CONSTRUCTIONS OF SCHEMES

Contents

1. Introduction

In this chapter we introduce ways of constructing schemes out of others. A basic reference is [DG67].

2. Relative glueing

The following lemma is relevant in case we are trying to construct a scheme X over S, and we already know how to construct the restriction of X to the affine opens of S. The actual result is completely general and works in the setting of (locally) ringed spaces, although our proof is written in the language of schemes.

Lemma 2.1. *Let S be a scheme. Let $\mathcal{B}$ be a basis for the topology of S. Suppose given the following data:*

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(1) *For every* $U \in \mathcal{B}$ *a scheme* $f_U : X_U \to U$ *over* U.
(2) *For every pair* $U, V \in \mathcal{B}$ *such that* $V \subset U$ *a morphism* $\rho^U_V : X_V \to X_U$.

Assume that

(a) *each* ρ^U_V *induces an isomorphism* $X_V \to f_U^{-1}(V)$ *of schemes over* V,
(b) *whenever* $W, V, U \in \mathcal{B}$, *with* $W \subset V \subset U$ *we have* $\rho^U_W = \rho^U_V \circ \rho^V_W$.

Then there exists a unique scheme $f : X \to S$ *over* S *and isomorphisms* $i_U : f^{-1}(U) \to X_U$ *over* U *such that for* $V \subset U \subset S$ *affine open the composition*

$$X_V \xrightarrow{i_V^{-1}} f^{-1}(V) \xrightarrow{inclusion} f^{-1}(U) \xrightarrow{i_U} X_U$$

is the morphism ρ^U_V.

Proof. To prove this we will use Schemes, Lemma 15.4. First we define a contravariant functor F from the category of schemes to the category of sets. Namely, for a scheme T we set

$$F(T) = \left\{ \begin{matrix} (g, \{h_U\}_{U \in \mathcal{B}}),\ g : T \to S,\ h_U : g^{-1}(U) \to X_U, \\ f_U \circ h_U = g|_{g^{-1}(U)},\ h_U|_{g^{-1}(V)} = \rho^U_V \circ h_V\ \forall\ V, U \in \mathcal{B}, V \subset U \end{matrix} \right\}.$$

The restriction mapping $F(T) \to F(T')$ given a morphism $T' \to T$ is just gotten by composition. For any $W \in \mathcal{B}$ we consider the subfunctor $F_W \subset F$ consisting of those systems $(g, \{h_U\})$ such that $g(T) \subset W$.

First we show F satisfies the sheaf property for the Zariski topology. Suppose that T is a scheme, $T = \bigcup V_i$ is an open covering, and $\xi_i \in F(V_i)$ is an element such that $\xi_i|_{V_i \cap V_j} = \xi_j|_{V_i \cap V_j}$. Say $\xi_i = (g_i, \{h_{i,U}\})$. Then we immediately see that the morphisms g_i glue to a unique global morphism $g : T \to S$. Moreover, it is clear that $g^{-1}(U) = \bigcup g_i^{-1}(U)$. Hence the morphisms $h_{i,U} : g_i^{-1}(U) \to X_U$ glue to a unique morphism $h_U : U \to X_U$. It is easy to verify that the system $(g, \{f_U\})$ is an element of $F(T)$. Hence F satisfies the sheaf property for the Zariski topology.

Next we verify that each F_W, $W \in \mathcal{B}$ is representable. Namely, we claim that the transformation of functors

$$F_W \longrightarrow \mathrm{Mor}(-, X_W),\ (g, \{h_U\}) \longmapsto h_W$$

is an isomorphism. To see this suppose that T is a scheme and $\alpha : T \to X_W$ is a morphism. Set $g = f_W \circ \alpha$. For any $U \in \mathcal{B}$ such that $U \subset W$ we can define $h_U : g^{-1}(U) \to X_U$ be the composition $(\rho^W_U)^{-1} \circ \alpha|_{g^{-1}(U)}$. This works because the image $\alpha(g^{-1}(U))$ is contained in $f_W^{-1}(U)$ and condition (a) of the lemma. It is clear that $f_U \circ h_U = g|_{g^{-1}(U)}$ for such a U. Moreover, if also $V \in \mathcal{B}$ and $V \subset U \subset W$, then $\rho^U_V \circ h_V = h_U|_{g^{-1}(V)}$ by property (b) of the lemma. We still have to define h_U for an arbitrary element $U \in \mathcal{B}$. Since $\mathcal{B}$ is a basis for the topology on S we can find an open covering $U \cap W = \bigcup U_i$ with $U_i \in \mathcal{B}$. Since g maps into W we have $g^{-1}(U) = g^{-1}(U \cap W) = \bigcup g^{-1}(U_i)$. Consider the morphisms $h_i = \rho^U_{U_i} \circ h_{U_i} : g^{-1}(U_i) \to X_U$. It is a simple matter to use condition (b) of the lemma to prove that $h_i|_{g^{-1}(U_i) \cap g^{-1}(U_j)} = h_j|_{g^{-1}(U_i) \cap g^{-1}(U_j)}$. Hence these morphisms glue to give the desired morphism $h_U : g^{-1}(U) \to X_U$. We omit the (easy) verification that the system $(g, \{h_U\})$ is an element of $F_W(T)$ which maps to α under the displayed arrow above.

Next, we verify each $F_W \subset F$ is representable by open immersions. This is clear from the definitions.

Finally we have to verify the collection $(F_W)_{W\in\mathcal{B}}$ covers F. This is clear by construction and the fact that $\mathcal{B}$ is a basis for the topology of S.

Let X be a scheme representing the functor F. Let $(f, \{i_U\}) \in F(X)$ be a "universal family". Since each F_W is representable by X_W (via the morphism of functors displayed above) we see that $i_W : f^{-1}(W) \to X_W$ is an isomorphism as desired. The lemma is proved. □

Lemma 2.2. *Let S be a scheme. Let $\mathcal{B}$ be a basis for the topology of S. Suppose given the following data:*

1. *For every $U \in \mathcal{B}$ a scheme $f_U : X_U \to U$ over U.*
2. *For every $U \in \mathcal{B}$ a quasi-coherent sheaf $\mathcal{F}_U$ over X_U.*
3. *For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\rho^U_V : X_V \to X_U$.*
4. *For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\theta^U_V : (\rho^U_V)^*\mathcal{F}_U \to \mathcal{F}_V$.*

Assume that

(a) *each ρ^U_V induces an isomorphism $X_V \to f_U^{-1}(V)$ of schemes over V,*
(b) *each θ^U_V is an isomorphism,*
(c) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\rho^U_W = \rho^U_V \circ \rho^V_W$,*
(d) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\theta^U_W = \theta^V_W \circ (\rho^V_W)^*\theta^U_V$.*

Then there exists a unique scheme $f : X \to S$ over S together with a unique quasi-coherent sheaf $\mathcal{F}$ on X and isomorphisms $i_U : f^{-1}(U) \to X_U$ and $\theta_U : i_U^*\mathcal{F}_U \to \mathcal{F}|_{f^{-1}(U)}$ over U such that for $V \subset U \subset S$ affine open the composition

$$X_V \xrightarrow{i_V^{-1}} f^{-1}(V) \xrightarrow{inclusion} f^{-1}(U) \xrightarrow{i_U} X_U$$

is the morphism ρ^U_V, and the composition

$$(\rho^U_V)^*\mathcal{F}_U = (i_V^{-1})^*((i_U^*\mathcal{F}_U)|_{f^{-1}(V)}) \xrightarrow{\theta_U|_{f^{-1}(V)}} (i_V^{-1})^*(\mathcal{F}|_{f^{-1}(V)}) \xrightarrow{\theta_V^{-1}} \mathcal{F}_V \tag{2.2.1}$$

is equal to θ^U_V.

Proof. By Lemma 2.1 we get the scheme X over S and the isomorphisms i_U. Set $\mathcal{F}'_U = i_U^*\mathcal{F}_U$ for $U \in \mathcal{B}$. This is a quasi-coherent $\mathcal{O}_{f^{-1}(U)}$-module. The maps

$$\mathcal{F}'_U|_{f^{-1}(V)} = i_U^*\mathcal{F}_U|_{f^{-1}(V)} = i_V^*(\rho^U_V)^*\mathcal{F}_U \xrightarrow{i_V^*\theta^U_V} i_V^*\mathcal{F}_V = \mathcal{F}'_V$$

define isomorphisms $(\theta')^U_V : \mathcal{F}'_U|_{f^{-1}(V)} \to \mathcal{F}'_V$ whenever $V \subset U$ are elements of $\mathcal{B}$. Condition (d) says exactly that this is compatible in case we have a triple of elements $W \subset V \subset U$ of $\mathcal{B}$. This allows us to get well defined isomorphisms

$$\varphi_{12} : \mathcal{F}'_{U_1}|_{f^{-1}(U_1\cap U_2)} \longrightarrow \mathcal{F}'_{U_2}|_{f^{-1}(U_1\cap U_2)}$$

whenever $U_1, U_2 \in \mathcal{B}$ by covering the intersection $U_1 \cap U_2 = \bigcup V_j$ by elements V_j of $\mathcal{B}$ and taking

$$\varphi_{12}|_{V_j} = \left((\theta')^{U_2}_{V_j}\right)^{-1} \circ (\theta')^{U_1}_{V_j}.$$

We omit the verification that these maps do indeed glue to a φ_{12} and we omit the verification of the cocycle condition of a glueing datum for sheaves (as in Sheaves, Section 33). By Sheaves, Lemma 33.2 we get our $\mathcal{F}$ on X. We omit the verification of (2.2.1). □

Remark 2.3. There is a functoriality property for the constructions explained in Lemmas 2.1 and 2.2. Namely, suppose given two collections of data $(f_U : X_U \to U, \rho_V^U)$ and $(g_U : Y_U \to U, \sigma_V^U)$ as in Lemma 2.1. Suppose for every $U \in \mathcal{B}$ given a morphism $h_U : X_U \to Y_U$ over U compatible with the restrictions ρ_V^U and σ_V^U. Functoriality means that this gives rise to a morphism of schemes $h : X \to Y$ over S restricting back to the morphisms h_U, where $f : X \to S$ is obtained from the datum $(f_U : X_U \to U, \rho_V^U)$ and $g : Y \to S$ is obtained from the datum $(g_U : Y_U \to U, \sigma_V^U)$.

Similarly, suppose given two collections of data $(f_U : X_U \to U, \mathcal{F}_U, \rho_V^U, \theta_V^U)$ and $(g_U : Y_U \to U, \mathcal{G}_U, \sigma_V^U, \eta_V^U)$ as in Lemma 2.2. Suppose for every $U \in \mathcal{B}$ given a morphism $h_U : X_U \to Y_U$ over U compatible with the restrictions ρ_V^U and σ_V^U, and a morphism $\tau_U : h_U^*\mathcal{G}_U \to \mathcal{F}_U$ compatible with the maps θ_V^U and η_V^U. Functoriality means that these give rise to a morphism of schemes $h : X \to Y$ over S restricting back to the morphisms h_U, and a morphism $h^*\mathcal{G} \to \mathcal{F}$ restricting back to the maps h_U where $(f : X \to S, \mathcal{F})$ is obtained from the datum $(f_U : X_U \to U, \mathcal{F}_U, \rho_V^U, \theta_V^U)$ and where $(g : Y \to S, \mathcal{G})$ is obtained from the datum $(g_U : Y_U \to U, \mathcal{G}_U, \sigma_V^U, \eta_V^U)$.

We omit the verifications and we omit a suitable formulation of "equivalence of categories" between relative glueing data and relative objects.

3. Relative spectrum via glueing

Situation 3.1. Here S is a scheme, and $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra.

In this section we outline how to construct a morphism of schemes

$$\underline{\mathrm{Spec}}_S(\mathcal{A}) \longrightarrow S$$

by glueing the spectra $\mathrm{Spec}(\Gamma(U, \mathcal{A}))$ where U ranges over the affine opens of S. We first show that the spectra of the values of $\mathcal{A}$ over affines form a suitable collection of schemes, as in Lemma 2.1.

Lemma 3.2. *In Situation 3.1. Suppose $U \subset U' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$ and $A' = \mathcal{A}(U')$. The map of rings $A' \to A$ induces a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$, and the diagram*

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathrm{Spec}(A') \arrow[d] \\ U \arrow[r] & U' \end{tikzcd}$$

is cartesian.

Proof. Let $R = \mathcal{O}_S(U)$ and $R' = \mathcal{O}_S(U')$. Note that the map $R \otimes_{R'} A' \to A$ is an isomorphism as $\mathcal{A}$ is quasi-coherent (see Schemes, Lemma 7.3 for example). The result follows from the description of the fibre product of affine schemes in Schemes, Lemma 6.7. $\square$

In particular the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ of the lemma is an open immersion.

Lemma 3.3. *In Situation 3.1. Suppose $U \subset U' \subset U'' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ and $A'' = \mathcal{A}(U'')$. The composition of the morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$, and $\mathrm{Spec}(A') \to \mathrm{Spec}(A'')$ of Lemma 3.2 gives the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A'')$ of Lemma 3.2.*

Proof. This follows as the map $A'' \to A$ is the composition of $A'' \to A'$ and $A' \to A$ (because $\mathcal{A}$ is a sheaf). $\square$

Lemma 3.4. *In Situation 3.1. There exists a morphism of schemes*
$$\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \longrightarrow S$$
with the following properties:

1. *for every affine open $U \subset S$ there exists an isomorphism $i_U : \pi^{-1}(U) \to \mathrm{Spec}(\mathcal{A}(U))$, and*
2. *for $U \subset U' \subset S$ affine open the composition*
$$\mathrm{Spec}(\mathcal{A}(U)) \xrightarrow{i_U^{-1}} \pi^{-1}(U) \xrightarrow{inclusion} \pi^{-1}(U') \xrightarrow{i_{U'}} \mathrm{Spec}(\mathcal{A}(U'))$$
is the open immersion of Lemma 3.2 above.

Proof. Follows immediately from Lemmas 2.1, 3.2, and 3.3. $\square$

4. Relative spectrum as a functor

We place ourselves in Situation 3.1. So S is a scheme and $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. (This means that $\mathcal{A}$ is a sheaf of $\mathcal{O}_S$-algebras which is quasi-coherent as an $\mathcal{O}_S$-module.)

For any $f : T \to S$ the pullback $f^*\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_T$-algebras. We are going to consider pairs $(f : T \to S, \varphi)$ where f is a morphism of schemes and $\varphi : f^*\mathcal{A} \to \mathcal{O}_T$ is a morphism of $\mathcal{O}_T$-algebras. Note that this is the same as giving a $f^{-1}\mathcal{O}_S$-algebra homomorphism $\varphi : f^{-1}\mathcal{A} \to \mathcal{O}_T$, see Sheaves, Lemma 20.2. This is also the same as giving a $\mathcal{O}_S$-algebra map $\varphi : \mathcal{A} \to f_*\mathcal{O}_T$, see Sheaves, Lemma 24.7. We will use all three ways of thinking about φ, without further mention.

Given such a pair $(f : T \to S, \varphi)$ and a morphism $a : T' \to T$ we get a second pair $(f' = f \circ a, \varphi' = a^*\varphi)$ which we call the pullback of (f, φ). One way to describe $\varphi' = a^*\varphi$ is as the composition $\mathcal{A} \to f_*\mathcal{O}_T \to f'_*\mathcal{O}_{T'}$ where the second map is $f_*a^\sharp$ with $a^\sharp : \mathcal{O}_T \to a_*\mathcal{O}_{T'}$. In this way we have defined a functor

$$\begin{array}{rcl} F : Sch^{opp} & \longrightarrow & Sets \\ T & \longmapsto & F(T) = \{\text{pairs } (f, \varphi) \text{ as above}\} \end{array} \tag{4.0.1}$$

Lemma 4.1. *In Situation 3.1. Let F be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F' be the functor associated to $(S', \mathcal{A}')$ above. Then there is a canonical isomorphism*
$$F' \cong h_{S'} \times_{h_S} F$$
of functors.

Proof. A pair $(f' : T \to S', \varphi' : (f')^*\mathcal{A}' \to \mathcal{O}_T)$ is the same as a pair $(f, \varphi : f^*\mathcal{A} \to \mathcal{O}_T)$ together with a factorization of f as $f = g \circ f'$. Namely with this notation we have $(f')^*\mathcal{A}' = (f')^*g^*\mathcal{A} = f^*\mathcal{A}$. Hence the lemma. $\square$

Lemma 4.2. *In Situation 3.1. Let F be the functor associated to $(S, \mathcal{A})$ above. If S is affine, then F is representable by the affine scheme* $\mathrm{Spec}(\Gamma(S, \mathcal{A}))$.

Proof. Write $S = \mathrm{Spec}(R)$ and $A = \Gamma(S, \mathcal{A})$. Then A is an R-algebra and $\mathcal{A} = \widetilde{A}$. The ring map $R \to A$ gives rise to a canonical map

$$f_{univ} : \mathrm{Spec}(A) \longrightarrow S = \mathrm{Spec}(R).$$

We have $f_{univ}^*\mathcal{A} = \widetilde{A \otimes_R A}$ by Schemes, Lemma 7.3. Hence there is a canonical map

$$\varphi_{univ} : f_{univ}^*\mathcal{A} = \widetilde{A \otimes_R A} \longrightarrow \widetilde{A} = \mathcal{O}_{\mathrm{Spec}(A)}$$

coming from the A-module map $A \otimes_R A \to A$, $a \otimes a' \mapsto aa'$. We claim that the pair $(f_{univ}, \varphi_{univ})$ represents F in this case. In other words we claim that for any scheme T the map

$$\mathrm{Mor}(T, \mathrm{Spec}(A)) \longrightarrow \{\text{pairs } (f, \varphi)\}, \quad a \longmapsto (a^* f_{univ}, a^*\varphi)$$

is bijective.

Let us construct the inverse map. For any pair $(f : T \to S, \varphi)$ we get the induced ring map

$$A = \Gamma(S, \mathcal{A}) \xrightarrow{f^*} \Gamma(T, f^*\mathcal{A}) \xrightarrow{\varphi} \Gamma(T, \mathcal{O}_T)$$

This induces a morphism of schemes $T \to \mathrm{Spec}(A)$ by Schemes, Lemma 6.4.

The verification that this map is inverse to the map displayed above is omitted. $\square$

Lemma 4.3. *In Situation 3.1. The functor F is representable by a scheme.*

Proof. We are going to use Schemes, Lemma 15.4.

First we check that F satisfies the sheaf property for the Zariski topology. Namely, suppose that T is a scheme, that $T = \bigcup_{i \in I} U_i$ is an open covering, and that $(f_i, \varphi_i) \in F(U_i)$ such that $(f_i, \varphi_i)|_{U_i \cap U_j} = (f_j, \varphi_j)|_{U_i \cap U_j}$. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{I_i} = f_i$, see Schemes, Section 14. Thus $f_i^*\mathcal{A} = f^*\mathcal{A}|_{U_i}$ and by assumption the morphisms φ_i agree on $U_i \cap U_j$. Hence by Sheaves, Section 33 these glue to a morphism of $\mathcal{O}_T$-algebras $f^*\mathcal{A} \to \mathcal{O}_T$. This proves that F satisfies the sheaf condition with respect to the Zariski topology.

Let $S = \bigcup_{i \in I} U_i$ be an affine open covering. Let $F_i \subset F$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subset U_i$.

We have to show each F_i is representable. This is the case because F_i is identified with the functor associated to U_i equipped with the quasi-coherent $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$, by Lemma 4.1. Thus the result follows from Lemma 4.2.

Next we show that $F_i \subset F$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of F_i that given $a : T' \to T$ we gave $a^*(f, \varphi) \in F_i(T')$ if and only if $a(T') \subset V_i$. This is what we were required to show.

Finally, we have to show that the collection $(F_i)_{i \in I}$ covers F. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i \in I} U_i$ is an open covering of S we see that $T = \bigcup_{i \in I} V_i$ is an open covering of T. Moreover $(f, \varphi)|_{V_i} \in F_i(V_i)$. This finishes the proof of the lemma. $\square$

Lemma 4.4. *In Situation 3.1. The scheme $\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ constructed in Lemma 3.4 and the scheme representing the functor F are canonically isomorphic as schemes over S.*

Proof. Let $X \to S$ be the scheme representing the functor F. Consider the sheaf of $\mathcal{O}_S$-algebras $\mathcal{R} = \pi_*\mathcal{O}_{\underline{\text{Spec}}_S(\mathcal{A})}$. By construction of $\underline{\text{Spec}}_S(\mathcal{A})$ we have isomorphisms $\mathcal{A}(U) \to \mathcal{R}(U)$ for every affine open $U \subset S$; this follows from Lemma 3.4 part (1). For $U \subset U' \subset S$ open these isomorphisms are compatible with the restriction mappings; this follows from Lemma 3.4 part (2). Hence by Sheaves, Lemma 30.13 these isomorphisms result from an isomorphism of $\mathcal{O}_S$-algebras $\varphi : \mathcal{A} \to \mathcal{R}$. Hence this gives an element $(\underline{\text{Spec}}_S(\mathcal{A}), \varphi) \in F(\underline{\text{Spec}}_S(\mathcal{A}))$. Since X represents the functor F we get a corresponding morphism of schemes $can : \underline{\text{Spec}}_S(\mathcal{A}) \to X$ over S.

Let $U \subset S$ be any affine open. Let $F_U \subset F$ be the subfunctor of F corresponding to pairs (f, φ) over schemes T with $f(T) \subset U$. Clearly the base change X_U represents F_U. Moreover, F_U is represented by $\text{Spec}(\mathcal{A}(U)) = \pi^{-1}(U)$ according to Lemma 4.2. In other words $X_U \cong \pi^{-1}(U)$. We omit the verification that this identification is brought about by the base change of the morphism can to U. □

Definition 4.5. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. The *relative spectrum of $\mathcal{A}$ over S*, or simply the *spectrum of $\mathcal{A}$ over S* is the scheme constructed in Lemma 3.4 which represents the functor F (4.0.1), see Lemma 4.4. We denote it $\pi : \underline{\text{Spec}}_S(\mathcal{A}) \to S$. The "universal family" is a morphism of $\mathcal{O}_S$-algebras

$$\mathcal{A} \longrightarrow \pi_*\mathcal{O}_{\underline{\text{Spec}}_S(\mathcal{A})}$$

The following lemma says among other things that forming the relative spectrum commutes with base change.

Lemma 4.6. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Let $\pi : \underline{\text{Spec}}_S(\mathcal{A}) \to S$ be the relative spectrum of $\mathcal{A}$ over S.*

1. *For every affine open $U \subset S$ the inverse image $\pi^{-1}(U)$ is affine.*
2. *For every morphism $g : S' \to S$ we have $S' \times_S \underline{\text{Spec}}_S(\mathcal{A}) = \underline{\text{Spec}}_{S'}(g^*\mathcal{A})$.*
3. *The universal map*
$$\mathcal{A} \longrightarrow \pi_*\mathcal{O}_{\underline{\text{Spec}}_S(\mathcal{A})}$$
is an isomorphism of $\mathcal{O}_S$-algebras.

Proof. Part (1) comes from the description of the relative spectrum by glueing, see Lemma 3.4. Part (2) follows immediately from Lemma 4.1. Part (3) follows because it is local on S and it is clear in case S is affine by Lemma 4.2 for example. □

Lemma 4.7. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. By Schemes, Lemma 24.1 the sheaf $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. There is a canonical morphism*

$$can : X \longrightarrow \underline{\text{Spec}}_S(f_*\mathcal{O}_X)$$

of schemes over S. For any affine open $U \subset S$ the restriction $can|_{f^{-1}(U)}$ is identified with the canonical morphism

$$f^{-1}(U) \longrightarrow \text{Spec}(\Gamma(f^{-1}(U), \mathcal{O}_X))$$

coming from Schemes, Lemma 6.4.

Proof. The morphism comes, via the definition of $\underline{\text{Spec}}$ as the scheme representing the functor F, from the canonical map $\varphi : f^*f_*\mathcal{O}_X \to \mathcal{O}_X$ (which by adjointness of push and pull corresponds to id : $f_*\mathcal{O}_X \to f_*\mathcal{O}_X$). The statement on the restriction to $f^{-1}(U)$ follows from the description of the relative spectrum over affines, see Lemma 4.2. $\square$

5. Affine n-space

As an application of the relative spectrum we define affine n-space over a base scheme S as follows. For any integer $n \geq 0$ we can consider the quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{O}_S[T_1, \ldots, T_n]$. It is quasi-coherent because as a sheaf of $\mathcal{O}_S$-modules it is just the direct sum of copies of $\mathcal{O}_S$ indexed by multi-indices.

Definition 5.1. Let S be a scheme and $n \geq 0$. The scheme

$$\mathbf{A}^n_S = \underline{\mathrm{Spec}}_S(\mathcal{O}_S[T_1, \ldots, T_n])$$

over S is called *affine n-space over S*. If $S = \mathrm{Spec}(R)$ is affine then we also call this *affine n-space over R* and we denote it $\mathbf{A}^n_R$.

Note that $\mathbf{A}^n_R = \mathrm{Spec}(R[T_1, \ldots, T_n])$. For any morphism $g : S' \to S$ of schemes we have $g^*\mathcal{O}_S[T_1, \ldots, T_n] = \mathcal{O}_{S'}[T_1, \ldots, T_n]$ and hence $\mathbf{A}^n_{S'} = S' \times_S \mathbf{A}^n_S$ is the base change. Therefore an alternative definition of affine n-space is the formula

$$\mathbf{A}^n_S = S \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{A}^n_{\mathbf{Z}}.$$

Also, a morphism from an S-scheme $f : X \to S$ to $\mathbf{A}^n_S$ is given by a homomorphism of $\mathcal{O}_S$-algebras $\mathcal{O}_S[T_1, \ldots, T_n] \to f_*\mathcal{O}_X$. This is clearly the same thing as giving the images of the T_i. In other words, a morphism from X to $\mathbf{A}^n_S$ over S is the same as giving n elements $h_1, \ldots, h_n \in \Gamma(X, \mathcal{O}_X)$.

6. Vector bundles

Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. By Modules, Lemma 18.6 the symmetric algebra $\mathrm{Sym}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{O}_S$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Hence it makes sense to apply the construction of the previous section to it.

Definition 6.1. Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module[1]. The *vector bundle associated to $\mathcal{E}$* is

$$\mathbf{V}(\mathcal{E}) = \underline{\mathrm{Spec}}_S(\mathrm{Sym}(\mathcal{E})).$$

The vector bundle associated to $\mathcal{E}$ comes with a bit of extra structure. Namely, we have a grading

$$\pi_*\mathcal{O}_{\mathbf{V}(\mathcal{E})} = \bigoplus\nolimits_{n \geq 0} \mathrm{Sym}^n(\mathcal{E}).$$

which turns $\pi_*\mathcal{O}_{\mathbf{V}(\mathcal{E})}$ into a graded $\mathcal{O}_S$-algebra. Conversely, we can recover $\mathcal{E}$ from the degree 1 part of this. Thus we define an abstract vector bundle as follows.

Definition 6.2. Let S be a scheme. A *vector bundle $\pi : V \to S$ over S* is an affine morphism of schemes such that $\pi_*\mathcal{O}_V$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_V = \bigoplus_{n \geq 0} \mathcal{E}_n$ such that $\mathcal{E}_0 = \mathcal{O}_S$ and such that the maps

$$\mathrm{Sym}^n(\mathcal{E}_1) \longrightarrow \mathcal{E}_n$$

[1]The reader may expect here the condition that $\mathcal{E}$ is finite locally free. We do not do so in order to be consistent with [DG67, II, Definition 1.7.8].

are isomorphisms for all $n \geq 0$. A *morphism of vector bundles over S* is a morphism $f : V \to V'$ such that the induced map

$$f^* : \pi'_*\mathcal{O}_{V'} \longrightarrow \pi_*\mathcal{O}_V$$

is compatible with the given gradings.

An example of a vector bundle over S is affine n-space $\mathbf{A}^n_S$ over S, see Definition 5.1. This is true because $\mathcal{O}_S[T_1, \ldots, T_n] = \text{Sym}(\mathcal{O}_S^{\oplus n})$.

Lemma 6.3. *The category of vector bundles over a scheme S is anti-equivalent to the category of quasi-coherent $\mathcal{O}_S$-modules.*

Proof. Omitted. Hint: In one direction one uses the functor $\underline{\text{Spec}}_S(-)$ and in the other the functor $(\pi : V \to S) \rightsquigarrow (\pi_*\mathcal{O}_V)_1$ (degree 1 part). $\square$

7. Cones

In algebraic geometry cones correspond to graded algebras. By our conventions a graded ring or algebra A comes with a grading $A = \bigoplus_{d \geq 0} A_d$ by the nonnegative integers, see Algebra, Section 54.

Definition 7.1. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Assume that $\mathcal{O}_S \to \mathcal{A}_0$ is an isomorphism[2]. The *cone associated to $\mathcal{A}$* or the *affine cone associated to $\mathcal{A}$* is

$$C(\mathcal{A}) = \underline{\text{Spec}}_S(\mathcal{A}).$$

The cone associated to a graded sheaf of $\mathcal{O}_S$-algebras comes with a bit of extra structure. Namely, we obtain a grading

$$\pi_*\mathcal{O}_{C(\mathcal{A})} = \bigoplus\nolimits_{n \geq 0} \mathcal{A}_n$$

Thus we can define an abstract cone as follows.

Definition 7.2. Let S be a scheme. A *cone $\pi : C \to S$ over S* is an affine morphism of schemes such that $\pi_*\mathcal{O}_C$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_C = \bigoplus_{n \geq 0} \mathcal{A}_n$ such that $\mathcal{A}_0 = \mathcal{O}_S$. A *morphism of cones* from $\pi : C \to S$ to $\pi' : C' \to S$ is a morphism $f : C \to C'$ such that the induced map

$$f^* : \pi'_*\mathcal{O}_{C'} \longrightarrow \pi_*\mathcal{O}_C$$

is compatible with the given gradings.

Any vector bundle is an example of a cone. In fact the category of vector bundles over S is a full subcategory of the category of cones over S.

8. Proj of a graded ring

Let S be a graded ring. Consider the topological space $\text{Proj}(S)$ associated to S, see Algebra, Section 55. We will endow this space with a sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ such that the resulting pair $(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})$ will be a scheme.

Recall that $\text{Proj}(S)$ has a basis of open sets $D_+(f)$, $f \in S_d$, $d \geq 1$ which we call *standard opens*, see Algebra, Section 55. This terminology will always imply that f is homogeneous of positive degree even if we forget to mention it. In addition,

[2]Often one imposes the assumption that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{O}_S$. We do not assume this in order to be consistent with [DG67, II, (8.3.1)].

the intersection of two standard opens is another: $D_+(f) \cap D_+(g) = D_+(fg)$, for $f, g \in S$ homogeneous of positive degree.

Lemma 8.1. *Let S be a graded ring. Let $f \in S$ homogeneous of positive degree.*

1. *If $g \in S$ homogeneous of positive degree and $D_+(g) \subset D_+(f)$, then*
   - (a) *f is invertible in S_g, and $f^{\deg(g)}/g^{\deg(f)}$ is invertible in $S_{(g)}$,*
   - (b) *$g^e = af$ for some $e \geq 1$ and $a \in S$ homogeneous,*
   - (c) *there is a canonical S-algebra map $S_f \to S_g$,*
   - (d) *there is a canonical S_0-algebra map $S_{(f)} \to S_{(g)}$ compatible with the map $S_f \to S_g$,*
   - (e) *the map $S_{(f)} \to S_{(g)}$ induces an isomorphism*
     $$(S_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong S_{(g)},$$
   - (f) *these maps induce a commutative diagram of topological spaces*
     $$\begin{tikzcd} D_+(g) \arrow[d] & \{\mathbf{Z}\text{-graded primes of } S_g\} \arrow[l] \arrow[r] \arrow[d] & \mathrm{Spec}(S_{(g)}) \arrow[d] \\ D_+(f) & \{\mathbf{Z}\text{-graded primes of } S_f\} \arrow[l] \arrow[r] & \mathrm{Spec}(S_{(f)}) \end{tikzcd}$$
     where the horizontal maps are homeomorphisms and the vertical maps are open immersions,
   - (g) *there are compatible canonical S_f and $S_{(f)}$-module maps $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ for any graded S-module M, and*
   - (h) *the map $M_{(f)} \to M_{(g)}$ induces an isomorphism*
     $$(M_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong M_{(g)}.$$
2. *Any open covering of $D_+(f)$ can be refined to a finite open covering of the form $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$.*
3. *Let $g_1, \ldots, g_n \in S$ be homogeneous of positive degree. Then $D_+(f) \subset \bigcup D_+(g_i)$ if and only if $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$.*

Proof. Recall that $D_+(g) = \mathrm{Spec}(S_{(g)})$ with identification given by the ring maps $S \to S_g \leftarrow S_{(g)}$, see Algebra, Lemma 55.3. Thus $f^{\deg(g)}/g^{\deg(f)}$ is an element of $S_{(g)}$ which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma 16.2. We conclude that (a) holds. Write the inverse of f in S_g as a/g^d. We may replace a by its homogeneous part of degree $d\deg(g) - \deg(f)$. This means $g^d - af$ is annihilated by a power of g, whence $g^e = af$ for some $a \in S$ homogeneous of degree $e\deg(g) - \deg(f)$. This proves (b). For (c), the map $S_f \to S_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. This clearly induces a map of the subrings $S_{(f)} \to S_{(g)}$ of degree zero elements as well. We can similarly define $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ by mapping x/f^n to $a^n x/g^{ne}$. The statements writing $S_{(g)}$ resp. $M_{(g)}$ as principal localizations of $S_{(f)}$ resp. $M_{(f)}$ are clear from the formulas above. The maps in the commutative diagram of topological spaces correspond to the ring maps given above. The horizontal arrows are homeomorphisms by Algebra, Lemma 55.3. The vertical arrows are open immersions since the left one is the inclusion of an open subset.

The open $D_+(f)$ is quasi-compact because it is homeomorphic to $\mathrm{Spec}(S_{(f)})$, see Algebra, Lemma 16.10. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma 16.2. □

In Sheaves, Section 30 we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas 30.6 and 30.9. Moreover, we showed in Sheaves, Lemma 30.4 that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

Definition 8.2. Let S be a graded ring. Suppose that $D_+(f) \subset \mathrm{Proj}(S)$ is a standard open. A *standard open covering* of $D_+(f)$ is a covering $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, where $g_1, \ldots, g_n \in S$ are homogeneous of positive degree.

Let S be a graded ring. Let M be a graded S-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \mathrm{Proj}(S)$ is a standard open. If $f, g \in S$ are homogeneous of positive degree such that $D_+(f) = D_+(g)$, then by Lemma 8.1 above there are canonical maps $M_{(f)} \to M_{(g)}$ and $M_{(g)} \to M_{(f)}$ which are mutually inverse. Hence we may choose any f such that $U = D_+(f)$ and define

$$\widetilde{M}(U) = M_{(f)}.$$

Note that if $D_+(g) \subset D_+(f)$, then by Lemma 8.1 above we have a canonical map

$$\widetilde{M}(D_+(f)) = M_{(f)} \longrightarrow M_{(g)} = \widetilde{M}(D_+(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = S$, then $\widetilde{S}$ is a presheaf of rings on the basis of standard opens. And for general M we see that $\widetilde{M}$ is a presheaf of $\widetilde{S}$-modules on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \mathrm{Proj}(S)$. Suppose that x corresponds to the homogeneous prime ideal $\mathfrak{p} \subset S$. By definition of the stalk we see that

$$\widetilde{M}_x = \mathrm{colim}_{f \in S_d, d > 0, f \not\in \mathfrak{p}} M_{(f)}$$

Here the set $\{f \in S_d, d > 0, f \not\in \mathfrak{p}\}$ is partially ordered by the rule $f \geq f' \Leftrightarrow D_+(f) \subset D_+(f')$. If $f_1, f_2 \in S \setminus \mathfrak{p}$ are homogeneous of positive degree, then we have $f_1 f_2 \geq f_1$ in this ordering. In Algebra, Section 55 we defined $M_{(\mathfrak{p})}$ as the ring whose elements are fractions x/f with x, f homogeneous, $\deg(x) = \deg(f)$, $f \not\in \mathfrak{p}$. Since $\mathfrak{p} \in \mathrm{Proj}(S)$ there exists at least one $f_0 \in S$ homogeneous of positive degree with $f_0 \not\in \mathfrak{p}$. Hence $x/f = f_0 x/f f_0$ and we see that we may always assume the denominator of an element in $M_{(\mathfrak{p})}$ has positive degree. From these remarks it follows easily that

$$\widetilde{M}_x = M_{(\mathfrak{p})}.$$

Next, we check the sheaf condition for the standard open coverings. If $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(g_i)} \to \bigoplus M_{(g_i g_j)}.$$

Note that $D_+(g_i) = D_+(fg_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(fg_i)} \to \bigoplus M_{(fg_ig_j)}.$$

By Lemma 8.1 we see that $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$, and that the modules $M_{(fg_i)}$, $M_{(fg_ig_j)}$ are the principal localizations of the $S_{(f)}$-module $M_{(f)}$ at these elements and their products. Thus we may apply Algebra, Lemma 22.2 to the module $M_{(f)}$ over $S_{(f)}$ and the elements $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section 30 that there exists a unique sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ which agrees with $\widetilde{S}$ on the standard opens. Note that by our computation of stalks above and Algebra, Lemma 55.5 the stalks of this sheaf of rings are all local rings.

Similarly, for any graded S-module M there exists a unique sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma 30.12.

Definition 8.3. Let S be a graded ring.

(1) The *structure sheaf $\mathcal{O}_{Proj(S)}$ of the homogeneous spectrum of S* is the unique sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ which agrees with $\widetilde{S}$ on the basis of standard opens.
(2) The locally ringed space $(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})$ is called the *homogeneous spectrum* of S and denoted $\text{Proj}(S)$.
(3) The sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules extending $\widetilde{M}$ to all opens of $\text{Proj}(S)$ is called the sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

Lemma 8.4. *Let S be a graded ring. Let M be a graded S-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{Proj(S)}$-modules associated to M.*

(1) *For every $f \in S$ homogeneous of positive degree we have*
$$\Gamma(D_+(f), \mathcal{O}_{Proj(S)}) = S_{(f)}.$$
(2) *For every $f \in S$ homogeneous of positive degree we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$ as an $S_{(f)}$-module.*
(3) *Whenever $D_+(g) \subset D_+(f)$ the restriction mappings on $\mathcal{O}_{Proj(S)}$ and $\widetilde{M}$ are the maps $S_{(f)} \to S_{(g)}$ and $M_{(f)} \to M_{(g)}$ from Lemma 8.1.*
(4) *Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in Proj(S)$ be the corresponding point. We have $\mathcal{O}_{Proj(S),x} = S_{(\mathfrak{p})}$.*
(5) *Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in Proj(S)$ be the corresponding point. We have $\mathcal{F}_x = M_{(\mathfrak{p})}$ as an $S_{(\mathfrak{p})}$-module.*
(6) *There is a canonical ring map $S_0 \longrightarrow \Gamma(Proj(S), \widetilde{S})$ and a canonical S_0-module map $M_0 \longrightarrow \Gamma(Proj(S), \widetilde{M})$ compatible with the descriptions of sections over standard opens and stalks above.*

Moreover, all these identifications are functorial in the graded S-module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of graded S-modules to the category of $\mathcal{O}_{Proj(S)}$-modules.

Proof. Assertions (1) - (5) are clear from the discussion above. We see (6) since there are canonical maps $M_0 \to M_{(f)}$, $x \mapsto x/1$ compatible with the restriction maps described in (3). The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{(\mathfrak p)}$ is exact (see Algebra, Lemma 55.5) and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma 3.1. $\square$

Remark 8.5. The map from M_0 to the global sections of $\widetilde{M}$ is generally far from being an isomorphism. A trivial example is to take $S = k[x, y, z]$ with $1 = \deg(x) = \deg(y) = \deg(z)$ (or any number of variables) and to take $M = S/(x^{100}, y^{100}, z^{100})$. It is easy to see that $\widetilde{M} = 0$, but $M_0 = k$.

Lemma 8.6. *Let S be a graded ring. Let $f \in S$ be homogeneous of positive degree. Suppose that $D(g) \subset \text{Spec}(S_{(f)})$ is a standard open. Then there exists a $h \in S$ homogeneous of positive degree such that $D(g)$ corresponds to $D_+(h) \subset D_+(f)$ via the homeomorphism of Algebra, Lemma 55.3. In fact we can take h such that $g = h/f^n$ for some n.*

Proof. Write $g = h/f^n$ for some h homogeneous of positive degree and some $n \geq 1$. If $D_+(h)$ is not contained in $D_+(f)$ then we replace h by hf and n by $n+1$. Then h has the required shape and $D_+(h) \subset D_+(f)$ corresponds to $D(g) \subset \text{Spec}(S_{(f)})$. $\square$

Lemma 8.7. *Let S be a graded ring. The locally ringed space Proj(S) is a scheme. The standard opens $D_+(f)$ are affine opens. For any graded S-module M the sheaf $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{Proj(S)}$-modules.*

Proof. Consider a standard open $D_+(f) \subset \text{Proj}(S)$. By Lemmas 8.1 and 8.4 we have $\Gamma(D_+(f), \mathcal{O}_{\text{Proj}(S)}) = S_{(f)}$, and we have a homeomorphism $\varphi : D_+(f) \to \text{Spec}(S_{(f)})$. For any standard open $D(g) \subset \text{Spec}(S_{(f)})$ we may pick a $h \in S_+$ as in Lemma 8.6. Then $\varphi^{-1}(D(g)) = D_+(h)$, and by Lemmas 8.4 and 8.1 we see

$$\Gamma(D_+(h), \mathcal{O}_{\text{Proj}(S)}) = S_{(h)} = (S_{(f)})_{h^{\deg(f)}/f^{\deg(h)}} = (S_{(f)})_g = \Gamma(D(g), \mathcal{O}_{\text{Spec}(S_{(f)})}).$$

Thus the restriction of $\mathcal{O}_{\text{Proj}(S)}$ to $D_+(f)$ corresponds via the homeomorphism φ exactly to the sheaf $\mathcal{O}_{\text{Spec}(S_{(f)})}$ as defined in Schemes, Section 5. We conclude that $D_+(f)$ is an affine scheme isomorphic to $\text{Spec}(S_{(f)})$ via φ and hence that Proj(S) is a scheme.

In exactly the same way we show that $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules. Namely, the argument above will show that

$$\widetilde{M}|_{D_+(f)} \cong \varphi^* \left(\widetilde{M_{(f)}}\right)$$

which shows that $\widetilde{M}$ is quasi-coherent. $\square$

Lemma 8.8. *Let S be a graded ring. The scheme Proj(S) is separated.*

Proof. We have to show that the canonical morphism $\text{Proj}(S) \to \text{Spec}(\mathbf{Z})$ is separated. We will use Schemes, Lemma 21.8. Thus it suffices to show given any pair of standard opens $D_+(f)$ and $D_+(g)$ that $D_+(f) \cap D_+(g) = D_+(fg)$ is affine (clear) and that the ring map

$$S_{(f)} \otimes_{\mathbf{Z}} S_{(g)} \longrightarrow S_{(fg)}$$

is surjective. Any element s in $S_{(fg)}$ is of the form $s = h/(f^n g^m)$ with $h \in S$ homogeneous of degree $n \deg(f) + m \deg(g)$. We may multiply h by a suitable monomial $f^i g^j$ and assume that $n = n' \deg(g)$, and $m = m' \deg(f)$. Then we can rewrite s as $s = h/f^{(n'+m')\deg(g)} \cdot f^{m'\deg(g)}/g^{m'\deg(f)}$. So s is indeed in the image of the displayed arrow. $\square$

Lemma 8.9. *Let S be a graded ring. The scheme Proj(S) is quasi-compact if and only if there exist finitely many homogeneous elements $f_1, \ldots, f_n \in S_+$ such that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$. In this case Proj$(S) = D_+(f_1) \cup \ldots \cup D_+(f_n)$.*

Proof. Given such a collection of elements the standard affine opens $D_+(f_i)$ cover $\text{Proj}(S)$ by Algebra, Lemma 55.3. Conversely, if $\text{Proj}(S)$ is quasi-compact, then we may cover it by finitely many standard opens $D_+(f_i)$, $i = 1, \ldots, n$ and we see that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$ by the lemma referenced above. $\square$

Lemma 8.10. *Let S be a graded ring. The scheme Proj(S) has a canonical morphism towards the affine scheme* $\text{Spec}(S_0)$*, agreeing with the map on topological spaces coming from Algebra, Definition 55.1.*

Proof. We saw above that our construction of $\widetilde{S}$, resp. $\widetilde{M}$ gives a sheaf of S_0-algebras, resp. S_0-modules. Hence we get a morphism by Schemes, Lemma 6.4. This morphism, when restricted to $D_+(f)$ comes from the canonical ring map $S_0 \to S_{(f)}$. The maps $S \to S_f$, $S_{(f)} \to S_f$ are S_0-algebra maps, see Lemma 8.1. Hence if the homogeneous prime $\mathfrak{p} \subset S$ corresponds to the $\mathbf{Z}$-graded prime $\mathfrak{p}' \subset S_f$ and the (usual) prime $\mathfrak{p}'' \subset S_{(f)}$, then each of these has the same inverse image in S_0. $\square$

Lemma 8.11. *Let S be a graded ring. If S is finitely generated as an algebra over S_0, then the morphism Proj$(S) \to$* $\text{Spec}(S_0)$ *satisfies the existence and uniqueness parts of the valuative criterion, see Schemes, Definition 20.3.*

Proof. The uniqueness part follows from the fact that $\text{Proj}(S)$ is separated (Lemma 8.8 and Schemes, Lemma 22.1). Choose $x_i \in S_+$ homogeneous, $i = 1, \ldots, n$ which generate S over S_0. Let $d_i = \deg(x_i)$ and set $d = \text{lcm}\{d_i\}$. Suppose we are given a diagram

$$\begin{tikzcd} \text{Spec}(K) \arrow[r] \arrow[d] & \text{Proj}(S) \arrow[d] \\ \text{Spec}(A) \arrow[r] & \text{Spec}(S_0) \end{tikzcd}$$

as in Schemes, Definition 20.3. Denote $v : K^* \to \Gamma$ the valuation of A, see Algebra, Definition 48.13. We may choose an $f \in S_+$ homogeneous such that $\text{Spec}(K)$ maps into $D_+(f)$. Then we get a commutative diagram of ring maps

$$\begin{tikzcd} K & S_{(f)} \arrow[l, "\varphi"'] \\ A \arrow[u] & S_0 \arrow[l] \arrow[u] \end{tikzcd}$$

Let $i_0 \in \{1, \ldots, n\}$ be an index minimizing the valuation $(d/d_i)v(\varphi(x_i^{\deg(f)}/f^{d_i}))$ where we temporarily use the convention that the valuation of zero is bigger than any element of the value group. For convenience set $x_0 = x_{i_0}$ and $d_0 = d_{i_0}$. Since the open sets $D_+(x_i)$ cover $\text{Proj}(S)$ we see that $\varphi(x_0) \neq 0$. This means that the ring map φ factors though a map $\varphi' : S_{(fx_0)} \to K$. We see that

$$\deg(f)v(\varphi'(x_i^{d_0}/x_0^{d_i})) = d_0 v(\varphi(x_i^{\deg(f)}/f^{d_i})) - d_i v(\varphi(x_0^{\deg(f)}/f^{d_0})) \geq 0$$

by our choice of i_0. This implies that the S_0-algebra $S_{(x_0)}$, which is generated by the elements $x_i^{d_0}/x_0^{d_i}$ over S_0, maps into A via φ'. The corresponding morphism of schemes $\text{Spec}(A) \to \text{Spec}(S_{(x_0)}) = D_+(x_0) \subset \text{Proj}(S)$ provides the morphism fitting into the first commutative diagram of this proof. □

We saw in the proof of Lemma 8.11 that, under the hypotheses of that lemma, the morphism $\text{Proj}(S) \to \text{Spec}(S_0)$ is quasi-compact as well. Hence (by Schemes, Proposition 20.6) we see that $\text{Proj}(S) \to \text{Spec}(S_0)$ is universally closed in the situation of the lemma. We give two examples showing these results do not hold without some assumption on the graded ring S.

Example 8.12. Let $\mathbf{C}[X_1, X_2, X_3, \ldots]$ be the graded $\mathbf{C}$-algebra with each X_i in degree 0. Consider the ring map

$$\mathbf{C}[X_1, X_2, X_3, \ldots] \longrightarrow \mathbf{C}[t^\alpha; \alpha \in \mathbf{Q}_{\geq 0}]$$

which maps X_i to $t^{1/i}$. The right hand side becomes a valuation ring A upon localization at the ideal $\mathfrak{m} = (t^\alpha; \alpha > 0)$. This gives a morphism from $\text{Spec}(f.f.(A))$ to $\text{Proj}(\mathbf{C}[X_1, X_2, X_3, \ldots])$ which does not extend to a morphism defined on all of $\text{Spec}(A)$. The reason is that the image of $\text{Spec}(A)$ would be contained in one of the $D_+(X_i)$ but then X_{i+1}/X_i would map to an element of A which it doesn't since it maps to $t^{1/(i+1)-1/i}$.

Example 8.13. Let $R = \mathbf{C}[t]$ and

$$S = R[X_1, X_2, X_3, \ldots]/(X_i^2 - tX_{i+1}).$$

The grading is such that $R = S_0$ and $\deg(X_i) = 2^{i-1}$. Note that if $\mathfrak{p} \in \text{Proj}(S)$ then $t \notin \mathfrak{p}$ (otherwise $\mathfrak{p}$ has to contain all of the X_i which is not allowed for an element of the homogeneous spectrum). Thus we see that $D_+(X_i) = D_+(X_{i+1})$ for all i. Hence $\text{Proj}(S)$ is quasi-compact; in fact it is affine since it is equal to $D_+(X_1)$. It is easy to see that the image of $\text{Proj}(S) \to \text{Spec}(R)$ is $D(t)$. Hence the morphism $\text{Proj}(S) \to \text{Spec}(R)$ is not closed. Thus the valuative criterion cannot apply because it would imply that the morphism is closed (see Schemes, Proposition 20.6).

Example 8.14. Let A be a ring. Let $S = A[T]$ as a graded A algebra with T in degree 1. Then the canonical morphism $\text{Proj}(S) \to \text{Spec}(A)$ (see Lemma 8.10) is an isomorphism.

9. Quasi-coherent sheaves on Proj

Let S be a graded ring. Let M be a graded S-module. We saw in Lemma 8.4 how to construct a quasi-coherent sheaf of modules $\widetilde{M}$ on $\text{Proj}(S)$ and a map

$$M_0 \longrightarrow \Gamma(\text{Proj}(S), \widetilde{M}) \tag{9.0.1}$$

of the degree 0 part of M to the global sections of $\widetilde{M}$. The degree 0 part of the nth twist $M(n)$ of the graded module M (see Algebra, Section 54) is equal to M_n. Hence we can get maps

$$M_n \longrightarrow \Gamma(\text{Proj}(S), \widetilde{M(n)}). \tag{9.0.2}$$

We would like to be able to perform this operation for any quasi-coherent sheaf $\mathcal{F}$ on $\text{Proj}(S)$. We will do this by tensoring with the nth twist of the structure sheaf, see Definition 10.1. In order to relate the two notions we will use the following lemma.

Lemma 9.1. *Let S be a graded ring. Let $(X, \mathcal{O}_X) = (Proj(S), \mathcal{O}_{Proj(S)})$ be the scheme of Lemma 8.7. Let $f \in S_+$ be homogeneous. Let $x \in X$ be a point corresponding to the homogeneous prime $\mathfrak{p} \subset S$. Let M, N be graded S-modules. There is a canonical map of $\mathcal{O}_{Proj(S)}$-modules*

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

which induces the canonical map $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$ on sections over $D_+(f)$ and the canonical map $M_{(\mathfrak{p})} \otimes_{S_{(\mathfrak{p})}} N_{(\mathfrak{p})} \to (M \otimes_S N)_{(\mathfrak{p})}$ on stalks at x. Moreover, the following diagram

$$\begin{tikzcd} M_0 \otimes_{S_0} N_0 \arrow[r] \arrow[d] & (M \otimes_S N)_0 \arrow[d] \\ \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) \arrow[r] & \Gamma(X, \widetilde{M \otimes_R N}) \end{tikzcd}$$

is commutative where the vertical maps are given by (9.0.1).

Proof. To construct a morphism as displayed is the same as constructing a $\mathcal{O}_X$-bilinear map

$$\widetilde{M} \times \widetilde{N} \longrightarrow \widetilde{M \otimes_R N}$$

see Modules, Section 15. It suffices to define this on sections over the opens $D_+(f)$ compatible with restriction mappings. On $D_+(f)$ we use the $S_{(f)}$-bilinear map $M_{(f)} \times N_{(f)} \to (M \otimes_S N)_{(f)}$, $(x/f^n, y/f^m) \mapsto (x \otimes y)/f^{n+m}$. Details omitted. $\square$

Remark 9.2. In general the map constructed in Lemma 9.1 above is not an isomorphism. Here is an example. Let k be a field. Let $S = k[x, y, z]$ with k in degree 0 and $\deg(x) = 1$, $\deg(y) = 2$, $\deg(z) = 3$. Let $M = S(1)$ and $N = S(2)$, see Algebra, Section 54 for notation. Then $M \otimes_S N = S(3)$. Note that

$$\begin{aligned} S_z &= k[x, y, z, 1/z] \\ S_{(z)} &= k[x^3/z, xy/z, y^3/z^2] \cong k[u, v, w]/(uw - v^3) \\ M_{(z)} &= S_{(z)} \cdot x + S_{(z)} \cdot y^2/z \subset S_z \\ N_{(z)} &= S_{(z)} \cdot y + S_{(z)} \cdot x^2 \subset S_z \\ S(3)_{(z)} &= S_{(z)} \cdot z \subset S_z \end{aligned}$$

Consider the maximal ideal $\mathfrak{m} = (u, v, w) \subset S_{(z)}$. It is not hard to see that both $M_{(z)}/\mathfrak{m}M_{(z)}$ and $N_{(z)}/\mathfrak{m}N_{(z)}$ have dimension 2 over $\kappa(\mathfrak{m})$. But $S(3)_{(z)}/\mathfrak{m}S(3)_{(z)}$ has dimension 1. Thus the map $M_{(z)} \otimes N_{(z)} \to S(3)_{(z)}$ is not an isomorphism.

10. Invertible sheaves on Proj

Recall from Algebra, Section 54 the construction of the twisted module $M(n)$ associated to a graded module over a graded ring.

Definition 10.1. Let S be a graded ring. Let $X = \text{Proj}(S)$.

1. We define $\mathcal{O}_X(n) = \widetilde{S(n)}$. This is called the nth *twist of the structure sheaf of Proj(S)*.
2. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

We are going to use Lemma 9.1 to construct some canonical maps. Since $S(n) \otimes_S S(m) = S(n+m)$ we see that there are canonical maps

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n+m). \tag{10.1.1}$$

These maps are not isomorphisms in general, see the example in Remark 9.2. The same example shows that $\mathcal{O}_X(n)$ is *not* an invertible sheaf on X in general. Tensoring with an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ we get maps

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \longrightarrow \mathcal{F}(n+m). \tag{10.1.2}$$

The maps (10.1.1) on global sections give a map of graded rings

$$S \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)). \tag{10.1.3}$$

And for an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ the maps (10.1.2) give a graded module structure

$$\bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)) \times \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m)) \longrightarrow \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m)) \tag{10.1.4}$$

and via (10.1.3) also a S-module structure. More generally, given any graded S-module M we have $M(n) = M \otimes_S S(n)$. Hence we get maps

$$\widetilde{M}(n) = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \longrightarrow \widetilde{M(n)}. \tag{10.1.5}$$

On global sections we get a map of graded S-modules

$$M \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M(n)}). \tag{10.1.6}$$

Here is an important fact which follows basically immediately from the definitions.

Lemma 10.2. *Let S be a graded ring. Set $X = Proj(S)$. Let $f \in S$ be homogeneous of degree $d > 0$. The sheaves $\mathcal{O}_X(nd)|_{D_+(f)}$ are invertible, and in fact trivial for all $n \in \mathbf{Z}$ (see Modules, Definition 21.1). The maps (10.1.1) restricted to $D_+(f)$*

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(nd+m)|_{D_+(f)},$$

the maps (10.1.2) restricted to $D_+(f)$

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{F}(m)|_{D_+(f)} \longrightarrow \mathcal{F}(nd+m)|_{D_+(f)},$$

and the maps (10.1.5) restricted to $D_+(f)$

$$\widetilde{M}(nd)|_{D_+(f)} = \widetilde{M}|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(nd)|_{D_+(f)} \longrightarrow \widetilde{M(nd)}|_{D_+(f)}$$

are isomorphisms for all $n, m \in \mathbf{Z}$.

Proof. The (not graded) S-module maps $S \to S(n)$, and $M \to M(n)$, given by $x \mapsto f^{n/d}x$ become isomorphisms after inverting f. The first shows that $S_{(f)} \cong S(n)_{(f)}$ which gives an isomorphism $\mathcal{O}_{D_+(f)} \cong \mathcal{O}_X(n)|_{D_+(f)}$. The second shows that the map $S(n)_{(f)} \otimes_{S_{(f)}} M_{(f)} \to M(n)_{(f)}$ is an isomorphism. The case of the map (10.1.2) is a consequence of the case of the map (10.1.1). $\square$

Lemma 10.3. *Let S be a graded ring. Let M be a graded S-module. Set $X = Proj(S)$. If S is generated by S_1 over S_0, then the sheaves $\mathcal{O}_X(n)$ are invertible and the maps (10.1.1), (10.1.2), and (10.1.5) are isomorphisms. In particular, these maps induce isomorphisms*

$$\mathcal{O}_X(1)^{\otimes n} \cong \mathcal{O}_X(n) \quad \textit{and} \quad \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \widetilde{M}(n) \cong \widetilde{M(n)}$$

Thus (9.0.2) becomes a map

$$M_n \longrightarrow \Gamma(X, \widetilde{M}(n)) \tag{10.3.1}$$

and (10.1.6) becomes a map

$$M \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M}(n)). \tag{10.3.2}$$

In fact these results hold more generally if X is covered by the standard opens $D_+(f)$ with $f \in S_1$.

Proof. Under the assumptions of the lemma X is covered by the open subsets $D_+(f)$ with $f \in S_1$ and the lemma is a consequence of Lemma 10.2 above. $\square$

Lemma 10.4. *Let S be a graded ring. Set $X = Proj(S)$. Fix $d \geq 1$ an integer. The following open subsets of X are equal:*

1. *The largest open subset $W = W_d \subset X$ such that each $\mathcal{O}_X(dn)|_W$ is invertible and all the multiplication maps $\mathcal{O}_X(nd)|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(md)|_W \to \mathcal{O}_X(nd + md)|_W$ (see 10.1.1) are isomorphisms.*
2. *The union of the open subsets $D_+(fg)$ with $f, g \in S$ homogeneous and* $\deg(f) = \deg(g) + d$.

Moreover, all the maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 10.1.5) are isomorphisms.

Proof. If $x \in D_+(fg)$ with $\deg(f) = \deg(g) + d$ then on $D_+(fg)$ the sheaves $\mathcal{O}_X(dn)$ are generated by the element $(f/g)^n = f^{2n}/(fg)^n$. This implies x is in the open subset W defined in (1) by arguing as in the proof of Lemma 10.2.

Conversely, suppose that $\mathcal{O}_X(d)$ is free of rank 1 in an open neighbourhood V of $x \in X$ and all the multiplication maps $\mathcal{O}_X(nd)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_X(md)|_V \to \mathcal{O}_X(nd+md)|_V$ are isomorphisms. We may choose $h \in S_+$ homogeneous such that $D_+(h) \subset V$. By the definition of the twists of the structure sheaf we conclude there exists an element s of $(S_h)_d$ such that s^n is a basis of $(S_h)_{nd}$ as a module over $S_{(h)}$ for all $n \in \mathbf{Z}$. We may write $s = f/h^m$ for some $m \geq 1$ and $f \in S_{d+m\deg(h)}$. Set $g = h^m$ so $s = f/g$. Note that $x \in D(g)$ by construction. Note that $g^d \in (S_h)_{-d\deg(g)}$. By assumption we can write this as a multiple of $s^{\deg(g)} = f^{\deg(g)}/g^{\deg(g)}$, say $g^d = a/g^e \cdot f^{\deg(g)}/g^{\deg(g)}$. Then we conclude that $g^{d+e+\deg(g)} = af^{\deg(g)}$ and hence also $x \in D_+(f)$. So x is an element of the set defined in (2).

The existence of the generating section $s = f/g$ over the affine open $D_+(fg)$ whose powers freely generate the sheaves of modules $\mathcal{O}_X(nd)$ easily implies that the multiplication maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 10.1.5) are isomorphisms. Compare with the proof of Lemma 10.2. $\square$

Recall from Modules, Lemma 21.7 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x\mathcal{L}_x\}$ is open.

Lemma 10.5. *Let S be a graded ring. Set $X = Proj(S)$. Fix $d \geq 1$ an integer. Let $W = W_d \subset X$ be the open subscheme defined in Lemma 10.4. Let $n \geq 1$ and $f \in S_{nd}$. Denote $s \in \Gamma(W, \mathcal{O}_W(nd))$ the section which is the image of f via (10.1.3) restricted to W. Then*

$$W_s = D_+(f) \cap W.$$

Proof. Let $D_+(ab) \subset W$ be a standard affine open with $a, b \in S$ homogeneous and $\deg(a) = \deg(b) + d$. Note that $D_+(ab) \cap D_+(f) = D_+(abf)$. On the other hand the restriction of s to $D_+(ab)$ corresponds to the element $f/1 = b^n f/a^n (a/b)^n \in (S_{ab})_{nd}$. We have seen in the proof of Lemma 10.4 that $(a/b)^n$ is a generator for $\mathcal{O}_W(nd)$ over $D_+(ab)$. We conclude that $W_s \cap D_+(ab)$ is the principal open associated to $b^n f/a^n \in \mathcal{O}_X(D_+(ab))$. Thus the result of the lemma is clear. $\square$

The following lemma states the properties that we will later use to characterize schemes with an ample invertible sheaf.

Lemma 10.6. *Let S be a graded ring. Let $X = Proj(S)$. Let $Y \subset X$ be a quasi-compact open subscheme. Denote $\mathcal{O}_Y(n)$ the restriction of $\mathcal{O}_X(n)$ to Y. There exists an integer $d \geq 1$ such that*

(1) *the subscheme Y is contained in the open W_d defined in Lemma 10.4,*
(2) *the sheaf $\mathcal{O}_Y(dn)$ is invertible for all $n \in \mathbf{Z}$,*
(3) *all the maps $\mathcal{O}_Y(nd) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \longrightarrow \mathcal{O}_Y(nd + m)$ of Equation (10.1.1) are isomorphisms,*
(4) *all the maps $\widetilde{M}(nd)|_Y = \widetilde{M}|_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_X(nd)|_Y \to \widetilde{M(nd)}|_Y$ (see 10.1.5) are isomorphisms,*
(5) *given $f \in S_{nd}$ denote $s \in \Gamma(Y, \mathcal{O}_Y(nd))$ the image of f via (10.1.3) restricted to Y, then $D_+(f) \cap Y = Y_s$,*
(6) *a basis for the topology on Y is given by the collection of opens Y_s, where $s \in \Gamma(Y, \mathcal{O}_Y(nd))$, $n \geq 1$, and*
(7) *a basis for the topology of Y is given by those opens $Y_s \subset Y$, for $s \in \Gamma(Y, \mathcal{O}_Y(nd))$, $n \geq 1$ which are affine.*

Proof. Since Y is quasi-compact there exist finitely many homogeneous $f_i \in S_+$, $i = 1, \ldots, n$ such that the standard opens $D_+(f_i)$ give an open covering of Y. Let $d_i = \deg(f_i)$ and set $d = d_1 \ldots d_n$. Note that $D_+(f_i) = D_+(f_i^{d/d_i})$ and hence we see immediately that $Y \subset W_d$, by characterization (2) in Lemma 10.4 or by (1) using Lemma 10.2. Note that (1) implies (2), (3) and (4) by Lemma 10.4. (Note that (3) is a special case of (4).) Assertion (5) follows from Lemma 10.5. Assertions (6) and (7) follow because the open subsets $D_+(f)$ form a basis for the topology of X and are affine. $\square$

11. Functoriality of Proj

A graded ring map $\psi : A \to B$ does not always give rise to a morphism of associated projective homogeneous spectra. The reason is that the inverse image $\psi^{-1}(\mathfrak{q})$ of a homogeneous prime $\mathfrak{q} \subset B$ may contain the irrelevant prime A_+ even if $\mathfrak{q}$ does not contain B_+. The correct result is stated as follows.

Lemma 11.1. *Let A, B be two graded rings. Set $X = Proj(A)$ and $Y = Proj(B)$. Let $\psi : A \to B$ be a graded ring map. Set*

$$U(\psi) = \bigcup\nolimits_{f \in A_+ \ homogeneous} D_+(\psi(f)) \subset Y.$$

Then there is a canonical morphism of schemes

$$r_\psi : U(\psi) \longrightarrow X$$

and a map of $\mathbf{Z}$*-graded* $\mathcal{O}_{U(\psi)}$*-algebras*

$$\theta = \theta_\psi : r_\psi^*\left(\bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_X(d)\right) \longrightarrow \bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ *is characterized by the following properties:*

1. *For every* $d \geq 0$ *the diagram*
$$\begin{tikzcd} A_d \arrow[rr, "\psi"] \arrow[d] & & B_d \arrow[d] \\ \Gamma(X, \mathcal{O}_X(d)) \arrow[r, "\theta"] & \Gamma(U(\psi), \mathcal{O}_Y(d)) & \Gamma(Y, \mathcal{O}_Y(d)) \arrow[l] \end{tikzcd}$$
is commutative.
2. *For any* $f \in A_+$ *homogeneous we have* $r_\psi^{-1}(D_+(f)) = D_+(\psi(f))$ *and the restriction of* r_ψ *to* $D_+(\psi(f))$ *corresponds to the ring map* $A_{(f)} \to B_{(\psi(f))}$ *induced by* ψ.

Proof. Clearly condition (2) uniquely determines the morphism of schemes and the open subset $U(\psi)$. Pick $f \in A_d$ with $d \geq 1$. Note that $\mathcal{O}_X(n)|_{D_+(f)}$ corresponds to the $A_{(f)}$-module $(A_f)_n$ and that $\mathcal{O}_Y(n)|_{D_+(\psi(f))}$ corresponds to the $B_{(\psi(f))}$-module $(B_{\psi(f)})_n$. In other words θ when restricted to $D_+(\psi(f))$ corresponds to a map of $\mathbf{Z}$-graded $B_{(\psi(f))}$-algebras

$$A_f \otimes_{A_{(f)}} B_{(\psi(f))} \longrightarrow B_{\psi(f)}$$

Condition (1) determines the images of all elements of A. Since f is an invertible element which is mapped to $\psi(f)$ we see that $1/f^m$ is mapped to $1/\psi(f)^m$. It easily follows from this that θ is uniquely determined, namely it is given by the rule

$$a/f^m \otimes b/\psi(f)^e \longmapsto \psi(a)b/\psi(f)^{m+e}.$$

To show existence we remark that the proof of uniqueness above gave a well defined prescription for the morphism r and the map θ when restricted to every standard open of the form $D_+(\psi(f)) \subset U(\psi)$ into $D_+(f)$. Call these r_f and θ_f. Hence we only need to verify that if $D_+(f) \subset D_+(g)$ for some $f, g \in A_+$ homogeneous, then the restriction of r_g to $D_+(\psi(f))$ matches r_f. This is clear from the formulas given for r and θ above. $\square$

Lemma 11.2. *Let* A*,* B*, and* C *be graded rings. Set* $X = Proj(A)$*,* $Y = Proj(B)$ *and* $Z = Proj(C)$*. Let* $\varphi : A \to B$*,* $\psi : B \to C$ *be graded ring maps. Then we have*

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi\circ\varphi} = r_\varphi \circ r_\psi|_{U(\psi\circ\varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^*\theta_\varphi = \theta_{\psi\circ\varphi}$$

with obvious notation.

Proof. Omitted. $\square$

Lemma 11.3. *With hypotheses and notation as in Lemma 11.1 above. Assume* $A_d \to B_d$ *is surjective for all* $d \gg 0$*. Then*

1. $U(\psi) = Y$*,*
2. $r_\psi : Y \to X$ *is a closed immersion, and*
3. *the maps* $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ *are surjective but not isomorphisms in general (even if* $A \to B$ *is surjective).*

Proof. Part (1) follows from the definition of $U(\psi)$ and the fact that $D_+(f) = D_+(f^n)$ for any $n > 0$. For $f \in A_+$ homogeneous we see that $A_{(f)} \to B_{(\psi(f))}$ is surjective because any element of $B_{(\psi(f))}$ can be represented by a fraction $b/\psi(f)^n$ with n arbitrarily large (which forces the degree of $b \in B$ to be large). This proves (2). The same argument shows the map

$$A_f \to B_{\psi(f)}$$

is surjective which proves the surjectivity of θ. For an example where this map is not an isomorphism consider the graded ring $A = k[x, y]$ where k is a field and $\deg(x) = 1$, $\deg(y) = 2$. Set $I = (x)$, so that $B = k[y]$. Note that $\mathcal{O}_Y(1) = 0$ in this case. But it is easy to see that $r_\psi^*\mathcal{O}_Y(1)$ is not zero. (There are less silly examples.) □

Lemma 11.4. *With hypotheses and notation as in Lemma 11.1 above. Assume $A_d \to B_d$ is an isomorphism for all $d \gg 0$. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is an isomorphism, and*
3. *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. We have (1) by Lemma 11.3. Let $f \in A_+$ be homogeneous. The assumption on ψ implies that $A_f \to B_f$ is an isomorphism (details omitted). Thus it is clear that r_ψ and θ restrict to isomorphisms over $D_+(f)$. The lemma follows. □

Lemma 11.5. *With hypotheses and notation as in Lemma 11.1 above. Assume $A_d \to B_d$ is surjective for $d \gg 0$ and that A is generated by A_1 over A_0. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is a closed immersion, and*
3. *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. By Lemmas 11.4 and 11.2 we may replace B by the image of $A \to B$ without changing X or the sheaves $\mathcal{O}_X(n)$. Thus we may assume that $A \to B$ is surjective. By Lemma 11.3 we get (1) and (2) and surjectivity in (3). By Lemma 10.3 we see that both $\mathcal{O}_X(n)$ and $\mathcal{O}_Y(n)$ are invertible. Hence θ is an isomorphism. □

Lemma 11.6. *With hypotheses and notation as in Lemma 11.1 above. Assume there exists a ring map $R \to A_0$ and a ring map $R \to R'$ such that $B = R' \otimes_R A$. Then*

1. *$U(\psi) = Y$,*
2. *the diagram*

$$\begin{tikzcd} Y = Proj(B) \arrow[r, "r_\psi"'] \arrow[d] & Proj(A) = X \arrow[d] \\ \mathrm{Spec}(R') \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

   is a fibre product square, and

3. *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. This follows immediately by looking at what happens over the standard opens $D_+(f)$ for $f \in A_+$. □

Lemma 11.7. *With hypotheses and notation as in Lemma 11.1 above. Assume there exists a $g \in A_0$ such that ψ induces an isomorphism $A_g \to B$. Then $U(\psi) = Y$, $r_\psi : Y \to X$ is an open immersion which induces an isomorphism of Y with the inverse image of $D(g) \subset \mathrm{Spec}(A_0)$. Moreover the map θ is an isomorphism.*

Proof. This is a special case of Lemma 11.6 above. $\square$

12. Morphisms into Proj

Let S be a graded ring. Let $X = \mathrm{Proj}(S)$ be the homogeneous spectrum of S. Let $d \geq 1$ be an integer. Consider the open subscheme

$$U_d = \bigcup\nolimits_{f \in S_d} D_+(f) \quad \subset \quad X = \mathrm{Proj}(S) \tag{12.0.1}$$

Note that $d|d' \Rightarrow U_d \subset U_{d'}$ and $X = \bigcup_d U_d$. Neither X nor U_d need be quasi-compact, see Algebra, Lemma 55.3. Let us write $\mathcal{O}_{U_d}(n) = \mathcal{O}_X(n)|_{U_d}$. By Lemma 10.2 we know that $\mathcal{O}_{U_d}(nd)$, $n \in \mathbf{Z}$ is an invertible $\mathcal{O}_{U_d}$-module and that all the multiplication maps $\mathcal{O}_{U_d}(nd) \otimes_{\mathcal{O}_{U_d}} \mathcal{O}_X(m) \to \mathcal{O}_{U_d}(nd+m)$ of (10.1.1) are isomorphisms. In particular we have $\mathcal{O}_{U_d}(nd) \cong \mathcal{O}_{U_d}(d)^{\otimes n}$. The graded ring map (10.1.3) on global sections combined with restriction to U_d give a homomorphism of graded rings

$$\psi^d : S^{(d)} \longrightarrow \Gamma_*(U_d, \mathcal{O}_{U_d}(d)). \tag{12.0.2}$$

For the notation $S^{(d)}$, see Algebra, Section 54. For the notation Γ_* see Modules, Definition 21.4. Moreover, since U_d is covered by the opens $D_+(f)$, $f \in S_d$ we see that $\mathcal{O}_{U_d}(d)$ is globally generated by the sections in the image of $\psi^d_1 : S^{(d)}_1 = S_d \to \Gamma(U_d, \mathcal{O}_{U_d}(d))$, see Modules, Definition 4.1.

Let Y be a scheme, and let $\varphi : Y \to X$ be a morphism of schemes. Assume the image $\varphi(Y)$ is contained in the open subscheme U_d of X. By the discussion following Modules, Definition 21.4 we obtain a homomorphism of graded rings

$$\Gamma_*(U_d, \mathcal{O}_{U_d}(d)) \longrightarrow \Gamma_*(Y, \varphi^*\mathcal{O}_X(d)).$$

The composition of this and ψ^d gives a graded ring homomorphism

$$\psi^d_\varphi : S^{(d)} \longrightarrow \Gamma_*(Y, \varphi^*\mathcal{O}_X(d)) \tag{12.0.3}$$

which has the property that the invertible sheaf $\varphi^*\mathcal{O}_X(d)$ is globally generated by the sections in the image of $(S^{(d)})_1 = S_d \to \Gamma(Y, \varphi^*\mathcal{O}_X(d))$.

Lemma 12.1. *Let S be a graded ring, and $X = Proj(S)$. Let $d \geq 1$ and $U_d \subset X$ as above. Let Y be a scheme. Let $\mathcal{L}$ be an invertible sheaf on Y. Let $\psi : S^{(d)} \to \Gamma_*(Y, \mathcal{L})$ be a graded ring homomorphism such that $\mathcal{L}$ is generated by the sections in the image of $\psi|_{S_d} : S_d \to \Gamma(Y, \mathcal{L})$. Then there exists a morphism $\varphi : Y \to X$ such that $\varphi(Y) \subset U_d$ and an isomorphism $\alpha : \varphi^*\mathcal{O}_{U_d}(d) \to \mathcal{L}$ such that ψ^d_φ agrees with ψ via α:*

$$\begin{tikzcd}
\Gamma_*(Y, \mathcal{L}) & \Gamma_*(Y, \varphi^*\mathcal{O}_{U_d}(d)) \arrow[l, "\alpha"'] & \Gamma_*(U_d, \mathcal{O}_{U_d}(d)) \arrow[l, "\varphi^*"'] \\
S^{(d)} \arrow[u, "\psi"] & & S^{(d)} \arrow[ll, "id"] \arrow[ul, "\psi^d_\varphi"] \arrow[u, "\psi^d"']
\end{tikzcd}$$

commutes. Moreover, the pair (φ, α) is unique.

Proof. Pick $f \in S_d$. Denote $s = \psi(f) \in \Gamma(Y, \mathcal{L})$. On the open set Y_s where s does not vanish multiplication by s induces an isomorphism $\mathcal{O}_{Y_s} \to \mathcal{L}|_{Y_s}$, see Modules, Lemma 21.7. We will denote the inverse of this map $x \mapsto x/s$, and similarly for powers of $\mathcal{L}$. Using this we define a ring map $\psi_{(f)} : S_{(f)} \to \Gamma(Y_s, \mathcal{O})$ by mapping the fraction a/f^n to $\psi(a)/s^n$. By Schemes, Lemma 6.4 this corresponds to a morphism $\varphi_f : Y_s \to \mathrm{Spec}(S_{(f)}) = D_+(f)$. We also introduce the isomorphism $\alpha_f : \varphi_f^*\mathcal{O}_{D_+(f)}(d) \to \mathcal{L}|_{Y_s}$ which maps the pullback of the trivializing section f over $D_+(f)$ to the trivializing section s over Y_s. With this choice the commutativity of the diagram in the lemma holds with Y replace by Y_s, φ replaced by φ_f, and α replaced by α_f; verification omitted.

Suppose that $f' \in S_d$ is a second element, and denote $s' = \psi(f') \in \Gamma(Y, \mathcal{L})$. Then $Y_s \cap Y_{s'} = Y_{ss'}$ and similarly $D_+(f) \cap D_+(f') = D_+(ff')$. In Lemma 10.6 we saw that $D_+(f') \cap D_+(f)$ is the same as the set of points of $D_+(f)$ where the section of $\mathcal{O}_X(d)$ defined by f' does not vanish. Hence $\varphi_f^{-1}(D_+(f') \cap D_+(f)) = Y_s \cap Y_{s'} = \varphi_{f'}^{-1}(D_+(f') \cap D_+(f))$. On $D_+(f) \cap D_+(f')$ the fraction f/f' is an invertible section of the structure sheaf with inverse f'/f. Note that $\psi_{(f')}(f/f') = \psi(f)/s' = s/s'$ and $\psi_{(f)}(f'/f) = \psi(f')/s = s'/s$. We claim there is a unique ring map $S_{(ff')} \to \Gamma(Y_{ss'}, \mathcal{O})$ making the following diagram commute

$$\begin{tikzcd}
\Gamma(Y_s, \mathcal{O}) \arrow[r] & \Gamma(Y_{ss'}, \mathcal{O}) & \Gamma(Y_{s,'} \mathcal{O}) \arrow[l] \\
S_{(f)} \arrow[u, "\psi_{(f)}"] \arrow[r] & S_{(ff')} \arrow[u] & S_{(f')} \arrow[l] \arrow[u, "\psi_{(f')}"']
\end{tikzcd}$$

It exists because we may use the rule $x/(ff')^n \mapsto \psi(x)/(ss')^n$, which "works" by the formulas above. Uniqueness follows as $\mathrm{Proj}(S)$ is separated, see Lemma 8.8 and its proof. This shows that the morphisms φ_f and $\varphi_{f'}$ agree over $Y_s \cap Y_{s'}$. The restrictions of α_f and $\alpha_{f'}$ agree over $Y_s \cap Y_{s'}$ because the regular functions s/s' and $\psi_{(f')}(f)$ agree. This proves that the morphisms ψ_f glue to a global morphism from Y into $U_d \subset X$, and that the maps α_f glue to an isomorphism satisfying the conditions of the lemma.

We still have to show the pair (φ, α) is unique. Suppose (φ', α') is a second such pair. Let $f \in S_d$. By the commutativity of the diagrams in the lemma we have that the inverse images of $D_+(f)$ under both φ and φ' are equal to $Y_{\psi(f)}$. Since the opens $D_+(f)$ are a basis for the topology on X, and since X is a sober topological space (see Schemes, Lemma 11.1) this means the maps φ and φ' are the same on underlying topological spaces. Let us use $s = \psi(f)$ to trivialize the invertible sheaf $\mathcal{L}$ over $Y_{\psi(f)}$. By the commutativity of the diagrams we have that $\alpha^{\otimes n}(\psi^d_\varphi(x)) = \psi(x) = (\alpha')^{\otimes n}(\psi^d_{\varphi'}(x))$ for all $x \in S_{nd}$. By construction of ψ^d_φ and $\psi^d_{\varphi'}$ we have $\psi^d_\varphi(x) = \varphi^\sharp(x/f^n)\psi^d_\varphi(f^n)$ over $Y_{\psi(f)}$, and similarly for $\psi^d_{\varphi'}$. by the commutativity of the diagrams of the lemma we deduce that $\varphi^\sharp(x/f^n) = (\varphi')^\sharp(x/f^n)$. This proves that φ and φ' induce the same morphism from $Y_{\psi(f)}$ into the affine scheme $D_+(f) = \mathrm{Spec}(S_{(f)})$. Hence φ and φ' are the same as morphisms. Finally, it remains to show that the commutativity of the diagram of the lemma singles out, given φ, a unique α. We omit the verification. $\square$

We continue the discussion from above the lemma. Let S be a graded ring. Let Y be a scheme. We will consider *triples* $(d, \mathcal{L}, \psi)$ where

(1) $d \geq 1$ is an integer,
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and
(3) $\psi : S^{(d)} \to \Gamma_*(Y, \mathcal{L})$ is a graded ring homomorphism such that $\mathcal{L}$ is generated by the global sections $\psi(f)$, with $f \in S_d$.

Given a morphism $h : Y' \to Y$ and a triple $(d, \mathcal{L}, \psi)$ over Y we can pull it back to the triple $(d, h^*\mathcal{L}, h^* \circ \psi)$. Given two triples $(d, \mathcal{L}, \psi)$ and $(d, \mathcal{L}', \psi')$ with the same integer d we say they are *strictly equivalent* if there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as graded ring maps $S^{(d)} \to \Gamma_*(Y, \mathcal{L}')$.

For each integer $d \geq 1$ we define

$$\begin{array}{rcl} F_d : Sch^{opp} & \longrightarrow & Sets, \\ Y & \longmapsto & \{\text{strict equivalence classes of triples } (d, \mathcal{L}, \psi) \text{ as above}\} \end{array}$$

with pullbacks as defined above.

Lemma 12.2. *Let S be a graded ring. Let $X = Proj(S)$. The open subscheme $U_d \subset X$ (12.0.1) represents the functor F_d and the triple $(d, \mathcal{O}_{U_d}(d), \psi^d)$ defined above is the universal family (see Schemes, Section 15).*

Proof. This is a reformulation of Lemma 12.1 □

Lemma 12.3. *Let S be a graded ring generated as an S_0-algebra by the elements of S_1. In this case the scheme $X = Proj(S)$ represents the functor which associates to a scheme Y the set of pairs $(\mathcal{L}, \psi)$, where*

(1) *$\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and*
(2) *$\psi : S \to \Gamma_*(Y, \mathcal{L})$ is a graded ring homomorphism such that $\mathcal{L}$ is generated by the global sections $\psi(f)$, with $f \in S_1$*

up to strict equivalence as above.

Proof. Under the assumptions of the lemma we have $X = U_1$ and the lemma is a reformulation of Lemma 12.2 above. □

We end this section with a discussion of a functor corresponding to Proj(S) for a general graded ring S. We advise the reader to skip the rest of this section.

Fix an arbitrary graded ring S. Let T be a scheme. We will say two triples $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ over T with possibly different integers d, d' are *equivalent* if there exists an isomorphism $\beta : \mathcal{L}^{\otimes d'} \to (\mathcal{L}')^{\otimes d}$ of invertible sheaves over T such that $\beta \circ \psi|_{S^{(dd')}}$ and $\psi'|_{S^{(dd')}}$ agree as graded ring maps $S^{(dd')} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes dd'})$.

Lemma 12.4. *Let S be a graded ring. Set $X = Proj(S)$. Let T be a scheme. Let $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ be two triples over T. The following are equivalent:*

(1) *Let $n = lcm(d, d')$. Write $n = ad = a'd'$. There exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{S^{(n)}}$ and $\psi'|_{S^{(n)}}$ agree as graded ring maps $S^{(n)} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes n})$.*
(2) *The triples $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ are equivalent.*
(3) *For some positive integer $n = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{S^{(n)}}$ and $\psi'|_{S^{(n)}}$ agree as graded ring maps $S^{(n)} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes n})$.*
(4) *The morphisms $\varphi : T \to X$ and $\varphi' : T \to X$ associated to $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ are equal.*

Proof. Clearly (1) implies (2) and (2) implies (3) by restricting to more divisible degrees and powers of invertible sheaves. Also (3) implies (4) by the uniqueness statement in Lemma 12.1. Thus we have to prove that (4) implies (1). Assume (4), in other words $\varphi = \varphi'$. Note that this implies that we may write $\mathcal{L} = \varphi^*\mathcal{O}_X(d)$ and $\mathcal{L}' = \varphi^*\mathcal{O}_X(d')$. Moreover, via these identifications we have that the graded ring maps ψ and ψ' correspond to the restriction of the canonical graded ring map

$$S \longrightarrow \bigoplus\nolimits_{n\geq 0} \Gamma(X, \mathcal{O}_X(n))$$

to $S^{(d)}$ and $S^{(d')}$ composed with pullback by φ (by Lemma 12.1 again). Hence taking β to be the isomorphism

$$(\varphi^*\mathcal{O}_X(d))^{\otimes a} = \varphi^*\mathcal{O}_X(n) = (\varphi^*\mathcal{O}_X(d'))^{\otimes a'}$$

works. □

Let S be a graded ring. Let $X = \text{Proj}(S)$. Over the open subscheme scheme $U_d \subset X = \text{Proj}(S)$ (12.0.1) we have the triple $(d, \mathcal{O}_{U_d}(d), \psi^d)$. Clearly, if $d|d'$ the triples $(d, \mathcal{O}_{U_d}(d), \psi^d)$ and $(d', \mathcal{O}_{U_{d'}}(d'), \psi^{d'})$ are equivalent when restricted to the open U_d (which is a subset of $U_{d'}$). This, combined with Lemma 12.1 shows that morphisms $Y \to X$ correspond roughly to equivalence classes of triples over Y. This is not quite true since if Y is not quasi-compact, then there may not be a single triple which works. Thus we have to be slightly careful in defining the corresponding functor.

Here is one possible way to do this. Suppose $d' = ad$. Consider the transformation of functors $F_d \to F_{d'}$ which assigns to the triple $(d, \mathcal{L}, \psi)$ over T the triple $(d', \mathcal{L}^{\otimes a}, \psi|_{S^{(d')}})$. One of the implications of Lemma 12.4 is that the transformation $F_d \to F_{d'}$ is injective! For a quasi-compact scheme T we define

$$F(T) = \bigcup\nolimits_{d \in \mathbf{N}} F_d(T)$$

with transition maps as explained above. This clearly defines a contravariant functor on the category of quasi-compact schemes with values in sets. For a general scheme T we define

$$F(T) = \lim_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. We omit the definition of the pullback map $F(T) \to F(T')$ for a morphism $T' \to T$ of schemes. Thus we have defined our functor

$$F : Sch^{opp} \longrightarrow Sets$$

Lemma 12.5. *Let S be a graded ring. Let $X = Proj(S)$. The functor F defined above is representable by the scheme X.*

Proof. We have seen above that the functor F_d corresponds to the open subscheme $U_d \subset X$. Moreover the transformation of functors $F_d \to F_{d'}$ (if $d|d'$) defined above corresponds to the inclusion morphism $U_d \to U_{d'}$ (see discussion above). Hence to show that F is represented by X it suffices to show that $T \to X$ for a quasi-compact scheme T ends up in some U_d, and that for a general scheme T we have

$$\text{Mor}(T, X) = \lim_{V \subset T \text{ quasi-compact open}} \text{Mor}(V, X).$$

These verifications are omitted. □

13. Projective space

Projective space is one of the fundamental objects studied in algebraic geometry. In this section we just give its construction as Proj of a polynomial ring. Later we will discover many of its beautiful properties.

Lemma 13.1. *Let $S = \mathbf{Z}[T_0, \ldots, T_n]$ with $\deg(T_i) = 1$. The scheme*
$$\mathbf{P}^n_{\mathbf{Z}} = \mathit{Proj}(S)$$
represents the functor which associates to a scheme Y the pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ where

1. *$\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and*
2. *$s_0, \ldots, s_n$ are global sections of $\mathcal{L}$ which generate $\mathcal{L}$*

up to the following equivalence: $(\mathcal{L}, (s_0, \ldots, s_n)) \sim (\mathcal{N}, (t_0, \ldots, t_n)) \Leftrightarrow$ there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{N}$ with $\beta(s_i) = t_i$ for $i = 0, \ldots, n$.

Proof. This is a special case of Lemma 12.3 above. Namely, for any graded ring A we have
$$\begin{array}{rcl} \mathrm{Mor}_{gradedrings}(\mathbf{Z}[T_0, \ldots, T_n], A) & = & A_1 \times \ldots \times A_1 \\ \psi & \mapsto & (\psi(T_0), \ldots, \psi(T_n)) \end{array}$$
and the degree 1 part of $\Gamma_*(Y, \mathcal{L})$ is just $\Gamma(Y, \mathcal{L})$. □

Definition 13.2. The scheme $\mathbf{P}^n_{\mathbf{Z}} = \mathrm{Proj}(\mathbf{Z}[T_0, \ldots, T_n])$ is called *projective n-space over $\mathbf{Z}$*. Its base change $\mathbf{P}^n_S$ to a scheme S is called *projective n-space over S*. If R is a ring the base change to $\mathrm{Spec}(R)$ is denoted $\mathbf{P}^n_R$ and called *projective n-space over R*.

Given a scheme Y over S and a pair $(\mathcal{L}, (s_0, \ldots, s_n))$ as in Lemma 13.1 the induced morphism to $\mathbf{P}^n_S$ is denoted
$$\varphi_{(\mathcal{L}, (s_0, \ldots, s_n))} : Y \longrightarrow \mathbf{P}^n_S$$
This makes sense since the pair defines a morphism into $\mathbf{P}^n_{\mathbf{Z}}$ and we already have the structure morphism into S so combined we get a morphism into $\mathbf{P}^n_S = \mathbf{P}^n_{\mathbf{Z}} \times S$. Note that this is the S-morphism characterized by
$$\mathcal{L} = \varphi^*_{(\mathcal{L}, (s_0, \ldots, s_n))} \mathcal{O}_{\mathbf{P}^n_R}(1) \quad \text{and} \quad s_i = \varphi^*_{(\mathcal{L}, (s_0, \ldots, s_n))} T_i$$
where we think of T_i as a global section of $\mathcal{O}_{\mathbf{P}^n_S}(1)$ via (10.1.3).

Lemma 13.3. *Projective n-space over $\mathbf{Z}$ is covered by $n+1$ standard opens*
$$\mathbf{P}^n_{\mathbf{Z}} = \bigcup\nolimits_{i=0, \ldots, n} D_+(T_i)$$
where each $D_+(T_i)$ is isomorphic to $\mathbf{A}^n_{\mathbf{Z}}$ affine n-space over $\mathbf{Z}$.

Proof. This is true because $\mathbf{Z}[T_0, \ldots, T_n]_+ = (T_0, \ldots, T_n)$ and since
$$\mathrm{Spec}\left(\mathbf{Z}\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]\right) \cong \mathbf{A}^n_{\mathbf{Z}}$$
in an obvious way. □

Lemma 13.4. *Let S be a scheme. The structure morphism $\mathbf{P}^n_S \to S$ is*

1. *separated,*
2. *quasi-compact,*

(3) *satisfies the existence and uniqueness parts of the valuative criterion, and*
(4) *universally closed.*

Proof. All these properties are stable under base change (this is clear for the last two and for the other two see Schemes, Lemmas 21.13 and 19.3). Hence it suffices to prove them for the morphism $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$. Separatedness is Lemma 8.8. Quasi-compactness follows from Lemma 13.3. Existence and uniqueness of the valuative criterion follow from Lemma 8.11. Universally closed follows from the above and Schemes, Proposition 20.6. □

Remark 13.5. What's missing in the list of properties above? Well to be sure the property of being of finite type. The reason we do not list this here is that we have not yet defined the notion of finite type at this point. (Another property which is missing is "smoothness". And I'm sure there are many more you can think of.)

We finish this section with two simple lemmas. These lemmas are special cases of more general results later, but perhaps it makes sense to prove these directly here now.

Lemma 13.6. *Let R be a ring. Let $Z \subset \mathbf{P}^n_R$ be a closed subscheme. Let*

$$I_d = Ker\big(R[T_0, \ldots, T_n]_d \longrightarrow \Gamma(Z, \mathcal{O}_{\mathbf{P}^n_R}(d)|_Z)\big)$$

Then $I = \bigoplus I_d \subset R[T_0, \ldots, T_n]$ is a graded ideal and $Z = Proj(R[T_0, \ldots, T_n]/I)$.

Proof. It is clear that I is a graded ideal. Set $Z' = \mathrm{Proj}(R[T_0, \ldots, T_n]/I)$. By Lemma 11.5 we see that Z' is a closed subscheme of $\mathbf{P}^n_R$. To see the equality $Z = Z'$ it suffices to check on an standard affine open $D_+(T_i)$. By renumbering the homogeneous coordinates we may assume $i = 0$. Say $Z \cap D_+(T_0)$, resp. $Z' \cap D_+(T_0)$ is cut out by the ideal J, resp. J' of $R[T_1/T_0, \ldots, T_n/T_0]$. Then J' is the ideal generated by the elements $F/T_0^{\deg(F)}$ where $F \in I$ is homogeneous. Suppose the degree of $F \in I$ is d. Since F vanishes as a section of $\mathcal{O}_{\mathbf{P}^n_R}(d)$ restricted to Z we see that F/T_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $f \in J$. If f has total degree d in $T_1/T_0, \ldots, T_n/T_0$, then we can write $f = F/T_0^d$ for some $F \in R[T_0, \ldots, T_n]_d$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(T_i)$ is cut out by some ideal $J_i \subset R[T_0/T_i, \ldots, T_n/T_i]$. Moreover,

$$J \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right] = J_i \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]$$

The left hand side is the localization of J with respect to the element T_i/T_0 and the right hand side is the localization of J_i with respect to the element T_0/T_i. It follows that $T_0^{d_i}F/T_i^{d+d_i}$ is an element of J_i for some d_i sufficiently large. This proves that $T_0^{\max(d_i)}F$ is an element of I, because its restriction to each standard affine open $D_+(T_i)$ vanishes on the closed subscheme $Z \cap D_+(T_i)$. Hence $f \in J'$ and we conclude $J \subset J'$ as desired. □

The following lemma is a special case of the more general Properties, Lemma 26.3.

Lemma 13.7. *Let R be a ring. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathbf{P}^n_R$. For $d \geq 0$ set*

$$M_d = \Gamma(\mathbf{P}^n_R, \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{O}_{\mathbf{P}^n_R}(d)) = \Gamma(\mathbf{P}^n_R, \mathcal{F}(d))$$

Then $M = \bigoplus_{d \geq 0} M_d$ is a graded $R[T_0, \ldots, R_n]$-module and there is a canonical isomorphism $\mathcal{F} = \widetilde{M}$.

Proof. The multiplication maps

$$R[T_0, \ldots, R_n]_e \times M_d \longrightarrow M_{d+e}$$

come from the natural isomorphisms

$$\mathcal{O}_{\mathbf{P}^n_R}(e) \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{F}(d) \longrightarrow \mathcal{F}(e+d)$$

see Equation (10.1.4). Let us construct the map $c : \widetilde{M} \to \mathcal{F}$. On each of the standard affines $U_i = D_+(T_i)$ we see that $\Gamma(U_i, \widetilde{M}) = (M[1/T_i])_0$ where the subscript $_0$ means degree 0 part. An element of this can be written as m/T_i^d with $m \in M_d$. Since T_i is a generator of $\mathcal{O}(1)$ over U_i we can always write $m|_{U_i} = m_i \otimes T_i^d$ where $m_i \in \Gamma(U_i, \mathcal{F})$ is a unique section. Thus a natural guess is $c(m/T_i^d) = m_i$. A small argument, which is omitted here, shows that this gives a well defined map $c : \widetilde{M} \to \mathcal{F}$ if we can show that

$$(T_i/T_j)^d m_i|_{U_i \cap U_j} = m_j|_{U_i \cap U_j}$$

in $M[1/T_iT_j]$. But this is clear since on the overlap the generators T_i and T_j of $\mathcal{O}(1)$ differ by the invertible function T_i/T_j.

Injectivity of c. We may check for injectivity over the affine opens U_i. Let $i \in \{0, \ldots, n\}$ and let s be an element $s = m/T_i^d \in \Gamma(U_i, \widetilde{M})$ such that $c(m/T_i^d) = 0$. By the description of c above this means that $m_i = 0$, hence $m|_{U_i} = 0$. Hence $T_i^e m = 0$ in M for some e. Hence $s = m/T_i^d = T_i^e/T_i^{e+d} = 0$ as desired.

Surjectivity of c. We may check for surjectivity over the affine opens U_i. By renumbering it suffices to check it over U_0. Let $s \in \mathcal{F}(U_0)$. Let us write $\mathcal{F}|_{U_i} = \widetilde{N_i}$ for some $R[T_0/T_i, \ldots, T_0/T_i]$-module N_i, which is possible because $\mathcal{F}$ is quasi-coherent. So s corresponds to an element $x \in N_0$. Then we have that

$$(N_i)_{T_j/T_i} \cong (N_j)_{T_i/T_j}$$

(where the subscripts mean "principal localization at") as modules over the ring

$$R\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}, \frac{T_0}{T_j}, \ldots, \frac{T_n}{T_j}\right].$$

This means that for some large integer d there exist elements $s_i \in N_i$, $i = 1, \ldots, n$ such that

$$s = (T_i/T_0)^d s_i$$

on $U_0 \cap U_i$. Next, we look at the difference

$$t_{ij} = s_i - (T_j/T_i)^d s_j$$

on $U_i \cap U_j$, $0 < i < j$. By our choice of s_i we know that $t_{ij}|_{U_0 \cap U_i \cap U_j} = 0$. Hence there exists a large integer e such that $(T_0/T_i)^e t_{ij} = 0$. Set $s_i' = (T_0/T_i)^e s_i$, and $s_0' = s$. Then we will have

$$s_a' = (T_b/T_a)^{e+d} s_b'$$

on $U_a \cap U_b$ for all a, b. This is exactly the condition that the elements s_a' glue to a global section $m \in \Gamma(\mathbf{P}^n_R, \mathcal{F}(e+d))$. And moreover $c(m/T_0^{e+d}) = s$ by construction. Hence c is surjective and we win. $\square$

14. Invertible sheaves and morphisms into Proj

Let T be a scheme and let $\mathcal{L}$ be an invertible sheaf on T. For a section $s \in \Gamma(T, \mathcal{L})$ we denote T_s the open subset of points where s does not vanish. See Modules, Lemma 21.7. We can view the following lemma as a slight generalization of Lemma 12.3. It also is a generalization of Lemma 11.1.

Lemma 14.1. *Let A be a graded ring. Set $X = Proj(A)$. Let T be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_T$-module. Let $\psi : A \to \Gamma_*(T, \mathcal{L})$ be a homomorphism of graded rings. Set*

$$U(\psi) = \bigcup\nolimits_{f \in A_+ \ homogeneous} T_{\psi(f)}$$

The morphism ψ induces a canonical morphism of schemes

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow X$$

together with a map of $\mathbf{Z}$-graded $\mathcal{O}_T$-algebras

$$\theta : r^*_{\mathcal{L},\psi}\left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d)\right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{L}^{\otimes d}|_{U(\psi)}.$$

The triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ is characterized by the following properties:

1. *For $f \in A_+$ homogeneous we have $r^{-1}_{\mathcal{L},\psi}(D_+(f)) = T_{\psi(f)}$.*
2. *For every $d \geq 0$ the diagram*

$$\begin{tikzcd} A_d \arrow[r, "\psi"'] \arrow[d, "(10.1.3)"'] & \Gamma(T, \mathcal{L}^{\otimes d}) \arrow[d, "restrict"] \\ \Gamma(X, \mathcal{O}_X(d)) \arrow[r, "\theta"] & \Gamma(U(\psi), \mathcal{L}^{\otimes d}) \end{tikzcd}$$

   is commutative.

Moreover, for any $d \geq 1$ and any open subscheme $V \subset T$ such that the sections in $\psi(A_d)$ generate $\mathcal{L}^{\otimes d}|_V$ the morphism $r_{\mathcal{L},\psi}|_V$ agrees with the morphism $\varphi : V \to Proj(A)$ and the map $\theta|_V$ agrees with the map $\alpha : \varphi^*\mathcal{O}_X(d) \to \mathcal{L}^{\otimes d}|_V$ where (φ, α) is the pair of Lemma 12.1 associated to $\psi|_{A^{(d)}} : A^{(d)} \to \Gamma_*(V, \mathcal{L}^{\otimes d})$.

Proof. Suppose that we have two triples $(U, r : U \to X, \theta)$ and $(U', r' : U' \to X, \theta')$ satisfying (1) and (2). Property (1) implies that $U = U' = U(\psi)$ and that $r = r'$ as maps of underlying topological spaces, since the opens $D_+(f)$ form a basis for the topology on X, and since X is a sober topological space (see Algebra, Section 55 and Schemes, Lemma 11.1). Let $f \in A_+$ be homogeneous. Note that $\Gamma(D_+(f), \bigoplus_{n \in \mathbf{Z}} \mathcal{O}_X(n)) = A_f$ as a $\mathbf{Z}$-graded algebra. Consider the two $\mathbf{Z}$-graded ring maps

$$\theta, \theta' : A_f \longrightarrow \Gamma(T_{\psi(f)}, \bigoplus \mathcal{L}^{\otimes n}).$$

We know that multiplication by f (resp. $\psi(f)$) is an isomorphism on the left (resp. right) hand side. We also know that $\theta(x/1) = \theta'(x/1) = \psi(x)|_{T_{\psi(f)}}$ by (2) for all $x \in A$. Hence we deduce easily that $\theta = \theta'$ as desired. Considering the degree 0 parts we deduce that $r^\sharp = (r')^\sharp$, i.e., that $r = r'$ as morphisms of schemes. This proves the uniqueness.

Now we come to existence. By the uniqueness just proved, it is enough to construct the pair (r, θ) locally on T. Hence we may assume that $T = \mathrm{Spec}(R)$ is affine, that $\mathcal{L} = \mathcal{O}_T$ and that for some $f \in A_+$ homogeneous we have $\psi(f)$ generates

$\mathcal{O}_T = \mathcal{O}_T^{\otimes \deg(f)}$. In other words, $\psi(f) = u \in R^*$ is a unit. In this case the map ψ is a graded ring map

$$A \longrightarrow R[x] = \Gamma_*(T, \mathcal{O}_T)$$

which maps f to $ux^{\deg(f)}$. Clearly this extends (uniquely) to a $\mathbf{Z}$-graded ring map $\theta : A_f \to R[x, x^{-1}]$ by mapping $1/f$ to $u^{-1}x^{-\deg(f)}$. This map in degree zero gives the ring map $A_{(f)} \to R$ which gives the morphism $r : T = \mathrm{Spec}(R) \to \mathrm{Spec}(A_{(f)}) = D_+(f) \subset X$. Hence we have constructed (r, θ) in this special case.

Let us show the last statement of the lemma. According to Lemma 12.1 the morphism constructed there is the unique one such that the displayed diagram in its statement commutes. The commutativity of the diagram in the lemma implies the commutativity when restricted to V and $A^{(d)}$. Whence the result. $\square$

Remark 14.2. Assumptions as in Lemma 14.1 above. The image of the morphism $r_{\mathcal{L},\psi}$ need not be contained in the locus where the sheaf $\mathcal{O}_X(1)$ is invertible. Here is an example. Let k be a field. Let $S = k[A, B, C]$ graded by $\deg(A) = 1$, $\deg(B) = 2$, $\deg(C) = 3$. Set $X = \mathrm{Proj}(S)$. Let $T = \mathbf{P}^2_k = \mathrm{Proj}(k[X_0, X_1, X_2])$. Recall that $\mathcal{L} = \mathcal{O}_T(1)$ is invertible and that $\mathcal{O}_T(n) = \mathcal{L}^{\otimes n}$. Consider the composition ψ of the maps

$$S \to k[X_0, X_1, X_2] \to \Gamma_*(T, \mathcal{L}).$$

Here the first map is $A \mapsto X_0^6$, $B \mapsto X_1^3$, $C \mapsto X_2^3$ and the second map is (10.1.3). By the lemma this corresponds to a morphism $r_{\mathcal{L},\psi} : T \to X = \mathrm{Proj}(S)$ which is easily seen to be surjective. On the other hand, in Remark 9.2 we showed that the sheaf $\mathcal{O}_X(1)$ is not invertible at all points of X.

15. Relative Proj via glueing

Situation 15.1. Here S is a scheme, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra.

In this section we outline how to construct a morphism of schemes

$$\underline{\mathrm{Proj}}_S(\mathcal{A}) \longrightarrow S$$

by glueing the homogeneous spectra $\mathrm{Proj}(\Gamma(U, \mathcal{A}))$ where U ranges over the affine opens of S. We first show that the homogeneous spectra of the values of $\mathcal{A}$ over affines form a suitable collection of schemes, as in Lemma 2.1.

Lemma 15.2. *In Situation 15.1. Suppose $U \subset U' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$ and $A' = \mathcal{A}(U')$. The map of graded rings $A' \to A$ induces a morphism $r : Proj(A) \to Proj(A')$, and the diagram*

$$\begin{tikzcd} Proj(A) \arrow[r] \arrow[d] & Proj(A') \arrow[d] \\ U \arrow[r] & U' \end{tikzcd}$$

is cartesian. Moreover there are canonical isomorphisms $\theta : r^*\mathcal{O}_{Proj(A')}(n) \to \mathcal{O}_{Proj(A)}(n)$ compatible with multiplication maps.

Proof. Let $R = \mathcal{O}_S(U)$ and $R' = \mathcal{O}_S(U')$. Note that the map $R \otimes_{R'} A' \to A$ is an isomorphism as $\mathcal{A}$ is quasi-coherent (see Schemes, Lemma 7.3 for example). Hence the lemma follows from Lemma 11.6. $\square$

In particular the morphism $\mathrm{Proj}(A) \to \mathrm{Proj}(A')$ of the lemma is an open immersion.

Lemma 15.3. *In Situation 15.1. Suppose $U \subset U' \subset U'' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ and $A'' = \mathcal{A}(U'')$. The composition of the morphisms $r : \mathit{Proj}(A) \to \mathit{Proj}(A')$, and $r' : \mathit{Proj}(A') \to \mathit{Proj}(A'')$ of Lemma 15.2 gives the morphism $r'' : \mathit{Proj}(A) \to \mathit{Proj}(A'')$ of Lemma 15.2. A similar statement holds for the isomorphisms θ.*

Proof. This follows from Lemma 11.2 since the map $A'' \to A$ is the composition of $A'' \to A'$ and $A' \to A$. □

Lemma 15.4. *In Situation 15.1. There exists a morphism of schemes*

$$\pi : \underline{\mathit{Proj}}_S(\mathcal{A}) \longrightarrow S$$

with the following properties:

1. *for every affine open $U \subset S$ there exists an isomorphism $i_U : \pi^{-1}(U) \to \mathit{Proj}(A)$ with $A = \mathcal{A}(U)$, and*
2. *for $U \subset U' \subset S$ affine open the composition*

$$\mathit{Proj}(A) \xrightarrow{i_U^{-1}} \pi^{-1}(U) \xrightarrow{\mathit{inclusion}} \pi^{-1}(U') \xrightarrow{i_{U'}} \mathit{Proj}(A')$$

with $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ is the open immersion of Lemma 15.2 above.

Proof. Follows immediately from Lemmas 2.1, 15.2, and 15.3. □

Lemma 15.5. *In Situation 15.1. The morphism $\pi : \underline{\mathit{Proj}}_S(\mathcal{A}) \to S$ of Lemma 15.4 comes with the following additional structure. There exists a quasi-coherent $\mathbf{Z}$-graded sheaf of $\mathcal{O}_{\underline{\mathit{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_{\underline{\mathit{Proj}}_S(\mathcal{A})}(n)$, and a morphism of graded $\mathcal{O}_S$-algebras*

$$\psi : \mathcal{A} \longrightarrow \bigoplus\nolimits_{n \geq 0} \pi_* \left(\mathcal{O}_{\underline{\mathit{Proj}}_S(\mathcal{A})}(n)\right)$$

uniquely determined by the following property: For every affine open $U \subset S$ with $A = \mathcal{A}(U)$ there is an isomorphism

$$\theta_U : i_U^* \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{O}_{\mathit{Proj}(A)}(n)\right) \longrightarrow \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{O}_{\underline{\mathit{Proj}}_S(\mathcal{A})}(n)\right)|_{\pi^{-1}(U)}$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\pi^{-1}(U)}$-algebras such that

$$\begin{tikzcd}
A_n \arrow[rr, "\psi"] \arrow[dr, "(10.1.3)"'] & & \Gamma(\pi^{-1}(U), \mathcal{O}_{\underline{\mathit{Proj}}_S(\mathcal{A})}(n)) \\
& \Gamma(\mathit{Proj}(A), \mathcal{O}_{\mathit{Proj}(A)}(n)) \arrow[ur, "\theta_U"'] &
\end{tikzcd}$$

is commutative.

Proof. We are going to use Lemma 2.2 to glue the sheaves of $\mathbf{Z}$-graded algebras $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_{\mathrm{Proj}(A)}(n)$ for $A = \mathcal{A}(U)$, $U \subset S$ affine open over the scheme $\underline{\mathrm{Proj}}_S(\mathcal{A})$. We have constructed the data necessary for this in Lemma 15.2 and we have checked condition (d) of Lemma 2.2 in Lemma 15.3. Hence we get the sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n)$ together with the isomorphisms θ_U for all $U \subset S$ affine open and all $n \in \mathbf{Z}$. For every affine open $U \subset S$ with $A = \mathcal{A}(U)$ we have a map $A \to \Gamma(\mathrm{Proj}(A), \bigoplus_{n \geq 0} \mathcal{O}_{\mathrm{Proj}(A)}(n))$. Hence the map ψ exists by functoriality of relative glueing, see Remark 2.3. The diagram of the lemma commutes by construction. This characterizes the sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}$-algebras

$\bigoplus \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$ because the proof of Lemma 11.1 shows that having these diagrams commute uniquely determines the maps θ_U. Some details omitted. $\square$

16. Relative Proj as a functor

We place ourselves in Situation 15.1. So S is a scheme and $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. In this section we relativize the construction of Proj by constructing a functor which the relative homogeneous spectrum will represent. As a result we will construct a morphism of schemes

$$\underline{\text{Proj}}_S(\mathcal{A}) \longrightarrow S$$

which above affine opens of S will look like the homogeneous spectrum of a graded ring. The discussion will be modeled after our discussion of the relative spectrum in Section 4. The easier method using glueing schemes of the form Proj(A), $A = \Gamma(U, \mathcal{A})$, $U \subset S$ affine open, is explained in Section 15, and the result in this section will be shown to be isomorphic to that one.

Fix for the moment an integer $d \geq 1$. We denote $\mathcal{A}^{(d)} = \bigoplus_{n \geq 0} \mathcal{A}_{nd}$ similarly to the notation in Algebra, Section 54. Let T be a scheme. Let us consider *quadruples* $(d, f : T \to S, \mathcal{L}, \psi)$ *over* T where

(1) d is the integer we fixed above,
(2) $f : T \to S$ is a morphism of schemes,
(3) $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and
(4) $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a homomorphism of graded $\mathcal{O}_T$-algebras such that $f^*\mathcal{A}_d \to \mathcal{L}$ is surjective.

Given a morphism $h : T' \to T$ and a quadruple $(d, f, \mathcal{L}, \psi)$ over T we can pull it back to the quadruple $(d, f \circ h, h^*\mathcal{L}, h^*\psi)$ over T'. Given two quadruples $(d, f, \mathcal{L}, \psi)$ and $(d, f', \mathcal{L}', \psi')$ over T with the same integer d we say they are *strictly equivalent* if $f = f'$ and there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as graded $\mathcal{O}_T$-algebra maps $f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes n}$.

For each integer $d \geq 1$ we define

$$\begin{array}{rcl} F_d : Sch^{opp} & \longrightarrow & Sets, \\ T & \longmapsto & \{\text{strict equivalence classes of } (d, f : T \to S, \mathcal{L}, \psi) \text{ as above}\} \end{array}$$

with pullbacks as defined above.

Lemma 16.1. *In Situation 15.1. Let $d \geq 1$. Let F_d be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F'_d be the functor associated to $(S', \mathcal{A}')$ above. Then there is a canonical isomorphism*

$$F'_d \cong h_{S'} \times_{h_S} F_d$$

of functors.

Proof. A quadruple $(d, f' : T \to S', \mathcal{L}', \psi' : (f')^*(\mathcal{A}')^{(d)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes n})$ is the same as a quadruple $(d, f, \mathcal{L}, \psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n})$ together with a factorization of f as $f = g \circ f'$. Namely, the correspondence is $f = g \circ f'$, $\mathcal{L} = \mathcal{L}'$ and $\psi = \psi'$ via the identifications $(f')^*(\mathcal{A}')^{(d)} = (f')^*g^*(\mathcal{A}^{(d)}) = f^*\mathcal{A}^{(d)}$. Hence the lemma. $\square$

Lemma 16.2. *In Situation 15.1. Let F_d be the functor associated to $(d, S, \mathcal{A})$ above. If S is affine, then F_d is representable by the open subscheme U_d (12.0.1) of the scheme Proj$(\Gamma(S, \mathcal{A}))$.*

Proof. Write $S = \mathrm{Spec}(R)$ and $A = \Gamma(S, \mathcal{A})$. Then A is a graded R-algebra and $\mathcal{A} = \widetilde{A}$. To prove the lemma we have to identify the functor F_d with the functor $F_d^{triples}$ of triples defined in Section 12.

Let $(d, f : T \to S, \mathcal{L}, \psi)$ be a quadruple. We may think of ψ as a $\mathcal{O}_S$-module map $\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n}$. Since $\mathcal{A}^{(d)}$ is quasi-coherent this is the same thing as an R-linear homomorphism of graded rings $A^{(d)} \to \Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n})$. Clearly, $\Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n}) = \Gamma_*(T, \mathcal{L})$. Thus we may associate to the quadruple the triple $(d, \mathcal{L}, \psi)$.

Conversely, let $(d, \mathcal{L}, \psi)$ be a triple. The composition $R \to A_0 \to \Gamma(T, \mathcal{O}_T)$ determines a morphism $f : T \to S = \mathrm{Spec}(R)$, see Schemes, Lemma 6.4. With this choice of f the map $A^{(d)} \to \Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n})$ is R-linear, and hence corresponds to a ψ which we can use for a quadruple $(d, f : T \to S, \mathcal{L}, \psi)$. We omit the verification that this establishes an isomorphism of functors $F_d = F_d^{triples}$. $\square$

Lemma 16.3. *In Situation 15.1. The functor F_d is representable by a scheme.*

Proof. We are going to use Schemes, Lemma 15.4.

First we check that F_d satisfies the sheaf property for the Zariski topology. Namely, suppose that T is a scheme, that $T = \bigcup_{i \in I} U_i$ is an open covering, and that $(d, f_i, \mathcal{L}_i, \psi_i) \in F_d(U_i)$ such that $(d, f_i, \mathcal{L}_i, \psi_i)|_{U_i \cap U_j}$ and $(d, f_j, \mathcal{L}_j, \psi_j)|_{U_i \cap U_j}$ are strictly equivalent. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{I_i} = f_i$, see Schemes, Section 14. Thus $f_i^*\mathcal{A}^{(d)} = f^*\mathcal{A}^{(d)}|_{U_i}$. It also implies there exist isomorphisms $\beta_{ij} : \mathcal{L}_i|_{U_i \cap U_j} \to \mathcal{L}_j|_{U_i \cap U_j}$ such that $\beta_{ij} \circ \psi_i = \psi_j$ on $U_i \cap U_j$. Note that the isomorphisms β_{ij} are uniquely determined by this requirement because the maps $f_i^*\mathcal{A}_d \to \mathcal{L}_i$ are surjective. In particular we see that $\beta_{jk} \circ \beta_{ij} = \beta_{ik}$ on $U_i \cap U_j \cap U_k$. Hence by Sheaves, Section 33 the invertible sheaves $\mathcal{L}_i$ glue to an invertible $\mathcal{O}_T$-module $\mathcal{L}$ and the morphisms ψ_i glue to morphism of $\mathcal{O}_T$-algebras $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$. This proves that F_d satisfies the sheaf condition with respect to the Zariski topology.

Let $S = \bigcup_{i \in I} U_i$ be an affine open covering. Let $F_{d,i} \subset F_d$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subset U_i$.

We have to show each $F_{d,i}$ is representable. This is the case because $F_{d,i}$ is identified with the functor associated to U_i equipped with the quasi-coherent graded $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$) by Lemma 16.1. Thus the result follows from Lemma 16.2.

Next we show that $F_{d,i} \subset F_d$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F_d(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of $F_{d,i}$ that given $a : T' \to T$ we gave $a^*(f, \varphi) \in F_{d,i}(T')$ if and only if $a(T') \subset V_i$. This is what we were required to show.

Finally, we have to show that the collection $(F_{d,i})_{i \in I}$ covers F_d. Let $(f : T \to S, \varphi) \in F_d(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i \in I} U_i$ is an open covering of S we see that $T = \bigcup_{i \in I} V_i$ is an open covering of T. Moreover $(f, \varphi)|_{V_i} \in F_{d,i}(V_i)$. This finishes the proof of the lemma. $\square$

At this point we can redo the material at the end of Section 12 in the current relative setting and define a functor which is representable by $\underline{\mathrm{Proj}}_S(\mathcal{A})$. To do this we introduce the notion of equivalence between two quadruples $(d, f : T \to S, \mathcal{L}, \psi)$ and $(d', f' : T \to S, \mathcal{L}', \psi')$ with possibly different values of the integers d, d'.

Namely, we say these are *equivalent* if $f = f'$, and there exists an isomorphism $\beta : \mathcal{L}^{\otimes d'} \to (\mathcal{L}')^{\otimes d}$ such that $\beta \circ \psi|_{f^*\mathcal{A}^{(dd')}} = \psi'|_{f^*\mathcal{A}^{(dd')}}$. The following lemma implies that this defines an equivalence relation. (This is not a complete triviality.)

Lemma 16.4. *In Situation 15.1. Let T be a scheme. Let $(d, f, \mathcal{L}, \psi)$, $(d', f', \mathcal{L}', \psi')$ be two quadruples over T. The following are equivalent:*

1. *Let $m = lcm(d, d')$. Write $m = ad = a'd'$. We have $f = f'$ and there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes mn}$.*
2. *The quadruples $(d, f, \mathcal{L}, \psi)$ and $(d', f', \mathcal{L}', \psi')$ are equivalent.*
3. *We have $f = f'$ and for some positive integer $m = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes mn}$.*

Proof. Clearly (1) implies (2) and (2) implies (3) by restricting to more divisible degrees and powers of invertible sheaves. Assume (3) for some integer $m = ad = a'd'$. Let $m_0 = \text{lcm}(d, d')$ and write it as $m_0 = a_0 d = a_0' d'$. We are given an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property described in (3). We want to find an isomorphism $\beta_0 : \mathcal{L}^{\otimes a_0} \to (\mathcal{L}')^{\otimes a_0'}$ having that property as well. Since by assumption the maps $\psi : f^*\mathcal{A}_d \to \mathcal{L}$ and $\psi' : (f')^*\mathcal{A}_{d'} \to \mathcal{L}'$ are surjective the same is true for the maps $\psi : f^*\mathcal{A}_{m_0} \to \mathcal{L}^{\otimes a_0}$ and $\psi' : (f')^*\mathcal{A}_{m_0} \to (\mathcal{L}')^{\otimes a_0}$. Hence if β_0 exists it is uniquely determined by the condition that $\beta_0 \circ \psi = \psi'$. This means that we may work locally on T. Hence we may assume that $f = f' : T \to S$ maps into an affine open, in other words we may assume that S is affine. In this case the result follows from the corresponding result for triples (see Lemma 12.4) and the fact that triples and quadruples correspond in the affine base case (see proof of Lemma 16.2). $\square$

Suppose $d' = ad$. Consider the transformation of functors $F_d \to F_{d'}$ which assigns to the quadruple $(d, f, \mathcal{L}, \psi)$ over T the quadruple $(d', f, \mathcal{L}^{\otimes a}, \psi|_{f^*\mathcal{A}^{(d')}})$. One of the implications of Lemma 16.4 is that the transformation $F_d \to F_{d'}$ is injective! For a quasi-compact scheme T we define

$$F(T) = \bigcup\nolimits_{d \in \mathbf{N}} F_d(T)$$

with transition maps as explained above. This clearly defines a contravariant functor on the category of quasi-compact schemes with values in sets. For a general scheme T we define

$$F(T) = \lim\nolimits_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. We omit the definition of the pullback map $F(T) \to F(T')$ for a morphism $T' \to T$ of schemes. Thus we have defined our functor

$$F : Sch^{opp} \longrightarrow Sets \tag{16.4.1}$$

Lemma 16.5. *In Situation 15.1. The functor F above is representable by a scheme.*

Proof. Let $U_d \to S$ be the scheme representing the functor F_d defined above. Let $\mathcal{L}_d$, $\psi^d : \pi_d^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}_d^{\otimes n}$ be the universal object. If $d | d'$, then we may consider the quadruple $(d', \pi_d, \mathcal{L}_d^{\otimes d'/d}, \psi^d|_{\mathcal{A}^{(d')}})$ which determines a canonical morphism

$U_d \to U_{d'}$ over S. By construction this morphism corresponds to the transformation of functors $F_d \to F_{d'}$ defined above.

For every affine open $\mathrm{Spec}(R) = V \subset S$ setting $A = \Gamma(V, \mathcal{A})$ we have a canonical identification of the base change $U_{d,V}$ with the corresponding open subscheme of $\mathrm{Proj}(A)$, see Lemma 16.2. Moreover, the morphisms $U_{d,V} \to U_{d',V}$ constructed above correspond to the inclusions of opens in $\mathrm{Proj}(A)$. Thus we conclude that $U_d \to U_{d'}$ is an open immersion.

This allows us to construct X by glueing the schemes U_d along the open immersions $U_d \to U_{d'}$. Technically, it is convenient to choose a sequence $d_1|d_2|d_3|\ldots$ such that every positive integer divides one of the d_i and to simply take $X = \bigcup U_{d_i}$ using the open immersions above. It is then a simple matter to prove that X represents the functor F. $\square$

Lemma 16.6. *In Situation 15.1. The scheme $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ constructed in Lemma 15.4 and the scheme representing the functor F are canonically isomorphic as schemes over S.*

Proof. Let X be the scheme representing the functor F. Note that X is a scheme over S since the functor F comes equipped with a natural transformation $F \to h_S$. Write $Y = \underline{\mathrm{Proj}}_S(\mathcal{A})$. We have to show that $X \cong Y$ as S-schemes. We give two arguments.

The first argument uses the construction of X as the union of the schemes U_d representing F_d in the proof of Lemma 16.5. Over each affine open of S we can identify X with the homogeneous spectrum of the sections of $\mathcal{A}$ over that open, since this was true for the opens U_d. Moreover, these identifications are compatible with further restrictions to smaller affine opens. On the other hand, Y was constructed by glueing these homogeneous spectra. Hence we can glue these isomorphisms to an isomorphism between X and $\underline{\mathrm{Proj}}_S(\mathcal{A})$ as desired. Details omitted.

Here is the second argument. Lemma 15.5 shows that there exists a morphism of graded algebras

$$\psi : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n \geq 0} \mathcal{O}_Y(n)$$

over Y which on sections over affine opens of S agrees with (10.1.3). Hence for every $y \in Y$ there exists an open neighbourhood $V \subset Y$ of y and an integer $d \geq 1$ such that for $d|n$ the sheaf $\mathcal{O}_Y(n)|_V$ is invertible and the multiplication maps $\mathcal{O}_Y(n)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_Y(m)|_V \to \mathcal{O}_Y(n+m)|_V$ are isomorphisms. Thus ψ restricted to the sheaf $\pi^*\mathcal{A}^{(d)}|_V$ gives an element of $F_d(V)$. Since the opens V cover Y we see "ψ" gives rise to an element of $F(Y)$. Hence a canonical morphism $Y \to X$ over S. Because this construction is completely canonical to see that it is an isomorphism we may work locally on S. Hence we reduce to the case S affine where the result is clear. $\square$

Definition 16.7. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The *relative homogeneous spectrum of $\mathcal{A}$ over S*, or the *homogeneous spectrum of $\mathcal{A}$ over S*, or the *relative Proj of $\mathcal{A}$ over S* is the scheme constructed in Lemma 15.4 which represents the functor F (16.4.1), see Lemma 16.6. We denote it $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$.

The relative Proj comes equipped with a quasi-coherent sheaf of $\mathbf{Z}$-graded algebras $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$ (the twists of the structure sheaf) and a "universal" homomorphism of graded algebras

$$\psi_{univ} : \mathcal{A} \longrightarrow \pi_* \left(\bigoplus\nolimits_{n \geq 0} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n) \right)$$

see Lemma 15.5. We may also think of this as a homomorphism

$$\psi_{univ} : \pi^* \mathcal{A} \longrightarrow \bigoplus\nolimits_{n \geq 0} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$$

if we like. The following lemma is a formulation of the universality of this object.

Lemma 16.8. *In Situation 15.1. Let $(f : T \to S, d, \mathcal{L}, \psi)$ be a quadruple. Let $r_{d,\mathcal{L},\psi} : T \to \underline{Proj}_S(\mathcal{A})$ be the associated S-morphism. There exists an isomorphism of $\mathbf{Z}$-graded $\mathcal{O}_T$-algebras*

$$\theta : r_{d,\mathcal{L},\psi}^* \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(nd) \right) \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$$

such that the following diagram commutes

$$\begin{tikzcd}[column sep=small]
\mathcal{A}^{(d)} \arrow[rr, "\psi"'] \arrow[dr, "\psi_{univ}"'] & & f_* \left(\bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n} \right) \\
& \pi_* \left(\bigoplus_{n \geq 0} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(nd) \right) \arrow[ur, "\theta"'] &
\end{tikzcd}$$

The commutativity of this diagram uniquely determines θ.

Proof. Note that the quadruple $(f : T \to S, d, \mathcal{L}, \psi)$ defines an element of $F_d(T)$. Let $U_d \subset \underline{\text{Proj}}_S(\mathcal{A})$ be the locus where the sheaf $\mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(d)$ is invertible and generated by the image of $\psi_{univ} : \pi^* \mathcal{A}_d \to \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(d)$. Recall that U_d represents the functor F_d, see the proof of Lemma 16.5. Hence the result will follow if we can show the quadruple $(U_d \to S, d, \mathcal{O}_{U_d}(d), \psi_{univ}|_{\mathcal{A}^{(d)}})$ is the universal family, i.e., the representing object in $F_d(U_d)$. We may do this after restricting to an affine open of S because (a) the formation of the functors F_d commutes with base change (see Lemma 16.1), and (b) the pair $(\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n), \psi_{univ})$ is constructed by glueing over affine opens in S (see Lemma 15.5). Hence we may assume that S is affine. In this case the functor of quadruples F_d and the functor of triples F_d agree (see proof of Lemma 16.2) and moreover Lemma 12.2 shows that $(d, \mathcal{O}_{U_d}(d), \psi^d)$ is the universal triple over U_d. Going backwards through the identifications in the proof of Lemma 16.2 shows that $(U_d \to S, d, \mathcal{O}_{U_d}(d), \psi_{univ}|_{\mathcal{A}^{(d)}})$ is the universal quadruple as desired. $\square$

Lemma 16.9. *Let S be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The morphism $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ is separated.*

Proof. To prove a morphism is separated we may work locally on the base, see Schemes, Section 21. By construction $\underline{\text{Proj}}_S(\mathcal{A})$ is over any affine $U \subset S$ isomorphic to $\text{Proj}(A)$ with $A = \mathcal{A}(U)$. By Lemma 8.8 we see that $\text{Proj}(A)$ is separated. Hence $\text{Proj}(A) \to U$ is separated (see Schemes, Lemma 21.14) as desired. $\square$

Lemma 16.10. *Let S be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. Let $g : S' \to S$ be any morphism of schemes. Then there is a canonical isomorphism*

$$r : \underline{Proj}_{S'}(g^*\mathcal{A}) \longrightarrow S' \times_S \underline{Proj}_S(\mathcal{A})$$

as well as a corresponding isomorphism

$$\theta : r^*pr_2^*\left(\bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d)\right) \longrightarrow \bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_{S'}(g^*\mathcal{A})}(d)$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{Proj}_{S'}(g^*\mathcal{A})}$-algebras.

Proof. This follows from Lemma 16.1 and the construction of $\underline{\mathrm{Proj}}_S(\mathcal{A})$ in Lemma 16.5 as the union of the schemes U_d representing the functors F_d. In terms of the construction of relative Proj via glueing this isomorphism is given by the isomorphisms constructed in Lemma 11.6 which provides us with the isomorphism θ. Some details omitted. $\square$

Lemma 16.11. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-modules generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. In this case the scheme $X = \underline{Proj}_S(\mathcal{A})$ represents the functor F_1 which associates to a scheme $f : T \to S$ over S the set of pairs $(\mathcal{L}, \psi)$, where*

1. *$\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and*
2. *$\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ is a graded $\mathcal{O}_T$-algebra homomorphism such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective*

up to strict equivalence as above. Moreover, in this case all the quasi-coherent sheaves $\mathcal{O}_{\underline{Proj}(\mathcal{A})}(n)$ are invertible $\mathcal{O}_{\underline{Proj}(\mathcal{A})}$-modules and the multiplication maps induce isomorphisms $\mathcal{O}_{\underline{Proj}(\mathcal{A})}(n) \otimes_{\mathcal{O}_{\underline{Proj}(\mathcal{A})}} \mathcal{O}_{\underline{Proj}(\mathcal{A})}(m) = \mathcal{O}_{\underline{Proj}(\mathcal{A})}(n+m)$.

Proof. Under the assumptions of the lemma the sheaves $\mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{A})}(n)$ are invertible and the multiplication maps isomorphisms by Lemma 16.5 and Lemma 12.3 over affine opens of S. Thus X actually represents the functor F_1, see proof of Lemma 16.5. $\square$

17. Quasi-coherent sheaves on relative Proj

We briefly discuss how to deal with graded modules in the relative setting.

We place ourselves in Situation 15.1. So S is a scheme, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{M} = \bigoplus_{n\in\mathbf{Z}} \mathcal{M}_n$ be a graded $\mathcal{A}$-module, quasi-coherent as an $\mathcal{O}_S$-module. We are going to describe the associated quasi-coherent sheaf of modules on $\underline{\mathrm{Proj}}_S(\mathcal{A})$. We first describe the value of this sheaf schemes T mapping into the relative Proj.

Let T be a scheme. Let $(d, f : T \to S, \mathcal{L}, \psi)$ be a quadruple over T, as in Section 16. We define a quasi-coherent sheaf $\widetilde{\mathcal{M}}_T$ of $\mathcal{O}_T$-modules as follows

$$\widetilde{\mathcal{M}}_T = \left(f^*\mathcal{M}^{(d)} \otimes_{f^*\mathcal{A}^{(d)}} \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}\right)\right)_0 \tag{17.0.1}$$

So $\widetilde{\mathcal{M}}_T$ is the degree 0 part of the tensor product of the graded $f^*\mathcal{A}^{(d)}$-modules $\mathcal{M}^{(d)}$ and $\bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$. Note that the sheaf $\widetilde{\mathcal{M}}_T$ depends on the quadruple even though we suppressed this in the notation. This construction has the pleasing property that given any morphism $g : T' \to T$ we have $\widetilde{\mathcal{M}}_{T'} = g^*\widetilde{\mathcal{M}}_T$ where $\widetilde{\mathcal{M}}_{T'}$ denotes the quasi-coherent sheaf associated to the pullback quadruple $(d, f \circ g, g^*\mathcal{L}, g^*\psi)$.

Since all sheaves in (17.0.1) are quasi-coherent we can spell out the construction over an affine open $\text{Spec}(C) = V \subset T$ which maps into an affine open $\text{Spec}(R) = U \subset S$. Namely, suppose that $\mathcal{A}|_U$ corresponds to the graded R-algebra A, that $\mathcal{M}|_U$ corresponds to the graded A-module M, and that $\mathcal{L}|_V$ corresponds to the invertible C-module L. The map ψ gives rise to a graded R-algebra map $\gamma : A^{(d)} \to \bigoplus_{n \geq 0} L^{\otimes n}$. (Tensor powers of L over C.) Then $(\widetilde{\mathcal{M}_T})|_V$ is the quasi-coherent sheaf associated to the C-module

$$N_{R,C,A,M,\gamma} = \left(M^{(d)} \otimes_{A^{(d)},\gamma} \left(\bigoplus\nolimits_{n \in \mathbf{Z}} L^{\otimes n}\right)\right)_0$$

By assumption we may even cover T by affine opens V such that there exists some $a \in A_d$ such that $\gamma(a) \in L$ is a C-basis for the module L. In that case any element of $N_{R,C,A,M,\gamma}$ is a sum of pure tensors $\sum m_i \otimes \gamma(a)^{-n_i}$ with $m \in M_{n_i d}$. In fact we may multiply each m_i with a suitable positive power of a and collect terms to see that each element of $N_{R,C,A,M,\gamma}$ can be written as $m \otimes \gamma(a)^{-n}$ with $m \in M_{nd}$ and $n \gg 0$. In other words we see that in this case

$$N_{R,C,A,M,\gamma} = M_{(a)} \otimes_{A_{(a)}} C$$

where the map $A_{(a)} \to C$ is the map $x/a^n \mapsto \gamma(x)/\gamma(a)^n$. In other words, this is the value of $\widetilde{M}$ on $D_+(a) \subset \text{Proj}(A)$ pulled back to $\text{Spec}(C)$ via the morphism $\text{Spec}(C) \to D_+(a)$ coming from γ.

Lemma 17.1. *In Situation 15.1. For any quasi-coherent sheaf of graded $\mathcal{A}$-modules $\mathcal{M}$ on S, there exists a canonical associated sheaf of $\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}$-modules $\widetilde{\mathcal{M}}$ with the following properties:*

1. *Given a scheme T and a quadruple $(T \to S, d, \mathcal{L}, \psi)$ over T corresponding to a morphism $h : T \to \underline{Proj}_S(\mathcal{A})$ there is a canonical isomorphism $\widetilde{\mathcal{M}}_T = h^*\widetilde{\mathcal{M}}$ where $\widetilde{\mathcal{M}}_T$ is defined by (17.0.1).*
2. *The isomorphisms of (1) are compatible with pullbacks.*
3. *There is a canonical map*
$$\pi^*\mathcal{M}_0 \longrightarrow \widetilde{\mathcal{M}}.$$
4. *The construction $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is functorial in $\mathcal{M}$.*
5. *The construction $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is exact.*
6. *There are canonical maps*
$$\widetilde{\mathcal{M}} \otimes_{\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}} \widetilde{\mathcal{N}} \longrightarrow \widetilde{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}$$
as in Lemma 9.1.
7. *There exist canonical maps*
$$\pi^*\mathcal{M} \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \widetilde{\mathcal{M}(n)}$$
generalizing (10.1.6).
8. *The formation of $\widetilde{\mathcal{M}}$ commutes with base change.*

Proof. Omitted. We should split this lemma into parts and prove the parts separately. □

18. Functoriality of relative Proj

This section is the analogue of Section 11 for the relative Proj. Let S be a scheme. A graded $\mathcal{O}_S$-algebra map $\psi : \mathcal{A} \to \mathcal{B}$ does not always give rise to a morphism of associated relative Proj. The correct result is stated as follows.

Lemma 18.1. *Let S be a scheme. Let $\mathcal{A}$, $\mathcal{B}$ be two graded quasi-coherent $\mathcal{O}_S$-algebras. Set $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ and $q : Y = \underline{Proj}_S(\mathcal{B}) \to S$. Let $\psi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of graded $\mathcal{O}_S$-algebras. There is a canonical open $U(\psi) \subset Y$ and a canonical morphism of schemes*

$$r_\psi : U(\psi) \longrightarrow X$$

over S and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^*\left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d)\right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the property that for any affine open $W \subset S$ the triple

$$(U(\psi) \cap p^{-1}W, \quad r_\psi|_{U(\psi)\cap p^{-1}W} : U(\psi) \cap p^{-1}W \to q^{-1}W, \quad \theta|_{U(\psi)\cap p^{-1}W})$$

is equal to the triple associated to $\psi : \mathcal{A}(W) \to \mathcal{B}(W)$ in Lemma 11.1 via the identifications $p^{-1}W = Proj(\mathcal{A}(W))$ and $q^{-1}W = Proj(\mathcal{B}(W))$ of Section 15.

Proof. This lemma proves itself by glueing the local triples. $\square$

Lemma 18.2. *Let S be a scheme. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be quasi-coherent graded $\mathcal{O}_S$-algebras. Set $X = \underline{Proj}_S(\mathcal{A})$, $Y = \underline{Proj}_S(\mathcal{B})$ and $Z = \underline{Proj}_S(\mathcal{C})$. Let $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{C}$ be graded $\mathcal{O}_S$-algebra maps. Then we have*

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi\circ\varphi} = r_\varphi \circ r_\psi|_{U(\psi\circ\varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^*\theta_\varphi = \theta_{\psi\circ\varphi}$$

with obvious notation.

Proof. Omitted. $\square$

Lemma 18.3. *With hypotheses and notation as in Lemma 18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is a closed immersion, and*
3. *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are surjective but not isomorphisms in general (even if $\mathcal{A} \to \mathcal{B}$ is surjective).*

Proof. Follows on combining Lemma 18.1 with Lemma 11.3. $\square$

Lemma 18.4. *With hypotheses and notation as in Lemma 18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is an isomorphism for all $d \gg 0$. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is an isomorphism, and*
3. *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. Follows on combining Lemma 18.1 with Lemma 11.4. $\square$

Lemma 18.5. *With hypotheses and notation as in Lemma 18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$ and that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Then*

(1) *$U(\psi) = Y$,*
(2) *$r_\psi : Y \to X$ is a closed immersion, and*
(3) *the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. Follows on combining Lemma 18.1 with Lemma 11.5. □

19. Invertible sheaves and morphisms into relative Proj

It seems that we may need the following lemma somewhere. The situation is the following:

(1) Let S be a scheme.
(2) Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra.
(3) Denote $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$ the relative homogeneous spectrum over S.
(4) Let $f : X \to S$ be a morphism of schemes.
(5) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.
(6) Let $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ be a homomorphism of graded $\mathcal{O}_X$-algebras.

Given this data set

$$U(\psi) = \bigcup\nolimits_{(U,V,a)} U_{\psi(a)}$$

where (U, V, a) satisfies:

(1) $V \subset S$ affine open,
(2) $U = f^{-1}(V)$, and
(3) $a \in \mathcal{A}(V)_+$ is homogeneous.

Namely, then $\psi(a) \in \Gamma(U, \mathcal{L}^{\otimes \deg(a)})$ and $U_{\psi(a)}$ is the corresponding open (see Modules, Lemma 21.7).

Lemma 19.1. *With assumptions and notation as above. The morphism ψ induces a canonical morphism of schemes over S*

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow \underline{Proj}_S(\mathcal{A})$$

together with a map of graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta : r_{\mathcal{L},\psi}^*\left(\bigoplus\nolimits_{d \geq 0} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d)\right) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{L}^{\otimes d}|_{U(\psi)}$$

characterized by the following properties:

(1) *For every open $V \subset S$ and every $d \geq 0$ the diagram*

$$\begin{tikzcd}
\mathcal{A}_d(V) \arrow[rr, "\psi"'] \arrow[d, "\psi"'] & & \Gamma(f^{-1}(V), \mathcal{L}^{\otimes d}) \arrow[d, "restrict"] \\
\Gamma(\pi^{-1}(V), \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d)) \arrow[rr, "\theta"] & & \Gamma(f^{-1}(V) \cap U(\psi), \mathcal{L}^{\otimes d})
\end{tikzcd}$$

is commutative.

(2) *For any $d \geq 1$ and any open subscheme $W \subset X$ such that $\psi|_W : f^*\mathcal{A}_d|_W \to \mathcal{L}^{\otimes d}|_W$ is surjective the restriction of the morphism $r_{\mathcal{L},\psi}$ agrees with the morphism $W \to \underline{Proj}_S(\mathcal{A})$ which exists by the construction of the relative homogeneous spectrum, see Definition 16.7.*

(3) *For any affine open $V \subset S$, the restriction*

$$(U(\psi) \cap f^{-1}(V), r_{\mathcal{L},\psi}|_{U(\psi)\cap f^{-1}(V)}, \theta|_{U(\psi)\cap f^{-1}(V)})$$

agrees via i_V (see Lemma 15.4) with the triple $(U(\psi'), r_{\mathcal{L},\psi'}, \theta')$ of Lemma 14.1 associated to the map $\psi' : A = \mathcal{A}(V) \to \Gamma_*(f^{-1}(V), \mathcal{L}|_{f^{-1}(V)})$ induced by ψ.

Proof. Use characterization (3) to construct the morphism $r_{\mathcal{L},\psi}$ and θ locally over S. Use the uniqueness of Lemma 14.1 to show that the construction glues. Details omitted. ☐

20. Twisting by invertible sheaves and relative Proj

Let S be a scheme. Let $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{L}$ be an invertible sheaf on S. In this situation we obtain another quasi-coherent graded $\mathcal{O}_S$-algebra, namely

$$\mathcal{B} = \bigoplus\nolimits_{d \geq 0} \mathcal{A}_d \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes d}$$

It turns out that $\mathcal{A}$ and $\mathcal{B}$ have isomorphic relative homogeneous spectra.

Lemma 20.1. *With notation S, $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{B}$ as above. There is a canonical isomorphism*

$$\begin{tikzcd}
P = \underline{Proj}_S(\mathcal{A}) \arrow[rr, "g"'] \arrow[dr, "\pi"'] & & \underline{Proj}_S(\mathcal{B}) = P' \arrow[dl, "\pi'"] \\
& S &
\end{tikzcd}$$

with the following properties

(1) *There are isomorphisms $\theta_n : g^*\mathcal{O}_{P'}(n) \to \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}$ which fit together to give an isomorphism of $\mathbf{Z}$-graded algebras*

$$\theta : g^* \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{O}_{P'}(n) \right) \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}$$

(2) *For every open $V \subset S$ the diagrams*

$$\begin{tikzcd}
\mathcal{A}_n(V) \otimes \mathcal{L}^{\otimes n}(V) \arrow[r, "multiply"'] \arrow[d, "\psi \otimes \pi^*"] & \mathcal{B}_n(V) \arrow[dd, "\psi"] \\
\Gamma(\pi^{-1}V, \mathcal{O}_P(n)) \otimes \Gamma(\pi^{-1}V, \pi^*\mathcal{L}^{\otimes n}) \arrow[d, "multiply"] & \\
\Gamma(\pi^{-1}V, \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}) & \Gamma(\pi'^{-1}V, \mathcal{O}_{P'}(n)) \arrow[l, "\theta_n"']
\end{tikzcd}$$

are commutative.

(3) *Add more here as necessary.*

Proof. This is the identity map when $\mathcal{L} \cong \mathcal{O}_S$. In general choose an open covering of S such that $\mathcal{L}$ is trivialized over the pieces and glue the corresponding maps. Details omitted. ☐

21. Projective bundles

Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. By Modules, Lemma 18.6 the symmetric algebra $\text{Sym}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{O}_S$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Note that it is generated in degree 1 over $\mathcal{O}_S$. Hence it makes sense to apply the construction of the previous section to it, specifically Lemmas 16.5 and 16.11.

Definition 21.1. Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module[3]. We denote

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E})) \longrightarrow S$$

and we call it the *projective bundle associated to* $\mathcal{E}$. The symbol $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n)$ indicates the invertible $\mathcal{O}_{\mathbf{P}(\mathcal{E})}$-modules introduced in Lemma 16.5 and is called the nth *twist of the structure sheaf*.

Note that according to Lemma 16.5 there are canonical $\mathcal{O}_S$-module homomorphisms

$$\text{Sym}^n(\mathcal{E}) \longrightarrow \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n))$$

for all $n \geq 0$. This, combined with the fact that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is the canonical relatively ample invertible sheaf on $\mathbf{P}(\mathcal{E})$, is a good way to remember how we have normalized our construction of $\mathbf{P}(\mathcal{E})$. Namely, in some references the space $\mathbf{P}(\mathcal{E})$ is only defined for $\mathcal{E}$ finite locally free on S, and sometimes $\mathbf{P}(\mathcal{E})$ is actually defined as our $\mathbf{P}(\mathcal{E}^\wedge)$ where $\mathcal{E}^\wedge$ is the dual of the sheaf $\mathcal{E}$.

Example 21.2. The map $\text{Sym}^n(\mathcal{E}) \to \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n))$ is an isomorphism if $\mathcal{E}$ is locally free, but in general need not be an isomorphism. In fact we will give an example where this map is not injective for $n = 1$. Set $S = \text{Spec}(A)$ with

$$A = k[u, v, s_1, s_2, t_1, t_2]/I$$

where k is a field and

$$I = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1).$$

Denote $\overline{u}$ the class of u in A and similarly for the other variables. Let $M = (Ax \oplus Ay)/A(\overline{u}x + \overline{v}y)$ so that

$$\text{Sym}(M) = A[x, y]/(\overline{u}x + \overline{v}y) = k[x, y, u, v, s_1, s_2, t_1, t_2]/J$$

where

$$J = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1, ux + vy).$$

In this case the projective bundle associated to the quasi-coherent sheaf $\mathcal{E} = \widetilde{M}$ on $S = \text{Spec}(A)$ is the scheme

$$P = \text{Proj}(\text{Sym}(M)).$$

Note that this scheme as an affine open covering $P = D_+(x) \cup D_+(y)$. Consider the element $m \in M$ which is the image of the element $us_1x + vt_2y$. Note that

$$x(us_1x + vt_2y) = (s_1x + s_2y)(ux + vy) \bmod I$$

and

$$y(us_1x + vt_2y) = (t_1x + t_2y)(ux + vy) \bmod I.$$

[3]The reader may expect here the condition that $\mathcal{E}$ is finite locally free. We do not do so in order to be consistent with [DG67, II, Definition 4.1.1].

The first equation implies that m maps to zero as a section of $\mathcal{O}_P(1)$ on $D_+(x)$ and the second that it maps to zero as a section of $\mathcal{O}_P(1)$ on $D_+(y)$. This shows that m maps to zero in $\Gamma(P, \mathcal{O}_P(1))$. On the other hand we claim that $m \neq 0$, so that m gives an example of a nonzero global section of $\mathcal{E}$ mapping to zero in $\Gamma(P, \mathcal{O}_P(1))$. Assume $m = 0$ to get a contradiction. In this case there exists an element $f \in k[u, v, s_1, s_2, t_1, t_2]$ such that

$$us_1x + vt_2y = f(ux + vy) \bmod I$$

Since I is generated by homogeneous polynomials of degree 2 we may decompose f into its homogeneous components and take the degree 1 component. In other words we may assume that

$$f = au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2$$

for some $a, b, \alpha_1, \alpha_2, \beta_1, \beta_2 \in k$. The resulting conditions are that

$$\begin{aligned} us_1 - u(au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) &\in I \\ vt_2 - v(au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) &\in I \end{aligned}$$

There are no terms u^2, uv, v^2 in the generators of I and hence we see $a = b = 0$. Thus we get the relations

$$\begin{aligned} us_1 - u(\alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) &\in I \\ vt_2 - v(\alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) &\in I \end{aligned}$$

We may use the first generator of I to replace any occurrence of us_1 by $vt_1 + ut_2$, the second generator of I to replace any occurrence of vs_1 by $-us_2 + vt_2$, the third generator to remove occurrences of vs_2 and the third to remove occurences of ut_1. Then we get the relations

$$\begin{aligned} (1 - \alpha_1)vt_1 + (1 - \alpha_1)ut_2 - \alpha_2 us_2 - \beta_2 ut_2 &= 0 \\ (1 - \alpha_1)vt_2 + \alpha_1 us_2 - \beta_1 vt_1 - \beta_2 vt_2 &= 0 \end{aligned}$$

This implies that α_1 should be both 0 and 1 which is a contradiction as desired.

Lemma 21.3. *Let S be a scheme. The structure morphism $\mathbf{P}(\mathcal{E}) \to S$ of a projective bundle over S is separated.*

Proof. Immediate from Lemma 16.9. $\square$

Lemma 21.4. *Let S be a scheme. Let $n \geq 0$. Then $\mathbf{P}^n_S$ is a projective bundle over S.*

Proof. Note that

$$\mathbf{P}^n_{\mathbf{Z}} = \text{Proj}(\mathbf{Z}[T_0, \ldots, T_n]) = \underline{\text{Proj}}_{\text{Spec}(\mathbf{Z})}\left(\widetilde{\mathbf{Z}[T_0, \ldots, T_n]}\right)$$

where the grading on the ring $\mathbf{Z}[T_0, \ldots, T_n]$ is given by $\deg(T_i) = 1$ and the elements of $\mathbf{Z}$ are in degree 0. Recall that $\mathbf{P}^n_S$ is defined as $\mathbf{P}^n_{\mathbf{Z}} \times_{\text{Spec}(\mathbf{Z})} S$. Moreover, forming the relative homogeneous spectrum commutes with base change, see Lemma 16.10. For any scheme $g : S \to \text{Spec}(\mathbf{Z})$ we have $g^*\mathcal{O}_{\text{Spec}(\mathbf{Z})}[T_0, \ldots, T_n] = \mathcal{O}_S[T_0, \ldots, T_n]$. Combining the above we see that

$$\mathbf{P}^n_S = \underline{\text{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n]).$$

Finally, note that $\mathcal{O}_S[T_0, \ldots, T_n] = \text{Sym}(\mathcal{O}_S^{\oplus n+1})$. Hence we see that $\mathbf{P}^n_S$ is a projective bundle over S. $\square$

22. Grassmannians

In this section we introduce the standard Grassmannian functors and we show that they are represented by schemes. Pick integers k, n with $0 < k < n$. We will construct a functor

$$G(k,n) : Sch \longrightarrow Sets \tag{22.0.1}$$

which will loosely speaking parametrize k-dimensional subspaces of n-space. However, for technical reasons it is more convenient to parametrize $(n-k)$-dimensional quotients and this is what we will do.

More precisely, $G(k,n)$ associates to a scheme S the set $G(k,n)(S)$ of isomorphism classes of surjections

$$q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{Q}$$

where $\mathcal{Q}$ is a finite locally free $\mathcal{O}_S$-module of rank $n-k$. Note that this is indeed a set, for example by Modules, Lemma 9.8 or by the observation that the isomorphism class of the surjection q is determined by the kernel of q (and given a sheaf there is a set of subsheaves). Given a morphism of schemes $f : T \to S$ we let $G(k,n)(f) : G(k,n)(S) \to G(k,n)(T)$ which sends the isomorphism class of $q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{Q}$ to the isomorphism class of $f^*q : \mathcal{O}_T^{\oplus n} \longrightarrow f^*\mathcal{Q}$. This makes sense since (1) $f^*\mathcal{O}_S = \mathcal{O}_T$, (2) f^* is additive, (3) f^* preserves locally free modules (Modules, Lemma 14.3), and (4) f^* is right exact (Modules, Lemma 3.3).

Lemma 22.1. *Let $0 < k < n$. The functor $G(k,n)$ of (22.0.1) is representable by a scheme.*

Proof. Set $F = G(k,n)$. To prove the lemma we will use the criterion of Schemes, Lemma 15.4. The reason F satisfies the sheaf property for the Zariski topology is that we can glue sheaves, see Sheaves, Section 33 (some details omitted).

The family of subfunctors F_i. Let I be the set of subsets of $\{1, \ldots, n\}$ of cardinality $n-k$. Given a scheme S and $j \in \{1, \ldots, n\}$ we denote e_j the global section

$$e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \quad (1 \text{ in } j\text{th spot})$$

of $\mathcal{O}_S^{\oplus n}$. Of course these sections freely generate $\mathcal{O}_S^{\oplus n}$. Similarly, for $j \in \{1, \ldots, k\}$ we denote f_j the global section of $\mathcal{O}_S^{\oplus k}$ which is zero in all summands except the jth where we put a 1. For $i \in I$ we let

$$s_i : \mathcal{O}_S^{\oplus n-k} \longrightarrow \mathcal{O}_S^{\oplus n}$$

which is the direct sum of the coprojections $\mathcal{O}_S \to \mathcal{O}_S^{\oplus n}$ corresponding to elements of i. More precisely, if $i = \{i_1, \ldots, i_{n-k}\}$ with $i_1 < i_2 < \ldots < i_{n-k}$ then s_i maps f_j to e_{i_j} for $j \in \{1, \ldots, n-k\}$. With this notation we can set

$$F_i(S) = \{q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q} \in F(S) \mid q \circ s_i \text{ is surjective}\} \subset F(S)$$

Given a morphism $f : T \to S$ of schemes the pullback f^*s_i is the corresponding map over T. Since f^* is right exact (Modules, Lemma 3.3) we conclude that F_i is a subfunctor of F.

Representability of F_i. To prove this we may assume (after renumbering) that $i = \{1, \ldots, n-k\}$. This means s_i is the inclusion of the first $n-k$ summands. Observe that if $q \circ s_i$ is surjective, then $q \circ s_i$ is an isomorphism as a surjective map between finite locally free modules of the same rank (Modules, Lemma 14.5). Thus if $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ is an element of $F_i(S)$, then we can use $q \circ s_i$ to identify $\mathcal{Q}$ with $\mathcal{O}_S^{\oplus n-k}$. After doing so we obtain

$$q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{O}_S^{\oplus n-k}$$

mapping e_j to f_j (notation as above) for $j = 1, \ldots, n-k$. To determine q completely we have to fix the images $q(e_{n-k+1}), \ldots, q(e_n)$ in $\Gamma(S, \mathcal{O}_S^{\oplus n-k})$. It follows that F_i is isomorphic to the functor

$$S \longmapsto \prod\nolimits_{j=n-k+1,\ldots,n} \Gamma(S, \mathcal{O}_S^{\oplus n-k})$$

This functor is isomorphic to the $k(n-k)$-fold self product of the functor $S \mapsto \Gamma(S, \mathcal{O}_S)$. By Schemes, Example 15.2 the latter is representable by $\mathbf{A}^1_{\mathbf{Z}}$. It follows F_i is representable by $\mathbf{A}^{k(n-k)}_{\mathbf{Z}}$ since fibred product over $\mathrm{Spec}(\mathbf{Z})$ is the product in the category of schemes.

The inclusion $F_i \subset F$ is representable by open immersions. Let S be a scheme and let $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ be an element of $F(S)$. By Modules, Lemma 9.4. the set $U_i = \{s \in S \mid (q \circ s_i)_s \text{ surjective}\}$ is open in S. Since $\mathcal{O}_{S,s}$ is a local ring and $\mathcal{Q}_s$ a finite $\mathcal{O}_{S,s}$-module by Nakayama's lemma (Algebra, Lemma 19.1) we have

$$s \in U_i \Leftrightarrow \big(\text{the map } \kappa(s)^{\oplus n-k} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s \text{ induced by } (q \circ s_i)_s \text{ is surjective}\big)$$

Let $f : T \to S$ be a morphism of schemes and let $t \in T$ be a point mapping to $s \in S$. We have $(f^*\mathcal{Q})_t = \mathcal{Q}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{T,t}$ (Sheaves, Lemma 26.4) and so on. Thus the map

$$\kappa(t)^{\oplus n-k} \to (f^*\mathcal{Q})_t/\mathfrak{m}_t(f^*\mathcal{Q})_t$$

induced by $(f^*q \circ f^*s_i)_t$ is the base change of the map $\kappa(s)^{\oplus n-k} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$ above by the field extension $\kappa(s) \subset \kappa(t)$. It follows that $s \in U_i$ if and only if t is in the corresponding open for f^*q. In particular $T \to S$ factors through U_i if and only if $f^*q \in F_i(T)$ as desired.

The collection F_i, $i \in I$ covers F. Let $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ be an element of $F(S)$. We have to show that for every point s of S there exists an $i \in I$ such that s_i is surjective in a neighbourhood of s. Thus we have to show that one of the compositions

$$\kappa(s)^{\oplus n-k} \xrightarrow{s_i} \kappa(s)^{\oplus n} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$$

is surjective (see previous paragraph). As $\mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$ is a vector space of dimension $n-k$ this follows from the theory of vector spaces. ☐

Definition 22.2. Let $0 < k < n$. The scheme $\mathbf{G}(k, n)$ representing the functor $G(k, n)$ is called *Grassmannian over* $\mathbf{Z}$. Its base change $\mathbf{G}(k, n)_S$ to a scheme S is called *Grassmannian over* S. If R is a ring the base change to $\mathrm{Spec}(R)$ is denoted $\mathbf{G}(k, n)_R$ and called *Grassmannian over* R.

The definition makes sense as we've shown in Lemma 22.1 that these functors are indeed representable.

Lemma 22.3. *Let $n \geq 1$. There is a canonical isomorphism $\mathbf{G}(n, n+1) = \mathbf{P}^n_{\mathbf{Z}}$.*

Proof. According to Lemma 13.1 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ represents the functor which assigns to a scheme S the set of isomorphisms classes of pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ consisting of an invertible module $\mathcal{L}$ and an $(n+1)$-tuple of global sections generating $\mathcal{L}$. Given such a pair we obtain a quotient

$$\mathcal{O}_S^{\oplus n+1} \longrightarrow \mathcal{L}, \quad (h_0, \ldots, h_n) \longmapsto \sum h_i s_i.$$

Conversely, given an element $q : \mathcal{O}_S^{\oplus n+1} \to \mathcal{Q}$ of $G(n, n+1)(S)$ we obtain such a pair, namely $(\mathcal{Q}, (q(e_1), \ldots, q(e_{n+1})))$. Here e_i, $i = 1, \ldots, n+1$ are the standard generating sections of the free module $\mathcal{O}_S^{\oplus n+1}$. We omit the verification that these constructions define mutually inverse transformations of functors. □

23. Other chapters

References

[DG67] Jean Dieudonné and Alexander Grothendieck, *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1961–1967).