

# DERIVED CATEGORIES OF STACKS

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## 1. Introduction

In this chapter we write about derived categories associated to algebraic stacks. This mean in particular derived categories of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapters entitled “Derived Categories of Schemes” and “Derived Categories of Spaces”. The results in this chapter are different from those in [LMB00] mainly because we consistently use the “big sites”. Before reading this chapter please take a quick look at the chapters “Sheaves on Algebraic Stacks” and “Cohomology of Algebraic Stacks” where the terminology we use here is introduced.

## 2. Conventions, notation, and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2. We use notation as explained in Cohomology of Stacks, Section 3.

## 3. The lisse-étale and the flat-fppf sites

The section is the analogue of Cohomology of Stacks, Section 11 for derived categories.

**Lemma 3.1.** *Let  $\mathcal{X}$  be an algebraic stack. Notation as in Cohomology of Stacks, Lemmas 11.2 and 11.3.*

- (1) *The functor  $g_! : Ab(\mathcal{X}_{\text{lisse},\text{étale}}) \rightarrow Ab(\mathcal{X}_{\text{étale}})$  has a left derived functor*

$$Lg_! : D(\mathcal{X}_{\text{lisse},\text{étale}}) \longrightarrow D(\mathcal{X}_{\text{étale}})$$

*which is left adjoint to  $g^{-1}$  and such that  $g^{-1}Lg_! = id$ .*

- (2) The functor  $g_! : \text{Mod}(\mathcal{X}_{\text{lisse}, \text{étale}}, \mathcal{O}_{\mathcal{X}_{\text{lisse}, \text{étale}}}) \rightarrow \text{Mod}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$  has a left derived functor

$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{\text{lisse}, \text{étale}}}) \longrightarrow D(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$$

which is left adjoint to  $g^*$  and such that  $g^*Lg_! = \text{id}$ .

- (3) The functor  $g_! : \text{Ab}(\mathcal{X}_{\text{flat}, \text{fppf}}) \rightarrow \text{Ab}(\mathcal{X}_{\text{fppf}})$  has a left derived functor

$$Lg_! : D(\mathcal{X}_{\text{flat}, \text{fppf}}) \longrightarrow D(\mathcal{X}_{\text{fppf}})$$

which is left adjoint to  $g^{-1}$  and such that  $g^{-1}Lg_! = \text{id}$ .

- (4) The functor  $g_! : \text{Mod}(\mathcal{X}_{\text{flat}, \text{fppf}}, \mathcal{O}_{\mathcal{X}_{\text{flat}, \text{fppf}}}) \rightarrow \text{Mod}(\mathcal{X}_{\text{fppf}}, \mathcal{O}_{\mathcal{X}})$  has a left derived functor

$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{\text{flat}, \text{fppf}}}) \longrightarrow D(\mathcal{O}_{\mathcal{X}})$$

which is left adjoint to  $g^*$  and such that  $g^*Lg_! = \text{id}$ .

*Warning:* It is not clear (a priori) that  $Lg_!$  on modules agrees with  $Lg_!$  on abelian sheaves, see *Cohomology on Sites*, Remark 27.2.

**Proof.** The existence of the functor  $Lg_!$  and adjointness to  $g^*$  is Cohomology on Sites, Lemma 27.1. (For the case of abelian sheaves use the constant sheaf  $\mathbf{Z}$  as the structure sheaves.) Moreover, it is computed on a complex  $\mathcal{H}^\bullet$  by taking a suitable left resolution  $\mathcal{K}^\bullet \rightarrow \mathcal{H}^\bullet$  and applying the functor  $g_!$  to  $\mathcal{K}^\bullet$ . Since  $g^{-1}g_!\mathcal{K}^\bullet = \mathcal{K}^\bullet$  by Cohomology of Stacks, Lemmas 11.3 and 11.2 we see that the final assertion holds in each case.  $\square$

**Lemma 3.2.** *With assumptions and notation as in Cohomology of Stacks, Lemma 11.6. We have*

$$g^{-1} \circ Rf_* = Rf'_* \circ (g')^{-1} \quad \text{and} \quad L(g')_! \circ (f')^{-1} = f^{-1} \circ Lg_!$$

on unbounded derived categories (both for the case of modules and for the case of abelian sheaves).

**Proof.** Let  $\mathcal{F}$  be an abelian sheaf on  $\mathcal{X}_{\text{étale}}$  (resp.  $\mathcal{X}_{\text{fppf}}$ ). We first show that the canonical (base change) map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism. To do this let  $y$  be an object of  $\mathcal{Y}_{\text{lisse}, \text{étale}}$  (resp.  $\mathcal{Y}_{\text{flat}, \text{fppf}}$ ). Say  $y$  lies over the scheme  $V$  such that  $y : V \rightarrow \mathcal{Y}$  is smooth (resp. flat). Since  $g^{-1}$  is the restriction we find that

$$(g^{-1}R^p f_*\mathcal{F})(y) = H_\tau^p(V \times_{y, \mathcal{Y}} \mathcal{X}, \text{pr}^{-1}\mathcal{F})$$

where  $\tau = \text{étale}$  (resp.  $\tau = \text{fppf}$ ), see Sheaves on Stacks, Lemma 20.2. By Cohomology of Stacks, Equation (11.6.1) for any sheaf  $\mathcal{H}$  on  $\mathcal{X}_{\text{lisse}, \text{étale}}$  (resp.  $\mathcal{X}_{\text{flat}, \text{fppf}}$ )

$$f'_*\mathcal{H}(y) = \Gamma((V \times_{y, \mathcal{Y}} \mathcal{X})', (\text{pr}')^{-1}\mathcal{H})$$

An object of  $(V \times_{y, \mathcal{Y}} \mathcal{X})'$  can be seen as a pair  $(x, \varphi)$  where  $x$  is an object of  $\mathcal{X}_{\text{lisse}, \text{étale}}$  (resp.  $\mathcal{X}_{\text{flat}, \text{fppf}}$ ) and  $\varphi : f(x) \rightarrow y$  is a morphism in  $\mathcal{Y}$ . We can also think of  $\varphi$  as a section of  $(f')^{-1}h_y$  over  $x$ . Thus  $(V \times_{y, \mathcal{Y}} \mathcal{X})'$  is the localization of the site  $\mathcal{X}_{\text{lisse}, \text{étale}}$  (resp.  $\mathcal{X}_{\text{flat}, \text{fppf}}$ ) at the sheaf of sets  $(f')^{-1}h_y$ , see Sites, Lemma 29.3. The morphism

$$\text{pr}' : (V \times_{y, \mathcal{Y}} \mathcal{X})' \rightarrow \mathcal{X}_{\text{lisse}, \text{étale}} \quad (\text{resp.} \quad \text{pr}' : (V \times_{y, \mathcal{Y}} \mathcal{X})' \rightarrow \mathcal{X}_{\text{flat}, \text{fppf}})$$

is the localization morphism. In particular, the pullback  $(\mathrm{pr}')^{-1}$  preserves injective abelian sheaves, see Cohomology on Sites, Lemma 13.3. At this point exactly the same argument as in Sheaves on Stacks, Lemma 20.2 shows that

$$(3.2.1) \quad R^p f'_* \mathcal{H}(y) = H_\tau^p((V \times_{y, \mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1} \mathcal{H})$$

where  $\tau = \textit{étale}$  (resp.  $\tau = \textit{fppf}$ ). Since  $(g')^{-1}$  is given by restriction we conclude that

$$(R^p f'_*(g')^* \mathcal{F})(y) = H_\tau^p((V \times_{y, \mathcal{Y}} \mathcal{X})', \mathrm{pr}^{-1} \mathcal{F}|_{(V \times_{y, \mathcal{Y}} \mathcal{X})'})$$

Finally, we can apply Sheaves on Stacks, Lemma 22.3 to see that

$$H_\tau^p((V \times_{y, \mathcal{Y}} \mathcal{X})', \mathrm{pr}^{-1} \mathcal{F}|_{(V \times_{y, \mathcal{Y}} \mathcal{X})'}) = H_\tau^p(V \times_{y, \mathcal{Y}} \mathcal{X}, \mathrm{pr}^{-1} \mathcal{F})$$

are equal as desired; although we omit the verification of the assumptions of the lemma we note that the fact that  $V \rightarrow \mathcal{Y}$  is smooth (resp. flat) is used to verify the second condition.

The rest of the proof is formal. Since cohomology of abelian groups and sheaves of modules agree we also conclude that  $g^{-1} Rf_* \mathcal{F} = Rf'_*(g')^{-1} \mathcal{F}$  when  $\mathcal{F}$  is a sheaf of modules on  $\mathcal{X}_{\textit{étale}}$  (resp.  $\mathcal{X}_{\textit{fppf}}$ ).

Next we show that for  $\mathcal{G}$  (either sheaf of modules or abelian groups) on  $\mathcal{Y}_{\textit{lisse, étale}}$  (resp.  $\mathcal{Y}_{\textit{flat, fppf}}$ ) the canonical map

$$L(g')_!(f')^{-1} \mathcal{G} \rightarrow f^{-1} Lg_! \mathcal{G}$$

is an isomorphism. To see this it is enough to prove for any injective sheaf  $\mathcal{I}$  on  $\mathcal{X}_{\textit{étale}}$  (resp.  $\mathcal{X}_{\textit{fppf}}$ ) that the induced map

$$\mathrm{Hom}(L(g')_!(f')^{-1} \mathcal{G}, \mathcal{I}[n]) \leftarrow \mathrm{Hom}(f^{-1} Lg_! \mathcal{G}, \mathcal{I}[n])$$

is an isomorphism for all  $n \in \mathbf{Z}$ . (Hom's taken in suitable derived categories.) By the adjointness of  $f^{-1}$  and  $Rf_*$ , the adjointness of  $Lg_!$  and  $g^{-1}$ , and their “primed” versions this follows from the isomorphism  $g^{-1} Rf_* \mathcal{I} \rightarrow Rf'_*(g')^{-1} \mathcal{I}$  proved above.

In the case of a bounded complex  $\mathcal{G}^\bullet$  (of modules or abelian groups) on  $\mathcal{Y}_{\textit{lisse, étale}}$  (resp.  $\mathcal{Y}_{\textit{fppf}}$ ) the canonical map

$$(3.2.2) \quad L(g')_!(f')^{-1} \mathcal{G}^\bullet \rightarrow f^{-1} Lg_! \mathcal{G}^\bullet$$

is an isomorphism as follows from the case of a sheaf by the usual arguments involving truncations and the fact that the functors  $L(g')_!(f')^{-1}$  and  $f^{-1} Lg_!$  are exact functors of triangulated categories.

Suppose that  $\mathcal{G}^\bullet$  is a bounded above complex (of modules or abelian groups) on  $\mathcal{Y}_{\textit{lisse, étale}}$  (resp.  $\mathcal{Y}_{\textit{fppf}}$ ). The canonical map (3.2.2) is an isomorphism because we can use the stupid truncations  $\sigma_{\geq -n}$  (see Homology, Section 13) to write  $\mathcal{G}^\bullet$  as a colimit  $\mathcal{G}^\bullet = \mathrm{colim} \mathcal{G}_n^\bullet$  of bounded complexes. This gives a distinguished triangle

$$\bigoplus_{n \geq 1} \mathcal{G}_n^\bullet \rightarrow \bigoplus_{n \geq 1} \mathcal{G}_n^\bullet \rightarrow \mathcal{G}^\bullet \rightarrow \dots$$

and each of the functors  $L(g')_!$ ,  $(f')^{-1}$ ,  $f^{-1}$ ,  $Lg_!$  commutes with direct sums (of complexes).

If  $\mathcal{G}^\bullet$  is an arbitrary complex (of modules or abelian groups) on  $\mathcal{Y}_{\textit{lisse, étale}}$  (resp.  $\mathcal{Y}_{\textit{fppf}}$ ) then we use the canonical truncations  $\tau_{\leq n}$  (see Homology, Section 13) to write  $\mathcal{G}^\bullet$  as a colimit of bounded above complexes and we repeat the argument of the paragraph above.

Finally, by the adjointness of  $f^{-1}$  and  $Rf_*$ , the adjointness of  $Lg_!$  and  $g^{-1}$ , and their “primed” versions we conclude that the first identity of the lemma follows from the second in full generality.  $\square$

**Lemma 3.3.** *Let  $\mathcal{X}$  be an algebraic stack. Notation as in Cohomology of Stacks, Lemma 11.2.*

- (1) *Let  $\mathcal{H}$  be a quasi-coherent  $\mathcal{O}_{\mathcal{X}_{\text{lis\`e-}\acute{e}t\`a}l\`e}}$ -module on the lisse-étale site of  $\mathcal{X}$ . For all  $p \in \mathbf{Z}$  the sheaf  $H^p(Lg_!\mathcal{H})$  is a locally quasi-coherent module with the flat base change property on  $\mathcal{X}$ .*
- (2) *Let  $\mathcal{H}$  be a quasi-coherent  $\mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}}$ -module on the flat-fppf site of  $\mathcal{X}$ . For all  $p \in \mathbf{Z}$  the sheaf  $H^p(Lg_!\mathcal{H})$  is a locally quasi-coherent module with the flat base change property on  $\mathcal{X}$ .*

**Proof.** Pick a scheme  $U$  and a surjective smooth morphism  $x : U \rightarrow \mathcal{X}$ . By Modules on Sites, Definition 23.1 there exists an étale (resp. fppf) covering  $\{U_i \rightarrow U\}_{i \in I}$  such that each pullback  $f_i^{-1}\mathcal{H}$  has a global presentation (see Modules on Sites, Definition 17.1). Here  $f_i : U_i \rightarrow \mathcal{X}$  is the composition  $U_i \rightarrow U \rightarrow \mathcal{X}$  which is a morphism of algebraic stacks. (Recall that the pullback “is” the restriction to  $\mathcal{X}/f_i$ , see Sheaves on Stacks, Definition 9.2 and the discussion following.) After refining the covering we may assume each  $U_i$  is an affine scheme. Since each  $f_i$  is smooth (resp. flat) by Lemma 3.2 we see that  $f_i^{-1}Lg_!\mathcal{H} = Lg_{i,!}(f'_i)^{-1}\mathcal{H}$ . Using Cohomology of Stacks, Lemma 7.5 we reduce the statement of the lemma to the case where  $\mathcal{H}$  has a global presentation and where  $\mathcal{X} = (\text{Sch}/X)_{\text{fppf}}$  for some affine scheme  $X = \text{Spec}(A)$ .

Say our presentation looks like

$$\bigoplus_{j \in J} \mathcal{O} \longrightarrow \bigoplus_{i \in I} \mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

where  $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{\text{lis\`e-}\acute{e}t\`a}l\`e}}$  (resp.  $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}}$ ). Note that the site  $\mathcal{X}_{\text{lis\`e-}\acute{e}t\`a}l\`e}$  (resp.  $\mathcal{X}_{\text{flat},\text{fppf}}$ ) has a final object, namely  $X/X$  which is quasi-compact (see Cohomology on Sites, Section 16). Hence we have

$$\Gamma\left(\bigoplus_{i \in I} \mathcal{O}\right) = \bigoplus_{i \in I} A$$

by Sites, Lemma 11.2. Hence the map in the presentation corresponds to a similar presentation

$$\bigoplus_{j \in J} A \longrightarrow \bigoplus_{i \in I} A \longrightarrow M \longrightarrow 0$$

of an  $A$ -module  $M$ . Moreover,  $\mathcal{H}$  is equal to the restriction to the lisse-étale (resp. flat-fppf) site of the quasi-coherent sheaf  $M^a$  associated to  $M$ . Choose a resolution

$$\dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

by free  $A$ -modules. The complex

$$\dots \mathcal{O} \otimes_A F_2 \rightarrow \mathcal{O} \otimes_A F_1 \rightarrow \mathcal{O} \otimes_A F_0 \rightarrow \mathcal{H} \rightarrow 0$$

is a resolution of  $\mathcal{H}$  by free  $\mathcal{O}$ -modules because for each object  $U/X$  of  $\mathcal{X}_{\text{lis\`e-}\acute{e}t\`a}l\`e}$  (resp.  $\mathcal{X}_{\text{flat},\text{fppf}}$ ) the structure morphism  $U \rightarrow X$  is flat. Hence by construction the value of  $Lg_!\mathcal{H}$  is

$$\dots \rightarrow \mathcal{O}_{\mathcal{X}} \otimes_A F_2 \rightarrow \mathcal{O}_{\mathcal{X}} \otimes_A F_1 \rightarrow \mathcal{O}_{\mathcal{X}} \otimes_A F_0 \rightarrow 0 \rightarrow \dots$$

Since this is a complex of quasi-coherent modules on  $\mathcal{X}_{\acute{e}t\`a}l\`e}$  (resp.  $\mathcal{X}_{\text{fppf}}$ ) it follows from Cohomology of Stacks, Proposition 7.4 that  $H^p(Lg_!\mathcal{H})$  is quasi-coherent.  $\square$

#### 4. Derived categories of quasi-coherent modules

Let  $\mathcal{X}$  be an algebraic stack. As the inclusion functor  $QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow Mod(\mathcal{O}_{\mathcal{X}})$  isn't exact, we cannot define  $D_{QCoh}(\mathcal{O}_{\mathcal{X}})$  as the full subcategory of  $D(\mathcal{O}_{\mathcal{X}})$  consisting of complexes with quasi-coherent cohomology sheaves. In stead we define the category as follows.

**Definition 4.1.** Let  $\mathcal{X}$  be an algebraic stack. Let  $\mathcal{M}_{\mathcal{X}} \subset Mod(\mathcal{O}_{\mathcal{X}})$  denote the category of locally quasi-coherent  $\mathcal{O}_{\mathcal{X}}$ -modules with the flat base change property. Let  $\mathcal{P}_{\mathcal{X}} \subset \mathcal{M}_{\mathcal{X}}$  be the full subcategory consisting of parasitic objects. We define the *derived category of  $\mathcal{O}_{\mathcal{X}}$ -modules with quasi-coherent cohomology sheaves* as the Verdier quotient<sup>1</sup>

$$D_{QCoh}(\mathcal{O}_{\mathcal{X}}) = D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}) / D_{\mathcal{P}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$$

This definition makes sense: By Cohomology of Stacks, Proposition 7.4 we see that  $\mathcal{M}_{\mathcal{X}}$  is a weak Serre subcategory of  $Mod(\mathcal{O}_{\mathcal{X}})$  hence  $D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$  is a strictly full, saturated triangulated subcategory of  $D(\mathcal{O}_{\mathcal{X}})$ , see Derived Categories, Lemma 13.1. Since parasitic modules form a Serre subcategory of  $Mod(\mathcal{O}_{\mathcal{X}})$  (by Cohomology of Stacks, Lemma 8.2) we see that  $\mathcal{P}_{\mathcal{X}} = \text{Parasitic} \cap \mathcal{M}_{\mathcal{X}}$  is a weak Serre subcategory of  $Mod(\mathcal{O}_{\mathcal{X}})$  and hence  $D_{\mathcal{P}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$  is a strictly full, saturated triangulated subcategory of  $D(\mathcal{O}_{\mathcal{X}})$ . Since clearly

$$D_{\mathcal{P}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}) \subset D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$$

we conclude that the first is a strictly full, saturated triangulated subcategory of the second. Hence the Verdier quotient exists. A morphism  $a : E \rightarrow E'$  of  $D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$  becomes an isomorphism in  $D_{QCoh}(\mathcal{O}_{\mathcal{X}})$  if and only if the cone  $C(a)$  has parasitic cohomology sheaves, see Derived Categories, Section 6 and especially Lemma 6.10.

Consider the functors

$$D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}) \xrightarrow{H^i} \mathcal{M}_{\mathcal{X}} \xrightarrow{Q} QCoh(\mathcal{O}_{\mathcal{X}})$$

Note that  $Q$  annihilates the subcategory  $\mathcal{P}_{\mathcal{X}}$ , see Cohomology of Stacks, Lemma 9.2. By Derived Categories, Lemma 6.8 we obtain a cohomological functor

$$(4.1.1) \quad H^i : D_{QCoh}(\mathcal{O}_{\mathcal{X}}) \longrightarrow QCoh(\mathcal{O}_{\mathcal{X}})$$

Moreover, note that  $E \in D_{QCoh}(\mathcal{O}_{\mathcal{X}})$  is zero if and only if  $H^i(E) = 0$  for all  $i \in \mathbf{Z}$ .

Note that the categories  $\mathcal{P}_{\mathcal{X}}$  and  $\mathcal{M}_{\mathcal{X}}$  are also weak Serre subcategories of the abelian category  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$  of modules in the étale topology, see Cohomology of Stacks, Proposition 7.4 and Lemma 8.2. Hence the statement of the following lemma makes sense.

**Lemma 4.2.** *Let  $\mathcal{X}$  be an algebraic stack. The comparison morphism  $\epsilon : \mathcal{X}_{fppf} \rightarrow \mathcal{X}_{\acute{e}tale}$  induces a commutative diagram*

$$\begin{array}{ccccc} D_{\mathcal{P}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}) & \longrightarrow & D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}) & \longrightarrow & D(\mathcal{O}_{\mathcal{X}}) \\ \epsilon^* \uparrow & & \epsilon^* \uparrow & & \epsilon^* \uparrow \\ D_{\mathcal{P}_{\mathcal{X}}}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}}) & \longrightarrow & D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}}) & \longrightarrow & D(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

<sup>1</sup>This definition is different from the one in the literature, see [Ols07, 6.3], but it agrees with that definition by Lemma 4.3.

Moreover, the left two vertical arrows are equivalences of triangulated categories, hence we also obtain an equivalence

$$D_{\mathcal{M}_X}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_X) / D_{\mathcal{P}_X}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

**Proof.** Since  $\epsilon^*$  is exact it is clear that we obtain a diagram as in the statement of the lemma. We will show the middle vertical arrow is an equivalence by applying Cohomology on Sites, Lemma 22.6 to the following situation:  $\mathcal{C} = \mathcal{X}$ ,  $\tau = fppf$ ,  $\tau' = \acute{e}tale$ ,  $\mathcal{O} = \mathcal{O}_X$ ,  $\mathcal{A} = \mathcal{M}_X$ , and  $\mathcal{B}$  is the set of objects of  $\mathcal{X}$  lying over affine schemes. To see the lemma applies we have to check conditions (1), (2), (3), (4). Conditions (1) and (2) are clear from the discussion above (explicitly this follows from Cohomology of Stacks, Proposition 7.4). Condition (3) holds because every scheme has a Zariski open covering by affines. Condition (4) follows from Descent, Lemma 8.4.

We omit the verification that the equivalence of categories  $\epsilon^* : D_{\mathcal{M}_X}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_X) \rightarrow D_{\mathcal{M}_X}(\mathcal{O}_X)$  induces an equivalence of the subcategories of complexes with parasitic cohomology sheaves.  $\square$

It turns out that  $D_{QCoh}(\mathcal{O}_X)$  is the same as the derived category of complexes of modules with quasi-coherent cohomology sheaves on the lisse-étale or flat-fppf site.

**Lemma 4.3.** *Let  $\mathcal{X}$  be an algebraic stack. Let  $\mathcal{F}^\bullet$  be an object of  $D_{\mathcal{M}_X}(\mathcal{O}_X)$ .*

(1) *With  $g$  as in Cohomology of Stacks, Lemma 11.2 for the lisse-étale site we have*

- (a)  $g^{-1}\mathcal{F}^\bullet$  *is in*  $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse, \acute{e}tale}})$ ,
- (b)  $g^{-1}\mathcal{F}^\bullet = 0$  *if and only if*  $\mathcal{F}^\bullet$  *is in*  $D_{\mathcal{P}_X}(\mathcal{O}_X)$ ,
- (c)  $Lg_!\mathcal{H}^\bullet$  *is in*  $D_{\mathcal{M}_X}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_X)$  *for*  $\mathcal{H}^\bullet$  *in*  $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse, \acute{e}tale}})$ , *and*
- (d) *the functors*  $g^{-1}$  *and*  $Lg_!$  *define mutually inverse functors*

$$D_{QCoh}(\mathcal{O}_X) \begin{array}{c} \xrightarrow{g^{-1}} \\ \xleftarrow{Lg_!} \end{array} D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse, \acute{e}tale}})$$

(2) *With  $g$  as in Cohomology of Stacks, Lemma 11.2 for the flat-fppf site we have*

- (a)  $g^{-1}\mathcal{F}^\bullet$  *is in*  $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse, \acute{e}tale}})$ ,
- (b)  $g^{-1}\mathcal{F}^\bullet = 0$  *if and only if*  $\mathcal{F}^\bullet$  *is in*  $D_{\mathcal{P}_X}(\mathcal{O}_X)$ ,
- (c)  $Lg_!\mathcal{H}^\bullet$  *is in*  $D_{\mathcal{M}_X}(\mathcal{O}_X)$  *for*  $\mathcal{H}^\bullet$  *in*  $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat, fppf}})$ , *and*
- (d) *the functors*  $g^{-1}$  *and*  $Lg_!$  *define mutually inverse functors*

$$D_{QCoh}(\mathcal{O}_X) \begin{array}{c} \xrightarrow{g^{-1}} \\ \xleftarrow{Lg_!} \end{array} D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat, fppf}})$$

**Proof.** The functor  $g^{-1}$  is exact, hence (a) and (b) follow from Cohomology of Stacks, Lemmas 12.3 and 11.5.

The construction of  $Lg_!$  in Lemma 3.1 (via Cohomology on Sites, Lemma 27.1 which in turn uses Derived Categories, Proposition 28.2) shows that  $Lg_!$  on any object  $\mathcal{H}^\bullet$  of  $D(\mathcal{O}_{\mathcal{X}_{lisse, \acute{e}tale}})$  is computed as

$$Lg_!\mathcal{H}^\bullet = \operatorname{colim} g_!\mathcal{K}_n^\bullet = g_! \operatorname{colim} \mathcal{K}_n^\bullet$$

(termwise colimits) where the quasi-isomorphism  $\operatorname{colim} \mathcal{K}_n^\bullet \rightarrow \mathcal{H}^\bullet$  induces quasi-isomorphisms  $\mathcal{K}_n^\bullet \rightarrow \tau_{\leq n}\mathcal{H}^\bullet$ . Since  $\mathcal{M}_X \subset \operatorname{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_X)$  (resp.  $\mathcal{M}_X \subset \operatorname{Mod}(\mathcal{O}_X)$ )

is preserved under colimits we see that it suffices to prove (c) on bounded above complexes  $\mathcal{H}^\bullet$  in  $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{\text{isse},\text{étale}}})$  (resp.  $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}})$ ). In this case to show that  $H^n(Lg_!\mathcal{H}^\bullet)$  is in  $\mathcal{M}_{\mathcal{X}}$  we can argue by induction on the integer  $m$  such that  $\mathcal{H}^i = 0$  for  $i > m$ . If  $m < n$ , then  $H^n(Lg_!\mathcal{H}^\bullet) = 0$  and the result holds. In general consider the distinguished triangle

$$\tau_{\leq m-1}\mathcal{H}^\bullet \rightarrow \mathcal{H}^\bullet \rightarrow H^m(\mathcal{H}^\bullet)[-m] \rightarrow \dots$$

(Derived Categories, Remark 12.4) and apply the functor  $Lg_!$ . Since  $\mathcal{M}_{\mathcal{X}}$  is a weak Serre subcategory of the module category it suffices to prove (c) for two out of three. We have the result for  $Lg_!\tau_{\leq m-1}\mathcal{H}^\bullet$  by induction and we have the result for  $Lg_!H^m(\mathcal{H}^\bullet)[-m]$  by Lemma 3.3. Whence (c) holds.

Let us prove (2)(d). By (a) and (b) the functor  $g^{-1} = g^*$  induces a functor

$$c : D_{QCoh}(\mathcal{O}_{\mathcal{X}}) \longrightarrow D_{QCoh}(\mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}})$$

see Derived Categories, Lemma 6.8. Thus we have the following diagram of triangulated categories

$$\begin{array}{ccc} D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}) & \xrightarrow{q} & D_{QCoh}(\mathcal{O}_{\mathcal{X}}) \\ & \searrow^{g^{-1}} & \swarrow^c \\ & D_{QCoh}(\mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}}) & \end{array}$$

$Lg_!$

where  $q$  is the quotient functor, the inner triangle is commutative, and  $g^{-1}Lg_! = \text{id}$ . For any object  $E$  of  $D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$  the map  $a : Lg_!g^{-1}E \rightarrow E$  maps to a quasi-isomorphism in  $D(\mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}})$ . Hence the cone on  $a$  maps to zero under  $g^{-1}$  and by (b) we see that  $q(a)$  is an isomorphism. Thus  $q \circ Lg_!$  is a quasi-inverse to  $c$ .

In the case of the lisse-étale site exactly the same argument as above proves that

$$D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}}) / D_{\mathcal{P}_{\mathcal{X}}}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$$

is equivalent to  $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{\text{isse},\text{étale}}})$ . Applying the last equivalence of Lemma 4.2 finishes the proof.  $\square$

The following lemma tells us that the quotient functor  $D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}) \rightarrow D_{QCoh}(\mathcal{O}_{\mathcal{X}})$  is a Bousfield colocalization (insert future reference here).

**Lemma 4.4.** *Let  $\mathcal{X}$  be an algebraic stack. Let  $E$  be an object of  $D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$ . There exists a canonical distinguished triangle*

$$E' \rightarrow E \rightarrow P \rightarrow E'[1]$$

in  $D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$  such that  $P$  is in  $D_{\mathcal{P}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$  and

$$\text{Hom}_{D(\mathcal{O}_{\mathcal{X}})}(E', P') = 0$$

for all  $P'$  in  $D_{\mathcal{P}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$ .

**Proof.** Consider the morphism of ringed topoi  $g : Sh(\mathcal{X}_{\text{flat},\text{fppf}}) \rightarrow Sh(\mathcal{X}_{\text{fppf}})$ . Set  $E' = Lg_!g^{-1}E$  and let  $P$  be the cone on the adjunction map  $E' \rightarrow E$ . Since  $g^{-1}E' \rightarrow g^{-1}E$  is an isomorphism we see that  $P$  is an object of  $D_{\mathcal{P}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}})$  by Lemma 4.3 (2)(b). Finally,  $\text{Hom}(E', P') = \text{Hom}(Lg_!g^{-1}E, P') = \text{Hom}(g^{-1}E, g^{-1}P') = 0$  as  $g^{-1}P' = 0$ .

Uniqueness. Suppose that  $E'' \rightarrow E \rightarrow P'$  is a second distinguished triangle as in the statement of the lemma. Since  $\text{Hom}(E', P') = 0$  the morphism  $E' \rightarrow E$  factors

as  $E' \rightarrow E'' \rightarrow E$ , see Derived Categories, Lemma 4.2. Similarly, the morphism  $E'' \rightarrow E$  factors as  $E'' \rightarrow E' \rightarrow E$ . Consider the composition  $\varphi : E' \rightarrow E'$  of the maps  $E' \rightarrow E''$  and  $E'' \rightarrow E'$ . Note that  $\varphi - 1 : E' \rightarrow E'$  fits into the commutative diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow \varphi - 1 & & \downarrow 0 \\ E' & \longrightarrow & E \end{array}$$

hence factors through  $P[-1] \rightarrow E$ . Since  $\text{Hom}(E', P[-1]) = 0$  we see that  $\varphi = 1$ . Whence the maps  $E' \rightarrow E''$  and  $E'' \rightarrow E'$  are inverse to each other.  $\square$

### 5. Derived pushforward of quasi-coherent modules

As a first application of the material above we construct the derived pushforward. In Examples, Section 51 the reader can find an example of a quasi-compact and quasi-separated morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  of algebraic stacks such that the direct image functor  $Rf_*$  does not induce a functor  $D_{QCoh}(\mathcal{O}_{\mathcal{X}}) \rightarrow D_{QCoh}(\mathcal{O}_{\mathcal{Y}})$ . Thus restricting to bounded below complexes is necessary.

**Proposition 5.1.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor  $Rf_*$  induces a commutative diagram*

$$\begin{array}{ccccc} D_{\mathcal{P}_{\mathcal{X}}}^+(\mathcal{O}_{\mathcal{X}}) & \longrightarrow & D_{\mathcal{M}_{\mathcal{X}}}^+(\mathcal{O}_{\mathcal{X}}) & \longrightarrow & D(\mathcal{O}_{\mathcal{X}}) \\ \downarrow Rf_* & & \downarrow Rf_* & & \downarrow Rf_* \\ D_{\mathcal{P}_{\mathcal{Y}}}^+(\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & D_{\mathcal{M}_{\mathcal{Y}}}^+(\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & D(\mathcal{O}_{\mathcal{Y}}) \end{array}$$

and hence induces a functor

$$Rf_{QCoh,*} : D_{QCoh}^+(\mathcal{O}_{\mathcal{X}}) \longrightarrow D_{QCoh}^+(\mathcal{O}_{\mathcal{Y}})$$

on quotient categories. Moreover, the functor  $R^i f_{QCoh}$  of Cohomology of Stacks, Proposition 10.1 are equal to  $H^i \circ Rf_{QCoh,*}$  with  $H^i$  as in (4.1.1).

**Proof.** We have to show that  $Rf_* E$  is an object of  $D_{\mathcal{M}_{\mathcal{Y}}}^+(\mathcal{O}_{\mathcal{Y}})$  for  $E$  in  $D_{\mathcal{M}_{\mathcal{X}}}^+(\mathcal{O}_{\mathcal{X}})$ . This follows from Cohomology of Stacks, Proposition 7.4 and the spectral sequence  $R^i f_* H^j(E) \Rightarrow R^{i+j} f_* E$ . The case of parasitic modules works the same way using Cohomology of Stacks, Lemma 8.3. The final statement is clear from the definition of  $H^i$  in (4.1.1).  $\square$

### 6. Derived pullback of quasi-coherent modules

Derived pullback of complexes with quasi-coherent cohomology sheaves exists in general.

**Proposition 6.1.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of algebraic stacks. The exact functor  $f^*$  induces a commutative diagram*

$$\begin{array}{ccc} D_{\mathcal{M}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}) & \longrightarrow & D(\mathcal{O}_{\mathcal{X}}) \\ f^* \uparrow & & \uparrow f^* \\ D_{\mathcal{M}_{\mathcal{Y}}}(\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & D(\mathcal{O}_{\mathcal{Y}}) \end{array}$$

The composition

$$D_{\mathcal{M}_Y}(\mathcal{O}_Y) \xrightarrow{f^*} D_{\mathcal{M}_X}(\mathcal{O}_X) \xrightarrow{q_X} D_{QCoh}(\mathcal{O}_X)$$

is left deriveable with respect to the localization  $D_{\mathcal{M}_Y}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_Y)$  and we may define  $Lf_{QCoh}^*$  as its left derived functor

$$Lf_{QCoh}^* : D_{QCoh}(\mathcal{O}_Y) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

(see *Derived Categories*, Definitions 15.2 and 15.9). If  $f$  is quasi-compact and quasi-separated, then  $Lf_{QCoh}^*$  and  $Rf_{QCoh,*}$  satisfy the following adjointness:

$$\mathrm{Hom}_{D_{QCoh}(\mathcal{O}_X)}(Lf_{QCoh}^*A, B) = \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_Y)}(A, Rf_{QCoh,*}B)$$

for  $A \in D_{QCoh}(\mathcal{O}_Y)$  and  $B \in D_{QCoh}^+(\mathcal{O}_X)$ .

**Proof.** To prove the first statement, we have to show that  $f^*E$  is an object of  $D_{\mathcal{M}_X}(\mathcal{O}_X)$  for  $E$  in  $D_{\mathcal{M}_Y}(\mathcal{O}_Y)$ . Since  $f^* = f^{-1}$  is exact this follows immediately from the fact that  $f^*$  maps  $\mathcal{M}_Y$  into  $\mathcal{M}_X$ .

Set  $\mathcal{D} = D_{\mathcal{M}_Y}(\mathcal{O}_Y)$ . Let  $S$  be the collection of morphisms in  $\mathcal{D}$  whose cone is an object of  $D_{\mathcal{P}_Y}(\mathcal{O}_Y)$ . Set  $\mathcal{D}' = D_{QCoh}(\mathcal{O}_X)$ . Set  $F = q_X \circ f^* : \mathcal{D} \rightarrow \mathcal{D}'$ . Then  $\mathcal{D}, S, \mathcal{D}', F$  are as in *Derived Categories*, Situation 15.1 and Definition 15.2. Let us prove that  $LF(E)$  is defined for any object  $E$  of  $\mathcal{D}$ . Namely, consider the triangle

$$E' \rightarrow E \rightarrow P \rightarrow E'[1]$$

constructed in Lemma 4.4. Note that  $s : E' \rightarrow E$  is an element of  $S$ . We claim that  $E'$  computes  $LF$ . Namely, suppose that  $s' : E'' \rightarrow E$  is another element of  $S$ , i.e., fits into a triangle  $E'' \rightarrow E \rightarrow P' \rightarrow E''[1]$  with  $P'$  in  $D_{\mathcal{P}_Y}(\mathcal{O}_Y)$ . By Lemma 4.4 (and its proof) we see that  $E' \rightarrow E$  factors through  $E'' \rightarrow E$ . Thus we see that  $E' \rightarrow E$  is cofinal in the system  $S/E$ . Hence it is clear that  $E'$  computes  $LF$ .

To see the final statement, write  $B = q_X(H)$  and  $A = q_Y(E)$ . Choose  $E' \rightarrow E$  as above. We will use on the one hand that  $Rf_{QCoh,*}(B) = q_Y(Rf_*H)$  and on the other that  $Lf_{QCoh}^*(A) = q_X(f^*E')$ .

$$\begin{aligned} \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_X)}(Lf_{QCoh}^*A, B) &= \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_X)}(q_X(f^*E'), q_X(H)) \\ &= \mathrm{colim}_{H \rightarrow H'} \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*E', H') \\ &= \mathrm{colim}_{H \rightarrow H'} \mathrm{Hom}_{D(\mathcal{O}_Y)}(E', Rf_*H') \\ &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(E', Rf_*H) \\ &= \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_Y)}(A, Rf_{QCoh,*}B) \end{aligned}$$

Here the colimit is over morphisms  $s : H \rightarrow H'$  in  $D_{\mathcal{M}_X}^+(\mathcal{O}_X)$  whose cone  $P(s)$  is an object of  $D_{\mathcal{P}_X}^+(\mathcal{O}_X)$ . The first equality we've seen above. The second equality holds by construction of the Verdier quotient. The third equality holds by Cohomology on Sites, Lemma 19.1. Since  $Rf_*P(s)$  is an object of  $D_{\mathcal{P}_Y}^+(\mathcal{O}_Y)$  by Proposition 5.1 we see that  $\mathrm{Hom}_{D(\mathcal{O}_Y)}(E', Rf_*P(s)) = 0$ . Thus the fourth equality holds. The final equality holds by construction of  $E'$ .  $\square$

## 7. Other chapters

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(2) Conventions

(3) Set Theory

(4) Categories

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  - (6) Sheaves on Spaces
  - (7) Sites and Sheaves
  - (8) Stacks
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