MORE ON MORPHISMS OF SPACES

Contents

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1. Introduction

In this chapter we continue our study of properties of morphisms of algebraic spaces. A fundamental reference is [Knu71].

2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

3. Radicial morphisms

It turns out that a radicial morphism is not the same thing as a universally injective morphism, contrary to what happens with morphisms of schemes. In fact it is a bit stronger.

Definition 3.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *radicial* if for any morphism $\text{Spec}(K) \to Y$ where K is a field the reduction $(\text{Spec}(K) \times_Y X)_{red}$ is either empty or representable by the spectrum of a purely inseparable field extension of K.

Lemma 3.2. *A radicial morphism of algebraic spaces is universally injective.*

Proof. Let S be a scheme. Let $f : X \to Y$ be a radicial morphism of algebraic spaces over S. It is clear from the definition that given a morphism $\text{Spec}(K) \to Y$ there is at most one lift of this morphism to a morphism into X. Hence we conclude that f is universally injective by Morphisms of Spaces, Lemma 19.2. $\square$

Example 3.3. It is no longer true that universally injective is equivalent to radicial. For example the morphism

$$X = [\text{Spec}(\overline{\mathbf{Q}})/\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})] \longrightarrow S = \text{Spec}(\mathbf{Q})$$

of Spaces, Example 14.7 is universally injective, but is not radicial in the sense above.

Nonetheless it is often the case that the reverse implication holds.

Lemma 3.4. *Let S be a scheme. Let $f : X \to Y$ be a universally injective morphism of algebraic spaces over S.*

1. *If f is decent then f is radicial.*
2. *If f is quasi-separated then f is radicial.*
3. *If f is locally separated then f is radicial.*

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is stable under base change and composition and holds for closed immersions. Assume $f : X \to Y$ has $\mathcal{P}$ and is universally injective. Then, in the situation of Definition 3.1 the morphism $(\mathrm{Spec}(K) \times_Y X)_{red} \to \mathrm{Spec}(K)$ is universally injective and has $\mathcal{P}$. This reduces the problem of proving

$$\mathcal{P} + \text{universally injective} \Rightarrow \text{radicial}$$

to the problem of proving that any nonempty reduced algebraic space X over field whose structure morphism $X \to \mathrm{Spec}(K)$ is universally injective and $\mathcal{P}$ is representable by the spectrum of a field. Namely, then $X \to \mathrm{Spec}(K)$ will be a morphism of schemes and we conclude by the equivalence of radicial and universally injective for morphisms of schemes, see Morphisms, Lemma 12.2.

Let us prove (1). Assume f is decent and universally injective. By Decent Spaces, Lemmas 15.4, 15.6, and 15.2 (to see that an immersion is decent) we see that the discussion in the first paragraph applies. Let X be a nonempty decent reduced algebraic space universally injective over a field K. In particular we see that $|X|$ is a singleton. By Decent Spaces, Lemma 12.1 we conclude that $X \cong \mathrm{Spec}(L)$ for some extension $K \subset L$ as desired.

A quasi-separated morphism is decent, see Decent Spaces, Lemma 15.2. Hence (1) implies (2).

Let us prove (3). Recall that the separation axioms are stable under base change and composition and that closed immersions are separated, see Morphisms of Spaces, Lemmas 4.4, 4.8, and 10.7. Thus the discussion in the first paragraph of the proof applies. Let X be a reduced algebraic space universally injective and locally separated over a field K. In particular $|X|$ is a singleton hence X is quasi-compact, see Properties of Spaces, Lemma 5.2. We can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 6.3. Consider the morphism of schemes

$$j : U \times_X U \longrightarrow U \times_{\mathrm{Spec}(K)} U$$

As $X \to \mathrm{Spec}(K)$ is universally injective j is surjective, and as $X \to \mathrm{Spec}(K)$ is locally separated j is an immersion. A surjective immersion is a closed immersion, see Schemes, Lemma 10.4. Hence $R = U \times_X U$ is affine as a closed subscheme of an affine scheme. In particular R is quasi-compact. It follows that $X = U/R$ is quasi-separated, and the result follows from (2). ☐

Remark 3.5. Let $X \to Y$ be a morphism of algebraic spaces. For some applications (of radicial morphisms) it is enough to require that for every $\mathrm{Spec}(K) \to Y$ where K is a field

1. the space $|\mathrm{Spec}(K) \times_Y X|$ is a singleton,
2. there exists a monomorphism $\mathrm{Spec}(L) \to \mathrm{Spec}(K) \times_Y X$, and
3. $K \subset L$ is purely inseparable.

If needed later we will may call such a morphism *weakly radicial*. For example if $X \to Y$ is a surjective weakly radicial morphism then $X(k) \to Y(k)$ is surjective for every algebraically closed field k. Note that the base change $X_{\overline{\mathbf{Q}}} \to \mathrm{Spec}(\overline{\mathbf{Q}})$ of the morphism in Example 3.3 is weakly radicial, but not radicial. The analogue of Lemma 3.4 is that if $X \to Y$ has property (β) and is universally injective, then it is weakly radicial (proof omitted).

Lemma 3.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

1. *f is locally of finite type,*
2. *for every étale morphism $V \to Y$ the map $|X \times_Y V| \to |V|$ is injective.*

Then f is universally injective.

Proof. The question is étale local on Y by Morphisms of Spaces, Lemma 19.6. Hence we may assume that Y is a scheme. Then Y is in particular decent and by Decent Spaces, Lemma 16.9 we see that f is locally quasi-finite. Let $y \in Y$ be a point and let X_y be the scheme theoretic fibre. Assume X_y is not empty. By Spaces over Fields, Lemma 7.2 we see that X_y is a scheme which is locally quasi-finite over $\kappa(y)$. Since $|X_y| \subset |X|$ is the fibre of $|X| \to |Y|$ over y we see that X_y has a unique point x. The same is true for $X_y \times_{\mathrm{Spec}(\kappa(y))} \mathrm{Spec}(k)$ for any finite separable extension $\kappa(y) \subset k$ because we can realize k as the residue field at a point lying over y in an étale scheme over Y, see see More on Morphisms, Lemma 27.2. Thus X_y is geometrically connected, see Varieties, Lemma 5.11. This implies that the finite extension $\kappa(y) \subset \kappa(x)$ is purely inseparable.

We conclude (in the case that Y is a scheme) that for every $y \in Y$ either the fibre X_y is empty, or $(X_y)_{red} = \mathrm{Spec}(\kappa(x))$ with $\kappa(y) \subset \kappa(x)$ purely inseparable. Hence f is radicial (some details omitted), whence universally injective by Lemma 3.2. □

4. Conormal sheaf of an immersion

Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 13.1. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Morphisms of Spaces, Lemma 14.1. This quasi-coherent $\mathcal{O}_Z$-module is the *conormal sheaf of Z in X* and is often denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Morphisms of Spaces, Section 14.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of Spaces, Remark 12.4. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 4.1. Let $i : Z \to X$ be an immersion. The *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X* or the *conormal sheaf of i* is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [DG67, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion.* It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

Lemma 4.2. *Let S be a scheme. Let $i : Z \to X$ be an immersion. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then*

$$\mathcal{C}_{Z/X}|_{Z_U} = \mathcal{C}_{Z_U/U}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma just states that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z_U \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 13.1. $\square$

Lemma 4.3. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 13.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 13.1). Thus we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] & i_*\mathcal{O}_Z \arrow[r] & 0 \\ 0 \arrow[r] & g^{-1}\mathcal{I}' \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_{X'} \arrow[r] \arrow[u] & g^{-1}i'_*\mathcal{O}_{Z'} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^2 = g^{-1}((\mathcal{I}')^2)$. Hence the diagram induces a map $g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{C}_{Z/X}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives a map $f^{-1}\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$, which induces the desired map. $\square$

Lemma 4.4. *Let S be a scheme. The conormal sheaf of Definition 4.1, and its functoriality of Lemma 4.3 satisfy the following properties:*

1. *If $Z \to X$ is an immersion of schemes over S, then the conormal sheaf agrees with the one from Morphisms, Definition 33.1.*
2. *If in Lemma 4.3 all the spaces are schemes, then the map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ is the same as the one constructed in Morphisms, Lemma 33.3.*

(3) *Given a commutative diagram*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] \arrow[d, "f'"'] & X' \arrow[d, "g'"] \\ Z'' \arrow[r, "i''"] & X'' \end{tikzcd}$$

then the map $(f' \circ f)^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z/X}$ is the same as the composition of $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ with the pullback by f of $(f')^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z'/X'}$

Proof. Omitted. Note that Part (1) is a special case of Lemma 4.2. □

Lemma 4.5. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S. Assume i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ of Lemma 4.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ U' \arrow[r] & X' \end{tikzcd}$$

where U, U' are schemes and the horizontal arrows are surjective and étale, see Spaces, Lemma 11.4. Then using Lemmas 4.2 and 4.4 we see that the question reduces to the case of a morphism of schemes. In the schemes case this is Morphisms, Lemma 33.4. □

Lemma 4.6. *Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 4.3 and $i : Z \to Y$ is the first morphism.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. Via Lemmas 4.2 and 4.4 the exactness of the sequence translates immediately into the exactness of the corresponding sequence for the immersions of schemes $Z \times_X U \to Y \times_X U \to U$. Hence the lemma follows from Morphisms, Lemma 33.5. □

5. The normal cone of an immersion

Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 13.1. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by

$\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms of Spaces, Lemma 14.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the *conormal algebra of* Z *in* X and is often simply denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms of Spaces, Section 14.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal algebra of i as the conormal algebra of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of Spaces, Remark 12.4. It is often denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 5.1. Let $i : Z \to X$ be an immersion. The *conormal algebra* $\mathcal{C}_{Z/X,*}$ *of* Z *in* X or the *conormal algebra of* i is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

$$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1} \tag{5.1.1}$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

$$\mathrm{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*} \tag{5.1.2}$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

Lemma 5.2. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then*

$$\mathcal{C}_{Z/X,*}|_{Z_U} = \mathcal{C}_{Z_U/U,*}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma follows from the fact that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z_U \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 13.1. $\square$

Lemma 5.3. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of graded $\mathcal{O}_Z$-algebras

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let

$\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 13.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 13.1). Thus we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] & i_*\mathcal{O}_Z \arrow[r] & 0 \\ 0 \arrow[r] & g^{-1}\mathcal{I}' \arrow[r] \arrow[u] & g^{-1}\mathcal{O}_{X'} \arrow[r] \arrow[u] & g^{-1}i'_*\mathcal{O}_{Z'} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^n = g^{-1}((\mathcal{I}')^n)$ for all $n \geq 1$. Hence the diagram induces a map $g^{-1}((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{C}_{Z/X,n}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives maps $f^{-1}\mathcal{C}_{Z'/X',n} \to \mathcal{C}_{Z/X,n}$, which induce the desired map. $\square$

Lemma 5.4. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a cartesion square of algebraic spaces over S with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ of Lemma 5.3 is surjective. If g is flat, then it is an isomorphism.

Proof. We may check the statement after étale localizing X'. In this case we may assume $X' \to X$ is a morphism of schemes, hence Z and Z' are schemes and the result follows from the case of schemes, see Divisors, Lemma 11.4. $\square$

We use the same conventions for cones and vector bundles over algebraic spaces as we do for schemes (where we use the conventions of EGA), see Constructions, Sections 7 and 6. In particular, a vector bundle is a very general gadget (and not locally isomorphic to an affine space bundle).

Definition 5.5. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. The *normal cone* C_ZX of Z in X is

$$C_ZX = \underline{\text{Spec}}_Z(\mathcal{C}_{Z/X,*})$$

see Morphisms of Spaces, Definition 20.8. The *normal bundle* of Z in X is the vector bundle

$$N_ZX = \underline{\text{Spec}}_Z(\text{Sym}(\mathcal{C}_{Z/X}))$$

Thus $C_ZX \to Z$ is a cone over Z and $N_ZX \to Z$ is a vector bundle over Z. Moreover, the canonical surjection (5.1.2) of graded algebras defines a canonical closed immersion

$$C_ZX \longrightarrow N_ZX \tag{5.5.1}$$

of cones over Z.

6. Sheaf of differentials of a morphism

We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 127), the corresponding section in the chapter on morphism of schemes (Morphisms, Section 34) as well as Modules on Sites, Section 32. We first show that the notion of sheaf of differentials for a morphism of schemes agrees with the corresponding morphism of small étale (ringed) sites.

To clearly state the following lemma we temporarily go back to denoting $\mathcal{F}^a$ the sheaf of $\mathcal{O}_{X_{étale}}$-modules associated to a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ on the scheme X, see Descent, Definition 7.2.

Lemma 6.1. *Let $f : X \to Y$ be a morphism of schemes. Let $f_{small} : X_{étale} \to Y_{étale}$ be the associated morphism of small étale sites, see Descent, Remark 7.4. Then there is a canonical isomorphism*

$$(\Omega_{X/Y})^a = \Omega_{X_{étale}/Y_{étale}}$$

compatible with universal derivations. Here the first module is the sheaf on $X_{étale}$ associated to the quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/Y}$, see Morphisms, Definition 34.1, and the second module is the one from Modules on Sites, Definition 32.3.

Proof. Let $h : U \to X$ be an étale morphism. In this case the natural map $h^*\Omega_{X/Y} \to \Omega_{U/Y}$ is an isomorphism, see More on Morphisms, Lemma 7.7. This means that there is a natural $\mathcal{O}_{Y_{étale}}$-derivation

$$\mathrm{d}^a : \mathcal{O}_{X_{étale}} \longrightarrow (\Omega_{X/Y})^a$$

since we have just seen that the value of $(\Omega_{X/Y})^a$ on any object U of $X_{étale}$ is canonically identified with $\Gamma(U, \Omega_{U/Y})$. By the universal property of $\mathrm{d}_{X/Y} : \mathcal{O}_{X_{étale}} \to \Omega_{X_{étale}/Y_{étale}}$ there is a unique $\mathcal{O}_{X_{étale}}$-linear map $c : \Omega_{X_{étale}/Y_{étale}} \to (\Omega_{X/Y})^a$ such that $\mathrm{d}^a = c \circ \mathrm{d}_{X/Y}$.

Conversely, suppose that $\mathcal{F}$ is an $\mathcal{O}_{X_{étale}}$-module and $D : \mathcal{O}_{X_{étale}} \to \mathcal{F}$ is a $\mathcal{O}_{Y_{étale}}$-derivation. Then we can simply restrict D to the small Zariski site X_{Zar} of X. Since sheaves on X_{Zar} agree with sheaves on X, see Descent, Remark 7.3, we see that $D|_{X_{Zar}} : \mathcal{O}_X \to \mathcal{F}|_{X_{Zar}}$ is just a "usual" Y-derivation. Hence we obtain a map $\psi : \Omega_{X/Y} \longrightarrow \mathcal{F}|_{X_{Zar}}$ such that $D|_{X_{Zar}} = \psi \circ \mathrm{d}$. In particular, if we apply this with $\mathcal{F} = \Omega_{X_{étale}/Y_{étale}}$ we obtain a map

$$c' : \Omega_{X/Y} \longrightarrow \Omega_{X_{étale}/Y_{étale}}|_{X_{Zar}}$$

Consider the morphism of ringed sites $\mathrm{id}_{small,étale,Zar} : X_{étale} \to X_{Zar}$ discussed in Descent, Remark 7.4 and Lemma 7.5. Since the restriction functor $\mathcal{F} \mapsto \mathcal{F}|_{X_{Zar}}$ is equal to $\mathrm{id}_{small,étale,Zar,*}$, since $\mathrm{id}^*_{small,étale,Zar}$ is left adjoint to $\mathrm{id}_{small,étale,Zar,*}$ and since $(\Omega_{X/Y})^a = \mathrm{id}^*_{small,étale,Zar}\Omega_{X/Y}$ we see that c' is adjoint to a map

$$c'' : (\Omega_{X/Y})^a \longrightarrow \Omega_{X_{étale}/Y_{étale}}.$$

We claim that c'' and c' are mutually inverse. This claim finishes the proof of the lemma. To see this it is enough to show that $c''(\mathrm{d}(f)) = \mathrm{d}_{X/Y}(f)$ and $c(\mathrm{d}_{X/Y}(f)) = \mathrm{d}(f)$ if f is a local section of $\mathcal{O}_X$ over an open of X. We omit the verification. □

This clears the way for the following definition. For an alternative, see Remark 6.5.

Definition 6.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *sheaf of differentials* $\Omega_{X/Y}$ *of* X *over* Y is sheaf of differentials (Modules on Sites, Definition 32.10) for the morphism of ringed topoi

$$(f_{small}, f^\sharp) : (X_{\acute{e}tale}, \mathcal{O}_X) \to (Y_{\acute{e}tale}, \mathcal{O}_Y)$$

of Properties of Spaces, Lemma 18.3. The *universal* Y*-derivation* will be denoted $\text{d}_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$.

By Lemma 6.1 this does not conflict with the already existing notion in case X and Y are representable. From now on, if X and Y are representable, we no longer distinguish between the sheaf of differentials defined above and the one defined in Morphisms, Definition 34.1. We want to relate this to the usual modules of differentials for morphisms of schemes. Here is the key lemma.

Lemma 6.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider any commutative diagram*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where the vertical arrows are étale morphisms of algebraic spaces. Then

$$\Omega_{X/Y}|_{U_{\acute{e}tale}} = \Omega_{U/V}$$

In particular, if U, V are schemes, then this is equal to the usual sheaf of differentials of the morphism of schemes $U \to V$.

Proof. By Properties of Spaces, Lemma 15.10 and Equation (15.10.1) we may think of the restriction of a sheaf on $X_{\acute{e}tale}$ to $U_{\acute{e}tale}$ as the pullback by a_{small}. Similarly for b. By Modules on Sites, Lemma 32.6 we have

$$\Omega_{X/Y}|_{U_{\acute{e}tale}} = \Omega_{\mathcal{O}_{U_{\acute{e}tale}}/a_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{\acute{e}tale}}}$$

Since $a_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{\acute{e}tale}} = \psi_{small}^{-1}b_{small}^{-1}\mathcal{O}_{Y_{\acute{e}tale}} = \psi_{small}^{-1}\mathcal{O}_{V_{\acute{e}tale}}$ we see that the lemma holds. $\square$

Lemma 6.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. Choose a diagram as in Lemma 6.3 with a and b surjective and U and V schemes. Then we see that $\Omega_{X/Y}|_U = \Omega_{U/V}$ which is quasi-coherent (for example by Morphisms, Lemma 34.7). Hence we conclude that $\Omega_{X/Y}$ is quasi-coherent by Properties of Spaces, Lemma 27.6. $\square$

Remark 6.5. Now that we know that $\Omega_{X/Y}$ is quasi-coherent we can attempt to construct it in another manner. For example we can use the result of Properties of Spaces, Section 30 to construct the sheaf of differentials by glueing. For example if Y is a scheme and if $U \to X$ is a surjective étale morphism from a scheme towards X, then we see that $\Omega_{U/Y}$ is a quasi-coherent $\mathcal{O}_U$-module, and since $s, t : R \to U$ are étale we get an isomorphism

$$\alpha : s^*\Omega_{U/Y} \to \Omega_{R/Y} \to t^*\Omega_{U/Y}$$

by using Morphisms, Lemma 35.16. You check that this satisfies the cocycle condition and you're done. If Y is not a scheme, then you define $\Omega_{U/Y}$ as the cokernel of the map $(U \to Y)^*\Omega_{Y/S} \to \Omega_{U/S}$, and proceed as before. This two step process is a little bit ugly. Another possibility is to glue the sheaves $\Omega_{U/V}$ for any diagram as in Lemma 6.3 but this is not very elegant either. Both approaches will work however, and will give a slightly more elementary construction of the sheaf of differentials.

Lemma 6.6. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*d_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^*d_{X/Y}(t)$ mapsto $d_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is a special case of Modules on Sites, Lemma 32.11. □

Lemma 6.7. *Let S be a scheme. Let*

$$\begin{tikzcd} X'' \arrow[r, "g"'] \arrow[d] & X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ Y'' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S. Then we have

$$c_{f\circ g} = c_g \circ g^*c_f$$

as maps $(f \circ g)^*\Omega_{X/Y} \to \Omega_{X''/Y''}$.

Proof. Omitted. Hint: Use the characterization of $c_f, c_g, c_{f\circ g}$ in terms of the effect these maps have on local sections. □

Lemma 6.8. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to B$ be morphisms of algebraic spaces over S. Then there is a canonical exact sequence*

$$f^*\Omega_{Y/B} \to \Omega_{X/B} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 6.6.

Proof. Follows from the schemes version, see Morphisms, Lemma 34.9, of this result via étale localization, see Lemma 6.3. □

Lemma 6.9. *Let S be a scheme. If $X \to Y$ is an immersion of algebraic spaces over S then $\Omega_{X/S}$ is zero.*

Proof. Follows from the schemes version, see Morphisms, Lemma 34.14, of this result via étale localization, see Lemma 6.3. □

Lemma 6.10. *Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of algebraic spaces over B. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

where the first arrow is induced by $\mathrm{d}_{X/B}$ and the second arrow comes from Lemma 6.6.

Proof. This is the algebraic spaces version of Morphisms, Lemma 34.15 and will be a consequence of that lemma by étale localization, see Lemmas 6.3 and 4.2. However, we should make sure we can define the first arrow globally. Hence we explain the meaning of "induced by $\mathrm{d}_{X/B}$" here. Namely, we may assume that i is a closed immersion after replacing X by an open subspace. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to $Z \subset X$. Then $\mathrm{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$-linear. By Morphisms of Spaces, Lemma 14.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. □

Lemma 6.11. *Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of schemes over B, and assume i (étale locally) has a left inverse. Then the canonical sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 6.10 is (étale locally) split exact.

Proof. Clarification: we claim that if $g : X \to Z$ is a left inverse of i, then i^*c_g is a right inverse of the map $i^*\Omega_{X/B} \to \Omega_{Z/B}$. Having said this, the result follows from the corresponding result for morphisms of schemes by étale localization, see Lemmas 6.3 and 4.2. □

Lemma 6.12. *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Let $X' = X_{Y'}$ be the base change of X. Denote $g' : X' \to X$ the projection. Then the map*

$$(g')^*\Omega_{X/Y} \to \Omega_{X'/Y'}$$

of Lemma 6.6 is an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 34.10 and étale localization, see Lemma 6.3. □

Lemma 6.13. *Let S be a scheme. Let $f : X \to B$ and $g : Y \to B$ be morphisms of algebraic spaces over S with the same target. Let $p : X \times_B Y \to X$ and $q : X \times_B Y \to Y$ be the projection morphisms. The maps from Lemma 6.6*

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \longrightarrow \Omega_{X \times_S Y/S}$$

give an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 34.11 and étale localization, see Lemma 6.3. □

Lemma 6.14. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $\Omega_{X/Y}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Follows from the schemes version, see Morphisms, Lemma 34.12 and étale localization, see Lemma 6.3. □

Lemma 6.15. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $\Omega_{X/Y}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Follows from the schemes version, see Morphisms, Lemma 34.13 and étale localization, see Lemma 6.3. $\square$

7. Topological invariance of the étale site

We show that the site $X_{spaces,étale}$ is a "topological invariant". It then follows that $X_{étale}$, which consists of the representable objects in $X_{spaces,étale}$, is a topological invariant too, see Lemma 7.2.

Theorem 7.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral, universally injective and surjective. The functor*

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$.

Proof. The morphism f is representable and a universal homeomorphism, see Morphisms of Spaces, Section 47.

We first prove that the functor is faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $a, b : V' \to V$ are distinct morphisms over Y. Since V', V are étale over Y the equalizer

$$E = V' \times_{(a,b), V \times_Y V, \Delta_{V/Y}} V$$

of a, b is étale over Y also. Hence $E \to V'$ is an étale monomorphism (i.e., an open immersion) which is an isomorphism if and only if it is surjective. Since $X \to Y$ is a universal homeomorphism we see that this is the case if and only if $E_X = V'_X$, i.e., if and only if $a_X = b_X$.

Next, we prove that the functor is fully faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $c : V'_X \to V_X$ is a morphism over X. We want to construct a morphism $a : V' \to V$ over Y such that $a_X = c$. Let $a' : V'' \to V'$ be a surjective étale morphism such that V'' is a separated algebraic space. If we can construct a morphism $a'' : V'' \to V$ such that $a''_X = c \circ a'_X$, then the two compositions

$$V'' \times_{V'} V'' \xrightarrow{\text{pr}_i} V'' \xrightarrow{a''} V$$

will be equal by the faithfulness of the functor proved in the first paragraph. Hence a'' will factor through a unique morphism $a : V' \to V$ as V' is (as a sheaf) the quotient of V'' by the equivalence relation $V'' \times_{V'} V''$. Hence we may assume that V' is separated. In this case the graph

$$\Gamma_c \subset (V' \times_Y V)_X$$

is open and closed (details omitted). Since $X \to Y$ is a universal homeomorphism, there exists an open and closed subspace $\Gamma \subset V' \times_Y V$ such that $\Gamma_X = \Gamma_c$. The projection $\Gamma \to V'$ is an étale morphism whose base change to X is an isomorphism. Hence $\Gamma \to V'$ is étale, universally injective, and surjective, so an isomorphism by Morphisms of Spaces, Lemma 45.2. Thus Γ is the graph of a morphism $a : V' \to V$ as desired.

Finally, we prove that the functor is essentially surjective. Suppose that U is an object of $X_{spaces,étale}$. We have to find an object V of $Y_{spaces,étale}$ such that $V_X \cong U$. Let $U' \to U$ be a surjective étale morphism such that $U' \cong V'_X$ and $U' \times_U U' \cong V''_X$ for some objects V'', V' of $Y_{spaces,étale}$. Then by fully faithfulness of the functor we obtain morphisms $s, t : V'' \to V'$ with $t_X = \text{pr}_0$ and $s_X = \text{pr}_1$ as morphisms $U' \times_U U' \to U'$. Using that $(\text{pr}_0, \text{pr}_1) : U' \times_U U' \to U' \times_S U'$ is an étale equivalence relation, and that $U' \to V'$ and $U' \times_U U' \to V''$ are universally injective and surjective we deduce that $(t, s) : V'' \to V' \times_S V'$ is an étale equivalence relation. Then the quotient $V = V'/V''$ (see Spaces, Theorem 10.5) is an algebraic space V over Y. There is a morphism $V' \to V$ such that $V'' = V' \times_V V'$. Thus we obtain a morphism $V \to Y$ (see Descent on Spaces, Lemma 6.2). On base change to X we see that we have a morphism $U' \to V_X$ and a compatible isomorphism $U' \times_{V_X} U' = U' \times_U U'$, which implies that $V_X \cong U$ (by the lemma just cited once more).

Pick a scheme W and a surjective étale morphism $W \to Y$. Pick a scheme U' and a surjective étale morphism $U' \to U \times_X W_X$. Note that U' and $U' \times_U U'$ are schemes étale over X whose structure morphism to X factors through the scheme W_X. Hence by Étale Cohomology, Theorem 46.1 there exist schemes V', V'' étale over W whose base change to W_X is isomorphic to respectively U' and $U' \times_U U'$. This finishes the proof. $\square$

Lemma 7.2. *With assumption and notation as in Theorem 7.1 the equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$ restricts to an equivalence of categories $Y_{étale} \to X_{étale}$.*

Proof. This is just the statement that given an object $V \in Y_{spaces,étale}$ we have V is a scheme if and only if $V \times_Y X$ is a scheme. Since $V \times_Y X \to V$ is integral, universally injective, and surjective (as a base change of $X \to Y$) this follows from Limits of Spaces, Lemma 15.4. $\square$

Remark 7.3. A universal homeomorphism of algebraic spaces need not be representable, see Morphisms of Spaces, Example 47.3. The argument in the proof of Theorem 7.1 above cannot be used in this case. In fact we do not know whether given a universal homeomorphism of algebraic spaces $f : X \to Y$ the categories $X_{spaces,étale}$ and $Y_{spaces,étale}$ are equivalent. If you do, please email stacks.project@gmail.com.

8. Thickenings

The following terminology may not be completely standard, but it is convenient.

Definition 8.1. Thickenings. Let S be a scheme.

(1) We say an algebraic space X' is a *thickening* of an algebraic space X if X is a closed subspace of X' and the associated topological spaces are equal.
(2) We say X' is a *first order thickening* of X if X is a closed subspace of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a *morphism of thickenings* is a morphism $f' : X' \to Y'$ such that $f(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subspace Y. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.

(4) Let B be an algebraic space. We similarly define *thickenings over* B, and *morphisms of thickenings over* B. This means that the spaces X, X', Y, Y' above are algebraic spaces endowed with a structure morphism to B, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over B.

The fundamental equivalence. Note that if $X \subset X'$ is a thickening, then $X \to X'$ is integral and universally bijective. This implies that

$$X_{spaces,étale} = X'_{spaces,étale} \tag{8.1.1}$$

via the pullback functor, see Theorem 7.1. Hence we may think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$. Thus a canonical equivalence of locally ringed topoi

$$(Sh(X'_{spaces,étale}), \mathcal{O}_{X'}) \cong (Sh(X_{spaces,étale}), \mathcal{O}_{X'}) \tag{8.1.2}$$

Below we will frequently combine this with the fully faithfulness result of Properties of Spaces, Theorem 26.4. For example the closed immersion $i_X : X \to X'$ corresponds to the surjective map $i_X^\sharp : \mathcal{O}_{X'} \to \mathcal{O}_X$.

Let S be a scheme, and let B be an algebraic space over S. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings over B. Note that the diagram of continuous functors

$$\begin{tikzcd}
X_{spaces,étale} & Y_{spaces,étale} \arrow[l] \\
X'_{spaces,étale} \arrow[u] & Y'_{spaces,étale} \arrow[l] \arrow[u]
\end{tikzcd}$$

is commutative and the vertical arrows are equivalences. Hence $f_{spaces,étale}$, f_{small}, $f'_{spaces,étale}$, and f'_{small} all define the same morphism of topoi. Thus we may think of

$$(f')^\sharp : f^{-1}_{spaces,étale}\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_{X'}$$

as a map of sheaves of $\mathcal{O}_B$-algebras fitting into the commutative diagram

$$\begin{tikzcd}
f^{-1}_{spaces,étale}\mathcal{O}_Y \arrow[r, "f^\sharp"'] & \mathcal{O}_X \\
f^{-1}_{spaces,étale}\mathcal{O}_{Y'} \arrow[u, "i_Y^\sharp"] \arrow[r, "(f')^\sharp"] & \mathcal{O}_{X'} \arrow[u, "i_X^\sharp"']
\end{tikzcd}$$

Here $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the names of the given closed immersions.

Lemma 8.2. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be thickenings of algebraic spaces over B. Let $f : X \to Y$ be a morphism of algebraic spaces over B. Given any map of $\mathcal{O}_B$-algebras*

$$\alpha : f^{-1}_{spaces,étale}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

such that

$$\begin{tikzcd}
f^{-1}_{spaces,étale}\mathcal{O}_Y \arrow[r, "f^\sharp"'] & \mathcal{O}_X \\
f^{-1}_{spaces,étale}\mathcal{O}_{Y'} \arrow[u, "i_Y^\sharp"] \arrow[r, "\alpha"] & \mathcal{O}_{X'} \arrow[u, "i_X^\sharp"']
\end{tikzcd}$$

commutes, there exists a unique morphism of (f, f') of thickenings over B such that $\alpha = (f')^\sharp$.

Proof. To find f', by Properties of Spaces, Theorem 26.4, all we have to do is show that the morphism of ringed topoi

$$(f_{spaces,\acute{e}tale}, \alpha) : (Sh(X_{spaces,\acute{e}tale}), \mathcal{O}_{X'}) \longrightarrow (Sh(Y_{spaces,\acute{e}tale}), \mathcal{O}_{Y'})$$

is a morphism of locally ringed topoi. This follows directly from the definition of morphisms of locally ringed topoi (Modules on Sites, Definition 39.8), the fact that $(f, f^\sharp)$ is a morphism of locally ringed topoi (Properties of Spaces, Lemma 26.1), that α fits into the given commutative diagram, and the fact that the kernels of $i_X^\sharp$ and $i_Y^\sharp$ are locally nilpotent. Finally, the fact that $f' \circ i_X = i_Y \circ f$ follows from the commutativity of the diagram and another application of Properties of Spaces, Theorem 26.4. We omit the verification that f' is a morphism over B. □

Lemma 8.3. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. For any open subspace $U \subset X$ there exists a unique open subspace $U' \subset X'$ such that $U = X \times_{X'} U'$.*

Proof. Let $U' \to X'$ be the object of $X'_{spaces,\acute{e}tale}$ corresponding to the object $U \to X$ of $X_{spaces,\acute{e}tale}$ via (8.1.1). The morphism $U' \to X'$ is étale and universally injective, hence an open immersion, see Morphisms of Spaces, Lemma 45.2. □

Finite order thickenings. Let $i_X : X \to X'$ be a thickening of algebraic spaces. Any local section of the kernel $\mathcal{I} = \mathrm{Ker}(i_X^\sharp) \subset \mathcal{O}_{X'}$ is locally nilpotent. Let us say that $X \subset X'$ is a *finite order thickening* if the ideal sheaf $\mathcal{I}$ is "globally" nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over B. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

Lemma 8.4. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. Let U be an affine object of $X_{spaces,\acute{e}tale}$. Then*

$$\Gamma(U, \mathcal{O}_{X'}) \to \Gamma(U, \mathcal{O}_X)$$

is surjective where we think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,\acute{e}tale}$ via (8.1.2).

Proof. Let $U' \to X'$ be the étale morphism of algebraic spaces such that $U = X \times_{X'} U'$, see Theorem 7.1. By Limits of Spaces, Lemma 15.1 we see that U' is an affine scheme. Hence $\Gamma(U, \mathcal{O}_{X'}) = \Gamma(U', \mathcal{O}_{U'}) \to \Gamma(U, \mathcal{O}_U)$ is surjective as $U \to U'$ is a closed immersion of affine schemes. Below we give a direct proof for finite order thickenings which is the case most used in practice. □

Proof for finite order thickenings. We may assume that $X \subset X'$ is a first order thickening by the principle explained above. Denote $\mathcal{I}$ the kernel of the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. As $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_{X'}$-module and since $\mathcal{I}^2 = 0$ by the definition of a first order thickening we may apply Morphisms of Spaces, Lemma 14.1 to see that $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module. Hence the lemma follows from the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and the fact that $H^1_{étale}(U, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 7.10 and Cohomology of Schemes, Lemma 2.2. □

Lemma 8.5. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. If X is (representable by) a scheme, then so is X'.*

Proof. Note that $X'_{red} = X_{red}$. Hence if X is a scheme, then X'_{red} is a scheme. Thus the result follows from Limits of Spaces, Lemma 15.3. Below we give a direct proof for finite order thickenings which is the case most often used in practice. □

Proof for finite order thickenings. It suffices to prove this when X' is a first order thickening of X. By Properties of Spaces, Lemma 10.1 there is a largest open subspace of X' which is a scheme. Thus we have to show that every point x of $|X'| = |X|$ is contained in an open subspace of X' which is a scheme. Using Lemma 8.3 we may replace $X \subset X'$ by $U \subset U'$ with $x \in U$ and U an affine scheme. Hence we may assume that X is affine. Thus we reduce to the case discussed in the next paragraph.

Assume $X \subset X'$ is a first order thickening where X is an affine scheme. Set $A = \Gamma(X, \mathcal{O}_X)$ and $A' = \Gamma(X', \mathcal{O}_{X'})$. By Lemma 8.4 the map $A \to A'$ is surjective. The kernel I is an ideal of square zero. By Properties of Spaces, Lemma 31.1 we obtain a canonical morphism $f : X' \to \mathrm{Spec}(A')$ which fits into the following commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X' \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(A') \end{tikzcd}$$

Because the horizontal arrows are thickenings it is clear that f is universally injective and surjective. Hence it suffices to show that f is étale, since then Morphisms of Spaces, Lemma 45.2 will imply that f is an isomorphism.

To prove that f is étale choose an affine scheme U' and an étale morphism $U' \to X'$. It suffices to show that $U' \to X' \to \mathrm{Spec}(A')$ is étale, see Properties of Spaces, Definition 13.2. Write $U' = \mathrm{Spec}(B')$. Set $U = X \times_{X'} U'$. Since U is a closed subspace of U', it is a closed subscheme, hence $U = \mathrm{Spec}(B)$ with $B' \to B$ surjective. Denote $J = \mathrm{Ker}(B' \to B)$ and note that $J = \Gamma(U, \mathcal{I})$ where $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X)$ on $X_{spaces,étale}$ as in the proof of Lemma 8.4. The morphism $U' \to X' \to \mathrm{Spec}(A')$ induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now, since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module we have $\mathcal{I} = (\widetilde{I})^a$, see Descent, Definition 7.2 for notation and Descent, Proposition 7.11 for why this is true. Hence we see that $J = I \otimes_A B$. Finally, note that $A \to B$ is étale as $U \to X$ is étale as the base change of the étale morphism $U' \to X'$. We conclude that $A' \to B'$ is étale by Algebra, Lemma 138.12. $\square$

Lemma 8.6. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. The functor*

$$V' \longmapsto V = X \times_{X'} V'$$

defines an equivalence of categories $X'_{\acute{e}tale} \to X_{\acute{e}tale}$.

Proof. The functor $V' \mapsto V$ defines an equivalence of categories $X'_{spaces,\acute{e}tale} \to X_{spaces,\acute{e}tale}$, see Theorem 7.1. Thus it suffices to show that V is a scheme if and only if V' is a scheme. This is the content of Lemma 8.5. $\square$

First order thickening are described as follows.

Lemma 8.7. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on $X_{\acute{e}tale}$ where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_B$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_B$-algebras, and $\mathcal{I}$ is its kernel. If

1. *$\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and*
2. *$\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module*

then there exists a first order thickening $X \subset X'$ over B and an isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ of $f^{-1}\mathcal{O}_B$-algebras compatible with the surjections to $\mathcal{O}_X$.

Proof. In this proof we redo some of the arguments used in the proofs of Lemmas 8.4 and 8.5. We first handle the case $B = S = \mathrm{Spec}(\mathbf{Z})$. Let U be an affine scheme, and let $U \to X$ be étale. Then

$$0 \to \mathcal{I}(U) \to \mathcal{A}(U) \to \mathcal{O}_X(U) \to 0$$

is exact as $H^1(U_{\acute{e}tale}, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 7.10 and Cohomology of Schemes, Lemma 2.2. If $V \to U$ is a morphism of affine objects of $X_{spaces,\acute{e}tale}$ then

$$\mathcal{I}(V) = \mathcal{I}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$$

since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module, see Descent, Proposition 7.11. Hence $\mathcal{A}(U) \to \mathcal{A}(V)$ is an étale ring map, see Algebra, Lemma 138.12. Hence we see that

$$U \longmapsto U' = \mathrm{Spec}(\mathcal{A}(U))$$

is a functor from $X_{affine,\acute{e}tale}$ to the category of affine schemes and étale morphisms. In fact, we claim that this functor can be extended to a functor $U \mapsto U'$ on all of $X_{\acute{e}tale}$. To see this, if U is an object of $X_{\acute{e}tale}$, note that

$$0 \to \mathcal{I}|_{U_{Zar}} \to \mathcal{A}|_{U_{Zar}} \to \mathcal{O}_X|_{U_{Zar}} \to 0$$

and $\mathcal{I}|_{U_{Zar}}$ is a quasi-coherent sheaf on U, see Descent, Proposition 7.14. Hence by More on Morphisms, Lemma 2.2 we obtain a first order thickening $U \subset U'$ of schemes such that $\mathcal{O}_{U'}$ is isomorphic to $\mathcal{A}|_{U_{Zar}}$. It is clear that this construction is compatible with the construction for affines above.

Choose a presentation $X = U/R$, see Spaces, Definition 9.3 so that $s, t : R \to U$ define an étale equivalence relation. Applying the functor above we obtain an étale equivalence relation $s', t' : R' \to U'$ in schemes. Consider the algebraic space $X' = U'/R'$ (see Spaces, Theorem 10.5). The morphism $X = U/R \to U'/R' = X'$ is a first order thickening. Consider $\mathcal{O}_{X'}$ viewed as a sheaf on $X_{\acute{e}tale}$. By construction we have an isomorphism

$$\gamma : \mathcal{O}_{X'}|_{U_{\acute{e}tale}} \longrightarrow \mathcal{A}|_{U_{\acute{e}tale}}$$

such that $s^{-1}\gamma$ agrees with $t^{-1}\gamma$ on $R_{\acute{e}tale}$. Hence by Properties of Spaces, Lemma 15.13 this implies that γ comes from a unique isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ as desired.

To handle the case of a general base algebraic space B, we first construct X' as an algebraic space over $\mathbf{Z}$ as above. Then we use the isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ to define $f^{-1}\mathcal{O}_B \to \mathcal{O}_{X'}$. According to Lemma 8.2 this defines a morphism $X' \to B$ compatible with the given morphism $X \to B$ and we are done. □

Lemma 8.8. *Let S be a scheme. Let $Y \subset Y'$ be a thickening of algebraic spaces over S. Let $X' \to Y'$ be a morphism and set $X = Y \times_{Y'} X'$. Then $(X \subset X') \to (Y \subset Y')$ is a morphism of thickenings. If $Y \subset Y'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.*

Proof. Omitted. □

Lemma 8.9. *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings of algebraic spaces over S. Then*

1. *f is an affine morphism if and only if f' is an affine morphism,*
2. *f is a surjective morphism if and only if f' is a surjective morphism,*
3. *f is quasi-compact if and only if f' quasi-compact,*
4. *f is universally closed if and only if f' is universally closed,*
5. *f is integral if and only if f' is integral,*
6. *f is (quasi-)separated if and only if f' is (quasi-)separated,*
7. *f is universally injective if and only if f' is universally injective,*
8. *f is universally open if and only if f' is universally open, and*
9. *add more here.*

Proof. Observe that $Y \to Y'$ and $X \to X'$ are integral and universal homeomorphisms. This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Limits of Spaces, Proposition 15.2 which tells us that there is a 1-to-1 correspondence between affine schemes étale over X and X' and between affine schemes étale over Y and Y'. Part (5) follows from (1) and (4) by Morphisms of Spaces, Lemma 41.7. Finally, note that

$$X \times_Y X = X \times_{Y'} X \to X \times_{Y'} X' \to X' \times_{Y'} X'$$

is a thickening (the two arrows are thickenings by Lemma 8.8). Hence applying (3) and (4) to the morphism $(X \subset X') \to (X \times_Y X \to X' \times_{Y'} X')$ we obtain (6). □

Lemma 8.10. *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings such that $X = S \times_{S'} X'$. If $S \subset S'$ is a finite order thickening, then*

1. *f is a closed immersion if and only if f' is a closed immersion,*
2. *f is locally of finite type if and only if f' is locally of finite type,*
3. *f is locally quasi-finite if and only if f' is locally quasi-finite,*
4. *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
5. *f is unramified if and only if f' is unramified,*
6. *f is proper if and only if f' is proper,*
7. *f is a finite morphism if and only if f' is an finite morphism, and*

(8) *add more here.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $V = Y \times_{Y'} V'$ and $U = X \times_{X'} U'$. Then for étale local properties of morphisms we can reduce to the morphism of thickenings of schemes $(U \subset U') \to (V \subset V')$ and apply More on Morphisms, Lemma 2.5. This proves (2), (3), (4), and (5).

The properties of morphisms in (1), (6), (7) are stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Spaces, Lemma 12.3, and Morphisms of Spaces, Lemmas 37.3, 41.5.

The interesting direction in (1), (6), (7) to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion on finite order thickenings above.

Proof of (1). Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$ and $U = X \times_Y V$. Then $U \to V$ is a closed immersion, which implies that U is a scheme, which in turn implies that U' is a scheme (Lemma 8.5). Thus we can apply the lemma in the case of schemes to $(U \subset U') \to (V \subset V')$ to conclude.

Proof of (6). Follows by combining (2) with results of Lemma 8.9 and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (7). Follows by combining (2) with results of Lemma 8.9 and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma 44.4). $\square$

9. First order infinitesimal neighbourhood

A natural construction of first order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of algebraic spaces. Choose an open subspace $U \subset X$ such that i identifies Z with a closed subspace $Z \subset U$ (see Morphisms of Spaces, Remark 12.4). Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U, see Morphisms of Spaces, Lemma 13.1. Then we can consider the closed subspace $Z' \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

Definition 9.1. Let $i : Z \to X$ be an immersion of algebraic spaces. The *first order infinitesimal neighbourhood* of Z in X is the first order thickening $Z \subset Z'$ over X described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

Lemma 9.2. *Let $i : Z \to X$ be an immersion of algebraic spaces. The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram*

$$\begin{tikzcd} Z \arrow[d, "i"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

Proof. Let $U \subset X$ be the open subspace used in the construction of Z', i.e., an open such that Z is identified with a closed subspace of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $|b|(|T'|) \subset |U|$. Hence we can think of b as a morphism into U, see Properties of Spaces, Lemma 4.9. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the square zero quasi-coherent sheaf of ideals cutting out T. By the commutativity of the diagram we have $b|_T = i \circ a$ where $i : Z \to U$ is the closed immersion. We conclude that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$ by Morphisms of Spaces, Lemma 13.1. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Morphisms of Spaces, Lemma 13.1 this implies that b factors through Z'. Letting $a' : T' \to Z'$ be this factorization we win. $\square$

Lemma 9.3. *Let $i : Z \to X$ be an immersion of algebraic spaces. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Lemma 4.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. $\square$

10. Formally smooth, étale, unramified transformations

Recall that a ring map $R \to A$ is called *formally smooth*, resp. *formally étale*, resp. *formally unramified* (see Algebra, Definition 133.1, resp. Definition 143.1, resp. Definition 141.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, there exists a, resp. exists a unique, resp. exists at most one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of algebraic spaces, and more generally functors.

Definition 10.1. Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Consider commutative solid diagrams of the form

$$\begin{tikzcd} F \arrow[d, "a"'] & T \arrow[l] \arrow[d, "i"] \\ G & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where T and T' are affine schemes and i is a closed immersion defined by an ideal of square zero.

(1) We say a is *formally smooth* if given any solid diagram as above there exists a dotted arrow making the diagram commute[1].
(2) We say a is *formally étale* if given any solid diagram as above there exists exactly one dotted arrow making the diagram commute.
(3) We say a is *formally unramified* if given any solid diagram as above there exists at most one dotted arrow making the diagram commute.

Lemma 10.2. *Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Then a is formally étale if and only if a is both formally smooth and formally unramified.*

Proof. Formal from the definition. $\square$

Lemma 10.3. *Composition.*
(1) *A composition of formally smooth transformations of functors is formally smooth.*
(2) *A composition of formally étale transformations of functors is formally étale.*
(3) *A composition of formally unramified transformations of functors is formally unramified.*

Proof. This is formal. $\square$

Lemma 10.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram*

$$\begin{tikzcd} H \times_{b,G,a} F \arrow[r, "b'"'] \arrow[d, "a'"'] & F \arrow[d, "a"] \\ H \arrow[r, "b"] & G \end{tikzcd}$$

(1) *If a is formally smooth, then the base change a' is formally smooth.*
(2) *If a is formally étale, then the base change a' is formally étale.*
(3) *If a is formally unramified, then the base change a' is formally unramified.*

Proof. This is formal. $\square$

Lemma 10.5. *Let S be a scheme. Let $F, G : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$ be a representable transformation of functors.*
(1) *If a is smooth then a is formally smooth.*
(2) *If a is étale, then a is formally étale.*
(3) *If a is unramified, then a is formally unramified.*

Proof. Consider a solid commutative diagram

$$\begin{tikzcd} F \arrow[d, "a"'] & T \arrow[l] \arrow[d, "i"] \\ G & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

[1]This is just one possible definition that one can make here. Another slightly weaker condition would be to require that the dotted arrow exists fppf locally on T'. This weaker notion has in some sense better formal properties.

as in Definition 10.1. Then $F \times_G T'$ is a scheme smooth (resp. étale, resp. unramified) over T'. Hence by More on Morphisms, Lemma 9.7 (resp. Lemma 6.9, resp. Lemma 4.8) we can fill in (resp. uniquely fill in, resp. fill in in at most one way) the dotted arrow in the diagram

$$\begin{tikzcd} F \times_G T' \arrow[d] & T \arrow[l] \arrow[d, "i"] \\ T' & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

an hence we also obtain the corresponding assertion in the first diagram. □

Lemma 10.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume that a is representable, surjective, and étale.*

1. *If b is formally smooth, then $b \circ a$ is formally smooth.*
2. *If b is formally étale, then $b \circ a$ is formally étale.*
3. *If b is formally unramified, then $b \circ a$ is formally unramified.*

Conversely, consider a solid commutative diagram

$$\begin{tikzcd} G \arrow[d, "b"] & T \arrow[l] \arrow[d, "i"] \\ H & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

with T' an affine scheme over S and $i : T \to T'$ a closed immersion defined by an ideal of square zero.

4. *If $b \circ a$ is formally smooth, then for every $t \in T$ there exists an étale morphism of affines $U' \to T'$ and a morphism $U' \to G$ such that*

$$\begin{tikzcd} G \arrow[d, "b"] & T \arrow[l] & T \times_{T'} U' \arrow[l] \arrow[d] \\ H & T' \arrow[l] & U' \arrow[l] \arrow[ull] \end{tikzcd}$$

commutes and t is in the image of $U' \to T'$.

5. *If $b \circ a$ is formally unramified, then there exists at most one dotted arrow in the diagram above, i.e., b is formally unramified.*
6. *If $b \circ a$ is formally étale, then there exists exactly one dotted arrow in the diagram above, i.e., b is formally étale.*

Proof. Assume b is formally smooth (resp. formally étale, resp. formally unramified). Since an étale morphism is both smooth and unramified we see that a is representable and smooth (resp. étale, resp. unramified). Hence parts (1), (2) and (3) follow from a combination of Lemma 10.5 and Lemma 10.3.

Assume that $b \circ a$ is formally smooth. Consider a diagram as in the statement of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Choose an affine open subscheme $U' \subset W'$ such that t is in the image of $U' \to T'$. Because $b \circ a$ is formally smooth we see that the exist morphisms $U' \to F$ such that

$$
\begin{tikzcd}
F \arrow[d, "b\circ a"'] & W \arrow[l] & T \times_{T'} U' \arrow[l] \arrow[d] \\
H & T' \arrow[l] & U' \arrow[l] \arrow[llu]
\end{tikzcd}
$$

commutes. Taking the composition $U' \to F \to G$ gives a map as in part (5) of the lemma.

Assume that $f, g : T' \to G$ are two dotted arrows fitting into the diagram of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Since a is formally étale the compositions

$$W' \to T' \xrightarrow{f} G \quad \text{and} \quad W' \to T' \xrightarrow{g} G$$

lift to morphisms $f', g' : W' \to F$ (lift on affine opens and glue by uniqueness). Now if $b \circ a : F \to H$ is formally unramified, then $f' = g'$ and hence $f = g$ as $W' \to T'$ is an étale covering. This proves part (6) of the lemma.

Assume that $b \circ a$ is formally étale. Then by part (4) we can étale locally on T' find a dotted arrow fitting into the diagram and by part (5) this dotted arrow is unique. Hence we may glue the local solutions to get assertion (6). Some details omitted. $\square$

Remark 10.7. It is tempting to think that in the situation of Lemma 10.6 we have "b formally smooth" $\Leftrightarrow$ "$b \circ a$ formally smooth". However, this is likely not true in general.

Lemma 10.8. *Let S be a scheme. Let $F, G, H : (Sch/S)_{fppf}^{opp} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume b is formally unramified.*

(1) *If $b \circ a$ is formally unramified then a is formally unramified.*
(2) *If $b \circ a$ is formally étale then a is formally étale.*
(3) *If $b \circ a$ is formally smooth then a is formally smooth.*

Proof. Let $T \subset T'$ be a closed immersion of affine schemes defined by an ideal of square zero. Let $g' : T' \to G$ and $f : T \to F$ be given such that $g'|_T = a \circ f$. Because b is formally unramified, there is a one to one correspondence between

$$\{f' : T' \to F \mid f = f'|_T \text{ and } a \circ f' = g'\}$$

and

$$\{f' : T' \to F \mid f = f'|_T \text{ and } b \circ a \circ f' = b \circ g'\}.$$

From this the lemma follows formally. $\square$

11. Formally unramified morphisms

In this section we work out what it means that a morphism of algebraic spaces is formally unramified.

Definition 11.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally unramified* if it is formally unramified as a transformation of functors as in Definition 10.1.

We will not restate the results proved in the more general setting of formally unramified transformations of functors in Section 10. It turns out we can characterize this property in terms of vanishing of the module of relative differentials, see Lemma 11.6.

Lemma 11.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally unramified,*
(2) *for every diagram*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally unramified (as in More on Morphisms, Definition 4.1), and
(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally unramified.*

Proof. Assume f is formally unramified. By Lemma 10.5 the morphisms $U \to X$ and $V \to Y$ are formally unramified. Thus by Lemma 10.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 10.8 that $U \to V$ is formally unramified. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 10.5 the morphism $V \to Y$ is formally unramified. Thus by Lemma 10.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 10.6 that $X \to Y$ is formally unramified, i.e., (1) holds. $\square$

Lemma 11.3. *Let S be a scheme. If $f : X \to Y$ is a formally unramified morphism of algebraic spaces over S, then given any solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of algebraic spaces over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 11.1 the condition that T be an affine scheme may be dropped.

Proof. This is true because there exists a surjective étale morphism $U' \to T'$ where U' is a disjoint union of affine schemes (see Properties of Spaces, Lemma 6.1) and a morphism $T' \to X$ is determined by its restriction to U'. $\square$

Lemma 11.4. *A composition of formally unramified morphisms is formally unramified.*

Proof. This is formal. $\square$

Lemma 11.5. *A base change of a formally unramified morphism is formally unramified.*

Proof. This is formal. $\square$

Lemma 11.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally unramified, and*
(2) *$\Omega_{X/Y} = 0$.*

Proof. This is a combination of Lemma 11.2, More on Morphisms, Lemma 4.7, and Lemma 6.3. $\square$

Lemma 11.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is unramified,*
(2) *the morphism f is locally of finite type and $\Omega_{X/Y} = 0$, and*
(3) *the morphism f is locally of finite type and formally unramified.*

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. Then we see

$$\begin{aligned} f \text{ unramified} &\Leftrightarrow \psi \text{ unramified} \\ &\Leftrightarrow \psi \text{ locally finite type and } \Omega_{U/V} = 0 \\ &\Leftrightarrow f \text{ locally finite type and } \Omega_{X/Y} = 0 \\ &\Leftrightarrow f \text{ locally finite type and formally unramified} \end{aligned}$$

Here we have used Morphisms, Lemma 36.2 and Lemma 11.6. $\square$

Lemma 11.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is unramified and a monomorphism,*
(2) *f is unramified and universally injective,*
(3) *f is locally of finite type and a monomorphism,*
(4) *f is universally injective, locally of finite type, and formally unramified.*

Moreover, in this case f is also representable, separated, and locally quasi-finite.

Proof. We have seen in Lemma 11.7 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_Y X$ is both an open immersion (Morphisms of Spaces, Lemma 35.9) and surjective (Morphisms of Spaces, Lemma 19.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1). Finally, we see that f is representable, separated, and locally quasi-finite by Morphisms of Spaces, Lemmas 26.10 and 45.1. $\square$

Lemma 11.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is a closed immersion,*
(2) *f is universally closed, unramified, and a monomorphism,*
(3) *f is universally closed, unramified, and universally injective,*

(4) *f is universally closed, locally of finite type, and a monomorphism,*
(5) *f is universally closed, universally injective, locally of finite type, and formally unramified.*

Proof. The equivalence of (2) – (5) follows immediately from Lemma 11.8. Moreover, if (2) – (5) are satisfied then f is representable. Similarly, if (1) is satisfied then f is representable. Hence the result follows from the case of schemes, see Étale Morphisms, Lemma 7.2. $\square$

12. Universal first order thickenings

Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. A *universal first order thickening* of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram

(12.0.1)
$$\begin{tikzcd} & Z \arrow[ld] & & T \arrow[ll, "a"'] \arrow[rd] & \\ Z' \arrow[rrd] & & & & T' \arrow[llll, dotted, "a'"'] \arrow[lld, "b"] \\ & & X & & \end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does. Before we prove this, let us show that a universal first order thickening in the category of schemes is a universal first order thickening in the category of algebraic spaces.

Lemma 12.1. *Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. Let $Z \subset Z'$ be a first order thickening over X. The following are equivalent*
(1) *$Z \subset Z'$ is a universal first order thickening,*
(2) *for any diagram (12.0.1) with T' a scheme a unique dotted arrow exists making the diagram commute, and*
(3) *for any diagram (12.0.1) with T' an affine scheme a unique dotted arrow exists making the diagram commute.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are formal. Assume (3) a assume given an arbitrary diagram (12.0.1). Choose a presentation $T' = U'/R'$, see Spaces, Definition 9.3. We may assume that $U' = \coprod U'_i$ is a disjoint union of affines, so $R' = U' \times_{T'} U' = \coprod_{i,j} U'_i \times'_T U'_j$. For each pair (i, j) choose an affine open covering $U'_i \times'_T U'_j = \bigcup_k R'_{ijk}$. Denote U_i, R_{ijk} the fibre products with T over T'. Then each $U_i \subset U'_i$ and $R_{ijk} \subset R'_{ijk}$ is a first order thickening of affine schemes. Denote $a_i : U_i \to Z$, resp. $a_{ijk} : R_{ijk} \to Z$ the composition of $a : T \to Z$ with the morphism $U_i \to T$, resp. $R_{ijk} \to T$. By (3) applied to $a_i : U_i \to Z$ we obtain unique morphisms $a'_i : U'_i \to Z'$. By (3) applied to a_{ijk} we see that the two compositions $R'_{ijk} \to R'_i \to Z'$ and $R'_{ijk} \to R'_j \to Z'$ are equal. Hence

$a' = \coprod a'_i : U' = \coprod U'_i \to Z'$ descends to the quotient sheaf $T' = U'/R'$ and we win. $\square$

Lemma 12.2. *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. If $Z \subset Z'$ is a universal first order thickening of Z over Y and $Y \to X$ is formally étale, then $Z \subset Z'$ is a universal first order thickening of Z over X.*

Proof. This is formal. Namely, by Lemma 12.1 it suffices to consider solid commutative diagrams (12.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y$ as $Y \to X$ is assumed formally étale. Hence the fact that $Z \subset Z'$ is a universal first order thickening over Y produces the desired morphism $a' : T' \to Z'$. $\square$

Lemma 12.3. *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. Assume $Z \to Y$ is étale.*

1. *If $Y \subset Y'$ is a universal first order thickening of Y over X, then the unique étale morphism $Z' \to Y'$ such that $Z = Y \times_{Y'} Z'$ (see Theorem 7.1) is a universal first order thickening of Z over X.*
2. *If $Z \to Y$ is surjective and $(Z \subset Z') \to (Y \subset Y')$ is an étale morphism of first order thickenings over X and Z' is a universal first order thickening of Z over X, then Y' is a universal first order thickening of Y over X.*

Proof. Proof of (1). By Lemma 12.1 it suffices to consider solid commutative diagrams (12.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y'$ as Y' is the universal first order thickening. Then the fact that $Z' \to Y'$ is étale implies (see Lemma 10.5) that $T' \to Y'$ lifts to the desired morphism $a' : T' \to Z'$.

Proof of (2). Let $T \subset T'$ be a first order thickening over X and let $a : T \to Y$ be a morphism. Set $W = T \times_Y Z$ and denote $c : W \to Z$ the projection Let $W' \to T'$ be the unique étale morphism such that $W = T \times_{T'} W'$, see Theorem 7.1. Note that $W' \to T'$ is surjective as $Z \to Y$ is surjective. By assumption we obtain a unique morphism $c' : W' \to Z'$ over X restricting to c on W. By uniqueness the two restrictions of c' to $W' \times_{T'} W'$ are equal (as the two restrictions of c to $W \times_T W$ are equal). Hence c' descends to a unique morphism $a' : T' \to Y'$ and we win. $\square$

Lemma 12.4. *Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S. There exists a universal first order thickening $Z \subset Z'$ of Z over X.*

Proof. Choose any commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

where V and U are schemes and the vertical arrows are étale. Note that $V \to U$ is a formally unramified morphism of schemes, see Lemma 11.2. Combining Lemma 12.1 and More on Morphisms, Lemma 5.1 we see that a universal first order thickening $V \subset V'$ of V over U exists. By Lemma 12.2 part (1) V' is a universal first order thickening of V over X.

Fix a scheme U and a surjective étale morphism $U \to X$. The argument above shows that for any $V \to Z$ étale with V a scheme such that $V \to Z \to X$ factors through U a universal first order thickening $V \subset V'$ of V over X exists (but does not depend on the chosen factorization of $V \to X$ through U). Now we may choose V such that $V \to Z$ is surjective étale (see Spaces, Lemma 11.4). Then $R = V \times_Z V$ a scheme étale over Z such that $R \to X$ factors through U also. Hence we obtain universal first order thickenings $V \subset V'$ and $R \subset R'$ over X. As $V \subset V'$ is a universal first order thickening, the two projections $s, t : R \to V$ lift to morphisms $s', t' : R' \to V'$. By Lemma 12.3 as R' is the universal first order thickening of R over X these morphisms are étale. Then $(t', s') : R' \to V'$ is an étale equivalence relation and we can set $Z' = V'/R'$. Since $V' \to Z'$ is surjective étale and v' is the universal first order thickening of V over X we conclude from Lemma 12.2 part (2) that Z' is a universal first order thickening of Z over X. $\square$

Definition 12.5. Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S.

(1) The *universal first order thickening* of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 12.4.
(2) The *conormal sheaf of Z over X* is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on $Z_{\acute{e}tale}$ and $\mathcal{C}_{Z/X}$ is a quasi-coherent $\mathcal{O}_Z$-module. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

Lemma 12.6. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then*

(1) *i is formally unramified,*
(2) *the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 9.1,*
(3) *the conormal sheaf of i in the sense of Definition 4.1 agrees with the conormal sheaf of i in the sense of Definition 12.5.*

Proof. An immersion of algebraic spaces is by definition a representable morphism. Hence by Morphisms, Lemmas 36.7 and 36.8 an immersion is unramified (via the abstract principle of Spaces, Lemma 5.8). Hence it is formally unramified by Lemma 11.7. The other assertions follow by combining Lemmas 9.2 and 9.3 and the definitions. $\square$

Lemma 12.7. *Let S be a scheme. Let $Z \to X$ be a formally unramified morphism of algebraic spaces over S. Then the universal first order thickening Z' is formally unramified over X.*

Proof. Let $T \subset T'$ be a first order thickening of affine schemes over X. Let

$$\begin{tikzcd} Z' \arrow[d] & T \arrow[l, "c"'] \arrow[d] \\ X & T' \arrow[l] \arrow[lu, "{a,b}"'] \end{tikzcd}$$

be a commutative diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. $\square$

Lemma 12.8. *Let S be a scheme Consider a commutative diagram of algebraic spaces over S*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y. There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram

$$\begin{tikzcd} & & Z' \arrow[ld] \arrow[d, "f'"] \\ Z \arrow[rru] \arrow[r] \arrow[d, "f"'] & X \arrow[d] & W' \arrow[ld] \\ W \arrow[r] \arrow[rru] & Y & \end{tikzcd}$$

In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Lemma 4.3 applied to $(Z \subset Z') \to (W \subset W')$. $\square$

Lemma 12.9. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening *of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 12.8 is surjective.*

Proof. The morphism h is formally unramified by Lemma 11.5. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 4.5 for why this implies that the map of conormal sheaves is surjective. $\square$

Lemma 12.10. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms of Spaces, Lemma 28.4. Hence the first statement follows from the description of W' in Lemma 12.9. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 4.5 for why this implies that the map of conormal sheaves is an isomorphism. $\square$

Lemma 12.11. *Taking the universal first order thickenings commutes with étale localization. More precisely, let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over a base scheme S. Let*

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

be a commutative diagram with étale vertical arrows. Let Z' be the universal first order thickening of Z over X. Then $V \to U$ is formally unramified and the universal first order thickening V' of V over U is étale over Z'. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.

Proof. The first statement is Lemma 11.2. The compatibility of universal first order thickenings is a consequence of Lemmas 12.2 and 12.3. $\square$

Lemma 12.12. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map*

$$dh' : (h')^*\Omega_{X/B} \to \Omega_{Z'/B}$$

induces an isomorphism $h^*\Omega_{X/B} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Lemma 6.6. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the case of schemes by étale localization, see Lemma 12.11 and Lemma 6.3. In this case the result is More on Morphisms, Lemma 5.9. $\square$

Lemma 12.13. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/B} \to \Omega_{Z/B} \to 0.$$

The first arrow is induced by $d_{Z'/B}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Lemma 6.10. Hence the result follows on applying Lemma 12.12. $\square$

Lemma 12.14. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 12.8 and the second from Lemma 12.13.

Proof. Since the maps have been defined, checking the sequence is exact reduces to the case of schemes by étale localization, see Lemma 12.11 and Lemma 6.3. In this case the result is More on Morphisms, Lemma 5.11. $\square$

Lemma 12.15. *Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S.*

1. *If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.*
2. *There is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 12.8 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 12.8. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Lemma 4.6 we have an exact sequence

$$(i')^*\mathcal{C}_{Y \times_{Y'} Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y \times_{Y'} Z'} \to 0$$

where $i' : Z \to Y \times_{Y'} Z'$ is the given morphism. By Lemma 4.5 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y \times_{Y'} Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y \times_{Y'} Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. $\square$

13. Formally étale morphisms

In this section we work out what it means that a morphism of algebraic spaces is formally étale.

Definition 13.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally étale* if it is formally étale as a transformation of functors as in Definition 10.1.

We will not restate the results proved in the more general setting of formally étale transformations of functors in Section 10.

Lemma 13.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally étale,*
(2) *for every diagram*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally étale (as in More on Morphisms, Definition 6.1), and
(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally étale.*

Proof. Assume f is formally étale. By Lemma 10.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. Thus by Lemma 10.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 10.8 that $U \to V$ is formally étale. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 10.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 10.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 10.6 that $X \to Y$ is formally étale, i.e., (1) holds. $\square$

Lemma 13.3. *Let S be a scheme. Let $f : X \to Y$ be a formally étale morphism of algebraic spaces over S. Then given any solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of algebraic spaces over Y there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 13.1 the condition that T be affine may be dropped.

Proof. Let $U' \to T'$ be a surjective étale morphism where $U' = \coprod U'_i$ is a disjoint union of affine schemes. Let $U_i = T \times_{T'} U'_i$. Then we get morphisms $a'_i : U'_i \to X$ such that $a'_i|_{U_i}$ equals the composition $U_i \to T \to X$. By uniqueness (see Lemma 11.3) we see that a'_i and a'_j agree on the fibre product $U'_i \times_{T'} U'_j$. Hence $\coprod a'_i : U' \to X$ descends to give a unique morphism $a' : T' \to X$. $\square$

Lemma 13.4. *A composition of formally étale morphisms is formally étale.*

Proof. This is formal. ☐

Lemma 13.5. *A base change of a formally étale morphism is formally étale.*

Proof. This is formal. ☐

Lemma 13.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S The following are equivalent:*

1. *f is formally étale,*
2. *f is formally unramified and the universal first order thickening of X over Y is equal to X,*
3. *f is formally unramified and $\mathcal{C}_{X/Y} = 0$, and*
4. *$\Omega_{X/Y} = 0$ and $\mathcal{C}_{X/Y} = 0$.*

Proof. Actually, the last assertion only make sense because $\Omega_{X/Y} = 0$ implies that $\mathcal{C}_{X/Y}$ is defined via Lemma 11.6 and Definition 12.5. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/Y} = 0$ since after all by definition $\mathcal{C}_{X/Y} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. ☐

Lemma 13.7. *An unramified flat morphism is formally étale.*

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 6.7 and étale localization, see Lemmas 11.2 and 13.2 and Morphisms of Spaces, Lemma 28.5. ☐

Lemma 13.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *The morphism f is étale, and*
2. *the morphism f is locally of finite presentation and formally étale.*

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 6.9 and étale localization, see Lemma 13.2 and Morphisms of Spaces, Lemmas 27.4 and 36.2. ☐

14. Infinitesimal deformations of maps

In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X, see Equations (8.1.1) and (8.1.2).

Lemma 14.1. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings of algebraic spaces over B. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over B. Assume that*

1. *$a = b$, and*
2. *the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Lemma 4.3) are equal.*

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_B$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutive diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}_{X/X'} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & a^{-1}\mathcal{C}_{Y/Y'} \arrow[r] \arrow[u] & a^{-1}\mathcal{O}_{Y'} \arrow[r] \arrow[u, shift left, "(a')^\sharp"'] \arrow[u, shift right, "(b')^\sharp"] & a^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_B$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_B$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_B$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write D as

(14.1.1) $$D = \mathrm{d}_{Y/B} \circ \theta$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/B} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$.

Lemma 14.2. *Let S be a scheme. Let B be an algebraic space over S. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over B. Let*

$$\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 14.1 hold and the derivation D and θ are related by Equation (14.1.1).

Proof. Consider the map

$$\alpha = (a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (14.1.1). As D is an $\mathcal{O}_B$-derivation it follows that α is a map of sheaves of $\mathcal{O}_B$-algebras. By construction we have $i_X^\sharp \circ \alpha = a^\sharp \circ i_Y^\sharp$ where $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the given closed immersions. By Lemma 8.2 we obtain a unique morphism $(a, b') : (X \subset X') \to (Y \subset Y')$ of thickenings over B such that $\alpha = (b')^\sharp$. Setting $b = a$ we win. □

Lemma 14.3. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over B. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an object U' of $(X')_{spaces,étale}$ with $U = X \times_{X'} U'$ consider morphisms $a' : U' \to Y'$ such that*

(1) *a' is a morphism over B,*
(2) *$a'|_U = a|_U$, and*
(3) *the induced map $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ is the restriction of A to U.*

Then the rule

(14.3.1) $$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\}$$

defines a sheaf of sets on $(X')_{spaces,étale}$.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms of algebraic spaces satisfy étale descent, see Descent on Spaces, Lemma 6.2. $\square$

Lemma 14.4. *Same notation and assumptions as in Lemma 14.3. We identify sheaves on X and X' via (8.1.1). There is an action of the sheaf*

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$$

on the sheaf (14.3.1). Moreover, the action is simply transitive for any object U' of $(X')_{spaces,étale}$ over which the sheaf (14.3.1) has a section.

Proof. This is a combination of Lemmas 14.1, 14.2, and 14.3. $\square$

Remark 14.5. A special case of Lemmas 14.1, 14.2, 14.3, and 14.4 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 14.3 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } S \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (14.1.1).

15. Infinitesimal deformations of algebraic spaces

The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

Lemma 15.1. *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings of algebraic spaces over S. Assume that f is flat. Then the following are equivalent*

1. *f' is flat and $X = Y \times_{Y'} X'$, and*
2. *the canonical map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $U = X \times_{X'} U'$ and $V = Y \times_{Y'} V'$. According to our definition of a flat morphism of algebraic spaces we see that the induced map $g : U \to V$ is a flat morphism of schemes and that f' is flat if and only if the corresponding morphism $g' : U' \to V'$ is flat. Also, $X = Y \times_{Y'} X'$ if and only if $U = V \times_{V'} V'$. Finally, the map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism if and only if $g^*\mathcal{C}_{V/V'} \to \mathcal{C}_{U/U'}$ is an isomorphism. Hence the lemma follows from its analogue for morphisms of schemes, see More on Morphisms, Lemma 8.1. $\square$

16. Formally smooth morphisms

In this section we introduce the notion of a formally smooth morphism $X \to Y$ of algebraic spaces. Such a morphism is characterized by the property that T-valued points of X lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 16.6. It turns out that this criterion is often easier to use than the Jacobian criterion.

Definition 16.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally smooth* if it is formally smooth as a transformation of functors as in Definition 10.1.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 11.3 and 13.3. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow étale locally on T'. In fact, analyzing the proof of Lemma 16.6 shows that this would be equivalent to the definition as it currently stands. It is also true that requiring the existence of the dotted arrow fppf locally on T' would be sufficient, but that is slightly more difficult to prove.

We will not restate the results proved in the more general setting of formally smooth transformations of functors in Section 10.

Lemma 16.2. *A composition of formally smooth morphisms is formally smooth.*

Proof. Omitted. ☐

Lemma 16.3. *A base change of a formally smooth morphism is formally smooth.*

Proof. Omitted, but see Algebra, Lemma 133.2 for the algebraic version. ☐

Lemma 16.4. *Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.*

Proof. Omitted. ☐

Here is a helper lemma which will be superseded by Lemma 16.9.

Lemma 16.5. *Let S be a scheme. Let*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a commutative diagram of morphisms of algebraic spaces over S. If the vertical arrows are étale and f is formally smooth, then ψ is formally smooth.

Proof. By Lemma 10.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. By Lemma 10.3 the composition $U \to Y$ is formally smooth. By Lemma 10.8 we see $\psi : U \to V$ is formally smooth. ☐

The following lemma is the main result of this section. It implies, combined with Limits of Spaces, Proposition 3.9, that we can recognize whether a morphism of algebraic spaces $f : X \to Y$ is smooth in terms of "simple" properties of the transformation of functors $X \to Y$.

Lemma 16.6 (Infinitesimal lifting criterion)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is smooth.*
(2) *The morphism f is locally of finite presentation, and formally smooth.*

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. By Lemma 16.5 we see $\psi : U \to V$ is formally smooth. By Morphisms of Spaces, Lemma 27.4 the morphism ψ is locally of finite presentation. Hence by the case of schemes the morphism ψ is smooth, see More on Morphisms, Lemma 9.7. Hence f is smooth, see Morphisms of Spaces, Lemma 34.4.

Conversely, assume that $f : X \to Y$ is smooth. Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 16.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 14.3, see also Remark 14.5. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{étale}$ introduced in Lemma 14.4. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 5.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 5.3.

First we prove (I). To see this choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes and the vertical arrows are étale and surjective. As f is assumed smooth we see that ψ is smooth and hence formally smooth by Lemma 10.5. By the same lemma the morphism $V \to Y$ is formally étale. Thus by Lemma 10.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 10.6 part (4).

Finally we prove (II). By Lemma 6.15 we see that $\Omega_{X/S}$ is of finite presentation. Hence $a^*\Omega_{X/S}$ is of finite presentation (see Properties of Spaces, Section 28). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$ is quasi-coherent by Properties of Spaces, Lemma 27.7. Thus by Descent, Proposition 7.10 and Cohomology of Schemes, Lemma 2.2 we have

$$H^1(T_{spaces,étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}) = H^1(T, \mathcal{H}) = 0$$

as desired. $\square$

We do a bit more work to show that being formally smooth is étale local on the source. To begin we show that a formally smooth morphism has a nice sheaf of differentials. The notion of a locally projective quasi-coherent module is defined in Properties of Spaces, Section 29.

Lemma 16.7. *Let S be a scheme. Let $f : X \to Y$ be a formally smooth morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is locally projective on X.*

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are affine(!) schemes and the vertical arrows are étale. By Lemma 16.5 we see $\psi : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see More on Morphisms, Lemma 9.6. Hence by Algebra, Lemma 133.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 19. Since $\Omega_{X/Y}|_U = \Omega_{U/V}$ we see that $\Omega_{X/Y}$ is locally projective too. (Because we can find an étale covering of X by the affine U's fitting into diagrams as above – details omitted.) $\square$

Lemma 16.8. *Let T be an affine scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_T$-modules on $T_{étale}$. Consider the internal hom sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}, \mathcal{G})$ on $T_{étale}$. If $\mathcal{F}$ is locally projective, then $H^1(T_{étale}, \mathcal{H}) = 0$.*

Proof. By the definition of a locally projective sheaf on an algebraic space (see Properties of Spaces, Definition 29.2) we see that $\mathcal{F}_{Zar} = \mathcal{F}|_{T_{Zar}}$ is a locally projective sheaf on the scheme T. Thus $\mathcal{F}_{Zar}$ is a direct summand of a free $\mathcal{O}_{T_{Zar}}$-module. Whereupon we conclude (as $\mathcal{F} = (\mathcal{F}_{Zar})^a$, see Descent, Proposition 7.11) that $\mathcal{F}$ is a direct summand of a free $\mathcal{O}_T$-module on $T_{étale}$. Hence we may assume that $\mathcal{F} = \bigoplus_{i \in I} \mathcal{O}_T$ is a free module. In this case $\mathcal{H} = \prod_{i \in I} \mathcal{G}$ is a product of quasi-coherent modules. By Cohomology on Sites, Lemma 12.5 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Descent, Proposition 7.10 and Cohomology of Schemes, Lemma 2.2. $\square$

Lemma 16.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is formally smooth,*
2. *for every diagram*

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

   where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally smooth (as in More on Morphisms, Definition 4.1), and
3. *for one such diagram with surjective vertical arrows the morphism ψ is formally smooth.*

Proof. We have seen that (1) implies (2) and (3) in Lemma 16.5. Assume (3). The proof that f is formally smooth is entirely similar to the proof of (1) $\Rightarrow$ (2) of Lemma 16.6.

Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 16.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 14.3, see also Remark 14.5. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{étale}$ introduced in Lemma 14.4. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 5.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 5.3.

First we prove (I). To see this consider a diagram (which exists because we are assuming (3))

$$\begin{tikzcd} U \arrow[r, "\psi"'] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes, the vertical arrows are étale and surjective, and ψ is formally smooth. By Lemma 10.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 10.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 10.6 part (4).

Finally we prove (II). By Lemma 16.7 we see that $\Omega_{U/V}$ locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Descent on Spaces, Lemma 5.5. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties of Spaces, Lemma 29.3. Hence

$$H^1(T_{étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'}) = 0$$

by Lemma 16.8 as desired. $\square$

Lemma 16.10. *The property $\mathcal{P}(f) =$ "f is formally smooth" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S. Choose an index set I and diagrams

$$\begin{tikzcd} U_i \arrow[r, "\psi_i"'] \arrow[d] & V_i \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with étale vertical arrows and U_i, V_i affine schemes. Moreover, assume that $\coprod U_i \to X$ and $\coprod V_i \to Y$ are surjective, see Properties of Spaces, Lemma 6.1. By Lemma 16.9 we see that f is formally smooth if and only if each of the morphisms ψ_i are formally smooth. Hence we reduce to the case of a morphism of affine schemes. In this case the result follows from Algebra, Lemma 133.15. Some details omitted. □

Lemma 16.11. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume f is formally smooth. Then*

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Z} \to 0$$

Lemma 6.8 is short exact.

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 9.9, by étale localization, see Lemmas 16.9 and 6.3. □

Lemma 16.12. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Assume that Z is formally smooth over B. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 12.13 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 12.13 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z \to \Omega_{Z/B} \to 0.$$

Since $Z \to S$ is formally smooth we can étale locally on Z' find a left inverse $Z' \to Z$ over B to the inclusion map $Z \to Z'$. Thus the sequence is étale locally split, see Lemma 6.11. □

Lemma 16.13. *Let S be a scheme. Let*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[dr, "j"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 12.14 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 12.13 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"'] \arrow[dr, "j'"'] & Z' \arrow[r, "a"'] \arrow[d, "k"] & X \arrow[d, "f"] \\ & Z'' \arrow[r, "b"] & Y \end{tikzcd}$$

The sequence above is identified with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

via our definitions concerning conormal sheaves of formally unramified morphisms. Let $U'' \to Z''$ be an étale morphism with U'' affine. Denote $U \to Z$ and $U' \to Z'$ the corresponding affine schemes étale over Z and Z'. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Z' \arrow[d, "k"] \\ U'' \arrow[r] \arrow[ur, "g"] & Z'' \end{tikzcd}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. ☐

17. Smoothness over a Noetherian base

This section is the analogue of More on Morphisms, Section 10.

Lemma 17.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Assume that Y is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth at x,*
(2) *for every solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ Y & \mathrm{Spec}(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

where $B' \to B$ is a surjection of local rings with $Ker(B' \to B)$ of square zero, and α mapping the closed point of $\mathrm{Spec}(B)$ to x there exists a dotted arrow making the diagram commute, and
(3) *same as in (2) but with $B' \to B$ ranging over small extensions (see Algebra, Definition 136.1).*

Proof. Condition (1) means there is an open subspace $X' \subset X$ such that $X' \to Y$ is smooth. Hence (1) implies conditions (2) and (3) by Lemma 16.6. Condition (2) implies condition (3) trivially. Assume (3). Choose a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] \\ Y & V \arrow[l] \end{tikzcd}$$

with U and V affine, horizontal arrows étale and such that there is a point $u \in U$ mapping to x. Next, consider a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ Y & V \arrow[l] & \mathrm{Spec}(B') \arrow[l, "\beta"] \end{tikzcd}$$

as in (3) but for $u \in U \to V$. Let $\gamma : \mathrm{Spec}(B') \to X$ be the arrow we get from our assumption that (3) holds for X. Because $U \to X$ is étale and hence formally étale (Lemma 13.8) the morphism γ has a unique lift to U compatible with α. Then because $V \to Y$ is étale hence formally étale this lift is compatible with β. Hence (3) holds for $u \in U \to V$ and we conclude that $U \to V$ is smooth at u by More on Morphisms, Lemma 10.1. This proves that $X \to Y$ is smooth at x, thereby finishing the proof. $\square$

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions "centered" at points of finite type (see Morphisms of Spaces, Section 25).

Lemma 17.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume Y is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth,*
(2) *for every solid commutative diagram*

$$\begin{tikzcd}
X \arrow[d, "f"'] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\
Y & \mathrm{Spec}(B') \arrow[l, "\beta"'] \arrow[lu, dashed]
\end{tikzcd}$$

where $B' \to B$ is a small extension of Artinian local rings and β of finite type (!) there exists a dotted arrow making the diagram commute.

Proof. If f is smooth, then the infinitesimal lifting criterion (Lemma 16.6) says f is formally smooth and (2) holds.

Assume f is not smooth. The set of points $x \in X$ where f is not smooth forms a closed subset T of $|X|$. By Morphisms of Spaces, Lemma 25.6, there exists a point $x \in T \subset X$ with $x \in X_{\text{ft-pts}}$. Choose a commutative diagram

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l] \arrow[d] & u \arrow[d, mapsto] \\
Y & V \arrow[l] & v
\end{tikzcd}$$

with U and V affine, horizontal arrows étale and such that there is a point $u \in U$ mapping to x. Then u is a finite type point of U. Since $U \to V$ is not smooth at the point u, by More on Morphisms, Lemma 10.1 there is a diagram

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l] \arrow[d] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\
Y & V \arrow[l] & \mathrm{Spec}(B') \arrow[l, "\beta"'] \arrow[lu, dashed]
\end{tikzcd}$$

with $B' \to B$ a small extension of (Artinian) local rings such that the residue field of B is equal to $\kappa(v)$ and such that the dotted arrow does not exist. Since $U \to V$ is of finite type, we see that v is a finite type point of V. By Morphisms, Lemma 17.2 the morphism β is of finite type, hence the composition $\mathrm{Spec}(B) \to Y$ is of finite type also. Arguing exactly as in the proof of Lemma 17.1 (using that $U \to X$ and $V \to Y$ are étale hence formally étale) we see that there cannot be an arrow $\mathrm{Spec}(B) \to X$ fitting into the outer rectangle of the last displayed diagram. In other words, (2) doesn't hold and the proof is complete. $\square$

Here is a useful application.

Lemma 17.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y locally Noetherian. Let $Z \subset Y$ be a closed subspace with nth infinitesimal neighbourhood $Z_n \subset Y$. Set $X_n = Z_n \times_Y X$.*

(1) *If $X_n \to Z_n$ is smooth for all n, then f is smooth at every point of $f^{-1}(Z)$.*
(2) *If $X_n \to Z_n$ is étale for all n, then f is étale at every point of $f^{-1}(Z)$.*

Proof. Assume $X_n \to Z_n$ is smooth for all n. Let $x \in X$ be a point lying over a point of Z. Given a small extension $B' \to B$ and morphisms α, β as in Lemma 17.1 part (3) the maximal ideal of B' is nilpotent (as B' is Artinian) and hence the morphism β factors through Z_n and α factors through X_n for a suitable n. Thus the lifting property for $X_n \to Z_n$ kicks in to get the desired dotted arrow in the diagram. This proves (1). Part (2) follows from (1) and the fact that a morphism is étale if and only if it is smooth of relative dimension 0. $\square$

18. Openness of the flat locus

This section is analogue of More on Morphisms, Section 12. Note that we have defined the notion of flatness for quasi-coherent modules on algebraic spaces in Morphisms of Spaces, Section 29.

Theorem 18.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume f is locally of finite presentation and that $\mathcal{F}$ is an $\mathcal{O}_X$-module which is locally of finite presentation. Then*

$$\{x \in |X| : \mathcal{F} \text{ is flat over } Y \text{ at } x\}$$

is open in $|X|$.

Proof. Choose a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d, "p"'] & V \arrow[d, "q"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

with U, V schemes and p, q surjective and étale as in Spaces, Lemma 11.4. By More on Morphisms, Theorem 12.1 the set $U' = \{u \in |U| : p^*\mathcal{F} \text{ is flat over } V \text{ at } u\}$ is open in U. By Morphisms of Spaces, Definition 29.2 the image of U' in $|X|$ is the set of the theorem. Hence we are done because the map $|U| \to |X|$ is open, see Properties of Spaces, Lemma 4.6. $\square$

Lemma 18.2. *Let S be a scheme. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a cartesian diagram of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation. Then

$$\{x' \in |X'| : (g')^*\mathcal{F} \text{ is flat over } Y' \text{ at } x'\}$$

is the inverse image of the open subset of Theorem 18.1 under the continuous map $|g'| : |X'| \to |X|$.

Proof. This follows from Morphisms of Spaces, Lemma 29.3. $\square$

19. Critère de platitude par fibres

Let S be a scheme. Consider a commutative diagram of algebraic spaces over S

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in |X|$ we consider the question as to whether $\mathcal{F}$ is flat over Y at x. If $\mathcal{F}$ is flat over Z at x, then the theorem below states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre of $X \to Z$ over $g(x)$ is flat over the fibre of $Y \to Z$ over $g(x)$. To make sense out of this we offer the following preliminary lemma.

Lemma 19.1. *In the situation above the following are equivalent*

(1) *Pick a geometric point* $\overline{x}$ *of* X *lying over* x. *Set* $\overline{y} = f \circ \overline{x}$ *and* $\overline{z} = g \circ \overline{x}$. *Then the module* $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}}$ *is flat over* $\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}$.
(2) *Pick a morphism* $x : \mathrm{Spec}(K) \to X$ *in the equivalence class of* x. *Set* $z = g \circ x$, $X_z = \mathrm{Spec}(K) \times_{z,Z} X$, $Y_z = \mathrm{Spec}(K) \times_{z,Z} Y$, *and* $\mathcal{F}_z$ *the pullback of* $\mathcal{F}$ *to* X_z. *Then* $\mathcal{F}_z$ *is flat at* x *over* Y_z *(as defined in Morphisms of Spaces, Definition 29.2).*
(3) *Pick a commutative diagram*

$$\begin{tikzcd} & & U \arrow[lld, "a"'] \arrow[rr] \arrow[dr] & & V \arrow[lld, "b"'] \arrow[dl] \\ X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[dl, "h"] & & W \arrow[llld, "c"] & \\ Z & & & & \end{tikzcd}$$

where U, V, W *are schemes, and* a, b, c *are étale, and a point* $u \in U$ *mapping to* x. *Let* $w \in W$ *be the image of* u. *Let* $\mathcal{F}_w$ *be the pullback of* $\mathcal{F}$ *to the fibre* U_w *of* $U \to W$ *at* w. *Then* $\mathcal{F}_w$ *is flat over* V_w *at* u.

Proof. Note that in (2) the morphism $x : \mathrm{Spec}(K) \to X$ defines a K-rational point of X_z, hence the statement makes sense. Moreover, the condition in (2) is independent of the choice of $\mathrm{Spec}(K) \to X$ in the equivalence class of x (details omitted; this will also follow from the arguments below because the other conditions do not depend on this choice). Also note that we can always choose a diagram as in (3) by: first choosing a scheme W and a surjective étale morphism $W \to Z$, then choosing a scheme V and a surjective étale morphism $V \to W \times_Z Y$, and finally choosing a scheme U and a surjective étale morphism $U \to V \times_Y X$. Having made these choices we set $U \to W$ equal to the composition $U \to V \to W$ and we can pick a point $u \in U$ mapping to x because the morphism $U \to X$ is surjective.

Suppose given both a diagram as in (3) and a geometric point $\overline{x} : \mathrm{Spec}(k) \to X$ as in (1). By Properties of Spaces, Lemma 16.4 we can choose a geometric point

$\overline{u} : \mathrm{Spec}(k) \to U$ lying over u such that $\overline{x} = a \circ \overline{u}$. Denote $\overline{v} : \mathrm{Spec}(k) \to V$ and $\overline{w} : \mathrm{Spec}(k) \to W$ the induced geometric points of V and W. In this setting we know that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,u}^{sh}$ and similarly for Y and Z, see Properties of Spaces, Lemma 19.1. In the same vein we have

$$\mathcal{F}_{\overline{x}} = (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U,u}^{sh}$$

see Properties of Spaces, Lemma 27.4. Note that the stalk of $\mathcal{F}_w$ at u is given by

$$(\mathcal{F}_w)_u = (a^*\mathcal{F})_u / \mathfrak{m}_w (a^*\mathcal{F})_u$$

and the local ring of V_w at v is given by

$$\mathcal{O}_{V_w,v} = \mathcal{O}_{V,v}/\mathfrak{m}_w\mathcal{O}_{V,v}.$$

Since $\mathfrak{m}_{\overline{z}} = \mathfrak{m}_w\mathcal{O}_{Z,\overline{z}} = \mathfrak{m}_w\mathcal{O}_{W,w}^{sh}$ we see that

$$\begin{aligned} \mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}} & = (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}} \\ & = (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}_{U,u}^{sh}/\mathfrak{m}_w\mathcal{O}_{U,u}^{sh} \\ & = (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}_{U_w,\overline{u}}^{sh} \\ & = (\mathcal{F}_w)_{\overline{u}} \end{aligned}$$

the penultimate equality by Algebra, Lemma 145.30 and the last equality by Properties of Spaces, Lemma 27.4. The same arguments applied to the structure sheaves of V and Y show that

$$\mathcal{O}_{V_w,\overline{v}}^{sh} = \mathcal{O}_{V,v}^{sh}/\mathfrak{m}_w\mathcal{O}_{V,v}^{sh} = \mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}.$$

OK, and now we can use Morphisms of Spaces, Lemma 29.1 to see that (1) is equivalent to (3).

Finally we prove the equivalence of (2) and (3). To do this we pick a field extension $\tilde{K}$ of K and and a morphism $\tilde{x} : \mathrm{Spec}(\tilde{K}) \to U$ which lies over u (this is possible because $u \times_{X,x} \mathrm{Spec}(K)$ is a nonempty scheme). Set $\tilde{z} : \mathrm{Spec}(\tilde{K}) \to U \to W$ be the composition. We obtain a commutative diagram

$$\begin{tikzcd} & & & U_w \times_w \tilde{z} \arrow[lll, "a"'] \arrow[rr] \arrow[dr] & & V_w \times_w \tilde{z} \arrow[dlll, "b"'] \arrow[dl] \\ X_z \arrow[rr, "f"] \arrow[dr, "g"'] & & Y_z \arrow[dl, "h"] & & \tilde{z} \arrow[dlll, "c"] & \\ & z & & & & \end{tikzcd}$$

where $z = \mathrm{Spec}(K)$ and $w = \mathrm{Spec}(\kappa(w))$. Now it is clear that $\mathcal{F}_w$ and $\mathcal{F}_z$ pull back to the same module on $U_w \times_w \tilde{z}$. This leads to a commutative diagram

$$\begin{tikzcd} X_z \arrow[d] & U_w \times_w \tilde{z} \arrow[l] \arrow[r] \arrow[d] & U_w \arrow[d] \\ Y_z & V_w \times_w \tilde{z} \arrow[l] \arrow[r] & V_w \end{tikzcd}$$

both of whose squares are cartesian and whose bottom horizontal arrows are flat: the lower left horizontal arrow is the composition of the morphism $Y \times_Z \tilde{z} \to Y \times_Z z = Y_z$ (base change of a flat morphism), the étale morphism $V \times_Z \tilde{z} \to Y \times_Z \tilde{z}$, and the étale morphism $V \times_W \tilde{z} \to V \times_Z \tilde{z}$. Thus it follows from Morphisms of Spaces, Lemma 29.3 that

$$\mathcal{F}_z \text{ flat at } x \text{ over } Y_z \Leftrightarrow \mathcal{F}|_{U_w \times_w \tilde{z}} \text{ flat at } \tilde{x} \text{ over } V_w \times_w \tilde{z} \Leftrightarrow \mathcal{F}_w \text{ flat at } u \text{ over } V_w$$

and we win. □

Definition 19.2. Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$ be a point and denote $z \in |Z|$ its image.

1. We say *the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z* if the equivalent conditions of Lemma 19.1 are satisfied.
2. We say *the fibre of X over z is flat at x over the fibre of Y over z* if the equivalent conditions of Lemma 19.1 hold with $\mathcal{F} = \mathcal{O}_X$.
3. We say *the fibre of X over z is flat over the fibre of Y over z* if for all $x \in |X|$ lying over z the fibre of X over z is flat at x over the fibre of Y over z

With this definition in hand we can state a version of the criterion as follows. The Noetherian version can be found in Section 20.

Theorem 19.3. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

1. *X is locally of finite presentation over Z,*
2. *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and*
3. *Y is locally of finite type over Z.*

Let $x \in |X|$ and let $y \in |Y|$ and $z \in |Z|$ be the images of x. If $\mathcal{F}_{\overline{x}} \neq 0$, then the following are equivalent:

1. *$\mathcal{F}$ is flat over Z at x and the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, and*
2. *Y is flat over Z at y and $\mathcal{F}$ is flat over Y at x.*

Moreover, the set of points x where (1) and (2) hold is open in $Supp(\mathcal{F})$.

Proof. Choose a diagram as in Lemma 19.1 part (3). It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. Thus the theorem follows from the corresponding result for schemes which is More on Morphisms, Theorem 13.2. □

Lemma 19.4. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

1. *X is locally of finite presentation over Z,*
2. *X is flat over Z,*
3. *for every $z \in |Z|$ the fibre of X over z is flat over the fibre of Y over z, and*
4. *Y is locally of finite type over Z.*

Then f is flat. If f is also surjective, then Y is flat over Z.

Proof. This is a special case of Theorem 19.3. □

Lemma 19.5. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

1. *X is locally of finite presentation over Z,*

(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,*
(3) *$\mathcal{F}$ is flat over Z, and*
(4) *Y is locally of finite type over Z.*

Then the set

$$A = \{x \in |X| : \mathcal{F} \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change: If $Z' \to Z$ is a morphism of algebraic spaces, and A' is the set of points of $X' = X \times_Z Z'$ where $\mathcal{F}' = \mathcal{F} \times_Z Z'$ is flat over $Y' = Y \times_Z Z'$, then A' is the inverse image of A under the continuous map $|X'| \to |X|$.

Proof. One way to prove this is to translate the proof as given in More on Morphisms, Lemma 13.4 into the category of algebraic spaces. Instead we will prove this by reducing to the case of schemes. Namely, choose a diagram as in Lemma 19.1 part (3) such that a, b, and c are surjective. It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. The only minor point to make is that given a morphism of algebraic spaces $Z' \to Z$ we choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Then we set $U' = W' \times_W U$ and $V' = W' \times_W V$. We write a', b', c' for the morphisms from U', V', W' to X', Y', Z'. In this case A, resp. A' are images of the open subsets of U, resp. U' associated to $a^*\mathcal{F}$, resp. $(a')^*\mathcal{F}'$. This indeed does reduce the lemma to More on Morphisms, Lemma 13.4. ☐

Lemma 19.6. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *X is flat over Z, and*
(3) *Y is locally of finite type over Z.*

Then the set

$$\{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change $Z' \to Z$.

Proof. This is a special case of Lemma 19.5. ☐

20. Flatness over a Noetherian base

Here is the "Critère de platitude par fibres" in the Noetherian case.

Theorem 20.1. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X, Y, Z locally Noetherian, and*
(2) *$\mathcal{F}$ a coherent $\mathcal{O}_X$-module.*

Let $x \in |X|$ and let $y \in |Y|$ and $z \in |Z|$ be the images of x. If $\mathcal{F}_{\overline{x}} \neq 0$, then the following are equivalent:

(1) *$\mathcal{F}$ is flat over Z at x and the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, and*
(2) *Y is flat over Z at y and $\mathcal{F}$ is flat over Y at x.*

Proof. Choose a diagram as in Lemma 19.1 part (3). It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. Thus the theorem follows from the corresponding result for schemes which is More on Morphisms, Theorem 13.1. □

Lemma 20.2. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

1. *X, Y, Z locally Noetherian,*
2. *X is flat over Z,*
3. *for every $z \in |Z|$ the fibre of X over z is flat over the fibre of Y over z.*

Then f is flat. If f is also surjective, then Y is flat over Z.

Proof. This is a special case of Theorem 20.1. □

Just like for checking smoothness, if the base is Noetherian it suffices to check flatness over Artinian rings. Here is a sample statement.

Lemma 20.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let X be an algebraic space locally of finite presentation over $S = \mathrm{Spec}(A)$. For $n \geq 1$ set $S_n = \mathrm{Spec}(A/I^n)$ and $X_n = S_n \times_S X$. Let $\mathcal{F}$ be coherent $\mathcal{O}_X$-module. If for every $n \geq 1$ the pullback $\mathcal{F}_n$ of $\mathcal{F}$ to X is flat over S_n, then the (open) locus where $\mathcal{F}$ is flat over X contains the inverse image of $V(I)$ under $X \to S$.*

Proof. The locus where $\mathcal{F}$ is flat over S is open in $|X|$ by Theorem 18.1. The statement is insensitive to replacing X by the members of an étale covering, hence we may assume X is an affine scheme. In this case the result follows immediately from Algebra, Lemma 95.10. Some details omitted. □

21. Normalization revisited

Normalization commutes with smooth base change.

Lemma 21.1. *Let S be a scheme. Let $f : Y \to X$ be a smooth morphism of algebraic spaces over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The integral closure of $\mathcal{O}_Y$ in $f^*\mathcal{A}$ is equal to $f^*\mathcal{A}'$ where $\mathcal{A}' \subset \mathcal{A}$ is the integral closure of $\mathcal{O}_X$ in $\mathcal{A}$.*

Proof. By our construction of the integral closure, see Morphisms of Spaces, Definition 43.2, this reduces immediately to the case where X and Y are affine. In this case the result is Algebra, Lemma 140.4. □

Lemma 21.2 (Normalization commutes with smooth base change)**.** *Let S be a scheme. Let*

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d] & Y_1 \arrow[d, "f"] \\ X_2 \arrow[r, "\varphi"] & X_1 \end{tikzcd}$$

be a fibre square of algebraic spaces over S. Assume f is quasi-compact and quasi-separated and φ is smooth. Let $Y_i \to X_i' \to X_i$ be the normalization of X_i in Y_i. Then $X_2' \cong X_2 \times_{X_1} X_1'$.

Proof. The base change of the factorization $Y_1 \to X_1' \to X_1$ to X_2 is a factorization $Y_2 \to X_2 \times_{X_1} X_1' \to X_1$ and $X_2 \times_{X_1} X_1' \to X_1$ is integral (Morphisms of Spaces, Lemma 41.5). Hence we get a morphism $h : X_2' \to X_2 \times_{X_1} X_1'$ by the universal property of Morphisms of Spaces, Lemma 43.5. Observe that X_2' is the relative spectrum of the integral closure of $\mathcal{O}_{X_2}$ in $f_{2,*}\mathcal{O}_{Y_2}$. If $\mathcal{A}' \subset f_{1,*}\mathcal{O}_{Y_1}$ denotes the integral closure of $\mathcal{O}_{X_2}$, then $X_2 \times_{X_1} X_1'$ is the relative spectrum of $\varphi^*\mathcal{A}'$ as the construction of the relative spectrum commutes with arbitrary base change. By Cohomology of Spaces, Lemma 10.1 we know that $f_{2,*}\mathcal{O}_{Y_2} = \varphi^* f_{1,*}\mathcal{O}_{Y_1}$. Hence the result follows from Lemma 21.1. $\square$

22. Slicing Cohen-Macaulay morphisms

Let S be a scheme. Let X be an algebraic space over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. In this case we denote $V(f_1, \ldots, f_r)$ the *closed subspace of X cut out by $f_1, \ldots, f_r$*. More precisely, we can define $V(f_1, \ldots, f_r)$ as the closed subspace of X corresponding to the quasi-coherent sheaf of ideals generated by $f_1, \ldots, f_r$, see Morphisms of Spaces, Lemma 13.1. Alternatively, we can choose a presentation $X = U/R$ and consider the closed subscheme $Z \subset U$ cut out by $f_1|U, \ldots, f_r|_U$. It is clear that Z is an R-invariant (see Groupoids, Definition 17.1) closed subscheme and we may set $V(f_1, \ldots, f_r) = Z/R_Z$.

Lemma 22.1. *Let S be a scheme. Consider a cartesian diagram*

$$\begin{tikzcd} X \arrow[d] & F \arrow[l, "p"'] \arrow[d] \\ Y & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is flat and locally of finite presentation, and where k is a field over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow Y$$

is flat and locally of finite presentation.

Proof. Set $Z = V(f_1, \ldots, f_r)$. It is clear that $Z \to X$ is locally of finite presentation, hence the composition $Z \to Y$ is locally of finite presentation, see Morphisms of Spaces, Lemma 27.2. Hence it suffices to show that $Z \to Y$ is flat in a neighbourhood of $p(z)$. Let $k \subset k'$ be an extension field. Then $F' = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is surjective and flat over F, hence we can find a point $z' \in |F'|$ mapping to z and the local ring map $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces, Lemma 28.8. Hence the image of $f_1, \ldots, f_r$ in $\mathcal{O}_{F',\overline{z}'}$ is a regular sequence too, see Algebra, Lemma 67.7. Thus, during the proof we may replace k by an extension field. In particular, we may assume that $z \in |F|$ comes from a section $z : \mathrm{Spec}(k) \to F$ of the structure morphism $F \to \mathrm{Spec}(k)$.

Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. After possibly enlarging k once more we may assume that $\mathrm{Spec}(k) \to F \to X$ factors through U (as $U \to X$ is surjective). Let $u : \mathrm{Spec}(k) \to U$ be such a factorization and denote $v \in V$ the image of u. Note that the morphisms

$$U_v \times_{\mathrm{Spec}(\kappa(v))} \mathrm{Spec}(k) = U \times_V \mathrm{Spec}(k) \to U \times_Y \mathrm{Spec}(k) \to F$$

are étale (the first as the base change of $V \to V \times_Y V$ and the second as the base change of $U \to X$). Moreover, by construction the point $u : \mathrm{Spec}(k) \to U$ gives a point of the left most space which maps to z on the right. Hence the elements $f_1, \ldots, f_r$ map to a regular sequence in the local ring on the right of the following map

$$\mathcal{O}_{U_v,u} \longrightarrow \mathcal{O}_{U_v \times_{\mathrm{Spec}(\kappa(v))} \mathrm{Spec}(k), \overline{u}} = \mathcal{O}_{U \times_V \mathrm{Spec}(k), \overline{u}}.$$

But since the displayed arrow is flat (combine More on Flatness, Lemma 2.5 and Morphisms of Spaces, Lemma 28.8) we see from Algebra, Lemma 67.7 that $f_1, \ldots, f_r$ maps to a regular sequence in $\mathcal{O}_{U_v,u}$. By More on Morphisms, Lemma 18.2 we conclude that the morphism of schemes

$$V(f_1, \ldots, f_r) \times_X U = V(f_1|_U, \ldots, f_r|_U) \to V$$

is flat in an open neighbourhood U' of u. Let $X' \subset X$ be the open subspace corresponding to the image of $|U'| \to |X|$ (see Properties of Spaces, Lemmas 4.6 and 4.8). We conclude that $V(f_1, \ldots, f_r) \cap X' \to Y$ is flat (see Morphisms of Spaces, Definition 28.1) as we have the commutative diagram

$$\begin{tikzcd} V(f_1, \ldots, f_r) \times_X U' \arrow[r] \arrow[d, "a"'] & V \arrow[d, "b"] \\ V(f_1, \ldots, f_r) \cap X' \arrow[r] & Y \end{tikzcd}$$

with a, b étale and a surjective. $\square$

23. Étale localization of morphisms

The section is the analogue of More on Morphisms, Section 30.

Lemma 23.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. Let $x_1, \ldots, x_n \in |X|$ mapping to y. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated,*
(3) *f is quasi-finite at $x_1, \ldots, x_n$, and*
(4) *f is quasi-compact or Y is decent.*

Then there exists an étale morphism $(U, u) \to (Y, y)$ of pointed algebraic spaces and a decomposition

$$U \times_Y X = W \amalg V$$

into open and closed subspaces such that the morphism $V \to U$ is finite, every point of the fibre of $|V| \to |U|$ over u maps to an x_i, and the fibre of $|W| \to |U|$ over u contains no point mapping to an x_i.

Proof. Let $(U, u) \to (Y, y)$ be an étale morphism of algebraic spaces and consider the set of $w \in |U \times_Y X|$ mapping to $u \in |U|$ and one of the $x_i \in |X|$. By Decent Spaces, Lemma 16.4 (if f is of finite type) or Decent Spaces, Lemma 16.5 (if Y is decent) this set is finite. It follows that we may replace f by the base change $U \times_Y X \to U$ and $x_1, \ldots, x_n$ by the set of these w. In particular we may and do assume that Y is an affine scheme, whence X is a separated algebraic space.

Choose an affine scheme Z and an étale morphism $Z \to X$ such that $x_1, \ldots, x_n$ are in the image of $|Z| \to |X|$. The fibres of $|Z| \to |X|$ are finite, see Properties of Spaces, Lemma 12.3 (or the more general discussion in Decent Spaces, Section 6). Let $\{z_1, \ldots, z_m\} \subset |Z|$ be the preimage of $\{x_1, \ldots, x_n\}$. By More on Morphisms, Lemma 30.4 there exists an étale morphism $(U, u) \to (Y, y)$ such that $U \times_Y Z = Z_1 \amalg Z_2$ with $Z_1 \to U$ finite and $(Z_1)_y = \{z_1, \ldots, z_m\}$. We may assume that U is affine and hence Z_1 is affine too.

Since f is separated, the image V of $Z_1 \to X$ is both open and closed (Morphisms of Spaces, Lemma 37.6). Set $W = X \setminus V$ to get a decomposition as in the lemma. To finish the proof we have to show that $V \to U$ is finite. As $Z_1 \to V$ is surjective and étale, V is the quotient of Z_1 by the étale equivalence relation $R = Z_1 \times_V Z_1$, see Spaces, Lemma 9.1. Since f is separated, $V \to U$ is separated and R is closed in $Z_1 \times_U Z_1$. Since $Z_1 \to U$ is finite, the projections $s, t : R \to Z_1$ are finite. Thus V is an affine scheme by Groupoids, Proposition 21.8. By Morphisms, Lemma 42.8 we conclude that $V \to U$ is proper and by Morphisms, Lemma 44.10 we conclude that $V \to U$ is finite, thereby finishing the proof. $\square$

Lemma 23.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$ with image $y \in |Y|$. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated, and*
(3) *f is quasi-finite at x.*

Then there exists an étale morphism $(U, u) \to (Y, y)$ of pointed algebraic spaces and a decomposition

$$U \times_Y X = W \amalg V$$

into open and closed subspaces such that the morphism $V \to U$ is finite and there exists a point $v \in |V|$ which maps to x in $|X|$ and u in $|U|$.

Proof. Pick a scheme U, a point $u \in U$, and an étale morphism $U \to Y$ mapping u to y. There exists a point $x' \in |U \times_Y X|$ mapping to x in $|X|$ and u in $|U|$ (Properties of Spaces, Lemma 4.3). To finish, apply Lemma 23.1 to the morphism $U \times_Y X \to U$ and the point x'. It applies because U is a scheme and hence u comes from the monomorphism $\mathrm{Spec}(\kappa(u)) \to U$. $\square$

24. Zariski's Main Theorem

In this section we apply the results of the previous section to prove Zariski's main theorem for morphisms of algebraic spaces. This section is the analogue of More on Morphisms, Section 31.

Lemma 24.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is of finite type and separated. Let Y' be the normalization of Y in X. Picture:*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' \arrow[dl, "\nu"] \\ & Y & \end{tikzcd}$$

Then there exists an open subspace $U' \subset Y'$ such that

(1) *$(f')^{-1}(U') \to U'$ is an isomorphism, and*

(2) $(f')^{-1}(U') \subset X$ *is the set of points at which* f *is quasi-finite.*

Proof. By Morphisms of Spaces, Lemma 32.7 there is an open subspace $U \subset X$ corresponding to the points of $|X|$ where f is quasi-finite. We have to prove

(a) the image of $|U| \to |Y'|$ is $|U'|$ for some open subspace U' of Y',
(b) $U = f^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Since formation of U commutes with arbitrary base change (Morphisms of Spaces, Lemma 32.7), since formation of the normalization Y' commutes with smooth base change (Lemma 21.2), since étale morphisms are open, and since "being an isomorphism" is fpqc local on the base (Descent on Spaces, Lemma 10.13), it suffices to prove (a), (b), (c) étale locally on Y (some details omitted). Thus we may assume Y is an affine scheme. This implies that Y' is an (affine) scheme as well.

Let $x \in |U|$. Claim: there exists an open neighbourhood $f'(x) \in V \subset Y'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is a scheme (as an open of Y'), locally of finite type over Y (as an open subspace of X), and for $v \in V$ the residue field extension $\kappa(v) \supset \kappa(\nu(v))$ is algebraic (as $V \subset Y'$ and Y' is integral over Y). Hence the fibres of $V \to Y$ are discrete (Morphisms, Lemma 21.2) and $(f')^{-1}V \to Y$ is locally quasi-finite (Morphisms, Lemma 21.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since x was arbitrary we see that (a), (b), and (c) are true.

Let $y = f(x) \in |Y|$. Let $(T, t) \to (Y, y)$ be an étale morphism of pointed schemes. Denote by a subscript $_T$ the base change to T. Let $z \in X_T$ be a point in the fibre X_t lying over x. Note that $U_T \subset X_T$ is the set of points where f_T is quasi-finite, see Morphisms of Spaces, Lemma 32.7. Note that

$$X_T \xrightarrow{f'_T} Y'_T \xrightarrow{\nu_T} T$$

is the normalization of T in X_T, see Lemma 21.2. Suppose that the claim holds for $z \in U_T \subset X_T \to Y'_T \to T$, i.e., suppose that we can find an open neighbourhood $f'_T(z) \in V' \subset Y'_T$ such that $(f'_T)^{-1}V' \to V'$ is an isomorphism. The morphism $Y'_T \to Y'$ is étale hence the image $V \subset Y'$ of V' is open. Observe that $f'(x) \in V$ as $f'_T(z) \in V'$. Observe that

$$\begin{tikzcd} (f'_T)^{-1}V' \arrow[r] \arrow[d] & (f')^{-1}(V) \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is a fibre square (as $Y'_T \times_{Y'} X = X_T$). Since the left vertical arrow is an isomorphism and $\{V' \to V\}$ is a étale covering, we conclude that the right vertical arrow is an isomorphism by Descent on Spaces, Lemma 10.13. In other words, the claim holds for $x \in U \subset X \to Y' \to Y$.

By the result of the previous paragraph to prove the claim for $x \in |U|$, we may replace Y by an étale neighbourhood T of $y = f(x)$ and x by any point lying over x in $T \times_Y X$. Thus we may assume there is a decomposition

$$X = V \amalg W$$

into open and closed subspaces where $V \to Y$ is finite and $x \in V$, see Lemma 23.1. Since X is a disjoint union of V and W over Y and since $V \to Y$ is finite we see that the normalization of Y in X is the morphism

$$X = V \amalg W \longrightarrow V \amalg W' \longrightarrow S$$

where W' is the normalization of Y in W, see Morphisms of Spaces, Lemmas 43.6, 41.6, and 43.7. The claim follows and we win. $\square$

The following lemma is a duplicate of Morphisms of Spaces, Lemma 46.2. The reason for having two copies of the same lemma is that the proofs are somewhat different. The proof given below rests on Zariski's Main Theorem for nonrepresentable morphisms of algebraic spaces as presented above, whereas the proof of Morphisms of Spaces, Lemma 46.2 rests on Morphisms of Spaces, Proposition 44.2 to reduce to the case of morphisms of schemes.

Lemma 24.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated. Let Y' be the normalization of Y in X. Picture:*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' \arrow[dl, "\nu"] \\ & Y & \end{tikzcd}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. This follows from Lemma 24.1. Namely, by that lemma there exists an open subspace $U' \subset Y'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : Y' \to Y$ is separated (Morphisms of Spaces, Lemma 8.8). Hence for every affine scheme Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is a quasi-compact open subscheme of the affine scheme $Z \times_Y Y'$. Hence f is quasi-affine by definition. $\square$

Lemma 24.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated and assume that Y is quasi-compact and quasi-separated. Then there exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

where j is a quasi-compact open immersion and π is finite.

Proof. Let $X \to Y' \to Y$ be as in the conclusion of Lemma 24.2. By Limits of Spaces, Lemma 9.7 we can write $\nu_*\mathcal{O}_{Y'} = \text{colim}_{i \in I} \mathcal{A}_i$ as a directed colimit of finite quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_i \subset \nu_*\mathcal{O}_{Y'}$. Then $\pi_i : T_i = \underline{\text{Spec}}_Y(\mathcal{A}_i) \to Y$ is a finite morphism for each i. Note that the transition morphisms $T_{i'} \to T_i$ are affine and that $Y' = \lim T_i$.

By Limits of Spaces, Lemma 5.5 there exists an i and a quasi-compact open $U_i \subset T_i$ whose inverse image in Y' equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of U_i in $T_{i'}$. Then $X \cong f'(X) = \lim_{i' \geq i} U_{i'}$, see Limits of Spaces, Lemma 4.1. By

Limits of Spaces, Lemma 5.10 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large i' but we don't need this.) Hence $X \to T_{i'}$ is an immersion. By Morphisms of Spaces, Lemma 12.6 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. $\square$

Lemma 24.4. *With notation and hypotheses as in Lemma 24.3. Assume moreover that f is locally of finite presentation. Then we can choose the factorization such that T is finite and of finite presentation over Y.*

Proof. By Limits of Spaces, Lemma 11.3 we can write $T = \lim T_i$ where all T_i are finite and of finite presentation over Y and the transition morphisms $T_{i'} \to T_i$ are closed immersions. By Limits of Spaces, Lemma 5.5 there exists an i and an open subscheme $U_i \subset T_i$ whose inverse image in T is X. By Limits of Spaces, Lemma 5.10 we see that $X \cong U_i$ for large enough i. Replacing T by T_i finishes the proof. $\square$

Lemma 24.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is finite,*
2. *f is proper and locally quasi-finite,*
3. *f is proper and $|X_k|$ is a discrete space for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field,*
4. *f is universally closed, separated, locally of finite type and $|X_k|$ is a discrete space for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field.*

Proof. We have (1) $\Rightarrow$ (2) by Morphisms of Spaces, Lemmas 41.9, 41.8. We have (2) $\Rightarrow$ (3) by Morphisms of Spaces, Lemma 26.5. By definition (3) implies (4).

Assume (4). Since f is universally closed it is quasi-compact (Morphisms of Spaces, Lemma 9.7). Pick a point y of $|Y|$. We represent y by a morphism $\mathrm{Spec}(k) \to Y$. Note that $|X_k|$ is finite discrete as a quasi-compact discrete space. The map $|X_k| \to |X|$ surjects onto the fibre of $|X| \to |Y|$ over y (Properties of Spaces, Lemma 4.3). By Morphisms of Spaces, Lemma 32.8 we see that $X \to Y$ is quasi-finite at all the points of the fibre of $|X| \to |Y|$ over y. Choose an elementary étale neighbourhood $(U, u) \to (Y, y)$ and decomposition $X_U = V \coprod W$ as in Lemma 23.1 adapted to all the points of $|X|$ lying over y. Note that $W_u = \emptyset$ because we used all the points in the fibre of $|X| \to |Y|$ over y. Since f is universally closed we see that the image of $|W|$ in $|U|$ is a closed set not containing u. After shrinking U we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over U. Since $y \in |Y|$ was arbitrary this means there exists a family $\{U_i \to Y\}$ of étale morphisms whose images cover Y such that the base changes $X_{U_i} \to U_i$ are finite. We conclude that f is finite by Morphisms of Spaces, Lemma 41.3. $\square$

As a consequence we have the following useful result.

Lemma 24.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. Assume*

1. *f is proper, and*
2. *f is quasi-finite at all $x \in |X|$ lying over y (Decent Spaces, Lemma 16.10).*

Then there exists an open neighbourhood $V \subset Y$ of y such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. By Morphisms of Spaces, Lemma 32.7 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $y \notin f(Z)$. Since f is proper the set $f(Z) \subset Y$ is closed. Choose any open neighbourhood $V \subset Y$ of y with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence $f^{-1}(V) \to V$ is finite by Lemma 24.5. $\square$

Lemma 24.7. *Let S be a scheme. Let*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

be a commutative diagram of morphism of algebraic spaces over S. Let $b \in B$ and let $\mathrm{Spec}(k) \to B$ be a morphism in the equivalence class of b. Assume

1. *$X \to B$ is a proper morphism,*
2. *$Y \to B$ is separated and locally of finite type,*
3. *one of the following is true*
   (a) *the image of $|X_k| \to |Y_k|$ is finite,*
   (b) *the image of $|f|^{-1}(\{b\})$ in $|Y|$ is finite and B is decent.*

Then there is an open subspace $B' \subset B$ containing b such that $X_{B'} \to Y_{B'}$ factors through a closed subspace $Z \subset Y_{B'}$ finite over B'.

Proof. Let $Z \subset Y$ be the scheme theoretic image of h, see Morphisms of Spaces, Section 16. By Morphisms of Spaces, Lemma 37.8 the morphism $X \to Z$ is surjective and $Z \to B$ is proper. Thus

$$\{x \in |X| \text{ lying over } b\} \to \{z \in |Z| \text{ lying over } b\}$$

and $|X_k| \to |Z_k|$ are surjective. We see that either (3)(a) or (3)(b) imply that $Z \to B$ is quasi-finite all all points of $|Z|$ lying over b by Decent Spaces, Lemma 16.10. Hence $Z \to B$ is finite in an open neighbourhood of b by Lemma 24.6. $\square$

25. Stein factorization

Stein factorization is the statement that a proper morphism $f : X \to S$ with $f_*\mathcal{O}_X = \mathcal{O}_S$ has connected fibres.

Lemma 25.1. *Let S be a scheme. Let $f : X \to Y$ be a universally closed, quasi-compact and quasi-separated morphism of algebraic spaces over S. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

with the following properties:

1. *the morphism f' is universally closed, quasi-compact, quasi-separated and surjective,*
2. *the morphism $\pi : Y' \to Y$ is integral,*
3. *we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,*
4. *we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and*
5. *Y' is the normalization of Y in X as defined in Morphisms of Spaces, Definition 43.3.*

Proof. We just define Y' as the normalization of Y in X, so (5) and (2) hold automatically. By Morphisms of Spaces, Lemma 43.8 we see that (4) holds. The morphism f' is universally closed by Morphisms of Spaces, Lemma 37.6. It is quasi-compact by Morphisms of Spaces, Lemma 8.8 and quasi-separated by Morphisms of Spaces, Lemma 4.10.

To show the remaining statements we may assume the base Y is affine (as taking normalization commutes with étale localization). Say $Y = \mathrm{Spec}(R)$. Then $Y' = \mathrm{Spec}(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral R-algebra. Thus it is clear that $f'_*\mathcal{O}_X$ is $\mathcal{O}_{Y'}$ (because $f'_*\mathcal{O}_X$ is quasi-coherent, by Morphisms of Spaces, Lemma 11.2, and hence equal to $\widetilde{A}$). This proves (3).

Let us show that f' is surjective. As f' is universally closed (see above) the image of f' is a closed subset $V(I) \subset S' = \mathrm{Spec}(A)$. Pick $h \in I$. Then $h|_X = f^\sharp(h)$ is a global section of the structure sheaf of X which vanishes at every point. As X is quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. In other words I is contained in the radical ideal of A and we conclude that $V(I) = S'$ as desired. $\square$

Let $f : X \to Y$ be a morphism of algebraic spaces and let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. Then the *fibre of f over $\overline{y}$* is the algebraic space $X_{\overline{y}} = X \times_{Y,\overline{y}} \mathrm{Spec}(k)$ over k. If Y is a scheme and $y \in Y$ is a point, then we denote $X_y = X \times_Y \mathrm{Spec}(\kappa(y))$ the fibre as usual.

Lemma 25.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{y}$ be a geometric point of Y. Then $X_{\overline{y}}$ is connected, if and only if for every étale neighbourhood $(V, \overline{v}) \to (Y, \overline{y})$ where V is a scheme the base change $X_V \to V$ has connected fibre X_v.*

Proof. Since the category of étale neighbourhoods of $\overline{y}$ is cofiltered and contains a cofinal collection of schemes (Properties of Spaces, Lemma 16.3) we may replace Y by one of these neighbourhoods and assume that Y is a scheme. Let $y \in Y$ be the point corresponding to $\overline{y}$. Then X_y is geometrically connected over $\kappa(y)$ if and only if $X_{\overline{y}}$ is connected and if and only if $(X_y)_{k'}$ is connected for every finite separable extension k' of $\kappa(y)$. See Spaces over Fields, Section 8 and especially Lemma 8.8. By More on Morphisms, Lemma 27.2 there exists an affine étale neighbourhood $(V, v) \to (Y, y)$ such that $\kappa(s) \subset \kappa(u)$ is identified with $\kappa(s) \subset k'$ any given finite separable extension. The lemma follows. $\square$

Theorem 25.3 (Stein factorization; Noetherian case)**.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y locally Noetherian. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & Y' \arrow[ld, "\pi"] \\ & Y & \end{tikzcd}$$

with the following properties:

1. *the morphism f' is proper with connected geometric fibres,*
2. *the morphism $\pi : Y' \to Y$ is finite,*
3. *we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,*
4. *we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and*

(5) *Y' is the normalization of Y in X, see Morphisms, Definition 48.3.*

Proof. Let $f = \pi \circ f'$ be the factorization of Lemma 25.1. Note that besides the conclusions of Lemma 25.1 we also have that f' is separated (Morphisms of Spaces, Lemma 4.10) and finite type (Morphisms of Spaces, Lemma 23.6). Hence f' is proper. By Cohomology of Spaces, Lemma 19.2 we see that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module. Hence we see that π is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that the geometric fibres of f' are connected. It is clear from the discussion above that we may replace Y by Y'. Then Y is locally Noetherian, $f : X \to Y$ is proper, and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let $\overline{y}$ be a geometric point of Y. At this point we apply the theorem on formal functions, more precisely Cohomology of Spaces, Lemma 20.7. It tells us that

$$\mathcal{O}^\wedge_{Y,\overline{y}} = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where $X_n = \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{y}}^n) \times_Y X$. Note that $X_1 = X_{\overline{y}} \to X_n$ is a (finite order) thickening and hence the underlying topological space of X_n is equal to that of $X_{\overline{y}}$. Thus, if $X_{\overline{y}} = T_1 \coprod T_2$ is a disjoint union of nonempty open and closed subspaces, then similarly $X_n = T_{1,n} \coprod T_{2,n}$ for all n. And this in turn means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent $e_{1,n}$, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that $e_{1,n+1}$ restricts to $e_{1,n}$ on X_n. Hence $e_1 = \lim e_{1,n}$ is a nontrivial idempotent of the limit. This contradicts the fact that $\mathcal{O}^\wedge_{Y,\overline{y}}$ is a local ring. Thus the assumption was wrong, i.e., $X_{\overline{y}}$ is connected as desired. □

Theorem 25.4 (Stein factorization; general case)**.** *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S. There exists a factorization*

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & Y' \arrow[ld, "\pi"] \\ & Y & \end{tikzcd}$$

with the following properties:

1. *the morphism f' is proper with connected geometric fibres,*
2. *the morphism $\pi : Y' \to Y$ is integral,*
3. *we have $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$,*
4. *we have $Y' = \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$, and*
5. *Y' is the normalization of Y in X, see Morphisms, Definition 48.3.*

Proof. We may apply Lemma 25.1 to get the morphism $f' : X \to Y'$. Note that besides the conclusions of Lemma 25.1 we also have that f' is separated (Morphisms of Spaces, Lemma 4.10) and finite type (Morphisms of Spaces, Lemma 23.6). Hence f' is proper. At this point we have proved all of the statements except for the statement that f' has connected geometric fibres.

It is clear from the discussion that we may replace Y by Y'. Then $f : X \to Y$ is proper and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Note that these conditions are preserved under flat base change (Morphisms of Spaces, Lemma 37.3 and Cohomology of Spaces, Lemma 10.1). Let $\overline{y}$ be a geometric point of Y. By Lemma 25.2 and the remark just made we reduce to the case where Y is a scheme, $y \in Y$ is a point, $f : X \to Y$ is a proper algebraic space over Y with $f_*\mathcal{O}_X = \mathcal{O}_Y$, and we have to show the fibre X_y

is connected. Replacing Y by an affine neighbourhood of y we may assume that $Y = \mathrm{Spec}(R)$ is affine. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ signifies that the ring map $R \to \Gamma(X, \mathcal{O}_X)$ is bijective.

By Limits of Spaces, Lemma 12.2 we can write $(X \to Y) = \lim(X_i \to Y_i)$ with $X_i \to Y_i$ proper and of finite presentation and Y_i Noetherian. For i large enough Y_i is affine (Limits of Spaces, Lemma 5.8). Say $Y_i = \mathrm{Spec}(R_i)$. Let $R'_i = \Gamma(X_i, \mathcal{O}_{X_i})$. Observe that we have ring maps $R_i \to R'_i \to R$. Namely, we have the first because X_i is an algebraic space over R_i and the second because we have $X \to X_i$ and $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \mathrm{colim}\, R'_i$ by Limits of Spaces, Lemma 5.4. Then

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[d] & \\ Y \arrow[r] & Y'_i \arrow[r] & Y_i \end{tikzcd}$$

is commutative with $Y'_i = \mathrm{Spec}(R'_i)$. Let $y'_i \in Y'_i$ be the image of y. We have $X_y = \lim X_{i,y'_i}$ because $X = \lim X_i$, $Y = \lim Y'_i$, and $\kappa(y) = \mathrm{colim}\, \kappa(y'_i)$. Now let $X_y = U \amalg V$ with U and V open and closed. Then U, V are the inverse images of opens U_i, V_i in X_{i,y'_i} (Limits of Spaces, Lemma 5.5). By Theorem 25.3 the fibres of $X_i \to Y'_i$ are connected, hence either U or V is empty. This finishes the proof. $\square$

26. Extending properties from an open

In this section we collect a number of results of the form: If $f : X \to Y$ is a flat morphism of algebraic spaces and f satisfies some property over a dense open of Y, then f satisfies the same property over all of Y.

Lemma 26.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Assume*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is of finite type and flat over Y,*
(3) *$V \to Y$ is quasi-compact and scheme theoretically dense,*
(4) *$\mathcal{F}|_{f^{-1}V}$ is of finite presentation.*

Then $\mathcal{F}$ is of finite presentation.

Proof. It suffices to prove the pullback of $\mathcal{F}$ to a scheme surjective and étale over X is of finite presentation. Hence we may assume X is a scheme. Similarly, we can replace Y by a scheme surjective and étale and over Y (the inverse image of V in this scheme is scheme theoretically dense, see Morphisms of Spaces, Section 17). Thus we reduce to the case of schemes which is More on Flatness, Lemma 10.11. $\square$

Lemma 26.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. Assume*

(1) *f is locally of finite type and flat,*
(2) *$V \to Y$ is quasi-compact and scheme theoretically dense,*
(3) *$f|_{f^{-1}V} : f^{-1}V \to V$ is locally of finite presentation.*

Then f is of locally of finite presentation.

Proof. The proof is identical to the proof of Lemma 26.1 except one uses More on Flatness, Lemma 10.12. □

Lemma 26.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. Let $d \geq 0$. Assume*

(1) *f is flat and locally of finite presentation,*
(2) *$V \subset Y$ is scheme theoretically dense, and*
(3) *$f|_{f^{-1}V} : f^{-1}V \to V$ has relative dimension $\leq d$.*

Then $f : X \to Y$ has relative dimension $\leq d$.

Proof. By definition the property of having relative dimension $\leq d$ can be checked on an étale covering, see Morphisms of Spaces, Sections 31. Thus it suffices to prove f has relative dimension $\leq d$ after replacing X by a scheme surjective and étale over X. Similarly, we can replace Y by a scheme surjective and étale and over Y. The inverse image of V in this scheme is scheme theoretically dense, see Morphisms of Spaces, Section 17. Since a scheme theoretically dense open of a scheme is in particular dense, we reduce to the case of schemes which is More on Morphisms, Lemma 17.8. □

Lemma 26.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. If*

(1) *f is separated, locally of finite type, and flat,*
(2) *$f^{-1}(V) \to V$ is an isomorphism, and*
(3) *$V \to Y$ is quasi-compact and scheme theoretically dense,*

then f is an open immersion.

Proof. Applying Lemma 26.2 we see that f is locally of finite presentation. Applying Lemma 26.3 we see that f has relative dimension ≤ 0. By Morphisms of Spaces, Lemma 32.6 this implies that f is locally quasi-finite. By Morphisms of Spaces, Lemma 45.1 this implies that f is representable. By Descent on Spaces, Lemma 10.12 we can check whether f is an open immersion étale locally on Y. Hence we may assume that Y is a scheme. Since f is representable, we reduce to the case of schemes which is More on Morphisms, Lemma 31.4. □

27. Blowing up and flatness

Instead of redoing the work in More on Flatness, Section 27 we prove an analogue of More on Flatness, Lemma 27.6 which tells us that the problem of finding a suitable blowup is often étale local on the base.

Lemma 27.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\varphi : W \to X$ be a quasi-compact separated étale morphism. Let $U \subset X$ be a quasi-compact open subspace. Let $\mathcal{I} \subset \mathcal{O}_U$ be a finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}) \cap \varphi^{-1}(U) = \emptyset$. Then there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that*

(1) *$V(\mathcal{J}) \cap U = \emptyset$, and*
(2) *$\varphi^{-1}(\mathcal{J})\mathcal{O}_W = \mathcal{I}\mathcal{I}'$ for some finite type quasi-coherent ideal $\mathcal{I}' \subset \mathcal{O}_W$.*

Proof. Choose a factorization $W \to Y \to X$ where $j : W \to Y$ is a quasi-compact open immersion and $\pi : Y \to X$ is a finite morphism of finite presentation (Lemma 24.4). Let $V = j(W) \cup \pi^{-1}(U) \subset Y$. Note that $\mathcal{I}$ on $W \cong j(W)$ and $\mathcal{O}_{\pi^{-1}(U)}$

glue to a finite type quasi-coherent sheaf of ideals $\mathcal{I}_1 \subset \mathcal{O}_V$. By Limits of Spaces, Lemma 9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_2 \subset \mathcal{O}_Y$ such that $\mathcal{I}_2|_V = \mathcal{I}_1$. In other words, $\mathcal{I}_2 \subset \mathcal{O}_Y$ is a finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_2)$ is disjoint from $\pi^{-1}(U)$ and $j^{-1}\mathcal{I}_2 = \mathcal{I}$. Denote $i : Z \to Y$ the corresponding closed immersion which is of finite presentation (Morphisms of Spaces, Lemma 27.12). In particular the composition $\tau = \pi \circ i : Z \to X$ is finite and of finite presentation (Morphisms of Spaces, Lemmas 27.2 and 41.4).

Let $\mathcal{F} = \tau_*\mathcal{O}_Z$ which we think of as a quasi-coherent $\mathcal{O}_X$-module. By Descent on Spaces, Lemma 5.7 we see that $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{J} = \mathrm{Fit}_0(\mathcal{F})$. (Insert reference to fitting modules on ringed topoi here.) This is a finite type quasi-coherent sheaf of ideals on X (as $\mathcal{F}$ is of finite presentation, see More on Algebra, Lemma 5.4). Part (1) of the lemma holds because $|\tau|(|Z|) \cap |U| = \emptyset$ by our choice of $\mathcal{I}_2$ and because the 0th fitting ideal of the trivial module equals the structure sheaf. To prove (2) note that $\varphi^{-1}(\mathcal{J})\mathcal{O}_W = \mathrm{Fit}_0(\varphi^*\mathcal{F})$ because taking fitting ideals commutes with base change. On the other hand, as $\varphi : W \to X$ is separated and étale we see that $(1, j) : W \to W \times_X Y$ is an open and closed immersion. Hence $W \times_Y Z = V(\mathcal{I}) \amalg Z'$ for some finite and finitely presented morphism of algebraic spaces $\tau' : Z' \to W$. Thus we see that

$$\begin{aligned}
\mathrm{Fit}_0(\varphi^*\mathcal{F}) &= \mathrm{Fit}_0((W \times_Y Z \to W)_*\mathcal{O}_{W \times_Y Z}) \\
&= \mathrm{Fit}_0(\mathcal{O}_W/\mathcal{I}) \cdot \mathrm{Fit}_0(\tau'_*\mathcal{O}_{Z'}) \\
&= \mathcal{I} \cdot \mathrm{Fit}_0(\tau'_*\mathcal{O}_{Z'})
\end{aligned}$$

the second equality by More on Algebra, Lemma 5.4 translated in sheaves on ringed topoi. Setting $\mathcal{I}' = \mathrm{Fit}_0(\tau'_*\mathcal{O}_{Z'})$ finishes the proof of the lemma. □

Theorem 27.2. *Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let X be an algebraic space over B. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset B$ be a quasi-compact open subspace. Assume*

1. *X is quasi-compact,*
2. *X is locally of finite presentation over B,*
3. *$\mathcal{F}$ is a module of finite type,*
4. *$\mathcal{F}_U$ is of finite presentation, and*
5. *$\mathcal{F}_U$ is flat over U.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{O}_{X \times_B B'}$-module of finite presentation and flat over B'.

Proof. Choose an affine scheme V and a surjective étale morphism $V \to X$. Because strict transform commutes with étale localization (Divisors on Spaces, Lemma 7.2) it suffices to prove the result with X replaced by V. Hence we may assume that $X \to B$ is representable (in addition to the hypotheses of the lemma).

Assume that $X \to B$ is representable. Choose an affine scheme W and a surjective étale morphism $\varphi : W \to B$. Note that $X \times_B W$ is a scheme. By the case of schemes (More on Flatness, Theorem 27.8) we can find a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_W$ such that (a) $|V(\mathcal{I})| \cap |\varphi^{-1}(U)| = \emptyset$ and (b) the strict transform of $\mathcal{F}|_{X \times_B W}$ with respect to the blowing up $W' \to W$ in $\mathcal{I}$ becomes flat over W' and is a module of finite presentation. Choose a finite type sheaf of ideals $\mathcal{J} \subset \mathcal{O}_B$ as in Lemma 27.1. Let $B' \to B$ be the blowing up of $\mathcal{J}$. We claim that this blow up works. Namely, it is clear that $B' \to B$ is U-admissible by our choice of ideal

$\mathcal{J}$. Moreover, the base change $B' \times_B W \to W$ is the blowup of W in $\varphi^{-1}\mathcal{J} = \mathcal{I}\mathcal{I}'$ (compatibility of blowup with flat base change, see Divisors on Spaces, Lemma 6.3). Hence there is a factorization

$$W \times_B B' \to W' \to W$$

where the first morphism is a blowup as well, see Divisors on Spaces, Lemma 6.9). The restriction of $\mathcal{F}'$ (which lives on $B' \times_B X$) to $W \times_B B' \times_B X$ is the strict transform of $\mathcal{F}|_{X \times_B W}$ (Divisors on Spaces, Lemma 7.2) and hence is the twice repeated strict transform of $\mathcal{F}|_{X \times_B W}$ by the two blowups displayed above (Divisors on Spaces, Lemma 7.7). After the first blow up our sheaf is already flat over the base and of finite presentation (by construction). Whence this holds after the second strict transform as well (since this is a pullback by Divisors on Spaces, Lemma 7.4). Thus we see that the restriction of $\mathcal{F}'$ to an étale cover of $B' \times_B X$ has the desired properties and the theorem is proved. □

28. Applications

In this section we apply the result on flattening by blowing up.

Lemma 28.1. *Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let X be an algebraic space over B. Let $U \subset B$ be a quasi-compact open subspace. Assume*

1. *$X \to B$ is of finite type and quasi-separated, and*
2. *$X_U \to U$ is flat and locally of finite presentation.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform of X is flat and of finite presentation over B'.

Proof. Let $B' \to B$ be a U-admissible blowup. Note that the strict transform of X is quasi-compact and quasi-separated over B' as X is quasi-compact and quasi-separated over B. Hence we only need to worry about finding a U-admissible blowup such that the strict transform becomes flat and locally of finite presentation. We cannot directly apply Theorem 27.2 because X is not locally of finite presentation over B.

Choose an affine scheme V and a surjective étale morphism $V \to X$. (This is possible as X is quasi-compact as a finite type space over the quasi-compact space B.) Then it suffices to show the result for the morphism $V \to B$ (as strict transform commutes with étale localization, see Divisors on Spaces, Lemma 7.2). Hence we may assume that $X \to B$ is separated as well as finite type. In this case we can find a closed immersion $i : X \to Y$ with $Y \to B$ separated and of finite presentation, see Limits of Spaces, Proposition 11.7.

Apply Theorem 27.2 to $\mathcal{F} = i_*\mathcal{O}_X$ on Y/B. We find a U-admissible blowup $B' \to B$ such that that strict transform of $\mathcal{F}$ is flat over B' and of finite presentation. Let X' be the strict transform of X under the blowup $B' \to B$. Let $i' : X' \to Y \times_B B'$ be the induced morphism. Since taking strict transform commutes with pushforward along affine morphisms (Divisors on Spaces, Lemma 7.5), we see that $i'_*\mathcal{O}_{X'}$ is flat over B' and of finite presentation as a $\mathcal{O}_{Y \times_B B'}$-module. Thus $X' \to B'$ is flat and locally of finite presentation. This implies the lemma by our earlier remarks. □

Lemma 28.2. *Let S be a scheme. Let $\varphi : X \to B$ be a morphism of algebraic spaces over S. Assume φ is of finite type with B quasi-compact and quasi-separated. Let*

$U \subset B$ be a quasi-compact open subspace such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $B' \to B$ such that U is scheme theoretically dense in B' and such that the strict transform X' of X is isomorphic to an open subspace of B'.

Proof. As the composition of U-admissible blowups is U-admissible (Divisors on Spaces, Lemma 8.2) we can proceed in stages. Pick a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_B$ with $|B| \setminus |U| = |V(\mathcal{I})|$. Replace B by the blowup of B in $\mathcal{I}$ and X by the strict transform of X. After this replacement $B \setminus U$ is the support of an effective Cartier divisor D (Divisors on Spaces, Lemma 6.4). In particular U is scheme theoretically dense in B (Divisors on Spaces, Lemma 2.4). Next, we do another U-admissible blowup to get to the situation where $X \to B$ is flat and of finite presentation, see Lemma 28.1. Note that U is still scheme theoretically dense in B. Hence $X \to B$ is an open immersion by Lemma 26.4. □

The following lemma says that a modification can be dominated by a blowup.

Lemma 28.3. *Let S be a scheme. Let $\varphi : X \to B$ be a proper morphism of algebraic spaces over S. Assume B quasi-compact and quasi-separated. Let $U \subset B$ be a quasi-compact open subspace such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $B' \to B$ which dominates X, i.e., such that there exists a factorization $B' \to X \to B$ of the blowup morphism.*

Proof. By Lemma 28.2 we may find a U-admissible blowup $B' \to B$ such that the strict transform X' is an open subspace of B' and U is scheme theoretically dense in B'. Since $X' \to B'$ is proper we see that $|X'|$ is closed in $|B'|$. As $U \subset B'$ is dense $X' = B'$. □

29. Chow's lemma

In this section we prove some variants of Chow's lemma. Since we have yet to define projective morphisms of algebraic spaces, the statements will involve representable proper morphisms, rather than projective ones.

Lemma 29.1. *Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $U \to X_1$ and $U \to X_2$ be open immersions of algebraic spaces over Y and assume U, X_1, X_2 of finite type and separated over Y. Then there exists a commutative diagram*

$$\begin{tikzcd}
X_1' \arrow[r] \arrow[d] & X & X_2' \arrow[l] \arrow[d] \\
X_1 & U \arrow[l] \arrow[r] \arrow[u] \arrow[ul] \arrow[ur] & X_2
\end{tikzcd}$$

of algebraic spaces over Y where $X_i' \to X_i$ is a U-admissible blowup, $X_i' \to X$ is an open immersion, and X is separated and finite type over Y.

Proof. Throughout the proof all the algebraic spaces will be separated of finite type over Y. This in particular implies these algebraic spaces and the morphisms between them will be quasi-compact and quasi-separated. We will use that if $U \to W$ is an immersion of such spaces over Y, then the scheme theoretic image Z of U in W is a closed subspace of W and $U \to Z$ is an open immersion, $U \subset Z$ is scheme theoretically dense, and $|U| \subset |Z|$ is dense. See Morphisms of Spaces, Lemma 17.7.

Let $X_{12} \subset X_1 \times_Y X_2$ be the scheme theoretic image of $U \to X_1 \times_Y X_2$. We claim the projections $p_i : X_{12} \to X_i$ induce isomorphisms $p_i^{-1}(U) \to U$. Namely, $p_i : X_{12} \to X_i$ is separated and $U \to X_{12}$ is a section of p_i. Hence $U \to p_i^{-1}(U)$ is a closed immersion (Morphisms of Spaces, Lemma 4.6) as well as scheme theoretically dense whence an isomorphism. Choose a U-admissible blowup $X_i^i \to X_i$ such that the strict transform X_{12}^i of X_{12} is isomorphic to an open subspace of X_i^i, see Lemma 28.2. Let $\mathcal{I}_i \subset \mathcal{O}_{X_i}$ be the corresponding finite type quasi-coherent sheaf of ideals. Recall that $X_{12}^i \to X_{12}$ is the blowup in $p_i^{-1}\mathcal{I}_i\mathcal{O}_{X_{12}}$. Let X_{12}' be the blowup of X_{12} in $p_1^{-1}\mathcal{I}_1 p_2^{-1}\mathcal{I}_2\mathcal{O}_{X_{12}}$. We obtain a commutative diagram

$$\begin{tikzcd} X_{12}' \arrow[r] \arrow[d] & X_{12}^2 \arrow[d] \\ X_{12}^1 \arrow[r] & X_{12} \end{tikzcd}$$

where all the morphisms are U-admissible blowing ups. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{J}_i$ on X_i^i extending the pull back of $\mathcal{I}_{1-i}$ to X_{12}^i (see Limits of Spaces, Lemma 9.8). Let $X_i' \to X_i^i$ be the blowing up in $\mathcal{J}_i$. By construction $X_{12}' \subset X_i'$ is an open subspace and the diagram

$$\begin{tikzcd} X_{12}' \arrow[r] \arrow[d] & X_i' \arrow[d] \\ X_{12}^i \arrow[r] & X_i^i \end{tikzcd}$$

is commutative with vertical arrows blowing ups and horizontal arrows open immersions. Note that $X_{12}' \to X_1' \times_Y X_2'$ is an immersion and proper (use that $X_{12}' \to X_{12}$ is proper and $X_{12} \to X_1 \times_Y X_2$ is closed and $X_1' \times_Y X_2' \to X_1 \times_Y X_2$ is separated and apply Morphisms of Spaces, Lemma 37.6). Thus $X_{12}' \to X_1' \times_Y X_2'$ is a closed immersion. It follows that if we define X by glueing X_1' and X_2' along the common open subspace X_{12}', then $X \to Y$ is of finite type and separated (details omitted). As compositions of U-admissible blowups are U-admissible blowups (Divisors on Spaces, Lemma 8.2) the lemma is proved. $\square$

Lemma 29.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $U \subset X$ be an open subscheme. Assume*

1. *U is quasi-compact,*
2. *Y is quasi-compact and quasi-separated,*
3. *there exists an immersion $U \to \mathbf{P}^n_Y$ over Y,*
4. *f is of finite type and separated.*

Then there exists a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \overline{X}' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $X' \to X$ is a U-admissible blowup, $X' \to \overline{X}'$ is an open immersion, and $\overline{X}' \to Y$ is a proper and representable morphism of algebraic spaces.

Proof. Let $Z \subset \mathbf{P}^n_Y$ be the scheme theoretic image of the immersion $U \to \mathbf{P}^n_Y$. Since $U \to \mathbf{P}^n_Y$ is quasi-compact we see that $U \subset Z$ is a (scheme theoretically) dense open subspace (Morphisms of Spaces, Lemma 17.7). Apply Lemma 29.1 to find a diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \overline{X}' & Z' \arrow[l] \arrow[d] \\ X & U \arrow[l] \arrow[r] \arrow[ul] \arrow[u] \arrow[ur] & Z \end{tikzcd}$$

with properties as listed in the statement of that lemma. Since $Z' \to Z \to Y$ is proper we see that $Z' \subset \overline{X}'$ is closed (see Morphisms of Spaces, Lemma 37.6). After replacing $\overline{X}'$ by a further U-admissible blowup we may assume that U is scheme theoretically dense in $\overline{X}'$ (details omitted; use Divisors on Spaces, Lemmas 6.4 and 2.4). It follows that $Z' = \overline{X}'$ and the lemma is proved. $\square$

Lemma 29.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated, of finite type, and Y Noetherian. Then there exists a commutative diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \overline{X}' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $X' \to X$ is a U-admissible blowup for some dense open $U \subset X$, the morphism $X' \to \overline{X}'$ is an open immersion, and $\overline{X}' \to Y$ is a proper and representable morphism of algebraic spaces.

Proof. By Limits of Spaces, Lemma 13.3 there exists a dense open subspace $U \subset X$ and an immersion $U \to \mathbf{A}^n_Y$ over Y. Composing with the open immersion $\mathbf{A}^n_Y \to \mathbf{P}^n_Y$ we obtain a situation as in Lemma 29.2 and the result follows. $\square$

Remark 29.4. In Lemma 29.2 the morphism $\overline{X}' \to Y$ is a composition

$$\overline{X}' \to Z \to \mathbf{P}^n_Y \to Y$$

where $b : \overline{X}' \to Z$ is a U-admissible blowing up (in particular $b|_U : U \to b(U)$ is an isomorphism onto an open subspace of Z) and where $Z \to \mathbf{P}^n_Y$ is a closed immersion. This is immediate from the proof. It follows that the morphism $\overline{X}' \to Y$ obtained in the statement of Lemma 29.3 has a factorization of this type as well.

The following result is [Knu71, IV Theorem 3.1]. Note that the immersion $X' \to \mathbf{P}^n_Y$ is quasi-compact, hence can be factored as $X' \to \overline{X}' \to \mathbf{P}^n_Y$ where the first morphism is an open immersion and the second morphism a closed immersion (Morphisms of Spaces, Lemma 17.7).

Lemma 29.5 (Chow's lemma)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated of finite type, and Y separated and Noetherian. Then there exists a commutative diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $X' \to X$ is a U-admissible blowup for some dense open $U \subset X$ and the morphism $X' \to \mathbf{P}^n_Y$ is an immersion.

Proof. In this first paragraph of the proof we reduce the lemma to the case where Y is of finite type over $\text{Spec}(\mathbf{Z})$. We may and do replace the base scheme S by $\text{Spec}(\mathbf{Z})$. We can write $Y = \lim Y_i$ as a directed limit of separated algebraic spaces of finite type over $\text{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 8.1 and Lemma 5.7. For all i sufficiently large we can find a separated finite type morphism $X_i \to Y_i$ such that $X = Y \times_{Y_i} X_i$, see Limits of Spaces, Lemmas 7.1 and 6.8. Let $\eta_1, \ldots, \eta_n$ be the generic points of the irreducible components of $|X|$ (X is Noetherian as a finite type separated algebraic space over the Noetherian algebraic space Y and therefore $|X|$ is a Noetherian topological space). By Limits of Spaces, Lemma 5.2 we find that the images of $\eta_1, \ldots, \eta_n$ in $|X_i|$ are distinct for i large enough. We may replace X_i by the scheme theoretic image of the (quasi-compact, in fact affine) morphism $X \to X_i$. After this replacement we see that the images of $\eta_1, \ldots, \eta_n$ in $|X_i|$ are the generic points of the irreducible components of $|X_i|$, see Morphisms of Spaces, Lemma 16.3. Having said this, suppose we can find a diagram

$$\begin{tikzcd} X'_i \arrow[r] \arrow[d] & \mathbf{P}^n_{Y_i} \arrow[d] \\ X_i \arrow[r] & Y \end{tikzcd}$$

where $X'_i \to X_i$ is a U_i-admissible blowup for some dense open $U_i \subset X_i$ and the morphism $X'_i \to \mathbf{P}^n_{Y_i}$ is an immersion. Then the strict transform $X' \to X$ of X relative to $X'_i \to X_i$ is a U-admissible blowing up where $U \subset X$ is the inverse image of U_i in X. Because of our carefuly chosen index i it follows that $\eta_1, \ldots, \eta_n \in |U|$ and $U \subset X$ is dense. Moreover, $X' \to \mathbf{P}^n_Y$ is an immersion as X' is closed in $X'_i \times_{X_i} X = X'_i \times_{Y_i} Y$ which comes with an immersion into $\mathbf{P}^n_Y$. Thus we have reduced to the situation of the following paragraph.

Assume that Y is separated of finite type over $\text{Spec}(\mathbf{Z})$. Then $X \to \text{Spec}(\mathbf{Z})$ is separated of finite type as well. We apply Lemma 29.3 to find a diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \overline{X}' \arrow[d] \\ X \arrow[r] & \text{Spec}(\mathbf{Z}) \end{tikzcd}$$

where $X' \to X$ is a U-admissible blowup for some dense open $U \subset X$ and $X' \to \overline{X}'$ is an open immersion and $\overline{X}' \to \text{Spec}(\mathbf{Z})$ is representable and proper. In fact, by Remark 29.4 we see that $\overline{X}' \to \text{Spec}(\mathbf{Z})$ can be factored as

$$\overline{X}' \to Z \to \mathbf{P}^n_{\mathbf{Z}} \to \text{Spec}(\mathbf{Z}).$$

where the first morphism is a U-admissible blowing up, the second morphism is a closed immersion, and the third morphism is the structure morphism. Note that Z has an ample invertible sheaf, namely $\mathcal{O}_{\mathbf{P}^n}(1)|_Z$. Hence $\overline{X}' \to Z$ is a H-projective morphism by Morphisms, Lemma 43.13. It follows that $\overline{X}' \to \text{Spec}(\mathbf{Z})$ is H-projective by Morphisms, Lemma 43.7. Thus there exists a closed immersion

$\overline{X}' \to \mathbf{P}^n_{\mathrm{Spec}(\mathbf{Z})}$. It follows that the diagonal map $X' \to Y \times \mathbf{P}^n_{\mathrm{Spec}(\mathbf{Z})} = \mathbf{P}^n_Y$ is an immersion and we win. □

30. Variants of Chow's Lemma

In this section we prove a number of variants of Chow's lemma dealing with morphisms between non-Noetherian algebraic spaces. The Noetherian versions are Lemma 29.3 and Lemma 29.5.

Lemma 30.1. *Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $f : X \to Y$ be a separated morphism of finite type. Then there exists a commutative diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \overline{X}' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $X' \to X$ is proper surjective, $X' \to \overline{X}'$ is an open immersion, and $\overline{X}' \to Y$ is proper and representable morphism of algebraic spaces.

Proof. By Limits of Spaces, Proposition 11.7 we can find a closed immersion $X \to X_1$ where X_1 is separated and of finite presentation over Y. Clearly, if we prove the assertion for $X_1 \to Y$, then the result follows for X. Hence we may assume that X is of finite presentation over Y.

We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. Write $Y = \lim_i Y_i$ as a directed limit of quasi-separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 8.1. By Limits of Spaces, Lemma 7.1 we can find an index $i \in I$ and a scheme $X_i \to Y_i$ of finite presentation so that $X = Y \times_{Y_i} X_i$. By Limits of Spaces, Lemma 6.8 we may assume that $X_i \to Y_i$ is separated. Clearly, if we prove the assertion for X_i over Y_i, then the assertion holds for X. The case $X_i \to Y_i$ is treated by Lemma 29.3. □

Lemma 30.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated of finite type, and Y separated and quasi-compact. Then there exists a commutative diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $X' \to X$ is proper surjective morphism and the morphism $X' \to \mathbf{P}^n_Y$ is an immersion.

Proof. By Limits of Spaces, Proposition 11.7 we can find a closed immersion $X \to X_1$ where X_1 is separated and of finite presentation over Y. Clearly, if we prove the assertion for $X_1 \to Y$, then the result follows for X. Hence we may assume that X is of finite presentation over Y.

We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. Write $Y = \lim_i Y_i$ as a directed limit of quasi-separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 8.1. By Limits of Spaces, Lemma 5.7 we may assume that Y_i is separated for all i. By Limits of Spaces, Lemma 7.1 we can find an index $i \in I$ and a scheme $X_i \to Y_i$ of finite presentation so that $X = Y \times_{Y_i} X_i$. By Limits of Spaces, Lemma 6.8 we may assume that $X_i \to Y_i$ is separated. Clearly, if we prove the assertion for X_i over Y_i, then the assertion holds for X. The case $X_i \to Y_i$ is treated by Lemma 29.5. □

31. Grothendieck's existence theorem

In this section we discuss Grothendieck's existence theorem for algebraic spaces. Instead of developing a theory of "formal algebraic spaces" we temporarily develop a bit of language that replaces the notion of a "coherent module on a Noetherian adic formal space".

Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Below we will consider inverse systems $(\mathcal{F}_n)$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_n$ is annihilated by $\mathcal{I}^n$, and
(2) the transition maps induce isomorphisms $\mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1} \to \mathcal{F}_n$.

A morphism $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ of such inverse systems is simply a compatible system of morphisms $\alpha_n : \mathcal{F}_n \to \mathcal{G}_n$. Let us denote the category of these inverse systems with $\mathit{Coh}(X, \mathcal{I})$. We will develop some theory regarding these systems that will parallel to the corresponding results in the case of schemes, see Cohomology of Schemes, Sections 21, 22, and 23.

Functoriality. Let $f : X \to Y$ be a morphism of Noetherian algebraic spaces over a scheme S, and let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. In this situation there is a functor

$$f^* : \mathit{Coh}(Y, \mathcal{J}) \longrightarrow \mathit{Coh}(X, \mathcal{I})$$

which sends $(\mathcal{G}_n)$ to $(f^*\mathcal{G}_n)$. Compare with Cohomology of Schemes, Lemma 22.1. If f is étale, then we may think of this as simply the restriction of the system to X, see Properties of Spaces, Equation 24.1.1.

Étale descent. Let S be a scheme. Let $U_0 \to X$ be a surjective étale morphism of Noetherian algebraic spaces. Set $U_1 = U_0 \times_X U_0$ and $U_2 = U_0 \times_X U_0 \times_X U_0$. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Set $\mathcal{I}_i = \mathcal{I}|_{U_i}$. In this situation we obtain a diagram of categories

$$\mathit{Coh}(X, \mathcal{I}) \longrightarrow \mathit{Coh}(U_0, \mathcal{I}_0) \rightrightarrows \mathit{Coh}(U_1, \mathcal{I}_1) \overset{\longrightarrow}{\rightrightarrows} \mathit{Coh}(U_2, \mathcal{I}_2)$$

an the first arrow presents $\mathit{Coh}(X, \mathcal{I})$ as the homotopy limit of the right part of the diagram. More precisely, given a *descent datum*, i.e., a pair $((\mathcal{G}_n), \varphi)$ where $(\mathcal{G}_n)$ is an object of $\mathit{Coh}(U_0, \mathcal{I}_0)$ and $\varphi : \mathrm{pr}_0^*(\mathcal{G}_n) \to \mathrm{pr}_1^*(\mathcal{G}_n)$ is an isomorphism in $\mathit{Coh}(U_1, \mathcal{I}_1)$ satisfying the cocycle condition in $\mathit{Coh}(U_2, \mathcal{I}_2)$, then there exists a unique object $(\mathcal{F}_n)$ of $\mathit{Coh}(X, \mathcal{I})$ whose associated canonical descent datum is isomorphic to $((\mathcal{G}_n), \varphi)$. Compare with Descent on Spaces, Definition 3.3. The proof of this statement follows immediately by applying Descent on Spaces, Proposition 4.1 to the descent data $(\mathcal{G}_n, \varphi_n)$ for varying n.

Lemma 31.1. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.*

(1) *The category $\mathit{Coh}(X, \mathcal{I})$ is abelian.*

(2) *Exactness in* $Coh(X, \mathcal{I})$ *can be checked étale locally.*
(3) *For any flat morphism* $f : X' \to X$ *of Noetherian algebraic spaces the functor* $f^* : Coh(X, \mathcal{I}) \to Coh(X', f^{-1}\mathcal{I}\mathcal{O}_{X'})$ *is exact.*

Proof. Proof of (1). Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X$. Set $U_1 = U_0 \times_X U_0$ and $U_2 = U_0 \times_X U_0 \times_X U_0$ as in our discussion of étale descent above. The categories $Coh(U_i, \mathcal{I}_i)$ are abelian (Cohomology of Schemes, Lemma 21.2) and the pullback functors are exact functors $Coh(U_0, \mathcal{I}_0) \to Coh(U_1, \mathcal{I}_1)$ and $Coh(U_1, \mathcal{I}_1) \to Coh(U_2, \mathcal{I}_2)$ (Cohomology of Schemes, Lemma 22.1). The lemma then follows formally from the description of $Coh(X, \mathcal{I})$ as a category of descent data. Some details omitted; compare with the proof of Groupoids, Lemma 12.6.

Part (2) follows immediately from the discussion in the previous paragraph. In the situation of (3) choose a commutative diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

where U' and U are affine schemes and the vertical morphisms are surjective étale. Then $U' \to U$ is a flat morphism of Noetherian schemes (Morphisms of Spaces, Lemma 28.5) whence the pullback functor $Coh(U, \mathcal{I}\mathcal{O}_U) \to Coh(U', \mathcal{I}\mathcal{O}_{U'})$ is exact by Cohomology of Schemes, Lemma 22.1. Since we can check exactness in $Coh(X, \mathcal{O}_X)$ on U and similarly for X', U' the assertion follows. $\square$

Lemma 31.2. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. A map $(\mathcal{F}_n) \to (\mathcal{G}_n)$ is surjective in $Coh(X, \mathcal{I})$ if and only if $\mathcal{F}_1 \to \mathcal{G}_1$ is surjective.*

Proof. We can check on an affine étale cover of X by Lemma 31.1. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 21.3. $\square$

Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There is a functor

$$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge \tag{31.2.1}$$

which associates to the coherent $\mathcal{O}_X$-module $\mathcal{F}$ the object $\mathcal{F}^\wedge = (\mathcal{F}/\mathcal{I}^n\mathcal{F})$ of $Coh(X, \mathcal{I})$.

Lemma 31.3. *The functor (31.2.1) is exact.*

Proof. It suffices to check this étale locally on X, see Lemma 31.1. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 21.5. $\square$

Lemma 31.4. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Then*

$$\lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge).$$

Proof. Since $\mathcal{H}$ is a sheaf on $X_{étale}$ and since we have étale descent for objects of $Coh(X, \mathcal{I})$ it suffices to prove this étale locally. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 21.6. $\square$

We introduce the setting that we will focus on throughout the rest of this section.

Situation 31.5. Here A is a Noetherian ring complete with respect to an ideal I. Also $f : X \to \mathrm{Spec}(A)$ is a finite type separated morphism of algebraic spaces and $\mathcal{I} = I\mathcal{O}_X$.

In this situation we denote

$$\mathit{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$$

be the full subcategory of $\mathit{Coh}(\mathcal{O}_X)$ consisting of those coherent $\mathcal{O}_X$-modules whose scheme theoretic support is proper over $\mathrm{Spec}(A)$. Similarly, we let

$$\mathit{Coh}_{\text{support proper over } A}(X, \mathcal{I})$$

be the full subcategory of $\mathit{Coh}(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the scheme theoretic support of $\mathcal{F}_1$ is proper over $\mathrm{Spec}(A)$. Since the support of a quotient module is contained in the support of the module, it follows that (31.2.1) induces a functor

$$\mathit{Coh}_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow \mathit{Coh}_{\text{support proper over } A}(X, \mathcal{I}) \tag{31.5.1}$$

Our first result is that this functor is fully faithful.

Lemma 31.6. *In Situation 31.5. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Assume that the intersection of the scheme theoretic supports of $\mathcal{F}$ and $\mathcal{G}$ is proper over* $\mathrm{Spec}(A)$*. Then the map*

$$\mathrm{Mor}_{\mathit{Coh}(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Mor}_{\mathit{Coh}(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge)$$

coming from (31.2.1) is a bijection. In particular, (31.5.1) is fully faithful.

Proof. Let $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$. This is a coherent $\mathcal{O}_X$-module because its restriction of schemes étale over X is coherent by Modules, Lemma 19.4. By Lemma 31.4 the map

$$\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) \to \mathrm{Mor}_{\mathit{Coh}(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that Z is a closed subspace contained in the intersection of the scheme theoretic supports of $\mathcal{F}$ and $\mathcal{G}$. Hence $Z \to \mathrm{Spec}(A)$ is proper by assumption. Write $\mathcal{H} = i_*\mathcal{H}'$ for some coherent $\mathcal{O}_Z$-module $\mathcal{H}'$. We have $i_*(\mathcal{H}'/I^n\mathcal{H}') = \mathcal{H}/I^n\mathcal{H}$. Hence we obtain

$$\begin{aligned} \lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) &= \lim_n H^0(Z, \mathcal{H}'/\mathcal{I}^n\mathcal{H}') \\ &= H^0(Z, \mathcal{H}') \\ &= H^0(X, \mathcal{H}) \\ &= \mathrm{Mor}_{\mathit{Coh}(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G}) \end{aligned}$$

the second equality by the theorem on formal functions functions (Cohomology of Spaces, Lemma 20.6). This proves the lemma. $\square$

Remark 31.7. Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $\mathit{Coh}(X, \mathcal{I})$. Given an affine scheme $U = \mathrm{Spec}(A)$ and a surjective étale morphism $U \to X$ denote $I, K \subset A$ the ideals corresponding to the restrictions $\mathcal{I}|_U, \mathcal{K}|_U$. Denote $\alpha_U : M \to N$ of finite $A^\wedge$-modules which corresponds to $\alpha|_U$ via Cohomology of Schemes, Lemma 21.1. We claim the following are equivalent

(1) there exists an integer $t \geq 1$ such that $\mathrm{Ker}(\alpha_n)$ and $\mathrm{Coker}(\alpha_n)$ are annihilated by $\mathcal{K}^t$ for all $n \geq 1$,

(2) for any (or some) affine open $\mathrm{Spec}(A) = U \subset X$ as above the modules $\mathrm{Ker}(\alpha_U)$ and $\mathrm{Coker}(\alpha_U)$ are annihilated by K^t for some integer $t \geq 1$.

If these equivalent conditions hold we will say that α is a *map whose kernel and cokernel are annihilated by a power of* $\mathcal{K}$. To see the equivalence we refer to Cohomology of Schemes, Remark 22.2.

Lemma 31.8. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module, $(\mathcal{F}_n)$ an object of $Coh(X, \mathcal{I})$, and $\alpha : (\mathcal{F}_n) \to \mathcal{G}^\wedge$ a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$. Then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where*

(1) *$\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,*
(2) *$a : \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,*
(3) *$\beta : (\mathcal{F}_n) \to \mathcal{F}^\wedge$ is an isomorphism, and*
(4) *$\alpha = a^\wedge \circ \beta$.*

Proof. The uniqueness and étale descent for objects of $Coh(X, \mathcal{I})$ and $Coh(\mathcal{O}_X)$ implies it suffices to construct $(\mathcal{F}, a, \beta)$ étale locally on X. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 22.3. □

Lemma 31.9. *In Situation 31.5. Let $\mathcal{K} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $X_e \subset X$ be the closed subspace cut out by $\mathcal{K}^e$. Let $\mathcal{I}_e = \mathcal{I}\mathcal{O}_{X_e}$. Let $(\mathcal{F}_n)$ be an object of $Coh_{support\ proper\ over\ A}(X, \mathcal{I})$. Assume*

(1) *the functor $Coh_{support\ proper\ over\ A}(\mathcal{O}_{X_e}) \to Coh_{support\ proper\ over\ A}(X_e, \mathcal{I}_e)$ is an equivalence for all $e \geq 1$, and*
(2) *there exists an object $\mathcal{H}$ of $Coh_{support\ proper\ over\ A}(\mathcal{O}_X)$ and a map $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.*

Then $(\mathcal{F}_n)$ is in the essential image of (31.5.1).

Proof. During this proof we will use without further mention that for a closed immersion $i : Z \to X$ the functor i_* gives an equivalence between the category of coherent modules on Z and coherent modules on X annihilated by the ideal sheaf of Z, see Cohomology of Spaces, Lemma 11.8. In particular we think of

$$Coh_{\text{support proper over } A}(\mathcal{O}_{X_e}) \subset Coh_{\text{support proper over } A}(\mathcal{O}_X)$$

as the full subcategory of consisting of modules annihilated by $\mathcal{K}^e$ and

$$Coh_{\text{support proper over } A}(X_e, \mathcal{I}_e) \subset Coh_{\text{support proper over } A}(X, \mathcal{I})$$

as the full subcategory of of objects annihilated by $\mathcal{K}^e$. Moreover (1) tells us these two categories are equivalent under the completion functor (31.5.1).

Applying this equivalence we get a coherent $\mathcal{O}_X$-module $\mathcal{G}_e$ annihilated by $\mathcal{K}^e$ corresponding to the system $(\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n)$ of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. The maps $\mathcal{F}_n/\mathcal{K}^{e+1}\mathcal{F}_n \to \mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n$ correspond to canonical maps $\mathcal{G}_{e+1} \to \mathcal{G}_e$ which induce isomorphisms $\mathcal{G}_{e+1}/\mathcal{K}^e\mathcal{G}_{e+1} \to \mathcal{G}_e$. We obtain an object $(\mathcal{G}_e)$ of the category $Coh_{\text{support proper over } A}(X, \mathcal{K})$. The map α induces a system of maps

$$\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \longrightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$$

whence maps $\mathcal{G}_e \to \mathcal{H}/\mathcal{K}^e\mathcal{H}$ (by the equivalence of categories again). Let $t \geq 1$ be an integer, which exists by assumption (2), such that $\mathcal{K}^t$ annihilates the kernel and cokernel of all the maps $\mathcal{F}_n \to \mathcal{H}/\mathcal{I}^n\mathcal{H}$. Then $\mathcal{K}^{2t}$ annihilates the kernel and cokernel of the maps $\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \to \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$ (details omitted; see Cohomology of Schemes, Remark 22.2). Whereupon we conclude that $\mathcal{K}^{4t}$ annihilates the kernel and the cokernel of the maps

$$\mathcal{G}_e \longrightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H},$$

(details omitted; see Cohomology of Schemes, Remark 22.2). We apply Lemma 31.8 to obtain a coherent $\mathcal{O}_X$-module $\mathcal{F}$, a map $a : \mathcal{F} \to \mathcal{H}$ and an isomorphism $\beta : (\mathcal{G}_e) \to (\mathcal{F}/\mathcal{K}^e\mathcal{F})$ in $Coh(X, \mathcal{K})$. Working backwards, for a given n the triple $(\mathcal{F}/\mathcal{I}^n\mathcal{F}, a \bmod \mathcal{I}^n, \beta \bmod \mathcal{I}^n)$ is a triple as in the lemma for the morphism $\alpha_n \bmod \mathcal{K}^e : (\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n) \to (\mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H})$ of $Coh(X, \mathcal{K})$. Thus the uniqueness in Lemma 31.8 gives a canonical isomorphism $\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$ compatible with all the morphisms in sight.

To finish the proof of the lemma we still have to show that the scheme theoretic support of $\mathcal{F}$ is proper over A. By construction the kernel of $a : \mathcal{F} \to \mathcal{H}$ is annihilated by a power of $\mathcal{K}$. Hence the support of this kernel is contained in the support of $\mathcal{G}_1$. Since $\mathcal{G}_1$ is an object of $Coh_{\text{support proper over } A}(\mathcal{O}_{X_1})$ we see this is proper over A. Combined with the fact that the support of $\mathcal{H}$ is proper over A we conclude that the support of $\mathcal{F}$ is proper over A (some details omitted). $\square$

Lemma 31.10. *Let S be a scheme. Let $f : X \to Y$ be a representable proper morphism of Noetherian algebraic spaces over S. Let $\mathcal{J}, \mathcal{K} \subset \mathcal{O}_Y$ be quasi-coherent sheaves of ideals. Assume f is an isomorphism over $V = Y \setminus V(\mathcal{K})$. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Let $(\mathcal{G}_n)$ be an object of $Coh(Y, \mathcal{J})$, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and let $\beta : (f^*\mathcal{G}_n) \to \mathcal{F}^\wedge$ be an isomorphism in $Coh(X, \mathcal{I})$. Then there exists a map*

$$\alpha : (\mathcal{G}_n) \longrightarrow (f_*\mathcal{F})^\wedge$$

in $Coh(Y, \mathcal{J})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Proof. Since f is a proper morphism we see that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module (Cohomology of Spaces, Lemma 19.2). Thus the statement of the lemma makes sense. Consider the compositions

$$\gamma_n : \mathcal{G}_n \to f_*f^*\mathcal{G}_n \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F}).$$

Here the first map is the adjunction map and the second is $f_*\beta_n$. We claim that there exists a unique α as in the lemma such that the compositions

$$\mathcal{G}_n \xrightarrow{\alpha_n} f_*\mathcal{F}/\mathcal{J}^n f_*\mathcal{F} \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F})$$

equal γ_n for all n. Because of the uniqueness and étale descent for $Coh(Y, \mathcal{J})$ it suffices to prove this étale locally on Y. Thus we may assume Y is the spectrum of a Noetherian ring. As f is representable we see that X is a scheme as well. Thus we reduce to the case of schemes, see proof of Cohomology of Schemes, Lemma 22.5. $\square$

Theorem 31.11 (Grothendieck's existence theorem)**.** *In Situation 31.5 the functor (31.5.1) is an equivalence.*

Proof. We will use the equivalence of categories of Cohomology of Spaces, Lemma 11.8 without further mention in the proof of the theorem. By Lemma 31.6 the functor is fully faithful. Thus we need to prove the functor is essentially surjective.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that the statement holds for every object $(\mathcal{F}_n)$ of $\textit{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by $\mathcal{K}$. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Cohomology of Spaces, Lemma 12.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. Let $(\mathcal{F}_n)$ be an object of $\textit{Coh}_{\text{support proper over } A}(X, \mathcal{I})$. We will show that this object is in the essential image, thereby completing the proof of the theorem.

Apply Chow's lemma (Lemma 29.5) to find a proper surjective morphism $f : Y \to X$ which is an isomorphism over a dense open $U \subset X$ such that Y is H-quasi-projective over A. Note that Y is a scheme and f representable. Choose an open immersion $j : Y \to Y'$ with Y' projective over A, see Morphisms, Lemma 43.11. Let T_n be the scheme theoretic support of $\mathcal{F}_n$. Note that $|T_n| = |T_1|$, hence T_n is proper over A for all n (Morphisms of Spaces, Lemma 37.7). Then $f^*\mathcal{F}_n$ is supported on the closed subscheme $f^{-1}T_n$ which is proper over A (by Morphisms of Spaces, Lemma 37.4 and properness of f). In particular, the composition $f^{-1}T_n \to Y \to Y'$ is closed (Morphisms, Lemma 42.7). Let $T'_n \subset Y'$ be the corresponding closed subscheme; it is contained in the open subscheme Y and equal to $f^{-1}T_n$ as a closed subscheme of Y. Let $\mathcal{F}'_n$ be the coherent $\mathcal{O}_{Y'}$-module corresponding to $f^*\mathcal{F}_n$ viewed as a coherent module on Y' via the closed immersion $f^{-1}T_n = T'_n \subset Y'$. Then $(\mathcal{F}'_n)$ is an object of $\textit{Coh}(Y', I\mathcal{O}_{Y'})$. By the projective case of Grothendieck's existence theorem (Cohomology of Schemes, Lemma 21.9) there exists a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}'$ and an isomorphism $(\mathcal{F}')^\wedge \cong (\mathcal{F}'_n)$ in $\textit{Coh}(Y', I\mathcal{O}_{Y'})$. Let $Z' \subset Y'$ be the scheme theoretic support of $\mathcal{F}'$. Since $\mathcal{F}'/I\mathcal{F}' = \mathcal{F}'_1$ we see that $Z' \cap V(I\mathcal{O}_{Y'}) = T'_1$ set-theoretically. The structure morphism $p' : Y' \to \text{Spec}(A)$ is proper, hence $p'(Z' \cap (Y' \setminus Y))$ is closed in $\text{Spec}(A)$. If nonempty, then it would contain a point of $V(I)$ as I is contained in the radical of A (Algebra, Lemma 93.11). But we've seen above that $Z' \cap (p')^{-1}V(I) = T'_1 \subset Y$ hence we conclude that $Z' \subset Y$. Thus $\mathcal{F}'|_Y$ is supported on a closed subscheme of Y proper over A.

Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By Cohomology of Spaces, Lemma 19.2 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 31.10 there exists a morphism $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ in the category $\textit{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. Let $Z_0 \subset X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that $|Z_0| \subset f(|Z'|)$. Hence $Z_0 \to \text{Spec}(A)$ is proper (Morphisms of Spaces, Lemma 37.7). Thus $\mathcal{H}$ is an object of $\textit{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$. Since each of the sheaves of ideals $\mathcal{K}^e$ is an element of Ξ we see that the assumptions of Lemma 31.9 are satisfied and we conclude. $\square$

Remark 31.12 (Unwinding Grothendieck's existence theorem)**.** Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Picture:

$$\begin{tikzcd} X_1 \arrow[r, "i_1"'] \arrow[d] & X_2 \arrow[r, "i_2"'] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots & X \arrow[d] \\ S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \ldots & S \end{tikzcd}$$

In this situation we consider systems $(\mathcal{F}_n, \varphi_n)$ where

(1) $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module,
(2) $\varphi_n : i_n^*\mathcal{F}_{n+1} \to \mathcal{F}_n$ is an isomorphism, and
(3) $\text{Supp}(\mathcal{F}_1)$ is proper over S_1.

Theorem 31.11 says that the completion functor

$$\begin{matrix} \text{coherent } \mathcal{O}_X\text{-modules } \mathcal{F} \\ \text{with support proper over } A \end{matrix} \longrightarrow \begin{matrix} \text{systems } (\mathcal{F}_n) \\ \text{as above} \end{matrix}$$

is an equivalence of categories. In the special case that X is proper over A we can omit the conditions on the supports.

32. Grothendieck's algebraization theorem

This section is the analogue of Cohomology of Schemes, Section 23. However, this section is missing the result on algebraization of deformations of proper algebraic spaces endowed with ample invertible sheaves, as a proper algebraic space which comes with an ample invertible sheaf is a scheme. Our first result is a translation of Grothendieck's existence theorem in terms of closed subschemes and finite morphisms.

Lemma 32.1. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{tikzcd} Z_1 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & Z_3 \arrow[r] \arrow[d] & \ldots \\ X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \ldots \end{tikzcd}$$

of algebraic spaces with cartesian squares. Assume that

(1) *$Z_1 \to X_1$ is a closed immersion, and*
(2) *$Z_1 \to S_1$ is proper.*

Then there exists a closed immersion of algebraic spaces $Z \to X$ such that $Z_n = Z \times_S S_n$ for all $n \geq 1$. Moreover, Z is proper over S.

Proof. Let's write $j_n : Z_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of j_n to X_1 is j_1. Thus Limits of Spaces, Lemma 15.5 shows that j_n is a closed immersion. Set $\mathcal{F}_n = j_{n,*}\mathcal{O}_{Z_n}$, so that $j_n^\sharp$ is a surjection $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Again using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 31.12, tells us there exists a map $\mathcal{O}_X \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to X_n recovers $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is exact (Lemma 31.3) we see that $\mathcal{O}_X \to \mathcal{F}$ is surjective. Thus $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$ for some quasi-coherent sheaf of ideals $\mathcal{J}$. Setting $Z = V(\mathcal{J})$ finishes the proof. $\square$

Lemma 32.2. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic* *spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & \ldots \\
X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \ldots
\end{tikzcd}$$

of algebraic spaces with cartesian squares. Assume that

1. *$Y_1 \to X_1$ is a finite morphism, and*
2. *$Y_1 \to S_1$ is proper.*

Then there exists a finite morphism of algebraic spaces $Y \to X$ such that $Y_n = Y \times_S S_n$ for all $n \geq 1$. Moreover, Y is proper over S.

Proof. Let's write $f_n : Y_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of f_n to X_1 is f_1. Thus Lemma 8.10 shows that f_n is a finite morphism. Set $\mathcal{F}_n = f_{n,*}\mathcal{O}_{Y_n}$. Using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 31.12, tells us there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose restriction to X_n recovers $\mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is fuly faithful (Theorem 31.11) we see that the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{F}_n \to \mathcal{F}_n$ fit together to give an algebra structure on $\mathcal{F}$. Setting $Y = \underline{\text{Spec}}_X(\mathcal{F})$ finishes the proof. $\square$

Lemma 32.3. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let X, Y be algebraic spaces over S. For $n \geq 1$ we set $X_n = X \times_S S_n$ and $Y_n = Y \times_S S_n$. Suppose given a compatible system of commutative diagrams*

$$\begin{tikzcd}
& & X_{n+1} \arrow[rr, "g_{n+1}"'] \arrow[dr] & & Y_{n+1} \arrow[dl] \\
X_n \arrow[urr] \arrow[rr, "g_n"'] \arrow[dr] & & Y_n \arrow[urr] \arrow[dl] & S_{n+1} & \\
& S_n \arrow[urr] & & &
\end{tikzcd}$$

Assume that

1. *$X \to S$ is proper, and*
2. *$Y \to S$ is separated of finite type.*

Then there exists a unique morphism of algebraic spaces $g : X \to Y$ over S such that g_n is the base change of g to S_n.

Proof. The morphisms $(1, g_n) : X_n \to X_n \times_S Y_n$ are closed immersions because $Y_n \to S_n$ is separated (Morphisms of Spaces, Lemma 4.7). Thus by Lemma 32.1 there exists a closed subspace $Z \subset X \times_S Y$ proper over S whose base change to S_n recovers $X_n \subset X_n \times_S Y_n$. The first projection $p : Z \to X$ is a proper morphism (as Z is proper over S, see Morphisms of Spaces, Lemma 37.6) whose base change to S_n is an isomorphism for all n. In particular, $p : Z \to X$ is quasi-finite on an open subspace of Z containing every point of Z_0 for example by Morphisms of Spaces, Lemma 32.7. As Z is proper over S this open neighbourhood is all of Z.

We conclude that $p : Z \to X$ is finite by Zariski's main theorem (for example apply Lemma 24.3 and use properness of Z over X to see that the immersion is a closed immersion). Applying the equivalence of Theorem 31.11 we see that $p_*\mathcal{O}_Z = \mathcal{O}_X$ as this is true modulo I^n for all n. Hence p is an isomorphism and we obtain the morphism g as the composition $X \cong Z \to Y$. We omit the proof of uniqueness. $\square$

33. Regular immersions

This section is the analogue of Divisors, Section 13 for morphisms of algebraic spaces. The reader is encouraged to read up on regular immersions of schemes in that section first.

In Divisors, Section 13 we defined four types of regular immersions for morphisms of schemes. Of these only three are (as far as we know) local on the target for the étale topology; as usual plain old regular immersions aren't. This is why for morphisms of algebraic spaces we cannot actually define regular immersions. (These kinds of annoyances prompted Grothendieck and his school to replace original notion of a regular immersion by a Koszul-regular immersions, see [BGI71, Exposee VII, Definition 1.4].) But we can define Koszul-regular, H_1-regular, and quasi-regular immersions. Another remark is that since Koszul-regular immersions are not preserved by arbitrary base change, we cannot use the strategy of Morphisms of Spaces, Section 3 to define them. Similarly, as Koszul-regular immersions are not étale local on the source, we cannot use Morphisms of Spaces, Lemma 22.1 to define them either. We replace this lemma instead by the following.

Lemma 33.1. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the target. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Consider commutative diagrams*

$$\begin{tikzcd} X \times_Y V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where V is a scheme and $V \to Y$ is étale. The following are equivalent

1. *for any diagram as above the projection $X \times_Y V \to V$ has property $\mathcal{P}$, and*
2. *for some diagram as above with $V \to Y$ surjective the projection $X \times_Y V \to V$ has property $\mathcal{P}$.*

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (2) is immediate. Assume

$$\begin{tikzcd} X \times_Y V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \times_Y V' \arrow[r] \arrow[d] & V' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

are two diagrams as in the lemma. Assume $V \to Y$ is surjective and $X \times_Y V \to V$ has property $\mathcal{P}$. To show that (2) implies (1) we have to prove that $X \times_Y V' \to V'$

has $\mathcal{P}$. To do this consider the diagram

$$\begin{tikzcd} X \times_Y V \arrow[d] & (X \times_Y V) \times_X (X \times_Y V') \arrow[l] \arrow[r] \arrow[d] & X \times_Y V' \arrow[d] \\ V & V \times_Y V' \arrow[l] \arrow[r] & V' \end{tikzcd}$$

By our assumption that $\mathcal{P}$ is étale local on the source, we see that $\mathcal{P}$ is preserved under étale base change, see Descent, Lemma 18.2. Hence if the left vertical arrow has $\mathcal{P}$ the so does the middle vertical arrow. Since $U \times_X U' \to U'$ is surjective and étale (hence defines an étale covering of U') this implies (as $\mathcal{P}$ is assumed local for the étale topology on the target) that the left vertical arrow has $\mathcal{P}$.

If X and Y are representable, then we can take $\mathrm{id}_Y : Y \to Y$ as our étale covering to see the final statement of the lemma is true. $\square$

Note that "being a Koszul-regular (resp. H_1-regular, resp. quasi-regular) immersion" is a property of morphisms of schemes which is fpqc local on the target, see Descent, Lemma 19.30. Hence the following definition now makes sense.

Definition 33.2. Let S be a scheme. Let $i : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say i is a *Koszul-regular immersion* if i is representable and the equivalent conditions of Lemma 33.1 hold with $\mathcal{P}(f) =$"f is a Koszul-regular immersion".
(2) We say i is an *H_1-regular immersion* if i is representable and the equivalent conditions of Lemma 33.1 hold with $\mathcal{P}(f) =$"f is an H_1-regular immersion".
(3) We say i is a *quasi-regular immersion* if i is representable and the equivalent conditions of Lemma 33.1 hold with $\mathcal{P}(f) =$"f is a quasi-regular immersion".

Lemma 33.3. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. We have the following implications: i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.*

Proof. Via the definition this lemma immediately reduces to Divisors, Lemma 13.2. $\square$

Lemma 33.4. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Assume X is locally Noetherian. Then i is Koszul-regular $\Leftrightarrow$ i is H_1-regular $\Leftrightarrow$ i is quasi-regular.*

Proof. Via Definition 33.2 (and the definition of a locally Noetherian algebraic space in Properties of Spaces, Section 7) this immediately translates to the case of schemes which is Divisors, Lemma 13.3. $\square$

Lemma 33.5. *Let S be a scheme. Let $i : Z \to X$ be a Koszul-regular, H_1-regular, or quasi-regular immersion of algebraic spaces over S. Let $X' \to X$ be a flat morphism of algebraic spaces over S. Then the base change $i' : Z \times_X X' \to X'$ is a Koszul-regular, H_1-regular, or quasi-regular immersion.*

Proof. Via Definition 33.2 (and the definition of a flat morphism of algebraic spaces in Morphisms of Spaces, Section 28) this lemma reduces to the case of schemes, see Divisors, Lemma 13.4. $\square$

Lemma 33.6. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then i is a quasi-regular immersion if and only if the following conditions are satisfied*

1. *i is locally of finite presentation,*
2. *the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and*
3. *the map (5.1.2) is an isomorphism.*

Proof. Follows from the case of schemes (Divisors, Lemma 13.5) via étale localization (use Definition 33.2 and Lemma 5.2). $\square$

Lemma 33.7. *Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces over S. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Lemma 4.6*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and (étale) locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 33.6 and Lemma 33.3) it suffices to prove that the sequence is exact. It suffices to show that the first map is injective as the sequence is already right exact in general. After étale localization on X this reduces to the case of schemes, see Divisors, Lemma 13.6. $\square$

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 68.8. The other types of regular immersions are preserved under composition.

Lemma 33.8. *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S.*

1. *If i and j are Koszul-regular immersions, so is $j \circ i$.*
2. *If i and j are H_1-regular immersions, so is $j \circ i$.*
3. *If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.*

Proof. Immediate from the case of schemes, see Divisors, Lemma 13.7. $\square$

Lemma 33.9. *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S. Assume that the sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 4.6 is exact and locally split.

1. *If $j \circ i$ is a quasi-regular immersion, so is i.*
2. *If $j \circ i$ is a H_1-regular immersion, so is i.*
3. *If both j and $j \circ i$ are Koszul-regular immersions, so is i.*

Proof. Immediate from the case of schemes, see Divisors, Lemma 13.8. $\square$

Lemma 33.10. *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S. Assume X is locally Noetherian. The following are equivalent*

1. *i and j are Koszul regular immersions,*
2. *i and $j \circ i$ are Koszul regular immersions,*

(3) *$j \circ i$ is a Koszul regular immersion and the conormal sequence*
$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$
is exact and locally split.

Proof. Immediate from the case of schemes, see Divisors, Lemma 13.9. □

34. Pseudo-coherent morphisms

This section is the analogue of More on Morphisms, Section 40 for morphisms of schemes. The reader is encouraged to read up on pseudo-coherent morphisms of schemes in that section first.

The property "pseudo-coherent" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 40.9 and 40.12 and Descent, Lemma 28.6. By Morphisms of Spaces, Lemma 22.1 we may define the notion of a pseudo-coherent morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 40 when the algebraic spaces in question are representable.

Definition 34.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *pseudo-coherent* if the equivalent conditions of Morphisms of Spaces, Lemma 22.1 hold with $\mathcal{P} =$"pseudo-coherent".
(2) Let $x \in |X|$. We say f is *pseudo-coherent at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is pseudo-coherent.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

Lemma 34.2. *A flat base change of a pseudo-coherent morphism is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 40.3. □

Lemma 34.3. *A composition of pseudo-coherent morphisms is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 40.4. □

Lemma 34.4. *A pseudo-coherent morphism is locally of finite presentation.*

Proof. Immediate from the definitions. □

Lemma 34.5. *A flat morphism which is locally of finite presentation is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 40.6. □

Lemma 34.6. *Let $f : X \to Y$ be a morphism of algebraic spaces pseudo-coherent over a base algebraic space B. Then f is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 40.7. □

Lemma 34.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If Y is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 40.8. $\square$

35. Perfect morphisms

This section is the analogue of More on Morphisms, Section 41 for morphisms of schemes. The reader is encouraged to read up on perfect morphisms of schemes in that section first.

The property "perfect" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 41.10 and 41.12 and Descent, Lemma 28.6. By Morphisms of Spaces, Lemma 22.1 we may define the notion of a perfect morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 41 when the algebraic spaces in question are representable.

Definition 35.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *perfect* if the equivalent conditions of Morphisms of Spaces, Lemma 22.1 hold with $\mathcal{P} =$"perfect".
(2) Let $x \in |X|$. We say f is *perfect at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is perfect.

Note that a perfect morphism is pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

Lemma 35.2. *A flat base change of a perfect morphism is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 41.3. $\square$

Lemma 35.3. *A composition of perfect morphisms is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 41.4. $\square$

Lemma 35.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is flat and perfect, and*
(2) *f is flat and locally of finite presentation.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 41.5. $\square$

36. Local complete intersection morphisms

This section is the analogue of More on Morphisms, Section 42 for morphisms of schemes. The reader is encouraged to read up on local complete intersection morphisms of schemes in that section first.

The property "being a local complete intersection morphism" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 42.12 and 42.13 and Descent, Lemma 28.6. By Morphisms of Spaces, Lemma 22.1 we may define the notion of a local complete intersection morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 42 when the algebraic spaces in question are representable.

Definition 36.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is a *Koszul morphism*, or that f is a *local complete intersection morphism* if the equivalent conditions of Morphisms of Spaces, Lemma 22.1 hold with $\mathcal{P}(f) =$"f is a local complete intersection morphism".
(2) Let $x \in |X|$. We say f is *Koszul at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is a local complete intersection morphism.

In some sense the defining property of a local complete intersection morphism is the result of the following lemma.

Lemma 36.2. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Let P be an algebraic space smooth over Y. Let $U \to X$ be an étale morphism of algebraic spaces and let $i : U \to P$ an immersion of algebraic spaces over Y. Picture:*

$$\begin{tikzcd} X \arrow[dr] & U \arrow[l] \arrow[r, "i"'] \arrow[d] & P \arrow[dl] \\ & Y & \end{tikzcd}$$

Then i is a Koszul-regular immersion of algebraic spaces.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme W and a surjective étale morphism $W \to P \times_Y V$. Set $U' = U \times_P W$, which is a scheme étale over U. We have to show that $U' \to W$ is a Koszul-regular immersion of schemes, see Definition 33.2. By Definition 36.1 above the morphism of schemes $U' \to V$ is a local complete intersection morphism. Hence the result follows from More on Morphisms, Lemma 42.3. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

Lemma 36.3. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then*

(1) *f is locally of finite presentation,*
(2) *f is pseudo-coherent, and*
(3) *f is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 42.4. □

Beware that a base change of a Koszul morphism is not Koszul in general.

Lemma 36.4. *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 42.6. ☐

Lemma 36.5. *A composition of local complete intersection morphisms is a local complete intersection morphism.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 42.7. ☐

Lemma 36.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is flat and a local complete intersection morphism, and*
(2) *f is syntomic.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 42.8. ☐

Lemma 36.7. *Let S be a scheme. Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces over S. Assume that both p and q are flat and locally of finite presentation. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is Koszul. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if $f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of More on Morphisms, Lemma 42.14 and in fact we will deduce the lemma from it. By Definition 36.1 the set $\{x \in |X| : f \text{ is Koszul at } x\}$ is open in $|X|$ hence by Properties of Spaces, Lemma 4.8 it corresponds to an open subspace $U(f)$ of X. Hence we only need to prove the final statement.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 13.7). Note that by definition $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism of schemes $U \to V$ is Koszul. Similarly, $U(f')$ is the image in $|X'|$ of the set T' of points in U' where the morphism of schemes $U' \to V'$ is Koszul. Now, by construction the diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is cartesian (in the category of schemes). Hence the aforementioned More on Morphisms, Lemma 42.14 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. $\square$

Lemma 36.8. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.*

Proof. This follows from the corresponding result for morphisms of schemes, see More on Morphisms, Lemma 42.15, by étale localization, see Lemma 12.11. (Note that in the situation of this lemma the morphism $V \to U$ is unramified and a local complete intersection morphism by definition.) $\square$

Lemma 36.9. *Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 4.6 is short exact.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a scheme V and a surjective étale morphism $V \to U \times_X Y$. Choose a scheme W and a surjective étale morphism $W \to V \times_Y Z$. By Lemma 12.11 the morphisms $W \to V$ and $V \to U$ are formally unramified. Moreover the sequence $i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$ restricts to the corresponding sequence $i^*\mathcal{C}_{V/U} \to \mathcal{C}_{W/U} \to \mathcal{C}_{W/V} \to 0$ for $W \to V \to U$. Hence the result follows from the result for schemes (More on Morphisms, Lemma 42.16) as by definition the morphism $W \to V$ is a local complete intersection morphism. $\square$

37. When is a morphism an isomorphism?

More generally we can ask: "When does a morphism have property $\mathcal{P}$?" A more precise question is the following. Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Does there exist a monomorphism of algebraic spaces $W \to Z$ with the following two properties:

(1) the base change $f_W : X_W \to Y_W$ has property $\mathcal{P}$, and
(2) any morphism $Z' \to Z$ of algebraic spaces factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ has property $\mathcal{P}$.

In many cases, if $W \to Z$ exists, then it is an immersion, open immersion, or closed immersion.

The answer to this question may depend on auxiliary properties of the morphisms f, p, and q. An example is $\mathcal{P}(f) =$"f is flat" which we have discussed for morphisms of schemes in the case $Y = S$ in great detail in the chapter "More on Flatness", starting with More on Flatness, Section 19.

Lemma 37.1. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that p is locally of finite type and closed. Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified.

Proof. By Morphisms of Spaces, Lemma 35.10 there exists an open subspace $U(f) \subset X$ which is the set of points where f is unramified. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 4.8) with underlying set of points

$$|W| = |Z| \setminus |p| \left(|X| \setminus |U(f)|\right)$$

i.e., $z \in |Z|$ is a point of W if and only if f is unramified at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of $U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 4.9) that W has the desired universal property. $\square$

Lemma 37.2. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

(1) *p is locally of finite type,*
(2) *p is closed, and*
(3) *$p_2 : X \times_Y X \to Z$ is closed.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified and universally injective.

Proof. After replacing Z by the open subspace found in Lemma 37.1 we may assume that f is already unramified; note that this does not destroy assumption (2) or (3). By Morphisms of Spaces, Lemma 35.9 we see that $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion. This remains true after any base change. Hence by Morphisms of Spaces, Lemma 19.2 we see that $f_{Z'}$ is universally injective if and only if the base change of the diagonal $X_{Z'} \to (X \times_Y X)_{Z'}$ is an isomorphism. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 4.8) with underlying set of points

$$|W| = |Z| \setminus |p_2| \left(|X \times_Y X| \setminus \mathrm{Im}(|\Delta_{X/Y}|)\right)$$

i.e., $z \in |Z|$ is a point of W if and only if the fibre of $|X \times_Y X| \to |Z|$ over z is in the image of $|X| \to |X \times_Y X|$. Then it is clear from the discussion above that the restriction $p^{-1}(W) \to q^{-1}(W)$ of f is unramified and universally injective.

Conversely, suppose that $f_{Z'}$ is unramified and universally injective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. The discussion above shows that

$$|X_{Z'}| \longrightarrow |(X \times_Y X)_{Z'}|$$

is surjective. By Properties of Spaces, Lemma 4.3 in the commutative diagram

$$\begin{tikzcd}
|X_{Z'}| \arrow[r] \arrow[d] & |(X \times_Y X)_{Z'}| \arrow[d] \\
|p|^{-1}(\{z\}) \arrow[r] & |p_2|^{-1}(\{z\})
\end{tikzcd}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. □

Lemma 37.3. *Consider a commutative diagram*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& Z &
\end{tikzcd}$$

of algebraic spaces. Assume that

1. *p is locally of finite type,*
2. *p is universally closed, and*
3. *$q : Y \to Z$ is separated.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a closed immersion.

Proof. We will use the characterization of closed immersions as universally closed, unramified, and universally injective morphisms, see Lemma 11.9. First, note that since p is universally closed and q is separated, we see that f is universally closed, see Morphisms of Spaces, Lemma 37.6. It follows that any base change of f is universally closed, see Morphisms of Spaces, Lemma 9.3. Thus to finish the proof of the lemma it suffices to prove that the assumptions of Lemma 37.2 are satisfied. The projection $\text{pr}_0 : X \times_Y X \to X$ is universally closed as a base change of f, see Morphisms of Spaces, Lemma 9.3. Hence $X \times_Y X \to Z$ is universally closed as a composition of universally closed morphisms (see Morphisms of Spaces, Lemma 9.4). This finishes the proof of the lemma. □

Lemma 37.4. *Consider a commutative diagram*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& Z &
\end{tikzcd}$$

of algebraic spaces. Assume that

1. *p is locally of finite presentation,*
2. *p is flat,*
3. *p is closed, and*
4. *q is locally of finite type.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is flat.

Proof. By Lemma 19.6 the set
$$A = \{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$
is open in $|X|$ and its formation commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 4.8) with underlying set of points
$$|W| = |Z| \setminus |p|\left(|X| \setminus A\right)$$
i.e., $z \in |Z|$ is a point of W if and only if the whole fibre of $|X| \to |Z|$ over z is contained in A. This is open because p is closed. Since the formation of A commutes with arbitrary base change it follows that W works. $\square$

Lemma 37.5. *Consider a commutative diagram*
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$
of algebraic spaces. Assume that

1. *p is locally of finite presentation,*
2. *p is flat,*
3. *p is closed,*
4. *q is locally of finite type, and*
5. *q is closed.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is surjective and flat.

Proof. By Lemma 37.4 we may assume that f is flat. Note that f is locally of finite presentation by Morphisms of Spaces, Lemma 27.9. Hence f is open, see Morphisms of Spaces, Lemma 28.6. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 4.8) with underlying set of points
$$|W| = |Z| \setminus |q|\left(|Y| \setminus |f|(|X|)\right).$$
in other words for $z \in |Z|$ we have $z \in |W|$ if and only if the whole fibre of $|Y| \to |Z|$ over z is in the image of $|X| \to |Y|$. Since q is closed this set is open in $|Z|$. The morphism $X_W \to Y_W$ is surjective by construction. Finally, suppose that $X_{Z'} \to Y_{Z'}$ is surjective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. By Properties of Spaces, Lemma 4.3 in the commutative diagram
$$\begin{tikzcd} {|X_{Z'}|} \arrow[r] \arrow[d] & {|Y_{Z'}|} \arrow[d] \\ {|p|^{-1}(\{z\})} \arrow[r] & {|q|^{-1}(\{z\})} \end{tikzcd}$$
the vertical arrows are surjective. It follows that $z \in |W|$ as desired. $\square$

Lemma 37.6. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

1. *p is locally of finite presentation,*
2. *p is flat,*
3. *p is universally closed,*
4. *q is locally of finite type,*
5. *q is closed, and*
6. *q is separated.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is an isomorphism.

Proof. By Lemma 37.5 there exists an open subspace $W_1 \subset Z$ such that $f_{Z'}$ is surjective and flat if and only if $Z' \to Z$ factors through W_1. By Lemma 37.3 there exists an open subspace $W_2 \subset Z$ such that $f_{Z'}$ is a closed immersion if and only if $Z' \to Z$ factors through W_2. We claim that $W = W_1 \cap W_2$ works. Certainly, if $f_{Z'}$ is an isomorphism, then $Z' \to Z$ factors through W. Hence it suffices to show that f_W is an isomorphism. By construction f_W is a surjective flat closed immersion. In particular f_W is representable. Since a surjective flat closed immersion of schemes is an isomorphism (see Morphisms, Lemma 27.1) we win. (Note that actually f_W is locally of finite presentation, whence open, so you can avoid the use of this lemma if you like.) $\square$

Lemma 37.7. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

of algebraic spaces. Assume that

1. *p is flat and locally of finite presentation,*
2. *p is closed, and*
3. *q is flat and locally of finite presentation,*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a local complete intersection morphism.

Proof. By Lemma 36.7 there exists an open subspace $U(f) \subset X$ which is the set of points where f is Koszul. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 4.8) with underlying set of points

$$|W| = |Z| \setminus |p| \left(|X| \setminus |U(f)|\right)$$

i.e., $z \in |Z|$ is a point of W if and only if f is Koszul at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of

$U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 4.9) that W has the desired universal property. $\square$

38. Exact sequences of differentials and conormal sheaves

In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to composable morphisms of algebraic spaces.

In the sequences below each of the maps are as constructed in either Lemma 6.6 or Lemma 12.8. Let S be a scheme. Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of algebraic spaces over S.

(1) There is a canonical exact sequence
$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$
see Lemma 6.8. If $g : Z \to Y$ is formally smooth, then this sequence is a short exact sequence, see Lemma 16.11.

(2) If g is formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$
see Lemma 12.13. If $f \circ g : Z \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 16.12.

(3) if g and $f \circ g$ are formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$
see Lemma 12.14. If $f : Y \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 16.13.

(4) if g and f are formally unramified, then there is a canonical exact sequence
$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$
see Lemma 12.15. If $g : Z \to Y$ is a local complete intersection morphism, then this sequence is a short exact sequence, see Lemma 36.9.

39. Other chapters

References

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