

GROUPOIDS IN ALGEBRAIC SPACES

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1. Introduction

This chapter is devoted to generalities concerning groupoids in algebraic spaces. We recommend reading the beautiful paper [KM97] by Keel and Mori.

A lot of what we say here is a repeat of what we said in the chapter on groupoid schemes, see Groupoids, Section 1. The discussion of quotient stacks is new here.

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2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf} . And all rings A considered have the property that $\text{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S . In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

We continue our convention to label projection maps starting with index 0, so we have $\text{pr}_0 : X \times_S Y \rightarrow X$ and $\text{pr}_1 : X \times_S Y \rightarrow Y$.

3. Notation

Let S be a scheme; this will be our base scheme and all algebraic spaces will be over S . Let B be an algebraic space over S ; this will be our base algebraic space, and often other algebraic spaces, and schemes will be over B . If we say that X is an algebraic space over B , then we mean that X is an algebraic space over S which comes equipped with structure morphism $X \rightarrow B$. Moreover, we try to reserve the letter T to denote a “test” scheme over B . In other words T is a scheme which comes equipped with a structure morphism $T \rightarrow B$. In this situation we denote $X(T)$ for the set of T -valued points of X over B . In a formula:

$$X(T) = \text{Mor}_B(T, X).$$

Similarly, given a second algebraic space Y over B we set

$$X(Y) = \text{Mor}_B(Y, X).$$

Suppose we are given algebraic spaces X, Y over B as above and a morphism $f : X \rightarrow Y$ over B . For any scheme T over B we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

which is functorial in the scheme T over B . As f is a map of sheaves on $(Sch/S)_{fppf}$ over the sheaf B it is clear that f determines and is determined by this rule. More generally, we use the same notation for maps between fibre products. For example, if X, Y, Z are algebraic spaces over B , and if $m : X \times_B Y \rightarrow Z \times_B Z$ is a morphism of algebraic spaces over B , then we think of m as corresponding to a collection of maps between T -valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

Finally, given two maps $f, g : X \rightarrow Y$ of algebraic spaces over B , if the induced maps $f, g : X(T) \rightarrow Y(T)$ are equal for every scheme T over B , then $f = g$, and hence also $f, g : X(Z) \rightarrow Y(Z)$ are equal for every third algebraic space Z over B . Hence, for example, to check the axioms for a group algebraic space G over B , it suffices to check commutativity of diagram on T -valued points where T is a scheme over B as we do in Definition 5.1 below.

4. Equivalence relations

Please refer to Groupoids, Section 3 for notation.

Definition 4.1. Let $B \rightarrow S$ as in Section 3. Let U be an algebraic space over B .

- (1) A *pre-relation* on U over B is any morphism $j : R \rightarrow U \times_B U$ of algebraic spaces over B . In this case we set $t = \text{pr}_0 \circ j$ and $s = \text{pr}_1 \circ j$, so that $j = (t, s)$.
- (2) A *relation* on U over B is a monomorphism $j : R \rightarrow U \times_B U$ of algebraic spaces over B .
- (3) A *pre-equivalence relation* is a pre-relation $j : R \rightarrow U \times_B U$ such that the image of $j : R(T) \rightarrow U(T) \times U(T)$ is an equivalence relation for all schemes T over B .
- (4) We say a morphism $R \rightarrow U \times_B U$ of algebraic spaces over B is an *equivalence relation on U over B* if and only if for every T over B the T -valued points of R define an equivalence relation on the set of T -valued points of U .

In other words, an equivalence relation is a pre-equivalence relation such that j is a relation.

Lemma 4.2. Let $B \rightarrow S$ as in Section 3. Let U be an algebraic space over B . Let $j : R \rightarrow U \times_B U$ be a pre-relation. Let $g : U' \rightarrow U$ be a morphism of algebraic spaces over B . Finally, set

$$R' = (U' \times_B U') \times_{U \times_B U} R \xrightarrow{j'} U' \times_B U'$$

Then j' is a pre-relation on U' over B . If j is a relation, then j' is a relation. If j is a pre-equivalence relation, then j' is a pre-equivalence relation. If j is an equivalence relation, then j' is an equivalence relation.

Proof. Omitted. □

Definition 4.3. Let $B \rightarrow S$ as in Section 3. Let U be an algebraic space over B . Let $j : R \rightarrow U \times_B U$ be a pre-relation. Let $g : U' \rightarrow U$ be a morphism of algebraic spaces over B . The pre-relation $j' : R' \rightarrow U' \times_B U'$ is called the *restriction*, or *pullback* of the pre-relation j to U' . In this situation we sometimes write $R' = R|_{U'}$.

Lemma 4.4. Let $B \rightarrow S$ as in Section 3. Let $j : R \rightarrow U \times_B U$ be a pre-relation of algebraic spaces over B . Consider the relation on $|U|$ defined by the rule

$$x \sim y \Leftrightarrow \exists r \in |R| : t(r) = x, s(r) = y.$$

If j is a pre-equivalence relation then this is an equivalence relation.

Proof. Suppose that $x \sim y$ and $y \sim z$. Pick $r \in |R|$ with $t(r) = x$, $s(r) = y$ and pick $r' \in |R|$ with $t(r') = y$, $s(r') = z$. We may pick a field K such that r and r' can be represented by morphisms $r, r' : \text{Spec}(K) \rightarrow R$ with $s \circ r = t \circ r'$. Denote $x = t \circ r$, $y = s \circ r = t \circ r'$, and $z = s \circ r'$, so $x, y, z : \text{Spec}(K) \rightarrow U$. By construction $(x, y) \in j(R(K))$ and $(y, z) \in j(R(K))$. Since j is a pre-equivalence relation we see that also $(x, z) \in j(R(K))$. This clearly implies that $x \sim z$.

The proof that \sim is reflexive and symmetric is omitted. □

5. Group algebraic spaces

Please refer to Groupoids, Section 4 for notation.

Definition 5.1. Let $B \rightarrow S$ as in Section 3.

- (1) A *group algebraic space over B* is a pair (G, m) , where G is an algebraic space over B and $m : G \times_B G \rightarrow G$ is a morphism of algebraic spaces over B with the following property: For every scheme T over B the pair $(G(T), m)$ is a group.
- (2) A *morphism $\psi : (G, m) \rightarrow (G', m')$ of group algebraic spaces over B* is a morphism $\psi : G \rightarrow G'$ of algebraic spaces over B such that for every T/B the induced map $\psi : G(T) \rightarrow G'(T)$ is a homomorphism of groups.

Let (G, m) be a group algebraic space over the algebraic space B . By the discussion in Groupoids, Section 4 we obtain morphisms of algebraic spaces over B (identity) $e : B \rightarrow G$ and (inverse) $i : B \rightarrow B$ such that for every T the quadruple $(G(T), m, e, i)$ satisfies the axioms of a group.

Let $(G, m), (G', m')$ be group algebraic spaces over B . Let $f : G \rightarrow G'$ be a morphism of algebraic spaces over B . It follows from the definition that f is a morphism of group algebraic spaces over B if and only if the following diagram is commutative:

$$\begin{array}{ccc} G \times_B G & \xrightarrow{f \times f} & G' \times_B G' \\ m \downarrow & & \downarrow m' \\ G & \xrightarrow{f} & G' \end{array}$$

Lemma 5.2. Let $B \rightarrow S$ as in Section 3. Let (G, m) be a group algebraic space over B . Let $B' \rightarrow B$ be a morphism of algebraic spaces. The pullback $(G_{B'}, m_{B'})$ is a group algebraic space over B' .

Proof. Omitted. □

6. Properties of group algebraic spaces

In this section we collect some simple properties of group algebraic spaces which hold over any base.

Lemma 6.1. Let S be a scheme. Let B be an algebraic space over S . Let G be a group algebraic space over B . Then $G \rightarrow B$ is separated (resp. quasi-separated, resp. locally separated) if and only if the identity morphism $e : B \rightarrow G$ is a closed immersion (resp. quasi-compact, resp. an immersion).

Proof. We recall that by Morphisms of Spaces, Lemma 4.7 we have that e is a closed immersion (resp. quasi-compact, resp. an immersion) if $G \rightarrow B$ is separated (resp. quasi-separated, resp. locally separated). For the converse, consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\Delta_{G/B}} & G \times_B G \\ \downarrow & & \downarrow (g, g') \mapsto m(i(g), g') \\ B & \xrightarrow{e} & G \end{array}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that $\Delta_{G/B}$ is a base change of

e. Hence if e is a closed immersion (resp. quasi-compact, resp. an immersion) so is $\Delta_{G/B}$, see Spaces, Lemma 12.3 (resp. Morphisms of Spaces, Lemma 8.3, resp. Spaces, Lemma 12.3). \square

7. Examples of group algebraic spaces

If $G \rightarrow S$ is a group scheme over the base scheme S , then the base change G_B to any algebraic space B over S is an group algebraic space over B by Lemma 5.2. We will frequently use this in the examples below.

Example 7.1 (Multiplicative group algebraic space). Let $B \rightarrow S$ as in Section 3. Consider the functor which associates to any scheme T over B the group $\Gamma(T, \mathcal{O}_T^*)$ of units in the global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathbf{G}_{m,B} = B \times_S \mathbf{G}_{m,S}$$

over B . Here $\mathbf{G}_{m,S}$ is the multiplicative group scheme over S , see Groupoids, Example 5.1.

Example 7.2 (Roots of unity as a group algebraic space). Let $B \rightarrow S$ as in Section 3. Let $n \in \mathbf{N}$. Consider the functor which associates to any scheme T over B the subgroup of $\Gamma(T, \mathcal{O}_T^*)$ consisting of n th roots of unity. This is representable by the group algebraic space

$$\mu_{n,B} = B \times_S \mu_{n,S}$$

over B . Here $\mu_{n,S}$ is the group scheme of n th roots of unity over S , see Groupoids, Example 5.2.

Example 7.3 (Additive group algebraic space). Let $B \rightarrow S$ as in Section 3. Consider the functor which associates to any scheme T over B the group $\Gamma(T, \mathcal{O}_T)$ of global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathbf{G}_{a,B} = B \times_S \mathbf{G}_{a,S}$$

over B . Here $\mathbf{G}_{a,S}$ is the additive group scheme over S , see Groupoids, Example 5.3.

Example 7.4 (General linear group algebraic space). Let $B \rightarrow S$ as in Section 3. Let $n \geq 1$. Consider the functor which associates to any scheme T over B the group

$$\mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible $n \times n$ matrices over the global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathrm{GL}_{n,B} = B \times_S \mathrm{GL}_{n,S}$$

over B . Here $\mathbf{G}_{m,S}$ is the general linear group scheme over S , see Groupoids, Example 5.4.

Example 7.5. Let $B \rightarrow S$ as in Section 3. Let $n \geq 1$. The determinant defines a morphisms of group algebraic spaces

$$\det : \mathrm{GL}_{n,B} \longrightarrow \mathbf{G}_{m,B}$$

over B . It is the base change of the determinant morphism over S from Groupoids, Example 5.5.

Example 7.6 (Constant group algebraic space). Let $B \rightarrow S$ as in Section 3. Let G be an abstract group. Consider the functor which associates to any scheme T over B the group of locally constant maps $T \rightarrow G$ (where T has the Zariski topology and G the discrete topology). This is representable by the group algebraic space

$$G_B = B \times_S G_S$$

over B . Here G_S is the constant group scheme introduced in Groupoids, Example 5.6.

8. Actions of group algebraic spaces

Please refer to Groupoids, Section 8 for notation.

Definition 8.1. Let $B \rightarrow S$ as in Section 3. Let (G, m) be a group algebraic space over B . Let X be an algebraic space over B .

- (1) An *action of G on the algebraic space X/B* is a morphism $a : G \times_B X \rightarrow X$ over B such that for every scheme T over B the map $a : G(T) \times X(T) \rightarrow X(T)$ defines the structure of a $G(T)$ -set on $X(T)$.
- (2) Suppose that X, Y are algebraic spaces over B each endowed with an action of G . An *equivariant* or more precisely a *G -equivariant* morphism $\psi : X \rightarrow Y$ is a morphism of algebraic spaces over B such that for every T over B the map $\psi : X(T) \rightarrow Y(T)$ is a morphism of $G(T)$ -sets.

In situation (1) this means that the diagrams

$$(8.1.1) \quad \begin{array}{ccc} G \times_B G \times_B X & \xrightarrow{1_G \times a} & G \times_B X \\ m \times 1_X \downarrow & & \downarrow a \\ G \times_B X & \xrightarrow{a} & X \end{array} \quad \begin{array}{ccc} G \times_B X & \xrightarrow{a} & X \\ e \times 1_X \uparrow & \nearrow 1_X & \\ X & & \end{array}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{array}{ccc} G \times_B X & \xrightarrow{\text{id} \times f} & G \times_B Y \\ a \downarrow & & \downarrow a \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Definition 8.2. Let $B \rightarrow S$, $G \rightarrow B$, and $X \rightarrow B$ as in Definition 8.1. Let $a : G \times_B X \rightarrow X$ be an action of G on X/B . We say the action is *free* if for every scheme T over B the action $a : G(T) \times X(T) \rightarrow X(T)$ is a free action of the group $G(T)$ on the set $X(T)$.

Lemma 8.3. *Situation as in Definition 8.2, The action a is free if and only if*

$$G \times_B X \rightarrow X \times_B X, \quad (g, x) \mapsto (a(g, x), x)$$

is a monomorphism of algebraic spaces.

Proof. Immediate from the definitions. □

9. Principal homogeneous spaces

This section is the analogue of Groupoids, Section 9. We suggest reading that section first.

Definition 9.1. Let S be a scheme. Let B be an algebraic space over S . Let (G, m) be a group algebraic space over B . Let X be an algebraic space over B , and let $a : G \times_B X \rightarrow X$ be an action of G on X .

- (1) We say X is a *pseudo G -torsor* or that X is *formally principally homogeneous under G* if the induced morphism $G \times_B X \rightarrow X \times_B X$, $(g, x) \mapsto (a(g, x), x)$ is an isomorphism.
- (2) A pseudo G -torsor X is called *trivial* if there exists an G -equivariant isomorphism $G \rightarrow X$ over B where G acts on G by left multiplication.

It is clear that if $B' \rightarrow B$ is a morphism of algebraic spaces then the pullback $X_{B'}$ of a pseudo G -torsor over B is a pseudo $G_{B'}$ -torsor over B' .

Lemma 9.2. *In the situation of Definition 9.1.*

- (1) *The algebraic space X is a pseudo G -torsor if and only if for every scheme T over B the set $X(T)$ is either empty or the action of the group $G(T)$ on $X(T)$ is simply transitive.*
- (2) *A pseudo G -torsor X is trivial if and only if the morphism $X \rightarrow B$ has a section.*

Proof. Omitted. □

Definition 9.3. Let S be a scheme. Let B be an algebraic space over S . Let (G, m) be a group algebraic space over B . Let X be a pseudo G -torsor over B .

- (1) We say X is a *principal homogeneous space*, or more precisely a *principal homogeneous G -space over B* if there exists a fpqc covering¹ $\{B_i \rightarrow B\}_{i \in I}$ such that each $X_{B_i} \rightarrow B_i$ has a section (i.e., is a trivial pseudo G_{B_i} -torsor).
- (2) Let $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$. We say X is a *G -torsor in the τ topology*, or a *τ G -torsor*, or simply a *τ torsor* if there exists a τ covering $\{B_i \rightarrow B\}_{i \in I}$ such that each $X_{B_i} \rightarrow B_i$ has a section.
- (3) If X is a G -torsor, then we say that it is *quasi-isotrivial* if it is a torsor for the étale topology.
- (4) If X is a G -torsor, then we say that it is *locally trivial* if it is a torsor for the Zariski topology.

We sometimes say “let X be a G -principal homogeneous space over B ” to indicate that X is an algebraic space over B equipped with an action of G which turns it into a principal homogeneous space over B . Next we show that this agrees with the notation introduced earlier when both apply.

Lemma 9.4. *Let S be a scheme. Let (G, m) be a group algebraic space over S . Let X be an algebraic space over S , and let $a : G \times_S X \rightarrow X$ be an action of G on X . Then X is a G -torsor in the fppf-topology in the sense of Definition 9.3 if*

¹The default type of torsor in Groupoids, Definition 9.3 is a pseudo torsor which is trivial on an fpqc covering. Since G , as an algebraic space, can be seen a sheaf of groups there already is a notion of a G -torsor which corresponds to fppf-torsor, see Lemma 9.4. Hence we use “principal homogeneous space” for a pseudo torsor which is fpqc locally trivial, and we try to avoid using the word torsor in this situation.

and only if X is a G -torsor on $(Sch/S)_{fppf}$ in the sense of *Cohomology on Sites*, Definition 5.1.

Proof. Omitted. □

10. Equivariant quasi-coherent sheaves

Please compare with Groupoids, Section 10.

Definition 10.1. Let $B \rightarrow S$ as in Section 3. Let (G, m) be a group algebraic space over B , and let $a : G \times_B X \rightarrow X$ be an action of G on the algebraic space X over B . An G -equivariant quasi-coherent \mathcal{O}_X -module, or simply a *equivariant quasi-coherent \mathcal{O}_X -module*, is a pair (\mathcal{F}, α) , where \mathcal{F} is a quasi-coherent \mathcal{O}_X -module, and α is a $\mathcal{O}_{G \times_B X}$ -module map

$$\alpha : a^* \mathcal{F} \longrightarrow \text{pr}_1^* \mathcal{F}$$

where $\text{pr}_1 : G \times_B X \rightarrow X$ is the projection such that

(1) the diagram

$$\begin{array}{ccc} (1_G \times a)^* \text{pr}_2^* \mathcal{F} & \xrightarrow{\text{pr}_{12}^* \alpha} & \text{pr}_2^* \mathcal{F} \\ (1_G \times a)^* \alpha \uparrow & & \uparrow (m \times 1_X)^* \alpha \\ (1_G \times a)^* a^* \mathcal{F} & \xlongequal{\quad} & (m \times 1_X)^* a^* \mathcal{F} \end{array}$$

is a commutative in the category of $\mathcal{O}_{G \times_B G \times_B X}$ -modules, and

(2) the pullback

$$(e \times 1_X)^* \alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

For explanation compare with the relevant diagrams of Equation (8.1.1).

Note that the commutativity of the first diagram guarantees that $(e \times 1_X)^* \alpha$ is an idempotent operator on \mathcal{F} , and hence condition (2) is just the condition that it is an isomorphism.

Lemma 10.2. *Let $B \rightarrow S$ as in Section 3. Let G be a group algebraic space over B . Let $f : X \rightarrow Y$ be a G -equivariant morphism between algebraic spaces over B endowed with G -actions. Then pullback f^* given by $(\mathcal{F}, \alpha) \mapsto (f^* \mathcal{F}, (1_G \times f)^* \alpha)$ defines a functor from the category of G -equivariant sheaves on X to the category of quasi-coherent G -equivariant sheaves on Y .*

Proof. Omitted. □

11. Groupoids in algebraic spaces

Please refer to Groupoids, Section 11 for notation.

Definition 11.1. Let $B \rightarrow S$ as in Section 3.

- (1) A *groupoid in algebraic spaces over B* is a quintuple (U, R, s, t, c) where U and R are algebraic spaces over B , and $s, t : R \rightarrow U$ and $c : R \times_{s, U, t} R \rightarrow R$ are morphisms of algebraic spaces over B with the following property: For any scheme T over B the quintuple

$$(U(T), R(T), s, t, c)$$

is a groupoid category.

- (2) A morphism $f : (U, R, s, t, c) \rightarrow (U', R', s', t', c')$ of groupoids in algebraic spaces over B is given by morphisms of algebraic spaces $f : U \rightarrow U'$ and $f : R \rightarrow R'$ over B with the following property: For any scheme T over B the maps f define a functor from the groupoid category $(U(T), R(T), s, t, c)$ to the groupoid category $(U'(T), R'(T), s', t', c')$.

Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . Note that there are unique morphisms of algebraic spaces $e : U \rightarrow R$ and $i : R \rightarrow R$ over B such that for every scheme T over B the induced map $e : U(T) \rightarrow R(T)$ is the identity, and $i : R(T) \rightarrow R(T)$ is the inverse of the groupoid category. The septuple (U, R, s, t, c, e, i) satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) of Groupoids, Section 11. Conversely given a septuple with this property the quintuple (U, R, s, t, c) is a groupoid in algebraic spaces over B . Note that i is an isomorphism, and e is a section of both s and t . Moreover, given a groupoid in algebraic spaces over B we denote

$$j = (t, s) : R \longrightarrow U \times_B U$$

which is compatible with our conventions in Section 4 above. We sometimes say “let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B ” to stress the existence of identity and inverse.

Lemma 11.2. *Let $B \rightarrow S$ as in Section 3. Given a groupoid in algebraic spaces (U, R, s, t, c) over B the morphism $j : R \rightarrow U \times_B U$ is a pre-equivalence relation.*

Proof. Omitted. This is a nice exercise in the definitions. □

Lemma 11.3. *Let $B \rightarrow S$ as in Section 3. Given an equivalence relation $j : R \rightarrow U$ over B there is a unique way to extend it to a groupoid in algebraic spaces (U, R, s, t, c) over B .*

Proof. Omitted. This is a nice exercise in the definitions. □

Lemma 11.4. *Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . In the commutative diagram*

$$\begin{array}{ccccc}
 & & U & & \\
 & t \nearrow & & \nwarrow t & \\
 R & \xleftarrow{pr_0} & R \times_{s,U,t} R & \xrightarrow{c} & R \\
 \downarrow s & & \downarrow pr_1 & & \downarrow s \\
 U & \xleftarrow{t} & R & \xrightarrow{s} & U
 \end{array}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

Lemma 11.5. *Let $B \rightarrow S$ be as in Section 3. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B . The diagram*

$$(11.5.1) \quad \begin{array}{ccccc} R \times_{t,U,t} R & \xrightarrow{pr_1} & R & \xrightarrow{t} & U \\ \downarrow pr_0 \times c \circ (i,1) & \searrow pr_0 & \downarrow id_R & & \downarrow id_U \\ R \times_{s,U,t} R & \xrightarrow{c} & R & \xrightarrow{t} & U \\ \downarrow pr_1 & & \downarrow s & & \\ R & \xrightarrow{s} & U & \xrightarrow{t} & \end{array}$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

Proof. The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms (α, β) with the same target, the pair of morphisms $(\alpha, \alpha^{-1} \circ \beta)$. In any groupoid this defines a bijection between $\text{Arrows} \times_{t, \text{Ob}, t} \text{Arrows}$ and $\text{Arrows} \times_{s, \text{Ob}, t} \text{Arrows}$. Hence the second assertion of the lemma. The last assertion follows from Lemma 11.4. \square

12. Quasi-coherent sheaves on groupoids

Please compare with Groupoids, Section 12.

Definition 12.1. Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . A *quasi-coherent module* on (U, R, s, t, c) is a pair (\mathcal{F}, α) , where \mathcal{F} is a quasi-coherent \mathcal{O}_U -module, and α is a \mathcal{O}_R -module map

$$\alpha : t^* \mathcal{F} \longrightarrow s^* \mathcal{F}$$

such that

- (1) the diagram

$$\begin{array}{ccc} & pr_1^* t^* \mathcal{F} \xrightarrow{pr_1^* \alpha} pr_1^* s^* \mathcal{F} & \\ & \parallel & \parallel \\ pr_0^* s^* \mathcal{F} & & c^* s^* \mathcal{F} \\ & \swarrow pr_0^* \alpha & \searrow c^* \alpha \\ & pr_0^* t^* \mathcal{F} \xlongequal{=} c^* t^* \mathcal{F} & \end{array}$$

is commutative in the category of $\mathcal{O}_{R \times_{s,U,t} R}$ -modules, and

- (2) the pullback

$$e^* \alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

Compare with the commutative diagrams of Lemma 11.4.

The commutativity of the first diagram forces the operator $e^* \alpha$ to be idempotent. Hence the second condition can be reformulated as saying that $e^* \alpha$ is an isomorphism. In fact, the condition implies that α is an isomorphism.

Lemma 12.2. *Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S . If (\mathcal{F}, α) is a quasi-coherent module on (U, R, s, t, c) then α is an isomorphism.*

Proof. Pull back the commutative diagram of Definition 12.1 by the morphism $(i, 1) : R \rightarrow R \times_{s, U, t} R$. Then we see that $i^* \alpha \circ \alpha = s^* e^* \alpha$. Pulling back by the morphism $(1, i)$ we obtain the relation $\alpha \circ i^* \alpha = t^* e^* \alpha$. By the second assumption these morphisms are the identity. Hence $i^* \alpha$ is an inverse of α . \square

Lemma 12.3. *Let $B \rightarrow S$ as in Section 3. Consider a morphism $f : (U, R, s, t, c) \rightarrow (U', R', s', t', c')$ of groupoid in algebraic spaces over B . Then pullback f^* given by*

$$(\mathcal{F}, \alpha) \mapsto (f^* \mathcal{F}, f^* \alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U', R', s', t', c') to the category of quasi-coherent sheaves on (U, R, s, t, c) .

Proof. Omitted. \square

Lemma 12.4. *Let $B \rightarrow S$ be as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . The category of quasi-coherent modules on (U, R, s, t, c) has colimits.*

Proof. Let $i \mapsto (\mathcal{F}_i, \alpha_i)$ be a diagram over the index category \mathcal{I} . We can form the colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ which is a quasi-coherent sheaf on U , see Properties of Spaces, Lemma 27.7. Since colimits commute with pullback we see that $s^* \mathcal{F} = \operatorname{colim} s^* \mathcal{F}_i$ and similarly $t^* \mathcal{F} = \operatorname{colim} t^* \mathcal{F}_i$. Hence we can set $\alpha = \operatorname{colim} \alpha_i$. We omit the proof that (\mathcal{F}, α) is the colimit of the diagram in the category of quasi-coherent modules on (U, R, s, t, c) . \square

Lemma 12.5. *Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . If s, t are flat, then the category of quasi-coherent modules on (U, R, s, t, c) is abelian.*

Proof. Let $\varphi : (\mathcal{F}, \alpha) \rightarrow (\mathcal{G}, \beta)$ be a homomorphism of quasi-coherent modules on (U, R, s, t, c) . Since s is flat we see that

$$0 \rightarrow s^* \operatorname{Ker}(\varphi) \rightarrow s^* \mathcal{F} \rightarrow s^* \mathcal{G} \rightarrow s^* \operatorname{Coker}(\varphi) \rightarrow 0$$

is exact and similarly for pullback by t . Hence α and β induce isomorphisms $\kappa : t^* \operatorname{Ker}(\varphi) \rightarrow s^* \operatorname{Ker}(\varphi)$ and $\lambda : t^* \operatorname{Coker}(\varphi) \rightarrow s^* \operatorname{Coker}(\varphi)$ which satisfy the cocycle condition. Then it is straightforward to verify that $(\operatorname{Ker}(\varphi), \kappa)$ and $(\operatorname{Coker}(\varphi), \lambda)$ are a kernel and cokernel in the category of quasi-coherent modules on (U, R, s, t, c) . Moreover, the condition $\operatorname{Coim}(\varphi) = \operatorname{Im}(\varphi)$ follows because it holds over U . \square

13. Crystals in quasi-coherent sheaves

Let (I, Φ, j) be a pair consisting of a set I and a pre-relation $j : \Phi \rightarrow I \times I$. Assume given for every $i \in I$ a scheme X_i and for every $\phi \in \Phi$ a morphisms of schemes $f_\phi : X_{i'} \rightarrow X_i$ where $j(\phi) = (i, i')$. Set $X = (\{X_i\}_{i \in I}, \{f_\phi\}_{\phi \in \Phi})$. Define a *crystal in quasi-coherent modules on X* as a rule which associates to every $i \in \operatorname{Ob}(\mathcal{I})$ a quasi-coherent sheaf \mathcal{F}_i on X_i and for every $\phi \in \Phi$ with $j(\phi) = (i, i')$ an isomorphism

$$\alpha_\phi : f_\phi^* \mathcal{F}_i \longrightarrow \mathcal{F}_{i'}$$

of quasi-coherent sheaves on $X_{i'}$. These crystals in quasi-coherent modules form an additive category $CQC(X)^2$. This category has colimits (proof is the same as the proof of Lemma 12.4). If all the morphisms f_ϕ are flat, then $CQC(X)$ is abelian (proof is the same as the proof of Lemma 12.5). Let κ be a cardinal. We say that a crystal in quasi-coherent modules \mathcal{F} on X is κ -generated if each \mathcal{F}_i is κ -generated (see Properties, Definition 21.1).

Lemma 13.1. *In the situation above, if all the morphisms f_ϕ are flat, then there exists a cardinal κ such that every object $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$ of $CQC(X)$ is the directed colimit of its κ -generated submodules.*

Proof. In the lemma and in this proof a *submodule* of $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$ means the data of a quasi-coherent submodule $\mathcal{G}_i \subset \mathcal{F}_i$ for all i such that $\alpha_\phi(f_\phi^* \mathcal{G}_i) = \mathcal{G}_{i'}$ as subsheaves of $\mathcal{F}_{i'}$ for all $\phi \in \Phi$. This makes sense because since f_ϕ is flat the pullback f_ϕ^* is exact, i.e., preserves subsheaves. The proof will be a variant to the proof of Properties, Lemma 21.3. We urge the reader to read that proof first.

We claim that it suffices to prove the lemma in case all the schemes X_i are affine. To see this let

$$J = \coprod_{i \in I} \{U \subset X_i \text{ affine open}\}$$

and let

$$\begin{aligned} \Psi = & \coprod_{\phi \in \Phi} \{(U, V) \mid U \subset X_i, V \subset X_{i'} \text{ affine open with } f_\phi(U) \subset V\} \\ & \coprod \coprod_{i \in I} \{(U, U') \mid U, U' \subset X_i \text{ affine open with } U \subset U'\} \end{aligned}$$

endowed with the obvious map $\Psi \rightarrow J \times J$. Then our (\mathcal{F}, α) induces a crystal in quasi-coherent sheaves $(\{\mathcal{H}_j\}_{j \in J}, \{\beta_\psi\}_{\psi \in \Psi})$ on $Y = (J, \Psi)$ by setting $\mathcal{H}_{(i, U)} = \mathcal{F}_i|_U$ for $(i, U) \in J$ and setting β_ψ for $\psi \in \Psi$ equal to the restriction of α_ϕ to U if $\psi = (\phi, U, V)$ and equal to $\text{id} : (\mathcal{F}_i|_{U'})|_U \rightarrow \mathcal{F}_i|_U$ when $\psi = (i, U, U')$. Moreover, submodules of $(\{\mathcal{H}_j\}_{j \in J}, \{\beta_\psi\}_{\psi \in \Psi})$ correspond 1-to-1 with submodules of $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$. We omit the proof (hint: use Sheaves, Section 30). Moreover, it is clear that if κ works for Y , then the same κ works for X (by the definition of κ -generated modules). Hence it suffices to prove the lemma for crystals in quasi-coherent sheaves on Y .

Assume that all the schemes X_i are affine. Let κ be an infinite cardinal larger than the cardinality of I or Φ . Let $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$ be an object of $CQC(X)$. For each i write $X_i = \text{Spec}(A_i)$ and $M_i = \Gamma(X_i, \mathcal{F}_i)$. For every $\phi \in \Phi$ with $j(\phi) = (i, i')$ the map α_ϕ translates into an $A_{i'}$ -module isomorphism

$$\alpha_\phi : M_i \otimes_{A_i} A_{i'} \longrightarrow M_{i'}$$

Using the axiom of choice choose a rule

$$(\phi, m) \longmapsto S(\phi, m')$$

²We could single out a set of triples $\phi, \phi', \phi'' \in \Phi$ with $j(\phi) = (i, i')$, $j(\phi') = (i', i'')$, and $j(\phi'') = (i, i'')$ such that $f_{\phi''} = f_\phi \circ f_{\phi'}$ and require that $\alpha_{\phi'} \circ f_{\phi'}^* \alpha_\phi = \alpha_{\phi''}$ for these triples. This would define an additive subcategory. For example the data (I, Φ) could be the set of objects and arrows of an index category and X could be a diagram of schemes over this index category. The result of Lemma 13.1 immediately gives the corresponding result in the subcategory.

where the source is the collection of pairs (ϕ, m') such that $\phi \in \Phi$ with $j(\phi) = (i, i')$ and $m' \in M_{i'}$ and where the output is a finite subset $S(\phi, m') \subset M_i$ so that

$$m' = \alpha_\phi \left(\sum_{m \in S(\phi, m')} m \otimes a'_m \right)$$

for some $a'_m \in A_{i'}$.

Having made these choices we claim that any section of any \mathcal{F}_i over any X_i is in a κ -generated submodule. To see this suppose that we are given a collection $\mathcal{S} = \{S_i\}_{i \in I}$ of subsets $S_i \subset M_i$ each with cardinality at most κ . Then we define a new collection $\mathcal{S}' = \{S'_i\}_{i \in I}$ with

$$S'_i = S_i \cup \bigcup_{(\phi, m'), j(\phi)=(i, i'), m' \in S_{i'}} S(\phi, m')$$

Note that each S'_i still has cardinality at most κ . Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $S_i^{(\infty)} = \bigcup_{n \geq 0} S_i^{(n)}$ and $\mathcal{S}^{(\infty)} = \{S_i^{(\infty)}\}_{i \in I}$. By construction, for every $\phi \in \Phi$ with $j(\phi) = (i, i')$ and every $m' \in S_{i'}^{(\infty)}$ we can write m' as a finite linear combination of images $\alpha_\phi(m \otimes 1)$ with $m \in S_i^{(\infty)}$. Thus we see that setting N_i equal to the A_i -submodule of M_i generated by $S_i^{(\infty)}$ the corresponding quasi-coherent submodules $\widetilde{N}_i \subset \mathcal{F}_i$ form a κ -generated submodule. This finishes the proof. \square

Lemma 13.2. *Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . If s, t are flat, then there exists a set T and a family of objects $(\mathcal{F}_t, \alpha_t)_{t \in T}$ of $QCoh(U, R, s, t, c)$ such that every object (\mathcal{F}, α) is the directed colimit of its submodules isomorphic to one of the objects $(\mathcal{F}_t, \alpha_t)$.*

Proof. This lemma is a generalization of Groupoids, Lemma 13.6 which deals with the case of a groupoid in schemes. We can't quite use the same argument, so we use the material on "crystals of quasi-coherent sheaves" we developed above.

Choose a scheme W and a surjective étale morphism $W \rightarrow U$. Choose a scheme V and a surjective étale morphism $V \rightarrow W \times_{U, s} R$. Choose a scheme V' and a surjective étale morphism $V' \rightarrow R \times_{t, U} W$. Consider the collection of schemes

$$I = \{W, W \times_U W, V, V', V \times_R V'\}$$

and the set of morphisms of schemes

$$\Phi = \{\text{pr}_i : W \times_U W \rightarrow W, V \rightarrow W, V' \rightarrow W, V \times_R V' \rightarrow V, V \times_R V' \rightarrow V'\}$$

Set $X = (I, \Phi)$. Recall that we have defined a category $CQC(X)$ of crystals of quasi-coherent sheaves on X . There is a functor

$$QCoh(U, R, s, t, c) \longrightarrow CQC(X)$$

which assigns to (\mathcal{F}, α) the sheaf $\mathcal{F}|_W$ on W , the sheaf $\mathcal{F}|_{W \times_U W}$ on $W \times_U W$, the pullback of \mathcal{F} via $V \rightarrow W \times_{U, s} R \rightarrow W \rightarrow U$ on V , the pullback of \mathcal{F} via $V' \rightarrow R \times_{t, U} W \rightarrow W \rightarrow U$ on V' , and finally the pullback of \mathcal{F} via $V \times_R V' \rightarrow V \rightarrow W \times_{U, s} R \rightarrow W \rightarrow U$ on $V \times_R V'$. As comparison maps $\{\alpha_\phi\}_{\phi \in \Phi}$ we use the obvious ones (coming from associativity of pullbacks) except for the map

$\phi = \text{pr}_{V'} : V \times_R V' \rightarrow V'$ we use the pullback of $\alpha : t^*\mathcal{F} \rightarrow s^*\mathcal{F}$ to $V \times_R V'$. This makes sense because of the following commutative diagram

$$\begin{array}{ccccc}
 & & V \times_R V' & & \\
 & \swarrow & & \searrow & \\
 V & & & & V' \\
 \downarrow & & \searrow & & \swarrow \\
 & & R & & \\
 \downarrow & & \updownarrow & & \downarrow \\
 W & & s \quad t & & W \\
 & \swarrow & & \searrow & \\
 & & U & &
 \end{array}$$

The functor displayed above isn't an equivalence of categories. However, since $W \rightarrow U$ is surjective étale it is faithful³. Since all the morphisms in the diagram above are flat we see that it is an exact functor of abelian categories. Moreover, we claim that given (\mathcal{F}, α) with image $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$ there is a 1-to-1 correspondence between quasi-coherent submodules of (\mathcal{F}, α) and $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$. Namely, given a submodule of $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$ compatibility of the submodule over W with the projection maps $W \times_U W \rightarrow W$ will guarantee the submodule comes from a quasi-coherent submodule of \mathcal{F} (by Properties of Spaces, Proposition 30.1) and compatibility with $\alpha_{\text{pr}_{V'}}$ will insure this subsheaf is compatible with α (details omitted).

Choose a cardinal κ as in Lemma 13.1 for the system $X = (I, \Phi)$. It is clear from Properties, Lemma 21.2 that there is a set of isomorphism classes of κ -generated crystals in quasi-coherent sheaves on X . Hence the result is clear. \square

14. Groupoids and group spaces

Please compare with Groupoids, Section 14.

Lemma 14.1. *Let $B \rightarrow S$ as in Section 3. Let (G, m) be a group algebraic space over B with identity e_G and inverse i_G . Let X be an algebraic space over B and let $a : G \times_B X \rightarrow X$ be an action of G on X over B . Then we get a groupoid in algebraic spaces (U, R, s, t, c, e, i) over B in the following manner:*

- (1) We set $U = X$, and $R = G \times_B X$.
- (2) We set $s : R \rightarrow U$ equal to $(g, x) \mapsto x$.
- (3) We set $t : R \rightarrow U$ equal to $(g, x) \mapsto a(g, x)$.
- (4) We set $c : R \times_{s, U, t} R \rightarrow R$ equal to $((g, x), (g', x')) \mapsto (m(g, g'), x')$.
- (5) We set $e : U \rightarrow R$ equal to $x \mapsto (e_G(x), x)$.
- (6) We set $i : R \rightarrow R$ equal to $(g, x) \mapsto (i_G(g), a(g, x))$.

Proof. Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing g as an arrow from v to $a(g, v)$. \square

³In fact the functor is fully faithful, but we won't need this.

Lemma 14.2. *Let $B \rightarrow S$ as in Section 3. Let (G, m) be a group algebraic space over B . Let X be an algebraic space over B and let $a : G \times_B X \rightarrow X$ be an action of G on X over B . Let (U, R, s, t, c) be the groupoid in algebraic spaces constructed in Lemma 14.1. The rule $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$ defines an equivalence of categories between G -equivariant \mathcal{O}_X -modules and the category of quasi-coherent modules on (U, R, s, t, c) .*

Proof. The assertion makes sense because $t = a$ and $s = \text{pr}_1$ as morphisms $R = G \times_B X \rightarrow X$, see Definitions 10.1 and 12.1. Using the translation in Lemma 14.1 the commutativity requirements of the two definitions match up exactly. \square

15. The stabilizer group algebraic space

Please compare with Groupoids, Section 15. Given a groupoid in algebraic spaces we get a group algebraic space as follows.

Lemma 15.1. *Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . The algebraic space G defined by the cartesian square*

$$\begin{array}{ccc} G & \longrightarrow & R \\ \downarrow & & \downarrow j=(t,s) \\ U & \xrightarrow{\Delta} & U \times_B U \end{array}$$

is a group algebraic space over U with composition law m induced by the composition law c .

Proof. This is true because in a groupoid category the set of self maps of any object forms a group. \square

Since Δ is a monomorphism we see that $G = j^{-1}(\Delta_{U/B})$ is a subsheaf of R . Thinking of it in this way, the structure morphism $G = j^{-1}(\Delta_{U/B}) \rightarrow U$ is induced by either s or t (it is the same), and m is induced by c .

Definition 15.2. Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . The group algebraic space $j^{-1}(\Delta_{U/B}) \rightarrow U$ is called the *stabilizer of the groupoid in algebraic spaces (U, R, s, t, c) .*

In the literature the stabilizer group algebraic space is often denoted S (because the word stabilizer starts with an “s” presumably); we cannot do this since we have already used S for the base scheme.

Lemma 15.3. *Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B , and let G/U be its stabilizer. Denote R_t/U the algebraic space R seen as an algebraic space over U via the morphism $t : R \rightarrow U$. There is a canonical left action*

$$a : G \times_U R_t \longrightarrow R_t$$

induced by the composition law c .

Proof. In terms of points over T/B we define $a(g, r) = c(g, r)$. \square

16. Restricting groupoids

Please refer to Groupoids, Section 16 for notation.

Lemma 16.1. *Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . Let $g : U' \rightarrow U$ be a morphism of algebraic spaces. Consider the following diagram*

$$\begin{array}{ccccc}
 & & & & s' \\
 & & & & \curvearrowright \\
 & & R' & \longrightarrow & R \times_{s,U} U' & \longrightarrow & U' \\
 & & \downarrow & & \downarrow & & \downarrow g \\
 & & U' \times_{U,t} R & \longrightarrow & R & \xrightarrow{s} & U \\
 & & \downarrow & & \downarrow t & & \\
 & & U' & \xrightarrow{g} & U & & \\
 & & & & & & \\
 & t & \curvearrowleft & & & &
 \end{array}$$

where all the squares are fibre product squares. Then there is a canonical composition law $c' : R' \times_{s',U',t'} R' \rightarrow R'$ such that (U', R', s', t', c') is a groupoid in algebraic spaces over B and such that $U' \rightarrow U$, $R' \rightarrow R$ defines a morphism $(U', R', s', t', c') \rightarrow (U, R, s, t, c)$ of groupoids in algebraic spaces over B . Moreover, for any scheme T over B the functor of groupoids

$$(U'(T), R'(T), s', t', c') \rightarrow (U(T), R(T), s, t, c)$$

is the restriction (see Groupoids, Section 16) of $(U(T), R(T), s, t, c)$ via the map $U'(T) \rightarrow U(T)$.

Proof. Omitted. □

Definition 16.2. Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . Let $g : U' \rightarrow U$ be a morphism of algebraic spaces over B . The morphism of groupoids in algebraic spaces $(U', R', s', t', c') \rightarrow (U, R, s, t, c)$ constructed in Lemma 16.1 is called the *restriction of (U, R, s, t, c) to U'* . We sometime use the notation $R' = R|_{U'}$ in this case.

Lemma 16.3. *The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 16.2 and 4.3 agree via the constructions of Lemmas 11.2 and 11.3.*

Proof. What we are saying here is that R' of Lemma 16.1 is also equal to

$$R' = (U' \times_B U') \times_{U \times_B U} R \longrightarrow U' \times_B U'$$

In fact this might have been a clearer way to state that lemma. □

17. Invariant subspaces

In this section we discuss briefly the notion of an invariant subspace.

Definition 17.1. Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over the base B .

- (1) We say an open subspace $W \subset U$ is *R -invariant* if $t(s^{-1}(W)) \subset W$.
- (2) A locally closed subspace $Z \subset U$ is called *R -invariant* if $t^{-1}(Z) = s^{-1}(Z)$ as locally closed subspaces of R .

- (3) A monomorphism of algebraic spaces $T \rightarrow U$ is R -invariant if $T \times_{U,t} R = R \times_{s,U} T$ as algebraic spaces over R .

For an open subspace $W \subset U$ the R -invariance is also equivalent to requiring that $s^{-1}(W) = t^{-1}(W)$. If $W \subset U$ is R -invariant then the restriction of R to W is just $R_W = s^{-1}(W) = t^{-1}(W)$. Similarly, if $Z \subset U$ is an R -invariant locally closed subspace, then the restriction of R to Z is just $R_Z = s^{-1}(Z) = t^{-1}(Z)$.

Lemma 17.2. *Let $B \rightarrow S$ as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B .*

- (1) *If s and t are open, then for every open $W \subset U$ the open $s(t^{-1}(W))$ is R -invariant.*
- (2) *If s and t are open and quasi-compact, then U has an open covering consisting of R -invariant quasi-compact open subspaces.*

Proof. Assume s and t open and $W \subset U$ open. Since s is open we see that $W' = s(t^{-1}(W))$ is an open subspace of U . Now it is quite easy to using the functorial point of view that this is an R -invariant open subset of U , but we are going to argue this directly by some diagrams, since we think it is instructive. Note that $t^{-1}(W')$ is the image of the morphism

$$A := t^{-1}(W) \times_{s|_{t^{-1}(W)}, U, t} R \xrightarrow{\text{pr}_1} R$$

and that $s^{-1}(W')$ is the image of the morphism

$$B := R \times_{s, U, s|_{t^{-1}(W)}} t^{-1}(W) \xrightarrow{\text{pr}_0} R.$$

The algebraic spaces A, B on the left of the arrows above are open subspaces of $R \times_{s, U, t} R$ and $R \times_{s, U, s} R$ respectively. By Lemma 11.4 the diagram

$$\begin{array}{ccc} R \times_{s, U, t} R & \xrightarrow{(\text{pr}_1, c)} & R \times_{s, U, s} R \\ & \searrow \text{pr}_1 & \swarrow \text{pr}_0 \\ & R & \end{array}$$

is commutative, and the horizontal arrow is an isomorphism. Moreover, it is clear that $(\text{pr}_1, c)(A) = B$. Hence we conclude $s^{-1}(W') = t^{-1}(W')$, and W' is R -invariant. This proves (1).

Assume now that s, t are both open and quasi-compact. Then, if $W \subset U$ is a quasi-compact open, then also $W' = s(t^{-1}(W))$ is a quasi-compact open, and invariant by the discussion above. Letting W range over images of affines étale over U we see (2). \square

18. Quotient sheaves

Let S be a scheme, and let B be an algebraic space over S . Let $j : R \rightarrow U \times_B U$ be a pre-relation over B . For each scheme S' over S we can take the equivalence relation $\sim_{S'}$ generated by the image of $j(S') : R(S') \rightarrow U(S') \times U(S')$. Hence we get a presheaf

$$(18.0.1) \quad \begin{array}{ccc} (Sch/S)_{fppf}^{opp} & \longrightarrow & Sets, \\ S' & \longmapsto & U(S') / \sim_{S'} \end{array}$$

Note that since j is a morphism of algebraic spaces over B and into $U \times_B U$ there is a canonical transformation of presheaves from the presheaf (18.0.1) to B .

Definition 18.1. Let $B \rightarrow S$ and the pre-relation $j : R \rightarrow U \times_B U$ be as above. In this setting the *quotient sheaf* U/R associated to j is the sheafification of the presheaf (18.0.1) on $(Sch/S)_{fppf}$. If $j : R \rightarrow U \times_B U$ comes from the action of a group algebraic space G over B on U as in Lemma 14.1 then we denote the quotient sheaf U/G .

This means exactly that the diagram

$$R \rightrightarrows U \longrightarrow U/R$$

is a coequalizer diagram in the category of sheaves of sets on $(Sch/S)_{fppf}$. Again there is a canonical map of sheaves $U/R \rightarrow B$ as j is a morphism of algebraic spaces over B into $U \times_B U$.

Remark 18.2. A variant of the construction above would have been to sheafify the functor

$$\begin{array}{ccc} (Spaces/B)_{fppf}^{opp} & \longrightarrow & Sets, \\ X & \longmapsto & U(X)/\sim_X \end{array}$$

where now $\sim_X \subset U(X) \times U(X)$ is the equivalence relation generated by the image of $j : R(X) \rightarrow U(X) \times U(X)$. Here of course $U(X) = \text{Mor}_B(X, U)$ and $R(X) = \text{Mor}_B(X, R)$. In fact, the result would have been the same, via the identifications of (insert future reference in Topologies of Spaces here).

Definition 18.3. In the situation of Definition 18.1. We say that the pre-relation j has a *quotient representable by an algebraic space* if the sheaf U/R is an algebraic space. We say that the pre-relation j has a *representable quotient* if the sheaf U/R is representable by a scheme. We will say a groupoid in algebraic spaces (U, R, s, t, c) over B has a *representable quotient* (resp. *quotient representable by an algebraic space*) if the quotient U/R with $j = (t, s)$ is representable (resp. an algebraic space).

If the quotient U/R is representable by M (either a scheme or an algebraic space over S), then it comes equipped with a canonical structure morphism $M \rightarrow B$ as we've seen above.

The following lemma characterizes M representing the quotient. It applies for example if $U \rightarrow M$ is flat, of finite presentation and surjective, and $R \cong U \times_M U$.

Lemma 18.4. *In the situation of Definition 18.1. Assume there is an algebraic space M over S , and a morphism $U \rightarrow M$ such that*

- (1) *the morphism $U \rightarrow M$ equalizes s, t ,*
- (2) *the map $U \rightarrow M$ is a surjection of sheaves, and*
- (3) *the induced map $(t, s) : R \rightarrow U \times_M U$ is a surjection of sheaves.*

In this case M represents the quotient sheaf U/R .

Proof. Condition (1) says that $U \rightarrow M$ factors through U/R . Condition (2) says that $U/R \rightarrow M$ is surjective as a map of sheaves. Condition (3) says that $U/R \rightarrow M$ is injective as a map of sheaves. Hence the lemma follows. \square

The following lemma is wrong if we do not require j to be a pre-equivalence relation (but just a pre-relation say).

Lemma 18.5. *Let S be a scheme. Let B be an algebraic space over S . Let $j : R \rightarrow U \times_B U$ be a pre-equivalence relation over B . For a scheme S' over S and $a, b \in U(S')$ the following are equivalent:*

- (1) a and b map to the same element of $(U/R)(S')$, and
- (2) there exists an fppf covering $\{f_i : S_i \rightarrow S'\}$ of S' and morphisms $r_i : S_i \rightarrow R$ such that $a \circ f_i = s \circ r_i$ and $b \circ f_i = t \circ r_i$.

In other words, in this case the map of sheaves

$$R \longrightarrow U \times_{U/R} U$$

is surjective.

Proof. Omitted. Hint: The reason this works is that the presheaf (18.0.1) in this case is really given by $T \mapsto U(T)/j(R(T))$ as $j(R(T)) \subset U(T) \times U(T)$ is an equivalence relation, see Definition 4.1. \square

Lemma 18.6. *Let S be a scheme. Let B be an algebraic space over S . Let $j : R \rightarrow U \times_B U$ be a pre-relation over B and $g : U' \rightarrow U$ a morphism of algebraic spaces over B . Let $j' : R' \rightarrow U' \times_B U'$ be the restriction of j to U' . The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If $U' \rightarrow U$ is surjective as a map of sheaves, for example if $\{g : U' \rightarrow U\}$ is an fppf covering (see Topologies on Spaces, Definition 4.1), then $U'/R' \rightarrow U/R$ is an isomorphism of sheaves.

Proof. Suppose $\xi, \xi' \in (U'/R')(S')$ are sections which map to the same section of U/R . Then we can find an fppf covering $\mathcal{S} = \{S_i \rightarrow S'\}$ of S' such that $\xi|_{S_i}, \xi'|_{S_i}$ are given by $a_i, a'_i \in U'(S_i)$. By Lemma 18.5 and the axioms of a site we may after refining \mathcal{T} assume there exist morphisms $r_i : S_i \rightarrow R$ such that $g \circ a_i = s \circ r_i$, $g \circ a'_i = t \circ r_i$. Since by construction $R' = R \times_{U \times_S U} (U' \times_S U')$ we see that $(r_i, (a_i, a'_i)) \in R'(S_i)$ and this shows that a_i and a'_i define the same section of U'/R' over S_i . By the sheaf condition this implies $\xi = \xi'$.

If $U' \rightarrow U$ is a surjective map of sheaves, then $U'/R' \rightarrow U/R$ is surjective also. Finally, if $\{g : U' \rightarrow U\}$ is a fppf covering, then the map of sheaves $U' \rightarrow U$ is surjective, see Topologies on Spaces, Lemma 4.4. \square

Lemma 18.7. *Let S be a scheme. Let B be an algebraic space over S . Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . Let $g : U' \rightarrow U$ a morphism of algebraic spaces over B . Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) to U' . The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If the composition

$$U' \times_{g, U, t} R \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{pr_1} R \xrightarrow{s} U \end{array}$$

is a surjection of fppf sheaves then the map is bijective. This holds for example if $\{h : U' \times_{g, U, t} R \rightarrow U\}$ is an fppf-covering, or if $U' \rightarrow U$ is a surjection of sheaves, or if $\{g : U' \rightarrow U\}$ is a covering in the fppf topology.

Proof. Injectivity follows on combining Lemmas 11.2 and 18.6. To see surjectivity (see Sites, Section 12 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that T is a scheme and $\sigma \in U/R(T)$. There exists a covering $\{T_i \rightarrow T\}$ such that $\sigma|_{T_i}$ is the image of some element $f_i \in U(T_i)$. Hence we

may assume that σ is the image of $f \in U(T)$. By the assumption that h is a surjection of sheaves, we can find an fppf covering $\{\varphi_i : T_i \rightarrow T\}$ and morphisms $f_i : T_i \rightarrow U' \times_{g,U,t} R$ such that $f \circ \varphi_i = h \circ f_i$. Denote $f'_i = \text{pr}_0 \circ f_i : T_i \rightarrow U'$. Then we see that $f'_i \in U'(T_i)$ maps to $g \circ f'_i \in U(T_i)$ and that $g \circ f'_i \sim_{T_i} h \circ f_i = f \circ \varphi_i$ notation as in (18.0.1). Namely, the element of $R(T_i)$ giving the relation is $\text{pr}_1 \circ f_i$. This means that the restriction of σ to T_i is in the image of $U'/R'(T_i) \rightarrow U/R(T_i)$ as desired.

If $\{h\}$ is an fppf covering, then it induces a surjection of sheaves, see Topologies on Spaces, Lemma 4.4. If $U' \rightarrow U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid scheme). \square

19. Quotient stacks

In this section and the next few sections we describe a kind of generalization of Section 18 above and Groupoids, Section 18. It is different in the following way: We are going to take quotient stacks instead of quotient sheaves.

Let us assume we have a scheme S , and algebraic space B over S and a groupoid in algebraic spaces (U, R, s, t, c) over B . Given these data we consider the functor

$$(19.0.1) \quad \begin{array}{ccc} (Sch/S)_{fppf}^{opp} & \longrightarrow & Groupoids \\ S' & \longmapsto & (U(S'), R(S'), s, t, c) \end{array}$$

By Categories, Example 35.1 this ‘‘presheaf in groupoids’’ corresponds to a category fibred in groupoids over $(Sch/S)_{fppf}$. In this chapter we will denote this

$$[U/_p R] \rightarrow (Sch/S)_{fppf}$$

where the subscript $_p$ is there to distinguish from the quotient stack.

Definition 19.1. Quotient stacks. Let $B \rightarrow S$ be as above.

- (1) Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . The *quotient stack*

$$p : [U/R] \longrightarrow (Sch/S)_{fppf}$$

of (U, R, s, t, c) is the stackification (see Stacks, Lemma 9.1) of the category fibred in groupoids $[U/_p R]$ over $(Sch/S)_{fppf}$ associated to (19.0.1).

- (2) Let (G, m) be a group algebraic space over B . Let $a : G \times_B X \rightarrow X$ be an action of G on an algebraic space over B . The *quotient stack*

$$p : [X/G] \longrightarrow (Sch/S)_{fppf}$$

is the quotient stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ over B of Lemma 14.1.

Thus $[U/R]$ and $[X/G]$ are stacks in groupoids over $(Sch/S)_{fppf}$. These stacks will be very important later on and hence it makes sense to give a detailed description. Recall that given an algebraic space X over S we use the notation $\mathcal{S}_X \rightarrow (Sch/S)_{fppf}$ to denote the stack in sets associated to the sheaf X , see Categories, Lemma 36.6 and Stacks, Lemma 6.2.

Lemma 19.2. Assume $B \rightarrow S$ and (U, R, s, t, c) as in Definition 19.1 (1). There are canonical 1-morphisms $\pi : \mathcal{S}_U \rightarrow [U/R]$, and $[U/R] \rightarrow \mathcal{S}_B$ of stacks in groupoids over $(Sch/S)_{fppf}$. The composition $\mathcal{S}_U \rightarrow \mathcal{S}_B$ is the 1-morphism associated to the structure morphism $U \rightarrow B$.

Proof. During this proof let us denote $[U/pR]$ the category fibred in groupoids associated to the presheaf in groupoids (19.0.1). By construction of the stackification there is a 1-morphism $[U/pR] \rightarrow [U/R]$. The 1-morphism $\mathcal{S}_U \rightarrow [U/R]$ is simply the composition $\mathcal{S}_U \rightarrow [U/pR] \rightarrow [U/R]$, where the first arrow associates to the scheme S'/S and morphism $x : S' \rightarrow U$ over S the object $x \in U(S')$ of the fibre category of $[U/pR]$ over S' .

To construct the 1-morphism $[U/R] \rightarrow \mathcal{S}_B$ it is enough to construct the 1-morphism $[U/pR] \rightarrow \mathcal{S}_B$, see Stacks, Lemma 9.2. On objects over S'/S we just use the map

$$U(S') \longrightarrow B(S')$$

coming from the structure morphism $U \rightarrow B$. And clearly, if $a \in R(S')$ is an “arrow” with source $s(a) \in U(S')$ and target $t(a) \in U(S')$, then since s and t are morphisms *over* B these both map to the same element \bar{a} of $B(S')$. Hence we can map an arrow $a \in R(S')$ to the identity morphism of \bar{a} . (This is good because the fibre category $(\mathcal{S}_B)_{S'}$ only contains identities.) We omit the verification that this rule is compatible with pullback on these split fibred categories, and hence defines a 1-morphism $[U/pR] \rightarrow \mathcal{S}_B$ as desired.

We omit the verification of the last statement. \square

Lemma 19.3. *Assumptions and notation as in Lemma 19.2. There exists a canonical 2-morphism $\alpha : \pi \circ s \rightarrow \pi \circ t$ making the diagram*

$$\begin{array}{ccc} \mathcal{S}_R & \xrightarrow{s} & \mathcal{S}_U \\ t \downarrow & & \downarrow \pi \\ \mathcal{S}_U & \xrightarrow{\pi} & [U/R] \end{array}$$

2-commutative.

Proof. Let S' be a scheme over S . Let $r : S' \rightarrow R$ be a morphism over S . Then $r \in R(S')$ is an isomorphism between the objects $s \circ r, t \circ r \in U(S')$. Moreover, this construction is compatible with pullbacks. This gives a canonical 2-morphism $\alpha_p : \pi_p \circ s \rightarrow \pi_p \circ t$ where $\pi_p : \mathcal{S}_U \rightarrow [U/pR]$ is as in the proof of Lemma 19.2. Thus even the diagram

$$\begin{array}{ccc} \mathcal{S}_R & \xrightarrow{s} & \mathcal{S}_U \\ t \downarrow & & \downarrow \pi_p \\ \mathcal{S}_U & \xrightarrow{\pi_p} & [U/pR] \end{array}$$

is 2-commutative. Thus a fortiori the diagram of the lemma is 2-commutative. \square

Remark 19.4. In future chapters we will use the ambiguous notation where instead of writing \mathcal{S}_X for the stack in sets associated to X we simply write X . Using this notation the diagram of Lemma 19.3 becomes the familiar diagram

$$\begin{array}{ccc} R & \xrightarrow{s} & U \\ t \downarrow & & \downarrow \pi \\ U & \xrightarrow{\pi} & [U/R] \end{array}$$

In the following sections we will show that this diagram has many good properties. In particular we will show that it is a 2-fibre product (Section 21) and that it is close to being a 2-coequalizer of s and t (Section 22).

20. Functoriality of quotient stacks

A morphism of groupoids in algebraic spaces gives an associated morphism of quotient stacks.

Lemma 20.1. *Let S be a scheme. Let B be an algebraic space over S . Let $f : (U, R, s, t, c) \rightarrow (U', R', s', t', c')$ be a morphism of groupoids in algebraic spaces over B . Then f induces a canonical 1-morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R'].$$

Proof. Denote $[U/pR]$ and $[U'/pR']$ the categories fibred in groupoids over the base site $(Sch/S)_{fppf}$ associated to the functors (19.0.1). It is clear that f defines a 1-morphism $[U/pR] \rightarrow [U'/pR']$ which we can compose with the stackyfication functor for $[U'/R']$ to get $[U/pR] \rightarrow [U'/R']$. Then, by the universal property of the stackyfication functor $[U/pR] \rightarrow [U/R]$, see Stacks, Lemma 9.2 we get $[U/R] \rightarrow [U'/R']$. \square

Let $B \rightarrow S$ and $f : (U, R, s, t, c) \rightarrow (U', R', s', t', c')$ be as in Lemma 20.1. In this situation, we define a third groupoid in algebraic spaces over B as follows, using the language of T -valued points where T is a (varying) scheme over B :

- (1) $U'' = U \times_{f, U', t'} R'$ so that a T -valued point is a pair (u, r') with $f(u) = t'(r')$,
- (2) $R'' = R \times_{f \circ s, U', t'} R'$ so that a T -valued point is a pair (r, r') with $f(s(r)) = t'(r')$,
- (3) $s'' : R'' \rightarrow U''$ is given by $s''(r, r') = (s(r), r')$,
- (4) $t'' : R'' \rightarrow U''$ is given by $t''(r, r') = (t(r), c'(f(r), r'))$,
- (5) $c'' : R'' \times_{s'', U'', t''} R'' \rightarrow R''$ is given by $c''((r_1, r'_1), (r_2, r'_2)) = (c(r_1, r_2), r'_2)$.

The formula for c'' makes sense as $s''(r_1, r'_1) = t''(r_2, r'_2)$. It is clear that c'' is associative. The identity e'' is given by $e''(u, r) = (e(u), r)$. The inverse of (r, r') is given by $(i(r), c'(f(r), r'))$. Thus we do indeed get a groupoid in algebraic spaces over B .

Clearly the maps $U'' \rightarrow U$ and $R'' \rightarrow R$ define a morphism $g : (U'', R'', s'', t'', c'') \rightarrow (U, R, s, t, c)$ of groupoids in algebraic spaces over B . Moreover, the maps $U'' \rightarrow U'$, $(u, r') \mapsto s'(r')$ and $R'' \rightarrow U'$, $(r, r') \mapsto s'(r')$ show that in fact $(U'', R'', s'', t'', c'')$ is a groupoid in algebraic spaces over U' .

Lemma 20.2. *Notation and assumption as in Lemma 20.1. Let $(U'', R'', s'', t'', c'')$ be the groupoid in algebraic spaces over B constructed above. There is a 2-commutative square*

$$\begin{array}{ccc} [U''/R''] & \xrightarrow{[g]} & [U/R] \\ \downarrow & & \downarrow [f] \\ \mathcal{S}_{U'} & \longrightarrow & [U'/R'] \end{array}$$

which identifies $[U''/R'']$ with the 2-fibre product.

Proof. The maps $[f]$ and $[g]$ come from an application of Lemma 20.1 and the other two maps come from Lemma 19.2 (and the fact that $(U'', R'', s'', t'', c'')$ lives over U'). To show the 2-fibre product property, it suffices to prove the lemma for the diagram

$$\begin{array}{ccc} [U''/{}_p R''] & \xrightarrow{[g]} & [U/{}_p R] \\ \downarrow & & \downarrow [f] \\ \mathcal{S}_{U'} & \longrightarrow & [U'/{}_p R'] \end{array}$$

of categories fibred in groupoids, see Stacks, Lemma 9.3. In other words, it suffices to show that an object of the 2-fibre product $\mathcal{S}_U \times_{[U'/{}_p R']} [U/{}_p R]$ over T corresponds to a T -valued point of U'' and similarly for morphisms. And of course this is exactly how we constructed U'' and R'' in the first place.

In detail, an object of $\mathcal{S}_U \times_{[U'/{}_p R']} [U/{}_p R]$ over T is a triple (u', u, r') where u' is a T -valued point of U' , u is a T -valued point of U , and r' is a morphism from u' to $f(u)$ in $[U'/R']_T$, i.e., r' is a T -valued point of R with $s'(r') = u'$ and $t'(r') = f(u)$. Clearly we can forget about u' without losing information and we see that these objects are in one-to-one correspondence with T -valued points of R'' .

Similarly for morphisms: Let (u'_1, u_1, r'_1) and (u'_2, u_2, r'_2) be two objects of the fibre product over T . Then a morphism from (u'_2, u_2, r'_2) to (u'_1, u_1, r'_1) is given by $(1, r)$ where $1 : u'_1 \rightarrow u'_2$ means simply $u'_1 = u'_2$ (this is so because \mathcal{S}_U is fibred in sets), and r is a T -valued point of R with $s(r) = u_2$, $t(r) = u_1$ and moreover $c'(f(r), r'_2) = r'_1$. Hence the arrow

$$(1, r) : (u'_2, u_2, r'_2) \rightarrow (u'_1, u_1, r'_1)$$

is completely determined by knowing the pair (r, r'_2) . Thus the functor of arrows is represented by R'' , and moreover the morphisms s'' , t'' , and c'' clearly correspond to source, target and composition in the 2-fibre product $\mathcal{S}_U \times_{[U'/{}_p R']} [U/{}_p R]$. \square

21. The 2-cartesian square of a quotient stack

In this section we compute the *Isom*-sheaves for a quotient stack and we deduce that the defining diagram of a quotient stack is a 2-fibre product.

Lemma 21.1. *Assume $B \rightarrow S$, (U, R, s, t, c) and $\pi : \mathcal{S}_U \rightarrow [U/R]$ are as in Lemma 19.2. Let S' be a scheme over S . Let $x, y \in \text{Ob}([U/R]_{S'})$ be objects of the quotient stack over S' . If $x = \pi(x')$ and $y = \pi(y')$ for some morphisms $x', y' : S' \rightarrow U$, then*

$$\text{Isom}(x, y) = S' \times_{(y', x'), U \times_S U} R$$

as sheaves over S' .

Proof. Let $[U/{}_p R]$ be the category fibred in groupoids associated to the presheaf in groupoids (19.0.1) as in the proof of Lemma 19.2. By construction the sheaf $\text{Isom}(x, y)$ is the sheaf associated to the presheaf $\text{Isom}(x', y')$. On the other hand, by definition of morphisms in $[U/{}_p R]$ we have

$$\text{Isom}(x', y') = S' \times_{(y', x'), U \times_S U} R$$

and the right hand side is an algebraic space, therefore a sheaf. \square

Lemma 21.2. *Assume $B \rightarrow S$, (U, R, s, t, c) , and $\pi : \mathcal{S}_U \rightarrow [U/R]$ are as in Lemma 19.2. The 2-commutative square*

$$\begin{array}{ccc} \mathcal{S}_R & \xrightarrow{s} & \mathcal{S}_U \\ t \downarrow & & \downarrow \pi \\ \mathcal{S}_U & \xrightarrow{\pi} & [U/R] \end{array}$$

of Lemma 19.3 is a 2-fibre product of stacks in groupoids of $(Sch/S)_{fppf}$.

Proof. According to Stacks, Lemma 5.6 the lemma makes sense. It also tells us that we have to show that the functor

$$\mathcal{S}_R \longrightarrow \mathcal{S}_U \times_{[U/R]} \mathcal{S}_U$$

which maps $r : T \rightarrow R$ to $(T, t(r), s(r), \alpha(r))$ is an equivalence, where the right hand side is the 2-fibre product as described in Categories, Lemma 30.3. This is, after spelling out the definitions, exactly the content of Lemma 21.1. (Alternative proof: Work out the meaning of Lemma 20.2 in this situation will give you the result also.) \square

Lemma 21.3. *Assume $B \rightarrow S$ and (U, R, s, t, c) are as in Definition 19.1 (1). For any scheme T over S and objects x, y of $[U/R]$ over T the sheaf $Isom(x, y)$ on $(Sch/T)_{fppf}$ has the following property: There exists a fppf covering $\{T_i \rightarrow T\}_{i \in I}$ such that $Isom(x, y)|_{(Sch/T_i)_{fppf}}$ is representable by an algebraic space.*

Proof. Follows immediately from Lemma 21.1 and the fact that both x and y locally in the fppf topology come from objects of \mathcal{S}_U by construction of the quotient stack. \square

22. The 2-coequalizer property of a quotient stack

On a groupoid we have the composition, which leads to a cocycle condition for the canonical 2-morphism of the lemma above. To give the precise formulation we will use the notation introduced in Categories, Sections 26 and 27.

Lemma 22.1. *Assumptions and notation as in Lemmas 19.2 and 19.3. The vertical composition of*

$$\begin{array}{ccc} & \xrightarrow{\pi \circ s \circ pr_1 = \pi \circ s \circ c} & \\ \mathcal{S}_{R \times_{s, U, t} R} & \xrightarrow{\hspace{10em}} & [U/R] \\ & \xrightarrow{\pi \circ t \circ pr_1 = \pi \circ s \circ pr_0} & \\ & \xrightarrow{\pi \circ t \circ pr_0 = \pi \circ t \circ c} & \end{array}$$

$\Downarrow \alpha \star id_{pr_1}$
 $\Downarrow \alpha \star id_{pr_0}$

is the 2-morphism $\alpha \star id_c$. In a formula $\alpha \star id_c = (\alpha \star id_{pr_0}) \circ (\alpha \star id_{pr_1})$.

Proof. We make two remarks:

- (1) The formula $\alpha \star id_c = (\alpha \star id_{pr_0}) \circ (\alpha \star id_{pr_1})$ only makes sense if you realize the equalities $\pi \circ s \circ pr_1 = \pi \circ s \circ c$, $\pi \circ t \circ pr_1 = \pi \circ s \circ pr_0$, and $\pi \circ t \circ pr_0 = \pi \circ t \circ c$. Namely, the second one implies the vertical composition \circ makes sense, and the other two guarantee the two sides of the formula are 2-morphisms with the same source and target.

- (2) The reason the lemma holds is that composition in the category fibred in groupoids $[U/pR]$ associated to the presheaf in groupoids (19.0.1) comes from the composition law $c : R \times_{s,U,t} R \rightarrow R$.

We omit the proof of the lemma. \square

Note that, in the situation of the lemma, we actually have the equalities $s \circ \text{pr}_1 = s \circ c$, $t \circ \text{pr}_1 = s \circ \text{pr}_0$, and $t \circ \text{pr}_0 = t \circ c$ before composing with π . Hence the formula in the lemma below makes sense in exactly the same way that the formula in the lemma above makes sense.

Lemma 22.2. *Assumptions and notation as in Lemmas 19.2 and 19.3. The 2-commutative diagram of Lemma 19.3 is a 2-coequalizer in the following sense: Given*

- (1) a stack in groupoids \mathcal{X} over $(\text{Sch}/S)_{fppf}$,
- (2) a 1-morphism $f : \mathcal{S}_U \rightarrow \mathcal{X}$, and
- (3) a 2-arrow $\beta : f \circ s \rightarrow f \circ t$

such that

$$\beta \star id_c = (\beta \star id_{\text{pr}_0}) \circ (\beta \star id_{\text{pr}_1})$$

then there exists a 1-morphism $[U/R] \rightarrow \mathcal{X}$ which makes the diagram

$$\begin{array}{ccc} \mathcal{S}_R & \xrightarrow{s} & \mathcal{S}_U \\ \downarrow t & & \downarrow \\ \mathcal{S}_U & \xrightarrow{\quad} & [U/R] \\ & \searrow f & \downarrow f \\ & & \mathcal{X} \end{array}$$

2-commute.

Proof. Suppose given \mathcal{X} , f and β as in the lemma. By Stacks, Lemma 9.2 it suffices to construct a 1-morphism $g : [U/pR] \rightarrow \mathcal{X}$. First we note that the 1-morphism $\mathcal{S}_U \rightarrow [U/pR]$ is bijective on objects. Hence on objects we can set $g(x) = f(x)$ for $x \in \text{Ob}(\mathcal{S}_U) = \text{Ob}([U/pR])$. A morphism $\varphi : x \rightarrow y$ of $[U/pR]$ arises from a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_2 & \xrightarrow{x} & U \\ \downarrow h & \searrow \varphi & \uparrow s \\ & & R \\ & & \downarrow t \\ \mathcal{S}_1 & \xrightarrow{y} & U. \end{array}$$

Thus we can set $g(\varphi)$ equal to the composition

$$\begin{array}{ccccccc} f(x) & \equiv & f(s \circ \varphi) & \equiv & (f \circ s)(\varphi) & \xrightarrow{\beta} & (f \circ t)(\varphi) & \equiv & f(\varphi \circ t) & \equiv & f(y \circ h) \\ & & & & & & & & & & \downarrow \\ & & & & & & & & & & f(y). \end{array}$$

The vertical arrow is the result of applying the functor f to the canonical morphism $y \circ h \rightarrow y$ in \mathcal{S}_U (namely, the strongly cartesian morphism lifting h with target y).

Let us verify that f so defined is compatible with composition, at least on fibre categories. So let S' be a scheme over S , and let $a : S' \rightarrow R \times_{s,U,t} R$ be a morphism. In this situation we set $x = s \circ \text{pr}_1 \circ a = s \circ c \circ a$, $y = t \circ \text{pr}_1 \circ a = s \circ \text{pr}_0 \circ a$, and $z = t \circ \text{pr}_0 \circ a = t \circ \text{pr}_0 \circ c$ to get a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{c \circ a} & z \\ \text{pr}_1 \circ a \searrow & & \nearrow \text{pr}_0 \circ a \\ & y & \end{array}$$

in the fibre category $[U/pR]_{S'}$. Moreover, any commutative triangle in this fibre category has this form. Then we see by our definitions above that f maps this to a commutative diagram if and only if the diagram

$$\begin{array}{ccccc} & & (f \circ s)(c \circ a) & \xrightarrow{\beta} & (f \circ t)(c \circ a) & & \\ & & // & & // & & \\ (f \circ s)(\text{pr}_1 \circ a) & & & & & & (f \circ t)(\text{pr}_0 \circ a) \\ & \searrow \beta & & & \nearrow \beta & & \\ & & (f \circ t)(\text{pr}_1 \circ a) & = & (f \circ s)(\text{pr}_0 \circ a) & & \end{array}$$

is commutative which is exactly the condition expressed by the formula in the lemma. We omit the verification that f maps identities to identities and is compatible with composition for arbitrary morphisms. \square

23. Explicit description of quotient stacks

In order to formulate the result we need to introduce some notation. Assume $B \rightarrow S$ and (U, R, s, t, c) are as in Definition 19.1 (1). Let T be a scheme over S . Let $\mathcal{T} = \{T_i \rightarrow T\}_{i \in I}$ be an fppf covering. A $[U/R]$ -descent datum relative to \mathcal{T} is given by a system (u_i, r_{ij}) where

- (1) for each i a morphism $u_i : T_i \rightarrow U$, and
- (2) for each i, j a morphism $r_{ij} : T_i \times_T T_j \rightarrow R$

such that

- (a) as morphisms $T_i \times_T T_j \rightarrow U$ we have

$$s \circ r_{ij} = u_i \circ \text{pr}_0 \quad \text{and} \quad t \circ r_{ij} = u_j \circ \text{pr}_1,$$

- (b) as morphisms $T_i \times_T T_j \times_T T_k \rightarrow R$ we have

$$c \circ (r_{jk} \circ \text{pr}_{12}, r_{ij} \circ \text{pr}_{01}) = r_{ik} \circ \text{pr}_{02}.$$

A morphism $(u_i, r_{ij}) \rightarrow (u'_i, r'_{ij})$ between two $[U/R]$ -descent data over the same covering \mathcal{T} is a collection $(r_i : T_i \rightarrow R)$ such that

- (α) as morphisms $T_i \rightarrow U$ we have

$$u_i = s \circ r_i \quad \text{and} \quad u'_i = t \circ r_i$$

- (β) as morphisms $T_i \times_T T_j \rightarrow R$ we have

$$c \circ (r'_{ij}, r_i \circ \text{pr}_0) = c \circ (r_j \circ \text{pr}_1, r_{ij}).$$

There is a natural composition law on morphisms of descent data relative to a fixed covering and we obtain a category of descent data. This category is a groupoid. Finally, if $\mathcal{T}' = \{T'_j \rightarrow T\}_{j \in J}$ is a second fppf covering which refines \mathcal{T} then there is a notion of pullback of descent data. This is particularly easy to describe explicitly in this case. Namely, if $\alpha : J \rightarrow I$ and $\varphi_j : T'_j \rightarrow T_{\alpha(j)}$ is the morphism of coverings, then the pullback of the descent datum $(u_i, r_{ii'})$ is simply

$$(u_{\alpha(i)} \circ \varphi_j, r_{\alpha(j)\alpha(j')} \circ \varphi_j \times \varphi_{j'}).$$

Pullback defined in this manner defines a functor from the category of descent data over \mathcal{T} to the category of descent data over \mathcal{T}' .

Lemma 23.1. *Assume $B \rightarrow S$ and (U, R, s, t, c) are as in Definition 19.1 (1). Let $\pi : \mathcal{S}_U \rightarrow [U/R]$ be as in Lemma 19.2. Let T be a scheme over S .*

- (1) *for every object x of the fibre category $[U/R]_T$ there exists an fppf covering $\{f_i : T_i \rightarrow T\}_{i \in I}$ such that $f_i^* x \cong \pi(u_i)$ for some $u_i \in U(T_i)$,*
- (2) *the composition of the isomorphisms*

$$\pi(u_i \circ pr_0) = pr_0^* \pi(u_i) \cong pr_0^* f_i^* x \cong pr_1^* f_j^* x \cong pr_1^* \pi(u_j) = \pi(u_j \circ pr_1)$$

- are of the form $\pi(r_{ij})$ for certain morphisms $r_{ij} : T_i \times_T T_j \rightarrow R$,*
- (3) *the system (u_i, r_{ij}) forms a $[U/R]$ -descent datum as defined above,*
- (4) *any $[U/R]$ -descent datum (u_i, r_{ij}) arises in this manner,*
- (5) *if x corresponds to (u_i, r_{ij}) as above, and $y \in \text{Ob}([U/R]_T)$ corresponds to (u'_i, r'_{ij}) then there is a canonical bijection*

$$\text{Mor}_{[U/R]_T}(x, y) \longleftrightarrow \left\{ \begin{array}{l} \text{morphisms } (u_i, r_{ij}) \rightarrow (u'_i, r'_{ij}) \\ \text{of } [U/R]\text{-descent data} \end{array} \right\}$$

- (6) *this correspondence is compatible with refinements of fppf coverings.*

Proof. Statement (1) is part of the construction of the stackyfication. Part (2) follows from Lemma 21.1. We omit the verification of (3). Part (4) is a translation of the fact that in a stack all descent data are effective. We omit the verifications of (5) and (6). \square

24. Restriction and quotient stacks

In this section we study what happens to the quotient stack when taking a restriction.

Lemma 24.1. *Notation and assumption as in Lemma 20.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

is fully faithful if and only if R is the restriction of R' via the morphism $f : U \rightarrow U'$.

Proof. Let x, y be objects of $[U/R]$ over a scheme T/S . Let x', y' be the images of x, y in the category $[U'/R']_T$. The functor $[f]$ is fully faithful if and only if the map of sheaves

$$\text{Isom}(x, y) \longrightarrow \text{Isom}(x', y')$$

is an isomorphism for every T, x, y . We may test this locally on T (in the fppf topology). Hence, by Lemma 23.1 we may assume that x, y come from $a, b \in U(T)$.

In that case we see that x', y' correspond to $f \circ a, f \circ b$. By Lemma 21.1 the displayed map of sheaves in this case becomes

$$T \times_{(a,b), U \times_B U} R \longrightarrow T \times_{f \circ a, f \circ b, U' \times_B U'} R'.$$

This is an isomorphism if R is the restriction, because in that case $R = (U \times_B U) \times_{U' \times_B U'} R'$, see Lemma 16.3 and its proof. Conversely, if the last displayed map is an isomorphism for all T, a, b , then it follows that $R = (U \times_B U) \times_{U' \times_B U'} R'$, i.e., R is the restriction of R' . \square

Lemma 24.2. *Notation and assumption as in Lemma 20.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

is an equivalence if and only if

- (1) (U, R, s, t, c) is the restriction of (U', R', s', t', c') via $f : U \rightarrow U'$, and
- (2) the map

$$U \times_{f, U', t'} R' \xrightarrow{\text{pr}_1} R' \xrightarrow{s'} U' \xleftarrow{h} U$$

is a surjection of sheaves.

Part (2) holds for example if $\{h : U \times_{f, U', t'} R' \rightarrow U'\}$ is an fppf covering, or if $f : U \rightarrow U'$ is a surjection of sheaves, or if $\{f : U \rightarrow U'\}$ is an fppf covering.

Proof. We already know that part (1) is equivalent to fully faithfulness by Lemma 24.1. Hence we may assume that (1) holds and that $[f]$ is fully faithful. Our goal is to show, under these assumptions, that $[f]$ is an equivalence if and only if (2) holds. We may use Stacks, Lemma 4.8 which characterizes equivalences.

Assume (2). We will use Stacks, Lemma 4.8 to prove $[f]$ is an equivalence. Suppose that T is a scheme and $x' \in \text{Ob}([U'/R']_T)$. There exists a covering $\{g_i : T_i \rightarrow T\}$ such that $g_i^* x'$ is the image of some element $a'_i \in U'(T_i)$, see Lemma 23.1. Hence we may assume that x' is the image of $a' \in U'(T)$. By the assumption that h is a surjection of sheaves, we can find an fppf covering $\{\varphi_i : T_i \rightarrow T\}$ and morphisms $b_i : T_i \rightarrow U \times_{g, U', t'} R'$ such that $a' \circ \varphi_i = h \circ b_i$. Denote $a_i = \text{pr}_0 \circ b_i : T_i \rightarrow U$. Then we see that $a_i \in U(T_i)$ maps to $f \circ a_i \in U'(T_i)$ and that $f \circ a_i \cong_{T_i} h \circ b_i = a' \circ \varphi_i$, where \cong_{T_i} denotes isomorphism in the fibre category $[U'/R']_{T_i}$. Namely, the element of $R'(T_i)$ giving the isomorphism is $\text{pr}_1 \circ b_i$. This means that the restriction of x to T_i is in the essential image of the functor $[U/R]_{T_i} \rightarrow [U'/R']_{T_i}$ as desired.

Assume $[f]$ is an equivalence. Let $\xi' \in [U'/R']_{U'}$ denote the object corresponding to the identity morphism of U' . Applying Stacks, Lemma 4.8 we see there exists an fppf covering $\mathcal{U}' = \{g'_i : U'_i \rightarrow U'\}$ such that $(g'_i)^* \xi' \cong [f](\xi_i)$ for some ξ_i in $[U/R]_{U'_i}$. After refining the covering \mathcal{U}' (using Lemma 23.1) we may assume ξ_i comes from a morphism $a_i : U'_i \rightarrow U$. The fact that $[f](\xi_i) \cong (g'_i)^* \xi'$ means that, after possibly refining the covering \mathcal{U}' once more, there exist morphisms $r'_i : U'_i \rightarrow R'$ with $t' \circ r'_i = f \circ a_i$ and $s' \circ r'_i = \text{id}_{U'} \circ g'_i$. Picture

$$\begin{array}{ccc} U & \xleftarrow{a_i} & U'_i \\ \downarrow f & & \downarrow g'_i \\ U' & \xleftarrow{t'} R' \xrightarrow{s'} U' & \end{array} \quad \begin{array}{c} \nearrow r'_i \\ \searrow \end{array}$$

Thus $(a_i, r'_i) : U'_i \rightarrow U \times_{g, U', t'} R'$ are morphisms such that $h \circ (a_i, r'_i) = g'_i$ and we conclude that $\{h : U \times_{g, U', t'} R' \rightarrow U'\}$ can be refined by the fppf covering \mathcal{U}' which means that h induces a surjection of sheaves, see Topologies on Spaces, Lemma 4.4.

If $\{h\}$ is an fppf covering, then it induces a surjection of sheaves, see Topologies on Spaces, Lemma 4.4. If $U' \rightarrow U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid in algebraic spaces). \square

Lemma 24.3. *Notation and assumption as in Lemma 20.1. Assume that*

$$\begin{array}{ccc} R & \longrightarrow & R' \\ s \downarrow & & \downarrow s' \\ U & \longrightarrow & U' \end{array}$$

is cartesian. Then

$$\begin{array}{ccc} \mathcal{S}_U & \longrightarrow & [U/R] \\ \downarrow & & \downarrow [f] \\ \mathcal{S}_{U'} & \longrightarrow & [U'/R'] \end{array}$$

is a 2-fibre product square.

Proof. Applying the inverse isomorphisms $i : R \rightarrow R$ and $i' : R' \rightarrow R'$ to the (first) cartesian diagram of the statement of the lemma we see that

$$\begin{array}{ccc} R & \longrightarrow & R' \\ t \downarrow & & \downarrow t' \\ U & \longrightarrow & U' \end{array}$$

is cartesian as well. By Lemma 20.2 we have a 2-fibre square

$$\begin{array}{ccc} [U''/R''] & \longrightarrow & [U/R] \\ \downarrow & & \downarrow \\ \mathcal{S}_{U'} & \longrightarrow & [U'/R'] \end{array}$$

where $U'' = U \times_{f, U', t'} R'$ and $R'' = R \times_{f \circ s, U', t'} R'$. By the above we see that $(t, f) : R \rightarrow U''$ is an isomorphism, and that

$$R'' = R \times_{f \circ s, U', t'} R' = R \times_{s, U} U \times_{f, U', t'} R' = R \times_{s, U, t} \times R.$$

Explicitly the isomorphism $R \times_{s, U, t} R \rightarrow R''$ is given by the rule $(r_0, r_1) \mapsto (r_0, f(r_1))$. Moreover, s'', t'', c'' translate into the maps

$$R \times_{s, U, t} R \rightarrow R, \quad s''(r_0, r_1) = r_1, \quad t''(r_0, r_1) = c(r_0, r_1)$$

and

$$\begin{aligned} c'' : (R \times_{s, U, t} R) \times_{s'', R, t''} (R \times_{s, U, t} R) &\longrightarrow R \times_{s, U, t} R, \\ ((r_0, r_1), (r_2, r_3)) &\longmapsto (c(r_0, r_2), r_3). \end{aligned}$$

Precomposing with the isomorphism

$$R \times_{s, U, s} R \rightarrow R \times_{s, U, t} R, \quad (r_0, r_1) \mapsto (c(r_0, i(r_1)), r_1)$$

we see that t'' and s'' turn into pr_0 and pr_1 and that c'' turns into $\text{pr}_{02} : R \times_{s, U, s} R \times_{s, U, s} R \rightarrow R \times_{s, U, s} R$. Hence we see that there is an isomorphism $[U''/R''] \cong$

$[R/R \times_{s,U,s} R]$ where as a groupoid in algebraic spaces $(R, R \times_{s,U,s} R, s'', t'', c'')$ is the restriction of the trivial groupoid $(U, U, \text{id}, \text{id}, \text{id})$ via $s : R \rightarrow U$. Since $s : R \rightarrow U$ is a surjection of fppf sheaves (as it has a right inverse) the morphism

$$[U''/R''] \cong [R/R \times_{s,U,s} R] \longrightarrow [U/U] = \mathcal{S}_U$$

is an equivalence by Lemma 24.2. This proves the lemma. \square

25. Inertia and quotient stacks

The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 32.

Lemma 25.1. *Assume $B \rightarrow S$ and (U, R, s, t, c) as in Definition 19.1 (1). Let G/U be the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i) , see Definition 15.2. Set $R' = R \times_{s,U} G$ and set*

- (1) $s' : R' \rightarrow G, (r, g) \mapsto g,$
- (2) $t' : R' \rightarrow G, (r, g) \mapsto c(r, c(g, i(r))),$
- (3) $c' : R' \times_{s',G,t'} R' \rightarrow R', ((r_1, g_1), (r_2, g_2)) \mapsto (c(r_1, r_2), g_1).$

Then (G, R', s', t', c') is a groupoid in algebraic spaces over B and

$$\mathcal{I}_{[U/R]} = [G/R'].$$

i.e., the associated quotient stack is the inertia stack of $[U/R]$.

Proof. By Stacks, Lemma 8.5 it suffices to prove that $\mathcal{I}_{[U/pR]} = [G/pR']$. Let T be a scheme over S . Recall that an object of the inertia fibred category of $[U/pR]$ over T is given by a pair (x, g) where x is an object of $[U/pR]$ over T and g is an automorphism of x in its fibre category over T . In other words, $x : T \rightarrow U$ and $g : T \rightarrow R$ such that $x = s \circ g = t \circ g$. This means exactly that $g : T \rightarrow G$. A morphism in the inertia fibred category from $(x, g) \rightarrow (y, h)$ over T is given by $r : T \rightarrow R$ such that $s(r) = x, t(r) = y$ and $c(r, g) = c(h, r)$, see the commutative diagram in Categories, Lemma 32.1. In a formula

$$h = c(r, c(g, i(r))) = c(c(r, g), i(r)).$$

The notation $s(r)$, etc is a short hand for $s \circ r$, etc. The composition of $r_1 : (x_2, g_2) \rightarrow (x_1, g_1)$ and $r_2 : (x_1, g_1) \rightarrow (x_2, g_2)$ is $c(r_1, r_2) : (x_1, g_1) \rightarrow (x_3, g_3)$.

Note that in the above we could have written g in stead of (x, g) for an object of $\mathcal{I}_{[U/pR]}$ over T as x is the image of g under the structure morphism $G \rightarrow U$. Then the morphisms $g \rightarrow h$ in $\mathcal{I}_{[U/pR]}$ over T correspond exactly to morphisms $r' : T \rightarrow R'$ with $s'(r') = g$ and $t'(r') = h$. Moreover, the composition corresponds to the rule explained in (3). Thus the lemma is proved. \square

Lemma 25.2. *Assume $B \rightarrow S$ and (U, R, s, t, c) as in Definition 19.1 (1). Let G/U be the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i) , see Definition 15.2. There is a canonical 2-cartesian diagram*

$$\begin{array}{ccc} \mathcal{S}_G & \longrightarrow & \mathcal{S}_U \\ \downarrow & & \downarrow \\ \mathcal{I}_{[U/R]} & \longrightarrow & [U/R] \end{array}$$

of stacks in groupoids of $(\text{Sch}/S)_{\text{fppf}}$.

Proof. By Lemma 24.3 it suffices to prove that the morphism $s' : R' \rightarrow G$ of Lemma 25.1 isomorphic to the base change of s by the structure morphism $G \rightarrow U$. This base change property is clear from the construction of s' . \square

26. Gerbes and quotient stacks

In this section we relate quotient stacks to the discussion Stacks, Section 11 and especially gerbes as defined in Stacks, Definition 11.4. The stacks in groupoids occurring in this section are generally speaking not algebraic stacks!

Lemma 26.1. *Notation and assumption as in Lemma 20.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

turns $[U/R]$ into a gerbe over $[U'/R']$ if $f : U \rightarrow U'$ and $R \rightarrow R'|_U$ are surjective maps of fppf sheaves. Here $R'|_U$ is the restriction of R' to U via $f : U \rightarrow U'$.

Proof. We will verify that Stacks, Lemma 11.3 properties (2) (a) and (2) (b) hold. Property (2)(a) holds because $U \rightarrow U'$ is a surjective map of sheaves (use Lemma 23.1 to see that objects in $[U'/R']$ locally come from U'). To prove (2)(b) let x, y be objects of $[U/R]$ over a scheme T/S . Let x', y' be the images of x, y in the category $[U'/R']_T$. Condition (2)(b) requires us to check the map of sheaves

$$\text{Isom}(x, y) \longrightarrow \text{Isom}(x', y')$$

on $(\text{Sch}/T)_{\text{fppf}}$ is surjective. To see this we may work fppf locally on T and assume that come from $a, b \in U(T)$. In that case we see that x', y' correspond to $f \circ a, f \circ b$. By Lemma 21.1 the displayed map of sheaves in this case becomes

$$T \times_{(a,b), U \times_B U} R \longrightarrow T \times_{f \circ a, f \circ b, U' \times_B U'} R' = T \times_{(a,b), U \times_B U} R'|_U.$$

Hence the assumption that $R \rightarrow R'|_U$ is a surjective map of fppf sheaves on $(\text{Sch}/S)_{\text{fppf}}$ implies the desired surjectivity. \square

Lemma 26.2. *Let S be a scheme. Let B be an algebraic space over S . Let G be a group algebraic space over B . Endow B with the trivial action of G . The morphism*

$$[B/G] \longrightarrow \mathcal{S}_B$$

(Lemma 19.2) turns $[B/G]$ into a gerbe over B .

Proof. Immediate from Lemma 26.1 as the morphisms $B \rightarrow B$ and $B \times_B G \rightarrow B$ are surjective as morphisms of sheaves. \square

27. Quotient stacks and change of big site

We suggest skipping this section on a first reading. Pullbacks of stacks are defined in Stacks, Section 12.

Lemma 27.1. *Suppose given big sites Sch_{fppf} and $\text{Sch}'_{\text{fppf}}$. Assume that Sch_{fppf} is contained in $\text{Sch}'_{\text{fppf}}$, see Topologies, Section 10. Let $S \in \text{Ob}(\text{Sch}_{\text{fppf}})$. Let $B, U, R \in \text{Sh}((\text{Sch}/S)_{\text{fppf}})$ be algebraic spaces, and let (U, R, s, t, c) be a groupoid in algebraic spaces over B . Let $f : (\text{Sch}'/S)_{\text{fppf}} \rightarrow (\text{Sch}/S)_{\text{fppf}}$ the morphism of sites corresponding to the inclusion functor $u : \text{Sch}_{\text{fppf}} \rightarrow \text{Sch}'_{\text{fppf}}$. Then we have a canonical equivalence*

$$[f^{-1}U/f^{-1}R] \longrightarrow f^{-1}[U/R]$$

of stacks in groupoids over $(\text{Sch}'/S)_{\text{fppf}}$.

Proof. Note that $f^{-1}B, f^{-1}U, f^{-1}R \in Sh((Sch'/S)_{fppf})$ are algebraic spaces by Spaces, Lemma 15.1 and hence $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a groupoid in algebraic spaces over $f^{-1}B$. Thus the statement makes sense.

The category $u_p[U/pR]$ is the localization of the category $u_{pp}[U/pR]$ at right multiplicative system I of morphisms. An object of $u_{pp}[U/pR]$ is a triple

$$(T', \phi : T' \rightarrow T, x)$$

where $T' \in \text{Ob}((Sch'/S)_{fppf})$, $T \in \text{Ob}((Sch/S)_{fppf})$, ϕ is a morphism of schemes over S , and $x : T \rightarrow U$ is a morphism of sheaves on $(Sch/S)_{fppf}$. Note that the morphism of schemes $\phi : T' \rightarrow T$ is the same thing as a morphism $\phi : T' \rightarrow u(T)$, and since $u(T)$ represents $f^{-1}T$ it is the same thing as a morphism $T' \rightarrow f^{-1}T$. Moreover, as f^{-1} on algebraic spaces is fully faithful, see Spaces, Lemma 15.2, we may think of x as a morphism $x : f^{-1}T \rightarrow f^{-1}U$ as well. From now on we will make such identifications without further mention. A morphism

$$(a, a', \alpha) : (T'_1, \phi_1 : T'_1 \rightarrow T_1, x_1) \longrightarrow (T'_2, \phi_2 : T'_2 \rightarrow T_2, x_2)$$

of $u_{pp}[U/pR]$ is a commutative diagram

$$\begin{array}{ccccc} & & & & U \\ & & & & \uparrow s \\ & & & & \nearrow x_1 \\ T'_1 & \xrightarrow{\phi_1} & T_1 & \xrightarrow{\alpha} & R \\ \downarrow a' & & \downarrow a & & \downarrow t \\ T'_2 & \xrightarrow{\phi_2} & T_2 & \xrightarrow{x_2} & U \end{array}$$

and such a morphism is an element of I if and only if $T'_1 = T'_2$ and $a' = \text{id}$. We define a functor

$$u_{pp}[U/pR] \longrightarrow [f^{-1}U/pf^{-1}R]$$

by the rules

$$(T', \phi : T' \rightarrow T, x) \longmapsto (x \circ \phi : T' \rightarrow f^{-1}U)$$

on objects and

$$(a, a', \alpha) \longmapsto (\alpha \circ \phi_1 : T'_1 \rightarrow f^{-1}R)$$

on morphisms as above. It is clear that elements of I are transformed into isomorphisms as $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a groupoid in algebraic spaces over $f^{-1}B$. Hence this functor factors in a canonical way through a functor

$$u_p[U/pR] \longrightarrow [f^{-1}U/pf^{-1}R]$$

Applying stackification we obtain a functor of stacks

$$f^{-1}[U/R] \longrightarrow [f^{-1}U/f^{-1}R]$$

over $(Sch'/S)_{fppf}$, as by Stacks, Lemma 12.11 the stack $f^{-1}[U/R]$ is the stackification of $u_p[U/pR]$.

At this point we have a morphism of stacks, and to verify that it is an equivalence it suffices to show that it is fully faithful and that objects are locally in the essential image, see Stacks, Lemmas 4.7 and 4.8. The statement on objects holds as $f^{-1}R$ admits a surjective étale morphism $f^{-1}W \rightarrow f^{-1}R$ for some object W of $(Sch/S)_{fppf}$. To show that the functor is “full”, it suffices to show that morphisms are locally in the image of the functor which holds as $f^{-1}U$ admits a surjective

étale morphism $f^{-1}W \rightarrow f^{-1}U$ for some object W of $(Sch/S)_{fppf}$. We omit the proof that the functor is faithful. \square

28. Separation conditions

This really means conditions on the morphism $j : R \rightarrow U \times_B U$ when given a groupoid in algebraic spaces (U, R, s, t, c) over B . As in the previous section we first formulate the corresponding diagram.

Lemma 28.1. *Let $B \rightarrow S$ be as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . Let $G \rightarrow U$ be the stabilizer group algebraic space. The commutative diagram*

$$\begin{array}{ccccc} R & \xrightarrow{f \mapsto (f, s(f))} & R \times_{s,U} U & \longrightarrow & U \\ \downarrow \Delta_{R/U \times_B U} & & \downarrow & & \downarrow \\ R \times_{(U \times_B U)} R & \xrightarrow{(f,g) \mapsto (f, f^{-1} \circ g)} & R \times_{s,U} G & \longrightarrow & G \end{array}$$

the two left horizontal arrows are isomorphisms and the right square is a fibre product square.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. \square

Lemma 28.2. *Let $B \rightarrow S$ be as in Section 3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . Let $G \rightarrow U$ be the stabilizer group algebraic space.*

- (1) *The following are equivalent*
 - (a) $j : R \rightarrow U \times_B U$ is separated,
 - (b) $G \rightarrow U$ is separated, and
 - (c) $e : U \rightarrow G$ is a closed immersion.
- (2) *The following are equivalent*
 - (a) $j : R \rightarrow U \times_B U$ is locally separated,
 - (b) $G \rightarrow U$ is locally separated, and
 - (c) $e : U \rightarrow G$ is an immersion.
- (3) *The following are equivalent*
 - (a) $j : R \rightarrow U \times_B U$ is quasi-separated,
 - (b) $G \rightarrow U$ is quasi-separated, and
 - (c) $e : U \rightarrow G$ is quasi-compact.

Proof. The group algebraic space $G \rightarrow U$ is the base change of $R \rightarrow U \times_B U$ by the diagonal morphism $U \rightarrow U \times_B U$, see Lemma 15.1. Hence if j is separated (resp. locally separated, resp. quasi-separated), then $G \rightarrow U$ is separated (resp. locally separated, resp. quasi-separated). See Morphisms of Spaces, Lemma 4.4. Thus (a) \Rightarrow (b) in (1), (2), and (3).

Conversely, if $G \rightarrow U$ is separated (resp. locally separated, resp. quasi-separated), then the morphism $e : U \rightarrow G$, as a section of the structure morphism $G \rightarrow U$ is a closed immersion (resp. an immersion, resp. quasi-compact), see Morphisms of Spaces, Lemma 4.7. Thus (b) \Rightarrow (c) in (1), (2), and (3).

If e is a closed immersion (resp. an immersion, resp. quasi-compact) then by the result of Lemma 28.1 (and Spaces, Lemma 12.3, and Morphisms of Spaces, Lemma

8.3) we see that $\Delta_{R/U \times_B U}$ is a closed immersion (resp. an immersion, resp. quasi-compact). Thus (c) \Rightarrow (a) in (1), (2), and (3). \square

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