.9. This finishes the proof. □

17. Other chapters

References

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