DEFORMATION THEORY

Contents

1. Introduction

The goal of this chapter is to give a (relatively) gentle introduction to deformation theory of modules, morphisms, etc. In this chapter we deal with those results that can be proven using the naive cotangent complex. In the chapter on the cotangent complex we will extend these results a little bit. The advanced reader may wish to consult the treatise by Illusie on this subject, see [Ill72].

2. Deformations of rings and the naive cotangent complex

In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a surjective ring map $A' \to A$ whose kernel is an ideal I of square zero. Moreover we assume given a ring map $A \to B$, a B-module N, and an A-module map $c : I \to N$. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

(2.0.1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & ? & \longrightarrow & B & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

and moreover how unique the solution is (if it exists). More precisely, we look for a surjection of A'-algebras $B' \to B$ whose kernel is identified with N such that $A' \to B'$ induces the given map c. We will say B' is a *solution* to (2.0.1).

Lemma 2.1. *Given a commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N_2 & \longrightarrow & B'_2 & \longrightarrow & B_2 & \longrightarrow & 0 \\
 & & \uparrow\uparrow & & \uparrow {\scriptstyle c_2} & & \uparrow\uparrow & & \uparrow \\
0 & \longrightarrow & I_2 & \longrightarrow & A'_2 & \longrightarrow & A_2 & \longrightarrow & 0 \\
 & & & & & & & & \\
0 & \longrightarrow & N_1 & \longrightarrow & B'_1 & \longrightarrow & B_1 & \longrightarrow & 0 \\
 & & \uparrow {\scriptstyle c_1} & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & I_1 & \longrightarrow & A'_1 & \longrightarrow & A_1 & \longrightarrow & 0
\end{array}
$$

with front and back solutions to (2.0.1) we have

(1) *There exist a canonical element in $\mathrm{Ext}^1_{B_1}(NL_{B_1/A_1}, N_2)$ whose vanishing is a necessary and sufficient condition for the existence of a ring map $B'_1 \to B'_2$ fitting into the diagram.*

(2) *If there exists a map $B'_1 \to B'_2$ fitting into the diagram the set of all such maps is a principal homogeneous space under $\mathrm{Hom}_{B_1}(\Omega_{B_1/A_1}, N_2)$.*

Proof. Let $E = B_1$ viewed as a set. Consider the surjection $A_1[E] \to B_1$ with kernel J used to define the naive cotangent complex by the formula

$$NL_{B_1/A_1} = (J/J^2 \to \Omega_{A_1[E]/A_1} \otimes_{A_1[E]} B_1)$$

in Algebra, Section 129. Since $\Omega_{A_1[E]/A_1} \otimes B_1$ is a free B_1-module we have

$$\mathrm{Ext}^1_{B_1}(NL_{B_1/A_1}, N_2) = \frac{\mathrm{Hom}_{B_1}(J/J^2, N_2)}{\mathrm{Hom}_{B_1}(\Omega_{A_1[E]/A_1} \otimes B_1, N_2)}$$

We will construct an obstruction in the module on the right. Let $J' = \mathrm{Ker}(A'_1[E] \to B_1)$. Note that there is a surjection $J' \to J$ whose kernel is $I_1 A_1[E]$. For every $e \in E$ denote $x_e \in A_1[E]$ the corresponding variable. Choose a lift $y_e \in B'_1$ of the image of x_e in B_1 and a lift $z_e \in B'_2$ of the image of x_e in B_2. These choices determine A'_1-algebra maps

$$A'_1[E] \to B'_1 \quad \text{and} \quad A'_1[E] \to B'_2$$

The first of these gives a map $J' \to N_1$, $f' \mapsto f'(y_e)$ and the second gives a map $J' \to N_2$, $f' \mapsto f'(z_e)$. A calculation shows that these maps annihilate $(J')^2$. Because the left square of the diagram (involving c_1 and c_2) commutes we see that these maps agree on $I_1 A_1[E]$ as maps into N_2. Observe that B'_1 is the pushout of $J' \to A'_1[B_1]$ and $J' \to N_1$. Thus, if the maps $J' \to N_1 \to N_2$ and $J' \to N_2$ agree, then we obtain a map $B'_1 \to B'_2$ fitting into the diagram. Thus we let the obstruction be the class of the map

$$J/J^2 \to N_2, \quad f \mapsto f'(z_e) - \nu(f'(y_e))$$

where $\nu : N_1 \to N_2$ is the given map and where $f' \in J'$ is a lift of f. This is well defined by our remarks above. Note that we have the freedom to modify our choices of z_e into $z_e + \delta_{2,e}$ and y_e into $y_e + \delta_{1,e}$ for some $\delta_{i,e} \in N_i$. This will modify the map above into

$$f \mapsto f'(z_e + \delta_{2,e}) - \nu(f'(y_e + \delta_{1,e})) = f'(z_e) - \nu(f'(z_e)) + \sum (\delta_{2,e} - \nu(\delta_{1,e})) \frac{\partial f}{\partial x_e}$$

This means exactly that we are modifying the map $J/J^2 \to N_2$ by the composition $J/J^2 \to \Omega_{A_1[E]/A_1} \otimes B_1 \to N_2$ where the second map sends dx_e to $\delta_{2,e} - \nu(\delta_{1,e})$. Thus our obstruction is well defined and is zero if and only if a lift exists.

Part (2) comes from the observation that given two maps $\varphi, \psi : B_1' \to B_2'$ fitting into the diagram, then $\varphi - \psi$ factors through a map $D : B_1 \to N_2$ which is an A_1-derivation:

$$\begin{aligned}
D(fg) &= \varphi(f'g') - \psi(f'g') \\
&= \varphi(f')\varphi(g') - \psi(f')\psi(g') \\
&= (\varphi(f') - \psi(f'))\varphi(g') + \psi(f')(\varphi(g') - \psi(g')) \\
&= gD(f) + fD(g)
\end{aligned}$$

Thus D corresponds to a unique B_1-linear map $\Omega_{B_1/A_1} \to N_2$. Conversely, given such a linear map we get a derivation D and given a ring map $\psi : B_1' \to B_2'$ fitting into the diagram the map $\psi + D$ is another ring map fitting into the diagram. $\square$

The naive cotangent complex isn't good enough to contain all information regarding obstructions to finding solutions to (2.0.1). However, if the ring map is a local complete intersection, then the obstruction vanishes. This is a kind of lifting result; observe that for syntomic ring maps we have proved a rather strong lifting result in Smoothing Ring Maps, Proposition 4.2.

Lemma 2.2. *If $A \to B$ is a local complete intersection ring map, then there exists a solution to (2.0.1).*

Proof. Write $B = A[x_1, \ldots, x_n]/J$. Let $J' \subset A'[x_1, \ldots, x_n]$ be the inverse image of J. Denote $I[x_1, \ldots, x_n]$ the kernel of $A'[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]$. By More on Algebra, Lemma 22.5 we have $I[x_1, \ldots, x_n] \cap (J')^2 = J'I[x_1, \ldots, x_n] = JI[x_1, \ldots, x_n]$. Hence we obtain a short exact sequence

$$0 \to I \otimes_A B \to J'/(J')^2 \to J/J^2 \to 0$$

Since J/J^2 is projective (More on Algebra, Lemma 22.3) we can choose a splitting of this sequence

$$J'/(J')^2 = I \otimes_A B \oplus J/J^2$$

Let $(J')^2 \subset J'' \subset J'$ be the elements which map to the second summand in the decomposition above. Then

$$0 \to I \otimes_A B \to A'[x_1, \ldots, x_n]/J'' \to B \to 0$$

is a solution to (2.0.1) with $N = I \otimes_A B$. The general case is obtained by doing a pushout along the given map $I \otimes_A B \to N$. $\square$

Lemma 2.3. *If there exists a solution to (2.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}_B^1(NL_{B/A}, N)$.*

Proof. We observe right away that given two solutions B_1' and B_2' to (2.0.1) we obtain by Lemma 2.1 an obstruction element $o(B_1', B_2') \in \mathrm{Ext}_B^1(NL_{B/A}, N)$ to the existence of a map $B_1' \to B_2'$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution B' and an element $\xi \in \mathrm{Ext}_B^1(NL_{B/A}, N)$ we can find a second solution B_ξ' such that $o(B', B_\xi') = \xi$.

Let $E = B$ viewed as a set. Consider the surjection $A[E] \to B$ with kernel J used to define the naive cotangent complex by the formula

$$NL_{B/A} = (J/J^2 \to \Omega_{A[E]/A} \otimes_{A[E]} B)$$

in Algebra, Section 129. Since $\Omega_{A[E]/A} \otimes B$ is a free B-module we have

$$\mathrm{Ext}^1_B(NL_{B/A}, N) = \frac{\mathrm{Hom}_B(J/J^2, N)}{\mathrm{Hom}_B(\Omega_{A[E]/A} \otimes B, N)}$$

Thus we may represent ξ as the class of a morphism $\delta : J/J^2 \to N$.

For every $e \in E$ denote $x_e \in A[E]$ the corresponding variable. Choose a lift $y_e \in B'$ of the image of x_e in B. These choices determine an A'-algebra map $\varphi : A'[E] \to B'$. Let $J' = \mathrm{Ker}(A'[E] \to B)$. Observe that φ induces a map $\varphi|_{J'} : J' \to N$ and that B' is the pushout, as in the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \varphi|_{J'}} & & \uparrow & & \uparrow{\scriptstyle =} & & \\
0 & \longrightarrow & J' & \longrightarrow & A'[E] & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

Let $\psi : J' \to N$ be the sum of the map $\varphi|_{J'}$ and the composition

$$J' \to J'/(J')^2 \to J/J^2 \xrightarrow{\delta} N.$$

Then the pushout along ψ is an other ring extension B'_ξ fitting into a diagram as above. A calculation shows that $o(B', B'_\xi) = \xi$ as desired. $\square$

Lemma 2.4. *Let A be a ring and let I be an A-module.*

(1) *The set of extensions of rings $0 \to I \to A' \to A \to 0$ where I is an ideal of square zero is canonically bijective to $\mathrm{Ext}^1_A(NL_{A/\mathbf{Z}}, I)$.*

(2) *Given a ring map $A \to B$, a B-module N, an A-module map $c : I \to N$, and given extensions of rings with square zero kernels:*
   (a) *$0 \to I \to A' \to A \to 0$ corresponding to $\alpha \in \mathrm{Ext}^1_A(NL_{A/\mathbf{Z}}, I)$, and*
   (b) *$0 \to N \to B' \to B \to 0$ corresponding to $\beta \in \mathrm{Ext}^1_B(NL_{B/\mathbf{Z}}, N)$*
   then there is a map $A' \to B'$ fitting into a diagram (2.0.1) if and only if β and α map to the same element of $\mathrm{Ext}^1_A(NL_{A/\mathbf{Z}}, N)$.

Proof. To prove this we apply the previous results where we work over $0 \to 0 \to \mathbf{Z} \to \mathbf{Z} \to 0$, in order words, we work over the extension of $\mathbf{Z}$ by 0. Part (1) follows from Lemma 2.3 and the fact that there exists a solution, namely $I \oplus A$. Part (2) follows from Lemma 2.1 and a compatibility between the constructions in the proofs of Lemmas 2.3 and 2.1 whose statement and proof we omit. $\square$

3. Thickenings of ringed spaces

In the following few sections we will use the following notions:

(1) A sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ on a ringed space $(X', \mathcal{O}_{X'})$ is *locally nilpotent* if any local section of $\mathcal{I}$ is locally nilpotent. Compare with Algebra, Item 29.

(2) A *thickening* of ringed spaces is a morphism $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of ringed spaces such that
   (a) i induces a homeomorphism $X \to X'$,
   (b) the map $i^\sharp : \mathcal{O}_{X'} \to i_* \mathcal{O}_X$ is surjective, and

> (c) the kernel of $i^\sharp$ is a locally nilpotent sheaf of ideals.
> (3) A *first order thickening* of ringed spaces is a thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of ringed spaces such that $\mathrm{Ker}(i^\sharp)$ has square zero.
> (4) It is clear how to define *morphisms of thickenings, morphisms of thickenings over a base ringed space*, etc.

If $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is a thickening of ringed spaces then we identify the underlying topological spaces and think of $\mathcal{O}_X$, $\mathcal{O}_{X'}$, and $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ as sheaves on $X = X'$. We obtain a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

of $\mathcal{O}_{X'}$-modules. By Modules, Lemma 13.4 the category of $\mathcal{O}_X$-modules is equivalent to the category of $\mathcal{O}_{X'}$-modules annihilated by $\mathcal{I}$. In particular, if i is a first order thickening, then $\mathcal{I}$ is a $\mathcal{O}_X$-module.

Situation 3.1. A morphism of thickenings (f, f') is given by a commutative diagram

(3.1.1)
$$
\begin{array}{ccc}
(X, \mathcal{O}_X) & \xrightarrow{\ i\ } & (X', \mathcal{O}_{X'}) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
(S, \mathcal{O}_S) & \xrightarrow{\ t\ } & (S', \mathcal{O}_{S'})
\end{array}
$$

of ringed spaces whose horizontal arrows are thickenings. In this situation we set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}_{X'}$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{S'}$. As $f = f'$ on underlying topological spaces we will identify the (topological) pullback functors f^{-1} and $(f')^{-1}$. Observe that $(f')^\sharp : f^{-1}\mathcal{O}_{S'} \to \mathcal{O}_{X'}$ induces in particular a map $f^{-1}\mathcal{J} \to \mathcal{I}$ and therefore a map of $\mathcal{O}_{X'}$-modules

$$(f')^*\mathcal{J} \longrightarrow \mathcal{I}$$

If i and t are first order thickenings, then $(f')^*\mathcal{J} = f^*\mathcal{J}$ and the map above becomes a map $f^*\mathcal{J} \to \mathcal{I}$.

Definition 3.2. In Situation 3.1 we say that (f, f') is a *strict morphism of thickenings* if the map $(f')^*\mathcal{J} \longrightarrow \mathcal{I}$ is surjective.

The following lemma in particular shows that a morphism $(f, f') : (X \subset X') \to (S \subset S')$ of thickenings of schemes is strict if and only if $X = S \times_{S'} X'$.

Lemma 3.3. *In Situation 3.1 the morphism (f, f') is a strict morphism of thickenings if and only if (3.1.1) is cartesian in the category of ringed spaces.*

Proof. Omitted. $\qquad\square$

4. Modules on first order thickenings of ringed spaces

In this section we discuss some preliminaries to the deformation theory of modules. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. We will freely use the notation introduced in Section 3, in particular we will identify the underlying topological spaces. In this section we consider short exact sequences

(4.0.1) $$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}_{X'}$-modules, where $\mathcal{F}$, $\mathcal{K}$ are $\mathcal{O}_X$-modules and $\mathcal{F}'$ is an $\mathcal{O}_{X'}$-module. In this situation we have a canonical $\mathcal{O}_X$-module map

$$c_{\mathcal{F}'} : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{K}$$

where $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. Namely, given local sections f of $\mathcal{I}$ and s of $\mathcal{F}$ we set $c_{\mathcal{F}'}(f \otimes s) = fs'$ where s' is a local section of $\mathcal{F}'$ lifting s.

Lemma 4.1. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad and \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (4.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$.

(1) *If there exists an $\mathcal{O}_{X'}$-module map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then the diagram*

$$
\begin{array}{ccc}
\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\ c_{\mathcal{F}'}\ } & \mathcal{K} \\
{\scriptstyle 1 \otimes \varphi} \downarrow & & \downarrow {\scriptstyle \psi} \\
\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} & \xrightarrow{\ c_{\mathcal{G}'}\ } & \mathcal{L}
\end{array}
$$

   is commutative.

(2) *The set of $\mathcal{O}_{X'}$-module maps $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$.*

Proof. Part (1) is immediate from the description of the maps. For (2), if φ' and φ'' are two maps $\mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then $\varphi' - \varphi''$ factors as

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to \mathcal{G}'$$

The map in the middle comes from a unique element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$ by Modules, Lemma 13.4. Conversely, given an element α of this group we can add the composition (as displayed above with α in the middle) to φ'. Some details omitted. $\square$

Lemma 4.2. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad and \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (4.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$. Assume the diagram

$$
\begin{array}{ccc}
\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\ c_{\mathcal{F}'}\ } & \mathcal{K} \\
{\scriptstyle 1 \otimes \varphi} \downarrow & & \downarrow {\scriptstyle \psi} \\
\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} & \xrightarrow{\ c_{\mathcal{G}'}\ } & \mathcal{L}
\end{array}
$$

is commutative. Then there exists an element

$$o(\varphi, \psi) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$$

whose vanishing is a necessary and sufficient condition for the existence of a map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ.

Proof. We can construct explicitly an extension

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{F} \to 0$$

by taking $\mathcal{H}$ to be the cohomology of the complex

$$\mathcal{K} \xrightarrow{\ 1, -\psi\ } \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{\ \varphi, 1\ } \mathcal{G}$$

in the middle (with obvious notation). A calculation with local sections using the assumption that the diagram of the lemma commutes shows that $\mathcal{H}$ is annihilated by $\mathcal{I}$. Hence $\mathcal{H}$ defines a class in

$$\text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L}) \subset \text{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}, \mathcal{L})$$

Finally, the class of $\mathcal{H}$ is the difference of the pushout of the extension $\mathcal{F}'$ via ψ and the pullback of the extension $\mathcal{G}'$ via φ (calculations omitted). Thus the vanishing of the class of $\mathcal{H}$ is equivalent to the existence of a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow{\psi} & & \downarrow{\varphi'} & & \downarrow{\varphi} & & \\
0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & 0
\end{array}
$$

as desired. $\qquad\square$

Lemma 4.3. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}_X$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$. If there exists a sequence (4.0.1) with $c_{\mathcal{F}'} = c$ then the set of isomorphism classes of these extensions is principal homogeneous under $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$.*

Proof. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}'_1 \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

with $c_{\mathcal{F}'_1} = c_{\mathcal{F}'_2} = c$. Then the difference (in the extension group, see Homology, Section 6) is an extension

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is annihilated by $\mathcal{I}$ (local computation omitted). Hence the sequence is an extension of $\mathcal{O}_X$-modules, see Modules, Lemma 13.4. Conversely, given such an extension $\mathcal{E}$ we can add the extension $\mathcal{E}$ to the $\mathcal{O}_{X'}$-extension $\mathcal{F}'$ without affecting the map $c_{\mathcal{F}'}$. Some details omitted. $\qquad\square$

Lemma 4.4. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}_X$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$. Then there exists an element*

$$o(\mathcal{F}, \mathcal{K}, c) \in \text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

whose vanishing is a necessary and sufficient condition for the existence of a sequence (4.0.1) with $c_{\mathcal{F}'} = c$.

Proof. We first show that if $\mathcal{K}$ is an injective $\mathcal{O}_X$-module, then there does exist a sequence (4.0.1) with $c_{\mathcal{F}'} = c$. To do this, choose a flat $\mathcal{O}_{X'}$-module $\mathcal{H}'$ and a surjection $\mathcal{H}' \to \mathcal{F}$ (Modules, Lemma 16.6). Let $\mathcal{J} \subset \mathcal{H}'$ be the kernel. Since $\mathcal{H}'$ is flat we have

$$\mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{H}' = \mathcal{I}\mathcal{H}' \subset \mathcal{J} \subset \mathcal{H}'$$

Observe that the map

$$\mathcal{I}\mathcal{H}' = \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{H}' \longrightarrow \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

annihilates $\mathcal{I}\mathcal{J}$. Namely, if f is a local section of $\mathcal{I}$ and s is a local section of $\mathcal{H}$, then fs is mapped to $f \otimes \overline{s}$ where $\overline{s}$ is the image of s in $\mathcal{F}$. Thus we obtain

$$\begin{array}{ccc} \mathcal{I}\mathcal{H}'/\mathcal{I}\mathcal{J} & \longhookrightarrow & \mathcal{J}/\mathcal{I}\mathcal{J} \\ \downarrow & & \downarrow{\scriptstyle \gamma} \\ \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\;\;c\;\;} & \mathcal{K} \end{array}$$

a diagram of $\mathcal{O}_X$-modules. If $\mathcal{K}$ is injective as an $\mathcal{O}_X$-module, then we obtain the dotted arrow. Denote $\gamma' : \mathcal{J} \to \mathcal{K}$ the composition of γ with $\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J}$. A local calculation shows the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{H}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma'} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

is a solution to the problem posed by the lemma.

General case. Choose an embedding $\mathcal{K} \subset \mathcal{K}'$ with $\mathcal{K}'$ an injective $\mathcal{O}_X$-module. Let $\mathcal{Q}$ be the quotient, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{K}' \to \mathcal{Q} \to 0$$

Denote $c' : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}'$ be the composition. By the paragraph above there exists a sequence

$$0 \to \mathcal{K}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

as in (4.0.1) with $c_{\mathcal{E}'} = c'$. Note that c' composed with the map $\mathcal{K}' \to \mathcal{Q}$ is zero, hence the pushout of $\mathcal{E}'$ by $\mathcal{K}' \to \mathcal{Q}$ is an extension

$$0 \to \mathcal{Q} \to \mathcal{D}' \to \mathcal{F} \to 0$$

as in (4.0.1) with $c_{\mathcal{D}'} = 0$. This means exactly that $\mathcal{D}'$ is annihilated by $\mathcal{I}$, in other words, the $\mathcal{D}'$ is an extension of $\mathcal{O}_X$-modules, i.e., defines an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) = \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

(the equality holds by the long exact cohomology sequence associated to the exact sequence above and the vanishing of higher ext groups into the injective module $\mathcal{K}'$). If $o(\mathcal{F}, \mathcal{K}, c) = 0$, then we can choose a splitting $s : \mathcal{F} \to \mathcal{D}'$ and we can set

$$\mathcal{F}' = \mathrm{Ker}(\mathcal{E}' \to \mathcal{D}'/s(\mathcal{F}))$$

so that we obtain the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

with exact rows which shows that $c_{\mathcal{F}'} = c$. Conversely, if $\mathcal{F}'$ exists, then the pushout of $\mathcal{F}'$ by the map $\mathcal{K} \to \mathcal{K}'$ is isomorphic to $\mathcal{E}'$ by Lemma 4.3 and the vanishing of higher ext groups into the injective module $\mathcal{K}'$. This gives a diagram as above, which implies that $\mathcal{D}'$ is split as an extension, i.e., the class $o(\mathcal{F}, \mathcal{K}, c)$ is zero. $\qquad\square$

Remark 4.5. Let $(X, \mathcal{O}_X)$ be a ringed space. A first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is said to be *trivial* if there exists a morphism of ringed spaces $\pi : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ which is a left inverse to i. The choice of such a morphism π is called a *trivialization* of the first order thickening. Given π we obtain a splitting

$$(4.5.1) \qquad\qquad \mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{I}$$

as sheaves of algebras on X by using $\pi^\sharp$ to split the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. Conversely, such a splitting determines a morphism π. The category of trivialized first order thickenings of $(X, \mathcal{O}_X)$ is equivalent to the category of $\mathcal{O}_X$-modules.

Remark 4.6. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a trivial first order thickening of ringed spaces and let $\pi : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a trivialization. Then given any triple $(\mathcal{F}, \mathcal{K}, c)$ consisting of a pair of $\mathcal{O}_X$-modules and a map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$ we may set

$$\mathcal{F}'_{c,triv} = \mathcal{F} \oplus \mathcal{K}$$

and use the splitting (4.5.1) associated to π and the map c to define the $\mathcal{O}_{X'}$-module structure and obtain an extension (4.0.1). We will call $\mathcal{F}'_{c,triv}$ the *trivial extension* of $\mathcal{F}$ by $\mathcal{K}$ corresponding to c and the trivialization π. Given any extension $\mathcal{F}'$ as in (4.0.1) we can use $\pi^\sharp : \mathcal{O}_X \to \mathcal{O}_{X'}$ to think of $\mathcal{F}'$ as an $\mathcal{O}_X$-module extension, hence a class $\xi_{\mathcal{F}'}$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$. Lemma 4.3 assures that $\mathcal{F}' \mapsto \xi_{\mathcal{F}'}$ induces a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of extensions} \\ \mathcal{F}' \text{ as in (4.0.1) with } c = c_{\mathcal{F}'} \end{array} \right\} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

Moreover, the trivial extension $\mathcal{F}'_{c,triv}$ maps to the zero class.

Remark 4.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(X, \mathcal{O}_X) \to (X'_i, \mathcal{O}_{X'_i})$, $i = 1, 2$ be first order thickenings with ideal sheaves $\mathcal{I}_i$. Let $h : (X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2})$ be a morphism of first order thickenings of $(X, \mathcal{O}_X)$. Picture

$$(X, \mathcal{O}_X)$$

$$(X'_1, \mathcal{O}_{X'_1}) \xrightarrow{\quad h \quad} (X'_2, \mathcal{O}_{X'_2})$$

Observe that $h^\sharp : \mathcal{O}_{X'_2} \to \mathcal{O}_{X'_1}$ in particular induces an $\mathcal{O}_X$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}_X$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_2} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. Then there is a canonical functoriality

$$\left\{ \begin{array}{c} \mathcal{F}'_2 \text{ as in (4.0.1) with} \\ c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \mathcal{F}'_1 \text{ as in (4.0.1) with} \\ c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1 \end{array} \right\}$$

Namely, thinking of all sheaves $\mathcal{O}_X$, $\mathcal{O}_{X'_i}$, $\mathcal{F}$, $\mathcal{K}_i$, etc as sheaves on X, we set given $\mathcal{F}'_2$ the sheaf $\mathcal{F}'_1$ equal to the pushout, i.e., fitting into the following diagram of

extensions

$$0 \longrightarrow \mathcal{K}_2 \longrightarrow \mathcal{F}'_2 \longrightarrow \mathcal{F} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{K}_1 \longrightarrow \mathcal{F}'_1 \longrightarrow \mathcal{F} \longrightarrow 0$$

We omit the construction of the $\mathcal{O}_{X'_1}$-module structure on the pushout (this uses the commutativity of the diagram involving c_1 and c_2).

Remark 4.8. Let $(X, \mathcal{O}_X)$, $(X, \mathcal{O}_X) \to (X'_i, \mathcal{O}_{X'_i})$, $\mathcal{I}_i$, and $h : (X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2})$ be as in Remark 4.7. Assume that we are given given trivializations $\pi_i : X'_i \to X$ such that $\pi_1 = h \circ \pi_2$. In other words, assume h is a morphism of trivialized first order thickening of $(X, \mathcal{O}_X)$. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}_X$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\ c_2\ } \mathcal{K}_2$$
$$\mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\ c_1\ } \mathcal{K}_1$$

is commutative. In this situation the construction of Remark 4.6 induces a commutative diagram

$$\{\mathcal{F}'_2 \text{ as in } (4.0.1) \text{ with } c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2\} \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_2)$$
$$\{\mathcal{F}'_1 \text{ as in } (4.0.1) \text{ with } c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1\} \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1)$$

where the vertical map on the right is given by functoriality of Ext and the map $\mathcal{K}_2 \to \mathcal{K}_1$ and the vertical map on the left is the one from Remark 4.7.

Remark 4.9. Let $(X, \mathcal{O}_X)$ be a ringed space. We define a sequence of morphisms of first order thickenings

$$(X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2}) \to (X'_3, \mathcal{O}_{X'_3})$$

of $(X, \mathcal{O}_X)$ to be a *complex* if the corresponding maps between the ideal sheaves $\mathcal{I}_i$ give a complex of $\mathcal{O}_X$-modules $\mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1$ (i.e., the composition is zero). In this case the composition $(X'_1, \mathcal{O}_{X'_1}) \to (X'_3, \mathcal{O}_{X'_3})$ factors through $(X, \mathcal{O}_X) \to (X'_3, \mathcal{O}_{X'_3})$, i.e., the first order thickening $(X'_1, \mathcal{O}_{X'_1})$ of $(X, \mathcal{O}_X)$ is trivial and comes with a canonical trivialization $\pi : (X'_1, \mathcal{O}_{X'_1}) \to (X, \mathcal{O}_X)$.

We say a sequence of morphisms of first order thickenings

$$(X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2}) \to (X'_3, \mathcal{O}_{X'_3})$$

of $(X, \mathcal{O}_X)$ is *a short exact sequence* if the corresponding maps between ideal sheaves is a short exact sequence

$$0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1 \to 0$$

of $\mathcal{O}_X$-modules.

Remark 4.10. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let

$$(X_1', \mathcal{O}_{X_1'}) \to (X_2', \mathcal{O}_{X_2'}) \to (X_3', \mathcal{O}_{X_3'})$$

be a complex first order thickenings of $(X, \mathcal{O}_X)$, see Remark 4.9. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2, 3$ be pairs consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume given a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$$

such that

$$
\begin{array}{ccc}
\mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{c_2} \mathcal{K}_2 & & \mathcal{I}_3 \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{c_3} \mathcal{K}_3 \\
\downarrow \qquad\qquad \downarrow & \text{and} & \downarrow \qquad\qquad \downarrow \\
\mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{c_1} \mathcal{K}_1 & & \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{c_2} \mathcal{K}_2
\end{array}
$$

are commutative. Finally, assume given an extension

$$0 \to \mathcal{K}_2 \to \mathcal{F}_2' \to \mathcal{F} \to 0$$

as in (4.0.1) with $\mathcal{K} = \mathcal{K}_2$ of $\mathcal{O}_{X_2'}$-modules with $c_{\mathcal{F}_2'} = c_2$. In this situation we can apply the functoriality of Remark 4.7 to obtain an extension $\mathcal{F}_1'$ on X_1' (we'll describe $\mathcal{F}_1'$ in this special case below). By Remark 4.6 using the canonical splitting $\pi : (X_1', \mathcal{O}_{X_1'}) \to (X, \mathcal{O}_X)$ of Remark 4.9 we obtain $\xi_{\mathcal{F}_1'} \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1)$. Finally, we have the obstruction

$$o(\mathcal{F}, \mathcal{K}_3, c_3) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_3)$$

see Lemma 4.4. In this situation we **claim** that the canonical map

$$\partial : \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_3)$$

coming from the short exact sequence $0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$ sends $\xi_{\mathcal{F}_1'}$ to the obstruction class $o(\mathcal{F}, \mathcal{K}_3, c_3)$.

To prove this claim choose an embedding $j : \mathcal{K}_3 \to \mathcal{K}$ where $\mathcal{K}$ is an injective $\mathcal{O}_X$-module. We can lift j to a map $j' : \mathcal{K}_2 \to \mathcal{K}$. Set $\mathcal{E}_2' = j_*' \mathcal{F}_2'$ equal to the pushout of $\mathcal{F}_2'$ by j' so that $c_{\mathcal{E}_2'} = j' \circ c_2$. Picture:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle j'} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

Set $\mathcal{E}_3' = \mathcal{E}_2'$ but viewed as an $\mathcal{O}_{X_3'}$-module via $\mathcal{O}_{X_3'} \to \mathcal{O}_{X_2'}$. Then $c_{\mathcal{E}_3'} = j \circ c_3$. The proof of Lemma 4.4 constructs $o(\mathcal{F}, \mathcal{K}_3, c_3)$ as the boundary of the class of the extension of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}/\mathcal{K}_3 \to \mathcal{E}_3'/\mathcal{K}_3 \to \mathcal{F} \to 0$$

On the other hand, note that $\mathcal{F}_1' = \mathcal{F}_2'/\mathcal{K}_3$ hence the class $\xi_{\mathcal{F}_1'}$ is the class of the extension

$$0 \to \mathcal{K}_2/\mathcal{K}_3 \to \mathcal{F}_2'/\mathcal{K}_3 \to \mathcal{F} \to 0$$

seen as a sequence of $\mathcal{O}_X$-modules using $\pi^\sharp$ where $\pi : (X_1', \mathcal{O}_{X_1'}) \to (X, \mathcal{O}_X)$ is the canonical splitting. Thus finally, the claim follows from the fact that we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F}_2'/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}/\mathcal{K}_3 & \longrightarrow & \mathcal{E}_3'/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

which is $\mathcal{O}_X$-linear (with the $\mathcal{O}_X$-module structures given above).

5. Infinitesimal deformations of modules on ringed spaces

Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. We freely use the notation introduced in Section 3. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module and set $\mathcal{F} = i^*\mathcal{F}'$. In this situation we have a short exact sequence

$$0 \to \mathcal{IF}' \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}_{X'}$-modules. Since $\mathcal{I}^2 = 0$ the $\mathcal{O}_{X'}$-module structure on $\mathcal{IF}'$ comes from a unique $\mathcal{O}_X$-module structure. Thus the sequence above is an extension as in (4.0.1). As a special case, if $\mathcal{F}' = \mathcal{O}_{X'}$ we have $i^*\mathcal{O}_{X'} = \mathcal{O}_X$ and $\mathcal{IO}_{X'} = \mathcal{I}$ and we recover the sequence of structure sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

Lemma 5.1. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules. Set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. The set of lifts of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{IG}')$.*

Proof. This is a special case of Lemma 4.1 but we also give a direct proof. We have short exact sequences of modules

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0 \quad \text{and} \quad 0 \to \mathcal{IG}' \to \mathcal{G}' \to \mathcal{G} \to 0$$

and similarly for $\mathcal{F}'$. Since $\mathcal{I}$ has square zero the $\mathcal{O}_{X'}$-module structure on $\mathcal{I}$ and $\mathcal{IG}'$ comes from a unique $\mathcal{O}_X$-module structure. It follows that

$$\mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{IG}') = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{IG}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

The lemma now follows from the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{IG}') \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G})$$

see Homology, Lemma 5.8. $\square$

Lemma 5.2. *Let (f, f') be a morphism of first order thickenings of ringed spaces as in Situation 3.1. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}$ is flat over S and that (f, f') is a strict morphism of thickenings (Definition 3.2). Then the following are equivalent*

(1) *$\mathcal{F}'$ is flat over S', and*
(2) *the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{IF}'$ is an isomorphism.*

Moreover, in this case the maps

$$f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{IF}'$$

are isomorphisms.

Proof. The map $f^*\mathcal{J} \to \mathcal{I}$ is surjective as (f, f') is a strict morphism of thickenings. Hence the final statement is a consequence of (2).

Proof of the equivalence of (1) and (2). We may check these conditions at stalks. Let $x \in X \subset X'$ be a point with image $s = f(x) \in S \subset S'$. Set $A' = \mathcal{O}_{S',s}$, $B' = \mathcal{O}_{X',x}$, $A = \mathcal{O}_{S,s}$, and $B = \mathcal{O}_{X,x}$. Then $A = A'/J$ and $B = B'/I$ for some square zero ideals. Since (f, f') is a strict morphism of thickenings we have $I = JB'$. Let $M' = \mathcal{F}'_x$ and $M = \mathcal{F}_x$. Then M' is a B'-module and M is a B-module. Since $\mathcal{F} = i^*\mathcal{F}'$ we see that the kernel of the surjection $M' \to M$ is $IM' = JM'$. Thus we have a short exact sequence

$$0 \to JM' \to M' \to M \to 0$$

Using Sheaves, Lemma 26.4 and Modules, Lemma 15.1 to identify stalks of pullbacks and tensor products we see that the stalk at x of the canonical map of the lemma is the map

$$(J \otimes_A B) \otimes_B M = J \otimes_A M = J \otimes_{A'} M' \longrightarrow JM'$$

The assumption that $\mathcal{F}$ is flat over S signifies that M is a flat A-module.

Assume (1). Flatness implies $\mathrm{Tor}_1^{A'}(M', A) = 0$ by Algebra, Lemma 72.7. This means $J \otimes_{A'} M' \to M'$ is injective by Algebra, Remark 72.8. Hence $J \otimes_A M \to JM'$ is an isomorphism.

Assume (2). Then $J \otimes_{A'} M' \to M'$ is injective. Hence $\mathrm{Tor}_1^{A'}(M', A) = 0$ by Algebra, Remark 72.8. Hence M' is flat over A' by Algebra, Lemma 95.8. $\square$

Lemma 5.3. *Let (f, f') be a morphism of first order thickenings as in Situation 3.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. Assume that $\mathcal{G}'$ is flat over S' and that (f, f') is a strict morphism of thickenings. The set of lifts of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

Proof. Combine Lemmas 5.1 and 5.2. $\square$

Lemma 5.4. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. There exists an element*

$$o(\varphi) \in \mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

Proof. It is clear from the proof of Lemma 5.1 that the vanishing of the boundary of φ via the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}')$$

is a necessary and sufficient condition for the existence of a lift. We conclude as

$$\mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

the adjointness of $i_* = Ri_*$ and Li^* on the derived category (Cohomology, Lemma 29.1). $\square$

Lemma 5.5. *Let (f, f') be a morphism of first order thickenings as in Situation 3.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^* \mathcal{F}'$ and $\mathcal{G} = i^* \mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. Assume that $\mathcal{F}'$ and $\mathcal{G}'$ are flat over S' and that (f, f') is a strict morphism of thickenings. There exists an element*

$$o(\varphi) \in Ext^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^* \mathcal{J})$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

First proof. This follows from Lemma 5.4 as we claim that under the assumptions of the lemma we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(Li^* \mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^* \mathcal{J})$$

Namely, we have $\mathcal{I}\mathcal{G}' = \mathcal{G} \otimes_{\mathcal{O}_X} f^* \mathcal{J}$ by Lemma 5.2. On the other hand, observe that

$$H^{-1}(Li^* \mathcal{F}') = \mathrm{Tor}_1^{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{O}_X)$$

(local computation omitted). Using the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

we see that this Tor_1 is computed by the kernel of the map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ which is zero by the final assertion of Lemma 5.2. Thus $\tau_{\geq -1} Li^* \mathcal{F}' = \mathcal{F}$. On the other hand, we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(Li^* \mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(\tau_{\geq -1} Li^* \mathcal{F}', \mathcal{I}\mathcal{G}')$$

by the dual of Derived Categories, Lemma 17.1. $\square$

Second proof. We can apply Lemma 4.2 as follows. Note that $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $\mathcal{L} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G}$ by Lemma 5.2, that $c_{\mathcal{F}'} = 1 \otimes 1$ and $c_{\mathcal{G}'} = 1 \otimes 1$ and taking $\psi = 1 \otimes \varphi$ the diagram of the lemma commutes. Thus $o(\varphi) = o(\varphi, 1 \otimes \varphi)$ works. $\square$

Lemma 5.6. *Let (f, f') be a morphism of first order thickenings as in Situation 3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over S. If there exists a pair $(\mathcal{F}', \alpha)$ consisting of an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' and an isomorphism $\alpha : i^* \mathcal{F}' \to \mathcal{F}$, then the set of isomorphism classes of such pairs is principal homogeneous under $Ext^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})$.*

Proof. If we assume there exists one such module, then the canonical map

$$f^* \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an isomorphism by Lemma 5.2. Apply Lemma 4.3 with $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $c = 1$. By Lemma 5.2 the corresponding extensions $\mathcal{F}'$ are all flat over S'. $\square$

Lemma 5.7. *Let (f, f') be a morphism of first order thickenings as in Situation 3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over S. There exists an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' with $i^* \mathcal{F}' \cong \mathcal{F}$, if and only if*

(1) *the canonical map $f^* \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ is an isomorphism, and*
(2) *the class $o(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}, 1) \in Ext^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})$ of Lemma 4.4 is zero.*

Proof. This follows immediately from the characterization of $\mathcal{O}_{X'}$-modules flat over S' of Lemma 5.2 and Lemma 4.4. $\square$

6. Application to flat modules on flat thickenings of ringed spaces

Consider a commutative diagram

$$
\begin{array}{ccc}
(X, \mathcal{O}_X) & \xrightarrow{\ i\ } & (X', \mathcal{O}_{X'}) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
(S, \mathcal{O}_S) & \xrightarrow{\ t\ } & (S', \mathcal{O}_{S'})
\end{array}
$$

of ringed spaces whose horizontal arrows are first order thickenings as in Situation 3.1. Set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}_{X'}$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{S'}$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that

(1) (f, f') is a strict morphism of thickenings,
(2) f' is flat, and
(3) $\mathcal{F}$ is flat over S.

Note that (1) + (2) imply that $\mathcal{I} = f^* \mathcal{J}$ (apply Lemma 5.2 to $\mathcal{O}_{X'}$). The theory of the preceding section is especially nice under these assumptions. We summarize the results already obtained in the following lemma.

Lemma 6.1. *In the situation above.*

(1) *There exists an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' with $i^* \mathcal{F}' \cong \mathcal{F}$, if and only if the class $o(\mathcal{F}, f^* \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F}, 1) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, f^* \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$ of Lemma 4.4 is zero.*
(2) *If such a module exists, then the set of isomorphism classes of lifts is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, f^* \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$.*
(3) *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $\mathrm{id}_\mathcal{F}$ is canonically isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X}(\mathcal{F}, f^* \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$.*

Proof. Part (1) follows from Lemma 5.7 as we have seen above that $\mathcal{I} = f^* \mathcal{J}$. Part (2) follows from Lemma 5.6. Part (3) follows from Lemma 5.3. $\qquad\square$

Situation 6.2. Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Consider a commutative diagram

$$
\begin{array}{ccc}
(X_1', \mathcal{O}_1') \xrightarrow{\ h\ } (X_2', \mathcal{O}_2') \longrightarrow (X_3', \mathcal{O}_3') \\
{\scriptstyle f_1'}\downarrow \qquad\qquad {\scriptstyle f_2'}\downarrow \qquad\qquad {\scriptstyle f_3'}\downarrow \\
(S_1', \mathcal{O}_{S_1'}) \longrightarrow (S_2', \mathcal{O}_{S_2'}) \longrightarrow (S_3', \mathcal{O}_{S_3'})
\end{array}
$$

where (a) the top row is a short exact sequence of first order thickenings of X, (b) the lower row is a short exact sequence of first order thickenings of S, (c) each f_i' restricts to f, (d) each pair (f, f_i') is a strict morphism of thickenings, and (e) each f_i' is flat. Finally, let $\mathcal{F}_2'$ be an $\mathcal{O}_2'$-module flat over S_2' and set $\mathcal{F} = \mathcal{F}_2'|_X$. Let $\pi : X_1' \to X$ be the canonical splitting (Remark 4.9).

Lemma 6.3. *In Situation 6.2 the modules $\pi^* \mathcal{F}$ and $h^* \mathcal{F}_2'$ are $\mathcal{O}_1'$-modules flat over S_1' restricting to $\mathcal{F}$ on X. Their difference (Lemma 6.1) is an element θ of $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, f^* \mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F})$ whose boundary in $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, f^* \mathcal{J}_3 \otimes_{\mathcal{O}_X} \mathcal{F})$ equals the obstruction (Lemma 6.1) to lifting $\mathcal{F}$ to an $\mathcal{O}_3'$-module flat over S_3'.*

Proof. Note that both $\pi^*\mathcal{F}$ and $h^*\mathcal{F}_2'$ restrict to $\mathcal{F}$ on X and that the kernels of $\pi^*\mathcal{F} \to \mathcal{F}$ and $h^*\mathcal{F}_2' \to \mathcal{F}$ are given by $f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F}$. Hence flatness by Lemma 5.2. Taking the boundary makes sense as the sequence of modules

$$0 \to f^*\mathcal{J}_3 \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{J}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \to 0$$

is short exact due to the assumptions in Situation 6.2 and the fact that $\mathcal{F}$ is flat over S. The statement on the obstruction class is a direct translation of the result of Remark 4.10 to this particular situation. $\qquad\square$

7. Deformations of ringed spaces and the naive cotangent complex

In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a first order thickening $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$ of ringed spaces. We denote $\mathcal{J} = \mathrm{Ker}(t^\sharp)$ and we identify the underlying topological spaces of S and S'. Moreover we assume given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, and an f-map $c : \mathcal{J} \to \mathcal{G}$ of sheaves of modules (Sheaves, Definition 21.7 and Section 26). In this section we ask ourselves whether we can find the question mark fitting into the following diagram

(7.0.1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \big\uparrow{\scriptstyle c} & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0
\end{array}
$$

(where the vertical arrows are f-maps) and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ and a morphism of thickenings (f, f') as in (3.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say X' is a *solution* to (7.0.1).

Lemma 7.1. *Assume given a commutative diagram of morphisms ringed spaces*

(7.1.1)

whose horizontal arrows are first order thickenings. Set $\mathcal{G}_j = Ker(i_j^\sharp)$ and assume given a g-map $\nu : \mathcal{G}_1 \to \mathcal{G}_2$ of modules giving rise to the commutative diagram

(7.1.2)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{O}_{X_2'} & \longrightarrow & \mathcal{O}_{X_2} & \longrightarrow & 0 \\
 & & \uparrow c_2 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{J}_2 & \longrightarrow & \mathcal{O}_{S_2'} & \longrightarrow & \mathcal{O}_{S_2} & \longrightarrow & 0 \\
 & & & & & & & & \\
0 & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{O}_{X_1'} & \longrightarrow & \mathcal{O}_{X_1} & \longrightarrow & 0 \\
 & & \uparrow c_1 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{J}_1 & \longrightarrow & \mathcal{O}_{S_1'} & \longrightarrow & \mathcal{O}_{S_1} & \longrightarrow & 0
\end{array}
$$

with front and back solutions to (7.0.1).

(1) *There exist a canonical element in $Ext^1_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2)$ whose vanishing is a necessary and sufficient condition for the existence of a morphism of ringed spaces $X_2' \to X_1'$ fitting into (7.1.1) compatibly with ν.*

(2) *If there exists a morphism $X_2' \to X_1'$ fitting into (7.1.1) compatibly with ν the set of all such morphisms is a principal homogeneous space under*

$$\text{Hom}_{\mathcal{O}_{X_1}}(\Omega_{X_1/S_1}, g_* \mathcal{G}_2) = \text{Hom}_{\mathcal{O}_{X_2}}(g^* \Omega_{X_1/S_1}, \mathcal{G}_2) = Ext^0_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2).$$

Proof. The naive cotangent complex NL_{X_1/S_1} is defined in Modules, Definition 24.4. The equalities in the last statement of the lemma follow from the fact that g^* is adjoint to g_*, the fact that $H^0(NL_{X_1/S_1}) = \Omega_{X_1/S_1}$ (by construction of the naive cotangent complex) and the fact that Lg^* is the left derived functor of g^*. Thus we will work with the groups $Ext^k_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2)$, $k = 0, 1$ in the rest of the proof. We first argue that we can reduce to the case where the underlying topological spaces of all ringed spaces in the lemma is the same.

To do this, observe that $g^{-1} NL_{X_1/S_1}$ is equal to the naive cotangent complex of the homomorphism of sheaves of rings $g^{-1}f_1^{-1}\mathcal{O}_{S_1} \to g^{-1}\mathcal{O}_{X_1}$, see Modules, Lemma 24.3. Moreover, the degree 0 term of NL_{X_1/S_1} is a flat $\mathcal{O}_{X_1}$-module, hence the canonical map

$$Lg^* NL_{X_1/S_1} \longrightarrow g^{-1} NL_{X_1/S_1} \otimes_{g^{-1}\mathcal{O}_{X_1}} \mathcal{O}_{X_2}$$

induces an isomorphism on cohomology sheaves in degrees 0 and -1. Thus we may replace the Ext groups of the lemma with

$$Ext^k_{g^{-1}\mathcal{O}_{X_1}}(g^{-1} NL_{X_1/S_1}, \mathcal{G}_2) = Ext^k_{g^{-1}\mathcal{O}_{X_1}}(NL_{g^{-1}\mathcal{O}_{X_1}/g^{-1}f_1^{-1}\mathcal{O}_{S_1}}, \mathcal{G}_2)$$

The set of morphism of ringed spaces $X_2' \to X_1'$ fitting into (7.1.1) compatibly with ν is in one-to-one bijection with the set of homomorphisms of $g^{-1}f_1^{-1}\mathcal{O}_{S_1'}$-algebras $g^{-1}\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ which are compatible with $f^\sharp$ and ν. In this way we see that we may assume we have a diagram (7.1.2) of sheaves on X and we are looking to find a homomorphism of sheaves of rings $\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into it.

In the rest of the proof of the lemma we assume all underlying topological spaces are the same, i.e., we have a diagram (7.1.2) of sheaves on a space X and we are looking for homomorphisms of sheaves of rings $\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into it. As ext groups we will use $Ext^k_{\mathcal{O}_{X_1}}(NL_{\mathcal{O}_{X_1}/\mathcal{O}_{S_1}}, \mathcal{G}_2)$, $k = 0, 1$.

Step 1. Construction of the obstruction class. Consider the sheaf of sets

$$\mathcal{E} = \mathcal{O}_{X_1'} \times_{\mathcal{O}_{X_2}} \mathcal{O}_{X_2'}$$

This comes with a surjective map $\alpha : \mathcal{E} \to \mathcal{O}_{X_1}$ and hence we can use $NL(\alpha)$ instead of $NL_{\mathcal{O}_{X_1}/\mathcal{O}_{S_1}}$, see Modules, Lemma 24.2. Set

$$\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_1}) \quad \text{and} \quad \mathcal{I} = \mathrm{Ker}(\mathcal{O}_{S_1}[\mathcal{E}] \to \mathcal{O}_{X_1})$$

There is a surjection $\mathcal{I}' \to \mathcal{I}$ whose kernel is $\mathcal{J}_1 \mathcal{O}_{S_1'}[\mathcal{E}]$. We obtain two homomorphisms of $\mathcal{O}_{S_2'}$-algebras

$$a : \mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_1'} \quad \text{and} \quad b : \mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_2'}$$

which induce maps $a|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_1$ and $b|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_2$. Both a and b annihilate $(\mathcal{I}')^2$. Moreover a and b agree on $\mathcal{J}_1 \mathcal{O}_{S_1'}[\mathcal{E}]$ as maps into $\mathcal{G}_2$ because the left hand square of (7.1.2) is commutative. Thus the difference $b|_{\mathcal{I}'} - \nu \circ a|_{\mathcal{I}'}$ induces a well defined $\mathcal{O}_{X_1}$-linear map

$$\xi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{G}_2$$

which sends the class of a local section f of $\mathcal{I}$ to $a(f') - \nu(b(f'))$ where f' is a lift of f to a local section of $\mathcal{I}'$. We let $[\xi] \in \mathrm{Ext}^1_{\mathcal{O}_{X_1}}(NL(\alpha), \mathcal{G}_2)$ be the image (see below).

Step 2. Vanishing of $[\xi]$ is necessary. Let us write $\Omega = \Omega_{\mathcal{O}_{S_1}[\mathcal{E}]/\mathcal{O}_{S_1}} \otimes_{\mathcal{O}_{S_1}[\mathcal{E}]} \mathcal{O}_{X_1}$. Observe that $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega)$ fits into a distinguished triangle

$$\Omega[0] \to NL(\alpha) \to \mathcal{I}/\mathcal{I}^2[1] \to \Omega[1]$$

Thus we see that $[\xi]$ is zero if and only if ξ is a composition $\mathcal{I}/\mathcal{I}^2 \to \Omega \to \mathcal{G}_2$ for some map $\Omega \to \mathcal{G}_2$. Suppose there exists a homomorphisms of sheaves of rings $\varphi : \mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into (7.1.2). In this case consider the map $\mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{G}_2$, $f' \mapsto b(f') - \varphi(a(f'))$. A calculation shows this annihilates $\mathcal{J}_1 \mathcal{O}_{S_1'}[\mathcal{E}]$ and induces a derivation $\mathcal{O}_{S_1}[\mathcal{E}] \to \mathcal{G}_2$. The resulting linear map $\Omega \to \mathcal{G}_2$ witnesses the fact that $[\xi] = 0$ in this case.

Step 3. Vanishing of $[\xi]$ is sufficient. Let $\theta : \Omega \to \mathcal{G}_2$ be a $\mathcal{O}_{X_1}$-linear map such that ξ is equal to $\theta \circ (\mathcal{I}/\mathcal{I}^2 \to \Omega)$. Then a calculation shows that

$$b + \theta \circ d : \mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_2'}$$

annihilates $\mathcal{I}'$ and hence defines a map $\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into (7.1.2).

Proof of (2) in the special case above. Omitted. Hint: This is exactly the same as the proof of (2) of Lemma 2.1. $\qquad\square$

Lemma 7.2. *Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{G}$ be a $\mathcal{B}$-module. Let $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL_{\mathcal{B}/\mathcal{A}}, \mathcal{G})$. There exists a map of sheaves of sets $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL(\alpha), \mathcal{G})$ is the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ (see proof for notation).*

Proof. Recall that given $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$ is surjective with kernel $\mathcal{I}$ the complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$ is canonically isomorphic to $NL_{\mathcal{B}/\mathcal{A}}$, see Modules, Lemma 24.2. Observe moreover, that $\Omega = \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{e \in \mathcal{E}(U)} \mathcal{B}(U)$. In other words, Ω is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}$ and in particular there is a canonical map $\mathcal{E} \to \Omega$.

Having said this, pick some $\mathcal{E}$ (for example $\mathcal{E} = \mathcal{B}$ as in the definition of the naive cotangent complex). The obstruction to writing ξ as the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ is an element in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega, \mathcal{G})$. Say this is represented by the extension $0 \to \mathcal{G} \to \mathcal{H} \to \Omega \to 0$ of $\mathcal{B}$-modules. Consider the sheaf of sets $\mathcal{E}' = \mathcal{E} \times_\Omega \mathcal{H}$ which comes with an induced map $\alpha' : \mathcal{E}' \to \mathcal{B}$. Let $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}[\mathcal{E}'] \to \mathcal{B})$ and $\Omega' = \Omega_{\mathcal{A}[\mathcal{E}']/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}']} \mathcal{B}$. The pullback of ξ under the quasi-isomorphism $NL(\alpha') \to NL(\alpha)$ maps to zero in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega', \mathcal{G})$ because the pullback of the extension $\mathcal{H}$ by the map $\Omega' \to \Omega$ is split as Ω' is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}'$ and since by construction there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \Omega \end{array}$$

This finishes the proof. $\square$

Lemma 7.3. *If there exists a solution to (7.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$.*

Proof. We observe right away that given two solutions X'_1 and X'_2 to (7.0.1) we obtain by Lemma 7.1 an obstruction element $o(X'_1, X'_2) \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ to the existence of a map $X'_1 \to X'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution X' and an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ we can find a second solution X'_ξ such that $o(X', X'_\xi) = \xi$.

Pick $\alpha : \mathcal{E} \to \mathcal{O}_X$ as in Lemma 7.2 for the class ξ. Consider the surjection $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_X$ with kernel $\mathcal{I}$ and corresponding naive cotangent complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{f^{-1}\mathcal{O}_S[\mathcal{E}]/f^{-1}\mathcal{O}_S} \otimes_{f^{-1}\mathcal{O}_S[\mathcal{E}]} \mathcal{O}_X)$. By the lemma ξ is the class of a morphism $\delta : \mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$. After replacing $\mathcal{E}$ by $\mathcal{E} \times_{\mathcal{O}_X} \mathcal{O}_{X'}$ we may also assume that α factors through a map $\alpha' : \mathcal{E} \to \mathcal{O}_{X'}$.

These choices determine an $f^{-1}\mathcal{O}_{S'}$-algebra map $\varphi : \mathcal{O}_{S'}[\mathcal{E}] \to \mathcal{O}_{X'}$. Let $\mathcal{I}' = \mathrm{Ker}(\varphi)$. Observe that φ induces a map $\varphi|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}$ and that $\mathcal{O}_{X'}$ is the pushout, as in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \varphi|_{\mathcal{I}'}} & & \uparrow & & \| & & \\ 0 & \longrightarrow & \mathcal{I}' & \longrightarrow & f^{-1}\mathcal{O}_{S'}[\mathcal{E}] & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \end{array}$$

Let $\psi : \mathcal{I}' \to \mathcal{G}$ be the sum of the map $\varphi|_{\mathcal{I}'}$ and the composition

$$\mathcal{I}' \to \mathcal{I}'/(\mathcal{I}')^2 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \mathcal{G}.$$

Then the pushout along ψ is an other ring extension $\mathcal{O}_{X'_\xi}$ fitting into a diagram as above. A calculation (omitted) shows that $o(X', X'_\xi) = \xi$ as desired. $\square$

Lemma 7.4. *Let $(S, \mathcal{O}_S)$ be a ringed space and let $\mathcal{J}$ be an $\mathcal{O}_S$-module.*

(1) *The set of extensions of sheaves of rings $0 \to \mathcal{J} \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$ where $\mathcal{J}$ is an ideal of square zero is canonically bijective to $\mathrm{Ext}^1_{\mathcal{O}_S}(NL_{S/\mathbf{Z}}, \mathcal{J})$.*

(2) *Given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module*
    $\mathcal{G}$, an f-map $c : \mathcal{J} \to \mathcal{G}$, and given extensions of sheaves of rings with
    square zero kernels:
    (a) $0 \to \mathcal{J} \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$ *corresponding to* $\alpha \in Ext^1_{\mathcal{O}_S}(NL_{S/\mathbf{Z}}, \mathcal{J})$,
    (b) $0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ *corresponding to* $\beta \in Ext^1_{\mathcal{O}_X}(NL_{X/\mathbf{Z}}, \mathcal{G})$
    then there is a morphism $X' \to S'$ fitting into a diagram (7.0.1) if and only
    if β and α map to the same element of $Ext^1_{\mathcal{O}_X}(Lf^* NL_{S/\mathbf{Z}}, \mathcal{G})$.

Proof. To prove this we apply the previous results where we work over the base
ringed space $(*, \mathbf{Z})$ with trivial thickening. Part (1) follows from Lemma 7.3 and
the fact that there exists a solution, namely $\mathcal{J} \oplus \mathcal{O}_S$. Part (2) follows from Lemma
7.1 and a compatibility between the constructions in the proofs of Lemmas 7.3 and
7.1 whose statement and proof we omit. $\square$

8. Thickenings of ringed topoi

This section is the analogue of Section 3 for ringed topoi. In the following few
sections we will use the following notions:

(1) A sheaf of ideals $\mathcal{I} \subset \mathcal{O}'$ on a ringed topos $(Sh(\mathcal{D}), \mathcal{O}')$ is *locally nilpotent*
    if any local section of $\mathcal{I}$ is locally nilpotent.
(2) A *thickening* of ringed topoi is a morphism $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ of
    ringed topoi such that
    (a) i_* is an equivalence $Sh(\mathcal{C}) \to Sh(\mathcal{D})$,
    (b) the map $i^\sharp : \mathcal{O}' \to i_*\mathcal{O}$ is surjective, and
    (c) the kernel of $i^\sharp$ is a locally nilpotent sheaf of ideals.
(3) A *first order thickening* of ringed topoi is a thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to$
    $(Sh(\mathcal{D}), \mathcal{O}')$ of ringed topoi such that $\mathrm{Ker}(i^\sharp)$ has square zero.
(4) It is clear how to define *morphisms of thickenings of ringed topoi, morphisms*
    of thickenings of ringed topoi over a base ringed topos, etc.

If $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is a thickening of ringed topoi then we identify the
underlying topoi and think of $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ as sheaves on $\mathcal{C}$. We obtain
a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

of $\mathcal{O}'$-modules. By Modules on Sites, Lemma 25.1 the category of $\mathcal{O}$-modules is
equivalent to the category of $\mathcal{O}'$-modules annihilated by $\mathcal{I}$. In particular, if i is a
first order thickening, then $\mathcal{I}$ is a $\mathcal{O}$-module.

Situation 8.1. A morphism of thickenings of ringed topoi (f, f') is given by a
commutative diagram

(8.1.1)
$$
\begin{array}{ccc}
(Sh(\mathcal{C}), \mathcal{O}) & \xrightarrow{\ i\ } & (Sh(\mathcal{D}), \mathcal{O}') \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) & \xrightarrow{\ t\ } & (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})
\end{array}
$$

of ringed topoi whose horizontal arrows are thickenings. In this situation we set
$\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}'$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{\mathcal{B}'}$. As $f = f'$ on underlying topoi we will
identify the pullback functors f^{-1} and $(f')^{-1}$. Observe that $(f')^\sharp : f^{-1}\mathcal{O}_{\mathcal{B}'} \to \mathcal{O}'$
induces in particular a map $f^{-1}\mathcal{J} \to \mathcal{I}$ and therefore a map of $\mathcal{O}'$-modules

$$(f')^*\mathcal{J} \longrightarrow \mathcal{I}$$

If i and t are first order thickenings, then $(f')^*\mathcal{J} = f^*\mathcal{J}$ and the map above becomes a map $f^*\mathcal{J} \to \mathcal{I}$.

Definition 8.2. In Situation 8.1 we say that (f, f') is a *strict morphism of thickenings* if the map $(f')^*\mathcal{J} \longrightarrow \mathcal{I}$ is surjective.

9. Modules on first order thickenings of ringed topoi

In this section we discuss some preliminaries to the deformation theory of modules. Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. We will freely use the notation introduced in Section 8, in particular we will identify the underlying topological topoi. In this section we consider short exact sequences

$$(9.0.1) \qquad\qquad 0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}'$-modules, where $\mathcal{F}$, $\mathcal{K}$ are $\mathcal{O}$-modules and $\mathcal{F}'$ is an $\mathcal{O}'$-module. In this situation we have a canonical $\mathcal{O}$-module map

$$c_{\mathcal{F}'} : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathcal{K}$$

where $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. Namely, given local sections f of $\mathcal{I}$ and s of $\mathcal{F}$ we set $c_{\mathcal{F}'}(f \otimes s) = fs'$ where s' is a local section of $\mathcal{F}'$ lifting s.

Lemma 9.1. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad and \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (9.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$.

(1) *If there exists an $\mathcal{O}'$-module map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then the diagram*

$$
\begin{array}{ccc}
\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{\ c_{\mathcal{F}'}\ } & \mathcal{K} \\
{\scriptstyle 1 \otimes \varphi} \downarrow & & \downarrow {\scriptstyle \psi} \\
\mathcal{I} \otimes_{\mathcal{O}} \mathcal{G} & \xrightarrow{\ c_{\mathcal{G}'}\ } & \mathcal{L}
\end{array}
$$

   is commutative.

(2) *The set of $\mathcal{O}'$-module maps $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$.*

Proof. Part (1) is immediate from the description of the maps. For (2), if φ' and φ'' are two maps $\mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then $\varphi' - \varphi''$ factors as

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to \mathcal{G}'$$

The map in the middle comes from a unique element of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$ by Modules on Sites, Lemma 25.1. Conversely, given an element α of this group we can add the composition (as displayed above with α in the middle) to φ'. Some details omitted. $\square$

Lemma 9.2. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad and \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (9.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$. Assume the diagram

$$
\begin{array}{ccc}
\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_{\mathcal{F}'}} & \mathcal{K} \\
{\scriptstyle 1 \otimes \varphi} \downarrow & & \downarrow {\scriptstyle \psi} \\
\mathcal{I} \otimes_{\mathcal{O}} \mathcal{G} & \xrightarrow{c_{\mathcal{G}'}} & \mathcal{L}
\end{array}
$$

is commutative. Then there exists an element

$$o(\varphi, \psi) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$$

whose vanishing is a necessary and sufficient condition for the existence of a map
$\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ.

Proof. We can construct explicitly an extension

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{F} \to 0$$

by taking $\mathcal{H}$ to be the cohomology of the complex

$$\mathcal{K} \xrightarrow{1, -\psi} \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{\varphi, 1} \mathcal{G}$$

in the middle (with obvious notation). A calculation with local sections using the assumption that the diagram of the lemma commutes shows that $\mathcal{H}$ is annihilated by $\mathcal{I}$. Hence $\mathcal{H}$ defines a class in

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{L}) \subset \mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}, \mathcal{L})$$

Finally, the class of $\mathcal{H}$ is the difference of the pushout of the extension $\mathcal{F}'$ via ψ and the pullback of the extension $\mathcal{G}'$ via φ (calculations omitted). Thus the vanishing of the class of $\mathcal{H}$ is equivalent to the existence of a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow {\scriptstyle \psi} & & \downarrow {\scriptstyle \varphi'} & & \downarrow {\scriptstyle \varphi} & & \\
0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & 0
\end{array}
$$

as desired. $\square$

Lemma 9.3. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$. If there exists a sequence (9.0.1) with $c_{\mathcal{F}'} = c$ then the set of isomorphism classes of these extensions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$.*

Proof. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}'_1 \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

with $c_{\mathcal{F}'_1} = c_{\mathcal{F}'_2} = c$. Then the difference (in the extension group, see Homology, Section 6) is an extension

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is annihilated by $\mathcal{I}$ (local computation omitted). Hence the sequence is an extension of $\mathcal{O}$-modules, see Modules on Sites, Lemma 25.1. Conversely, given such an extension $\mathcal{E}$ we can add the extension $\mathcal{E}$ to the $\mathcal{O}'$-extension $\mathcal{F}'$ without affecting the map $c_{\mathcal{F}'}$. Some details omitted. $\square$

Lemma 9.4. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$. Then there exists an element*

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

whose vanishing is a necessary and sufficient condition for the existence of a sequence (9.0.1) with $c_{\mathcal{F}'} = c$.

Proof. We first show that if $\mathcal{K}$ is an injective $\mathcal{O}$-module, then there does exist a sequence (9.0.1) with $c_{\mathcal{F}'} = c$. To do this, choose a flat $\mathcal{O}'$-module $\mathcal{H}'$ and a surjection $\mathcal{H}' \to \mathcal{F}$ (Modules on Sites, Lemma 28.6). Let $\mathcal{J} \subset \mathcal{H}'$ be the kernel. Since $\mathcal{H}'$ is flat we have

$$\mathcal{I} \otimes_{\mathcal{O}'} \mathcal{H}' = \mathcal{I}\mathcal{H}' \subset \mathcal{J} \subset \mathcal{H}'$$

Observe that the map

$$\mathcal{I}\mathcal{H}' = \mathcal{I} \otimes_{\mathcal{O}'} \mathcal{H}' \longrightarrow \mathcal{I} \otimes_{\mathcal{O}'} \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$$

annihilates $\mathcal{I}\mathcal{J}$. Namely, if f is a local section of $\mathcal{I}$ and s is a local section of $\mathcal{H}$, then fs is mapped to $f \otimes \bar{s}$ where $\bar{s}$ is the image of s in $\mathcal{F}$. Thus we obtain

a diagram of $\mathcal{O}$-modules. If $\mathcal{K}$ is injective as an $\mathcal{O}$-module, then we obtain the dotted arrow. Denote $\gamma' : \mathcal{J} \to \mathcal{K}$ the composition of γ with $\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J}$. A local calculation shows the pushout

is a solution to the problem posed by the lemma.

General case. Choose an embedding $\mathcal{K} \subset \mathcal{K}'$ with $\mathcal{K}'$ an injective $\mathcal{O}$-module. Let $\mathcal{Q}$ be the quotient, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{K}' \to \mathcal{Q} \to 0$$

Denote $c' : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}'$ be the composition. By the paragraph above there exists a sequence

$$0 \to \mathcal{K}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

as in (9.0.1) with $c_{\mathcal{E}'} = c'$. Note that c' composed with the map $\mathcal{K}' \to \mathcal{Q}$ is zero, hence the pushout of $\mathcal{E}'$ by $\mathcal{K}' \to \mathcal{Q}$ is an extension

$$0 \to \mathcal{Q} \to \mathcal{D}' \to \mathcal{F} \to 0$$

as in (9.0.1) with $c_{\mathcal{D}'} = 0$. This means exactly that $\mathcal{D}'$ is annihilated by $\mathcal{I}$, in other words, the $\mathcal{D}'$ is an extension of $\mathcal{O}$-modules, i.e., defines an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{Q}) = \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

(the equality holds by the long exact cohomology sequence associated to the exact sequence above and the vanishing of higher ext groups into the injective module $\mathcal{K}'$). If $o(\mathcal{F}, \mathcal{K}, c) = 0$, then we can choose a splitting $s : \mathcal{F} \to \mathcal{D}'$ and we can set

$$\mathcal{F}' = \mathrm{Ker}(\mathcal{E}' \to \mathcal{D}'/s(\mathcal{F}))$$

so that we obtain the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

with exact rows which shows that $c_{\mathcal{F}'} = c$. Conversely, if $\mathcal{F}'$ exists, then the pushout of $\mathcal{F}'$ by the map $\mathcal{K} \to \mathcal{K}'$ is isomorphic to $\mathcal{E}'$ by Lemma 9.3 and the vanishing of higher ext groups into the injective module $\mathcal{K}'$. This gives a diagram as above, which implies that $\mathcal{D}'$ is split as an extension, i.e., the class $o(\mathcal{F}, \mathcal{K}, c)$ is zero. $\square$

Remark 9.5. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. A first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is said to be *trivial* if there exists a morphism of ringed topoi $\pi : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ which is a left inverse to i. The choice of such a morphism π is called a *trivialization* of the first order thickening. Given π we obtain a splitting

$$(9.5.1) \qquad\qquad\qquad\qquad \mathcal{O}' = \mathcal{O} \oplus \mathcal{I}$$

as sheaves of algebras on $\mathcal{C}$ by using $\pi^{\sharp}$ to split the surjection $\mathcal{O}' \to \mathcal{O}$. Conversely, such a splitting determines a morphism π. The category of trivialized first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$ is equivalent to the category of $\mathcal{O}$-modules.

Remark 9.6. Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a trivial first order thickening of ringed topoi and let $\pi : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a trivialization. Then given any triple $(\mathcal{F}, \mathcal{K}, c)$ consisting of a pair of $\mathcal{O}$-modules and a map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$ we may set

$$\mathcal{F}'_{c,triv} = \mathcal{F} \oplus \mathcal{K}$$

and use the splitting (9.5.1) associated to π and the map c to define the $\mathcal{O}'$-module structure and obtain an extension (9.0.1). We will call $\mathcal{F}'_{c,triv}$ the *trivial extension* of $\mathcal{F}$ by $\mathcal{K}$ corresponding to c and the trivialization π. Given any extension $\mathcal{F}'$ as in (9.0.1) we can use $\pi^{\sharp} : \mathcal{O} \to \mathcal{O}'$ to think of $\mathcal{F}'$ as an $\mathcal{O}$-module extension, hence a class $\xi_{\mathcal{F}'}$ in $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$. Lemma 9.3 assures that $\mathcal{F}' \mapsto \xi_{\mathcal{F}'}$ induces a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of extensions} \\ \mathcal{F}' \text{ as in (9.0.1) with } c = c_{\mathcal{F}'} \end{array} \right\} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

Moreover, the trivial extension $\mathcal{F}'_{c,triv}$ maps to the zero class.

Remark 9.7. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}'_i)$, $i = 1, 2$ be first order thickenings with ideal sheaves $\mathcal{I}_i$. Let $h : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2)$ be a morphism of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$. Picture

$$
\begin{array}{ccc}
& (Sh(\mathcal{C}), \mathcal{O}) & \\
\swarrow & & \searrow \\
(Sh(\mathcal{D}_1), \mathcal{O}'_1) \xrightarrow{\quad h \quad} & & (Sh(\mathcal{D}_2), \mathcal{O}'_2)
\end{array}
$$

Observe that $h^\sharp : \mathcal{O}'_2 \to \mathcal{O}'_1$ in particular induces an $\mathcal{O}$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_\mathcal{O} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$
\begin{array}{ccc}
\mathcal{I}_2 \otimes_\mathcal{O} \mathcal{F} & \xrightarrow{\ c_2\ } & \mathcal{K}_2 \\
\downarrow & & \downarrow \\
\mathcal{I}_1 \otimes_\mathcal{O} \mathcal{F} & \xrightarrow{\ c_1\ } & \mathcal{K}_1
\end{array}
$$

is commutative. Then there is a canonical functoriality

$$
\left\{ \begin{array}{l} \mathcal{F}'_2 \text{ as in (9.0.1) with} \\ c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \mathcal{F}'_1 \text{ as in (9.0.1) with} \\ c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1 \end{array} \right\}
$$

Namely, thinking of all sheaves $\mathcal{O}$, $\mathcal{O}'_i$, $\mathcal{F}$, $\mathcal{K}_i$, etc as sheaves on $\mathcal{C}$, we set given $\mathcal{F}'_2$ the sheaf $\mathcal{F}'_1$ equal to the pushout, i.e., fitting into the following diagram of extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \mathcal{F}'_1 & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

We omit the construction of the $\mathcal{O}'_1$-module structure on the pushout (this uses the commutativity of the diagram involving c_1 and c_2).

Remark 9.8. Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}'_i)$, $\mathcal{I}_i$, and $h : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2)$ be as in Remark 9.7. Assume that we are given given trivializations $\pi_i : (Sh(\mathcal{D}_i), \mathcal{O}'_i) \to (Sh(\mathcal{C}), \mathcal{O})$ such that $\pi_1 = h \circ \pi_2$. In other words, assume h is a morphism of trivialized first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_\mathcal{O} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$
\begin{array}{ccc}
\mathcal{I}_2 \otimes_\mathcal{O} \mathcal{F} & \xrightarrow{\ c_2\ } & \mathcal{K}_2 \\
\downarrow & & \downarrow \\
\mathcal{I}_1 \otimes_\mathcal{O} \mathcal{F} & \xrightarrow{\ c_1\ } & \mathcal{K}_1
\end{array}
$$

is commutative. In this situation the construction of Remark 9.6 induces a commutative diagram

$$
\begin{array}{ccc}
\{\mathcal{F}'_2 \text{ as in (9.0.1) with } c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2\} & \longrightarrow & \mathrm{Ext}^1_\mathcal{O}(\mathcal{F}, \mathcal{K}_2) \\
\downarrow & & \downarrow \\
\{\mathcal{F}'_1 \text{ as in (9.0.1) with } c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1\} & \longrightarrow & \mathrm{Ext}^1_\mathcal{O}(\mathcal{F}, \mathcal{K}_1)
\end{array}
$$

where the vertical map on the right is given by functoriality of Ext and the map $\mathcal{K}_2 \to \mathcal{K}_1$ and the vertical map on the left is the one from Remark 9.7.

Remark 9.9. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. We define a sequence of morphisms of first order thickenings

$$
(Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2) \to (Sh(\mathcal{D}_3), \mathcal{O}'_3)
$$

of $(Sh(\mathcal{C}), \mathcal{O})$ to be a *complex* if the corresponding maps between the ideal sheaves $\mathcal{I}_i$ give a complex of $\mathcal{O}$-modules $\mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1$ (i.e., the composition is zero). In this case the composition $(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$ factors through $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$, i.e., the first order thickening $(Sh(\mathcal{D}_1), \mathcal{O}_1')$ of $(Sh(\mathcal{C}), \mathcal{O})$ is trivial and comes with a canonical trivialization $\pi : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{C}), \mathcal{O})$.

We say a sequence of morphisms of first order thickenings

$$(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$$

of $(Sh(\mathcal{C}), \mathcal{O})$ is *a short exact sequence* if the corresponding maps between ideal sheaves is a short exact sequence

$$0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1 \to 0$$

of $\mathcal{O}$-modules.

Remark 9.10. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let

$$(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$$

be a complex first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, see Remark 9.9. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2, 3$ be pairs consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume given a short exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$$

such that

$$
\begin{array}{ccc}
\mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{\ c_2\ } \mathcal{K}_2 & & \mathcal{I}_3 \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{\ c_3\ } \mathcal{K}_3 \\
\downarrow \qquad\qquad \downarrow & \text{and} & \downarrow \qquad\qquad \downarrow \\
\mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{\ c_1\ } \mathcal{K}_1 & & \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{\ c_2\ } \mathcal{K}_2
\end{array}
$$

are commutative. Finally, assume given an extension

$$0 \to \mathcal{K}_2 \to \mathcal{F}_2' \to \mathcal{F} \to 0$$

as in (9.0.1) with $\mathcal{K} = \mathcal{K}_2$ of $\mathcal{O}_2'$-modules with $c_{\mathcal{F}_2'} = c_2$. In this situation we can apply the functoriality of Remark 9.7 to obtain an extension $\mathcal{F}_1'$ of $\mathcal{O}_1'$-modules (we'll describe $\mathcal{F}_1'$ in this special case below). By Remark 9.6 using the canonical splitting $\pi : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{C}), \mathcal{O})$ of Remark 9.9 we obtain $\xi_{\mathcal{F}_1'} \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1)$. Finally, we have the obstruction

$$o(\mathcal{F}, \mathcal{K}_3, c_3) \in \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_3)$$

see Lemma 9.4. In this situation we **claim** that the canonical map

$$\partial : \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_3)$$

coming from the short exact sequence $0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$ sends $\xi_{\mathcal{F}_1'}$ to the obstruction class $o(\mathcal{F}, \mathcal{K}_3, c_3)$.

To prove this claim choose an embedding $j : \mathcal{K}_3 \to \mathcal{K}$ where $\mathcal{K}$ is an injective $\mathcal{O}$-module. We can lift j to a map $j' : \mathcal{K}_2 \to \mathcal{K}$. Set $\mathcal{E}_2' = j_*' \mathcal{F}_2'$ equal to the pushout of $\mathcal{F}_2'$ by j' so that $c_{\mathcal{E}_2'} = j' \circ c_2$. Picture:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle j'} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

Set $\mathcal{E}_3' = \mathcal{E}_2'$ but viewed as an $\mathcal{O}_3'$-module via $\mathcal{O}_3' \to \mathcal{O}_2'$. Then $c_{\mathcal{E}_3'} = j \circ c_3$. The proof of Lemma 9.4 constructs $o(\mathcal{F}, \mathcal{K}_3, c_3)$ as the boundary of the class of the extension of $\mathcal{O}$-modules

$$0 \to \mathcal{K}/\mathcal{K}_3 \to \mathcal{E}_3'/\mathcal{K}_3 \to \mathcal{F} \to 0$$

On the other hand, note that $\mathcal{F}_1' = \mathcal{F}_2'/\mathcal{K}_3$ hence the class $\xi_{\mathcal{F}_1'}$ is the class of the extension

$$0 \to \mathcal{K}_2/\mathcal{K}_3 \to \mathcal{F}_2'/\mathcal{K}_3 \to \mathcal{F} \to 0$$

seen as a sequence of $\mathcal{O}$-modules using $\pi^\sharp$ where $\pi : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{C}), \mathcal{O})$ is the canonical splitting. Thus finally, the claim follows from the fact that we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F}_2'/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}/\mathcal{K}_3 & \longrightarrow & \mathcal{E}_3'/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

which is $\mathcal{O}$-linear (with the $\mathcal{O}$-module structures given above).

10. Infinitesimal deformations of modules on ringed topi

Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. We freely use the notation introduced in Section 8. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. In this situation we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{F}' \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}'$-modules. Since $\mathcal{I}^2 = 0$ the $\mathcal{O}'$-module structure on $\mathcal{I}\mathcal{F}'$ comes from a unique $\mathcal{O}$-module structure. Thus the sequence above is an extension as in (9.0.1). As a special case, if $\mathcal{F}' = \mathcal{O}'$ we have $i^*\mathcal{O}' = \mathcal{O}$ and $\mathcal{I}\mathcal{O}' = \mathcal{I}$ and we recover the sequence of structure sheaves

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

Lemma 10.1. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules. Set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. The set of lifts of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{I}\mathcal{G}')$.*

Proof. This is a special case of Lemma 9.1 but we also give a direct proof. We have short exact sequences of modules

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0 \quad \text{and} \quad 0 \to \mathcal{I}\mathcal{G}' \to \mathcal{G}' \to \mathcal{G} \to 0$$

and similarly for $\mathcal{F}'$. Since $\mathcal{I}$ has square zero the $\mathcal{O}'$-module structure on $\mathcal{I}$ and $\mathcal{I}\mathcal{G}'$ comes from a unique $\mathcal{O}$-module structure. It follows that

$$\mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{I}\mathcal{G}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

The lemma now follows from the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G})$$

see Homology, Lemma 5.8. $\square$

Lemma 10.2. *Let (f, f') be a morphism of first order thickenings of ringed topoi as in Situation 8.1. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}$ is flat over $\mathcal{O}_\mathcal{B}$ and that (f, f') is a strict morphism of thickenings (Definition 8.2). Then the following are equivalent*

(1) *$\mathcal{F}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$, and*
(2) *the canonical map $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ is an isomorphism.*

Moreover, in this case the maps

$$f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I}\mathcal{F}'$$

are isomorphisms.

Proof. The map $f^*\mathcal{J} \to \mathcal{I}$ is surjective as (f, f') is a strict morphism of thickenings. Hence the final statement is a consequence of (2).

Proof of the equivalence of (1) and (2). By definition flatness over $\mathcal{O}_\mathcal{B}$ means flatness over $f^{-1}\mathcal{O}_\mathcal{B}$. Similarly for flatness over $f^{-1}\mathcal{O}_{\mathcal{B}'}$. Note that the strictness of (f, f') and the assumption that $\mathcal{F} = i^*\mathcal{F}'$ imply that

$$\mathcal{F} = \mathcal{F}'/(f^{-1}\mathcal{J})\mathcal{F}'$$

as sheaves on $\mathcal{C}$. Moreover, observe that $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} = f^{-1}\mathcal{J} \otimes_{f^{-1}\mathcal{O}_\mathcal{B}} \mathcal{F}$. Hence the equivalence of (1) and (2) follows from Modules on Sites, Lemma 28.13. $\square$

Lemma 10.3. *Let (f, f') be a morphism of first order thickenings of ringed topoi as in Situation 8.1. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. Then the following are equivalent*

(1) *$\mathcal{F}'$ is an $\mathcal{O}'$-module of finite presentation, and*
(2) *$\mathcal{F}$ is an $\mathcal{O}$-module of finite presentation.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Modules on Sites, Lemma 23.4. For the converse, assume $\mathcal{F}$ of finite presentation. We may and do assume that $\mathcal{C} = \mathcal{C}'$. By Lemma 10.2 we have a short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}' \to \mathcal{F} \to 0$$

Let U be an object of $\mathcal{C}$ such that $\mathcal{F}|_U$ has a presentation

$$\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

After replacing U by the members of a covering we may assume the map $\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$ lifts to a map $(\mathcal{O}_U')^{\oplus n} \to \mathcal{F}'|_U$. The induced map $\mathcal{I}^{\oplus n} \to \mathcal{I} \otimes \mathcal{F}$ is surjective by right exactness of $\otimes$. Thus after replacing U by the members of a covering we can find a lift $(\mathcal{O}'|_U)^{\oplus m} \to (\mathcal{O}'|_U)^{\oplus n}$ of the given map $\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n}$ such that

$$(\mathcal{O}_U')^{\oplus m} \to (\mathcal{O}_U')^{\oplus n} \to \mathcal{F}'|_U \to 0$$

is a complex. Using right exactness of $\otimes$ once more it is seen that this complex is exact. $\square$

Lemma 10.4. *Let (f, f') be a morphism of first order thickenings as in Situation 8.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. Assume that $\mathcal{G}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. The set of lifts of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under*

$$\mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G} \otimes_\mathcal{O} f^*\mathcal{J})$$

Proof. Combine Lemmas 10.1 and 10.2. □

Lemma 10.5. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. There exists an element*

$$o(\varphi) \in Ext^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

Proof. It is clear from the proof of Lemma 10.1 that the vanishing of the boundary of φ via the map

$$\text{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}) \longrightarrow \text{Ext}^1_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}')$$

is a necessary and sufficient condition for the existence of a lift. We conclude as

$$\text{Ext}^1_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \text{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

the adjointness of $i_* = Ri_*$ and Li^* on the derived category (Cohomology on Sites, Lemma 19.1). □

Lemma 10.6. *Let (f, f') be a morphism of first order thickenings as in Situation 8.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. Assume that $\mathcal{F}'$ and $\mathcal{G}'$ are flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. There exists an element*

$$o(\varphi) \in Ext^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

First proof. This follows from Lemma 10.5 as we claim that under the assumptions of the lemma we have

$$\text{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \text{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

Namely, we have $\mathcal{I}\mathcal{G}' = \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J}$ by Lemma 10.2. On the other hand, observe that

$$H^{-1}(Li^*\mathcal{F}') = \text{Tor}_1^{\mathcal{O}'}(\mathcal{F}', \mathcal{O})$$

(local computation omitted). Using the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

we see that this Tor_1 is computed by the kernel of the map $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ which is zero by the final assertion of Lemma 10.2. Thus $\tau_{\geq -1} Li^*\mathcal{F}' = \mathcal{F}$. On the other hand, we have

$$\text{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \text{Ext}^1_{\mathcal{O}}(\tau_{\geq -1} Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

by the dual of Derived Categories, Lemma 17.1. □

Second proof. We can apply Lemma 9.2 as follows. Note that $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ and $\mathcal{L} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G}$ by Lemma 10.2, that $c_{\mathcal{F}'} = 1 \otimes 1$ and $c_{\mathcal{G}'} = 1 \otimes 1$ and taking $\psi = 1 \otimes \varphi$ the diagram of the lemma commutes. Thus $o(\varphi) = o(\varphi, 1 \otimes \varphi)$ works. □

Lemma 10.7. *Let (f, f') be a morphism of first order thickenings as in Situation 8.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over $\mathcal{O}_{\mathcal{B}}$. If there exists a pair $(\mathcal{F}', \alpha)$ consisting of an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ and an isomorphism $\alpha : i^* \mathcal{F}' \to \mathcal{F}$, then the set of isomorphism classes of such pairs is principal homogeneous under $\operatorname{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F})$.*

Proof. If we assume there exists one such module, then the canonical map

$$f^* \mathcal{J} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$$

is an isomorphism by Lemma 10.2. Apply Lemma 9.3 with $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ and $c = 1$. By Lemma 10.2 the corresponding extensions $\mathcal{F}'$ are all flat over $\mathcal{O}_{\mathcal{B}'}$. $\square$

Lemma 10.8. *Let (f, f') be a morphism of first order thickenings as in Situation 8.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over $\mathcal{O}_{\mathcal{B}}$. There exists an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ with $i^* \mathcal{F}' \cong \mathcal{F}$, if and only if*

    (1) *the canonical map $f^* \mathcal{J} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ is an isomorphism, and*
    (2) *the class $o(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}, 1) \in \operatorname{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F})$ of Lemma 9.4 is zero.*

Proof. This follows immediately from the characterization of $\mathcal{O}'$-modules flat over $\mathcal{O}_{\mathcal{B}'}$ of Lemma 10.2 and Lemma 9.4. $\square$

11. Application to flat modules on flat thickenings of ringed topoi

Consider a commutative diagram

$$
\begin{array}{ccc}
(Sh(\mathcal{C}), \mathcal{O}) & \xrightarrow{\ \ i\ \ } & (Sh(\mathcal{D}), \mathcal{O}') \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) & \xrightarrow{\ \ t\ \ } & (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})
\end{array}
$$

of ringed topoi whose horizontal arrows are first order thickenings as in Situation 8.1. Set $\mathcal{I} = \operatorname{Ker}(i^\sharp) \subset \mathcal{O}'$ and $\mathcal{J} = \operatorname{Ker}(t^\sharp) \subset \mathcal{O}_{\mathcal{B}'}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume that

    (1) (f, f') is a strict morphism of thickenings,
    (2) f' is flat, and
    (3) $\mathcal{F}$ is flat over $\mathcal{O}_{\mathcal{B}}$.

Note that (1) + (2) imply that $\mathcal{I} = f^* \mathcal{J}$ (apply Lemma 10.2 to $\mathcal{O}'$). The theory of the preceding section is especially nice under these assumptions. We summarize the results already obtained in the following lemma.

Lemma 11.1. *In the situation above.*

    (1) *There exists an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ with $i^* \mathcal{F}' \cong \mathcal{F}$, if and only if the class $o(\mathcal{F}, f^* \mathcal{J} \otimes_{\mathcal{O}} \mathcal{F}, 1) \in \operatorname{Ext}^2_{\mathcal{O}}(\mathcal{F}, f^* \mathcal{J} \otimes_{\mathcal{O}} \mathcal{F})$ of Lemma 9.4 is zero.*
    (2) *If such a module exists, then the set of isomorphism classes of lifts is principal homogeneous under $\operatorname{Ext}^1_{\mathcal{O}}(\mathcal{F}, f^* \mathcal{J} \otimes_{\mathcal{O}} \mathcal{F})$.*
    (3) *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $\operatorname{id}_{\mathcal{F}}$ is canonically isomorphic to $\operatorname{Ext}^0_{\mathcal{O}}(\mathcal{F}, f^* \mathcal{J} \otimes_{\mathcal{O}} \mathcal{F})$.*

Proof. Part (1) follows from Lemma 10.8 as we have seen above that $\mathcal{I} = f^* \mathcal{J}$. Part (2) follows from Lemma 10.7. Part (3) follows from Lemma 10.4. $\square$

Situation 11.2. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$ be a morphism of ringed topoi. Consider a commutative diagram

$$
\begin{array}{ccccc}
(Sh(\mathcal{C}'_1), \mathcal{O}'_1) & \xrightarrow{\ h\ } & (Sh(\mathcal{C}'_2), \mathcal{O}'_2) & \longrightarrow & (Sh(\mathcal{C}'_3), \mathcal{O}'_3) \\
f'_1 \downarrow & & f'_2 \downarrow & & f'_3 \downarrow \\
(Sh(\mathcal{B}'_1), \mathcal{O}_{\mathcal{B}'_1}) & \longrightarrow & (Sh(\mathcal{B}'_2), \mathcal{O}_{\mathcal{B}'_2}) & \longrightarrow & (Sh(\mathcal{B}'_3), \mathcal{O}_{\mathcal{B}'_3})
\end{array}
$$

where (a) the top row is a short exact sequence of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, (b) the lower row is a short exact sequence of first order thickenings of $(Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, (c) each f'_i restricts to f, (d) each pair (f, f'_i) is a strict morphism of thickenings, and (e) each f'_i is flat. Finally, let $\mathcal{F}'_2$ be an $\mathcal{O}'_2$-module flat over $\mathcal{O}_{\mathcal{B}'_2}$ and set $\mathcal{F} = \mathcal{F}'_2 \otimes \mathcal{O}$. Let $\pi : (Sh(\mathcal{C}'_1), \mathcal{O}'_1) \to (Sh(\mathcal{C}), \mathcal{O})$ be the canonical splitting (Remark 9.9).

Lemma 11.3. *In Situation 11.2 the modules $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ are $\mathcal{O}'_1$-modules flat over $\mathcal{O}_{\mathcal{B}'_1}$ restricting to $\mathcal{F}$ on $(Sh(\mathcal{C}), \mathcal{O})$. Their difference (Lemma 11.1) is an element θ of $Ext^1_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F})$ whose boundary in $Ext^2_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J}_3 \otimes_{\mathcal{O}} \mathcal{F})$ equals the obstruction (Lemma 11.1) to lifting $\mathcal{F}$ to an $\mathcal{O}'_3$-module flat over $\mathcal{O}_{\mathcal{B}'_3}$.*

Proof. Note that both $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ restrict to $\mathcal{F}$ on $(Sh(\mathcal{C}), \mathcal{O})$ and that the kernels of $\pi^*\mathcal{F} \to \mathcal{F}$ and $h^*\mathcal{F}'_2 \to \mathcal{F}$ are given by $f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F}$. Hence flatness by Lemma 10.2. Taking the boundary makes sense as the sequence of modules

$$0 \to f^*\mathcal{J}_3 \otimes_{\mathcal{O}} \mathcal{F} \to f^*\mathcal{J}_2 \otimes_{\mathcal{O}} \mathcal{F} \to f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F} \to 0$$

is short exact due to the assumptions in Situation 11.2 and the fact that $\mathcal{F}$ is flat over $\mathcal{O}_{\mathcal{B}}$. The statement on the obstruction class is a direct translation of the result of Remark 9.10 to this particular situation. $\qquad\square$

12. Deformations of ringed topoi and the naive cotangent complex

In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a first order thickening $t : (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ of ringed topoi. We denote $\mathcal{J} = \operatorname{Ker}(t^\sharp)$ and we identify the underlying topoi of $\mathcal{B}$ and $\mathcal{B}'$. Moreover we assume given a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, an $\mathcal{O}$-module $\mathcal{G}$, and a map $f^{-1}\mathcal{J} \to \mathcal{G}$ of sheaves of $f^{-1}\mathcal{O}_{\mathcal{B}}$-modules. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

(12.0.1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & f^{-1}\mathcal{J} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}'} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}} & \longrightarrow & 0
\end{array}
$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and a morphism of thickenings (f, f') as in (8.1.1) where $\operatorname{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say $(Sh(\mathcal{C}'), \mathcal{O}')$ is a *solution* to (12.0.1).

Lemma 12.1. *Assume given a commutative diagram of morphisms ringed topoi*

(12.1.1)

$$
\begin{array}{ccc}
(Sh(\mathcal{C}_2), \mathcal{O}_2) & \xrightarrow{\ i_2\ } & (Sh(\mathcal{C}_2'), \mathcal{O}_2') \\
\ \ \ \ f_2 \downarrow & & \downarrow f_2' \\
g \ \ \ (Sh(\mathcal{B}_2), \mathcal{O}_{\mathcal{B}_2}) & \xrightarrow{\ t_2\ } & (Sh(\mathcal{B}_2'), \mathcal{O}_{\mathcal{B}_2'}) \\
(Sh(\mathcal{C}_1), \mathcal{O}_1) & \xrightarrow{\ i_1\ } & (Sh(\mathcal{C}_1'), \mathcal{O}_1') \\
f_1 \downarrow & & \downarrow f_1' \\
(Sh(\mathcal{B}_1), \mathcal{O}_{\mathcal{B}_1}) & \xrightarrow{\ t_1\ } & (Sh(\mathcal{B}_1'), \mathcal{O}_{\mathcal{B}_1'})
\end{array}
$$

whose horizontal arrows are first order thickenings. Set $\mathcal{G}_j = Ker(i_j^\sharp)$ and assume given a map of $g^{-1}\mathcal{O}_1$-modules $\nu : g^{-1}\mathcal{G}_1 \to \mathcal{G}_2$ giving rise to the commutative diagram

(12.1.2)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{O}_2' & \longrightarrow & \mathcal{O}_2 & \longrightarrow & 0 \\
& & \uparrow c_2 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & f_2^{-1}\mathcal{J}_2 & \longrightarrow & f_2^{-1}\mathcal{O}_{\mathcal{B}_2'} & \longrightarrow & f_2^{-1}\mathcal{O}_{\mathcal{B}_2} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{O}_1' & \longrightarrow & \mathcal{O}_1 & \longrightarrow & 0 \\
& & \uparrow c_1 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & f_1^{-1}\mathcal{J}_1 & \longrightarrow & f_1^{-1}\mathcal{O}_{\mathcal{B}_1'} & \longrightarrow & f_1^{-1}\mathcal{O}_{\mathcal{B}_1} & \longrightarrow & 0
\end{array}
$$

with front and back solutions to (12.0.1). (The north-north-west arrows are maps on $\mathcal{C}_2$ after applying g^{-1} to the source.)

(1) *There exist a canonical element in $Ext^1_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$ whose vanishing is a necessary and sufficient condition for the existence of a morphism of ringed topoi $(Sh(\mathcal{C}_2'), \mathcal{O}_2') \to (Sh(\mathcal{C}_1'), \mathcal{O}_1')$ fitting into (12.1.1) compatibly with ν.*

(2) *If there exists a morphism $(Sh(\mathcal{C}_2'), \mathcal{O}_2') \to (Sh(\mathcal{C}_1'), \mathcal{O}_1')$ fitting into (12.1.1) compatibly with ν the set of all such morphisms is a principal homogeneous space under*

$$
\mathrm{Hom}_{\mathcal{O}_1}(\Omega_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, g_*\mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}_2}(g^*\Omega_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2) = Ext^0_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2).
$$

Proof. The proof of this lemma is identical to the proof of Lemma 7.1. We urge the reader to read that proof instead of this one. We will identify the underlying topoi for every thickening in sight (we have already used this convention in the statement). The equalities in the last statement of the lemma are immediate from the definitions. Thus we will work with the groups $Ext^k_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$, $k = 0, 1$ in the rest of the proof. We first argue that we can reduce to the case where the underlying topos of all ringed topoi in the lemma is the same.

To do this, observe that $g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$ is equal to the naive cotangent complex of the homomorphism of sheaves of rings $g^{-1} f_1^{-1} \mathcal{O}_{\mathcal{B}_1} \to g^{-1} \mathcal{O}_1$, see Modules on Sites, Lemma 32.5. Moreover, the degree 0 term of $NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$ is a flat $\mathcal{O}_1$-module, hence the canonical map

$$Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}} \longrightarrow g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}} \otimes_{g^{-1}\mathcal{O}_1} \mathcal{O}_2$$

induces an isomorphism on cohomology sheaves in degrees 0 and -1. Thus we may replace the Ext groups of the lemma with

$$\mathrm{Ext}^k_{g^{-1}\mathcal{O}_1}(g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2) = \mathrm{Ext}^k_{g^{-1}\mathcal{O}_1}(NL_{g^{-1}\mathcal{O}_1/g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$$

The set of morphism of ringed topoi $(Sh(\mathcal{C}_2'), \mathcal{O}_2') \to (Sh(\mathcal{C}_1'), \mathcal{O}_1')$ fitting into (12.1.1) compatibly with ν is in one-to-one bijection with the set of homomorphisms of $g^{-1} f_1^{-1} \mathcal{O}_{\mathcal{B}_1'}$-algebras $g^{-1} \mathcal{O}_1' \to \mathcal{O}_2'$ which are compatible with $f^\sharp$ and ν. In this way we see that we may assume we have a diagram (12.1.2) of sheaves on a site $\mathcal{C}$ (with $f_1 = f_2 = $ id on underlying topoi) and we are looking to find a homomorphism of sheaves of rings $\mathcal{O}_1' \to \mathcal{O}_2'$ fitting into it.

In the rest of the proof of the lemma we assume all underlying topological spaces are the same, i.e., we have a diagram (12.1.2) of sheaves on a site $\mathcal{C}$ (with $f_1 = f_2 = $ id on underlying topoi) and we are looking for homomorphisms of sheaves of rings $\mathcal{O}_1' \to \mathcal{O}_2'$ fitting into it. As ext groups we will use $\mathrm{Ext}^k_{\mathcal{O}_1}(NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$, $k = 0, 1$.

Step 1. Construction of the obstruction class. Consider the sheaf of sets

$$\mathcal{E} = \mathcal{O}_1' \times_{\mathcal{O}_2} \mathcal{O}_2'$$

This comes with a surjective map $\alpha : \mathcal{E} \to \mathcal{O}_1$ and hence we can use $NL(\alpha)$ instead of $NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$, see Modules on Sites, Lemma 34.2. Set

$$\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_{\mathcal{B}_1'}[\mathcal{E}] \to \mathcal{O}_1) \quad \text{and} \quad \mathcal{I} = \mathrm{Ker}(\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}] \to \mathcal{O}_1)$$

There is a surjection $\mathcal{I}' \to \mathcal{I}$ whose kernel is $\mathcal{J}_1 \mathcal{O}_{\mathcal{B}_1'}[\mathcal{E}]$. We obtain two homomorphisms of $\mathcal{O}_{\mathcal{B}_2'}$-algebras

$$a : \mathcal{O}_{\mathcal{B}_1'}[\mathcal{E}] \to \mathcal{O}_1' \quad \text{and} \quad b : \mathcal{O}_{\mathcal{B}_1'}[\mathcal{E}] \to \mathcal{O}_2'$$

which induce maps $a|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_1$ and $b|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_2$. Both a and b annihilate $(\mathcal{I}')^2$. Moreover a and b agree on $\mathcal{J}_1 \mathcal{O}_{\mathcal{B}_1'}[\mathcal{E}]$ as maps into $\mathcal{G}_2$ because the left hand square of (12.1.2) is commutative. Thus the difference $b|_{\mathcal{I}'} - \nu \circ a|_{\mathcal{I}'}$ induces a well defined $\mathcal{O}_1$-linear map

$$\xi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{G}_2$$

which sends the class of a local section f of $\mathcal{I}$ to $a(f') - \nu(b(f'))$ where f' is a lift of f to a local section of $\mathcal{I}'$. We let $[\xi] \in \mathrm{Ext}^1_{\mathcal{O}_1}(NL(\alpha), \mathcal{G}_2)$ be the image (see below).

Step 2. Vanishing of $[\xi]$ is necessary. Let us write $\Omega = \Omega_{\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}]/\mathcal{O}_{\mathcal{B}_1}} \otimes_{\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}]} \mathcal{O}_1$. Observe that $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega)$ fits into a distinguished triangle

$$\Omega[0] \to NL(\alpha) \to \mathcal{I}/\mathcal{I}^2[1] \to \Omega[1]$$

Thus we see that $[\xi]$ is zero if and only if ξ is a composition $\mathcal{I}/\mathcal{I}^2 \to \Omega \to \mathcal{G}_2$ for some map $\Omega \to \mathcal{G}_2$. Suppose there exists a homomorphisms of sheaves of rings $\varphi : \mathcal{O}_1' \to \mathcal{O}_2'$ fitting into (12.1.2). In this case consider the map $\mathcal{O}_1'[\mathcal{E}] \to \mathcal{G}_2$, $f' \mapsto b(f') - \varphi(a(f'))$. A calculation shows this annihilates $\mathcal{J}_1 \mathcal{O}_{\mathcal{B}_1'}[\mathcal{E}]$ and induces a derivation $\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}] \to \mathcal{G}_2$. The resulting linear map $\Omega \to \mathcal{G}_2$ witnesses the fact that $[\xi] = 0$ in this case.

Step 3. Vanishing of $[\xi]$ is sufficient. Let $\theta : \Omega \to \mathcal{G}_2$ be a $\mathcal{O}_1$-linear map such that ξ is equal to $\theta \circ (\mathcal{I}/\mathcal{I}^2 \to \Omega)$. Then a calculation shows that

$$b + \theta \circ d : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \longrightarrow \mathcal{O}'_2$$

annihilates $\mathcal{I}'$ and hence defines a map $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into (12.1.2).

Proof of (2) in the special case above. Omitted. Hint: This is exactly the same as the proof of (2) of Lemma 2.1. $\square$

Lemma 12.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Let $\mathcal{G}$ be a $\mathcal{B}$-module. Let $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL_{\mathcal{B}/\mathcal{A}}, \mathcal{G})$. There exists a map of sheaves of sets $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\xi \in \mathrm{Ext}^1_{\mathcal{B}}(NL(\alpha), \mathcal{G})$ is the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ (see proof for notation).*

Proof. Recall that given $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$ is surjective with kernel $\mathcal{I}$ the complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$ is canonically isomorphic to $NL_{\mathcal{B}/\mathcal{A}}$, see Modules on Sites, Lemma 34.2. Observe moreover, that $\Omega = \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{e \in \mathcal{E}(U)} \mathcal{B}(U)$. In other words, Ω is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}$ and in particular there is a canonical map $\mathcal{E} \to \Omega$.

Having said this, pick some $\mathcal{E}$ (for example $\mathcal{E} = \mathcal{B}$ as in the definition of the naive cotangent complex). The obstruction to writing ξ as the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ is an element in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega, \mathcal{G})$. Say this is represented by the extension $0 \to \mathcal{G} \to \mathcal{H} \to \Omega \to 0$ of $\mathcal{B}$-modules. Consider the sheaf of sets $\mathcal{E}' = \mathcal{E} \times_{\Omega} \mathcal{H}$ which comes with an induced map $\alpha' : \mathcal{E}' \to \mathcal{B}$. Let $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}[\mathcal{E}'] \to \mathcal{B})$ and $\Omega' = \Omega_{\mathcal{A}[\mathcal{E}']/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}']} \mathcal{B}$. The pullback of ξ under the quasi-isomorphism $NL(\alpha') \to NL(\alpha)$ maps to zero in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega', \mathcal{G})$ because the pullback of the extension $\mathcal{H}$ by the map $\Omega' \to \Omega$ is split as Ω' is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}'$ and since by construction there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}' & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{H} & \longrightarrow & \Omega
\end{array}
$$

This finishes the proof. $\square$

Lemma 12.3. *If there exists a solution to (12.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$.*

Proof. We observe right away that given two solutions $\mathcal{O}'_1$ and $\mathcal{O}'_2$ to (12.0.1) we obtain by Lemma 12.1 an obstruction element $o(\mathcal{O}'_1, \mathcal{O}'_2) \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$ to the existence of a map $\mathcal{O}'_1 \to \mathcal{O}'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution $\mathcal{O}'$ and an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_{\mathcal{B}}}, \mathcal{G})$ we can find a second solution $\mathcal{O}'_\xi$ such that $o(\mathcal{O}', \mathcal{O}'_\xi) = \xi$.

Pick $\alpha : \mathcal{E} \to \mathcal{O}$ as in Lemma 12.2 for the class ξ. Consider the surjection $f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}] \to \mathcal{O}$ with kernel $\mathcal{I}$ and corresponding naive cotangent complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}]/f^{-1}\mathcal{O}_{\mathcal{B}}} \otimes_{f^{-1}\mathcal{O}_{\mathcal{B}}[\mathcal{E}]} \mathcal{O})$. By the lemma ξ is the class of a morphism $\delta : \mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$. After replacing $\mathcal{E}$ by $\mathcal{E} \times_{\mathcal{O}} \mathcal{O}'$ we may also assume that α factors through a map $\alpha' : \mathcal{E} \to \mathcal{O}'$.

These choices determine an $f^{-1}\mathcal{O}_{\mathcal{B}'}$-algebra map $\varphi : \mathcal{O}_{\mathcal{B}'}[\mathcal{E}] \to \mathcal{O}'$. Let $\mathcal{I}' = \mathrm{Ker}(\varphi)$. Observe that φ induces a map $\varphi|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}$ and that $\mathcal{O}'$ is the pushout, as in the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}' & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle\varphi|_{\mathcal{I}'}} & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathcal{I}' & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}'}[\mathcal{E}] & \longrightarrow & \mathcal{O} & \longrightarrow & 0
\end{array}
$$

Let $\psi : \mathcal{I}' \to \mathcal{G}$ be the sum of the map $\varphi|_{\mathcal{I}'}$ and the composition

$$\mathcal{I}' \to \mathcal{I}'/(\mathcal{I}')^2 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \mathcal{G}.$$

Then the pushout along ψ is an other ring extension $\mathcal{O}'_\xi$ fitting into a diagram as above. A calculation (omitted) shows that $o(\mathcal{O}', \mathcal{O}'_\xi) = \xi$ as desired. $\square$

Lemma 12.4. *Let $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a ringed topos and let $\mathcal{J}$ be an $\mathcal{O}_\mathcal{B}$-module.*
  (1) *The set of extensions of sheaves of rings $0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{B}'} \to \mathcal{O}_\mathcal{B} \to 0$ where $\mathcal{J}$ is an ideal of square zero is canonically bijective to $Ext^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{J})$.*
  (2) *Given a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, an $\mathcal{O}$-module $\mathcal{G}$, an $f^{-1}\mathcal{O}_\mathcal{B}$-module map $c : f^{-1}\mathcal{J} \to \mathcal{G}$, and given extensions of sheaves of rings with square zero kernels:*
    (a) *$0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{B}'} \to \mathcal{O}_\mathcal{B} \to 0$ corresponding to $\alpha \in Ext^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{J})$,*
    (b) *$0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$ corresponding to $\beta \in Ext^1_\mathcal{O}(NL_{\mathcal{O}/\mathbf{Z}}, \mathcal{G})$*
    then there is a morphism $(Sh(\mathcal{C}), \mathcal{O}') \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}'})$ fitting into a diagram (12.0.1) if and only if β and α map to the same element of $Ext^1_\mathcal{O}(Lf^* NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{G})$.

Proof. To prove this we apply the previous results where we work over the base ringed topos $(Sh(*), \mathbf{Z})$ with trivial thickening. Part (1) follows from Lemma 12.3 and the fact that there exists a solution, namely $\mathcal{J} \oplus \mathcal{O}_\mathcal{B}$. Part (2) follows from Lemma 12.1 and a compatibility between the constructions in the proofs of Lemmas 12.3 and 12.1 whose statement and proof we omit. $\square$

13. Other chapters

References

[Ill72] Luc Illusie, *Complexe cotangent et déformations I and II*, Lecture Notes in Mathematics, Vol. 239 and 283, Springer-Verlag, Berlin, 1971/1972.