PRO-ÉTALE COHOMOLOGY

Contents

1. Introduction

The material in this chapter and more can be found in the preprint [BS13].

The goal of this chapter is to introduce the pro-étale topology and show how it simplifies the introduction of ℓ-adic cohomology in algebraic geometry.

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A brief overview of the history of this material as we have understood it. In [Gro77, Exposés V and VI] Grothendieck et al developed a theory for dealing with ℓ-adic sheaves as inverse systems of sheaves of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules. In his second paper on the Weil conjectures ([Del74]) Deligne introduced a derived category of ℓ-adic sheaves as a certain 2-limit of categories of complexes of sheaves of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules on the étale site of a scheme X. This approach is used in the paper by Beilinson, Bernstein, and Deligne ([BBD82]) as the basis for their beautiful theory of perverse sheaves. In a paper entitled "Continuous Étale Cohomology" ([Jan88]) Uwe Jannsen discusses an important variant of the cohomology of a ℓ-adic sheaf on a variety over a field. His paper is followed up by a paper of Torsten Ekedahl ([Eke90]) who discusses the adic formalism needed to work comfortably with derived categories defined as limits.

The goal of this chapter is to show that, if we work with the pro-étale site of a scheme, then one can avoid some of the technicalities these authors encountered. This comes at the expense of having to work with non-Noetherian schemes, even when one is only interested in working with ℓ-adic sheaves and cohomology of such on varieties over an algebraically closed field.

2. Some topology

Some preliminaries. We have defined *spectral spaces* and *spectral maps* of spectral spaces in Topology, Section 22. The spectrum of a ring is a spectral space, see Algebra, Lemma 25.2.

Lemma 2.1. *Let X be a spectral space. Let $X_0 \subset X$ be the set of closed points. The following are equivalent*

(1) *Every open covering of X can be refined by a finite disjoint union decomposition $X = \coprod U_i$ with U_i open and closed in X.*
(2) *The composition $X_0 \to X \to \pi_0(X)$ is bijective.*

Moreover, if X_0 is closed in X and every point of X specializes to a unique point of X_0, then these conditions are satisfied.

Proof. We will use without further mention that X_0 is quasi-compact (Topology, Lemma 11.9) and $\pi_0(X)$ is profinite (Topology, Lemma 22.8). Picture

$$\begin{tikzcd} X_0 \arrow[r] \arrow[dr, "f"'] & X \arrow[d, "\pi"] \\ & \pi_0(X) \end{tikzcd}$$

If (2) holds, the continuous bijective map $f : X_0 \to \pi_0(X)$ is a homeomorphism by Topology, Lemma 16.8. Given an open covering $X = \bigcup U_i$, we get an open covering $\pi_0(X) = \bigcup f(X_0 \cap U_i)$. By Topology, Lemma 21.3 we can find a finite open covering of the form $\pi_0(X) = \coprod V_j$ which refines this covering. Since $X_0 \to \pi_0(X)$ is bijective each connected component of X has a unique closed point, whence is equal to the set of points specializing to this closed point. Hence $\pi^{-1}(V_j)$ is the set of points specializing to the points of $f^{-1}(V_j)$. Now, if $f^{-1}(V_j) \subset X_0 \cap U_i \subset U_i$, then it follows that $\pi^{-1}(V_j) \subset U_i$ (because the open set U_i is closed under generalizations). In this way we see that the open covering $X = \coprod \pi^{-1}(V_j)$ refines the covering we started out with. In this way we see that (2) implies (1).

Assume (1). Let $x, y \in X$ be closed points. Then we have the open covering $X = (X \setminus \{x\}) \cup (X \setminus \{y\})$. It follows from (1) that there exists a disjoint union decomposition $X = U \amalg V$ with U and V open (and closed) and $x \in U$ and $y \in V$. In particular we see that every connected component of X has at most one closed point. By Topology, Lemma 11.8 every connected component (being closed) also does have a closed point. Thus $X_0 \to \pi_0(X)$ is bijective. In this way we see that (1) implies (2).

Assume X_0 is closed in X and every point specializes to a unique point of X_0. Then X_0 is a spectral space (Topology, Lemma 22.4) consisting of closed points, hence profinite (Topology, Lemma 22.7). Let $x, y \in X_0$ be distinct. By Topology, Lemma 21.3 we can find a disjoint union decomposition $X_0 = U_0 \amalg V_0$ with U_0 and V_0 open and closed. Let $\{U_i\}$ be the set of quasi-compact open subsets of X such that $U_0 = X_0 \cap U_i$. Similarly, let $\{V_j\}$ be the set of quasi-compact open subsets of X such that and $V_0 = X_0 \cap V_j$. If $U_i \cap V_j$ is nonempty for all i, j, then there exists a point ξ contained in all of them (use the $U_i \cap V_j$ is constructible, hence closed in the constructible topology, and use Topology, Lemmas 22.2 and 11.6). However, since X is sober and V_0 is closed in X, the intersection $\bigcap U_i$ is the set of points specializing to U_0. Similarly, $\bigcap V_j$ is the set of points specializing to V_0. Since $U_0 \cap V_0$ is empty this is a contradiction. Thus we find disjoint quasi-compact opens $U, V \subset X$ such that $U_0 = X_0 \cap U$ and $V_0 = X_0 \cap V$. Observe that $X = U \cup V = U \amalg V$ as $X_0 \subset U \cup V$ (use Topology, Lemma 11.8). This proves that x, y are not in the same connected component of X. In other words, $X_0 \to \pi_0(X)$ is injective. The map is also surjective by Topology, Lemma 11.8 and the fact that connected components are closed. In this way we see that the final condition implies (1). □

Example 2.2. Let T be a profinite space. Let $t \in T$ be a point and assume that $T \setminus \{t\}$ is not quasi-compact. Let $X = T \times \{0, 1\}$. Consider the topology on X with a subbase given by the sets $U \times \{0, 1\}$ for $U \subset T$ open, $X \setminus \{(t, 0)\}$, and $U \times \{1\}$ for $U \subset T$ open with $t \notin U$. The set of closed points of X is $X_0 = T \times \{0\}$ and $(t, 1)$ is in the closure of X_0. Moreover, $X_0 \to \pi_0(X)$ is a bijection. This example shows that conditions (1) and (2) of Lemma 2.1 do no imply the set of closed points is closed.

It turns out it is more convenient to work with spectral spaces which have the slightly stronger property mentioned in the final statement of Lemma 2.1. We give this property a name.

Definition 2.3. A spectral space X is *w-local* if the set of closed points X_0 is closed and every point of X specializes to a unique closed point. A continuous map $f : X \to Y$ of w-local spaces is *w-local* if it is spectral and maps any closed point of X to a closed point of Y.

We have seen in the proof of Lemma 2.1 that in this case $X_0 \to \pi_0(X)$ is a homeomorphism and that $X_0 \cong \pi_0(X)$ is a profinite space. Moreover, a connected component of X is exactly the set of points specializing to a given $x \in X_0$.

Lemma 2.4. *Let X be a w-local spectral space. If $Y \subset X$ is closed, then Y is w-local.*

Proof. The subset $Y_0 \subset Y$ of closed points is closed because $Y_0 = X_0 \cap Y$. Since X is w-local, every $y \in Y$ specializes to a unique point of X_0. This specialization is in Y, and hence also in Y_0, because $\overline{\{y\}} \subset Y$. In conclusion, Y is w-local. $\square$

Lemma 2.5. *Let X be a spectral space. Let*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & \pi_0(X) \end{tikzcd}$$

be a cartesian diagram in the category of topological spaces with T profinite. Then Y is spectral and $T = \pi_0(Y)$. If moreover X is w-local, then Y is w-local, $Y \to X$ is w-local, and the set of closed points of Y is the inverse image of the set of closed points of X.

Proof. Note that Y is a closed subspace of $X \times T$ as $\pi_0(X)$ is a profinite space hence Hausdorff (use Topology, Lemmas 22.8 and 3.4). Since $X \times T$ is spectral (Topology, Lemma 22.9) it follows that Y is spectral (Topology, Lemma 22.4). Let $Y \to \pi_0(Y) \to T$ be the canonical factorization (Topology, Lemma 6.8). It is clear that $\pi_0(Y) \to T$ is surjective. The fibres of $Y \to T$ are homeomorphic to the fibres of $X \to \pi_0(X)$. Hence these fibres are connected. It follows that $\pi_0(Y) \to T$ is injective. We conclude that $\pi_0(Y) \to T$ is a homeomorphism by Topology, Lemma 16.8.

Next, assume that X is w-local and let $X_0 \subset X$ be the set of closed points. The inverse image $Y_0 \subset Y$ of X_0 in Y maps bijectively onto T as $X_0 \to \pi_0(X)$ is a bijection by Lemma 2.1. Moreover, Y_0 is quasi-compact as a closed subset of the spectral space Y. Hence $Y_0 \to \pi_0(Y) = T$ is a homeomorphism by Topology, Lemma 16.8. It follows that all points of Y_0 are closed in Y. Conversely, if $y \in Y$ is a closed point, then it is closed in the fibre of $Y \to \pi_0(Y) = T$ and hence its image x in X is closed in the (homeomorphic) fibre of $X \to \pi_0(X)$. This implies $x \in X_0$ and hence $y \in Y_0$. Thus Y_0 is the collection of closed points of Y and for each $y \in Y_0$ the set of generalizations of y is the fibre of $Y \to \pi_0(Y)$. The lemma follows. $\square$

3. Local isomorphisms

We start with a definition.

Definition 3.1. Let $\varphi : A \to B$ be a ring map.

(1) We say $A \to B$ is a *local isomorphism* if for every prime $\mathfrak{q} \subset B$ there exists a $g \in B$, $g \notin \mathfrak{q}$ such that $A \to B_g$ induces an open immersion $\mathrm{Spec}(B_g) \to \mathrm{Spec}(A)$.
(2) We say $A \to B$ *identifies local rings* if for every prime $\mathfrak{q} \subset B$ the canonical map $A_{\varphi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ is an isomorphism.

We list some elementary properties.

Lemma 3.2. *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B.*

(1) *If $A \to B$ is a local isomorphism, then $A' \to B'$ is a local isomorphism.*
(2) *If $A \to B$ identifies local rings, then $A' \to B'$ identifies local rings.*

Proof. Omitted. □

Lemma 3.3. *Let $A \to B$ and $B \to C$ be ring maps.*

1. *If $A \to B$ and $B \to C$ are local isomorphisms, then $A \to C$ is a local isomorphism.*
2. *If $A \to B$ and $B \to C$ identify local rings, then $A \to C$ identifies local rings.*

Proof. Omitted. □

Lemma 3.4. *Let A be a ring. Let $B \to C$ be an A-algebra homomorphism.*

1. *If $A \to B$ and $A \to C$ are local isomorphisms, then $B \to C$ is a local isomorphism.*
2. *If $A \to B$ and $A \to C$ identify local rings, then $B \to C$ identifies local rings.*

Proof. Omitted. □

Lemma 3.5. *Let $A \to B$ be a local isomorphism. Then*

1. *$A \to B$ is étale,*
2. *$A \to B$ identifies local rings,*
3. *$A \to B$ is quasi-finite.*

Proof. Omitted. □

Lemma 3.6. *Let $A \to B$ be a local isomorphism. Then there exist $n \geq 0$, $g_1, \ldots, g_n \in B$, $f_1, \ldots, f_n \in A$ such that $(g_1, \ldots, g_n) = B$ and $A_{f_i} \cong B_{g_i}$.*

Proof. Omitted. □

Lemma 3.7. *Let $p : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and $q : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be morphisms of locally ringed spaces. If $\mathcal{O}_Y = p^{-1}\mathcal{O}_X$, then*

$$\mathrm{Mor}_{LRS/(X,\mathcal{O}_X)}((Z, \mathcal{O}_Z), (Y, \mathcal{O}_Y)) \longrightarrow \mathrm{Mor}_{Top/X}(Z, Y), \quad (f, f^\sharp) \longmapsto f$$

is bijective. Here $LRS/(X, \mathcal{O}_X)$ is the category of locally ringed spaces over X and Top/X is the category of topological spaces over X.

Proof. This is immediate from the definitions. □

Lemma 3.8. *Let A be a ring. Set $X = \mathrm{Spec}(A)$. The functor*

$$B \longmapsto \mathrm{Spec}(B)$$

from the category of A-algebras B such that $A \to B$ identifies local rings to the category of topological spaces over X is fully faithful.

Proof. This follows from Lemma 3.7 and the fact that if $A \to B$ identifies local rings, then the pullback of the structure sheaf of $\mathrm{Spec}(A)$ via $p : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is equal to the structure sheaf of $\mathrm{Spec}(B)$. □

4. Ind-Zariski algebra

We start with a definition; please see Remark 6.9 for a comparison with the corresponding definition of the article [BS13].

Definition 4.1. A ring map $A \to B$ is said to be *ind-Zariski* if B can be written as a filtered colimit $B = \operatorname{colim} B_i$ with each $A \to B_i$ a local isomorphism.

An example of an Ind-Zariski map is a localization $A \to S^{-1}A$, see Algebra, Lemma 9.9. The category of ind-Zariski algebras is closed under several natural operations.

Lemma 4.2. *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B. If $A \to B$ is ind-Zariski, then $A' \to B'$ is ind-Zariski.*

Proof. Omitted. $\square$

Lemma 4.3. *Let $A \to B$ and $B \to C$ be ring maps. If $A \to B$ and $B \to C$ are ind-Zariski, then $A \to C$ is ind-Zariski.*

Proof. Omitted. $\square$

Lemma 4.4. *Let A be a ring. Let $B \to C$ be an A-algebra homomorphism. If $A \to B$ and $A \to C$ are ind-Zariski, then $B \to C$ is ind-Zariski.*

Proof. Omitted. $\square$

Lemma 4.5. *A filtered colimit of ind-Zariski A-algebras is ind-Zariski over A.*

Proof. Omitted. $\square$

Lemma 4.6. *Let $A \to B$ be ind-Zariski. Then $A \to B$ identifies local rings,*

Proof. Omitted. $\square$

5. Constructing w-local affine schemes

An affine scheme X is called *w-local* if its underlying topological space is w-local (Definition 2.3). It turns out given any ring A there is a canonical faithfully flat ind-Zariski ring map $A \to A_w$ such that $\operatorname{Spec}(A_w)$ is w-local. The key to constructing A_w is the following simple lemma.

Lemma 5.1. *Let A be a ring. Set $X = \operatorname{Spec}(A)$. Let $Z \subset X$ be a locally closed subscheme which of the form $D(f) \cap V(I)$ for some $f \in A$ and ideal $I \subset A$. Then*

1. *there exists a multiplicative subset $S \subset A$ such that $\operatorname{Spec}(S^{-1}A)$ maps by a homeomorphism to the set of points of X specializing to Z,*
2. *the A-algebra $A_Z^\sim = S^{-1}A$ depends only on the underlying locally closed subset $Z \subset X$,*
3. *Z is a closed subscheme of $\operatorname{Spec}(A_Z^\sim)$,*

If $A \to A'$ is a ring map and $Z' \subset X' = \operatorname{Spec}(A')$ is a locally closed subscheme of the same form which maps into Z, then there is a unique A-algebra map $A_Z^\sim \to (A')_{Z'}^\sim$.

Proof. Let $S \subset A$ be the multiplicative set of elements which map to invertible elements of $\Gamma(Z, \mathcal{O}_Z) = (A/I)_f$. If $\mathfrak{p}$ is a prime of A which does not specialize to Z, then $\mathfrak{p}$ generates the unit ideal in $(A/I)_f$. Hence we can write $f^n = g + h$ for some $n \geq 0$, $g \in \mathfrak{p}$, $h \in I$. Then $g \in S$ and we see that $\mathfrak{p}$ is not in the spectrum of $S^{-1}A$. Conversely, if $\mathfrak{p}$ does specialize to Z, say $\mathfrak{p} \subset \mathfrak{q} \supset I$ with $f \notin \mathfrak{q}$, then we see that $S^{-1}A$ maps to $A_\mathfrak{q}$ and hence $\mathfrak{p}$ is in the spectrum of $S^{-1}A$. This proves (1).

The isomorphism class of the localization $S^{-1}A$ depends only on the corresponding subset $\mathrm{Spec}(S^{-1}A) \subset \mathrm{Spec}(A)$, whence (2) holds. By construction $S^{-1}A$ maps surjectively onto $(A/I)_f$, hence (3). The final statement follows as the multiplicative subset $S' \subset A'$ corresponding to Z' contains the image of the multiplicative subset S. □

Let A be a ring. Let $E \subset A$ be a finite subset. We get a stratification of $X = \mathrm{Spec}(A)$ into locally closed subschemes by looking at the vanishing behaviour of the elements of E. More precisely, given a disjoint union decomposition $E = E' \amalg E''$ we set

$$(5.1.1)\quad Z(E', E'') = \bigcap\nolimits_{f \in E'} D(f) \cap \bigcap\nolimits_{f \in E''} V(f) = D(\prod\nolimits_{f \in E'} f) \cap V(\sum\nolimits_{f \in E''} fA)$$

The points of $Z(E', E'')$ are exactly those $x \in X$ such that $f \in E'$ maps to a nonzero element in $\kappa(x)$ and $f \in E''$ maps to zero in $\kappa(x)$. Thus it is clear that

$$(5.1.2)\qquad X = \coprod\nolimits_{E = E' \amalg E''} Z(E', E'')$$

set theoretically. Observe that each stratum is constructible.

Lemma 5.2. *Let $X = \mathrm{Spec}(A)$ as above. Given any finite stratification $X = \coprod T_i$ by constructible subsets, there exists a finite subset $E \subset A$ such that the stratification (5.1.2) refines $X = \coprod T_i$.*

Proof. We may write $T_i = \bigcup_j U_{i,j} \cap V_{i,j}^c$ as a finite union for some $U_{i,j}$ and $V_{i,j}$ quasi-compact open in X. Then we may write $U_{i,j} = \bigcup D(f_{i,j,k})$ and $V_{i,j} = \bigcup D(g_{i,j,l})$. Then we set $E = \{f_{i,j,k}\} \cup \{g_{i,j,l}\}$. This does the job, because the stratification (5.1.2) is the one whose strata are labeled by the vanishing pattern of the elements of E which clearly refines the given stratification. □

We continue the discussion. Given a finite subset $E \subset A$ we set

$$(5.2.1)\qquad A_E = \prod\nolimits_{E = E' \amalg E''} A^{\sim}_{Z(E', E'')}$$

with notation as in Lemma 5.1. This makes sense because (5.1.1) shows that each $Z(E', E'')$ has the correct shape. We take the spectrum of this ring and denote it

$$(5.2.2)\qquad X_E = \mathrm{Spec}(A_E) = \coprod\nolimits_{E = E' \amalg E''} X_{E', E''}$$

with $X_{E', E''} = \mathrm{Spec}(A^{\sim}_{Z(E', E'')})$. Note that

$$(5.2.3)\qquad Z_E = \coprod\nolimits_{E = E' \amalg E''} Z(E', E'') \longrightarrow X_E$$

is a closed subscheme. By construction the closed subscheme Z_E contains all the closed points of the affine scheme X_E as every point of $X_{E', E''}$ specializes to a point of $Z(E', E'')$.

Let $I(A)$ be the partially ordered set of all finite subsets of A. This is a directed partially ordered set. For $E_1 \subset E_2$ there is a canonical transition map $A_{E_1} \to A_{E_2}$ of A-algebras. Namely, given a decomposition $E_2 = E_2' \amalg E_2''$ we set $E_1' = E_1 \cap E_2'$ and $E_1'' = E_1 \cap E_2''$. Then observe that $Z(E_1', E_1'') \subset Z(E_2', E_2'')$ hence a unique A-algebra map $A^{\sim}_{Z(E_1', E_1'')} \to A^{\sim}_{Z(E_2', E_2'')}$ by Lemma 5.1. Using these maps collectively we obtain the desired ring map $A_{E_1} \to A_{E_2}$. Observe that the corresponding map of affine schemes

$$(5.2.4)\qquad X_{E_2} \longrightarrow X_{E_1}$$

maps Z_{E_2} into Z_{E_1}. By uniqueness we obtain a system of A-algebras over $I(A)$ and we set

$$A_w = \text{colim}_{E \in I(A)} A_E \tag{5.2.5}$$

This A-algebra is ind-Zariski and faithfully flat over A. Finally, we set $X_w = \text{Spec}(A_w)$ and endow it with the closed subscheme $Z = \lim_{E \in I(A)} Z_E$. In a formula

$$X_w = \lim_{E \in I(A)} X_E \supset Z = \lim_{E \in I(A)} Z_E \tag{5.2.6}$$

Lemma 5.3. *Let $X = \text{Spec}(A)$ be an affine scheme. With $A \to A_w$, $X_w = \text{Spec}(A_w)$, and $Z \subset X_w$ as above.*

1. *$A \to A_w$ is ind-Zariski and faithfully flat,*
2. *$X_w \to X$ induces a bijection $Z \to X$,*
3. *Z is the set of closed points of X_w,*
4. *Z is a reduced scheme, and*
5. *every point of X_w specializes to a unique point of Z.*

In particular, X_w is w-local (Definition 2.3).

Proof. The map $A \to A_w$ is ind-Zariski by construction. For every E the morphism $Z_E \to X$ is a bijection, hence (2). As $Z \subset X_w$ we conclude $X_w \to X$ is surjective and $A \to A_w$ is faithfully flat by Algebra, Lemma 38.15. This proves (1).

Suppose that $y \in X_w$, $y \notin Z$. Then there exists an E such that the image of y in X_E is not contained in Z_E. Then for all $E \subset E'$ also y maps to an element of $X_{E'}$ not contained in $Z_{E'}$. Let $T_{E'} \subset X_{E'}$ be the reduced closed subscheme which is the closure of the image of y. It is clear that $T = \lim_{E \subset E'} T_{E'}$ is the closure of y in X_w. For every $E \subset E'$ the scheme $T_{E'} \cap Z_{E'}$ is nonempty by construction of $X_{E'}$. Hence $\lim T_{E'} \cap Z_{E'}$ is nonempty and we conclude that $T \cap Z$ is nonempty. Thus y is not a closed point. It follows that every closed point of X_w is in Z.

Suppose that $y \in X_w$ specializes to $z, z' \in Z$. We will show that $z = z'$ which will finish the proof of (3) and will imply (5). Let $x, x' \in X$ be the images of z and z'. Since $Z \to X$ is bijective it suffices to show that $x = x'$. If $x \neq x'$, then there exists an $f \in A$ such that $x \in D(f)$ and $x' \in V(f)$ (or vice versa). Set $E = \{f\}$ so that

$$X_E = \text{Spec}(A_f) \amalg \text{Spec}(A^\sim_{V(f)})$$

Then we see that z and z' map x_E and x'_E which are in different parts of the given decomposition of X_E above. But then it impossible for x_E and x'_E to be specializations of a common point. This is the desired contradiction.

Recall that given a finite subset $E \subset A$ we have Z_E is a disjoint union of the locally closed subschemes $Z(E', E'')$ each isomorphic to the spectrum of $(A/I)_f$ where I is the ideal generated by E'' and f the product of the elements of E'. Any nilpotent element b of $(A/I)_f$ is the class of g/f^n for some $g \in A$. Then setting $E' = E \cup \{g\}$ the reader verifies that b is pulls back to zero under the transition map $Z_{E'} \to Z_E$ of the system. This proves (4). $\square$

Remark 5.4. Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of A_w (Lemma 5.3) is at most κ. Namely, each A_E has cardinality at most κ and the set of finite subsets of A has cardinality at most κ as well. Thus the result follows as $\kappa \otimes \kappa = \kappa$, see Sets, Section 6.

Lemma 5.5 (Universal property of the construction)**.** *Let A be a ring. Let $A \to A_w$ be the ring map constructed in Lemma 5.3. For any ring map $A \to B$ such that $\text{Spec}(B)$ is w-local, there is a unique factorization $A \to A_w \to B$ such that $\text{Spec}(B) \to \text{Spec}(A_w)$ is w-local.*

Proof. Denote $Y = \text{Spec}(B)$ and $Y_0 \subset Y$ the set of closed points. Denote $f : Y \to X$ the given morphism. Recall that Y_0 is profinite, in particular every constructible subset of Y_0 is open and closed. Let $E \subset A$ be a finite subset. Recall that $A_w = \text{colim}\, A_E$ and that the set of closed points of $\text{Spec}(A_w)$ is the limit of the closed subsets $Z_E \subset X_E = \text{Spec}(A_E)$. Thus it suffices to show there is a unique factorization $A \to A_E \to B$ such that $Y \to X_E$ maps Y_0 into Z_E. Since $Z_E \to X = \text{Spec}(A)$ is bijective, and since the strata $Z(E', E'')$ are constructible we see that

$$Y_0 = \coprod f^{-1}(Z(E', E'')) \cap Y_0$$

is a disjoint union decomposition into open and closed subsets. As $Y_0 = \pi_0(Y)$ we obtain a corresponding decomposition of Y into open and closed pieces. Thus it suffices to construct the factorization in case $f(Y_0) \subset Z(E', E'')$ for some decomposition $E = E' \amalg E''$. In this case $f(Y)$ is contained in the set of points of X specializing to $Z(E', E'')$ which is homeomorphic to $X_{E',E''}$. Thus we obtain a unique continuous map $Y \to X_{E',E''}$ over X. By Lemma 3.7 this corresponds to a unique morphism of schemes $Y \to X_{E',E''}$ over X. This finishes the proof. □

Recall that the spectrum of a ring is profinite if and only if every point is closed. There are in fact a whole slew of equivalent conditions that imply this. See Algebra, Lemma 25.5 or Topology, Lemma 22.7.

Lemma 5.6. *Let A be a ring such that $\text{Spec}(A)$ is profinite. Let $A \to B$ be a ring map. Then $\text{Spec}(B)$ is profinite in each of the following cases:*

1. *if $\mathfrak{q}, \mathfrak{q}' \subset B$ lie over the same prime of A, then neither $\mathfrak{q} \subset \mathfrak{q}'$, nor $\mathfrak{q}' \subset \mathfrak{q}$,*
2. *$A \to B$ induces algebraic extensions of residue fields,*
3. *$A \to B$ is a local isomorphism,*
4. *$A \to B$ identifies local rings,*
5. *$A \to B$ is weakly étale,*
6. *$A \to B$ is quasi-finite,*
7. *$A \to B$ is unramified,*
8. *$A \to B$ is étale,*
9. *B is a filtered colimit of A-algebras as in (1) – (8),*
10. *etc.*

Proof. By the references mentioned above (Algebra, Lemma 25.5 or Topology, Lemma 22.7) there are no specializations between distinct points of $\text{Spec}(A)$ and $\text{Spec}(B)$ is profinite if and only if there are no specializations between distinct points of $\text{Spec}(B)$. These specializations can only happen in the fibres of $\text{Spec}(B) \to \text{Spec}(A)$. In this way we see that (1) is true.

The assumption in (2) implies all primes of B are maximal by Algebra, Lemma 34.9. Thus (2) holds. If $A \to B$ is a local isomorphism or identifies local rings, then the residue field extensions are trivial, so (3) and (4) follow from (2). If $A \to B$ is weakly étale, then More on Algebra, Lemma 67.16 tells us it induces separable algebraic residue field extensions, so (5) follows from (2). If $A \to B$ is quasi-finite, then the fibres are finite discrete topological spaces. Hence (6) follows from (1).

Hence (3) follows from (1). Cases (7) and (8) follow from this as unramified and étale ring map are quasi-finite (Algebra, Lemmas 144.6 and 138.6). If $B = \operatorname{colim} B_i$ is a filtered colimit of A-algebras, then $\operatorname{Spec}(B) = \operatorname{colim} \operatorname{Spec}(B_i)$, hence if each $\operatorname{Spec}(B_i)$ is profinite, so is $\operatorname{Spec}(B)$. This proves (9). □

Lemma 5.7. *Let A be a ring. Let $V(I) \subset \operatorname{Spec}(A)$ be a closed subset which is a profinite topological space. Then there exists an ind-Zariski ring map $A \to B$ such that $\operatorname{Spec}(B)$ is w-local, the set of closed points is $V(IB)$, and $A/I \cong B/IB$.*

Proof. Let $A \to A_w$ and $Z \subset Y = \operatorname{Spec}(A_w)$ as in Lemma 5.3. Let $T \subset Z$ be the inverse image of $V(I)$. Then $T \to V(I)$ is a homeomorphism by Topology, Lemma 16.8. Let $B = (A_w)^\sim_T$, see Lemma 5.1. It is clear that B is w-local with closed points $V(IB)$. The ring map $A/I \to B/IB$ is ind-Zariski and induces a homeomorphism on underlying topological spaces. Hence it is an isomorphism by Lemma 3.8. □

Lemma 5.8. *Let A be a ring such that $X = \operatorname{Spec}(A)$ is w-local. Let $I \subset A$ be the radical ideal cutting out the set X_0 of closed points in X. Let $A \to B$ be a ring map inducing algebraic extensions on residue fields at primes. Then*

1. *every point of $Z = V(IB)$ is a closed point of $\operatorname{Spec}(B)$,*
2. *there exists an ind-Zariski ring map $B \to C$ such that*
   - (a) *$B/IB \to C/IC$ is an isomorphism,*
   - (b) *the space $Y = \operatorname{Spec}(C)$ is w-local,*
   - (c) *the induced map $p : Y \to X$ is w-local, and*
   - (d) *$p^{-1}(X_0)$ is the set of closed points of Y.*

Proof. By Lemma 5.6 applied to $A/I \to B/IB$ all points of $Z = V(IB) = \operatorname{Spec}(B/IB)$ are closed, in fact $\operatorname{Spec}(B/IB)$ is a profinite space. To finish the proof we apply Lemma 5.7 to $IB \subset B$. □

6. Identifying local rings versus ind-Zariski

An ind-Zariski ring map $A \to B$ identifies local rings (Lemma 4.6). The converse does not hold (Examples, Section 37). However, it turns out that there is a kind of structure theorem for ring maps which identify local rings in terms of ind-Zariski ring maps, see Proposition 6.6.

Let A be a ring. Let $X = \operatorname{Spec}(A)$. The space of connected components $\pi_0(X)$ is a profinite space by Topology, Lemma 22.8 (and Algebra, Lemma 25.2).

Lemma 6.1. *Let A be a ring. Let $X = \operatorname{Spec}(A)$. Let $T \subset \pi_0(X)$ be a closed subset. There exists a surjective ind-Zariski ring map $A \to B$ such that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ induces a homeomorphism of $\operatorname{Spec}(B)$ with the inverse image of T in X.*

Proof. Let $Z \subset X$ be the inverse image of T. Then Z is the intersection $Z = \bigcap Z_\alpha$ of the open and closed subsets of X containing Z, see Topology, Lemma 11.12. For each α we have $Z_\alpha = \operatorname{Spec}(A_\alpha)$ where $A \to A_\alpha$ is a local isomorphism (a localization at an idempotent). Setting $B = \operatorname{colim} A_\alpha$ proves the lemma. □

Lemma 6.2. *Let A be a ring and let $X = \operatorname{Spec}(A)$. Let T be a profinite space and let $T \to \pi_0(X)$ be a continuous map. There exists an ind-Zariski ring map $A \to B$*

such that with $Y = \mathrm{Spec}(B)$ *the diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \pi_0(Y) \arrow[d] \\ X \arrow[r] & \pi_0(X) \end{tikzcd}$$

is cartesian in the category of topological spaces and such that $\pi_0(Y) = T$ *as spaces over* $\pi_0(X)$.

Proof. Namely, write $T = \lim T_i$ as the limit of an inverse system finite discrete spaces over a directed partially ordered set (see Topology, Lemma 21.2). For each i let $Z_i = \mathrm{Im}(T \to \pi_0(X) \times T_i)$. This is a closed subset. Observe that $X \times T_i$ is the spectrum of $A_i = \prod_{t \in T_i} A$ and that $A \to A_i$ is a local isomorphism. By Lemma 6.1 we see that $Z_i \subset \pi_0(X \times T_i) = \pi_0(X) \times T_i$ corresponds to a surjection $A_i \to B_i$ which is ind-Zariski such that $\mathrm{Spec}(B_i) = X \times_{\pi_0(X)} Z_i$ as subsets of $X \times T_i$. The transition maps $T_i \to T_{i'}$ induce maps $Z_i \to Z_{i'}$ and $X \times_{\pi_0(X)} Z_i \to X \times_{\pi_0(X)} Z_{i'}$. Hence ring maps $B_{i'} \to B_i$ (Lemmas 3.8 and 4.6). Set $B = \mathrm{colim}\, B_i$. Because $T = \lim Z_i$ we have $X \times_{\pi_0(X)} T = \lim X \times_{\pi_0(X)} Z_i$ and hence $Y = \mathrm{Spec}(B) = \lim \mathrm{Spec}(B_i)$ fits into the cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & \pi_0(X) \end{tikzcd}$$

of topological spaces. By Lemma 2.5 we conclude that $T = \pi_0(Y)$. □

Example 6.3. Let k be a field. Let T be a profinite topological space. There exists an ind-Zariski ring map $k \to A$ such that $\mathrm{Spec}(A)$ is homeomorphic to T. Namely, just apply Lemma 6.2 to $T \to \pi_0(\mathrm{Spec}(k)) = \{*\}$. In fact, in this case we have

$$A = \mathrm{colim}\, \mathrm{Map}(T_i, k)$$

whenever we write $T = \lim T_i$ as a filtered limit with each T_i finite.

Lemma 6.4. *Let* $A \to B$ *be ring map such that*

1. $A \to B$ *identifies local rings,*
2. *the topological spaces* $\mathrm{Spec}(B)$, $\mathrm{Spec}(A)$ *are w-local,*
3. $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is w-local, and*
4. $\pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$ *is bijective.*

Then $A \to B$ *is an isomorphism*

Proof. Let $X_0 \subset X = \mathrm{Spec}(A)$ and $Y_0 \subset Y = \mathrm{Spec}(B)$ be the sets of closed points. By assumption Y_0 maps into X_0 and the induced map $Y_0 \to X_0$ is a bijection. As a space $\mathrm{Spec}(A)$ is the disjoint union of the spectra of the local rings of A at closed points. Similarly for B. Hence $X \to Y$ is a bijection. Since $A \to B$ is flat we have going down (Algebra, Lemma 38.17). Thus Algebra, Lemma 40.11 shows for any prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ we have $B_\mathfrak{q} = B_\mathfrak{p}$. Since $B_\mathfrak{q} = A_\mathfrak{p}$ by assumption, we see that $A_\mathfrak{p} = B_\mathfrak{p}$ for all primes $\mathfrak{p}$ of A. Thus $A = B$ by Algebra, Lemma 23.1. □

Lemma 6.5. *Let* $A \to B$ *be ring map such that*

1. $A \to B$ *identifies local rings,*
2. *the topological spaces* $\mathrm{Spec}(B)$, $\mathrm{Spec}(A)$ *are w-local, and*

(3) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is w-local.*

Then $A \to B$ is ind-Zariski.

Proof. Set $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Let $X_0 \subset X$ and $Y_0 \subset Y$ be the set of closed points. Let $A \to A'$ be the ind-Zariski morphism of affine schemes such that with $X' = \mathrm{Spec}(A')$ the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \pi_0(X') \arrow[d] \\ X \arrow[r] & \pi_0(X) \end{tikzcd}$$

is cartesian in the category of topological spaces and such that $\pi_0(X') = \pi_0(Y)$ as spaces over $\pi_0(X)$, see Lemma 6.2. By Lemma 2.5 we see that X' is w-local and the set of closed points $X'_0 \subset X'$ is the inverse image of X_0.

We obtain a continuous map $Y \to X'$ of underlying topological spaces over X identifying $\pi_0(Y)$ with $\pi_0(X')$. By Lemma 3.8 (and Lemma 4.6) this is corresponds to a morphism of affine schemes $Y \to X'$ over X. Since $Y \to X$ maps Y_0 into X_0 we see that $Y \to X'$ maps Y_0 into X'_0, i.e., $Y \to X'$ is w-local. By Lemma 6.4 we see that $Y \cong X'$ and we win. □

The following proposition is a warm up for the type of result we will prove later.

Proposition 6.6. *Let $A \to B$ be a ring map which identifies local rings. Then there exists a faithfully flat, ind-Zariski ring map $B \to B'$ such that $A \to B'$ is ind-Zariski.*

Proof. Let $A \to A_w$, resp. $B \to B_w$ be the faithfully flat, ind-Zariski ring map constructed in Lemma 5.3 for A, resp. B. Since $\mathrm{Spec}(B_w)$ is w-local, there exists a unique factorization $A \to A_w \to B_w$ such that $\mathrm{Spec}(B_w) \to \mathrm{Spec}(A_w)$ is w-local by Lemma 5.5. Note that $A_w \to B_w$ identifies local rings, see Lemma 3.4. By Lemma 6.5 this means $A_w \to B_w$ is ind-Zariski. Since $B \to B_w$ is faithfully flat, ind-Zariski (Lemma 5.3) and the composition $A \to B \to B_w$ is ind-Zariski (Lemma 4.3) the proposition is proved. □

The proposition above allows us to characterize the affine, weakly contractible objects in the pro-Zariski site of an affine scheme.

Lemma 6.7. *Let A be a ring. The following are equivalent*

(1) *every faithfully flat ring map $A \to B$ identifying local rings has a section,*
(2) *every faithfully flat ind-Zariski ring map $A \to B$ has a section, and*
(3) *A satisfies*
    (a) $\mathrm{Spec}(A)$ *is w-local, and*
    (b) $\pi_0(\mathrm{Spec}(A))$ *is extremally disconnected.*

Proof. The equivalence of (1) and (2) follows immediately from Proposition 6.6.

Assume (3)(a) and (3)(b). Let $A \to B$ be faithfully flat and ind-Zariski. We will use without further mention the fact that a flat map $A \to B$ is faithfully flat if and only if every closed point of $\mathrm{Spec}(A)$ is in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ We will show that $A \to B$ has a section.

Let $I \subset A$ be an ideal such that $V(I) \subset \mathrm{Spec}(A)$ is the set of closed points of $\mathrm{Spec}(A)$. We may replace B by the ring C constructed in Lemma 5.8 for $A \to B$

and $I \subset A$. Thus we may assume $\text{Spec}(B)$ is w-local such that the set of closed points of $\text{Spec}(B)$ is $V(IB)$.

Assume $\text{Spec}(B)$ is w-local and the set of closed points of $\text{Spec}(B)$ is $V(IB)$. Choose a continuous section to the surjective continuous map $V(IB) \to V(I)$. This is possible as $V(I) \cong \pi_0(\text{Spec}(A))$ is extremally disconnected, see Topology, Proposition 25.6. The image is a closed subspace $T \subset \pi_0(\text{Spec}(B)) \cong V(JB)$ mapping homeomorphically onto $\pi_0(A)$. Replacing B by the ind-Zariski quotient ring constructed in Lemma 6.1 we see that we may assume $\pi_0(\text{Spec}(B)) \to \pi_0(\text{Spec}(A))$ is bijective. At this point $A \to B$ is an isomorphism by Lemma 6.4.

Assume (1) or equivalently (2). Let $A \to A_w$ be the ring map constructed in Lemma 5.3. By (1) there is a section $A_w \to A$. Thus $\text{Spec}(A)$ is homeomorphic to a closed subset of $\text{Spec}(A_w)$. By Lemma 2.4 we see (3)(a) holds. Finally, let $T \to \pi_0(A)$ be a surjective map with T an extremally disconnected, quasi-compact, Hausdorff topological space (Topology, Lemma 25.9). Choose $A \to B$ as in Lemma 6.2 adapted to $T \to \pi_0(\text{Spec}(A))$. By (1) there is a section $B \to A$. Thus we see that $T = \pi_0(\text{Spec}(B)) \to \pi_0(\text{Spec}(A))$ has a section. A formal categorical argument, using Topology, Proposition 25.6, implies that $\pi_0(\text{Spec}(A))$ is extremally disconnected. □

Lemma 6.8. *Let A be a ring. There exists a faithfully flat, ind-Zariski ring map $A \to B$ such that B satisfies the equivalent conditions of Lemma 6.7.*

Proof. We first apply Lemma 5.3 to see that we may assume that $\text{Spec}(A)$ is w-local. Choose an extremally disconnected space T and a surjective continuous map $T \to \pi_0(\text{Spec}(A))$, see Topology, Lemma 25.9. Note that T is profinite. Apply Lemma 6.2 to find an ind-Zariski ring map $A \to B$ such that $\pi_0(\text{Spec}(B)) \to \pi_0(\text{Spec}(A))$ realizes $T \to \pi_0(\text{Spec}(A))$ and such that

$$\begin{tikzcd} \text{Spec}(B) \arrow[r] \arrow[d] & \pi_0(\text{Spec}(B)) \arrow[d] \\ \text{Spec}(A) \arrow[r] & \pi_0(\text{Spec}(A)) \end{tikzcd}$$

is cartesian in the category of topological spaces. Note that $\text{Spec}(B)$ is w-local, that $\text{Spec}(B) \to \text{Spec}(A)$ is w-local, and that the set of closed points of $\text{Spec}(B)$ is the inverse image of the set of closed points of $\text{Spec}(A)$, see Lemma 2.5. Thus condition (3) of Lemma 6.7 holds for B. □

Remark 6.9. In each of Lemmas 6.1, 6.2, Proposition 6.6, and Lemma 6.8 we find an ind-Zariski ring map with some properties. In the paper [BS13] the authors use the notion of an ind-(Zariski localization) which is a filtered colimit of finite products of principal localizations. It is possible to replace ind-Zariski by ind-(Zariski localization) in each of the results listed above. However, we do not need this and the notion of an ind-Zariski homomorphism of rings as defined here has slightly better formal properties. Moreover, the notion of an ind-Zariski ring map is the natural analogue of the notion of an ind-étale ring map defined in the next section.

7. Ind-étale algebra

We start with a definition.

Definition 7.1. A ring map $A \to B$ is said to be *ind-étale* if B can be written as a filtered colimit of étale A-algebras.

The category of ind-étale algebras is closed under a number of natural operations.

Lemma 7.2. *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B. If $A \to B$ is ind-étale, then $A' \to B'$ is ind-étale.*

Proof. Omitted. $\square$

Lemma 7.3. *Let $A \to B$ and $B \to C$ be ring maps. If $A \to B$ and $B \to C$ are ind-étale, then $A \to C$ is ind-étale.*

Proof. Omitted. $\square$

Lemma 7.4. *A filtered colimit of ind-étale A-algebras is ind-étale over A.*

Proof. Omitted. $\square$

Lemma 7.5. *Let A be a ring. Let $B \to C$ be an A-algebra map of ind-étale A-algebras. Then C is an ind-étale B-algebra.*

Proof. Write $B = \operatorname{colim} B_i$ and $C = \operatorname{colim} C_j$ as filtered colimits of étale A-algebras. Then

$$C = B \otimes_B C = \operatorname{colim}_{(i,j)} B \otimes_{B_i} C_j$$

where the colimit is over the partially ordered set of pairs (i, j) such that $B_i \to B \to C$ factors through $C_j \to C$. Note that the factorization $B_i \to C_j$ is étale by Algebra, Lemma 138.9. Some details omitted. $\square$

Lemma 7.6. *Let $A \to B$ be ind-étale. Then $A \to B$ is weakly étale (More on Algebra, Definition 67.1).*

Proof. This follows from More on Algebra, Lemma 67.13. $\square$

Lemma 7.7. *Let A be a ring and let $I \subset A$ be an ideal. The base change functor*

$$\textit{ind-étale } A\textit{-algebras} \longrightarrow \textit{ind-étale } A/I\textit{-algebras}, \quad C \longmapsto C/IC$$

has a fully faithful right adjoint v. In particular, given an ind-étale A/I-algebra $\overline{C}$ there exists an ind-étale A-algebra $C = v(\overline{C})$ such that $\overline{C} = C/IC$.

Proof. Let $\overline{C}$ be an ind-étale A/I-algebra. Consider the category $\mathcal{C}$ of factorizations $A \to B \to \overline{C}$ where $A \to B$ is étale. (We ignore some set theoretical issues in this proof.) We will show that this category is directed and that $C = \operatorname{colim}_{\mathcal{C}} B$ is an ind-étale A-algebra such that $\overline{C} = C/IC$.

We first prove that $\mathcal{C}$ is directed (Categories, Definition 19.1). The category is nonempty as $A \to A \to \overline{C}$ is an object. Suppose that $A \to B \to \overline{C}$ and $A \to B' \to \overline{C}$ are two objects of $\mathcal{C}$. Then $A \to B \otimes_A B' \to \overline{C}$ is another (use Algebra, Lemma 138.3). Suppose that $f, g : B \to B'$ are two maps between objects $A \to B \to \overline{C}$ and $A \to B' \to \overline{C}$ of $\mathcal{C}$. Then a coequalizer is $A \to B' \otimes_{f,B,g} B' \to \overline{C}$. This is an object of $\mathcal{C}$ by Algebra, Lemmas 138.3 and 138.9. Thus the category $\mathcal{C}$ is directed.

Write $\overline{C} = \operatorname{colim} \overline{B_i}$ as a filtered colimit with $\overline{B_i}$ étale over A/I. For every i there exists $A \to B_i$ étale with $\overline{B_i} = B_i/IB_i$, see Algebra, Lemma 138.11. Thus $C \to \overline{C}$ is surjective. Since $C/IC \to \overline{C}$ is ind-étale (Lemma 7.5) we see that it is flat. Hence $\overline{C}$ is a localization of C/IC at some multiplicative subset $S \subset C/IC$ (Algebra, Lemma 104.2). Take an $f \in C$ mapping to an element of $S \subset C/IC$. Choose $A \to B \to \overline{C}$ in $\mathcal{C}$ and $g \in B$ mapping to f in the colimit. Then we see that $A \to B_g \to \overline{C}$ is an object of $\mathcal{C}$ as well. Thus f is an invertible element of C. It follows that $C/IC = \overline{C}$.

Next, we claim that for an ind-étale algebra D over A we have

$$\mathrm{Mor}_A(D, C) = \mathrm{Mor}_{A/I}(D/ID, \overline{C})$$

Namely, let $D/ID \to \overline{C}$ be an A/I-algebra map. Write $D = \mathrm{colim}_{i \in I} D_i$ as a filtered colimit over a partially ordered set I with D_i étale over A. By choice of $\mathcal{C}$ we obtain a transformation $I \to \mathcal{C}$ and hence a map $D \to C$ compatible with maps to $\overline{C}$. Whence the claim.

It follows that the functor v defined by the rule

$$\overline{C} \longmapsto v(\overline{C}) = \mathrm{colim}_{A \to B \to \overline{C}} B$$

is a right adjoint to the base change functor u as required by the lemma. The functor v is fully faithful because $u \circ v = \mathrm{id}$ by construction, see Categories, Lemma 24.3. □

8. Constructing ind-étale algebras

Let A be a ring. Recall that any étale ring map $A \to B$ is isomorphic to a standard smooth ring map of relative dimension 0. Such a ring map is of the form

$$A \longrightarrow A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

where the determinant of the $n \times n$-matrix with entries $\partial f_i / \partial x_j$ is invertible in the quotient ring. See Algebra, Lemma 138.2.

Let $S(A)$ be the set of all *faithfully flat*[1] standard smooth A-algebras of relative dimension 0. Let $I(A)$ be the partially ordered (by inclusion) set of finite subsets E of $S(A)$. Note that $I(A)$ is a directed partially ordered set. For $E = \{A \to B_1, \ldots, A \to B_n\}$ set

$$B_E = B_1 \otimes_A \ldots \otimes_A B_n$$

Observe that B_E is a faithfully flat étale A-algebra. For $E \subset E'$, there is a canonical transition map $B_E \to B_{E'}$ of étale A-algebras. Namely, say $E = \{A \to B_1, \ldots, A \to B_n\}$ and $E' = \{A \to B_1, \ldots, A \to B_{n+m}\}$ then $B_E \to B_{E'}$ sends $b_1 \otimes \ldots \otimes b_n$ to the element $b_1 \otimes \ldots \otimes b_n \otimes 1 \otimes \ldots \otimes 1$ of $B_{E'}$. This construction defines a system of faithfully flat étale A-algebras over $I(A)$ and we set

$$T(A) = \mathrm{colim}_{E \in I(A)} B_E$$

Observe that $T(A)$ is a faithfully flat ind-étale A-algebra (Algebra, Lemma 38.20). By construction given any faithfully flat étale A-algebra B there is a (non-unique) A-algebra map $B \to T(A)$. Namely, pick some $(A \to B_0) \in S(A)$ and an isomorphism $B \cong B_0$. Then the canonical coprojection

$$B \to B_0 \to T(A) = \mathrm{colim}_{E \in I(A)} B_E$$

is the desired map.

[1] In the presence of flatness, e.g., for smooth or étale ring maps, this just means that the induced map on spectra is surjective. See Algebra, Lemma 38.15.

Lemma 8.1. *Given a ring A there exists a faithfully flat ind-étale A-algebra C such that every faithfully flat étale ring map $C \to B$ has a section.*

Proof. Set $T^1(A) = T(A)$ and $T^{n+1}(A) = T(T^n(A))$. Let

$$C = \operatorname{colim} T^n(A)$$

This algebra is faithfully flat over each $T^n(A)$ and in particular over A, see Algebra, Lemma 38.20. Moreover, C is ind-étale over A by Lemma 7.4. If $C \to B$ is étale, then there exists an n and an étale ring map $T^n(A) \to B'$ such that $B = C \otimes_{T^n(A)} B'$, see Algebra, Lemma 138.3. If $C \to B$ is faithfully flat, then $\operatorname{Spec}(B) \to \operatorname{Spec}(C) \to \operatorname{Spec}(T^n(A))$ is surjective, hence $\operatorname{Spec}(B') \to \operatorname{Spec}(T^n(A))$ is surjective. In other words, $T^n(A) \to B'$ is faithfully flat. By our construction, there is a $T^n(A)$-algebra map $B' \to T^{n+1}(A)$. This induces a C-algebra map $B \to C$ which finishes the proof. $\square$

Remark 8.2. Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of $T(A)$ is at most κ. Namely, each B_E has cardinality at most κ and the index set $I(A)$ has cardinality at most κ as well. Thus the result follows as $\kappa \otimes \kappa = \kappa$, see Sets, Section 6. It follows that the ring constructed in the proof of Lemma 8.1 has cardinality at most κ as well.

Remark 8.3. The construction $A \mapsto T(A)$ is functorial in the following sense: If $A \to A'$ is a ring map, then we can construct a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & T(A) \arrow[d] \\ A' \arrow[r] & T(A') \end{tikzcd}$$

Namely, given $(A \to A[x_1, \ldots, x_n]/(f_1, \ldots, f_n))$ in $S(A)$ we can use the ring map $\varphi : A \to A'$ to obtain a corresponding element $(A' \to A'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_n^\varphi))$ of $S(A')$ where f^φ means the polynomial obtained by applying φ to the coefficients of the polynomial f. Moreover, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \arrow[d] \\ A' \arrow[r] & A'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_n^\varphi) \end{tikzcd}$$

which is a in the category of rings. For $E \subset S(A)$ finite, set $E' = \varphi(E)$ and define $B_E \to B_{E'}$ in the obvious manner. Taking the colimit gives the desired map $T(A) \to T(A')$, see Categories, Lemma 14.7.

Lemma 8.4. *Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a section. Then the same is true for every quotient ring A/I.*

Proof. Omitted. $\square$

Lemma 8.5. *Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a section. Then every local ring of A at a maximal ideal is strictly henselian.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of A. Let $A \to B$ be an étale ring map and let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{m}$. By the description of the strict henselization $A_{\mathfrak{m}}^{sh}$ in Algebra, Lemma 145.27 it suffices to show that $A_{\mathfrak{m}} = B_{\mathfrak{q}}$. Note that there are finitely many primes $\mathfrak{q} = \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_n$ lying over $\mathfrak{m}$ and there are no specializations between them as an étale ring map is quasi-finite, see Algebra, Lemma 138.6. Thus $\mathfrak{q}_i$ is a maximal ideal and we can find $g \in \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_n$, $g \notin \mathfrak{q}$ (Algebra, Lemma 14.2). After replacing B by B_g we see that $\mathfrak{q}$ is the only prime of B lying over $\mathfrak{m}$. The image $U \subset \mathrm{Spec}(A)$ of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is open (Algebra, Proposition 40.8). Thus the complement $\mathrm{Spec}(A) \setminus U$ is closed and we can find $f \in A$, $f \notin \mathfrak{p}$ such that $\mathrm{Spec}(A) = U \cup D(f)$. The ring map $A \to B \times A_f$ is faithfully flat and étale, hence has a section $\sigma : B \times A_f \to A$ by assumption on A. Observe that σ is étale, hence flat as a map between étale A-algebras (Algebra, Lemma 138.9). Since $\mathfrak{q}$ is the only prime of $B \times A_f$ lying over A we find that $A_\mathfrak{p} \to B_\mathfrak{q}$ has a section which is also flat. Thus $A_\mathfrak{p} \to B_\mathfrak{q} \to A_\mathfrak{p}$ are flat local ring maps whose composition is the identity. Since a flat local homomorphism of local rings is injective we conclude these maps are isomorphisms as desired. □

Lemma 8.6. *Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a section. Let $Z \subset \mathrm{Spec}(A)$ be a closed subscheme of the form $D(f) \cap V(I)$ and let $A \to A_Z^\sim$ be as constructed in Lemma 5.1. Then every faithfully flat étale ring map $A_Z^\sim \to C$ has a section.*

Proof. There exists an étale ring map $A \to B'$ such that $C = B' \otimes_A A_Z^\sim$ as $A_Z^\sim$-algebras. The image $U' \subset \mathrm{Spec}(A)$ of $\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is open and contains $V(I)$, hence we can find $f \in I$ such that $\mathrm{Spec}(A) = U' \cup D(f)$. Then $A \to B' \times A_f$ is étale and faithfully flat. By assumption there is a section $B' \times A_f \to A$. Localizing we obtain the desired section $C \to A_Z^\sim$. □

Lemma 8.7. *Let $A \to B$ be a ring map inducing algebraic extensions on residue fields. There exists a commutative diagram*

$$\begin{tikzcd} B \arrow[r] & D \\ A \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

with the following properties:

(1) *$A \to C$ is faithfully flat and ind-étale,*
(2) *$B \to D$ is faithfully flat and ind-étale,*
(3) $\mathrm{Spec}(C)$ *is w-local,*
(4) $\mathrm{Spec}(D)$ *is w-local,*
(5) $\mathrm{Spec}(D) \to \mathrm{Spec}(C)$ *is w-local,*
(6) *the set of closed points of* $\mathrm{Spec}(D)$ *is the inverse image of the set of closed points of* $\mathrm{Spec}(C)$,
(7) *the set of closed points of* $\mathrm{Spec}(C)$ *surjects onto* $\mathrm{Spec}(A)$,
(8) *the set of closed points of* $\mathrm{Spec}(D)$ *surjects onto* $\mathrm{Spec}(B)$,
(9) *for $\mathfrak{m} \subset C$ maximal the local ring $C_\mathfrak{m}$ is strictly henselian.*

Proof. There is a faithfully flat, ind-Zariski ring map $A \to A'$ such that $\mathrm{Spec}(A')$ is w-local and such that the set of closed points of $\mathrm{Spec}(A')$ maps onto $\mathrm{Spec}(A)$, see Lemma 5.3. Let $I \subset A'$ be the ideal such that $V(I)$ is the set of closed points of $\mathrm{Spec}(A')$. Choose $A' \to C'$ as in Lemma 8.1. Note that the local rings $C'_{\mathfrak{m}'}$ at maximal ideals $\mathfrak{m}' \subset C'$ are strictly henselian by Lemma 8.5. We apply Lemma 5.8 to $A' \to C'$ and $I \subset A'$ to get $C' \to C$ with $C'/IC' \cong C/IC$. Note that since $A' \to C'$ is faithfully flat, $\mathrm{Spec}(C'/IC')$ surjects onto the set of closed points of

A' and in particular onto $\operatorname{Spec}(A)$. Moreover, as $V(IC) \subset \operatorname{Spec}(C)$ is the set of closed points of C and $C' \to C$ is ind-Zariski (and identifies local rings) we obtain properties (1), (3), (7), and (9).

Denote $J \subset C$ the ideal such that $V(J)$ is the set of closed points of $\operatorname{Spec}(C)$. Set $D' = B \otimes_A C$. The ring map $C \to D'$ induces algebraic residue field extensions. Keep in mind that since $V(J) \to \operatorname{Spec}(A)$ is surjective the map $T = V(JD) \to \operatorname{Spec}(B)$ is surjective too. Apply Lemma 5.8 to $C \to D'$ and $J \subset C$ to get $D' \to D$ with $D'/JD' \cong D/JD$. All of the remaining properties given in the lemma are immediate from the results of Lemma 5.8. $\square$

9. Weakly étale versus pro-étale

Recall that a ring homomorphism $A \to B$ is *weakly étale* if $A \to B$ is flat and $B \otimes_A B \to B$ is flat. We have proved some properties of such ring maps in More on Algebra, Section 67. In particular, if $A \to B$ is a local homomorphism, and A is a strictly henselian local rings, then $A = B$, see More on Algebra, Theorem 67.24. Using this theorem and the work we've done above we obtain the following structure theorem for weakly étale ring maps.

Proposition 9.1. *Let $A \to B$ be a weakly étale ring map. Then there exists a faithfully flat, ind-étale ring map $B \to B'$ such that $A \to B'$ is ind-étale.*

Proof. The ring map $A \to B$ induces (separable) algebraic extensions of residue fields, see More on Algebra, Lemma 67.16. Thus we may apply Lemma 8.7 and choose a diagram

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

with the properties as listed in the lemma. Note that $C \to D$ is weakly étale by More on Algebra, Lemma 67.11. Pick a maximal ideal $\mathfrak{m} \subset D$. By construction this lies over a maximal ideal $\mathfrak{m}' \subset C$. By More on Algebra, Theorem 67.24 the ring map $C_{\mathfrak{m}'} \to D_{\mathfrak{m}}$ is an isomorphism. As every point of $\operatorname{Spec}(C)$ specializes to a closed point we conclude that $C \to D$ identifies local rings. Thus Proposition 6.6 applies to the ring map $C \to D$. Pick $D \to D'$ faithfully flat and ind-Zariski such that $C \to D'$ is ind-Zariski. Then $B \to D'$ is a solution to the problem posed in the proposition. $\square$

10. Constructing w-contractible covers

In this section we construct w-contractible covers of affine schemes.

Definition 10.1. Let A be a ring. We say A is *w-contractible* if every faithfully flat weakly-etale ring map $A \to B$ has a section.

We remark that by Proposition 9.1 an equivalent definition would be to ask that every faithfully flat, ind-étale ring map $A \to B$ has a section. Here is a key observation that will allow us to construct w-contractible rings.

Lemma 10.2. *Let A be a ring. The following are equivalent*

(1) *A is w-contractible,*
(2) *every faithfully flat, ind-étale ring map $A \to B$ has a section, and*

(3) *A satisfies*
   (a) $\mathrm{Spec}(A)$ *is w-local,*
   (b) $\pi_0(\mathrm{Spec}(A))$ *is extremally disconnected, and*
   (c) *for every maximal ideal $\mathfrak{m} \subset A$ the local ring $A_\mathfrak{m}$ is strictly henselian.*

Proof. The equivalence of (1) and (2) follows immediately from Proposition 9.1.

Assume (3)(a), (3)(b), and (3)(c). Let $A \to B$ be faithfully flat and ind-étale. We will use without further mention the fact that a flat map $A \to B$ is faithfully flat if and only if every closed point of $\mathrm{Spec}(A)$ is in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ We will show that $A \to B$ has a section.

Let $I \subset A$ be an ideal such that $V(I) \subset \mathrm{Spec}(A)$ is the set of closed points of $\mathrm{Spec}(A)$. We may replace B by the ring C constructed in Lemma 5.8 for $A \to B$ and $I \subset A$. Thus we may assume $\mathrm{Spec}(B)$ is w-local such that the set of closed points of $\mathrm{Spec}(B)$ is $V(IB)$. In this case $A \to B$ identifies local rings by condition (3)(c) as it suffices to check this at maximal ideals of B which lie over maximal ideals of A. Thus $A \to B$ has a section by Lemma 6.7.

Assume (1) or equivalently (2). We have (3)(c) by Lemma 8.5. Properties (3)(a) and (3)(b) follow from Lemma 6.7. □

Proposition 10.3. *For every ring A there exists a faithfully flat, ind-étale ring map $A \to D$ such that D is w-contractible.*

Proof. Applying Lemma 8.7 to $\mathrm{id}_A : A \to A$ we find a faithfully flat, ind-étale ring map $A \to C$ such that C is w-local and such that every local ring at a maximal ideal of C is strictly henselian. Choose an extremally disconnected space T and a surjective continuous map $T \to \pi_0(\mathrm{Spec}(C))$, see Topology, Lemma 25.9. Note that T is profinite. Apply Lemma 6.2 to find an ind-Zariski ring map $C \to D$ such that $\pi_0(\mathrm{Spec}(D)) \to \pi_0(\mathrm{Spec}(C))$ realizes $T \to \pi_0(\mathrm{Spec}(C))$ and such that

$$\begin{tikzcd} \mathrm{Spec}(D) \arrow[r] \arrow[d] & \pi_0(\mathrm{Spec}(D)) \arrow[d] \\ \mathrm{Spec}(C) \arrow[r] & \pi_0(\mathrm{Spec}(C)) \end{tikzcd}$$

is cartesian in the category of topological spaces. Note that $\mathrm{Spec}(D)$ is w-local, that $\mathrm{Spec}(D) \to \mathrm{Spec}(C)$ is w-local, and that the set of closed points of $\mathrm{Spec}(D)$ is the inverse image of the set of closed points of $\mathrm{Spec}(C)$, see Lemma 2.5. Thus it is still true that the local rings of D at its maximal ideals are strictly henselian (as they are isomorphic to the local rings at the corresponding maximal ideals of C). It follows from Lemma 10.2 that D is w-contractible. □

Remark 10.4. Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of the ring D constructed in Proposition 10.3 is at most

$$\kappa^{2^{2^{2^\kappa}}}.$$

Namely, the ring map $A \to D$ is constructed as a composition

$$A \to A_w = A' \to C' \to C \to D.$$

Here the first three steps of the construction are carried out in the first paragraph of the proof of Lemma 8.7. For the first step we have $|A_w| \leq \kappa$ by Remark 5.4. We have $|C'| \leq \kappa$ by Remark 8.2. Then $|C| \leq \kappa$ because C is a localization of $(C')_w$ (it is constructed from C' by an application of Lemma 5.7 in the proof of Lemma 5.8). Thus C has at most 2^κ maximal ideals. Finally, the ring map $C \to D$ identifies local rings and the cardinality of the set of maximal ideals of D is at most $2^{2^{2^\kappa}}$ by Topology, Remark 25.10. Since $D \subset \prod_{\mathfrak m \subset D} D_{\mathfrak m}$ we see that D has at most the size displayed above.

Lemma 10.5. *Let $A \to B$ be a quasi-finite and finitely presented ring map. If the residue fields of A are separably algebraically closed and* $\text{Spec}(A)$ *is extremally disconnected, then* $\text{Spec}(B)$ *is extremally disconnected.*

Proof. Set $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. Choose a finite partition $X = \coprod X_i$ and $X'_i \to X_i$ as in Étale Cohomology, Lemma 70.3. Because X is extremally disconnected, every constructible locally closed subset is open and closed, hence we see that X is topologically the disjoint union of the strata X_i. Thus we may replace X by the X_i and assume there exists a surjective finite locally free morphism $X' \to X$ such that $(X' \times_X Y)_{red}$ is isomorphic to a finite disjoint union of copies of X'_{red}. Picture

$$\begin{array}{ccc} \coprod_{i=1,\ldots,r} X' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

The assumption on the residue fields of A implies that this diagram is a fibre product diagram on underlying sets of points (details omitted). Since X is extremally disconnected and X' is Hausdorff (Lemma 5.6), the continuous map $X' \to X$ has a continuous section σ. Then $\coprod_{i=1,\ldots,r} \sigma(X) \to Y$ is a bijective continuous map. By Topology, Lemma 16.8 we see that it is a homeomorphism and the proof is done. $\square$

Lemma 10.6. *Let $A \to B$ be a finite and finitely presented ring map. If A is w-contractible, so is B.*

Proof. We will use the criterion of Lemma 10.2. Set $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. As $Y \to X$ is a finite morphism, we see that the set of closed points Y_0 of Y is the inverse image of the set of closed points X_0 of X. Moreover, every point of Y specializes to a unique point of Y_0 as (a) this is true for X and (b) the map $X \to Y$ is separated. For every $y \in Y_0$ with image $x \in X_0$ we see that $\mathcal{O}_{Y,y}$ is strictly henselian by Algebra, Lemma 145.4 applied to $\mathcal{O}_{X,x} \to B \otimes_A \mathcal{O}_{X,x}$. It remains to show that Y_0 is extremally disconnected. To do this we look at $X_0 \times_X Y \to X_0$ where $X_0 \subset X$ is the reduced induced scheme structure. Note that the underlying topological space of $X_0 \times_X Y$ agrees with Y_0. Now the desired result follows from Lemma 10.5. $\square$

Lemma 10.7. *Let A be a ring. Let $Z \subset \text{Spec}(A)$ be a closed subset of the form $Z = V(f_1, \ldots, f_r)$. Set $B = A_Z^\sim$, see Lemma 5.1. If A is w-contractible, so is B.*

Proof. Let $A_Z^\sim \to B$ be a weakly étale faithfully flat ring map. Consider the ring map

$$A \longrightarrow A_{f_1} \times \ldots \times A_{f_r} \times B$$

this is faithful flat and weakly étale. If A is w-contractible, then there is a section σ. Consider the morphism

$$\mathrm{Spec}(A_Z^\sim) \to \mathrm{Spec}(A) \xrightarrow{\mathrm{Spec}(\sigma)} \coprod \mathrm{Spec}(A_{f_i}) \amalg \mathrm{Spec}(B)$$

Every point of $Z \subset \mathrm{Spec}(A_Z^\sim)$ maps into the component $\mathrm{Spec}(B)$. Since every point of $\mathrm{Spec}(A_Z^\sim)$ specializes to a point of Z we find a morphism $\mathrm{Spec}(A_Z^\sim) \to \mathrm{Spec}(B)$ as desired. □

11. The pro-étale site

The (small) pro-étale site of a scheme has some remarkable properties. In particular, it has enough w-contractible objects which implies a number of useful consequences for the derived category of abelian sheaves and for inverse systems of sheaves. Thus it is well adapted to deal with some of the intricacies of working with ℓ-adic sheaves.

On the other hand, the pro-étale topology is a bit like the fpqc topology (see Topologies, Section 8) in that the topos of sheaves on the small pro-étale site of a scheme depends on the choice of the underlying category of schemes. Thus we cannot speak of *the* pro-étale topos of a scheme. However, it will be true that the cohomology groups of a sheaf are unchanged if we enlarge our underlying category of schemes.

Another curiosity is that we define pro-étale coverings using weakly étale morphisms of schemes, see More on Morphisms, Section 44. The reason is that, on the one hand, it is somewhat awkward to define the notion of a pro-étale morphism of schemes, and on the other, Proposition 9.1 assures us that we obtain the same sheaves with the definition that follows.

Definition 11.1. Let T be a scheme. A *pro-étale covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is weakly-étale and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \ldots, n$ with $\bigcup_{j=1}^n f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. Here is a lemma that will allow us to recognize pro-étale coverings. It will also allow us to reduce many lemmas about pro-étale coverings to the corresponding results for fpqc coverings.

Lemma 11.2. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

1. *$\{f_i : T_i \to T\}_{i \in I}$ is a pro-étale covering,*
2. *each f_i is weakly étale and $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering,*
3. *each f_i is weakly étale and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,*
4. *each f_i is weakly étale and there exists an affine open covering $T = \bigcup_{\alpha \in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \ldots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U_\alpha = \bigcup_{j=1,\ldots,n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$.*

If T is quasi-separated, these are also equivalent to

5. *each f_i is weakly étale, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T.*

Proof. The equivalence of (1) and (2) is immediate from the definitions. Hence the lemma follows from Topologies, Lemma 8.2. $\square$

Lemma 11.3. *Any étale covering and any Zariski covering is a pro-étale covering.*

Proof. This follows from the corresponding result for fpqc coverings (Topologies, Lemma 8.6), Lemma 11.2, and the fact that an étale morphism is a weakly étale morphism, see More on Morphisms, Lemma 44.9. $\square$

Lemma 11.4. *Let T be a scheme.*

1. *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a pro-étale covering of T.*
2. *If $\{T_i \to T\}_{i\in I}$ is a pro-étale covering and for each i we have a pro-étale covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a pro-étale covering.*
3. *If $\{T_i \to T\}_{i\in I}$ is a pro-étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a pro-étale covering.*

Proof. This follows from the fact that composition and base changes of weakly étale morphisms are weakly étale (More on Morphisms, Lemmas 44.5 and 44.6), Lemma 11.2, and the corresponding results for fpqc coverings, see Topologies, Lemma 8.7. $\square$

Lemma 11.5. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a pro-étale covering of T. Then there exists a pro-étale covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definition. $\square$

Thus we define the corresponding standard coverings of affines as follows.

Definition 11.6. Let T be an affine scheme. A *standard pro-étale covering* of T is a family $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ with each T_j is affine, each f_i is weakly étale, and $T = \bigcup f_i(T_i)$.

We interrupt the discussion for an explanation of the notion of w-contractible rings in terms of pro-étale coverings.

Lemma 11.7. *Let $T = \mathrm{Spec}(A)$ be an affine scheme. The following are equivalent*

1. *A is w-contractible, and*
2. *every pro-étale covering of T can be refined by a Zariski covering of the form $T = \coprod_{i=1,\ldots,n} U_i$.*

Proof. Assume A is w-contractible. By Lemma 11.5 it suffices to prove we can refine every standard pro-étale covering $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ by a Zariski covering of T. The morphism $\coprod T_i \to T$ is a surjective weakly étale morphism of affine schemes. Hence by Definition 10.1 there exists a morphism $\sigma : T \to \coprod T_i$ over T. Then the Zariski covering $T = \coprod \sigma^{-1}(T_i)$ refines $\{f_i : T_i \to T\}$.

Conversely, assume (2). If $A \to B$ is faithfully flat and weakly étale, then $\{\mathrm{Spec}(B) \to T\}$ is a pro-étale covering. Hence there exists a Zariski covering $T = \coprod U_i$ and morphisms $U_i \to \mathrm{Spec}(B)$ over T. Since $T = \coprod U_i$ we obtain $T \to \mathrm{Spec}(B)$, i.e., an A-algebra map $B \to A$. This means A is w-contractible. $\square$

We follow the general outline given in Topologies, Section 2 for constructing the big pro-étale site we will be working with. However, because we need a bit larger rings to accommodate for the size of certain constructions we modify the constructions slightly.

Definition 11.8. A *big pro-étale site* is any site $Sch_{pro\text{-}étale}$ as in Sites, Definition 6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of pro-étale coverings Cov_0 among these schemes.
(2) Change the function *Bound* of Sets, Equation (9.1.1) into
$$Bound(\kappa) = \max\{\kappa^{2^{2^{2^\kappa}}}, \kappa^{\aleph_0}, \kappa^+\}.$$
(3) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0 and the function *Bound*.
(4) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of pro-étale coverings, and the set Cov_0 chosen above.

See the remarks following Topologies, Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big and small pro-étale sites of a scheme, let us point out that (1) our category contains many weakly contractible objects, and (2) the topology on a big pro-étale site $Sch_{pro\text{-}étale}$ is in some sense induced from the pro-étale topology on the category of all schemes.

Lemma 11.9. *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site as in Definition 11.8. Let $T = \text{Spec}(A)$ be an affine object of $Sch_{pro\text{-}étale}$. If A is w-contractible, then T is a weakly contractible (Sites, Definition 39.2) object of $Sch_{pro\text{-}étale}$.*

Proof. Let $\mathcal{F} \to \mathcal{G}$ be a surjection of sheaves on $Sch_{pro\text{-}étale}$. Let $s \in \mathcal{G}(T)$. We have to show that s is in the image of $\mathcal{F}(T) \to \mathcal{G}(T)$. We can find a covering $\{T_i \to T\}$ of $Sch_{pro\text{-}étale}$ such that s lifts to a section of $\mathcal{F}$ over T_i (Sites, Definition 12.1). By Lemma 11.7 we can refine $\{T_i \to T\}$ by a Zariski covering of the form $T = \coprod_{j=1,\ldots,m} V_j$. Hence we get $t_j \in \mathcal{F}(U_j)$ mapping to $s|_{U_j}$. Since Zariski coverings are coverings in $Sch_{pro\text{-}étale}$ (Lemma 11.3) we conclude that $\mathcal{F}(T) = \prod \mathcal{F}(U_j)$. Thus, taking $t = (t_1, \ldots, t_m) \in \mathcal{F}(T)$ is a section mapping to s. □

Lemma 11.10. *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site as in Definition 11.8. For every object T of $Sch_{pro\text{-}étale}$ there exists a covering $\{T_i \to T\}$ in $Sch_{pro\text{-}étale}$ with each T_i affine and the spectrum of a w-contractible ring. In particular, T_i is weakly contractible in $Sch_{pro\text{-}étale}$.*

Proof. For those readers who do not care about set-theoretical issues this lemma is a trivial consequence of Lemma 11.9 and Proposition 10.3. Here are the details. Choose an affine open covering $T = \bigcup U_i$. Write $U_i = \text{Spec}(A_i)$. Choose faithfully flat, ind-étale ring maps $A_i \to D_i$ such that D_i is w-contractible as in Proposition 10.3. The family of morphisms $\{\text{Spec}(D_i) \to T\}$ is a pro-étale covering. If we can show that $\text{Spec}(D_i)$ is isomorphic to an object, say T_i, of $Sch_{pro\text{-}étale}$, then $\{T_i \to T\}$ will be combinatorially equivalent to a covering of $Sch_{pro\text{-}étale}$ by the construction of $Sch_{pro\text{-}étale}$ in Definition 11.8 and more precisely the application of Sets, Lemma 11.1 in the last step. To prove $\text{Spec}(D_i)$ is isomorphic to an object of $Sch_{pro\text{-}étale}$, it suffices to prove that $|D_i| \leq Bound(\text{Size}(T))$ by the construction of $Sch_{pro\text{-}étale}$ in Definition 11.8 and more precisely the application of Sets, Lemma 9.2 in step (3). Since $|A_i| \leq \text{size}(U_i) \leq \text{size}(T)$ by Sets, Lemmas 9.4 and 9.7 we get $|D_i| \leq \kappa^{2^{2^{2^\kappa}}}$ where $\kappa = \text{size}(T)$ by Remark 10.4. Thus by our choice of the function *Bound* in Definition 11.8 we win. $\square$

Lemma 11.11. *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site as in Definition 11.8. Let $T \in \text{Ob}(Sch_{pro\text{-}étale})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary pro-étale covering of T. There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{pro\text{-}étale}$ which refines $\{T_i \to T\}_{i\in I}$.*

Proof. Namely, we first let $\{V_k \to T\}$ be a covering as in Lemma 11.10. Then the pro-étale coverings $\{T_i \times_T V_k \to V_k\}$ can be refined by a finite disjoint open covering $V_k = V_{k,1} \amalg \ldots \amalg V_{k,n_k}$, see Lemma 11.7. Then $\{V_{k,i} \to T\}$ is a covering of $Sch_{pro\text{-}étale}$ which refines $\{T_i \to T\}_{i\in I}$. $\square$

Definition 11.12. Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S.

(1) The *big pro-étale site of S*, denoted $(Sch/S)_{pro\text{-}étale}$, is the site $Sch_{pro\text{-}étale}/S$ introduced in Sites, Section 24.
(2) The *small pro-étale site of S*, which we denote $S_{pro\text{-}étale}$, is the full subcategory of $(Sch/S)_{pro\text{-}étale}$ whose objects are those U/S such that $U \to S$ is weakly étale. A covering of $S_{pro\text{-}étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{pro\text{-}étale}$ with $U \in \text{Ob}(S_{pro\text{-}étale})$.
(3) The *big affine pro-étale site of S*, denoted $(Aff/S)_{pro\text{-}étale}$, is the full subcategory of $(Sch/S)_{pro\text{-}étale}$ whose objects are affine U/S. A covering of $(Aff/S)_{pro\text{-}étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{pro\text{-}étale}$ which is a standard pro-étale covering.

It is not completely clear that the small pro-étale site and the big affine pro-étale site are sites. We check this now.

Lemma 11.13. *Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. Both $S_{pro\text{-}étale}$ and $(Aff/S)_{pro\text{-}étale}$ are sites.*

Proof. Let us show that $S_{pro\text{-}étale}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 6.2. Since $(Sch/S)_{pro\text{-}étale}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{pro\text{-}étale}$ with $U \in \text{Ob}(S_{pro\text{-}étale})$ we also have $U_i \in \text{Ob}(S_{pro\text{-}étale})$. This follows from the definitions as the composition of weakly étale morphisms is weakly étale.

To show that $(Aff/S)_{pro\text{-}étale}$ is a site, reasoning as above, it suffices to show that the collection of standard pro-étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This follows from Lemma 11.2 and the corresponding result for standard fpqc coverings (Topologies, Lemma 8.10). $\square$

Lemma 11.14. *Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. Let Sch be the category of all schemes.*

(1) *The categories $Sch_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ have fibre products agreeing with fibre products in Sch.*

(2) *The categories $Sch_{pro\text{-}\acute{e}tale}$, $(Sch/S)_{pro\text{-}\acute{e}tale}$, $S_{pro\text{-}\acute{e}tale}$ have equalizers agreeing with equalizers in Sch.*
(3) *The categories $(Sch/S)_{pro\text{-}\acute{e}tale}$, and $S_{pro\text{-}\acute{e}tale}$ both have a final object, namely S/S.*
(4) *The category $Sch_{pro\text{-}\acute{e}tale}$ has a final object agreeing with the final object of Sch, namely* $\mathrm{Spec}(\mathbf{Z})$.

Proof. The category $Sch_{pro\text{-}\acute{e}tale}$ contains $\mathrm{Spec}(\mathbf{Z})$ and is closed under products and fibre products by construction, see Sets, Lemma 9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{pro\text{-}\acute{e}tale})$. The fibre product $V \times_U W$ in $Sch_{pro\text{-}\acute{e}tale}$ is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{pro\text{-}\acute{e}tale}$. This proves the result for $(Sch/S)_{pro\text{-}\acute{e}tale}$. If $U \to S$, $V \to U$ and $W \to U$ are weakly étale then so is $V \times_U W \to S$ (see More on Morphisms, Section 44) and hence we get fibre products for $S_{pro\text{-}\acute{e}tale}$. If U, V, W are affine, so is $V \times_U W$ and hence we get fibre products for $(Aff/S)_{pro\text{-}\acute{e}tale}$.

Let $a, b : U \to V$ be two morphisms in $Sch_{pro\text{-}\acute{e}tale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/\mathrm{Spec}(\mathbf{Z})}, V \times_{\mathrm{Spec}(\mathbf{Z})} V, (a,b)} (U \times_{\mathrm{Spec}(\mathbf{Z})} U)$$

which is an object of $Sch_{pro\text{-}\acute{e}tale}$ by what we saw above. Thus $Sch_{pro\text{-}\acute{e}tale}$ has equalizers. If a and b are morphisms over S, then the equalizer (in the category of schemes) is also given by

$$V \times_{\Delta_{V/S}, V \times_S V, (a,b)} (U \times_S U)$$

hence we see that $(Sch/S)_{pro\text{-}\acute{e}tale}$ has equalizers. Moreover, if U and V are weakly-étale over S, then so is the equalizer above as a fibre product of schemes weakly étale over S. Thus $S_{pro\text{-}\acute{e}tale}$ has equalizers. The statements on final objects is clear. $\square$

Next, we check that the big affine pro-étale site defines the same topos as the big pro-étale site.

Lemma 11.15. *Let S be a scheme. Let $Sch_{pro\text{-}\acute{e}tale}$ be a big pro-étale site containing S. The functor $(Aff/S)_{pro\text{-}\acute{e}tale} \to (Sch/S)_{pro\text{-}\acute{e}tale}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((Aff/S)_{pro\text{-}\acute{e}tale})$ to $Sh((Sch/S)_{pro\text{-}\acute{e}tale})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (Aff/S)_{pro\text{-}\acute{e}tale} \to (Sch/S)_{pro\text{-}\acute{e}tale}$. Being cocontinuous just means that any pro-étale covering of T/S, T affine, can be refined by a standard pro-étale covering of T. This is the content of Lemma 11.5. Hence (1) holds. We see u is continuous simply because a standard pro-étale covering is a pro-étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. $\square$

Lemma 11.16. *Let $Sch_{pro\text{-}\acute{e}tale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}\acute{e}tale}$. The functor $T_{pro\text{-}\acute{e}tale} \to (Sch/S)_{pro\text{-}\acute{e}tale}$ is cocontinuous and* *induces a morphism of topoi*

$$i_f : Sh(T_{pro\text{-}\acute{e}tale}) \longrightarrow Sh((Sch/S)_{pro\text{-}\acute{e}tale})$$

For a sheaf $\mathcal{G}$ *on* $(Sch/S)_{pro\text{-}\acute{e}tale}$ *we have the formula* $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. *The functor* i_f^{-1} *also has a left adjoint* $i_{f,!}$ *which commutes with fibre products and equalizers.*

Proof. Denote the functor $u : T_{pro\text{-}\acute{e}tale} \to (Sch/S)_{pro\text{-}\acute{e}tale}$. In other words, given a weakly étale morphism $j : U \to T$ corresponding to an object of $T_{pro\text{-}\acute{e}tale}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 11.14. Moreover, $T_{pro\text{-}\acute{e}tale}$ has equalizers and u commutes with them by Lemma 11.14. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 20.5 and 20.6. $\square$

Lemma 11.17. *Let S be a scheme. Let $Sch_{pro\text{-}\acute{e}tale}$ be a big pro-étale site containing S. The inclusion functor $S_{pro\text{-}\acute{e}tale} \to (Sch/S)_{pro\text{-}\acute{e}tale}$ satisfies the hypotheses of Sites, Lemma 20.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{pro\text{-}\acute{e}tale} \longrightarrow S_{pro\text{-}\acute{e}tale}$$

and a morphism of topoi

$$i_S : Sh(S_{pro\text{-}\acute{e}tale}) \longrightarrow Sh((Sch/S)_{pro\text{-}\acute{e}tale})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 11.16. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{pro\text{-}\acute{e}tale} \to (Sch/S)_{pro\text{-}\acute{e}tale}$, in addition to the properties seen in the proof of Lemma 11.16 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 20.8. $\square$

Definition 11.18. In the situation of Lemma 11.17 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small pro-étale site*, and for a sheaf $\mathcal{F}$ on the big pro-étale site we denote $\mathcal{F}|_{S_{pro\text{-}\acute{e}tale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\begin{aligned}
\mathrm{Mor}_{Sh(S_{pro\text{-}\acute{e}tale})}(\mathcal{F}|_{S_{pro\text{-}\acute{e}tale}}, \mathcal{G}) &= \mathrm{Mor}_{Sh((Sch/S)_{pro\text{-}\acute{e}tale})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\
\mathrm{Mor}_{Sh(S_{pro\text{-}\acute{e}tale})}(\mathcal{G}, \mathcal{F}|_{S_{pro\text{-}\acute{e}tale}}) &= \mathrm{Mor}_{Sh((Sch/S)_{pro\text{-}\acute{e}tale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})
\end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{pro\text{-}\acute{e}tale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{pro\text{-}\acute{e}tale}} = \mathcal{G}$.

Lemma 11.19. *Let $Sch_{pro\text{-}\acute{e}tale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}\acute{e}tale}$. The functor*

$$u : (Sch/T)_{pro\text{-}\acute{e}tale} \longrightarrow (Sch/S)_{pro\text{-}\acute{e}tale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{pro\text{-}\acute{e}tale} \longrightarrow (Sch/T)_{pro\text{-}\acute{e}tale}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{pro\text{-}\acute{e}tale}) \longrightarrow Sh((Sch/S)_{pro\text{-}\acute{e}tale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 11.16). Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. □

Lemma 11.20. *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}étale}$.*

1. *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 11.16 and i_T as in Lemma 11.17.*
2. *The functor $S_{pro\text{-}étale} \to T_{pro\text{-}étale}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi*
$$f_{small} : Sh(T_{pro\text{-}étale}) \longrightarrow Sh(S_{pro\text{-}étale}).$$
We have $f_{small,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.
3. *We have a commutative diagram of morphisms of sites*
$$\begin{array}{ccc} T_{pro\text{-}étale} & \xleftarrow[\pi_T]{} & (Sch/T)_{pro\text{-}étale} \\ \downarrow{\scriptstyle f_{small}} & & \downarrow{\scriptstyle f_{big}} \\ S_{pro\text{-}étale} & \xleftarrow{\pi_S} & (Sch/S)_{pro\text{-}étale} \end{array}$$
so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.
4. *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{pro\text{-}étale} \to T_{pro\text{-}étale}$, $u(U \to S) = (U \times_S T \to T)$ transforms coverings into coverings and commutes with fibre products, see Lemmas 11.4 and 11.14. Moreover, both $S_{pro\text{-}étale}$, $T_{pro\text{-}étale}$ have final objects, namely S/S and T/T and $u(S/S) = T/T$. Hence by Sites, Proposition 15.6 the functor u corresponds to a morphism of sites $T_{pro\text{-}étale} \to S_{pro\text{-}étale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 16.2. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 11.18 we have: for $\mathcal{F}$ a sheaf on the big pro-étale site of T
$$(f_{big,*}\mathcal{F})|_{S_{pro\text{-}étale}} = f_{small,*}(\mathcal{F}|_{T_{pro\text{-}étale}}),$$
This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small pro-étale site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 11.21. *Given schemes X, Y, Y in $Sch_{pro\text{-}étale}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 11.19. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 11.20. □

We can think about a sheaf on the big pro-étale site of S as a collection of sheaves on the small pro-étale site on schemes over S.

Lemma 11.22. *Let S be a scheme contained in a big pro-étale site $Sch_{pro\text{-}étale}$. A sheaf $\mathcal{F}$ on the big pro-étale site $(Sch/S)_{pro\text{-}étale}$ is given by the following data:*

1. *for every $T/S \in \mathrm{Ob}((Sch/S)_{pro\text{-}étale})$ a sheaf $\mathcal{F}_T$ on $T_{pro\text{-}étale}$,*
2. *for every $f : T' \to T$ in $(Sch/S)_{pro\text{-}étale}$ a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

- (i) *given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{pro\text{-}étale}$ the composition $g_{small}^{-1}c_f \circ c_g$ is equal to $c_{f\circ g}$, and*
- (ii) *if $f : T' \to T$ in $(Sch/S)_{pro\text{-}étale}$ is weakly étale then c_f is an isomorphism.*

Proof. Identical to the proof of Topologies, Lemma 4.18. □

Lemma 11.23. *Let S be a scheme. Let $S_{affine,pro\text{-}étale}$ denote the full subcategory of $S_{pro\text{-}étale}$ consisting of affine objects. A covering of $S_{affine,pro\text{-}étale}$ will be a standard étale covering, see Definition 11.6. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{affine,étale}}$$

defines an equivalence of topoi $Sh(S_{pro\text{-}étale}) \cong Sh(S_{affine,pro\text{-}étale})$.

Proof. This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 28.1 by checking that the inclusion functor $S_{affine,pro\text{-}étale} \to S_{pro\text{-}étale}$ is a special cocontinuous functor (see Sites, Definition 28.2). □

Lemma 11.24. *Let S be an affine scheme. Let S_{app} denote the full subcategory of $S_{pro\text{-}étale}$ consisting of affine objects U such that $\mathcal{O}(S) \to \mathcal{O}(U)$ is ind-étale. A covering of S_{app} will be a standard pro-étale covering, see Definition 11.6. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{app}}$$

defines an equivalence of topoi $Sh(S_{pro\text{-}étale}) \cong Sh(S_{app})$.

Proof. By Lemma 11.23 we may replace $S_{pro\text{-}étale}$ by $S_{affine,pro\text{-}étale}$. The lemma follows from Sites, Lemma 28.1 by checking that the inclusion functor $S_{app} \to S_{affine,pro\text{-}étale}$ is a special cocontinuous functor, see Sites, Definition 28.2. The conditions of Sites, Lemma 28.1 follow immediately from the definition and the facts (a) any object U of $S_{affine,pro\text{-}étale}$ has a covering $\{V \to U\}$ with V ind-étale over X (Proposition 9.1) and (b) the functor u is fully faithful. □

Next we show that cohomology of sheaves is independent of the choice of a partial universe. Namely, the functor g_* of the lemma below is an embedding of pro-étale topoi which does not change cohomology.

Lemma 11.25. *Let S be a scheme. Let $S_{pro\text{-}étale} \subset S'_{pro\text{-}étale}$ be two small pro-étale sites of S as constructed in Definition 11.12. Then the inclusion functor satisfies the assumptions of Sites, Lemma 20.8. Hence there exist morphisms of topoi*

$$Sh(S_{pro\text{-}étale}) \xrightarrow{g} Sh(S'_{pro\text{-}étale}) \xrightarrow{f} Sh(S_{pro\text{-}étale})$$

whose composition is isomorphic to the identity and with $f_* = g^{-1}$. Moreover,

(1) *for $\mathcal{F}' \in Ab(S'_{pro\text{-}étale})$ we have $H^p(S'_{pro\text{-}étale}, \mathcal{F}') = H^p(S_{pro\text{-}étale}, g^{-1}\mathcal{F}')$,*
(2) *for $\mathcal{F} \in Ab(S_{pro\text{-}étale})$ we have*
$$H^p(S_{pro\text{-}étale}, \mathcal{F}) = H^p(S'_{pro\text{-}étale}, g_*\mathcal{F}) = H^p(S'_{pro\text{-}étale}, f^{-1}\mathcal{F}).$$

Proof. The inclusion functor is fully faithful and continuous. We have seen that $S_{pro\text{-}étale}$ and $S'_{pro\text{-}étale}$ have fibre products and final objects and that our functor commutes with these (Lemma 11.14). It follows from Lemma 11.11 that the inclusion functor is cocontinuous. Hence the existence of f and g follows from Sites, Lemma 20.8. The equality in (1) is Cohomology on Sites, Lemma 8.2. Part (2) follows from (1) as $\mathcal{F} = g^{-1}g_*\mathcal{F} = g^{-1}f^{-1}\mathcal{F}$. □

Lemma 11.26. *Let S be a scheme. The topology on each of the pro-étale sites $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{affine,pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ is subcanonical.*

Proof. Combine Lemma 11.2 and Descent, Lemma 9.3. □

Lemma 11.27. *Let S be a scheme. The pro-étale sites $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{affine,pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ and if S is affine S_{app} have enough quasi-compact, weakly contractible objects, see Sites, Definition 39.2.*

Proof. Follows immediately from Lemma 11.10. □

12. Points of the pro-étale site

We first apply Deligne's criterion to show that there are enough points.

Lemma 12.1. *Let S be a scheme. The pro-étale sites $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{affine,pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ have enough points.*

Proof. The big topos is equivalent to the topos defined by $(Aff/S)_{pro\text{-}étale}$, see Lemma 11.15. The topos of sheaves on $S_{pro\text{-}étale}$ is equivalent to the topos associated to $S_{affine,pro\text{-}étale}$, see Lemma 11.23. The result for the sites $(Aff/S)_{pro\text{-}étale}$ and $S_{affine,pro\text{-}étale}$ follows immediately from Deligne's result Sites, Proposition 38.3. □

Let S be a scheme. Let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point. We define a *pro-étale neighbourhood* of $\overline{s}$ to be a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "\overline{u}"'] \arrow[rd, "\overline{s}"'] & U \arrow[d] \\ & S \end{tikzcd}$$

with $U \to S$ weakly étale. In exactly the same manner as in the chapter on étale cohomology one shows that the category of pro-étale neighbourhoods of $\overline{s}$ is cofiltered. Moreover, if $(U, \overline{u})$ is a pro-étale neighbourhood, and if $\{U_i \to U\}$ is a pro-étale covering, then there exists an i and a lift of $\overline{u}$ to a geometric point $\overline{u}_i$ of U_i. For $\mathcal{F}$ in $Sh(S_{pro\text{-}étale})$ define the *stalk of $\mathcal{F}$ at $\overline{s}$* by the formula

$$\mathcal{F}_{\overline{s}} = \operatorname{colim}_{(U,\overline{u})} \mathcal{F}(U)$$

where the colimit is over all pro-étale neighbourhoods $(U, \overline{u})$ of $\overline{s}$ with $U \in \mathrm{Ob}(S_{pro\text{-}étale})$. A formal argument using the facts above shows the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ defines a point of the topos $Sh(S_{pro\text{-}étale})$: it is an exact functor which commutes with arbitrary colimits. In fact, this functor has another description.

Lemma 12.2. *In the situation above the scheme* $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ *is an object of* $X_{pro\text{-}étale}$ *and there is a canonical isomorphism*

$$\mathcal{F}(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})) = \mathcal{F}_{\overline{s}}$$

functorial in $\mathcal{F}$.

Proof. The first statement is clear from the construction of the strict henselization as a filtered colimit of étale algebras over S, or by the characterization of weakly étale morphisms of More on Morphisms, Lemma 44.11. The second statement follows as by Olivier's theorem (More on Algebra, Theorem 67.24) the scheme $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is an initial object of the category of pro-étale neighbourhoods of $\overline{s}$. $\square$

Contrary to the situation with the étale topos of S it is not true that every point of $Sh(S_{pro\text{-}étale})$ is of this form, and it is not true that the collection of points associated to geometric point is conservative. Namely, suppose that $S = \mathrm{Spec}(k)$ where k is an algebraically closed field. Let A be an abelian group. Consider the sheaf $\mathcal{F}$ on $S_{pro\text{-}étale}$ defined by the rule

$$\mathcal{F}(U) = \frac{\{\text{functions } U \to A\}}{\{\text{locally constant functions}\}}$$

Then $\mathcal{F}(U) = 0$ if $U = S = \mathrm{Spec}(k)$ but in general $\mathcal{F}$ is not zero. Namely, $S_{pro\text{-}étale}$ contains objects with infinitely many points. For example, let $E = \lim E_n$ be an inverse limit of finite sets with surjective transition maps, e.g., $E = \lim \mathbf{Z}/n\mathbf{Z}$. The scheme $\mathrm{Spec}(\operatorname{colim} \mathrm{Map}(E_n, k))$ is an object of $S_{pro\text{-}étale}$ because $\operatorname{colim} \mathrm{Map}(E_n, k)$ is weakly étale (even ind-Zariski) over k. Thus $\mathcal{F}$ is a nonzero abelian sheaf whose stalk at the unique geometric point of S is zero.

The solution is to use the existence of quasi-compact, weakly contractible objects. First, there are enough quasi-compact, weakly contractible objects by Lemma 11.27. Second, if $W \in \mathrm{Ob}(S_{pro\text{-}étale})$ is quasi-compact, weakly contractible, then the functor

$$Sh(S_{pro\text{-}étale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}(W)$$

is an exact functor $Sh(S_{pro\text{-}étale}) \to Sets$ which commutes with all limits. The functor

$$Ab(S_{pro\text{-}étale}) \longrightarrow Ab, \quad \mathcal{F} \longmapsto \mathcal{F}(W)$$

is exact and commutes with direct sums (as W is quasi-compact, see Sites, Lemma 11.2), hence commutes with all limits and colimits. Moreover, we can check exactness of a complex of abelian sheaves by evaluation at the quasi-compact, weakly contractible objects of $S_{pro\text{-}étale}$, see Cohomology on Sites, Proposition 38.2.

13. Compact generation

Let S be a scheme. The site $S_{pro\text{-}\acute{e}tale}$ has enough quasi-compact, weakly contractible objects U. For any sheaf of rings $\mathcal{A}$ on $S_{pro\text{-}\acute{e}tale}$ the corresponding objects $j_{U!}\mathcal{A}_U$ are compact objects of the derived category $D(\mathcal{A})$, see Cohomology on Sites, Lemma 39.5. Since every complex of $\mathcal{A}$-modules is quasi-isomorphic to a complex whose terms are direct sums of the modules $j_{U!}\mathcal{A}_U$ (details omitted). Thus we see that $D(\mathcal{A})$ is generated by its compact objects.

The same argument works for the big pro-étale site of S.

14. Generalities on derived completion

We urge the reader to skip this section on a first reading.

The algebra version of this material can be found in More on Algebra, Section 64. Let $\mathcal{O}$ be a sheaf of rings on a site $\mathcal{C}$. Let f be a global section of $\mathcal{O}$. We denote $\mathcal{O}_f$ the sheaf associated to the presheaf of localizations $U \mapsto \mathcal{O}(U)_f$.

Lemma 14.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$.*

(1) *For $L, N \in D(\mathcal{O}_f)$ we have $R\mathcal{H}om_{\mathcal{O}}(L, N) = R\mathcal{H}om_{\mathcal{O}_f}(L, N)$. In particular the two $\mathcal{O}_f$-structures on $R\mathcal{H}om_{\mathcal{O}}(L, N)$ agree.*
(2) *For $K \in D(\mathcal{O})$ and $L \in D(\mathcal{O}_f)$ we have*
$$R\mathcal{H}om_{\mathcal{O}}(L, K) = R\mathcal{H}om_{\mathcal{O}_f}(L, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K))$$
In particular $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$.
(3) *If g is a second global section of $\mathcal{O}$, then*
$$R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, K)) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{gf}, K).$$

Proof. Proof of (1). Let $\mathcal{J}^\bullet$ be a K-injective complex of $\mathcal{O}_f$-modules representing N. By Cohomology on Sites, Lemma 20.3 it follows that $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules as well. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then
$$R\mathcal{H}om_{\mathcal{O}}(L, N) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{J}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)$$
by Modules on Sites, Lemma 11.4 because $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$ and of $\mathcal{O}_f$-modules.

Proof of (2). Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing K. Then $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ is represented by $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet)$ which is a K-injective complex of $\mathcal{O}_f$-modules and of $\mathcal{O}$-modules by Cohomology on Sites, Lemmas 20.4 and 20.3. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then
$$R\mathcal{H}om_{\mathcal{O}}(L, K) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet))$$
by Modules on Sites, Lemma 27.5 and because $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}_f$-modules.

Proof of (3). This follows from the fact that $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, \mathcal{I}^\bullet)$ is K-injective as a complex of $\mathcal{O}$-modules and the fact that $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, \mathcal{H})) = \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{gf}, \mathcal{H})$ for all sheaves of $\mathcal{O}$-modules $\mathcal{H}$. $\square$

Let $K \in D(\mathcal{O})$. We denote $T(K, f)$ a derived limit (Derived Categories, Definition 32.1) of the system
$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$
in $D(\mathcal{O})$.

Lemma 14.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$. Let $K \in D(\mathcal{O})$. The following are equivalent*

(1) $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K) = 0$,
(2) $R\mathcal{H}om_\mathcal{O}(L, K) = 0$ *for all* L *in* $D(\mathcal{O}_f)$,
(3) $T(K, f) = 0$.

Proof. It is clear that (2) implies (1). The implication (1) $\Rightarrow$ (2) follows from Lemma 14.1. A free resolution of the $\mathcal{O}$-module $\mathcal{O}_f$ is given by

$$0 \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \mathcal{O}_f \to 0$$

where the first map sends a local section $(x_0, x_1, \ldots)$ to $(fx_0 - x_1, fx_1 - x_2, \ldots)$ and the second map sends $(x_0, x_1, \ldots)$ to $x_0 + x_1/f + x_2/f^2 + \ldots$. Applying $\mathcal{H}om_\mathcal{O}(-, \mathcal{I}^\bullet)$ where $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules representing K we get a short exact sequence of complexes

$$0 \to \mathcal{H}om_\mathcal{O}(\mathcal{O}_f, \mathcal{I}^\bullet) \to \prod \mathcal{I}^\bullet \to \prod \mathcal{I}^\bullet \to 0$$

because $\mathcal{I}^n$ is an injective $\mathcal{O}$-module. The products are products in $D(\mathcal{O})$, see Injectives, Lemma 13.4. This means that the object $T(K, f)$ is a representative of $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K)$ in $D(\mathcal{O})$. Thus the equivalence of (1) and (3). $\square$

Lemma 14.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K \in D(\mathcal{O})$. The rule which associates to U the set $\mathcal{I}(U)$ of sections $f \in \mathcal{O}(U)$ such that $T(K|_U, f) = 0$ is a sheaf of ideals in $\mathcal{O}$.*

Proof. We will use the results of Lemma 14.2 without further mention. If $f \in \mathcal{I}(U)$, and $g \in \mathcal{O}(U)$, then $\mathcal{O}_{U,gf}$ is an $\mathcal{O}_{U,f}$-module hence $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_{U,gf}, K|_U) = 0$, hence $gf \in \mathcal{I}(U)$. Suppose $f, g \in \mathcal{O}(U)$. Then there is a short exact sequence

$$0 \to \mathcal{O}_{U,f+g} \to \mathcal{O}_{U,f(f+g)} \oplus \mathcal{O}_{U,g(f+g)} \to \mathcal{O}_{U,gf(f+g)} \to 0$$

because f, g generate the unit ideal in $\mathcal{O}(U)_{f+g}$. This follows from Algebra, Lemma 22.1 and the easy fact that the last arrow is surjective. Because $R\mathcal{H}om_\mathcal{O}(-, K|_U)$ is an exact functor of triangulated categories the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f(f+g)}, K|_U)$, $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g(f+g)}, K|_U)$, and $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,gf(f+g)}, K|_U)$, implies the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f+g}, K|_U)$. We omit the verification of the sheaf condition. $\square$

We can make the following definition for any ringed site.

Definition 14.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let $K \in D(\mathcal{O})$. We say that K is *derived complete with respect to* $\mathcal{I}$ if for every object U of $\mathcal{C}$ and $f \in \mathcal{I}(U)$ the object $T(K|_U, f)$ of $D(\mathcal{O}_U)$ is zero.

It is clear that the full subcategory $D_{comp}(\mathcal{O}) = D_{comp}(\mathcal{O}, \mathcal{I}) \subset D(\mathcal{O})$ consisting of derived complete objects is a saturated triangulated subcategory, see Derived Categories, Definitions 3.4 and 6.1. This subcategory is preserved under products and homotopy limits in $D(\mathcal{O})$. But it is not preserved under countable direct sums in general.

Lemma 14.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. If $K \in D(\mathcal{O})$ and $L \in D_{comp}(\mathcal{O})$, then $R\mathcal{H}om_\mathcal{O}(K, L) \in D_{comp}(\mathcal{O})$.*

Proof. Let U be an object of $\mathcal{C}$ and let $f \in \mathcal{I}(U)$. Recall that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K,L)|_U) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{O}_{U,f}, L|_U)$$

by Cohomology on Sites, Lemma 26.2. The right hand side is zero by Lemma 14.2 and the relationship between internal hom and actual hom, see Cohomology on Sites, Lemma 26.1. The same vanishing holds for all U'/U. Thus the object $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K,L)|_U)$ of $D(\mathcal{O}_U)$ has vanishing 0th cohomology sheaf (by locus citatus). Similarly for the other cohomology sheaves, i.e., $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K,L)|_U)$ is zero in $D(\mathcal{O}_U)$. By Lemma 14.2 we conclude. $\square$

Lemma 14.6. *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. The inverse image of $D_{comp}(\mathcal{O}, \mathcal{I})$ under the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ is $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$.*

Proof. Using Lemma 14.3 we see that $K' \in D(\mathcal{O}')$ is in $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ if and only if $T(K'|_U, f)$ is zero for every local section $f \in \mathcal{I}(U)$. Observe that the cohomology sheaves of $T(K'|_U, f)$ are computed in the category of abelian sheaves, so it doesn't matter whether we think of f as a section of $\mathcal{O}$ or take the image of f as a section of $\mathcal{O}'$. The lemma follows immediately from this and the definition of derived complete objects. $\square$

Lemma 14.7. *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. We may assume f is given by a morphism of ringed sites corresponding to a continuous functor $\mathcal{C} \to \mathcal{D}$ (Modules on Sites, Lemma 7.2). Let U be an object of $\mathcal{C}$ and let g be a section of $\mathcal{I}$ over U. We have to show that $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,g}, Rf_*K|_U) = 0$ whenever K is derived complete with respect to $\mathcal{I}'$. Namely, by Cohomology on Sites, Lemma 26.1 this, applied to all objects over U and all shifts of K, will imply that $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g}, Rf_*K|_U)$ is zero, which implies that $T(Rf_*K|_U, g)$ is zero (Lemma 14.2) which is what we have to show (Definition 14.4). Let V in $\mathcal{D}$ be the image of U. Then

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,g}, Rf_*K|_U) = \mathrm{Hom}_{D(\mathcal{O}'_V)}(\mathcal{O}'_{V,g'}, K|_V) = 0$$

where $g' = f^\sharp(g) \in \mathcal{I}'(V)$. The second equality because K is derived complete and the first equality because the derived pullback of $\mathcal{O}_{U,g}$ is $\mathcal{O}'_{V,g'}$ and Cohomology on Sites, Lemma 19.1. $\square$

The following lemma is the simplest case where one has derived completion.

Lemma 14.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Then the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$ has a left adjoint, i.e., given any object K of $D(\mathcal{O})$ there exists a map $K \to K^\wedge$ with $K^\wedge$ in $D_{comp}(\mathcal{O})$ such that the map*

$$\mathrm{Hom}_{D(\mathcal{O})}(K^\wedge, E) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(K, E)$$

is bijective whenever E is in $D_{comp}(\mathcal{O})$. In fact we have

$$K^\wedge = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \ldots \to \mathcal{O}_{f_1 \ldots f_r}, K)$$

functorially in K.

Proof. Define $K^\wedge$ by the last displayed formula of the lemma. There is a map of complexes

$$(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r}) \longrightarrow \mathcal{O}$$

which induces a map $K \to K^\wedge$. It suffices to prove that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete.

Let f be a global section of $\mathcal{O}$. By Lemma 14.1 the object $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K^\wedge)$ is equal to

$$R\mathcal{H}om_\mathcal{O}((\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{ff_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{ff_1\ldots f_r}), K)$$

If $f = f_i$ for some i, then $f_1, \ldots, f_r$ generate the unit ideal in $\mathcal{O}_f$, hence the extended alternating Čech complex

$$\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{ff_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{ff_1\ldots f_r}$$

is zero (even homotopic to zero). In this way we see that $K^\wedge$ is derived complete.

If K is derived complete, then $R\mathcal{H}om_\mathcal{O}(\mathcal{O}_f, K)$ is zero for all $f = f_{i_0} \ldots f_{i_p}$, $p \geq 0$. Thus $K \to K^\wedge$ is an isomorphism in $D(\mathcal{O})$. □

Next we explain why derived completion is a completion.

Lemma 14.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Let $K \in D(\mathcal{O})$. The derived completion $K^\wedge$ of Lemma 14.8 is given by the formula*

$$K^\wedge = R\lim K \otimes^\mathbf{L}_\mathcal{O} K_n$$

where $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$ is the Koszul complex on $f_1^n, \ldots, f_r^n$ over $\mathcal{O}$.

Proof. In More on Algebra, Lemma 20.13 we have seen that the extended alternating Čech complex

$$\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r}$$

is a colimit of the Koszul complexes $K^n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$ sitting in degrees $0, \ldots, r$. Note that K^n is a finite chain complex of finite free $\mathcal{O}$-modules with dual $\mathcal{H}om_\mathcal{O}(K^n, \mathcal{O}) = K_n$ where K_n is the Koszul cochain complex sitting in degrees $-r, \ldots, 0$ (as usual). By Lemma 14.8 the functor $K \mapsto K^\wedge$ is gotten by taking $R\mathcal{H}om$ from the extended alternating Čech complex into K:

$$K^\wedge = R\mathcal{H}om(\operatorname{colim} K^n, K)$$

This is equal to $R\lim(K \otimes^\mathbf{L}_\mathcal{O} K_n)$ by Cohomology on Sites, Lemma 36.10. □

Lemma 14.10. *There exist a way to construct*

1. *for every pair (A, I) consisting of a ring A and a finitely generated ideal $I \subset A$ a complex $K(A, I)$ of A-modules,*
2. *a map $K(A, I) \to A$ of complexes of A-modules,*
3. *for every ring map $A \to B$ and finitely generated ideal $I \subset A$ a map of complexes $K(A, I) \to K(B, IB)$,*

such that

(a) *for $A \to B$ and $I \subset A$ finitely generated the diagram*

$$\begin{tikzcd} K(A,I) \arrow[r] \arrow[d] & A \arrow[d] \\ K(B,IB) \arrow[r] & B \end{tikzcd}$$

commutes,

(b) *for $A \to B \to C$ and $I \subset A$ finitely generated the composition of the maps $K(A,I) \to K(B,IB) \to K(C,IC)$ is the map $K(A,I) \to K(C,IC)$.*

(c) *for $A \to B$ and a finitely generated ideal $I \subset A$ the induced map $K(A,I) \otimes_A^{\mathbf{L}} B \to K(B,IB)$ is an isomorphism in $D(B)$, and*

(d) *if $I = (f_1, \ldots, f_r) \subset A$ then there is a commutative diagram*

$$\begin{tikzcd} (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \arrow[r] \arrow[d] & K(A,I) \arrow[d] \\ A \arrow[r, "1"] & A \end{tikzcd}$$

in $D(A)$ whose horizontal arrows are isomorphisms.

Proof. Let S be the set of rings A_0 of the form $A_0 = \mathbf{Z}[x_1, \ldots, x_n]/J$. Every finite type $\mathbf{Z}$-algebra is isomorphic to an element of S. Let $\mathcal{A}_0$ be the category whose objects are pairs (A_0, I_0) where $A_0 \in S$ and $I_0 \subset A_0$ is an ideal and whose morphisms $(A_0, I_0) \to (B_0, J_0)$ are ring maps $\varphi : A_0 \to B_0$ such that $J_0 = \varphi(I_0)B_0$.

Suppose we can construct $K(A_0, I_0) \to A_0$ functorially for objects of $\mathcal{A}_0$ having properties (a), (b), (c), and (d). Then we take

$$K(A,I) = \mathrm{colim}_{\varphi:(A_0,I_0)\to(A,I)} K(A_0, I_0)$$

where the colimit is over ring maps $\varphi : A_0 \to A$ such that $\varphi(I_0)A = I$ with (A_0, I_0) in $\mathcal{A}_0$. A morphism between $(A_0, I_0) \to (A, I)$ and $(A_0', I_0') \to (A, I)$ are given by maps $(A_0, I_0) \to (A_0', I_0')$ in $\mathcal{A}_0$ commuting with maps to A. The category of these $(A_0, I_0) \to (A, I)$ is filtered (details omitted). Moreover, $\mathrm{colim}_{\varphi:(A_0,I_0)\to(A,I)} A_0 = A$ so that $K(A, I)$ is a complex of A-modules. Finally, given $\varphi : A \to B$ and $I \subset A$ for every $(A_0, I_0) \to (A, I)$ in the colimit, the composition $(A_0, I_0) \to (B, IB)$ lives in the colimit for (B, IB). In this way we get a map on colimits. Properties (a), (b), (c), and (d) follow readily from this and the corresponding properties of the complexes $K(A_0, I_0)$.

Endow $\mathcal{C}_0 = \mathcal{A}_0^{opp}$ with the chaotic topology. We equip $\mathcal{C}_0$ with the sheaf of rings $\mathcal{O} : (A, I) \mapsto A$. The ideals I fit together to give a sheaf of ideals $\mathcal{I} \subset \mathcal{O}$. Choose an injective resolution $\mathcal{O} \to \mathcal{J}^\bullet$. Consider the object

$$\mathcal{F}^\bullet = \bigcup\nolimits_n \mathcal{J}^\bullet[\mathcal{I}^n]$$

Let $U = (A, I) \in \mathrm{Ob}(\mathcal{C}_0)$. Since the topology in $\mathcal{C}_0$ is chaotic, the value $\mathcal{J}^\bullet(U)$ is a resolution of A by injective A-modules. Hence the value $\mathcal{F}^\bullet(U)$ is an object of $D(A)$ representing the image of $R\Gamma_I(A)$ in $D(A)$, see Dualizing Complexes, Section

8. Choose a complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ and a commutative diagram

$$\begin{tikzcd} \mathcal{O} \arrow[r] & \mathcal{J}^\bullet \\ \mathcal{K}^\bullet \arrow[u] \arrow[r] & \mathcal{F}^\bullet \arrow[u] \end{tikzcd}$$

where the horizontal arrows are quasi-isomorphisms. This is possible by the construction of the derived category $D(\mathcal{O})$. Set $K(A, I) = \mathcal{K}^\bullet(U)$ where $U = (A, I)$. Properties (a) and (b) are clear and properties (c) and (d) follow from Dualizing Complexes, Lemmas 8.10 and 8.12. ☐

Lemma 14.11. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a map $K \to \mathcal{O}$ in $D(\mathcal{O})$ such that for every $U \in \mathrm{Ob}(\mathcal{C})$ such that $\mathcal{I}|_U$ is generated by $f_1, \ldots, f_r \in \mathcal{I}(U)$ there is an isomorphism*

$$(\mathcal{O}_U \to \prod\nolimits_{i_0} \mathcal{O}_{U,f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{U,f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{U,f_1\ldots f_r}) \longrightarrow K|_U$$

compatible with maps to $\mathcal{O}_U$.

Proof. Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of objects U such that $\mathcal{I}|_U$ is generated by finitely many sections. Then $\mathcal{C}' \to \mathcal{C}$ is a special cocontinuous functor (Sites, Definition 28.2). Hence it suffices to work with $\mathcal{C}'$, see Sites, Lemma 28.1. in other words we may assume that for every object U of $\mathcal{C}$ there exists a finitely generated ideal $I \subset \mathcal{I}(U)$ such that $\mathcal{I}|_U = \mathrm{Im}(I \otimes \mathcal{O}_U \to \mathcal{O}_U)$. We will say that I generates $\mathcal{I}|_U$. Warning: We do not know that $\mathcal{I}(U)$ is a finitely generated ideal in $\mathcal{O}(U)$.

Let U be an object and $I \subset \mathcal{O}(U)$ a finitely generated ideal which generates $\mathcal{I}|_U$. On the category $\mathcal{C}/U$ consider the complex of presheaves

$$K_{U,I}^\bullet : U'/U \longmapsto K(\mathcal{O}(U'), I\mathcal{O}(U'))$$

with $K(-,-)$ as in Lemma 14.10. We claim that the sheafification of this is independent of the choice of I. Indeed, if $I' \subset \mathcal{O}(U)$ is a finitely generated ideal which also generates $\mathcal{I}|_U$, then there exists a covering $\{U_j \to U\}$ such that $I\mathcal{O}(U_j) = I'\mathcal{O}(U_j)$. (Hint: this works because both I and I' are finitely generated and generate $\mathcal{I}|_U$.) Hence $K_{U,I}^\bullet$ and $K_{U,I'}^\bullet$ are the *same* for any object lying over one of the U_j. The statement on sheafifications follows. Denote $K_U^\bullet$ the common value.

The independence of choice of I also shows that $K_U^\bullet|_{\mathcal{C}/U'} = K_{U'}^\bullet$ whenever we are given a morphism $U' \to U$ and hence a localization morphism $\mathcal{C}/U' \to \mathcal{C}/U$. Thus the complexes $K_U^\bullet$ glue to give a single well defined complex $K^\bullet$ of $\mathcal{O}$-modules. The existence of the map $K^\bullet \to \mathcal{O}$ and the quasi-isomorphism of the lemma follow immediately from the corresponding properties of the complexes $K(-,-)$ in Lemma 14.10. ☐

Proposition 14.12. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a left adjoint to the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$.*

Proof. Let $K \to \mathcal{O}$ in $D(\mathcal{O})$ be as constructed in Lemma 14.11. Let $E \in D(\mathcal{O})$. Then $E^\wedge = R\mathcal{H}om(K, E)$ together with the map $E \to E^\wedge$ will do the job. Namely, locally on the site $\mathcal{C}$ we recover the adjoint of Lemma 14.8. This shows that $E^\wedge$ is always derived complete and that $E \to E^\wedge$ is an isomorphism if E is derived complete. ☐

Remark 14.13 (Localization and derived completion)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $K \mapsto K^\wedge$ be the derived completion functor of Proposition 14.12. It follows from the construction in the proof of the proposition that $K^\wedge|_U$ is the derived completion of $K|_U$ for any $U \in \mathrm{Ob}(\mathcal{C})$. But we can also prove this as follows. From the definition of derived complete objects it follows that $K^\wedge|_U$ is derived complete. Thus we obtain a canonical map $a : (K|_U)^\wedge \to K^\wedge|_U$. On the other hand, if E is a derived complete object of $D(\mathcal{O}_U)$, then Rj_*E is a derived complete object of $D(\mathcal{O})$ by Lemma 14.7. Here j is the localization morphism (Modules on Sites, Section 19). Hence we also obtain a canonical map $b : K^\wedge \to Rj_*((K|_U)^\wedge)$. We omit the (formal) verification that the adjoint of b is the inverse of a.

Remark 14.14 (Completed tensor product)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Denote $K \mapsto K^\wedge$ the adjoint of Proposition 14.12. Then we set

$$K \otimes_{\mathcal{O}}^\wedge L = (K \otimes_{\mathcal{O}}^{\mathbf{L}} L)^\wedge$$

This *completed tensor product* defines a functor $D_{comp}(\mathcal{O}) \times D_{comp}(\mathcal{O}) \to D_{comp}(\mathcal{O})$ such that we have

$$\mathrm{Hom}_{D_{comp}(\mathcal{O})}(K, R\mathcal{H}om_{\mathcal{O}}(L, M)) = \mathrm{Hom}_{D_{comp}(\mathcal{O})}(K \otimes_{\mathcal{O}}^\wedge L, M)$$

for $K, L, M \in D_{comp}(\mathcal{O})$. Note that $R\mathcal{H}om_{\mathcal{O}}(L, M) \in D_{comp}(\mathcal{O})$ by Lemma 14.5.

Lemma 14.15. *Let $\mathcal{C}$ be a site. Assume $\varphi : \mathcal{O} \to \mathcal{O}'$ is a flat homomorphism of sheaves of rings. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$ such that $\mathcal{O}/(f_1, \ldots, f_r) \cong \mathcal{O}'/(f_1, \ldots, f_r)$. Then the map of extended alternating Čech complexes*

$$\begin{array}{c} \mathcal{O} \to \prod_{i_0} \mathcal{O}_{f_{i_0}} \to \prod_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r} \\ \downarrow \\ \mathcal{O}' \to \prod_{i_0} \mathcal{O}'_{f_{i_0}} \to \prod_{i_0<i_1} \mathcal{O}'_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}'_{f_1\ldots f_r} \end{array}$$

is a quasi-isomorphism.

Proof. Observe that the second complex is the tensor product of the first complex with $\mathcal{O}'$. We can write the first extended alternating Čech complex as a colimit of the Koszul complexes $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$, see More on Algebra, Lemma 20.13. Hence it suffices to prove $K_n \to K_n \otimes_{\mathcal{O}} \mathcal{O}'$ is a quasi-isomorphism. Since $\mathcal{O} \to \mathcal{O}'$ is flat it suffices to show that $H^i \to H^i \otimes_{\mathcal{O}} \mathcal{O}'$ is an isomorphism where H^i is the ith cohomology sheaf $H^i = H^i(K_n)$. These sheaves are annihilated by $f_1^n, \ldots, f_r^n$, see More on Algebra, Lemma 20.6. Thus it suffices to show that $\mathcal{O}/(f_1^n, \ldots, f_r^n) \to \mathcal{O}'/(f_1^n, \ldots, f_r^n)$ is an isomorphism. Equivalently, we will show that $\mathcal{O}/(f_1, \ldots, f_r)^n \to \mathcal{O}'/(f_1, \ldots, f_r)^n$ is an isomorphism for all n. This holds for $n = 1$ by assumption. It follows for all n by induction using Modules on Sites, Lemma 28.13 applied to the ring map $\mathcal{O}/(f_1, \ldots, f_r)^{n+1} \to \mathcal{O}/(f_1, \ldots, f_r)^n$ and the module $\mathcal{O}'/(f_1, \ldots, f_r)^{n+1}$. □

Lemma 14.16. *Let $\mathcal{C}$ be a site with enough points. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. If $\mathcal{O} \to \mathcal{O}'$ is flat and $\mathcal{O}/\mathcal{I} \cong \mathcal{O}'/\mathcal{I}\mathcal{O}'$, then the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ induces an equivalence $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}') \to D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. Let $K \to \mathcal{O}$ be the morphism of $D(\mathcal{O})$ constructed in Lemma 14.11. Set $K' = K \otimes_{\mathcal{O}} \mathcal{O}'$. Then $K' \to \mathcal{O}'$ is a map in $D(\mathcal{O}')$ satisfying the same condition with respect to $\mathcal{I}' = \mathcal{I}\mathcal{O}'$. The map $K \to K'$ is a quasi-isomorphism by Lemma 14.15. Now, let $E \in D_{comp}(\mathcal{O}, \mathcal{I})$. By the proof of Proposition 14.12 we have the first equality in

$$E = R\mathcal{H}om_{\mathcal{O}}(K, E) = R\mathcal{H}om_{\mathcal{O}}(K', E)$$

Since K' is a complex of $\mathcal{O}'$ modules, this shows that E is the image of some $E' \in D(\mathcal{O}')$. By Lemma 14.6 we have $E' \in D_{comp}(\mathcal{O}')$. Thus the functor is essentially surjective. In fact the functor $E \mapsto R\mathcal{H}om_{\mathcal{O}}(K', E)$ is a quasi-inverse to the restriction functor. The formula above shows this in one direction. The other direction hinges on the fact that for $E' \in D_{comp}(\mathcal{O}')$ the map

$$R\mathcal{H}om_{\mathcal{O}}(K', E') \to R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}', E')$$

is an isomorphism in $D(\mathcal{O}')$ for both $\mathcal{O}'$-module structures and the existence of a map $E' \to R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}', E')$ which is $\mathcal{O}'$-linear for both $\mathcal{O}'$-module structures on the target. Details omitted. □

Lemma 14.17. *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be finite type sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$ and has a left adjoint Lf^*_{comp} which is Lf^* followed by derived completion.*

Proof. The first statement we have seen in Lemma 14.7. Note that the second statement makes sense as we have a derived completion functor $D(\mathcal{O}') \to D_{comp}(\mathcal{O}', \mathcal{I}')$ by Proposition 14.12. OK, so now let $K \in D_{comp}(\mathcal{O}, \mathcal{I})$ and $M \in D_{comp}(\mathcal{O}', \mathcal{I}')$. Then we have

$$R\operatorname{Hom}(K, Rf_*M) = R\operatorname{Hom}(Lf^*K, M) = R\operatorname{Hom}(Lf^*_{comp}K, M)$$

by the universal property of derived completion. □

Lemma 14.18. *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Let $\mathcal{I}' \subset \mathcal{O}'$ be the ideal generated by $f^\sharp(f^{-1}\mathcal{I})$. Then Rf_* commutes with derived completion, i.e., $Rf_*(K^\wedge) = (Rf_*K)^\wedge$.*

Proof. By Proposition 14.12 the derived completion functors exist. By Lemma 14.7 the object $Rf_*(K^\wedge)$ is derived complete, and hence we obtain a canonical map $(Rf_*K)^\wedge \to Rf_*(K^\wedge)$ by the universal property of derived completion. We may check this map is an isomorphism locally on $\mathcal{C}$. Thus, since derived completion commutes with localization (Remark 14.13) we may assume that $\mathcal{I}$ is generated by global sections $f_1, \ldots, f_r$. Then $\mathcal{I}'$ is generated by $g_i = f^\sharp(f_i)$. By Lemma 14.9 we have to prove that

$$R\lim\left(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}} K(\mathcal{O}, f_1^n, \ldots, f_r^n)\right) = Rf_*\left(R\lim K \otimes^{\mathbf{L}}_{\mathcal{O}'} K(\mathcal{O}', g_1^n, \ldots, g_r^n)\right)$$

Because Rf_* commutes with $R\lim$ (Cohomology on Sites, Lemma 21.2) it suffices to prove that

$$Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}} K(\mathcal{O}, f_1^n, \ldots, f_r^n) = Rf_*\left(K \otimes^{\mathbf{L}}_{\mathcal{O}'} K(\mathcal{O}', g_1^n, \ldots, g_r^n)\right)$$

This follows from the projection formula (Cohomology on Sites, Lemma 37.1) and the fact that $Lf^*K(\mathcal{O}, f_1^n, \ldots, f_r^n) = K(\mathcal{O}', g_1^n, \ldots, g_r^n)$. □

15. Application to theorem on formal functions

We interrupt the flow of the exposition to talk a little bit about derived completion in the setting of quasi-coherent modules on schemes and to use this to give a somewhat different proof of the theorem on formal functions. We give some pointers to the literature in Remark 15.5.

Lemma 14.18 is a (very formal) derived version of the theorem on formal functions (Cohomology of Schemes, Theorem 18.5). To make this more explicit, suppose $f : X \to S$ is a morphism of schemes, $\mathcal{I} \subset \mathcal{O}_S$ is a quasi-coherent sheaf of ideals, and $\mathcal{F}$ is a quasi-coherent sheaf on X. Then the lemma says that

$$Rf_*(\mathcal{F}^\wedge) = (Rf_*\mathcal{F})^\wedge \tag{15.0.1}$$

where $\mathcal{F}^\wedge$ is the derived completion of $\mathcal{F}$ with respect to $f^{-1}\mathcal{I} \cdot \mathcal{O}_X$ and the right hand side is the derived completion of $\mathcal{F}$ with respect to $\mathcal{I}$. To see that this gives back the theorem on formal functions we have to do a bit of work. (We will work out what it means in the setting of the usual theorem on formal functions, but dear reader, we encourage you to try and discover new variants of the theorem on formal functions by using this in other cases.)

Lemma 15.1. *Let X be a scheme. Let (K_n) be an inverse system of $D_{QCoh}(\mathcal{O}_X)$ such that the maps $H^q(K_{n+1}) \to H^q(K_n)$ are surjective for all $q \in \mathbf{Z}$ and $n \geq 1$. Then the derived limit $K = R\lim K_n$ in $D(\mathcal{O}_X)$ has cohomology sheaves $H^q(K) = \lim H^q(K_n)$. Moreover, $R\lim H^q(K_n) = \lim H^q(K_n)$.*

Proof. This follows from Cohomology on Sites, Lemma 22.5. Namely, let $\mathcal{B}$ be the set of affine opens of X_{Zar}. The required vanishing follows from Cohomology of Schemes, Lemma 2.2 and the vanishing of $R^1\lim$ because the transition maps $H^0(U, H^q(K_{n+1})) \to H^0(U, H^q(K_n))$ are surjective for affine open subschemes of X by Schemes, Lemma 7.5. $\square$

Lemma 15.2. *Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.*

1. *A coherent $\mathcal{O}_X$-module $\mathcal{F}$ has derived completion $\mathcal{F}^\wedge$ equal to $\lim \mathcal{F}/\mathcal{I}^n\mathcal{F}$.*
2. *A pseudo-coherent object K of $D(\mathcal{O}_X)$ has derived completion $K^\wedge$ with cohomology sheaves $H^q(K^\wedge)$ equal to $H^q(K)^\wedge$.*

Proof. Proof of (1). Since derived completion commutes with localization (Remark 14.13) we may assume $X = \mathrm{Spec}(A)$ and $\mathcal{I} = \widetilde{I}$ for an ideal $I \subset A$. Say $I = (f_1, \ldots, f_r)$. Let $K_n = K(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex. By Lemma 14.9 the derived completion of $\mathcal{F}$ is given by $R\lim \mathcal{F} \otimes_A K_n$. Let $U = \mathrm{Spec}(B) \subset X$ be an affine open. Since $R\Gamma(U, -)$ commutes with $R\lim$ (Injectives, Lemma 13.6) we see that

$$R\Gamma(U, \mathcal{F}^\wedge) = R\lim \mathcal{F}(U) \otimes_A K_n$$

This is the derived completion of $\mathcal{F}(U)$ with respect to IB by More on Algebra, Lemma 64.16 and the fact that $K_n \otimes_A B = K(B, f_1^n, \ldots, f_r^n)$. By More on Algebra, Lemma 64.20 we conclude that $R\Gamma(U, \mathcal{F}^\wedge)$ has vanishing cohomology in degrees different from 0 and $H^0(U, \mathcal{F}^\wedge)$ is the completion of $\mathcal{F}(U)$ in degree 0. Since the affine opens form a basis for the topology, the lemma follows.

Part (2) can either be proved in exactly the same manner as part (1) or it can be deduced from part (1) using the derived completion is an exact functor between triangulated categories. Details omitted. $\square$

Lemma 15.3. *Let $S = \mathrm{Spec}(A)$ be an affine Noetherian scheme. Let $I \subset A$ be an ideal and let $\mathcal{I} \subset \mathcal{O}_S$ be the corresponding quasi-coherent sheaf of ideals. Let K be a pseudo-coherent object of $D(\mathcal{O}_S)$ with derived completion $K^\wedge$. Then*

$$H^p(S, K^\wedge) = H^p(S, K)^\wedge = \lim H^p(S, K)/I^n H^p(S, K)$$

Proof. Follows from Lemma 15.2 and the fact that $R\Gamma(S, -)$ commutes with derived limits. Alternately one could prove this by applying Lemma 14.18 to the morphism of ringed spaces $(S, \mathcal{O}_S) \to (pt, A)$ and using More on Algebra, Lemma 64.20. ☐

Lemma 15.4. *Let $f : X \to S$ be a morphism of Noetherian schemes with $S = \mathrm{Spec}(A)$ affine. Let $I \subset A$ be an ideal. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that $Rf_*\mathcal{F}$ is a bounded complex with coherent cohomology sheaves. Then there are short exact sequences*

$$0 \to R^1 \lim H^{p-1}(X, \mathcal{F}/I^n\mathcal{F}) \to H^p(X, \mathcal{F})^\wedge \to \lim H^p(X, \mathcal{F}/I^n\mathcal{F}) \to 0$$

of A-modules. If f is proper, then the $R^1 \lim$ term is zero.

Proof. We are going to prove this by working out what (15.0.1) means in this setting. Let us apply $H^p(-)$ to obtain

$$H^p(X, \mathcal{F}^\wedge) = H^p(S, (Rf_*\mathcal{F})^\wedge)$$

Lemma 15.3 tells us that the right hand side is equal to

$$H^p(S, Rf_*\mathcal{F})^\wedge = H^p(X, \mathcal{F})^\wedge = \lim H^p(X, \mathcal{F}/I^n\mathcal{F}).$$

On the other hand, Lemmas 15.2 and 15.1 tell us that $\mathcal{F}^\wedge = \lim \mathcal{F}/I^n\mathcal{F} = R\lim \mathcal{F}/I^n\mathcal{F}$. Since $R\Gamma(X, -)$ commutes with derived limits we obtain for LHS

$$H^p(X, \mathcal{F}^\wedge) = H^p(R\lim R\Gamma(X, \mathcal{F}/I^n\mathcal{F}))$$

By More on Algebra, Remark 61.16 we obtain exact sequences as in the statement of the lemma. The vanishing of the $R^1 \lim$ term follows from Cohomology of Schemes, Lemma 18.4. ☐

Remark 15.5. Here are some references to discussions of related material the literature. It seems that a "derived formal functions theorem" for proper maps goes back to [Lur04, Theorem 6.3.1]. There is the discussion in [Lur11], especially Chapter 4 which discusses the affine story, see More on Algebra, Section 64. In [GR13, Section 2.9] one finds a discussion of proper base change and derived completion using (ind) coherent modules. An analogue of (15.0.1) for complexes of quasi-coherent modules can be found as [HLP14, Theorem 6.5]

16. Derived completion in the constant Noetherian case

Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Recall from Modules on Sites, Lemma 41.4 that

$$\underline{\Lambda}^\wedge = \lim \underline{\Lambda/I^n}$$

is a flat $\underline{\Lambda}$-algebra and that the map $\underline{\Lambda} \to \underline{\Lambda}^\wedge$ identifies quotients by I. Hence Lemma 14.16 tells us that

$$D_{comp}(\mathcal{C}, \Lambda) = D_{comp}(\mathcal{C}, \underline{\Lambda}^\wedge)$$

In particular the cohomology sheaves $H^i(K)$ of an object K of $D_{comp}(\mathcal{C}, \Lambda)$ are sheaves of $\underline{\Lambda}^\wedge$-modules. For notational convenience we often work with $D_{comp}(\mathcal{C}, \Lambda)$.

Lemma 16.1. *Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. The left adjoint to the inclusion functor $D_{comp}(\mathcal{C}, \Lambda) \to D(\mathcal{C}, \Lambda)$ of Proposition 14.12 sends K to*

$$K^\wedge = R\lim(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$$

In particular, K is derived complete if and only if $K = R\lim(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$.

Proof. Choose generators $f_1, \ldots, f_r$ of I. By Lemma 14.9 we have

$$K^\wedge = R\lim(K \otimes_\Lambda^{\mathbf{L}} \underline{K_n})$$

where $K_n = K(\Lambda, f_1^n, \ldots, f_r^n)$. In More on Algebra, Lemma 64.18 we have seen that the pro-systems $\{K_n\}$ and $\{\Lambda/I^n\}$ of $D(\Lambda)$ are isomorphic. Thus the lemma follows. ☐

Lemma 16.2. *Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Then*

1. *Rf_* sends $D_{comp}(\mathcal{D}, \Lambda)$ into $D_{comp}(\mathcal{C}, \Lambda)$,*
2. *the map $Rf_* : D_{comp}(\mathcal{D}, \Lambda) \to D_{comp}(\mathcal{C}, \Lambda)$ has a left adjoint $Lf^*_{comp} : D_{comp}(\mathcal{C}, \Lambda) \to D_{comp}(\mathcal{D}, \Lambda)$ which is Lf^* followed by derived completion,*
3. *Rf_* commutes with derived completion,*
4. *for K in $D_{comp}(\mathcal{D}, \Lambda)$ we have $Rf_*K = R\lim Rf_*(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$.*
5. *for M in $D_{comp}(\mathcal{C}, \Lambda)$ we have $Lf^*_{comp}M = R\lim Lf^*(M \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$.*

Proof. We have seen (1) and (2) in Lemma 14.17. Part (3) follows from Lemma 14.18. For (4) let K be derived complete. Then

$$Rf_*K = Rf_*(R\lim K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}) = R\lim Rf_*(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$$

the first equality by Lemma 16.1 and the second because Rf_* commutes with $R\lim$ (Cohomology on Sites, Lemma 21.2). This proves (4). To prove (5), by Lemma 16.1 we have

$$Lf^*_{comp}M = R\lim(Lf^*M \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$$

Since Lf^* commutes with derived tensor product by Cohomology on Sites, Lemma 18.4 and since $Lf^*\underline{\Lambda/I^n} = \underline{\Lambda/I^n}$ we get (5). ☐

17. Derived completion on the pro-étale site

Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Although the general theory (see Sections 14 and 16) concerning $D_{comp}(\mathcal{C}, \Lambda)$ is quite satisfactory it is somewhat useless as it is hard to explicitly give examples of derived complete complexes. We know that

1. every object M of $D(\mathcal{C}, \Lambda/I^n)$ restricts to a derived complete object of $D(\mathcal{C}, \Lambda)$, and
2. for every $K \in D(\mathcal{C}, \Lambda)$ the derived completion $K^\wedge = R\lim(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n})$ is derived complete.

The first type of objects are trivially complete and perhaps not interesting. The problem with (2) is that derived completion in general is somewhat mysterious, even in case $K = \underline{\Lambda}$. Namely, by definition of homotopy limits there is a distinguished triangle

$$R\lim(\underline{\Lambda/I^n}) \to \prod \underline{\Lambda/I^n} \to \prod \underline{\Lambda/I^n} \to R\lim(\underline{\Lambda/I^n})[1]$$

in $D(\mathcal{C}, \Lambda)$ where the products are in $D(\mathcal{C}, \Lambda)$. These are computed by taking products of injective resolutions (Injectives, Lemma 13.4), so we see that the sheaf $H^p(\prod \underline{\Lambda/I^n})$ is the sheafification of the presheaf

$$U \longmapsto \prod H^p(U, \Lambda/I^n).$$

As an explicit example, if $X = \text{Spec}(\mathbf{C}[t, t^{-1}])$, $\mathcal{C} = X_{\acute{e}tale}$, $\Lambda = \mathbf{Z}$, $I = (2)$, and $p = 1$, then we get the sheafification of the presheaf

$$U \mapsto \prod H^1(U_{\acute{e}tale}, \mathbf{Z}/2^n\mathbf{Z})$$

for U étale over X. Note that $H^1(X_{\acute{e}tale}, \mathbf{Z}/m\mathbf{Z})$ is cyclic of order m with generator α_m given by the finite étale $\mathbf{Z}/m\mathbf{Z}$-covering given by the equation $t = s^m$ (see Étale Cohomology, Section 6). Then the section

$$\alpha = (\alpha_{2^n}) \in \prod H^1(X_{\acute{e}tale}, \mathbf{Z}/2^n\mathbf{Z})$$

of the presheaf above does not restrict to zero on any nonempty étale scheme over X, whence the sheaf associated to the presheaf is not zero.

However, on the pro-étale site this phenomenon does not occur. The reason is that we have enough (quasi-compact) weakly contractible objects. In the following proposition we collect some results about derived completion in the Noetherian constant case for sites having enough weakly contractible objects (see Sites, Definition 39.2).

Proposition 17.1. *Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has enough weakly contractible objects. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal.*

1. *The category of derived complete sheaves Λ-modules is a weak Serre subcategory of $Mod(\mathcal{C}, \Lambda)$.*
2. *A sheaf $\mathcal{F}$ of Λ-modules satisfies $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$ if and only if $\mathcal{F}$ is derived complete and $\bigcap I^n\mathcal{F} = 0$.*
3. *The sheaf $\underline{\Lambda}^\wedge$ is derived complete.*
4. *If $\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$ is an inverse system of derived complete sheaves of Λ-modules, then $\lim \mathcal{F}_n$ is derived complete.*
5. *An object $K \in D(\mathcal{C}, \Lambda)$ is derived complete if and only if each cohomology sheaf $H^p(K)$ is derived complete.*
6. *An object $K \in D_{comp}(\mathcal{C}, \Lambda)$ is bounded above if and only if $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ is bounded above.*
7. *An object $K \in D_{comp}(\mathcal{C}, \Lambda)$ is bounded if $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ has finite tor dimension.*

Proof. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset such that every $U \in \mathcal{B}$ is weakly contractible and every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$. We will use the results of Cohomology on Sites, Lemma 38.1 and Proposition 38.2 without further mention.

Recall that $R\lim$ commutes with $R\Gamma(U, -)$, see Injectives, Lemma 13.6. Let $f \in I$. Recall that $T(K, f)$ is the homotopy limit of the system

$$\ldots K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(\mathcal{C}, \Lambda)$. Thus

$$R\Gamma(U, T(K, f)) = T(R\Gamma(U, K), f).$$

Since we can test isomorphisms of maps between objects of $D(\mathcal{C}, \Lambda)$ by evaluating at $U \in \mathcal{B}$ we conclude an object K of $D(\mathcal{C}, \Lambda)$ is derived complete if and only if for every $U \in \mathcal{B}$ the object $R\Gamma(U, K)$ is derived complete as an object of $D(\Lambda)$.

The remark above implies that items (1), (5) follow from the corresponding results for modules over rings, see More on Algebra, Lemmas 64.1 and 64.6. In the same way (2) can be deduced from More on Algebra, Proposition 64.5 as $(I^n\mathcal{F})(U) = I^n \cdot \mathcal{F}(U)$ for $U \in \mathcal{B}$ (by exactness of evaluating at U).

Proof of (4). The homotopy limit $R\lim \mathcal{F}_n$ is in $D_{comp}(X, \Lambda)$ (see discussion following Definition 14.4). By part (5) just proved we conclude that $\lim \mathcal{F}_n = H^0(R\lim \mathcal{F}_n)$ is derived complete. Part (3) is a special case of (4).

Proof of (6) and (7). Follows from Lemma 16.1 and Cohomology on Sites, Lemma 35.8 and the computation of homotopy limits in Cohomology on Sites, Proposition 38.2. □

18. Comparison with the étale site

Let X be a scheme. With suitable choices of sites (as in Topologies, Remark 9.1) the functor $u : X_{étale} \to X_{pro\text{-}étale}$ sending U/X to U/X defines a morphism of sites

$$\epsilon : X_{pro\text{-}étale} \longrightarrow X_{étale}$$

This follows from Sites, Proposition 15.6. A fundamental fact about this comparison morphism is the following.

Lemma 18.1. *Let X be a scheme. Let $Y = \lim Y_i$ be an inverse limit of quasi-compact and quasi-separated schemes étale over X with affine transition morphisms. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\epsilon^{-1}\mathcal{F}(Y) = \operatorname{colim} \mathcal{F}(Y_i)$.*

Proof. Let $\mathcal{F} = h_U$ be a representable sheaf on $X_{étale}$ with U an object of $X_{étale}$. In this case $\epsilon^{-1}h_U = h_{u(U)}$ where $u(U)$ is U viewed as an object of $X_{pro\text{-}étale}$ (Sites, Lemma 14.5). Then

$$h_{u(U)}(Y) = \operatorname{Mor}_X(Y, U) = \operatorname{colim} \operatorname{Mor}_X(Y_i, U) = \operatorname{colim} h_U(Y_i)$$

by Limits, Proposition 5.1. Hence the lemma holds for every representable sheaf. Since every sheaf is a coequalizer of a map of coproducts of representable sheaves (Sites, Lemma 13.5) we obtain the result in general. □

Lemma 18.2. *Let X be a scheme. For every sheaf $\mathcal{F}$ on $X_{étale}$ the adjunction map $\mathcal{F} \to \epsilon_*\epsilon^{-1}\mathcal{F}$ is an isomorphism.*

Proof. Suppose that U is a quasi-compact and quasi-separated scheme étale over X. Then

$$\epsilon_*\epsilon^{-1}\mathcal{F}(U) = \epsilon^{-1}\mathcal{F}(U) = \mathcal{F}(U)$$

the second equality by (a special case of) Lemma 18.1. Since every object of $X_{étale}$ has a covering by quasi-compact and quasi-separated objects we conclude. □

Lemma 18.3. *Let X be an affine scheme. For injective abelian sheaf $\mathcal{I}$ on $X_{étale}$ we have $H^p(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{I}) = 0$ for $p > 0$.*

Proof. We are going to use Cohomology on Sites, Lemma 11.9 to prove this. The idea is simple: We show that every standard pro-étale covering of X is a limit of coverings in $X_{\acute{e}tale}$. If this holds then Lemma 18.1 will kick in to show the Čech cohomology groups of $\epsilon^{-1}\mathcal{I}$ are colimits of those of $\mathcal{I}$ which are zero in positive degree.

Here are the details. Let $\mathcal{B} \subset \text{Ob}(X_{pro\text{-}\acute{e}tale})$ be the set of affine schemes U over X such that $\mathcal{O}(X) \to \mathcal{O}(U)$ is ind-étale. Let Cov be the set of pro-étale coverings $\{U_i \to U\}_{i=1,\ldots,n}$ with $U, U_i \in \mathcal{B}$ such that $\mathcal{O}(U) \to \mathcal{O}(U_i)$ is ind-étale for $i = 1, \ldots, n$. Properties (1) and (2) of Cohomology on Sites, Lemma 11.9 hold for $\mathcal{B}$ and Cov by Proposition 9.1 (it also follows from Lemma 11.10).

To check condition (3) suppose that $\{U_i \to U\}_{i=1,\ldots,n}$ is an element of Cov. Then we can write $U_i = \lim_{a \in A_i} U_{i,a}$ with $U_{i,a} \to U$ étale and $U_{i,a}$ affine. Next we write $U = \lim_{b \in B} U_b$ with U_b affine and $U_b \to U$ étale. By Limits, Lemma 9.1 for each i and $a \in A_i$ we can choose a $b(i,a) \in B$ and for all $b \geq b(i,a)$ an affine scheme $U_{i,a,b}$ étale over U_b such that $U_{i,a} = \lim_{b \geq b(i,a)} U_{i,a,b}$[2]. Moreover, any transition map $U_{i,a} \to U_{i,a'}$ comes from an essentially unique morphism $U_{i,a,b} \to U_{i,a',b}$ for b large enough (by the same reference). Finally, given $a_1 \in A_1, \ldots, a_n \in A_n$ the morphism $U_{1,a_1} \amalg \ldots \amalg U_{n,a_n} \to U$ is surjective, hence for b large enough the map $U_{1,a_1,b} \amalg \ldots \amalg U_{n,a_n,b} \to U_b$ is surjective by Limits, Lemma 7.11. Let $\mathcal{D}$ be the category of coverings $\{U_{i,a_i,b} \to U_b\}_{i=1,\ldots,n}$ so obtained. This category is cofiltered. We claim that, given $i_0, \ldots, i_p \in \{1, \ldots, n\}$ we have

$$U_{i_0} \times_U U_{i_1} \times_U \ldots \times_U U_{i_p} = \lim_{\mathcal{D}} U_{i_0,a_{i_0},b} \times_{U_b} U_{i_1,a_{i_1},b} \times_{U_b} \ldots \times_{U_b} U_{i_p,a_{i_p},b}$$

This is clear from the fact that it holds for $p = -1$ (i.e., $U = \lim_{\mathcal{D}} U_b$) and for $p = 0$ (i.e., $U_i = \lim_{\mathcal{D}} U_{i,a_i,b}$) and the fact that fibre products commute with limits. Then finally it follows from Lemma 18.1 that

$$\check{\mathcal{C}}^\bullet(\{U_i \to U\}, \epsilon^{-1}\mathcal{I}) = \text{colim}_{\mathcal{D}^{opp}} \check{\mathcal{C}}^\bullet(\{U_{i,a_i,b} \to U_b\}, \mathcal{I})$$

Since each of the Čech complexes on the right hand side is acyclic in positive degrees (Cohomology on Sites, Lemma 11.2) it follows that the one on the left is too. This prove condition (3) of Cohomology on Sites, Lemma 11.9. Since $X \in \mathcal{B}$ the lemma follows. □

Lemma 18.4. *Let X be a scheme. For an abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have $R\epsilon_*(\epsilon^{-1}\mathcal{F}) = \mathcal{F}$.*

Proof. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{\acute{e}tale}$. Recall that $R^q\epsilon_*(\epsilon^{-1}\mathcal{I})$ is the sheaf associated to $U \mapsto H^q(U_{pro\text{-}\acute{e}tale}, \epsilon^{-1}\mathcal{I})$, see Cohomology on Sites, Lemma 8.4. By Lemma 18.3 we see that this is zero for $q > 0$ and U affine and étale over X. Since every object of $X_{\acute{e}tale}$ has a covering by affine objects, it follows that $R^q\epsilon_*(\epsilon^{-1}\mathcal{I}) = 0$ for $q > 0$. Combined with Lemma 18.2 we conclude that $R\epsilon_*\epsilon^{-1}\mathcal{I} = \mathcal{I}$ for every injective abelian sheaf. Since every abelian sheaf has a resolution by injective sheaves, the result follows. (Hint: use Leray acyclicity theorem – Derived Categories, Lemma 17.7.) □

[2] To be sure, we pick $U_{i,a,b} = U_b \times_{U_{b(i,a)}} U_{i,a,b(i,a)}$ although this isn't necessary for what follows.

Lemma 18.5. *Let X be a scheme. For an abelian sheaf $\mathcal{F}$ on $X_{\acute{e}tale}$ we have*
$$H^i(X_{\acute{e}tale}, \mathcal{F}) = H^i(X_{pro\text{-}\acute{e}tale}, \epsilon^{-1}\mathcal{F})$$
for all i.

Proof. Immediate consequence of Lemma 18.4 and the Leray spectral sequence (Cohomology on Sites, Lemma 14.6). □

Lemma 18.6. *Let X be a scheme. Let $\mathcal{G}$ be a sheaf of (possibly noncommutative) groups on $X_{\acute{e}tale}$. We have*
$$H^1(X_{\acute{e}tale}, \mathcal{G}) = H^1(X_{pro\text{-}\acute{e}tale}, \epsilon^{-1}\mathcal{G})$$
where H^1 is defined as the set of isomorphism classes of torsors (see Cohomology on Sites, Section 5).

Proof. Since the functor ϵ^{-1} is fully faithful by Lemma 18.2 it is clear that the map $H^1(X_{\acute{e}tale}, \mathcal{G}) \to H^1(X_{pro\text{-}\acute{e}tale}, \epsilon^{-1}\mathcal{G})$ is injective. To show surjectivity it suffices to show that any $\epsilon^{-1}\mathcal{G}$-torsor $\mathcal{F}$ is étale locally trivial. To do this we may assume that X is affine. Thus we reduce to proving surjectivity for X affine.

Choose a covering $\{U \to X\}$ with (a) U affine, (b) $\mathcal{O}(X) \to \mathcal{O}(U)$ ind-étale, and (c) $\mathcal{F}(U)$ nonempty. We can do this by Proposition 9.1 and the fact that standard pro-étale coverings of X are cofinal among all pro-étale coverings of X (Lemma 11.5). Write $U = \lim U_i$ as a limit of affine schemes étale over X. Pick $s \in \mathcal{F}(U)$. Let $g \in \epsilon^{-1}\mathcal{G}(U \times_X U)$ be the unique section such that $g \cdot \text{pr}_1^* s = \text{pr}_2^* s$ in $\mathcal{F}(U \times_X U)$. Then g satisfies the cocycle condition
$$\text{pr}_{12}^* g \cdot \text{pr}_{23}^* g = \text{pr}_{13}^* g$$
in $\epsilon^{-1}\mathcal{G}(U \times_X U \times_X U)$. By Lemma 18.1 we have
$$\epsilon^{-1}\mathcal{G}(U \times_X U) = \text{colim}\, \mathcal{G}(U_i \times_X U_i)$$
and
$$\epsilon^{-1}\mathcal{G}(U \times_X U \times_X U) = \text{colim}\, \mathcal{G}(U_i \times_X U_i \times_X U_i)$$
hence we can find an i and an element $g_i \in \mathcal{G}(U_i)$ mapping to g satisfying the cocycle condition. The cocycle g_i then defines a torsor for $\mathcal{G}$ on $X_{\acute{e}tale}$ whose pullback is isomorphic to $\mathcal{F}$ by construction. Some details omitted (namely, the relationship between torsors and 1-cocycles which should be added to the chapter on cohomology on sites). □

Lemma 18.7. *Let X be a scheme. Let Λ be a ring.*

(1) *The essential image of $\epsilon^{-1} : Mod(X_{\acute{e}tale}, \Lambda) \to Mod(X_{pro\text{-}\acute{e}tale}, \Lambda)$ is a weak Serre subcategory $\mathcal{C}$.*
(2) *The functor ϵ^{-1} defines an equivalence of categories of $D^+(X_{\acute{e}tale}, \Lambda)$ with $D^+_{\mathcal{C}}(X_{pro\text{-}\acute{e}tale}, \Lambda)$.*

Proof. To prove (1) we will prove conditions (1) – (4) of Homology, Lemma 9.3. Since ϵ^{-1} is fully faithful (Lemma 18.2) and exact, everything is clear except for condition (4). However, if
$$0 \to \epsilon^{-1}\mathcal{F}_1 \to \mathcal{G} \to \epsilon^{-1}\mathcal{F}_2 \to 0$$
is a short exact sequence of sheaves of Λ-modules on $X_{pro\text{-}\acute{e}tale}$, then we get
$$0 \to \epsilon_*\epsilon^{-1}\mathcal{F}_1 \to \epsilon_*\mathcal{G} \to \epsilon_*\epsilon^{-1}\mathcal{F}_2 \to R^1\epsilon_*\epsilon^{-1}\mathcal{F}_1$$

which by Lemma 18.4 is the same as a short exact sequence

$$0 \to \mathcal{F}_1 \to \epsilon_*\mathcal{G} \to \mathcal{F}_2 \to 0$$

Pulling pack we find that $\mathcal{G} = \epsilon^{-1}\epsilon_*\mathcal{G}$. This proves (1).

By (1) and the discussion in Derived Categories, Section 13 we obtain a strictly full, saturated, triangulated subcategory $D_\mathcal{C}(X_{pro\text{-}\acute{e}tale}, \Lambda)$. It is clear that ϵ^{-1} maps $D(X_{\acute{e}tale}, \Lambda)$ into $D_\mathcal{C}(X_{pro\text{-}\acute{e}tale}, \Lambda)$. If M is in $D^+(X_{\acute{e}tale}, \Lambda)$, then Lemma 18.4 shows that $M \to R\epsilon_*\epsilon^{-1}M$ is an isomorphism. If K is in $D^+_\mathcal{C}(X_{pro\text{-}\acute{e}tale}, \Lambda)$, then the spectral sequence

$$R^q\epsilon_*H^p(K) \Rightarrow H^{p+q}(R\epsilon_*K)$$

and the vanishing in Lemma 18.4 shows that $H^p(R\epsilon_*K) = R\epsilon_*H^p(K)$. Since ϵ is a flat morphism of ringed sites (ringed by the constant sheaf $\underline{\Lambda}$) we see that $\epsilon^{-1}R\epsilon_*K$ has cohomology sheaves $\epsilon^{-1}R\epsilon_*H^p(K)$. Since we've assumed $H^p(K)$ is in $\mathcal{C}$ we conclude by Lemma 18.4 once more that $\epsilon^{-1}R\epsilon_*K \to K$ is an isomorphism. In this way we see that ϵ^{-1} and $R\epsilon_*$ are quasi-inverse functors proving (2). ☐

Let Λ be a ring. In Modules on Sites, Section 42 we have defined the notion of a locally constant sheaf of Λ-modules on a site. If M is a Λ-module, then $\underline{M}$ is of finite presentation as a sheaf of $\underline{\Lambda}$-modules if and only if M is a finitely presented Λ-module, see Modules on Sites, Lemma 41.5.

Lemma 18.8. *Let X be a scheme. Let Λ be a ring. The functor ϵ^{-1} defines an equivalence of categories*

$$\left\{\begin{matrix}\textit{locally constant sheaves}\\ \textit{of } \Lambda\textit{-modules on } X_{\acute{e}tale}\\ \textit{of finite presentation}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{locally constant sheaves}\\ \textit{of } \Lambda\textit{-modules on } X_{pro\text{-}\acute{e}tale}\\ \textit{of finite presentation}\end{matrix}\right\}$$

Proof. Let $\mathcal{F}$ be a locally constant sheaf of Λ-modules on $X_{pro\text{-}\acute{e}tale}$ of finite presentation. Choose a pro-étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is constant, say $\mathcal{F}|_{U_i} \cong \underline{M_i}_{U_i}$. Observe that $U_i \times_X U_j$ is empty if M_i is not isomorphic to M_j. For each Λ-module M let $I_M = \{i \in I \mid M_i \cong M\}$. As pro-étale coverings are fpqc coverings and by Descent, Lemma 9.2 we see that $U_M = \bigcup_{i \in I_M} \text{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_M$ is a disjoint open covering of X. We may replace X by U_M for some M and assume that $M_i = M$ for all i.

Consider the sheaf $\mathcal{I} = \mathit{Isom}(\underline{M}, \mathcal{F})$. This sheaf is a torsor for $\mathcal{G} = \mathit{Isom}(\underline{M}, \underline{M})$. By Modules on Sites, Lemma 42.4 we have $\mathcal{G} = \underline{G}$ where $G = \mathit{Isom}_\Lambda(M, M)$. Since torsors for the étale topology and the pro-étale topology agree by Lemma 18.6 it follows that $\mathcal{I}$ has sections étale locally on X. Thus $\mathcal{F}$ is étale locally a constant sheaf which is what we had to show. ☐

Lemma 18.9. *Let X be a scheme. Let Λ be a Noetherian ring. Let $D_{flc}(X_{\acute{e}tale}, \Lambda)$, resp. $D_{flc}(X_{pro\text{-}\acute{e}tale}, \Lambda)$ be the full subcategory of $D(X_{\acute{e}tale}, \Lambda)$, resp. $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$ consisting of those complexes whose cohomology sheaves are locally constant sheaves of Λ-modules of finite type. Then*

$$\epsilon^{-1} : D^+_{flc}(X_{\acute{e}tale}, \Lambda) \longrightarrow D^+_{flc}(X_{pro\text{-}\acute{e}tale}, \Lambda)$$

is an equivalence of categories.

Proof. The categories $D_{flc}(X_{étale}, \Lambda)$ and $D_{flc}(X_{pro\text{-}étale}, \Lambda)$ are strictly full, saturated, triangulated subcategories of $D(X_{étale}, \Lambda)$ and $D(X_{pro\text{-}étale}, \Lambda)$ by Modules on Sites, Lemma 42.5 and Derived Categories, Section 13 The statement of the lemma follows by combining Lemmas 18.7 and 18.8. $\square$

Lemma 18.10. *Let X be a scheme. Let Λ be a Noetherian ring. Let K be an object of $D(X_{pro\text{-}étale}, \Lambda)$. Set $K_n = K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. If K_1 is*

1. *in the essential image of $\epsilon^{-1} : D(X_{étale}, \Lambda/I) \to D(X_{pro\text{-}étale}, \Lambda/I)$, and*
2. *has tor amplitude in $[a, \infty)$ for some $a \in \mathbf{Z}$,*

then (1) and (2) hold for K_n as an object of $D(X_{pro\text{-}étale}, \Lambda/I^n)$.

Proof. For assertion (2) this follows from the more general Cohomology on Sites, Lemma 35.8. The second assertion follows from the fact that the essential image of ϵ^{-1} is a triangulated subcategory of $D^+(X_{pro\text{-}étale}, \Lambda/I^n)$ (Lemma 18.7), the distinguished triangles

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} \to K_{n+1} \to K_n \to K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}}[1]$$

and the isomorphism

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} = K_1 \otimes_{\Lambda/I}^{\mathbf{L}} \underline{I^n/I^{n+1}}$$

$\square$

19. Cohomology of a point

Let Λ be a Noetherian ring complete with respect to an ideal $I \subset \Lambda$. Let k be a field. In this section we "compute"

$$H^i(\mathrm{Spec}(k)_{pro\text{-}étale}, \underline{\Lambda}^\wedge)$$

where $\underline{\Lambda}^\wedge = \lim \underline{\Lambda/I^n}$ as before. Let k^{sep} be a separable algebraic closure of k. Then

$$\mathcal{U} = \{\mathrm{Spec}(k^{sep}) \to \mathrm{Spec}(k)\}$$

is a pro-étale covering of $\mathrm{Spec}(k)$. We will use the Čech to cohomology spectral sequence with respect to this covering. Set $U_0 = \mathrm{Spec}(k^{sep})$ and

$$\begin{aligned} U_n &= \mathrm{Spec}(k^{sep}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{sep}) \times_{\mathrm{Spec}(k)} \ldots \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{sep}) \\ &= \mathrm{Spec}(k^{sep} \otimes_k k^{sep} \otimes_k \ldots \otimes_k k^{sep}) \end{aligned}$$

($n+1$ factors). Note that the underlying topological space $|U_0|$ of U_0 is a singleton and for $n \geq 1$ we have

$$|U_n| = G \times \ldots \times G \quad (n \text{ factors})$$

as profinite spaces where $G = \mathrm{Gal}(k^{sep}/k)$. Namely, every point of U_n has residue field k^{sep} and we identify $(\sigma_1, \ldots, \sigma_n)$ with the point corresponding to the surjection

$$k^{sep} \otimes_k k^{sep} \otimes_k \ldots \otimes_k k^{sep} \longrightarrow k^{sep}, \quad \lambda_0 \otimes \lambda_1 \otimes \ldots \lambda_n \longmapsto \lambda_0 \sigma_1(\lambda_1) \ldots \sigma_n(\lambda_n)$$

Then we compute

$$\begin{aligned} R\Gamma((U_n)_{pro\text{-}étale}, \underline{\Lambda}^\wedge) &= R\lim R\Gamma((U_n)_{pro\text{-}étale}, \underline{\Lambda/I^n}) \\ &= R\lim R\Gamma((U_n)_{étale}, \underline{\Lambda/I^n}) \\ &= \lim H^0(U_n, \underline{\Lambda/I^n}) \\ &= \mathrm{Maps}_{cont}(G \times \ldots \times G, \Lambda) \end{aligned}$$

The first equality because $R\Gamma$ commutes with derived limits and as $\Lambda^\wedge$ is the derived limit of the sheaves $\underline{\Lambda/I^n}$ by Proposition 17.1. The second equality by Lemma 18.5. The third equality by Étale Cohomology, Lemma 55.7. The fourth equality uses Étale Cohomology, Remark 23.2 to identify sections of the constant sheaf $\underline{\Lambda/I^n}$. Then it uses the fact that Λ is complete with respect to I and hence equal to $\lim \Lambda/I^n$ as a topological space, to see that $\lim \text{Map}_{cont}(G, \Lambda/I^n) = \text{Map}_{cont}(G, \Lambda)$ and similarly for higher powers of G. At this point Cohomology on Sites, Lemmas 11.3 and 11.7 tell us that

$$\Lambda \to \text{Maps}_{cont}(G, \Lambda) \to \text{Maps}_{cont}(G \times G, \Lambda) \to \ldots$$

computes the pro-étale cohomology. In other words, we see that

$$H^i(\text{Spec}(k)_{pro\text{-}\acute{e}tale}, \underline{\Lambda}^\wedge) = H^i_{cont}(G, \Lambda)$$

where the right hand side is continuous cohomology as defined by Tate in [Tat76]. Of course, this is as it should be.

Lemma 19.1. *Let k be a field. Let $G = Gal(k^{sep}/k)$ be its absolute Galois group. Further,*

(1) *let M be a profinite abelian group with a continuous G-action, or*
(2) *let Λ be a Noetherian ring and $I \subset \Lambda$ an ideal an let M be an I-adically complete Λ-module with continuous G-action.*

Then there is a canonical sheaf $\underline{M}^\wedge$ on $\text{Spec}(k)_{pro\text{-}\acute{e}tale}$ associated to M such that

$$H^i(\text{Spec}(k), \underline{M}^\wedge) = H^i_{cont}(G, M)$$

as abelian groups or Λ-modules.

Proof. Proof in case (2). Set $M_n = M/I^nM$. Then $M = \lim M_n$ as M is assumed I-adically complete. Since the action of G is continuous we get continuous actions of G on M_n. By Étale Cohomology, Theorem 57.3 this action corresponds to a (locally constant) sheaf $\underline{M_n}$ of Λ/I^n-modules on $\text{Spec}(k)_{\acute{e}tale}$. Pull back to $\text{Spec}(k)_{pro\text{-}\acute{e}tale}$ by the comparison morphism ϵ and take the limit

$$\underline{M}^\wedge = \lim \epsilon^{-1}\underline{M_n}$$

to get the sheaf promised in the lemma. Exactly the same argument as given in the introduction of this section gives the comparison with Tate's continuous Galois cohomology. $\square$

20. Weakly contractible hypercoverings

Let X be a scheme. For every object $U \in \text{Ob}(X_{pro\text{-}\acute{e}tale})$ there exists a covering $\{V \to U\}$ of $X_{pro\text{-}\acute{e}tale}$ with V weakly contractible. This follows from Lemma 11.10 and the elementary fact that a disjoint union of weakly contractible objects in $X_{pro\text{-}\acute{e}tale}$ is weakly contractible (discussion of set theoretic issues omitted). This observation leads to the existence of hypercoverings made up out weakly contractible objects.

Lemma 20.1. *Let X be a scheme.*

(1) *For every object U of $X_{pro\text{-}\acute{e}tale}$ there exists a hypercovering K of U in $X_{pro\text{-}\acute{e}tale}$ such that each term K_n consists of a single weakly contractible object of $X_{pro\text{-}\acute{e}tale}$ covering U.*

(2) *For every quasi-compact and quasi-separated object U of $X_{pro\text{-}\acute{e}tale}$ there exists a hypercovering K of U in $X_{pro\text{-}\acute{e}tale}$ such that each term K_n consists of a single affine and weakly contractible object of $X_{pro\text{-}\acute{e}tale}$ covering U.*

Proof. Let $\mathcal{B} \subset \text{Ob}(X_{pro\text{-}\acute{e}tale})$ be the set of weakly contractible objects of $X_{pro\text{-}\acute{e}tale}$. We have seen above that every object of $X_{pro\text{-}\acute{e}tale}$ has a covering by an element of $\mathcal{B}$. Apply Hypercoverings, Lemma 11.1 to get (1).

Let $X_{qcqs,pro\text{-}\acute{e}tale} \subset X_{pro\text{-}\acute{e}tale}$ be the full subcategory consisting of quasi-compact and quasi-separated objects. Note that $X_{qcqs,pro\text{-}\acute{e}tale}$ is preserved under fibre products. A covering of $X_{qcqs,pro\text{-}\acute{e}tale}$ will be a finite pro-étale covering. Then $X_{qcqs,pro\text{-}\acute{e}tale} \to X_{pro\text{-}\acute{e}tale}$ is a special cocontinuous functor hence $X_{qcqs,pro\text{-}\acute{e}tale}$ defines the same topos as $X_{pro\text{-}\acute{e}tale}$. Details omitted; see Sites, Definition 28.2 and Lemma 28.1. In particular, if K is a hypercovering of an object U in $X_{qcqs,pro\text{-}\acute{e}tale}$ then K is a hypercovering of $X_{pro\text{-}\acute{e}tale}$. Let $\mathcal{B} \subset \text{Ob}(X_{qcqs,pro\text{-}\acute{e}tale})$ be the set of affine and weakly contractible objects. By Lemma 11.10 and the fact that finite unions of affines are affine, for every object U of $X_{qcqs,pro\text{-}\acute{e}tale}$ there exists a covering $\{V \to U\}$ of $X_{qcqs,pro\text{-}\acute{e}tale}$ with $V \in \mathcal{B}$. Apply Hypercoverings, Lemma 11.1 to get (2). □

In the following lemma we use the Čech complex $\mathcal{F}(K)$ associated to a hypercovering K in a site. See Hypercoverings, Section 4. If K is a hypercovering of U and $K_n = \{U_n \to U\}$, then the Čech complex looks like this:

$$\mathcal{F}(K) = (\mathcal{F}(U_0) \to \mathcal{F}(U_1) \to \mathcal{F}(U_2) \to \ldots)$$

Lemma 20.2. *Let X be a scheme. Let $E \in D^+(X_{pro\text{-}\acute{e}tale})$ be represented by a bounded below complex $\mathcal{E}^\bullet$ of abelian sheaves. Let K be a hypercovering of $U \in \text{Ob}(X_{pro\text{-}\acute{e}tale})$ with $K_n = \{U_n \to U\}$ where U_n is a weakly contractible object of $X_{pro\text{-}\acute{e}tale}$. Then*

$$R\Gamma(U, E) = Tot(\mathcal{E}^\bullet(K))$$

in $D(Ab)$.

Proof. If $E = \mathcal{E}[n]$ is the object associated to a single abelian sheaf on $X_{pro\text{-}\acute{e}tale}$, then the spectral sequence of Hypercoverings, Lemma 4.3 implies that

$$R\Gamma(X_{pro\text{-}\acute{e}tale}, \mathcal{E}) = \mathcal{E}(K)$$

because the higher cohomology groups of any sheaf over U_n vanish, see Cohomology on Sites, Lemma 38.1.

If $\mathcal{E}^\bullet$ is bounded below, then we can choose an injective resolution $\mathcal{E}^\bullet \to \mathcal{I}^\bullet$ and consider the map of complexes

$$\text{Tot}(\mathcal{E}^\bullet(K)) \longrightarrow \text{Tot}(\mathcal{I}^\bullet(K))$$

For every n the map $\mathcal{E}^\bullet(U_n) \to \mathcal{I}^\bullet(U_n)$ is a quasi-isomorphism because taking sections over U_n is exact. Hence the displayed map is a quasi-isomorphism by one of the spectral sequences of Homology, Lemma 22.6. Using the result of the first paragraph we see that for every p the complex $\mathcal{I}^p(K)$ is acyclic in degrees $n > 0$ and computes $\mathcal{I}^p(U)$ in degree 0. Thus the other spectral sequence of Homology, Lemma 22.6 shows $\text{Tot}(\mathcal{I}^\bullet(K))$ computes $R\Gamma(U, E) = \mathcal{I}^\bullet(U)$. □

Lemma 20.3. *Let X be a quasi-compact and quasi-separated scheme. The functor $R\Gamma(X, -) : D^+(X_{pro\text{-}\acute{e}tale}) \to D(Ab)$ commutes with direct sums and homotopy colimits.*

Proof. The statement means the following: Suppose we have a family of objects E_i of $D^+(X_{\textit{pro-étale}})$ such that $\bigoplus E_i$ is an object of $D^+(X_{\textit{pro-étale}})$. Then $R\Gamma(X, \bigoplus E_i) = \bigoplus R\Gamma(X, E_i)$. To see this choose a hypercovering K of X with $K_n = \{U_n \to X\}$ where U_n is an affine and weakly contractible scheme, see Lemma 20.1. Let N be an integer such that $H^p(E_i) = 0$ for $p < N$. Choose a complex of abelian sheaves $\mathcal{E}_i^\bullet$ representing E_i with $\mathcal{E}_i^p = 0$ for $p < N$. The termwise direct sum $\bigoplus \mathcal{E}_i^\bullet$ represents $\bigoplus E_i$ in $D(X_{\textit{pro-étale}})$, see Injectives, Lemma 13.4. By Lemma 20.2 we have

$$R\Gamma(X, \bigoplus E_i) = \mathrm{Tot}((\bigoplus \mathcal{E}_i^\bullet)(K))$$

and

$$R\Gamma(X, E_i) = \mathrm{Tot}(\mathcal{E}_i^\bullet(K))$$

Since each U_n is quasi-compact we see that

$$\mathrm{Tot}((\bigoplus \mathcal{E}_i^\bullet)(K)) = \bigoplus \mathrm{Tot}(\mathcal{E}_i^\bullet(K))$$

by Modules on Sites, Lemma 29.2. The statement on homotopy colimits is a formal consequence of the fact that $R\Gamma$ is an exact functor of triangulated categories and the fact (just proved) that it commutes with direct sums. $\square$

Remark 20.4. Let X be a scheme. Because $X_{\textit{pro-étale}}$ has enough weakly contractible objects for all K in $D(X_{\textit{pro-étale}})$ we have $K = R\lim \tau_{\geq -n} K$ by Cohomology on Sites, Proposition 38.2. Since $R\Gamma$ commutes with $R\lim$ by Injectives, Lemma 13.6 we see that

$$R\Gamma(X, K) = R\lim R\Gamma(X, \tau_{\geq -n} K)$$

in $D(Ab)$. This will allows us to extend some results from bounded below complexes to all complexes.

21. Functoriality of the pro-étale site

Let $f : X \to Y$ be a morphism of schemes. The functor $Y_{\textit{pro-étale}} \to X_{\textit{pro-étale}}$, $V \mapsto X \times_Y V$ induces a morphism of sites $f_{\textit{pro-étale}} : X_{\textit{pro-étale}} \to Y_{\textit{pro-étale}}$, see Sites, Proposition 15.6. In fact, we obtain a commutative diagram of morphisms of sites

$$\begin{tikzcd} X_{\textit{pro-étale}} \arrow[r, "\epsilon"'] \arrow[d, "f_{\textit{pro-étale}}"'] & X_{\textit{étale}} \arrow[d, "f_{\textit{étale}}"] \\ Y_{\textit{pro-étale}} \arrow[r, "\epsilon"] & Y_{\textit{étale}} \end{tikzcd}$$

where ϵ is as in Section 18. In particular we have $\epsilon^{-1} f_{\textit{étale}}^{-1} = f_{\textit{pro-étale}}^{-1} \epsilon^{-1}$. Here is the corresponding result for pushforward.

Lemma 21.1. *Let $f : X \to Y$ be a morphism of schemes.*

(1) *Let $\mathcal{F}$ be a sheaf of sets on $X_{\textit{étale}}$. Then we have $f_{\textit{pro-étale},*}\epsilon^{-1}\mathcal{F} = \epsilon^{-1} f_{\textit{étale},*}\mathcal{F}$.*
(2) *Let $\mathcal{F}$ be an abelian sheaf on $X_{\textit{étale}}$. Then we have $Rf_{\textit{pro-étale},*}\epsilon^{-1}\mathcal{F} = \epsilon^{-1} Rf_{\textit{étale},*}\mathcal{F}$.*

Proof. Proof of (1). Let $\mathcal{F}$ be a sheaf of sets on $X_{\acute{e}tale}$. There is a canonical map $\epsilon^{-1} f_{\acute{e}tale,*}\mathcal{F} \to f_{pro\text{-}\acute{e}tale,*}\epsilon^{-1}\mathcal{F}$, see Sites, Section 44. To show it is an isomorphism we may work (Zariski) locally on Y, hence we may assume Y is affine. In this case every object of $Y_{pro\text{-}\acute{e}tale}$ has a covering by objects $V = \lim V_i$ which are limits of affine schemes V_i étale over Y (by Proposition 9.1 for example). Evaluating the map $\epsilon^{-1} f_{\acute{e}tale,*}\mathcal{F} \to f_{pro\text{-}\acute{e}tale,*}\epsilon^{-1}\mathcal{F}$ on V we obtain a map

$$\operatorname{colim} \Gamma(X \times_Y V_i, \mathcal{F}) \longrightarrow \Gamma(X \times_Y V, \epsilon^*\mathcal{F}).$$

see Lemma 18.1 for the left hand side. By Lemma 18.1 we have

$$\Gamma(X \times_Y V, \epsilon^*\mathcal{F}) = \Gamma(X \times_Y V, \mathcal{F})$$

Hence the result holds by Étale Cohomology, Lemma 52.3.

Proof of (2). Arguing in exactly the same manner as above we see that it suffices to show that

$$\operatorname{colim} H^i_{\acute{e}tale}(X \times_Y V_i, \mathcal{F}) \longrightarrow H^i_{\acute{e}tale}(X \times_Y V, \mathcal{F})$$

which follows once more from Étale Cohomology, Lemma 52.3. $\square$

22. Finite morphisms and pro-étale sites

It is not clear that a finite morphism of schemes determines an exact pushforward on abelian pro-étale sheaves.

Lemma 22.1. *Let $f : Z \to X$ be a finite morphism of schemes which is locally of finite presentation. Then $f_{pro\text{-}\acute{e}tale,*} : Ab(Z_{pro\text{-}\acute{e}tale}) \to Ab(X_{pro\text{-}\acute{e}tale})$ is exact.*

Proof. The prove this we may work (Zariski) locally on X and assume that X is affine, say $X = \operatorname{Spec}(A)$. Then $Z = \operatorname{Spec}(B)$ for some finite A-algebra B of finite presentation. The construction in the proof of Proposition 10.3 produces a faithfully flat, ind-étale ring map $A \to D$ with D w-contractible. We may check exactness of a sequence of sheaves by evaluating on $U = \operatorname{Spec}(D)$ be such an object. Then $f_{pro\text{-}\acute{e}tale,*}\mathcal{F}$ evaluated at U is equal to $\mathcal{F}$ evaluated at $V = \operatorname{Spec}(D \otimes_A B)$. Since $D \otimes_A B$ is w-contractible by Lemma 10.6 evaluation at V is exact. $\square$

23. Closed immersions and pro-étale sites

It is not clear (and likely false) that a closed immersion of schemes determines an exact pushforward on abelian pro-étale sheaves.

Lemma 23.1. *Let $i : Z \to X$ be a closed immersion morphism of affine schemes. Denote X_{app} and Z_{app} the sites introduced in Lemma 11.24. The base change functor*

$$u : X_{app} \to Z_{app}, \quad U \longmapsto u(U) = U \times_X Z$$

is continuous and has a fully faithful left adjoint v. For V in Z_{app} the morphism $V \to v(V)$ is a closed immersion identifying V with $u(v(V)) = v(V) \times_X Z$ and every point of $v(V)$ specializes to a point of V. The functor v is cocontinuous and sends coverings to coverings.

Proof. The existence of the adjoint follows immediately from Lemma 7.7 and the definitions. It is clear that u is continuous from the definition of coverings in X_{app}.

Write $X = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(A/I)$. Let $V = \operatorname{Spec}(\overline{C})$ be an object of Z_{app} and let $v(V) = \operatorname{Spec}(C)$. We have seen in the statement of Lemma 7.7 that

V equals $v(V) \times_X Z = \mathrm{Spec}(C/IC)$. Any $g \in C$ which maps to an invertible element of $C/IC = \overline{C}$ is invertible in C. Namely, we have the A-algebra maps $C \to C_g \to C/IC$ and by adjointness we obtain an C-algebra map $C_g \to C$. Thus every point of $v(V)$ specializes to a point of V.

Suppose that $\{V_i \to V\}$ is a covering in Z_{app}. Then $\{v(V_i) \to v(V)\}$ is a finite family of morphisms of Z_{app} such that every point of $V \subset v(V)$ is in the image of one of the maps $v(V_i) \to v(V)$. As the morphisms $v(V_i) \to v(V)$ are flat (since they are weakly étale) we conclude that $\{v(V_i) \to v(V)\}$ is jointly surjective. This proves that v sends coverings to coverings.

Let V be an object of Z_{app} and let $\{U_i \to v(V)\}$ be a covering in X_{app}. Then we see that $\{u(U_i) \to u(v(V)) = V\}$ is a covering of Z_{app}. By adjointness we obtain morphisms $v(u(U_i)) \to U_i$. Thus the family $\{v(u(U_i)) \to v(V)\}$ refines the given covering and we conclude that v is cocontinuous. □

Lemma 23.2. *Let $Z \to X$ be a closed immersion morphism of affine schemes. The corresponding morphism of topoi $i = i_{pro\text{-}\acute{e}tale}$ is equal to the morphism of topoi associated to the fully faithful cocontinuous functor $v : Z_{app} \to X_{app}$ of Lemma 23.1. It follows that*

1. *$i^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto \mathcal{F}(v(V))$,*
2. *for a weakly contractible object V of Z_{app} we have $i^{-1}\mathcal{F}(V) = \mathcal{F}(v(V))$,*
3. *$i^{-1} : Sh(X_{pro\text{-}\acute{e}tale}) \to Sh(Z_{pro\text{-}\acute{e}tale})$ has a left adjoint $i_!^{Sh}$,*
4. *$i^{-1} : Ab(X_{pro\text{-}\acute{e}tale}) \to Ab(Z_{pro\text{-}\acute{e}tale})$ has a left adjoint $i_!$,*
5. *$id \to i^{-1}i_!^{Sh}$, $id \to i^{-1}i_!$, and $i^{-1}i_* \to id$ are isomorphisms, and*
6. *i_*, $i_!^{Sh}$ and $i_!$ are fully faithful.*

Proof. By Lemma 11.24 we may describe $i_{pro\text{-}\acute{e}tale}$ in terms of the morphism of sites $u : X_{app} \to Z_{app}$, $V \mapsto V \times_X Z$. The first statement of the lemma follows from Sites, Lemma 21.2 (but with the roles of u and v reversed).

Proof of (1). By the description of i as the morphism of topoi associated to v this holds by the construction, see Sites, Lemma 20.1.

Proof of (2). Since the functor v sends coverings to coverings by Lemma 23.1 we see that the presheaf $\mathcal{G} : V \mapsto \mathcal{F}(v(V))$ is a separated presheaf (Sites, Definition 10.9). Hence the sheafification of $\mathcal{G}$ is $\mathcal{G}^+$, see Sites, Theorem 10.10. Next, let V be a weakly contractible object of Z_{app}. Let $\mathcal{V} = \{V_i \to V\}_{i=1,\ldots,n}$ be any covering in Z_{app}. Set $\mathcal{V}' = \{\coprod V_i \to V\}$. Since v commutes with finite disjoint unions (as a left adjoint or by the construction) and since $\mathcal{F}$ sends finite disjoint unions into products, we see that

$$H^0(\mathcal{V}, \mathcal{G}) = H^0(\mathcal{V}', \mathcal{G})$$

(notation as in Sites, Section 10; compare with Étale Cohomology, Lemma 22.1). Thus we may assume the covering is given by a single morphism, like so $\{V' \to V\}$. Since V is weakly contractible, this covering can be refined by the trivial covering $\{V \to V\}$. It therefore follows that the value of $\mathcal{G}^+ = i^{-1}\mathcal{F}$ on V is simply $\mathcal{F}(v(V))$ and (2) is proved.

Proof of (3). Every object of Z_{app} has a covering by weakly contractible objects (Lemma 11.27). By the above we see that we would have $i_!^{Sh}h_V = h_{v(V)}$ for V weakly contractible if $i_!^{Sh}$ existed. The existence of $i_!^{Sh}$ then follows from Sites, Lemma 23.1.

Proof of (4). Existence of $i_!$ follows in the same way by setting $i_!\mathbf{Z}_V = \mathbf{Z}_{v(V)}$ for V weakly contractible in Z_{app}, using similar for direct sums, and applying Homology, Lemma 25.6. Details omitted.

Proof of (5). Let V be a contractible object of Z_{app}. Then $i^{-1}i_!^{Sh}h_V = i^{-1}h_{v(V)} = h_{u(v(V))} = h_V$. (It is a general fact that $i^{-1}h_U = h_{u(U)}$.) Since the sheaves h_V for V contractible generate $Sh(Z_{app})$ (Sites, Lemma 13.5) we conclude $\text{id} \to i^{-1}i_!^{Sh}$ is an isomorphism. Similarly for the map $\text{id} \to i^{-1}i_!$. Then $(i^{-1}i_*\mathcal{H})(V) = i_*\mathcal{H}(v(V)) = \mathcal{H}(u(v(V))) = \mathcal{H}(V)$ and we find that $i^{-1}i_* \to \text{id}$ is an isomorphism.

The fully faithfulness statements of (6) now follow from Categories, Lemma 24.3. $\square$

Lemma 23.3. *Let $i : Z \to X$ be a closed immersion of schemes. Then*

1. *$i^{-1}_{pro\text{-}\acute{e}tale}$ commutes with limits,*
2. *$i_{pro\text{-}\acute{e}tale,*}$ is fully faithful, and*
3. *$i^{-1}_{pro\text{-}\acute{e}tale} i_{pro\text{-}\acute{e}tale,*} \cong id_{Sh(Z_{pro\text{-}\acute{e}tale})}$.*

Proof. Assertions (2) and (3) are equivalent by Sites, Lemma 40.1. Parts (1) and (3) are (Zariski) local on X, hence we may assume that X is affine. In this case the result follows from Lemma 23.2. $\square$

Lemma 23.4. *Let $i : Z \to X$ be an integral universally injective and surjective morphism of schemes. Then $i_{pro\text{-}\acute{e}tale,*}$ and $i^{-1}_{pro\text{-}\acute{e}tale}$ are quasi-inverse equivalences of categories of pro-étale topoi.*

Proof. There is an immediate reduction to the case that X is affine. Then Z is affine too. Set $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Z)$. Then the categories of étale algebras over A and B are equivalent, see Étale Cohomology, Theorem 46.1 and Remark 46.2. Thus the categories of ind-étale algebras over A and B are equivalent. In other words the categories X_{app} and Z_{app} of Lemma 11.24 are equivalent. We omit the verification that this equivalence sends coverings to coverings and vice versa. Thus the result as Lemma 11.24 tells us the pro-étale topos is the topos of sheaves on X_{app}. $\square$

Lemma 23.5. *Let $i : Z \to X$ be a closed immersion of schemes. Let $U \to X$ be an object of $X_{pro\text{-}\acute{e}tale}$ such that*

1. *U is affine and weakly contractible, and*
2. *every point of U specializes to a point of $U \times_X Z$.*

Then $i^{-1}_{pro\text{-}\acute{e}tale}\mathcal{F}(U \times_X Z) = \mathcal{F}(U)$ for all abelian sheaves on $X_{pro\text{-}\acute{e}tale}$.

Proof. Since pullback commutes with restriction, we may replace X by U. Thus we may assume that X is affine and weakly contractible and that every point of X specializes to a point of Z. By Lemma 23.2 part (1) it suffices to show that $v(Z) = X$ in this case. Thus we have to show: If A is a w-contractible ring, $I \subset A$ an ideal contained in the radical of A and $A \to B \to A/I$ is a factorization with $A \to B$ ind-étale, then there is a unique section $B \to A$ compatible with maps to A/I. Observe that $B/IB = A/I \times R$ as A/I-algebras. After replacing B by a localization we may assume $B/IB = A/I$. Note that $\text{Spec}(B) \to \text{Spec}(A)$ is surjective as the image contains $V(I)$ and hence all closed points and is closed under specialization. Since A is w-contractible there is a section $B \to A$. Since $B/IB = A/I$ this section is compatible with the map to A/I. We omit the proof of uniqueness (hint: use that A and B have isomorphic local rings at maximal ideals of A). $\square$

Lemma 23.6. *Let $i : Z \to X$ be a closed immersion of schemes. If $X \setminus i(Z)$ is a retrocompact open of X, then $i_{pro\text{-}\acute{e}tale,*}$ is exact.*

Proof. The question is local on X hence we may assume X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. There exist $f_1, \ldots, f_r \in I$ such that $Z = V(f_1, \ldots, f_r)$ set theoretically, see Algebra, Lemma 28.1. By Lemma 23.4 we may assume that $Z = \mathrm{Spec}(A/(f_1, \ldots, f_r))$. In this case the functor $i_{pro\text{-}\acute{e}tale,*}$ is exact by Lemma 22.1. $\square$

24. Extension by zero

The general material in Modules on Sites, Section 19 allows us to make the following definition.

Definition 24.1. Let $j : U \to X$ be a weakly étale morphism of schemes.

(1) The restriction functor $j^{-1} : Sh(X_{pro\text{-}\acute{e}tale}) \to Sh(U_{pro\text{-}\acute{e}tale})$ has a left adjoint $j_!^{Sh} : Sh(X_{pro\text{-}\acute{e}tale}) \to Sh(U_{pro\text{-}\acute{e}tale})$.
(2) The restriction functor $j^{-1} : Ab(X_{pro\text{-}\acute{e}tale}) \to Ab(U_{pro\text{-}\acute{e}tale})$ has a left adjoint which is denoted $j_! : Ab(U_{pro\text{-}\acute{e}tale}) \to Ab(X_{pro\text{-}\acute{e}tale})$ and called *extension by zero*.
(3) Let Λ be a ring. The functor $j^{-1} : Mod(X_{pro\text{-}\acute{e}tale}, \Lambda) \to Mod(U_{pro\text{-}\acute{e}tale}, \Lambda)$ has a left adjoint $j_! : Mod(U_{pro\text{-}\acute{e}tale}, \Lambda) \to Mod(X_{pro\text{-}\acute{e}tale}, \Lambda)$ and called *extension by zero*.

As usual we compare this to what happens in the étale case.

Lemma 24.2. *Let $j : U \to X$ be an étale morphism of schemes. Let $\mathcal{G}$ be an abelian sheaf on $U_{\acute{e}tale}$. Then $\epsilon^{-1} j_! \mathcal{G} = j_! \epsilon^{-1} \mathcal{G}$ as sheaves on $X_{pro\text{-}\acute{e}tale}$.*

Proof. This is true because both are left adjoints to $j_{pro\text{-}\acute{e}tale,*}\epsilon^{-1} = \epsilon^{-1} j_{\acute{e}tale,*}$, see Lemma 21.1. $\square$

Lemma 24.3. *Let $j : U \to X$ be a weakly étale morphism of schemes. Let $i : Z \to X$ be a closed immersion such that $U \times_X Z = \emptyset$. Let $V \to X$ be an affine object of $X_{pro\text{-}\acute{e}tale}$ such that every point of V specializes to a point of $V_Z = Z \times_X V$. Then $j_! \mathcal{F}(V) = 0$ for all abelian sheaves on $U_{pro\text{-}\acute{e}tale}$.*

Proof. Let $\{V_i \to V\}$ be a pro-étale covering. The lemma follows if we can refine this covering to a covering where the members have no morphisms into U over X (see construction of $j_!$ in Modules on Sites, Section 19). First refine the covering to get a finite covering with V_i affine. For each i let $V_i = \mathrm{Spec}(A_i)$ and let $Z_i \subset V_i$ be the inverse image of Z. Set $W_i = \mathrm{Spec}(A_{i,Z_i}^\sim)$ with notation as in Lemma 5.1. Then $\coprod W_i \to V$ is weakly étale and the image contains all points of V_Z. Hence the image contains all points of V by our assumption on specializations. Thus $\{W_i \to V\}$ is a pro-étale covering refining the given one. But each point in W_i specializes to a point lying over Z, hence there are no morphisms $W_i \to U$ over X. $\square$

Lemma 24.4. *Let $j : U \to X$ be an open immersion of schemes. Then $id \cong j^{-1} j_!$ and $j^{-1} j_* \cong id$ and the functors $j_!$ and j_* are fully faithful.*

Proof. See Sites, Lemma 26.4 and Categories, Lemma 24.3. $\square$

Here is the relationship between extension by zero and restriction to the complementary closed subscheme.

Lemma 24.5. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. Assume that j is a quasi-compact morphism. For every abelian sheaf on $X_{pro\text{-}\acute{e}tale}$ there is a canonical short exact sequence*

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{pro\text{-}\acute{e}tale}$ where all the functors are for the pro-étale topology.

Proof. We obtain the maps by the adjointness properties of the functors involved. It suffices to show that $X_{pro\text{-}\acute{e}tale}$ has enough objects (Sites, Definition 39.2) on which the sequence evaluates to a short exact sequence. Let $V = \text{Spec}(A)$ be an affine object of $X_{pro\text{-}\acute{e}tale}$ such that A is w-contractible (there are enough objects of this type). Then $V \times_X Z$ is cut out by an ideal $I \subset A$. The assumption that j is quasi-compact implies there exist $f_1, \ldots, f_r \in I$ such that $V(I) = V(f_1, \ldots, f_r)$. We obtain a faithfully flat, ind-Zariski ring map

$$A \longrightarrow A_{f_1} \times \ldots \times A_{f_r} \times A^\sim_{V(I)}$$

with $A^\sim_{V(I)}$ as in Lemma 5.1. Since $V_i = \text{Spec}(A_{f_i}) \to X$ factors through U we have

$$j_!j^{-1}\mathcal{F}(V_i) = \mathcal{F}(V_i) \quad \text{and} \quad i_*i^{-1}\mathcal{F}(V_i) = 0$$

On the other hand, for the scheme $V^\sim = \text{Spec}(A^\sim_{V(I)})$ we have

$$j_!j^{-1}\mathcal{F}(V^\sim) = 0 \quad \text{and} \quad \mathcal{F}(V^\sim) = i_*i^{-1}\mathcal{F}(V^\sim)$$

the first equality by Lemma 24.3 and the second by Lemmas 23.5 and 10.7. Thus the sequence evaluates to an exact sequence on $\text{Spec}(A_{f_1} \times \ldots \times A_{f_r} \times A^\sim_{V(I)})$ and the lemma is proved. □

Lemma 24.6. *Let $j : U \to X$ be a quasi-compact open immersion morphism of schemes. The functor $j_! : Ab(U_{pro\text{-}\acute{e}tale}) \to Ab(X_{pro\text{-}\acute{e}tale})$ commutes with limits.*

Proof. Since $j_!$ is exact it suffices to show that $j_!$ commutes with products. The question is local on X, hence we may assume X affine. Let $\mathcal{G}$ be an abelian sheaf on $U_{pro\text{-}\acute{e}tale}$. Note that there always is a canonical map

$$j_!\mathcal{G} \to j_*\mathcal{G}$$

see Modules on Sites, Remark 19.7. In our particular case this map can be obtained from the fact that $j^{-1}j_*\mathcal{G} = \mathcal{G}$. Hence applying the exact sequence of Lemma 24.5 we get

$$0 \to j_!\mathcal{G} \to j_*\mathcal{G} \to i_*i^{-1}j_*\mathcal{G} \to 0$$

where $i : Z \to X$ is the inclusion of the reduced induced scheme structure on the complement $Z = X \setminus U$. The functors j_* and i_* commute with products as right adjoints. The functor i^{-1} commutes with products by Lemma 23.3. Hence $j_!$ does because on the pro-étale site products are exact (Cohomology on Sites, Proposition 38.2). □

25. Constructible sheaves on the pro-étale site

We stick to constructible sheaves of Λ-modules for a Noetherian ring. In the future we intend to discuss constructible sheaves of sets, groups, etc.

Definition 25.1. Let X be a scheme. Let Λ be a Noetherian ring. A sheaf of Λ-modules on $X_{pro\text{-}étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all i.

Again this does not give anything "new".

Lemma 25.2. *Let X be a scheme. Let Λ be a Noetherian ring. The functor ϵ^{-1} defines an equivalence of categories*

$$\left\{\begin{matrix}\textit{constructible sheaves of}\\ \Lambda\textit{-modules on } X_{étale}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{constructible sheaves of}\\ \Lambda\textit{-modules on } X_{pro\text{-}étale}\end{matrix}\right\}$$

between constructible sheaves of Λ-modules on $X_{étale}$ and constructible sheaves of Λ-modules on $X_{pro\text{-}étale}$.

Proof. By Lemma 18.2 the functor ϵ^{-1} is fully faithful and commutes with pullback (restriction) to the strata. Hence ϵ^{-1} of a constructible étale sheaf is a constructible pro-étale sheaf. To finish the proof let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{pro\text{-}étale}$ as in Definition 25.1. There is a canonical map

$$\epsilon^{-1}\epsilon_*\mathcal{F} \longrightarrow \mathcal{F}$$

We will show this map is an isomorphism. This will prove that $\mathcal{F}$ is in the essential image of ϵ^{-1} and finish the proof (details omitted).

To prove this we may assume that X is affine. In this case we have a finite partition $X = \coprod_i X_i$ by constructible locally closed strata such that $\mathcal{F}|_{X_i}$ is locally constant of finite type. Let $U \subset X$ be one of the open strata in the partition and let $Z \subset X$ be the reduced induced structure on the complement. By Lemma 24.5 we have a short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{pro\text{-}étale}$. Functoriality gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \epsilon^{-1}\epsilon_*j_!j^{-1}\mathcal{F} \arrow[r] \arrow[d] & \epsilon^{-1}\epsilon_*\mathcal{F} \arrow[r] \arrow[d] & \epsilon^{-1}\epsilon_*i_*i^{-1}\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & j_!j^{-1}\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & i_*i^{-1}\mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

By induction on the length of the partition we know that on the one hand $\epsilon^{-1}\epsilon_*i^{-1}\mathcal{F} \to i^{-1}\mathcal{F}$ and $\epsilon^{-1}\epsilon_*j^{-1}\mathcal{F} \to j^{-1}\mathcal{F}$ are isomorphisms and on the other that $i^{-1}\mathcal{F} = \epsilon^{-1}\mathcal{A}$ and $j^{-1}\mathcal{F} = \epsilon^{-1}\mathcal{B}$ for some constructible sheaves of Λ-modules $\mathcal{A}$ on $Z_{étale}$ and $\mathcal{B}$ on $U_{étale}$. Then

$$\epsilon^{-1}\epsilon_*j_!j^{-1}\mathcal{F} = \epsilon^{-1}\epsilon_*j_!\epsilon^{-1}\mathcal{B} = \epsilon^{-1}\epsilon_*\epsilon^{-1}j_!\mathcal{B} = \epsilon^{-1}j_!\mathcal{B} = j_!\epsilon^{-1}\mathcal{B} = j_!j^{-1}\mathcal{F}$$

the second equality by Lemma 24.2, the third equality by Lemma 18.2, and the fourth equality by Lemma 24.2 again. Similarly, we have

$$\epsilon^{-1}\epsilon_*i_*i^{-1}\mathcal{F} = \epsilon^{-1}\epsilon_*i_*\epsilon^{-1}\mathcal{A} = \epsilon^{-1}\epsilon_*\epsilon^{-1}i_*\mathcal{A} = \epsilon^{-1}i_*\mathcal{A} = i_*\epsilon^{-1}\mathcal{A} = i_*i^{-1}\mathcal{F}$$

this time using Lemma 21.1. By the five lemma we conclude the vertical map in the middle of the big diagram is an isomorphism. $\square$

Lemma 25.3. *Let X be a scheme. Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{pro\text{-}\acute{e}tale}$ is a weak Serre subcategory of $Mod(X_{pro\text{-}\acute{e}tale}, \Lambda)$.*

Proof. This is a formal consequence of Lemmas 25.2 and 18.7 and the result for the étale site (Étale Cohomology, Lemma 69.6). $\square$

Lemma 25.4. *Let X be a scheme. Let Λ be a Noetherian ring. Let $D_c(X_{\acute{e}tale}, \Lambda)$, resp. $D_c(X_{pro\text{-}\acute{e}tale}, \Lambda)$ be the full subcategory of $D(X_{\acute{e}tale}, \Lambda)$, resp. $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$ consisting of those complexes whose cohomology sheaves are constructible sheaves of Λ-modules. Then*

$$\epsilon^{-1} : D_c^+(X_{\acute{e}tale}, \Lambda) \longrightarrow D_c^+(X_{pro\text{-}\acute{e}tale}, \Lambda)$$

is an equivalence of categories.

Proof. The categories $D_c(X_{\acute{e}tale}, \Lambda)$ and $D_c(X_{pro\text{-}\acute{e}tale}, \Lambda)$ are strictly full, saturated, triangulated subcategories of $D(X_{\acute{e}tale}, \Lambda)$ and $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$ by Étale Cohomology, Lemma 69.6 and Lemma 25.3 and Derived Categories, Section 13. The statement of the lemma follows by combining Lemmas 18.7 and 25.2. $\square$

Lemma 25.5. *Let X be a scheme. Let Λ be a Noetherian ring. Let $K, L \in D_c^-(X_{pro\text{-}\acute{e}tale}, \Lambda)$. Then $K \otimes_\Lambda^{\mathbf{L}} L$ is in $D_c^-(X_{pro\text{-}\acute{e}tale}, \Lambda)$.*

Proof. Note that $H^i(K \otimes_\Lambda^{\mathbf{L}} L)$ is the same as $H^i(\tau_{\geq i-1} K \otimes_\Lambda^{\mathbf{L}} \tau_{\geq i-1} L)$. Thus we may assume K and L are bounded. In this case we can apply Lemma 25.4 to reduce to the case of the étale site, see Étale Cohomology, Lemma 90.6. $\square$

Lemma 25.6. *Let X be a scheme. Let Λ be a Noetherian ring. Let K be an object of $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$. Set $K_n = K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. If K_1 is in $D_c^-(X_{pro\text{-}\acute{e}tale}, \Lambda/I)$, then K_n is in $D_c^-(X_{pro\text{-}\acute{e}tale}, \Lambda/I^n)$ for all n.*

Proof. Consider the distinguished triangles

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} \to K_{n+1} \to K_n \to K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}}[1]$$

and the isomorphisms

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} = K_1 \otimes_{\Lambda/I}^{\mathbf{L}} \underline{I^n/I^{n+1}}$$

By Lemma 25.5 we see that this tensor product has constructible cohomology sheaves (and vanishing when K_1 has vanishing cohomology). Hence by induction on n using Lemma 25.3 we see that each K_n has constructible cohomology sheaves. $\square$

26. Constructible adic sheaves

In this section we define the notion of a constructible Λ-sheaf as well as some variants.

Definition 26.1. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let X be a scheme. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}\acute{e}tale}$.

(1) We say $\mathcal{F}$ is a *constructible Λ-sheaf* if $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$ and each $\mathcal{F}/I^n\mathcal{F}$ is a constructible sheaf of Λ/I^n-modules.

(2) If $\mathcal{F}$ is a constructible Λ-sheaf, then we say $\mathcal{F}$ is *lisse* if each $\mathcal{F}/I^n\mathcal{F}$ is locally constant.
(3) We say $\mathcal{F}$ is *adic lisse*[3] if there exists a I-adically complete Λ-module M with M/IM finite such that $\mathcal{F}$ is locally isomorphic to

$$\underline{M}^\wedge = \lim \underline{M/I^nM}.$$

(4) We say $\mathcal{F}$ is *adic constructible*[4] if for every affine open $U \subset X$ there exists a decomposition $U = \coprod U_i$ into constructible locally closed subschemes such that $\mathcal{F}|_{U_i}$ is adic lisse.

The definition of a constructible Λ-sheaf is equivalent to the one in [Gro77, Exposé VI, Definition 1.1.1] when $\Lambda = \mathbf{Z}_\ell$ and $I = (\ell)$. It is clear that we have the implications

$$\begin{tikzcd} \text{lisse adic} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{adic constructible} \arrow[d, Rightarrow] \\ \text{lisse constructible } \Lambda\text{-sheaf} \arrow[r, Rightarrow] & \text{constructible } \Lambda\text{-sheaf} \end{tikzcd}$$

The vertical arrows can be inverted in some cases (see Lemmas 26.2 and 26.5). In general neither the category of adic constructible sheaves nor the category of adic constructible sheaves is closed under kernels and cokernels.

Namely, let X be an affine scheme whose underlying topological space $|X|$ is homeomorphic to $\Lambda = \mathbf{Z}_\ell$, see Example 6.3. Denote $f : |X| \to \mathbf{Z}_\ell = \Lambda$ a homeomorphism. We can think of f as a section of $\underline{\Lambda}^\wedge$ over X and multiplication by f then defines a two term complex

$$\underline{\Lambda}^\wedge \xrightarrow{f} \underline{\Lambda}^\wedge$$

on $X_{pro\text{-}étale}$. The sheaf $\underline{\Lambda}^\wedge$ is adic lisse. However, the cokernel of the map above, is not adic constructible, as the isomorphism type of the stalks of this cokernel attains infinitely many values: $\mathbf{Z}/\ell^n\mathbf{Z}$ and $\mathbf{Z}_\ell$. The cokernel is a constructible $\mathbf{Z}_\ell$-sheaf. However, the kernel is not even a constructible $\mathbf{Z}_\ell$-sheaf as it is zero a non-quasi-compact open but not zero.

Lemma 26.2. *Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a constructible Λ-sheaf on $X_{pro\text{-}étale}$. Then there exists a finite partition $X = \coprod X_i$ by locally closed subschemes such that the restriction $\mathcal{F}|_{X_i}$ is lisse.*

Proof. Let $R = \bigoplus I^n/I^{n+1}$. Observe that R is a Noetherian ring. Since each of the sheaves $\mathcal{F}/I^n\mathcal{F}$ is a constructible sheaf of $\Lambda/I^n\Lambda$-modules also $I^n\mathcal{F}/I^{n+1}\mathcal{F}$ is a constructible sheaf of Λ/I-modules and hence the pullback of a constructible sheaf $\mathcal{G}_n$ on $X_{étale}$ by Lemma 25.2. Set $\mathcal{G} = \bigoplus \mathcal{G}_n$. This is a sheaf of R-modules on $X_{étale}$ and the map

$$\mathcal{G}_0 \otimes_{\Lambda/I} \underline{R} \longrightarrow \mathcal{G}$$

is surjective because the maps

$$\mathcal{F}/I\mathcal{F} \otimes \underline{I^n/I^{n+1}} \to I^n\mathcal{F}/I^{n+1}\mathcal{F}$$

[3] This may be nonstandard notation.
[4] This may be nonstandard notation.

are surjective. Hence $\mathcal{G}$ is a constructible sheaf of R-modules by Étale Cohomology, Proposition 72.1. Choose a partition $X = \coprod X_i$ such that $\mathcal{G}|_{X_i}$ is a locally constant sheaf of R-modules of finite type (Étale Cohomology, Lemma 69.2). We claim this is a partition as in the lemma. Namely, replacing X by X_i we may assume $\mathcal{G}$ is locally constant. It follows that each of the sheaves $I^n\mathcal{F}/I^{n+1}\mathcal{F}$ is locally constant. Using the short exact sequences

$$0 \to I^n\mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$

induction and Modules on Sites, Lemma 42.5 the lemma follows. □

Lemma 26.3. *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}\acute{e}tale}$ such that*

1. *$\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$,*
2. *$\mathcal{F}/I^n\mathcal{F}$ is a constant sheaf of Λ/I^n-modules,*
3. *$\mathcal{F}/I\mathcal{F}$ is of finite type.*

Then $\mathcal{F} \cong \underline{M}^\wedge$ where M is a finite $\Lambda^\wedge$-module.

Proof. Pick a Λ/I^n-module M_n such that $\mathcal{F}/I^n\mathcal{F} \cong \underline{M_n}$. Since we have the surjections $\mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^n\mathcal{F}$ we conclude that there exist surjections $M_{n+1} \to M_n$ inducing isomorphisms $M_{n+1}/I^nM_{n+1} \to M_n$. Fix a choice of such surjections and set $M = \lim M_n$. Then M is an I-adically complete Λ-module with $M/I^nM = M_n$, see Algebra, Lemma 94.1. Since M_1 is a finite type Λ-module (Modules on Sites, Lemma 41.5) we see that M is a finite $\Lambda^\wedge$-module. Consider the sheaves

$$\mathcal{I}_n = \mathit{Isom}(\underline{M_n}, \mathcal{F}/I^n\mathcal{F})$$

on $X_{pro\text{-}\acute{e}tale}$. Modding out by I^n defines a transition map

$$\mathcal{I}_{n+1} \longrightarrow \mathcal{I}_n$$

By our choice of M_n the sheaf $\mathcal{I}_n$ is a torsor under

$$\mathit{Isom}(\underline{M_n}, \underline{M_n}) = \underline{\mathrm{Isom}_\Lambda(M_n, M_n)}$$

(Modules on Sites, Lemma 42.4) since $\mathcal{F}/I^n\mathcal{F}$ is (étale) locally isomorphic to $\underline{M_n}$. It follows from More on Algebra, Lemma 66.1 that the system of sheaves $(\mathcal{I}_n)$ is Mittag-Leffler. For each n let $\mathcal{I}'_n \subset \mathcal{I}_n$ be the image of $\mathcal{I}_N \to \mathcal{I}_n$ for all $N \gg n$. Then

$$\ldots \to \mathcal{I}'_3 \to \mathcal{I}'_2 \to \mathcal{I}'_1 \to *$$

is a sequence of sheaves of sets on $X_{pro\text{-}\acute{e}tale}$ with surjective transition maps. Since $*(X)$ is a singleton (not empty) and since evaluating at X transforms surjective maps of sheaves of sets into surjections of sets, we can pick $s \in \lim \mathcal{I}'_n(X)$. The sections define isomorphisms $\underline{M}^\wedge \to \lim \mathcal{F}/I^n\mathcal{F} = \mathcal{F}$ and the proof is done. □

Lemma 26.4. *Let X be a connected scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. If $\mathcal{F}$ is a lisse constructible Λ-sheaf on $X_{pro\text{-}\acute{e}tale}$, then $\mathcal{F}$ is adic lisse.*

Proof. By Lemma 18.8 we have $\mathcal{F}/I^n\mathcal{F} = \epsilon^{-1}\mathcal{G}_n$ for some locally constant sheaf $\mathcal{G}_n$ of Λ/I^n-modules. By Étale Cohomology, Lemma 68.8 there exists a finite Λ/I^n-module M_n such that $\mathcal{G}_n$ is locally isomorphic to $\underline{M_n}$. Choose a covering $\{W_t \to X\}_{t\in T}$ with each W_t affine and weakly contractible. Then $\mathcal{F}|_{W_t}$ satisfies the assumptions of Lemma 26.3 and hence $\mathcal{F}|_{W_t} \cong \underline{N_t}^\wedge$ for some finite $\Lambda^\wedge$-module

N_t. Note that $N_t/I^nN_t \cong M_n$ for all t and n. Hence $N_t \cong N_{t'}$ for all $t, t' \in T$, see More on Algebra, Lemma 66.2. This proves that $\mathcal{F}$ is adic lisse. $\square$

Lemma 26.5. *Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a constructible Λ-sheaf on $X_{pro\text{-}étale}$. Then $\mathcal{F}$ is adic constructible.*

Proof. This is a consequence of Lemmas 26.2 and 26.4, the fact that a Noetherian scheme is locally connected (Topology, Lemma 8.6), and the definitions. $\square$

It will be useful to identify the constructible Λ-sheaves inside the category of derived complete sheaves of Λ-modules. It turns out that the naive analogue of More on Algebra, Lemma 64.22 is wrong in this setting. However, here is the analogue of More on Algebra, Lemma 64.21.

Lemma 26.6. *Let X be a scheme. Let Λ be a ring and let $I \subset \Lambda$ be a finitely generated ideal. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}étale}$. If $\mathcal{F}$ is derived complete and $\mathcal{F}/I\mathcal{F} = 0$, then $\mathcal{F} = 0$.*

Proof. Assume that $\mathcal{F}/I\mathcal{F}$ is zero. Let $I = (f_1, \ldots, f_r)$. Let $i < r$ be the largest integer such that $\mathcal{G} = \mathcal{F}/(f_1, \ldots, f_i)\mathcal{F}$ is nonzero. If i does not exist, then $\mathcal{F} = 0$ which is what we want to show. Then $\mathcal{G}$ is derived complete as a cokernel of a map between derived complete modules, see Proposition 17.1. By our choice of i we have that $f_{i+1} : \mathcal{G} \to \mathcal{G}$ is surjective. Hence

$$\lim(\ldots \to \mathcal{G} \xrightarrow{f_{i+1}} \mathcal{G} \xrightarrow{f_{i+1}} \mathcal{G})$$

is nonzero, contradicting the derived completeness of $\mathcal{G}$. $\square$

Lemma 26.7. *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a derived complete sheaf of Λ-modules on $X_{pro\text{-}étale}$ with $\mathcal{F}/I\mathcal{F}$ a locally constant sheaf of Λ/I-modules of finite type. Then there exists an integer t and a surjective map*

$$(\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}$$

Proof. Since X is weakly contractible, there exists a finite disjoint open covering $X = \coprod U_i$ such that $\mathcal{F}/I\mathcal{F}|_{U_i}$ is isomorphic to the constant sheaf associated to a finite Λ/I-module M_i. Choose finitely many generators m_{ij} of M_i. We can find sections $s_{ij} \in \mathcal{F}(X)$ restricting to m_{ij} viewed as a section of $\mathcal{F}/I\mathcal{F}$ over U_i. Let t be the total number of s_{ij}. Then we obtain a map

$$\alpha : \underline{\Lambda}^{\oplus t} \longrightarrow \mathcal{F}$$

which is surjective modulo I by construction. By Lemma 16.1 the derived completion of $\underline{\Lambda}^{\oplus t}$ is the sheaf $(\underline{\Lambda}^\wedge)^{\oplus t}$. Since $\mathcal{F}$ is derived complete we see that α factors through a map

$$\alpha^\wedge : (\underline{\Lambda}^\wedge)^{\oplus t} \longrightarrow \mathcal{F}$$

Then $\mathcal{Q} = \mathrm{Coker}(\alpha^\wedge)$ is a derived complete sheaf of Λ-modules by Proposition 17.1. By construction $\mathcal{Q}/I\mathcal{Q} = 0$. It follows from Lemma 26.6 that $\mathcal{Q} = 0$ which is what we wanted to show. $\square$

27. A suitable derived category

Let X be a scheme. It will turn out that for many schemes X a suitable derived category of ℓ-adic sheaves can be gotten by considering the derived complete objects K of $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$ with the property that $K \otimes_\Lambda^{\mathbf{L}} \mathbf{F}_\ell$ is bounded with constructible cohomology sheaves. Here is the general definition.

Definition 27.1. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let X be a scheme. An object K of $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$ is called *constructible* if

(1) K is derived complete with respect to I,
(2) $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ has constructible cohomology sheaves and locally has finite tor dimension.

We denote $D_{cons}(X, \Lambda)$ the full subcategory of constructible K in $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$.

Recall that with our conventions a complex of finite tor dimension is bounded (Cohomology on Sites, Definition 35.1). In fact, let's collect everything proved so far in a lemma.

Lemma 27.2. *In the situation above suppose K is in $D_{cons}(X, \Lambda)$ and X is quasi-compact. Set $K_n = K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. There exist a, b such that*

(1) *$K = R\lim K_n$ and $H^i(K) = 0$ for $i \notin [a, b]$,*
(2) *each K_n has tor amplitude in $[a, b]$,*
(3) *each K_n has constructible cohomology sheaves,*
(4) *each $K_n = \epsilon^{-1} L_n$ for some $L_n \in D_{ctf}(X_{\acute{e}tale}, \Lambda/I^n)$ (Étale Cohomology, Definition 90.7).*

Proof. By definition of local having finite tor dimension, we can find a, b such that K_1 has tor amplitude in $[a, b]$. Part (2) follows from Cohomology on Sites, Lemma 35.8. Then (1) follows as K is derived complete by the description of limits in Cohomology on Sites, Proposition 38.2 and the fact that $H^b(K_{n+1}) \to H^b(K_n)$ is surjective as $K_n = K_{n+1} \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. Part (3) follows from Lemma 25.6, Part (4) follows from Lemma 25.4 and the fact that L_n has finite tor dimension because K_n does (small argument omitted). $\square$

Lemma 27.3. *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$ such that the cohomology sheaves of $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ are locally constant. Then there exists a finite disjoint open covering $X = \coprod U_i$ and for each i a finite collection of finite projective $\Lambda^\wedge$-modules $M_a, \ldots, M_b$ such that $K|_{U_i}$ is represented by a complex*

$$(\underline{M^a})^\wedge \to \ldots \to (\underline{M^b})^\wedge$$

in $D(U_{i,pro\text{-}\acute{e}tale}, \Lambda)$ for some maps of sheaves of Λ-modules $(\underline{M^i})^\wedge \to (\underline{M^{i+1}})^\wedge$.

Proof. We freely use the results of Lemma 27.2. Choose a, b as in that lemma. We will prove the lemma by induction on $b - a$. Let $\mathcal{F} = H^b(K)$. Note that $\mathcal{F}$ is a derived complete sheaf of Λ-modules by Proposition 17.1. Moreover $\mathcal{F}/I\mathcal{F}$ is a locally constant sheaf of Λ/I-modules of finite type. Apply Lemma 26.7 to get a surjection $\rho : (\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}$.

If $a = b$, then $K = \mathcal{F}[-b]$. In this case we see that

$$\mathcal{F} \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I} = \mathcal{F}/I\mathcal{F}$$

As X is weakly contractible and $\mathcal{F}/I\mathcal{F}$ locally constant, we can find a finite disjoint union decomposition $X = \coprod U_i$ by affine opens U_i and Λ/I-modules $\overline{M}_i$ such that $\mathcal{F}/I\mathcal{F}$ restricts to $\underline{\overline{M}_i}$ on U_i. After refining the covering we may assume the map

$$\rho|_{U_i} \bmod I : \underline{\Lambda/I}^{\oplus t} \longrightarrow \underline{\overline{M}_i}$$

is equal to $\underline{\alpha_i}$ for some surjective module map $\alpha_i : \Lambda/I^{\oplus t} \to \overline{M}_i$, see Modules on Sites, Lemma 42.3. Note that each $\overline{M}_i$ is a finite Λ/I-module. Since $\mathcal{F}/I\mathcal{F}$ has tor amplitude in $[0, 0]$ we conclude that $\overline{M}_i$ is a flat Λ/I-module. Hence $\overline{M}_i$ is finite projective (Algebra, Lemma 75.2). Hence we can find a projector $\overline{p}_i : (\Lambda/I)^{\oplus t} \to (\Lambda/I)^{\oplus t}$ whose image maps isomorphically to $\overline{M}_i$ under the map α_i. We can lift $\overline{p}_i$ to a projector $p_i : (\Lambda^\wedge)^{\oplus t} \to (\Lambda^\wedge)^{\oplus t}$[5]. Then $M_i = \mathrm{Im}(p_i)$ is a finite I-adically complete $\Lambda^\wedge$-module with $M_i/IM_i = \overline{M}_i$. Over U_i consider the maps

$$\underline{M_i}^\wedge \to (\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}|_{U_i}$$

By construction the composition induces an isomorphism modulo I. The source and target are derived complete, hence so are the cokernel $\mathcal{Q}$ and the kernel $\mathcal{K}$. We have $\mathcal{Q}/I\mathcal{Q} = 0$ by construction hence $\mathcal{Q}$ is zero by Lemma 26.6. Then

$$0 \to \mathcal{K}/I\mathcal{K} \to \underline{\overline{M}_i} \to \mathcal{F}/I\mathcal{F} \to 0$$

is exact by the vanishing of Tor_1 see at the start of this paragraph; also use that $\underline{\Lambda}^\wedge/I\overline{\Lambda}^\wedge$ by Modules on Sites, Lemma 41.4 to see that $\underline{M_i}^\wedge/I\underline{M_i}^\wedge = \underline{\overline{M}_i}$. Hence $\mathcal{K}/I\mathcal{K} = 0$ by construction and we conclude that $\mathcal{K} = 0$ as before. This proves the result in case $a = b$.

If $b > a$, then we lift the map ρ to a map

$$\tilde{\rho} : (\underline{\Lambda}^\wedge)^{\oplus t}[-b] \longrightarrow K$$

in $D(X_{pro\text{-}\acute{e}tale}, \Lambda)$. This is possible as we can think of K as a complex of $\underline{\Lambda}^\wedge$-modules by discussion in the introduction to Section 16 and because $X_{pro\text{-}\acute{e}tale}$ is weakly contractible hence there is no obstruction to lifting the elements $\rho(e_s) \in H^0(X, \mathcal{F})$ to elements of $H^b(X, K)$. Fitting $\tilde{\rho}$ into a distinguished triangle

$$(\underline{\Lambda}^\wedge)^{\oplus t}[-b] \to K \to L \to (\underline{\Lambda}^\wedge)^{\oplus t}[-b+1]$$

we see that L is an object of $D_{cons}(X, \Lambda)$ such that $L \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ has tor amplitude contained in $[a, b-1]$ (details omitted). By induction we can describe L locally as stated in the lemma, say L is isomorphic to

$$(\underline{M^a})^\wedge \to \ldots \to (\underline{M^{b-1}})^\wedge$$

The map $L \to (\underline{\Lambda}^\wedge)^{\oplus t}[-b+1]$ corresponds to a map $(\underline{M^{b-1}})^\wedge \to (\underline{\Lambda}^\wedge)^{\oplus t}$ which allows us to extend the complex by one. The corresponding complex is isomorphic to K in the derived category by the properties of triangulated categories. This finishes the proof. $\square$

Motivated by what happens for constructible Λ-sheaves we introduce the following notion.

Definition 27.4. Let X be a scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $K \in D(X_{pro\text{-}\acute{e}tale}, \Lambda)$.

[5]Proof: by Algebra, Lemma 31.6 we can lift $\overline{p}_i$ to a compatible system of projectors $p_{i,n} : (\Lambda/I^n)^{\oplus t} \to (\Lambda/I^n)^{\oplus t}$ and then we set $p_i = \lim p_{i,n}$ which works because $\Lambda^\wedge = \lim \Lambda/I^n$.

(1) We say K is *adic lisse*[6] if there exists a finite complex of finite projective $\Lambda^\wedge$-modules $M^\bullet$ such that K is locally isomorphic to

$$\underline{M^{a\wedge}} \to \ldots \to \underline{M^{b\wedge}}$$

(2) We say K is *adic constructible*[7] if for every affine open $U \subset X$ there exists a decomposition $U = \coprod U_i$ into constructible locally closed subschemes such that $K|_{U_i}$ is adic lisse.

The difference between the local structure obtained in Lemma 27.3 and the structure of an adic lisse complex is that the maps $\underline{M^{i\wedge}} \to \underline{M^{i+1\wedge}}$ in Lemma 27.3 need not be constant, whereas in the definition above they are required to be constant.

Lemma 27.5. *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$ such that $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$ is isomorphic in $D(X_{pro\text{-}\acute{e}tale}, \Lambda/I^n)$ to a complex of constant sheaves of Λ/I^n-modules. Then*

$$H^0(X, K \otimes_\Lambda^{\mathbf{L}} \Lambda/I^n)$$

has the Mittag-Leffler condition.

Proof. Say $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$ is isomorphic to $\underline{E_n}$ for some object E_n of $D(\Lambda/I^n)$. Since $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ has finite tor dimension and has finite type cohomology sheaves we see that E_1 is perfect (see More on Algebra, Lemma 56.2). The transition maps

$$K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^{n+1}} \to K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$$

locally come from (possibly many distinct) maps of complexes $E_{n+1} \to E_n$ in $D(\Lambda/I^{n+1})$ see Cohomology on Sites, Lemma 40.3. For each n choose one such map and observe that it induces an isomorphism $E_{n+1} \otimes_{\Lambda/I^{n+1}}^{\mathbf{L}} \Lambda/I^n \to E_n$ in $D(\Lambda/I^n)$. By More on Algebra, Lemma 65.3 we can find a finite complex $M^\bullet$ of finite projective $\Lambda^\wedge$-modules and isomorphisms $M^\bullet/I^nM^\bullet \to E_n$ in $D(\Lambda/I^n)$ compatible with the transition maps.

Now observe that for each finite collection of indices $n > m > k$ the triple of maps

$$H^0(X, K \otimes_\Lambda^{\mathbf{L}} \Lambda/I^n) \to H^0(X, K \otimes_\Lambda^{\mathbf{L}} \Lambda/I^m) \to H^0(X, K \otimes_\Lambda^{\mathbf{L}} \Lambda/I^k)$$

is isomorphic to

$$H^0(X, \underline{M^\bullet/I^nM^\bullet}) \to H^0(X, \underline{M^\bullet/I^mM^\bullet}) \to H^0(X, \underline{M^\bullet/I^kM^\bullet})$$

Namely, choose any isomorphism

$$\underline{M^\bullet/I^nM^\bullet} \to K \otimes_\Lambda^{\mathbf{L}} \Lambda/I^n$$

induces similar isomorphisms module I^m and I^k and we see that the assertion is true. Thus to prove the lemma it suffices to show that the system $H^0(X, \underline{M^\bullet/I^nM^\bullet})$ has Mittag-Leffler. Since taking sections over X is exact, it suffices to prove that the system of Λ-modules

$$H^0(M^\bullet/I^nM^\bullet)$$

has Mittag-Leffler. Set $A = \Lambda^\wedge$ and consider the spectral sequence

$$\mathrm{Tor}_{-p}^A(H^q(M^\bullet), A/I^nA) \Rightarrow H^{p+q}(M^\bullet/I^nM^\bullet)$$

[6] This may be nonstandard notation

[7] This may be nonstandard notation.

By More on Algebra, Lemma 19.3 the pro-systems $\{\mathrm{Tor}^A_{-p}(H^q(M^\bullet), A/I^nA)\}$ are zero for $p > 0$. Thus the pro-system $\{H^0(M^\bullet/I^nM^\bullet)\}$ is equal to the pro-system $\{H^0(M^\bullet)/I^nH^0(M^\bullet)\}$ and the lemma is proved. $\square$

Lemma 27.6. *Let X be a connected scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. If K is in $D_{cons}(X, \Lambda)$ such that $K \otimes_\Lambda \underline{\Lambda/I}$ has locally constant cohomology sheaves, then K is adic lisse (Definition 27.4).*

Proof. Write $K_n = K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$. We will use the results of Lemma 27.2 without further mention. By Cohomology on Sites, Lemma 40.5 we see that K_n has locally constant cohomology sheaves for all n. We have $K_n = \epsilon^{-1}L_n$ some L_n in $D_{ctf}(X_{étale}, \Lambda/I^n)$ with locally constant cohomology sheaves. By Étale Cohomology, Lemma 90.14 there exist perfect $M_n \in D(\Lambda/I^n)$ such that L_n is étale locally isomorphic to $\underline{M_n}$. The maps $L_{n+1} \to L_n$ corresponding to $K_{n+1} \to K_n$ induces isomorphisms $L_{n+1} \otimes^{\mathbf{L}}_{\Lambda/I^{n+1}} \underline{\Lambda/I^n} \to L_n$. Looking locally on X we conclude that there exist maps $M_{n+1} \to M_n$ in $D(\Lambda/I^{n+1})$ inducing isomorphisms $M_{n+1} \otimes_{\Lambda/I^{n+1}} \Lambda/I^n \to M_n$, see Cohomology on Sites, Lemma 40.3. Fix a choice of such maps. By More on Algebra, Lemma 65.3 we can find a finite complex $M^\bullet$ of finite projective $\Lambda^\wedge$-modules and isomorphisms $M^\bullet/I^nM^\bullet \to M_n$ in $D(\Lambda/I^n)$ compatible with the transition maps. To finish the proof we will show that K is locally isomorphic to

$$\underline{M^\bullet}^\wedge = \lim \underline{M^\bullet/I^nM^\bullet} = R\lim \underline{M^\bullet/I^nM^\bullet}$$

Let $E^\bullet$ be the dual complex to $M^\bullet$, see More on Algebra, Lemma 56.21 and its proof. Consider the objects

$$H_n = R\mathcal{H}om_{\Lambda/I^n}(\underline{M^\bullet/I^nM^\bullet}, K_n) = \underline{E^\bullet/I^nE^\bullet} \otimes^{\mathbf{L}}_{\Lambda/I^n} K_n$$

of $D(X_{pro\text{-}étale}, \Lambda/I^n)$. Modding out by I^n defines a transition map $H_{n+1} \to H_n$. Set $H = R\lim H_n$. Then H is an object of $D_{cons}(X, \Lambda)$ (details omitted) with $H \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n} = H_n$. Choose a covering $\{W_t \to X\}_{t \in T}$ with each W_t affine and weakly contractible. By our choice of $M^\bullet$ we see that

$$\begin{aligned} H_n|_{W_t} &\cong R\mathcal{H}om_{\Lambda/I^n}(\underline{M^\bullet/I^nM^\bullet}, \underline{M^\bullet/I^nM^\bullet}) \\ &= \underline{\mathrm{Tot}(E^\bullet/I^nE^\bullet \otimes_{\Lambda/I^n} M^\bullet/I^nM^\bullet)} \end{aligned}$$

Thus we may apply Lemma 27.5 to $H = R\lim H_n$. We conclude the system $H^0(W_t, H_n)$ satisfies Mittag-Leffler. Since for all $n \gg 1$ there is an element of $H^0(W_t, H_n)$ which maps to an isomorphism in

$$H^0(W_t, H_1) = \mathrm{Hom}(\underline{M^\bullet/IM^\bullet}, K_1)$$

we find an element $(\varphi_{t,n})$ in the inverse limit which produces an isomorphism mod I. Then

$$R\lim \varphi_{t,n} : \underline{M^\bullet}^\wedge|_{W_t} = R\lim \underline{M^\bullet/I^nM^\bullet}|_{W_t} \longrightarrow R\lim K_n|_{W_t} = K|_{W_t}$$

is an isomorphism. This finishes the proof. $\square$

Proposition 27.7. *Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$. Then K is adic constructible (Definition 27.4).*

Proof. This is a consequence of Lemma 27.6 and the fact that a Noetherian scheme is locally connected (Topology, Lemma 8.6), and the definitions. $\square$

28. Proper base change

In this section we explain how to prove the proper base change theorem for derived complete objects on the pro-étale site using the proper base change theorem for étale cohomology following the general theme that we use the pro-étale topology only to deal with "limit issues" and we use results proved for the étale topology to handle everything else.

Theorem 28.1. *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes giving rise to the base change diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal such that Λ/I is torsion. Let K be an object of $D(X_{pro\text{-}étale})$ such that

1. *K is derived complete, and*
2. *$K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$ is bounded below with cohomology sheaves coming from $X_{étale}$,*
3. *Λ/I^n is a perfect Λ-module*[8].

Then the base change map

$$Lg^*_{comp}Rf_*K \longrightarrow Rf'_*L(g')^*_{comp}K$$

is an isomorphism.

Proof. We omit the construction of the base change map (this uses only formal properties of derived pushforward and completed derived pullback, compare with Cohomology on Sites, Remark 19.2). Write $K_n = K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$. By Lemma 16.1 we have $K = R\lim K_n$ because K is derived complete. By Lemmas 16.2 and 16.1 we can unwind the left hand side

$$Lg^*_{comp}Rf_*K = R\lim Lg^*(Rf_*K) \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n} = R\lim Lg^*Rf_*K_n$$

the last equality because Λ/I^n is a perfect module and the projection formula (Cohomology on Sites, Lemma 37.1). Using Lemma 16.2 we can unwind the right hand side

$$Rf'_*L(g')^*_{comp}K = Rf'_*R\lim L(g')^*K_n = R\lim Rf'_*L(g')^*K_n$$

the last equality because Rf'_* commutes with $R\lim$ (Cohomology on Sites, Lemma 21.2). Thus it suffices to show the maps

$$Lg^*Rf_*K_n \longrightarrow Rf'_*L(g')^*K_n$$

are isomorphisms. By Lemma 18.7 and our second condition we can write $K_n = \epsilon^{-1}L_n$ for some $L_n \in D^+(X_{étale}, \Lambda/I^n)$. By Lemma 21.1 and the fact that ϵ^{-1} commutes with pullbacks we obtain

$$Lg^*Rf_*K_n = Lg^*Rf_*\epsilon^*L_n = Lg^*\epsilon^{-1}Rf_*L_n = \epsilon^{-1}Lg^*Rf_*L_n$$

and

$$Rf'_*L(g')^*K_n = Rf'_*L(g')^*\epsilon^{-1}L_n = Rf'_*\epsilon^{-1}L(g')^*L_n = \epsilon^{-1}Rf'_*L(g')^*L_n$$

[8] This assumption can be removed if K is a constructible complex, see [BS13].

(this also uses that L_n is bounded below). Finally, by the proper base change theorem for étale cohomology (Étale Cohomology, Theorem 77.11) we have

$$Lg^*Rf_*L_n = Rf'_*L(g')^*L_n$$

(again using that L_n is bounded below) and the theorem is proved. □

29. Other chapters

Miscellany

References

[BBD82] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[BS13] Bhargav Bhatt and Peter Scholze, *The pro-étale topology for schemes*, preprint, 2013.

[Del74] Pierre Deligne, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 273–307.

[Eke90] Torsten Ekedahl, *On the adic formalism*, 197–218.

[GR13] Dennis Gaitsgory and Nick Rozenblyum, *Dg indschemes*, 2013.

[Gro77] Alexandre Grothendieck, *Séminaire de géométrie algébrique du bois-marie 1965-66, cohomologie l-adique et fonctions l, sga5*, Springer Lecture Notes, vol. 589, Springer-Verlag, 1977.

[HLP14] Daniel Halpern-Leistner and Anatoly Preygel, *Mapping stacks and categorical notions of properness*, 2014.

[Jan88] Uwe Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), 207–245.

[Lur04] Jacob Lurie, *Derived algebraic geometry*, 2004.

[Lur11] ———, *Derived algebraic geometry XII: Proper morphisms, completions, and the grothendieck existence theorem.*

[Tat76] John Tate, *Relations between k2 and galois cohomology*, Inventiones mathematicae **36** (1976), 257–274.