CRYSTALLINE COHOMOLOGY

Contents

1. Introduction

This chapter is based on a lecture series given by Johan de Jong held in 2012 at Columbia University. The goals of this chapter are to give a quick introduction to crystalline cohomology. A reference is the book [Ber74].

We have moved the more elementary purely algebraic discussion of divided power rings to a preliminary chapter as it is also useful in discussing Tate resolutions in commutative algebra. Please see Divided Power Algebra, Section 1.

2. Divided power envelope

The construction of the following lemma will be dubbed the divided power envelope. It will play an important role later.

Lemma 2.1. *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. There exists a homomorphism of divided power rings*

$$(A, I, \gamma) \longrightarrow (D, \bar{J}, \bar{\gamma})$$

such that

$$\mathrm{Hom}_{(A,I,\gamma)}((D, \bar{J}, \bar{\gamma}), (C, K, \delta)) = \mathrm{Hom}_{(A,I)}((B, J), (C, K))$$

functorially in the divided power algebra (C, K, δ) over (A, I, γ). Here the LHS is morphisms of divided power rings over (A, I, γ) and the RHS is morphisms of (ring, ideal) pairs over (A, I).

Proof. Denote $\mathcal{C}$ the category of divided power rings (C, K, δ). Consider the functor $F : \mathcal{C} \longrightarrow Sets$ defined by

$$F(C, K, \delta) = \left\{ (\varphi, \psi) \,\middle|\, \begin{array}{l} \varphi : (A, I, \gamma) \to (C, K, \delta) \text{ homomorphism of divided power rings} \\ \psi : (B, J) \to (C, K) \text{ an } A\text{-algebra homomorphism with } \psi(J) \subset K \end{array} \right\}$$

We will show that Divided Power Algebra, Lemma 3.3 applies to this functor which will prove the lemma. Suppose that $(\varphi, \psi) \in F(C, K, \delta)$. Let $C' \subset C$ be the subring generated by $\varphi(A)$, $\psi(B)$, and $\delta_n(\psi(f))$ for all $f \in J$. Let $K' \subset K \cap C'$ be the ideal of C' generated by $\varphi(I)$ and $\delta_n(\psi(f))$ for $f \in J$. Then $(C', K', \delta|_{K'})$ is a divided power ring and C' has cardinality bounded by the cardinal $\kappa = |A| \otimes |B|^{\aleph_0}$. Moreover, φ factors as $A \to C' \to C$ and ψ factors as $B \to B' \to B$. This proves assumption (1) of Divided Power Algebra, Lemma 3.3 holds. Assumption (2) is clear as limits in the category of divided power rings commute with the forgetful functor $(C, K, \delta) \mapsto (C, K)$, see Divided Power Algebra, Lemma 3.2 and its proof. $\square$

Definition 2.2. Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. The divided power algebra $(D, \bar{J}, \bar{\gamma})$ constructed in Lemma 2.1 is called the *divided power envelope of J in B relative to* (A, I, γ) and is denoted $D_B(J)$ or $D_{B,\gamma}(J)$.

Let $(A, I, \gamma) \to (C, K, \delta)$ be a homomorphism of divided power rings. The universal property of $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$ is

$$\begin{array}{ccc} \text{ring maps } B \to C & & \text{divided power homomorphisms} \\ \text{which map } J \text{ into } K & \longleftrightarrow & (D, \bar{J}, \bar{\gamma}) \to (C, K, \delta) \end{array}$$

and the correspondence is given by precomposing with the map $B \to D$ which corresponds to id_D. Here are some properties of $(D, \bar{J}, \bar{\gamma})$ which follow directly from the universal property. There are A-algebra maps

$$(2.2.1) \qquad\qquad\qquad B \longrightarrow D \longrightarrow B/J$$

The first arrow maps J into $\bar{J}$ and $\bar{J}$ is the kernel of the second arrow. The elements $\bar{\gamma}_n(x)$ where $n > 0$ and x is an element in the image of $J \to D$ generate $\bar{J}$ as an ideal in D and generate D as a B-algebra.

Lemma 2.3. *Let (A, I, γ) be a divided power ring. Let $\varphi : B' \to B$ be a surjection of A-algebras with kernel K. Let $IB \subset J \subset B$ be an ideal. Let $J' \subset B'$ be the inverse image of J. Write $D_{B', \gamma}(J') = (D', \bar{J}', \bar{\gamma})$. Then $D_{B, \gamma}(J) = (D'/K', \bar{J}'/K', \bar{\gamma})$ where K' is the ideal generated by the elements $\bar{\gamma}_n(k)$ for $n \geq 1$ and $k \in K$.*

Proof. Write $D_{B, \gamma}(J) = (D, \bar{J}, \bar{\gamma})$. The universal property of D' gives us a homomorphism $D' \to D$ of divided power algebras. As $B' \to B$ and $J' \to J$ are surjective, we see that $D' \to D$ is surjective (see remarks above). It is clear that $\bar{\gamma}_n(k)$ is in the kernel for $n \geq 1$ and $k \in K$, i.e., we obtain a homomorphism $D'/K' \to D$. Conversely, there exists a divided power structure on $\bar{J}'/K' \subset D'/K'$, see Divided Power Algebra, Lemma 4.3. Hence the universal property of D gives an inverse $D \to D'/K'$ and we win. $\square$

In the situation of Definition 2.2 we can choose a surjection $P \to B$ where P is a polynomial algebra over A and let $J' \subset P$ be the inverse image of J. The previous lemma describes $D_{B, \gamma}(J)$ in terms of $D_{P, \gamma}(J')$. Note that γ extends to a divided power structure γ' on IP by Divided Power Algebra, Lemma 4.2. Hence $D_{P, \gamma}(J') = D_{P, \gamma'}(J')$ is an example of a special case of divided power envelopes we describe in the following lemma.

Lemma 2.4. *Let (B, I, γ) be a divided power algebra. Let $I \subset J \subset B$ be an ideal. Let $(D, \bar{J}, \bar{\gamma})$ be the divided power envelope of J relative to γ. Choose elements $f_t \in J$, $t \in T$ such that $J = I + (f_t)$. Then there exists a surjection*

$$\Psi : B\langle x_t \rangle \longrightarrow D$$

of divided power rings mapping x_t to the image of f_t in D. The kernel of Ψ is generated by the elements $x_t - f_t$ and all

$$\delta_n \left(\sum r_t x_t - r_0 \right)$$

whenever $\sum r_t f_t = r_0$ in B for some $r_t \in B$, $r_0 \in I$.

Proof. In the statement of the lemma we think of $B\langle x_t \rangle$ as a divided power ring with ideal $J' = IB\langle x_t \rangle + B\langle x_t \rangle_+$, see Divided Power Algebra, Remark 5.2. The existence of Ψ follows from the universal property of divided power polynomial rings. Surjectivity of Ψ follows from the fact that its image is a divided power subring of D, hence equal to D by the universal property of D. It is clear that $x_t - f_t$ is in the kernel. Set

$$\mathcal{R} = \{(r_0, r_t) \in I \oplus \bigoplus_{t \in T} B \mid \sum r_t f_t = r_0 \text{ in } B\}$$

If $(r_0, r_t) \in \mathcal{R}$ then it is clear that $\sum r_t x_t - r_0$ is in the kernel. As Ψ is a homomorphism of divided power rings and $\sum r_t x_t = r_0 \in J'$ it follows that $\delta_n(\sum r_t x_t - r_0)$ is in the kernel as well. Let $K \subset B\langle x_t \rangle$ be the ideal generated by $x_t - f_t$ and the elements $\delta_n(\sum r_t x_t - r_0)$ for $(r_0, r_t) \in \mathcal{R}$. To show that $K = \text{Ker}(\Psi)$ it suffices to show that δ extends to $B\langle x_t \rangle / K$. Namely, if so the universal property of D gives a map $D \to B\langle x_t \rangle / K$ inverse to Ψ. Hence we have to show that $K \cap J'$ is preserved by δ_n, see Divided Power Algebra, Lemma 4.3. Let $K' \subset B\langle x_t \rangle$ be the ideal generated by the elements

(1) $\delta_m(\sum r_t x_t - r_0)$ where $m > 0$ and $(r_0, r_t) \in \mathcal{R}$,
(2) $x_{t'}^{[m]}(x_t - f_t)$ where $m > 0$ and $t', t \in I$.

We claim that $K' = K \cap J'$. The claim proves that $K \cap J'$ is preserved by δ_n, $n > 0$ by the criterion of Divided Power Algebra, Lemma 4.3 (2)(c) and a computation of δ_n of the elements listed which we leave to the reader. To prove the claim note that $K' \subset K \cap J'$. Conversely, if $h \in K \cap J'$ then, modulo K' we can write

$$h = \sum r_t(x_t - f_t)$$

for some $r_t \in B$. As $h \in K \cap J' \subset J'$ we see that $r_0 = \sum r_t f_t \in I$. Hence $(r_0, r_t) \in \mathcal{R}$ and we see that

$$h = \sum r_t x_t - r_0$$

is in K' as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Lemma 2.5. *Let (A, I, γ) be a divided power ring. Let B be an A-algebra and $IB \subset J \subset B$ an ideal. Let x_i be a set of variables. Then*

$$D_{B[x_i], \gamma}(JB[x_i] + (x_i)) = D_{B, \gamma}(J)\langle x_i \rangle$$

Proof. One possible proof is to deduce this from Lemma 2.4 as any relation between x_i in $B[x_i]$ is trivial. On the other hand, the lemma follows from the universal property of the divided power polynomial algebra and the universal property of divided power envelopes. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Conditions (1) and (2) of the following lemma hold if $B \to B'$ is flat at all primes of $V(IB') \subset \mathrm{Spec}(B')$ and is very closely related to that condition, see Algebra, Lemma 95.8. It in particular says that taking the divided power envelope commutes with localization.

Lemma 2.6. *Let (A, I, γ) be a divided power ring. Let $B \to B'$ be a homomorphism of A-algebras. Assume that*

(1) *$B/IB \to B'/IB'$ is flat, and*
(2) *$Tor_1^B(B', B/IB) = 0$.*

Then for any ideal $IB \subset J \subset B$ the canonical map

$$D_B(J) \otimes_B B' \longrightarrow D_{B'}(JB')$$

is an isomorphism.

Proof. Set $D = D_B(J)$ and denote $\bar{J} \subset D$ its divided power ideal with divided power structure $\bar{\gamma}$. The universal property of D produces a B-algebra map $D \to D_{B'}(JB')$, whence a map as in the lemma. It suffices to show that the divided powers $\bar{\gamma}$ extend to $D \otimes_B B'$ since then the universal property of $D_{B'}(JB')$ will produce a map $D_{B'}(JB') \to D \otimes_B B'$ inverse to the one in the lemma.

Choose a surjection $P \to B'$ where P is a polynomial algebra over B. In particular $B \to P$ is flat, hence $D \to D \otimes_B P$ is flat by Algebra, Lemma 38.6. Then $\bar{\gamma}$ extends to $D \otimes_B P$ by Divided Power Algebra, Lemma 4.2; we will denote this extension $\bar{\gamma}$ also. Set $\mathfrak{a} = \mathrm{Ker}(P \to B')$ so that we have the short exact sequence

$$0 \to \mathfrak{a} \to P \to B' \to 0$$

Thus $\operatorname{Tor}_1^B(B', B/IB) = 0$ implies that $\mathfrak{a} \cap IP = I\mathfrak{a}$. Now we have the following commutative diagram

$$
\begin{array}{ccccc}
B/J \otimes_B \mathfrak{a} & \xrightarrow{\ \beta\ } & B/J \otimes_B P & \longrightarrow & B/J \otimes_B B' \\
\uparrow & & \uparrow & & \uparrow \\
D \otimes_B \mathfrak{a} & \xrightarrow{\ \alpha\ } & D \otimes_B P & \longrightarrow & D \otimes_B B' \\
\uparrow & & \uparrow & & \uparrow \\
\bar{J} \otimes_B \mathfrak{a} & \longrightarrow & \bar{J} \otimes_B P & \longrightarrow & \bar{J} \otimes_B B'
\end{array}
$$

This diagram is exact even with 0's added at the top and the right. We have to show the divided powers on the ideal $\bar{J} \otimes_B P$ preserve the ideal $\operatorname{Im}(\alpha) \cap \bar{J} \otimes_B P$, see Divided Power Algebra, Lemma 4.3. Consider the exact sequence

$$
0 \to \mathfrak{a}/I\mathfrak{a} \to P/IP \to B'/IB' \to 0
$$

(which uses that $\mathfrak{a} \cap IP = I\mathfrak{a}$ as seen above). As B'/IB' is flat over B/IB this sequence remains exact after applying $B/J \otimes_{B/IB} -$, see Algebra, Lemma 38.11. Hence

$$
\operatorname{Ker}(B/J \otimes_{B/IB} \mathfrak{a}/I\mathfrak{a} \to B/J \otimes_{B/IB} P/IP) = \operatorname{Ker}(\mathfrak{a}/J\mathfrak{a} \to P/JP)
$$

is zero. Thus β is injective. It follows that $\operatorname{Im}(\alpha) \cap \bar{J} \otimes_B P$ is the image of $\bar{J} \otimes \mathfrak{a}$. Now if $f \in \bar{J}$ and $a \in \mathfrak{a}$, then $\bar{\gamma}_n(f \otimes a) = \bar{\gamma}_n(f) \otimes a^n$ hence the result is clear. $\square$

The following lemma is a special case of [dJ95, Proposition 2.1.7] which in turn is a generalization of [Ber74, Proposition 2.8.2].

Lemma 2.7. *Let $(B, I, \gamma) \to (B', I', \gamma')$ be a homomorphism of divided power rings. Let $I \subset J \subset B$ and $I' \subset J' \subset B'$ be ideals. Assume*

  (1) *$B/I \to B'/I'$ is flat, and*
  (2) *$J' = JB' + I'$.*

Then the canonical map

$$
D_{B,\gamma}(J) \otimes_B B' \longrightarrow D_{B',\gamma'}(J')
$$

is an isomorphism.

Proof. Set $D = D_B(J)$ and denote $\bar{J} \subset D$ its divided power ideal with divided power structure $\bar{\gamma}$. The universal property of D produces a homomorphism of divided power rings $D \to D_{B'}(J')$, whence a map as in the lemma. It suffices to show that there exist divided powers on the image of $D \otimes_B I' + \bar{J} \otimes_B B' \to D \otimes_B B'$ compatible with $\bar{\gamma}$ and γ' since then the universal property of $D_{B'}(J')$ will produce a map $D_{B'}(J') \to D \otimes_B B'$ inverse to the one in the lemma.

Choose elements $f_t \in J$ which generate J/I. Set $\mathcal{R} = \{(r_0, r_t) \in I \oplus \bigoplus_{t \in T} B \mid \sum r_t f_t = r_0 \text{ in } B\}$ as in the proof of Lemma 2.4. This lemma shows that

$$
D = B\langle x_t \rangle / K
$$

where K is generated by the elements $x_t - f_t$ and $\delta_n(\sum r_t x_t - r_0)$ for $(r_0, r_t) \in \mathcal{R}$. Thus we see that

$$
(2.7.1) \qquad\qquad D \otimes_B B' = B'\langle x_t \rangle / K'
$$

where K' is generated by the images in $B'\langle x_t \rangle$ of the generators of K listed above. Let $f'_t \in B'$ be the image of f_t. By assumption (1) we see that the elements $f'_t \in J'$ generate J'/I' and we see that $x_t - f'_t \in K'$. Set

$$\mathcal{R}' = \{(r'_0, r'_t) \in I' \oplus \bigoplus_{t \in T} B' \mid \sum r'_t f'_t = r'_0 \text{ in } B'\}$$

To finish the proof we have to show that $\delta'_n(\sum r'_t x_t - r'_0) \in K'$ for $(r'_0, r'_t) \in \mathcal{R}'$, because then the presentation (2.7.1) of $D \otimes_B B'$ is identical to the presentation of $D_{B',\gamma'}(J')$ obtain in Lemma 2.4 from the generators f'_t. Suppose that $(r'_0, r'_t) \in \mathcal{R}'$. Then $\sum r'_t f'_t = 0$ in B'/I'. As $B/I \to B'/I'$ is flat by assumption (1) we can apply the equational criterion of flatness (Algebra, Lemma 38.10) to see that there exist an $m > 0$ and $r_{jt} \in B$ and $c_j \in B'$, $j = 1, \ldots, m$ such that

$$r_{j0} = \sum r_{jt} f_t \in I \text{ for } j = 1, \ldots, m, \quad \text{and} \quad r'_t = \sum c_j r_{jt}.$$

Note that this also implies that $r'_0 = \sum c_j r_{j0}$. Then we have

$$\delta'_n(\sum r'_t x_t - r'_0) = \delta'_n(\sum c_j(\sum r_{jt} x_t - r_{j0}))$$
$$= \sum c_1^{n_1} \ldots c_m^{n_m} \delta_{n_1}(\sum r_{1t} x_t - r_{10}) \ldots \delta_{n_m}(\sum r_{mt} x_t - r_{m0})$$

where the sum is over $n_1 + \ldots + n_m = n$. This proves what we want. $\square$

3. Some explicit divided power thickenings

The constructions in this section will help us to define the connection on a crystal in modules on the crystalline site.

Lemma 3.1. *Let (A, I, γ) be a divided power ring. Let M be an A-module. Let $B = A \oplus M$ as an A-algebra where M is an ideal of square zero and set $J = I \oplus M$. Set*

$$\delta_n(x + z) = \gamma_n(x) + \gamma_{n-1}(x)z$$

for $x \in I$ and $z \in M$. Then δ is a divided power structure and $A \to B$ is a homomorphism of divided power rings from (A, I, γ) to (B, J, δ).

Proof. We have to check conditions (1) – (5) of Divided Power Algebra, Definition 2.1. We will prove this directly for this case, but please see the proof of the next lemma for a method which avoids calculations. Conditions (1) and (3) are clear. Condition (2) follows from

$$\delta_n(x + z)\delta_m(x + z) = (\gamma_n(x) + \gamma_{n-1}(x)z)(\gamma_m(x) + \gamma_{m-1}(x)z)$$
$$= \gamma_n(x)\gamma_m(x) + \gamma_n(x)\gamma_{m-1}(x)z + \gamma_{n-1}(x)\gamma_m(x)z$$
$$= \frac{(n+m)!}{n!m!}\gamma_{n+m}(x) + \left(\frac{(n+m-1)!}{n!(m-1)!} + \frac{(n+m-1)!}{(n-1)!m!}\right)\gamma_{n+m-1}(x)z$$
$$= \frac{(n+m)!}{n!m!}\delta_{n+m}(x + z)$$

Condition (5) follows from

$$\delta_n(\delta_m(x+z)) = \delta_n(\gamma_m(x) + \gamma_{m-1}(x)z)$$
$$= \gamma_n(\gamma_m(x)) + \gamma_{n-1}(\gamma_m(x))\gamma_{m-1}(x)z$$
$$= \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x) + \frac{((n-1)m)!}{(n-1)!(m!)^{n-1}}\gamma_{(n-1)m}(x)\gamma_{m-1}(x)z$$
$$= \frac{(nm)!}{n!(m!)^n}(\gamma_{nm}(x) + \gamma_{nm-1}(x)z)$$

by elementary number theory. To prove (4) we have to see that

$$\delta_n(x + x' + z + z') = \gamma_n(x + x') + \gamma_{n-1}(x + x')(z + z')$$

is equal to

$$\sum_{i=0}^{n}(\gamma_i(x) + \gamma_{i-1}(x)z)(\gamma_{n-i}(x') + \gamma_{n-i-1}(x')z')$$

This follows easily on collecting the coefficients of 1, z, and z' and using condition (4) for γ. $\qquad\square$

Lemma 3.2. *Let (A, I, γ) be a divided power ring. Let M, N be A-modules. Let $q : M \times M \to N$ be an A-bilinear map. Let $B = A \oplus M \oplus N$ as an A-algebra with multiplication*

$$(x, z, w) \cdot (x', z', w') = (xx', xz' + x'z, xw' + x'w + q(z, z') + q(z', z))$$

and set $J = I \oplus M \oplus N$. Set

$$\delta_n(x, z, w) = (\gamma_n(x), \gamma_{n-1}(x)z, \gamma_{n-1}(z)w + \gamma_{n-2}(x)q(z, z))$$

for $(a, m, n) \in J$. Then δ is a divided power structure and $A \to B$ is a homomorphism of divided power rings from (A, I, γ) to (B, J, δ).

Proof. Suppose we want to prove that property (4) of Divided Power Algebra, Definition 2.1 is satisfied. Pick (x, z, w) and (x', z', w') in J. Pick a map

$$A_0 = \mathbf{Z}\langle s, s' \rangle \longrightarrow A, \quad s \longmapsto x, s' \longmapsto x'$$

which is possible by the universal property of divided power polynomial rings. Set $M_0 = A_0 \oplus A_0$ and $N_0 = A_0 \oplus A_0 \oplus M_0 \otimes_{A_0} M_0$. Let $q_0 : M_0 \times M_0 \to N_0$ be the obvious map. Define $M_0 \to M$ as the A_0-linear map which sends the basis vectors of M_0 to z and z'. Define $N_0 \to N$ as the A_0 linear map which sends the first two basis vectors of N_0 to w and w' and uses $M_0 \otimes_{A_0} M_0 \to M \otimes_A M \xrightarrow{q} N$ on the last summand. Then we see that it suffices to prove the identity (4) for the situation (A_0, M_0, N_0, q_0). Similarly for the other identities. This reduces us to the case of a $\mathbf{Z}$-torsion free ring A and A-torsion free modules. In this case all we have to do is show that

$$n!\delta_n(x, z, w) = (x, z, w)^n$$

in the ring A, see Divided Power Algebra, Lemma 2.2. To see this note that

$$(x, z, w)^2 = (x^2, 2xz, 2xw + 2q(z, z))$$

and by induction

$$(x, z, w)^n = (x^n, nx^{n-1}z, nx^{n-1}w + n(n-1)x^{n-2}q(z, z))$$

On the other hand,

$$n!\delta_n(x, z, w) = (n!\gamma_n(x), n!\gamma_{n-1}(x)z, n!\gamma_{n-1}(x)w + n!\gamma_{n-2}(x)q(z, z))$$

which matches. This finishes the proof. $\qquad\qquad\square$

4. Compatibility

This section isn't required reading; it explains how our discussion fits with that of [Ber74]. Consider the following technical notion.

Definition 4.1. Let (A, I, γ) and (B, J, δ) be divided power rings. Let $A \to B$ be a ring map. We say δ *is compatible with* γ if there exists a divided power structure $\bar{\gamma}$ on $J + IB$ such that

$$(A, I, \gamma) \to (B, J + IB, \bar{\gamma}) \quad \text{and} \quad (B, J, \delta) \to (B, J + IB, \bar{\gamma})$$

are homomorphisms of divided power rings.

Let p be a prime number. Let (A, I, γ) be a divided power ring. Let $A \to C$ be a ring map with p nilpotent in C. Assume that γ extends to IC (see Divided Power Algebra, Lemma 4.2). In this situation, the (big affine) crystalline site of $\mathrm{Spec}(C)$ over $\mathrm{Spec}(A)$ as defined in [Ber74] is the opposite of the category of systems

$$(B, J, \delta, A \to B, C \to B/J)$$

where

(1) (B, J, δ) is a divided power ring with p nilpotent in B,
(2) δ is compatible with γ, and
(3) the diagram

$$\begin{array}{ccc} B & \longrightarrow & B/J \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

is commutative.

The conditions "γ extends to C and δ compatible with γ" are used in [Ber74] to insure that the crystalline cohomology of $\mathrm{Spec}(C)$ is the same as the crystalline cohomology of $\mathrm{Spec}(C/IC)$. We will avoid this issue by working exclusively with C such that $IC = 0$[1]. In this case, for a system $(B, J, \delta, A \to B, C \to B/J)$ as above, the commutativity of the displayed diagram above implies $IB \subset J$ and compatibility is equivalent to the condition that $(A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings.

5. Affine crystalline site

In this section we discuss the algebraic variant of the crystalline site. Our basic situation in which we discuss this material will be as follows.

Situation 5.1. Here p is a prime number, (A, I, γ) is a divided power ring such that A is a $\mathbf{Z}_{(p)}$-algebra, and $A \to C$ is a ring map such that $IC = 0$ and such that p is nilpotent in C.

Usually the prime number p will be contained in the divided power ideal I.

Definition 5.2. In Situation 5.1.

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[1]Of course there will be a price to pay.

(1) A *divided power thickening* of C over (A, I, γ) is a homomorphism of divided power algebras $(A, I, \gamma) \to (B, J, \delta)$ such that p is nilpotent in B and a ring map $C \to B/J$ such that

$$
\begin{array}{ccc}
B & \longrightarrow & B/J \\
\uparrow & & \uparrow \\
& C & \\
& \uparrow & \\
A & \longrightarrow & A/I
\end{array}
$$

   is commutative.

(2) A *homomorphism of divided power thickenings*

$$(B, J, \delta, C \to B/J) \longrightarrow (B', J', \delta', C \to B'/J')$$

   is a homomorphism $\varphi : B \to B'$ of divided power A-algebras such that $C \to B/J \to B'/J'$ is the given map $C \to B'/J'$.

(3) We denote $\mathrm{CRIS}(C/A, I, \gamma)$ or simply $\mathrm{CRIS}(C/A)$ the category of divided power thickenings of C over (A, I, γ).

(4) We denote $\mathrm{Cris}(C/A, I, \gamma)$ or simply $\mathrm{Cris}(C/A)$ the full subcategory consisting of $(B, J, \delta, C \to B/J)$ such that $C \to B/J$ is an isomorphism. We often denote such an object $(B \to C, \delta)$ with $J = \mathrm{Ker}(B \to C)$ being understood.

Note that for a divided power thickening (B, J, δ) as above the ideal J is locally nilpotent, see Divided Power Algebra, Lemma 2.6. There is a canonical functor

(5.2.1) $\qquad \mathrm{CRIS}(C/A) \longrightarrow C\text{-algebras}, \quad (B, J, \delta) \longmapsto B/J$

This category does not have equalizers or fibre products in general. It also doesn't have an initial object (= empty colimit) in general.

Lemma 5.3. *In Situation 5.1.*

   (1) *$\mathrm{CRIS}(C/A)$ has products,*
   (2) *$\mathrm{CRIS}(C/A)$ has all finite nonempty colimits and (5.2.1) commutes with these, and*
   (3) *$\mathrm{Cris}(C/A)$ has all finite nonempty colimits and $\mathrm{Cris}(C/A) \to \mathrm{CRIS}(C/A)$ commutes with them.*

Proof. The empty product is $(C, 0, \emptyset)$. If (B_t, J_t, δ_t) is a family of objects of $\mathrm{CRIS}(C/A)$ then we can form the product $(\prod B_t, \prod J_t, \prod \delta_t)$ as in Divided Power Algebra, Lemma 3.4. The map $C \to \prod B_t / \prod J_t = \prod B_t / J_t$ is clear.

Given two objects (B, J, γ) and (B', J', γ') of $\mathrm{CRIS}(C/A)$ we can form a cocartesian diagram

$$
\begin{array}{ccc}
(B, J, \delta) & \longrightarrow & (B'', J'', \delta'') \\
\uparrow & & \uparrow \\
(A, I, \gamma) & \longrightarrow & (B', J', \delta')
\end{array}
$$

in the category of divided power rings. Then we see that we have

$$B''/J'' = B/J \otimes_{A/I} B'/J' \longleftarrow C \otimes_{A/I} C$$

see Divided Power Algebra, Remark 3.5. Denote $J'' \subset K \subset B''$ the ideal such that

$$
\begin{array}{ccc}
B''/J'' & \longrightarrow & B''/K \\
\uparrow & & \uparrow \\
\\
C \otimes_{A/I} C & \longrightarrow & C
\end{array}
$$

is a pushout, i.e., $B''/K \cong B/J \otimes_C B'/J'$. Let $D_{B''}(K) = (D, \bar{K}, \bar{\delta})$ be the divided power envelope of K in B'' relative to (B'', J'', δ''). Then it is easily verified that $(D, \bar{K}, \bar{\delta})$ is a coproduct of (B, J, δ) and (B', J', δ') in $\mathrm{CRIS}(C/A)$.

Next, we come to coequalizers. Let $\alpha, \beta : (B, J, \delta) \to (B', J', \delta')$ be morphisms of $\mathrm{CRIS}(C/A)$. Consider $B'' = B'/(\alpha(b) - \beta(b))$. Let $J'' \subset B''$ be the image of J'. Let $D_{B''}(J'') = (D, \bar{J}, \bar{\delta})$ be the divided power envelope of J'' in B'' relative to (B', J', δ'). Then it is easily verified that $(D, \bar{J}, \bar{\delta})$ is the coequalizer of (B, J, δ) and (B', J', δ') in $\mathrm{CRIS}(C/A)$.

By Categories, Lemma 18.6 we have all finite nonempty colimits in $\mathrm{CRIS}(C/A)$. The constructions above shows that (5.2.1) commutes with them. This formally implies part (3) as $\mathrm{Cris}(C/A)$ is the fibre category of (5.2.1) over C. $\qquad\square$

Remark 5.4. In Situation 5.1 we denote $\mathrm{Cris}^\wedge(C/A)$ the category whose objects are pairs $(B \to C, \delta)$ such that

(1) B is a p-adically complete A-algebra,
(2) $B \to C$ is a surjection of A-algebras,
(3) δ is a divided power structure on $\mathrm{Ker}(B \to C)$,
(4) $A \to B$ is a homomorphism of divided power rings.

Morphisms are defined as in Definition 5.2. Then $\mathrm{Cris}(C/A) \subset \mathrm{Cris}^\wedge(C/A)$ is the full subcategory consisting of those B such that p is nilpotent in B. Conversely, any object $(B \to C, \delta)$ of $\mathrm{Cris}^\wedge(C/A)$ is equal to the limit

$$(B \to C, \delta) = \lim_e (B/p^e B \to C, \delta)$$

where for $e \gg 0$ the object $(B/p^e B \to C, \delta)$ lies in $\mathrm{Cris}(C/A)$, see Divided Power Algebra, Lemma 4.5. In particular, we see that $\mathrm{Cris}^\wedge(C/A)$ is a full subcategory of the category of pro-objects of $\mathrm{Cris}(C/A)$, see Categories, Remark 22.4.

Lemma 5.5. *In Situation 5.1. Let $P \to C$ be a surjection of A-algebras with kernel J. Write $D_{P,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$. Let $(D^\wedge, J^\wedge, \bar{\gamma}^\wedge)$ be the p-adic completion of D, see Divided Power Algebra, Lemma 4.5. For every $e \geq 1$ set $P_e = P/p^e P$ and $J_e \subset P_e$ the image of J and write $D_{P_e, \gamma}(J_e) = (D_e, \bar{J}_e, \bar{\gamma})$. Then for all e large enough we have*

(1) *$p^e D \subset \bar{J}$ and $p^e D^\wedge \subset \bar{J}^\wedge$ are preserved by divided powers,*
(2) *$D^\wedge/p^e D^\wedge = D/p^e D = D_e$ as divided power rings,*
(3) *$(D_e, \bar{J}_e, \bar{\gamma})$ is an object of $\mathrm{Cris}(C/A)$,*
(4) *$(D^\wedge, \bar{J}^\wedge, \bar{\gamma}^\wedge)$ is equal to $\lim_e (D_e, \bar{J}_e, \bar{\gamma})$, and*
(5) *$(D^\wedge, \bar{J}^\wedge, \bar{\gamma}^\wedge)$ is an object of $\mathrm{Cris}^\wedge(C/A)$.*

Proof. Part (1) follows from Divided Power Algebra, Lemma 4.5. It is a general property of p-adic completion that $D/p^e D = D^\wedge/p^e D^\wedge$. Since $D/p^e D$ is a divided power ring and since $P \to D/p^e D$ factors through P_e, the universal property of D_e produces a map $D_e \to D/p^e D$. Conversely, the universal property of D produces a map $D \to D_e$ which factors through $D/p^e D$. We omit the verification that these

maps are mutually inverse. This proves (2). If e is large enough, then $p^e C = 0$, hence we see (3) holds. Part (4) follows from Divided Power Algebra, Lemma 4.5. Part (5) is clear from the definitions. $\square$

Lemma 5.6. *In Situation 5.1. Let P be a polynomial algebra over A and let $P \to C$ be a surjection of A-algebras with kernel J. With $(D_e, \bar{J}_e, \bar{\gamma})$ as in Lemma 5.5: for every object (B, J_B, δ) of $\mathrm{CRIS}(C/A)$ there exists an e and a morphism $D_e \to B$ of $\mathrm{CRIS}(C/A)$.*

Proof. We can find an A-algebra homomorphism $P \to B$ lifting the map $C \to B/J_B$. By our definition of $\mathrm{CRIS}(C/A)$ we see that $p^e B = 0$ for some e hence $P \to B$ factors as $P \to P_e \to B$. By the universal property of the divided power envelope we conclude that $P_e \to B$ factors through D_e. $\square$

Lemma 5.7. *In Situation 5.1. Let P be a polynomial algebra over A and let $P \to C$ be a surjection of A-algebras with kernel J. Let $(D, \bar{J}, \bar{\gamma})$ be the p-adic completion of $D_{P,\gamma}(J)$. For every object $(B \to C, \delta)$ of $\mathrm{Cris}^\wedge(C/A)$ there exists a morphism $D \to B$ of $\mathrm{Cris}^\wedge(C/A)$.*

Proof. We can find an A-algebra homomorphism $P \to B$ compatible with maps to C. By our definition of $\mathrm{Cris}(C/A)$ we see that $P \to B$ factors as $P \to D_{P,\gamma}(J) \to B$. As B is p-adically complete we can factor this map through D. $\square$

6. Module of differentials

In this section we develop a theory of modules of differentials for divided power rings.

Definition 6.1. Let A be a ring. Let (B, J, δ) be a divided power ring. Let $A \to B$ be a ring map. Let M be an B-module. A *divided power A-derivation* into M is a map $\theta : B \to M$ which is additive, annihilates the elements of A, satisfies the Leibniz rule $\theta(bb') = b\theta(b') + b'\theta(b)$ and satisfies

$$\theta(\gamma_n(x)) = \gamma_{n-1}(x)\theta(x)$$

for all $n \geq 1$ and all $x \in J$.

In the situation of the definition, just as in the case of usual derivations, there exists a *universal divided power A-derivation*

$$\mathrm{d}_{B/A,\delta} : B \to \Omega_{B/A,\delta}$$

such that any divided power A-derivation $\theta : B \to M$ is equal to $\theta = \xi \circ d_{B/A,\delta}$ for some B-linear map $\Omega_{B/A,\delta} \to M$. If $(A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings, then we can forget the divided powers on A and consider the divided power derivations of B over A. Here are some basic properties of the divided power module of differentials.

Lemma 6.2. *Let A be a ring. Let (B, J, δ) be a divided power ring and $A \to B$ a ring map.*

(1) *Consider $B[x]$ with divided power ideal $(JB[x], \delta')$ where δ' is the extension of δ to $B[x]$. Then*

$$\Omega_{B[x]/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B[x] \oplus B[x]dx.$$

(2) *Consider $B\langle x\rangle$ with divided power ideal $(JB\langle x\rangle + B\langle x\rangle_+, \delta')$. Then*

$$\Omega_{B\langle x\rangle/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B\langle x\rangle \oplus B\langle x\rangle\,dx.$$

(3) *Let $K \subset J$ be an ideal preserved by δ_n for all $n > 0$. Set $B' = B/K$ and denote δ' the induced divided power on J/K. Then $\Omega_{B'/A,\delta'}$ is the quotient of $\Omega_{B/A,\delta} \otimes_B B'$ by the B'-submodule generated by dk for $k \in K$.*

Proof. These are proved directly from the construction of $\Omega_{B/A,\delta}$ as the free B-module on the elements db modulo the relations

(1) $d(b + b') = db + db'$, $b, b' \in B$,
(2) $da = 0$, $a \in A$,
(3) $d(bb') = bdb' + b'db$, $b, b' \in B$,
(4) $d\delta_n(f) = \delta_{n-1}(f)df$, $f \in J$, $n > 1$.

Note that the last relation explains why we get "the same" answer for the divided power polynomial algebra and the usual polynomial algebra: in the first case x is an element of the divided power ideal and hence $dx^{[n]} = x^{[n-1]}dx$. $\qquad\square$

Let (A, I, γ) be a divided power ring. In this setting the correct version of the powers of I is given by the divided powers

$$I^{[n]} = \text{ideal generate by } \gamma_{e_1}(x_1)\ldots\gamma_{e_t}(x_t) \text{ with } \sum e_j \geq n \text{ and } x_j \in I.$$

Of course we have $I^n \subset I^{[n]}$. Note that $I^{[1]} = I$. Sometimes we also set $I^{[0]} = A$.

Lemma 6.3. *Let $(A, I, \gamma) \to (B, J, \delta)$ be a homomorphism of divided power rings. Let $(B(1), J(1), \delta(1))$ be the coproduct of (B, J, δ) with itself over (A, I, γ), i.e., such that*

$$
\begin{array}{ccc}
(B, J, \delta) & \longrightarrow & (B(1), J(1), \delta(1)) \\
\uparrow & & \uparrow \\
(A, I, \gamma) & \longrightarrow & (B, J, \delta)
\end{array}
$$

is cocartesian. Denote $K = \mathrm{Ker}(B(1) \to B)$. Then $K \cap J(1) \subset J(1)$ is preserved by the divided power structure and

$$\Omega_{B/A,\delta} = K/\left(K^2 + (K \cap J(1))^{[2]}\right)$$

canonically.

Proof. The fact that $K \cap J(1) \subset J(1)$ is preserved by the divided power structure follows from the fact that $B(1) \to B$ is a homomorphism of divided power rings.

Recall that K/K^2 has a canonical B-module structure. Denote $s_0, s_1 : B \to B(1)$ the two coprojections and consider the map $d : B \to K/K^2 + (K \cap J(1))^{[2]}$ given by $b \mapsto s_1(b) - s_0(b)$. It is clear that d is additive, annihilates A, and satisfies the Leibniz rule. We claim that d is an A-derivation. Let $x \in J$. Set $y = s_1(x)$ and $z = s_0(x)$. Denote δ the divided power structure on $J(1)$. We have to show that $\delta_n(y) - \delta_n(z) = \delta_{n-1}(y)(y - z)$ modulo $K^2 + (K \cap J(1))^{[2]}$ for $n \geq 1$. We will show this by induction on n. It is true for $n = 1$. Let $n > 1$ and that it holds for all smaller values. Note that

$$\delta_n(z - y) = \sum_{i=0}^{n}(-1)^{n-i}\delta_i(z)\delta_{n-i}(y)$$

is an element of $K^2 + (K \cap J(1))^{[2]}$. From this and induction we see that working modulo $K^2 + (K \cap J(1))^{[2]}$ we have

$$\delta_n(y) - \delta_n(z)$$
$$= \delta_n(y) + \sum_{i=0}^{n-1} (-1)^{n-i} \delta_i(z) \delta_{n-i}(y)$$
$$= \delta_n(y) + (-1)^n \delta_n(y) + \sum_{i=1}^{n-1} (-1)^{n-i} (\delta_i(y) - \delta_{i-1}(y)(y-z)) \delta_{n-i}(y)$$

Using that $\delta_i(y)\delta_{n-i}(y) = \binom{n}{i} \delta_n(y)$ and that $\delta_{i-1}(y)\delta_{n-i}(y) = \binom{n-1}{i} \delta_{n-1}(y)$ the reader easily verifies that this expression comes out to give $\delta_{n-1}(y)(y-z)$ as desired.

Let M be a B-module. Let $\theta : B \to M$ be a divided power A-derivation. Set $D = B \oplus M$ where M is an ideal of square zero. Define a divided power structure on $J \oplus M \subset D$ by setting $\delta_n(x + m) = \delta_n(x) + \delta_{n-1}(x)m$ for $n > 1$, see Lemma 3.1. There are two divided power algebra homomorphisms $B \to D$: the first is given by the inclusion and the second by the map $b \mapsto b + \theta(b)$. Hence we get a canonical homomorphism $B(1) \to D$ of divided power algebras over (A, I, γ). This induces a map $K \to M$ which annihilates K^2 (as M is an ideal of square zero) and $(K \cap J(1))^{[2]}$ as $M^{[2]} = 0$. The composition $B \to K/K^2 + (K \cap J(1))^{[2]} \to M$ equals θ by construction. It follows that d is a universal divided power A-derivation and we win. $\square$

Remark 6.4. Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. The universal module $\Omega_{B/A,\delta}$ comes with a little bit of extra structure, namely the B-submodule N of $\Omega_{B/A,\delta}$ generated by $d_{B/A,\delta}(J)$. In terms of the isomorphism given in Lemma 6.3 this corresponds to the image of $K \cap J(1)$ in $\Omega_{B/A,\delta}$. Consider the A-algebra $D = B \oplus \Omega^1_{B/A,\delta}$ with ideal $\bar{J} = J \oplus N$ and divided powers $\bar{\delta}$ as in the proof of the lemma. Then $(D, \bar{J}, \bar{\delta})$ is a divided power ring and the two maps $B \to D$ given by $b \mapsto b$ and $b \mapsto b + d_{B/A,\delta}(b)$ are homomorphisms of divided power rings over A. Moreover, N is the smallest submodule of $\Omega_{B/A,\delta}$ such that this is true.

Lemma 6.5. *In Situation 5.1. Let (B, J, δ) be an object of CRIS(C/A). Let $(B(1), J(1), \delta(1))$ be the coproduct of (B, J, δ) with itself in CRIS(C/A). Denote $K = Ker(B(1) \to B)$. Then $K \cap J(1) \subset J(1)$ is preserved by the divided power structure and*

$$\Omega_{B/A,\delta} = K / \left(K^2 + (K \cap J(1))^{[2]} \right)$$

canonically.

Proof. Word for word the same as the proof of Lemma 6.3. The only point that has to be checked is that the divided power ring $D = B \oplus M$ is an object of CRIS(C/A) and that the two maps $B \to C$ are morphisms of CRIS(C/A). Since $D/(J \oplus M) = B/J$ we can use $C \to B/J$ to view D as an object of CRIS(C/A) and the statement on morphisms is clear from the construction. $\square$

Lemma 6.6. *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map and let $IB \subset J \subset B$ be an ideal. Let $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$ be the divided power envelope. Then we have*

$$\Omega_{D/A,\bar{\gamma}} = \Omega_{B/A} \otimes_B D$$

Proof. We will prove this first when B is flat over A. In this case γ extends to a divided power structure γ' on IB, see Divided Power Algebra, Lemma 4.2. Hence $D = D_{B',\gamma'}(J)$ is equal to a quotient of the divided power ring (D', J', δ) where $D' = B\langle x_t \rangle$ and $J' = IB\langle x_t \rangle + B\langle x_t \rangle_+$ by the elements $x_t - f_t$ and $\delta_n(\sum r_t x_t - r_0)$, see Lemma 2.4 for notation and explanation. Write $\mathrm{d} : D' \to \Omega_{D'/A,\delta}$ for the universal derivation. Note that

$$\Omega_{D'/A,\delta} = \Omega_{B/A} \otimes_B D' \oplus \bigoplus D' \mathrm{d}x_t,$$

see Lemma 6.2. We conclude that $\Omega_{D/A,\bar{\gamma}}$ is the quotient of $\Omega_{D'/A,\delta} \otimes_{D'} D$ by the submodule generated by d applied to the generators of the kernel of $D' \to D$ listed above, see Lemma 6.2. Since $\mathrm{d}(x_t - f_t) = -\mathrm{d}f_t + \mathrm{d}x_t$ we see that we have $\mathrm{d}x_t = \mathrm{d}f_t$ in the quotient. In particular we see that $\Omega_{B/A} \otimes_B D \to \Omega_{D/A,\gamma}$ is surjective with kernel given by the images of d applied to the elements $\delta_n(\sum r_t x_t - r_0)$. However, given a relation $\sum r_t f_t - r_0 = 0$ in B with $r_t \in B$ and $r_0 \in IB$ we see that

$$\begin{aligned}
\mathrm{d}\delta_n(\sum r_t x_t - r_0) &= \delta_{n-1}(\sum r_t x_t - r_0)\mathrm{d}(\sum r_t x_t - r_0) \\
&= \delta_{n-1}(\sum r_t x_t - r_0)\left(\sum r_t \mathrm{d}(x_t - f_t) + \sum(x_t - f_t)\mathrm{d}r_t\right)
\end{aligned}$$

because $\sum r_t f_t - r_0 = 0$ in B. Hence this is already zero in $\Omega_{B/A} \otimes_A D$ and we win in the case that B is flat over A.

In the general case we write B as a quotient of a polynomial ring $P \to B$ and let $J' \subset P$ be the inverse image of J. Then $D = D'/K'$ with notation as in Lemma 2.3. By the case handled in the first paragraph of the proof we have $\Omega_{D'/A,\bar{\gamma}'} = \Omega_{P/A} \otimes_P D'$. Then $\Omega_{D/A,\bar{\gamma}}$ is the quotient of $\Omega_{P/A} \otimes_P D$ by the submodule generated by $\mathrm{d}\bar{\gamma}'_n(k)$ where k is an element of the kernel of $P \to B$, see Lemma 6.2 and the description of K' from Lemma 2.3. Since $\mathrm{d}\bar{\gamma}'_n(k) = \bar{\gamma}'_{n-1}(k)\mathrm{d}k$ we see again that it suffices to divided by the submodule generated by $\mathrm{d}k$ with $k \in \mathrm{Ker}(P \to B)$ and since $\Omega_{B/A}$ is the quotient of $\Omega_{P/A} \otimes_A B$ by these elements (Algebra, Lemma 127.9) we win. $\square$

Remark 6.7. Let B be a ring. Write $\Omega_B = \Omega_{B/\mathbf{Z}}$ for the absolute[2] module of differentials of B. Let $\mathrm{d} : B \to \Omega_B$ denote the universal derivation. Set $\Omega_B^i = \wedge_B^i(\Omega_B)$ as in Algebra, Section 12. The absolute *de Rham complex*

$$\Omega_B^0 \to \Omega_B^1 \to \Omega_B^2 \to \ldots$$

Here $\mathrm{d} : \Omega_B^p \to \Omega_B^{p+1}$ is defined by the rule

$$\mathrm{d}(b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p) = \mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$$

which we will show is well defined; note that $\mathrm{d} \circ \mathrm{d} = 0$ so we get a complex. Recall that Ω_B is the B-module generated by elements $\mathrm{d}b$ subject to the relations $\mathrm{d}(a + b) = \mathrm{d}a + \mathrm{d}b$ and $\mathrm{d}(ab) = b\mathrm{d}a + a\mathrm{d}b$ for $a, b \in B$. To prove that our map is well defined for $p = 1$ we have to show that the elements

$$a\mathrm{d}(b + c) - a\mathrm{d}b - a\mathrm{d}c \quad \text{and} \quad a\mathrm{d}(bc) - ac\mathrm{d}b - ab\mathrm{d}c, \quad a, b, c \in B$$

---

[2]This actually makes sense: if Ω_B is the module of differentials where we only assume the Leibniz rule and not the vanishing of $\mathrm{d}1$, then the Leibniz rule gives $\mathrm{d}1 = \mathrm{d}(1 \cdot 1) = 1\mathrm{d}1 + 1\mathrm{d}1 = 2\mathrm{d}1$ and hence $\mathrm{d}1 = 0$ in Ω_B.

are mapped to zero by our rule. This is clear by direct computation (using the Leibniz rule). Thus we get a map

$$\Omega_B \otimes_{\mathbf{Z}} \ldots \otimes_{\mathbf{Z}} \Omega_B \longrightarrow \Omega_B^{p+1}$$

defined by the formula

$$\omega_1 \otimes \ldots \otimes \omega_p \longmapsto \sum (-1)^{i+1} \omega_1 \wedge \ldots \wedge \mathrm{d}(\omega_i) \wedge \ldots \wedge \omega_p$$

which matches our rule above on elements of the form $b_0 \mathrm{d}b_1 \otimes \mathrm{d}b_2 \otimes \ldots \otimes \mathrm{d}b_p$. It is clear that this map is alternating. To finish we have to show that

$$\omega_1 \otimes \ldots \otimes f\omega_i \otimes \ldots \otimes \omega_p \quad \text{and} \quad \omega_1 \otimes \ldots \otimes f\omega_j \otimes \ldots \otimes \omega_p$$

are mapped to the same element. By $\mathbf{Z}$-linearity and the alternating property, it is enough to show this for $p = 2$, $i = 1$, $j = 2$, $\omega_1 = a_1 \mathrm{d}b_1$ and $\omega_2 = a_2 \mathrm{d}b_2$. Thus we need to show that

$$\mathrm{d}fa_1 \wedge \mathrm{d}b_1 \wedge a_2 \mathrm{d}b_2 - fa_1 \mathrm{d}b_1 \wedge \mathrm{d}a_2 \wedge \mathrm{d}b_2$$
$$= \mathrm{d}a_1 \wedge \mathrm{d}b_1 \wedge fa_2 \mathrm{d}b_2 - a_1 \mathrm{d}b_1 \wedge \mathrm{d}fa_2 \wedge \mathrm{d}b_2$$

in other words that

$$(a_2 \mathrm{d}fa_1 + fa_1 \mathrm{d}a_2 - fa_2 \mathrm{d}a_1 - a_1 \mathrm{d}fa_2) \wedge \mathrm{d}b_1 \wedge \mathrm{d}b_2 = 0.$$

This follows from the Leibniz rule.

Lemma 6.8. *Let B be a ring. Let $\pi : \Omega_B \to \Omega$ be a surjective B-module map. Denote $d : B \to \Omega$ the composition of π with $d_B : B \to \Omega_B$. Set $\Omega^i = \wedge_B^i(\Omega)$. Assume that the kernel of π is generated, as a B-module, by elements $\omega \in \Omega_B$ such that $d_B(\omega) \in \Omega_B^2$ maps to zero in Ω^2. Then there is a de Rham complex*

$$\Omega^0 \to \Omega^1 \to \Omega^2 \to \ldots$$

whose differential is defined by the rule

$$d : \Omega^p \to \Omega^{p+1}, \quad d\left(f_0 df_1 \wedge \ldots \wedge df_p\right) = df_0 \wedge df_1 \wedge \ldots \wedge df_p$$

Proof. We will show that there exists a commutative diagram

$$
\begin{array}{ccccccc}
\Omega_B^0 & \xrightarrow{\ \mathrm{d}_B\ } & \Omega_B^1 & \xrightarrow{\ \mathrm{d}_B\ } & \Omega_B^2 & \xrightarrow{\ \mathrm{d}_B\ } & \ldots \\
\downarrow & & \pi \downarrow & & \wedge^2 \pi \downarrow & & \\
\Omega^0 & \xrightarrow{\ \mathrm{d}\ } & \Omega^1 & \xrightarrow{\ \mathrm{d}\ } & \Omega^2 & \xrightarrow{\ \mathrm{d}\ } & \ldots
\end{array}
$$

the description of the map d will follow from the construction of d_B in Remark 6.7. Since the left most vertical arrow is an isomorphism we have the first square. Because π is surjective, to get the second square it suffices to show that d_B maps the kernel of π into the kernel of $\wedge^2 \pi$. We are given that any element of the kernel of π is of the form $\sum b_i \omega_i$ with $\pi(\omega_i) = 0$ and $\wedge^2 \pi(\mathrm{d}_B(\omega_i)) = 0$. By the Leibniz rule for d_B we have $\mathrm{d}_B(\sum b_i \omega_i) = \sum b_i \mathrm{d}_B(\omega_i) + \sum \mathrm{d}_B(b_i) \wedge \omega_i$. Hence this maps to zero under $\wedge^2 \pi$.

For $i > 1$ we note that $\wedge^i \pi$ is surjective with kernel the image of $\mathrm{Ker}(\pi) \wedge \Omega_B^{i-1} \to \Omega_B^i$. For $\omega_1 \in \mathrm{Ker}(\pi)$ and $\omega_2 \in \Omega_B^{i-1}$ we have

$$\mathrm{d}_B(\omega_1 \wedge \omega_2) = \mathrm{d}_B(\omega_1) \wedge \omega_2 - \omega_1 \wedge \mathrm{d}_B(\omega_2)$$

which is in the kernel of $\wedge^{i+1} \pi$ by what we just proved above. Hence we get the $(i+1)$st square in the diagram above. This concludes the proof. $\square$

Remark 6.9. Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. Set $\Omega^i_{B/A,\delta} = \wedge^i_B \Omega_{B/A,\delta}$ where $\Omega_{B/A,\delta}$ is the target of the universal divided power A-derivation $\mathrm{d} = \mathrm{d}_{B/A} : B \to \Omega_{B/A,\delta}$. Note that $\Omega_{B/A,\delta}$ is the quotient of Ω_B by the B-submodule generated by the elements $\mathrm{d}a = 0$ for $a \in A$ and $\mathrm{d}\delta_n(x) - \delta_{n-1}(x)\mathrm{d}x$ for $x \in J$. We claim Lemma 6.8 applies. To see this it suffices to verify the elements $\mathrm{d}a$ and $\mathrm{d}\delta_n(x) - \delta_{n-1}(x)\mathrm{d}x$ of Ω_B are mapped to zero in $\Omega^2_{B/A,\delta}$. This is clear for the first, and for the last we observe that

$$\mathrm{d}(\delta_{n-1}(x)) \wedge \mathrm{d}x = \delta_{n-2}(x)\mathrm{d}x \wedge \mathrm{d}x = 0$$

in $\Omega^2_{B/A,\delta}$ as desired. Hence we obtain a *divided power de Rham complex*

$$\Omega^0_{B/A,\delta} \to \Omega^1_{B/A,\delta} \to \Omega^2_{B/A,\delta} \to \dots$$

which will play an important role in the sequel.

Remark 6.10. Let B be a ring. Let $\Omega_B \to \Omega$ be a quotient satisfying the assumptions of Lemma 6.8. Let M be a B-module. A *connection* is an additive map

$$\nabla : M \longrightarrow M \otimes_B \Omega$$

such that $\nabla(bm) = b\nabla(m) + m \otimes \mathrm{d}b$ for $b \in B$ and $m \in M$. In this situation we can define maps

$$\nabla : M \otimes_B \Omega^i \longrightarrow M \otimes_B \Omega^{i+1}$$

by the rule $\nabla(m \otimes \omega) = \nabla(m) \wedge \omega + m \otimes \mathrm{d}\omega$. This works because if $b \in B$, then

$$\nabla(bm \otimes \omega) - \nabla(m \otimes b\omega) = \nabla(bm) \otimes \omega + bm \otimes \mathrm{d}\omega - \nabla(m) \otimes b\omega - m \otimes \mathrm{d}(b\omega)$$
$$= b\nabla(m) \otimes \omega + m \otimes \mathrm{d}b \wedge \omega + bm \otimes \mathrm{d}\omega$$
$$- b\nabla(m) \otimes \omega - bm \otimes \mathrm{d}(\omega) - m \otimes \mathrm{d}b \wedge \omega = 0$$

As is customary we say the connection is *integrable* if and only if the composition

$$M \xrightarrow{\nabla} M \otimes_B \Omega^1 \xrightarrow{\nabla} M \otimes_B \Omega^2$$

is zero. In this case we obtain a complex

$$M \xrightarrow{\nabla} M \otimes_B \Omega^1 \xrightarrow{\nabla} M \otimes_B \Omega^2 \xrightarrow{\nabla} M \otimes_B \Omega^3 \xrightarrow{\nabla} M \otimes_B \Omega^4 \to \dots$$

which is called the de Rham complex of the connection.

Remark 6.11. Let $\varphi : B \to B'$ be a ring map. Let $\Omega_B \to \Omega$ and $\Omega_{B'} \to \Omega'$ be quotients satisfying the assumptions of Lemma 6.8. Assume that the map $\Omega_B \to \Omega_{B'}$, $b_1\mathrm{d}b_2 \mapsto \varphi(b_1)\mathrm{d}\varphi(b_2)$ fits into a commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & \Omega_B & \longrightarrow & \Omega \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\varphi} \\ B' & \longrightarrow & \Omega_{B'} & \longrightarrow & \Omega' \end{array}$$

In this situation, given any pair (M, ∇) where M is a B-module and $\nabla : M \to M \otimes_B \Omega$ is a connection we obtain a *base change* $(M \otimes_B B', \nabla')$ where

$$\nabla' : M \otimes_B B' \longrightarrow (M \otimes_B B') \otimes_{B'} \Omega' = M \otimes_B \Omega'$$

is defined by the rule

$$\nabla'(m \otimes b') = \sum m_i \otimes b'\mathrm{d}\varphi(b_i) + m \otimes \mathrm{d}b'$$

if $\nabla(m) = \sum m_i \otimes \mathrm{d}b_i$. If ∇ is integrable, then so is ∇', and in this case there is a canonical map of de Rham complexes

$$(6.11.1) \qquad M \otimes_B \Omega^\bullet \longrightarrow (M \otimes_B B') \otimes_{B'} (\Omega')^\bullet = M \otimes_B (\Omega')^\bullet$$

which maps $m \otimes \eta$ to $m \otimes \varphi(\eta)$.

Lemma 6.12. *Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. Let p be a prime number. Assume that A is a $\mathbf{Z}_{(p)}$-algebra and that p is nilpotent in B/J. Then we have*

$$\lim_e \Omega_{B_e/A,\bar{\delta}} = \lim_e \Omega_{B/A,\delta}/p^e \Omega_{B/A,\delta} = \lim_e \Omega_{B^\wedge/A,\delta^\wedge}/p^e \Omega_{B^\wedge/A,\delta^\wedge}$$

see proof for notation and explanation.

Proof. By Divided Power Algebra, Lemma 4.5 we see that δ extends to $B_e = B/p^e B$ for all sufficiently large e. Hence the first limit make sense. The lemma also produces a divided power structure $\delta^\wedge$ on the completion $B^\wedge = \lim_e B_e$, hence the last limit makes sense. By Lemma 6.2 and the fact that $\mathrm{d}p^e = 0$ (always) we see that the surjection $\Omega_{B/A,\delta} \to \Omega_{B_e/A,\bar{\delta}}$ has kernel $p^e \Omega_{B/A,\delta}$. Similarly for the kernel of $\Omega_{B^\wedge/A,\delta^\wedge} \to \Omega_{B_e/A,\bar{\delta}}$. Hence the lemma is clear. $\qquad\square$

7. Divided power schemes

Some remarks on how to globalize the previous notions.

Definition 7.1. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. A *divided power structure* γ on $\mathcal{I}$ is a sequence of maps $\gamma_n : \mathcal{I} \to \mathcal{I}$, $n \geq 1$ such that for any object U of $\mathcal{C}$ the triple

$$(\mathcal{O}(U), \mathcal{I}(U), \gamma)$$

is a divided power ring.

To be sure this applies in particular to sheaves of rings on topological spaces. But it's good to be a little bit more general as the structure sheaf of the crystalline site lives on a... site! A triple $(\mathcal{C}, \mathcal{I}, \gamma)$ as in the definition above is sometimes called a *divided power topos* in this chapter. Given a second $(\mathcal{C}', \mathcal{I}', \gamma')$ and given a morphism of ringed topoi $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ we say that $(f, f^\sharp)$ induces a *morphism of divided power topoi* if $f^\sharp(f^{-1}\mathcal{I}') \subset \mathcal{I}$ and the diagrams

$$\begin{array}{ccc} f^{-1}\mathcal{I}' & \xrightarrow{\ f^\sharp\ } & \mathcal{I} \\ {\scriptstyle f^{-1}\gamma'_n}\downarrow & & \downarrow{\scriptstyle \gamma_n} \\ f^{-1}\mathcal{I}' & \xrightarrow{\ f^\sharp\ } & \mathcal{I} \end{array}$$

are commutative for all $n \geq 1$. If f comes from a morphism of sites induced by a functor $u : \mathcal{C}' \to \mathcal{C}$ then this just means that

$$(\mathcal{O}'(U'), \mathcal{I}'(U'), \gamma') \longrightarrow (\mathcal{O}(u(U')), \mathcal{I}(u(U')), \gamma)$$

is a homomorphism of divided power rings for all $U' \in \mathrm{Ob}(\mathcal{C}')$.

In the case of schemes we require the divided power ideal to be **quasi-coherent**. But apart from this the definition is exactly the same as in the case of topoi. Here it is.

Definition 7.2. A *divided power scheme* is a triple $(S, \mathcal{I}, \gamma)$ where S is a scheme, $\mathcal{I}$ is a quasi-coherent sheaf of ideals, and γ is a divided power structure on $\mathcal{I}$. A *morphism of divided power schemes* $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ is a morphism of schemes $f : S \to S'$ such that $f^{-1}\mathcal{I}'\mathcal{O}_S \subset \mathcal{I}$ and such that

$$(\mathcal{O}_S(U'), \mathcal{I}(U'), \gamma) \longrightarrow (\mathcal{O}_{S'}(f^{-1}U'), \mathcal{I}(f^{-1}U'), \gamma)$$

is a homomorphism of divided power rings for all $U' \subset S'$ open.

Recall that there is a 1-to-1 correspondence between quasi-coherent sheaves of ideals and closed immersions, see Morphisms, Section 2. Thus given a divided power scheme $(T, \mathcal{J}, \gamma)$ we get a canonical closed immersion $U \to T$ defined by $\mathcal{J}$. Conversely, given a closed immersion $U \to T$ and a divided power structure γ on the sheaf of ideals $\mathcal{J}$ associated to $U \to T$ we obtain a divided power scheme $(T, \mathcal{J}, \gamma)$. In many situations we only want to consider such triples (U, T, γ) when the morphism $U \to T$ is a thickening, see More on Morphisms, Definition 2.1.

Definition 7.3. A triple (U, T, γ) as above is called a *divided power thickening* if $U \to T$ is a thickening.

Fibre products of divided power schemes exist when one of the three is a divided power thickening. Here is a formal statement.

Lemma 7.4. *Let* $(U', T', \delta') \to (S_0', S', \gamma')$ *and* $(S_0, S, \gamma) \to (S_0', S', \gamma')$ *be morphisms of divided power schemes. If* (U', T', δ') *is a divided power thickening, then there exists a divided power scheme* (T_0, T, δ) *and*

$$
\begin{array}{ccc}
T & \longrightarrow & T' \\
\downarrow & & \downarrow \\
S & \longrightarrow & S'
\end{array}
$$

which is a cartesian diagram in the category of divided power schemes.

Proof. Omitted. Hints: If T exists, then $T_0 = S_0 \times_{S_0'} U'$ (argue as in Divided Power Algebra, Remark 3.5). Since T' is a divided power thickening, we see that T (if it exists) will be a divided power thickening too. Hence we can define T as the scheme with underlying topological space the underlying topological space of $T_0 = S_0 \times_{S_0'} U'$ and as structure sheaf on affine pieces the ring given by Lemma 5.3. $\qquad\square$

We make the following observation. Suppose that (U, T, γ) is triple as above. Assume that T is a scheme over $\mathbf{Z}_{(p)}$ and that p is locally nilpotent on U. Then

  (1) p locally nilpotent on T $\Leftrightarrow$ $U \to T$ is a thickening (see Divided Power Algebra, Lemma 2.6), and
  (2) $p^e\mathcal{O}_T$ is locally on T preserved by γ for $e \gg 0$ (see Divided Power Algebra, Lemma 4.5).

This suggest that good results on divided power thickenings will be available under the following hypotheses.

Situation 7.5. Here p is a prime number and $(S, \mathcal{I}, \gamma)$ is a divided power scheme over $\mathbf{Z}_{(p)}$. We set $S_0 = V(\mathcal{I}) \subset S$. Finally, $X \to S_0$ is a morphism of schemes such that p is locally nilpotent on X.

It is in this situation that we will define the big and small crystalline sites.

8. The big crystalline site

We first define the big site. Given a divided power scheme $(S, \mathcal{I}, \gamma)$ we say $(T, \mathcal{J}, \delta)$ is a divided power scheme over $(S, \mathcal{I}, \gamma)$ if T comes endowed with a morphism $T \to S$ of divided power schemes. Similarly, we say a divided power thickening (U, T, δ) is a divided power thickening over $(S, \mathcal{I}, \gamma)$ if T comes endowed with a morphism $T \to S$ of divided power schemes.

Definition 8.1. In Situation 7.5.

(1) A *divided power thickening of X relative to $(S, \mathcal{I}, \gamma)$* is given by a divided power thickening (U, T, δ) over $(S, \mathcal{I}, \gamma)$ and an S-morphism $U \to X$.
(2) A *morphism of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$* is defined in the obvious manner.

The category of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$ is denoted $\text{CRIS}(X/S, \mathcal{I}, \gamma)$ or simply $\text{CRIS}(X/S)$.

For any (U, T, δ) in $\text{CRIS}(X/S)$ we have that p is locally nilpotent on T, see discussion after Definition 7.3. A good way to visualize all the data associated to (U, T, δ) is the commutative diagram

$$
\begin{array}{ccc}
T & \longleftarrow & U \\
\downarrow & & \downarrow \\
 & & X \\
\downarrow & & \downarrow \\
S & \longleftarrow & S_0
\end{array}
$$

where $S_0 = V(\mathcal{I}) \subset S$. Morphisms of $\text{CRIS}(X/S)$ can be similarly visualized as huge commutative diagrams. In particular, there is a canonical forgetful functor

(8.1.1) $\text{CRIS}(X/S) \longrightarrow Sch/X, \quad (U, T, \delta) \longmapsto U$

as well as its one sided inverse (and left adjoint)

(8.1.2) $Sch/X \longrightarrow \text{CRIS}(X/S), \quad U \longmapsto (U, U, \emptyset)$

which is sometimes useful.

Lemma 8.2. *In Situation 7.5. The category $CRIS(X/S)$ has all finite nonempty limits, in particular products of pairs and fibre products. The functor (8.1.1) commutes with limits.*

Proof. Omitted. Hint: See Lemma 5.3 for the affine case. See also Divided Power Algebra, Remark 3.5. $\square$

Lemma 8.3. *In Situation 7.5. Let*

$$
\begin{array}{ccc}
(U_3, T_3, \delta_3) & \longrightarrow & (U_2, T_2, \delta_2) \\
\downarrow & & \downarrow \\
(U_1, T_1, \delta_1) & \longrightarrow & (U, T, \delta)
\end{array}
$$

be a fibre square in the category of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$. If $T_2 \to T$ is flat, then $T_3 = T_1 \times_T T_2$ (as schemes).

Proof. This is true because a divided power structure extends uniquely along a flat ring map. See Divided Power Algebra, Lemma 4.2. $\square$

The lemma above means that the base change of a flat morphism of divided power thickenings is another flat morphism, and in fact is the "usual" base change of the morphism. This implies that the following definition makes sense.

Definition 8.4. In Situation 7.5.

(1) A family of morphisms $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$ of divided power thickenings of X/S is a *Zariski, étale, smooth, syntomic, or fppf covering* if and only if the family of morphisms of schemes $\{T_i \to T\}$ is one.
(2) The *big crystalline site* of X over $(S, \mathcal{I}, \gamma)$, is the category $\mathrm{CRIS}(X/S)$ endowed with the Zariski topology.
(3) The topos of sheaves on $\mathrm{CRIS}(X/S)$ is denoted $(X/S)_{\mathrm{CRIS}}$ or sometimes $(X/S, \mathcal{I}, \gamma)_{\mathrm{CRIS}}$[3].

There are some obvious functorialities concerning these topoi.

Remark 8.5 (Functoriality). Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Set $S_0 = V(\mathcal{I})$ and $S_0' = V(\mathcal{I}')$. Let

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow & & \downarrow \\
S_0 & \longrightarrow & S_0'
\end{array}
$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and Y. Then we get a continuous and cocontinuous functor

$$\mathrm{CRIS}(X/S) \longrightarrow \mathrm{CRIS}(Y/S')$$

by letting (U, T, δ) correspond to (U, T, δ) with $U \to X \to Y$ as the S'-morphism from U to Y. Hence we get a morphism of topoi

$$f_{\mathrm{CRIS}} : (X/S)_{\mathrm{CRIS}} \longrightarrow (Y/S')_{\mathrm{CRIS}}$$

see Sites, Section 20.

Remark 8.6 (Comparison with Zariski site). In Situation 7.5. The functor (8.1.1) is continuous, cocontinuous, and commutes with products and fibred products. Hence we obtain a morphism of topoi

$$U_{X/S} : (X/S)_{\mathrm{CRIS}} \longrightarrow Sh((Sch/X)_{Zar})$$

from the big crystalline topos of X/S to the big Zariski topos of X. See Sites, Section 20.

Remark 8.7 (Structure morphism). In Situation 7.5. Consider the closed subscheme $S_0 = V(\mathcal{I}) \subset S$. If we assume that p is locally nilpotent on S_0 (which is always the case in practice) then we obtain a situation as in Definition 8.1 with S_0 instead of X. Hence we get a site $\mathrm{CRIS}(S_0/S)$. If $f : X \to S_0$ is the structure

---

[3]This clashes with our convention to denote the topos associated to a site $\mathcal{C}$ by $Sh(\mathcal{C})$.

morphism of X over S, then we get a commutative diagram of morphisms of ringed topoi

$$
\begin{array}{ccc}
(X/S)_{\mathrm{CRIS}} & \xrightarrow{\;f_{\mathrm{CRIS}}\;} & (S_0/S)_{\mathrm{CRIS}} \\
{\scriptstyle U_{X/S}}\big\downarrow & & \big\downarrow{\scriptstyle U_{S_0/S}} \\
Sh((Sch/X)_{Zar}) & \xrightarrow{\;f_{big}\;} & Sh((Sch/S_0)_{Zar}) \\
& & \qquad\searrow \\
& & \qquad\qquad Sh((Sch/S)_{Zar})
\end{array}
$$

by Remark 8.5. We think of the composition $(X/S)_{\mathrm{CRIS}} \to Sh((Sch/S)_{Zar})$ as the structure morphism of the big crystalline site. Even if p is not locally nilpotent on S_0 the structure morphism

$$(X/S)_{\mathrm{CRIS}} \longrightarrow Sh((Sch/S)_{Zar})$$

is defined as we can take the lower route through the diagram above. Thus it is the morphism of topoi corresponding to the cocontinuous functor $\mathrm{CRIS}(X/S) \to (Sch/S)_{Zar}$ given by the rule $(U,T,\delta)/S \mapsto T/S$, see Sites, Section 20.

Remark 8.8 (Compatibilities). The morphisms defined above satisfy numerous compatibilities. For example, in the situation of Remark 8.5 we obtain a commutative diagram of ringed topoi

$$
\begin{array}{ccc}
(X/S)_{\mathrm{CRIS}} & \longrightarrow & (Y/S')_{\mathrm{CRIS}} \\
\big\downarrow & & \big\downarrow \\
Sh((Sch/S)_{Zar}) & \longrightarrow & Sh((Sch/S')_{Zar})
\end{array}
$$

where the vertical arrows are the structure morphisms.

9. The crystalline site

Since (8.1.1) commutes with products and fibre products, we see that looking at those (U,T,δ) such that $U \to X$ is an open immersion defines a full subcategory preserved under fibre products (and more generally finite nonempty limits). Hence the following definition makes sense.

Definition 9.1. In Situation 7.5.
  (1) The (small) *crystalline site* of X over $(S,\mathcal{I},\gamma)$, denoted $\mathrm{Cris}(X/S,\mathcal{I},\gamma)$ or simply $\mathrm{Cris}(X/S)$ is the full subcategory of $\mathrm{CRIS}(X/S)$ consisting of those (U,T,δ) in $\mathrm{CRIS}(X/S)$ such that $U \to X$ is an open immersion. It comes endowed with the Zariski topology.
  (2) The topos of sheaves on $\mathrm{Cris}(X/S)$ is denoted $(X/S)_{\mathrm{cris}}$ or sometimes $(X/S,\mathcal{I},\gamma)_{\mathrm{cris}}$[4].

For any (U,T,δ) in $\mathrm{Cris}(X/S)$ the morphism $U \to X$ defines an object of the small Zariski site X_{Zar} of X. Hence a canonical forgetful functor

(9.1.1) $$\mathrm{Cris}(X/S) \longrightarrow X_{Zar}, \quad (U,T,\delta) \longmapsto U$$

---

[4]This clashes with our convention to denote the topos associated to a site $\mathcal{C}$ by $Sh(\mathcal{C})$.

and a left adjoint

(9.1.2) $X_{Zar} \longrightarrow \mathrm{Cris}(X/S), \quad U \longmapsto (U, U, \emptyset)$

which is sometimes useful.

We can compare the small and big crystalline sites, just like we can compare the small and big Zariski sites of a scheme, see Topologies, Lemma 3.13.

Lemma 9.2. *Assumptions as in Definition 8.1. The inclusion functor*

$$Cris(X/S) \to CRIS(X/S)$$

commutes with finite nonempty limits, is fully faithful, continuous, and cocontinuous. There are morphisms of topoi

$$(X/S)_{cris} \xrightarrow{i} (X/S)_{CRIS} \xrightarrow{\pi} (X/S)_{cris}$$

whose composition is the identity and of which the first is induced by the inclusion functor. Moreover, $\pi_* = i^{-1}$.

Proof. For the first assertion see Lemma 8.2. This gives us a morphism of topoi $i : (X/S)_{\mathrm{cris}} \to (X/S)_{\mathrm{CRIS}}$ and a left adjoint $i_!$ such that $i^{-1}i_! = i^{-1}i_* = \mathrm{id}$, see Sites, Lemmas 20.5, 20.6, and 20.7. We claim that $i_!$ is exact. If this is true, then we can define π by the rules $\pi^{-1} = i_!$ and $\pi_* = i^{-1}$ and everything is clear. To prove the claim, note that we already know that $i_!$ is right exact and preserves fibre products (see references given). Hence it suffices to show that $i_!* = *$ where $*$ indicates the final object in the category of sheaves of sets. To see this it suffices to produce a set of objects (U_i, T_i, δ_i), $i \in I$ of $\mathrm{Cris}(X/S)$ such that

$$\coprod_{i \in I} h_{(U_i, T_i, \delta_i)} \to *$$

is surjective in $(X/S)_{\mathrm{CRIS}}$ (details omitted; hint: use that $\mathrm{Cris}(X/S)$ has products and that the functor $\mathrm{Cris}(X/S) \to \mathrm{CRIS}(X/S)$ commutes with them). In the affine case this follows from Lemma 5.6. We omit the proof in general. $\square$

Remark 9.3 (Functoriality). Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Let

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
\downarrow & & \downarrow \\
S_0 & \longrightarrow & S_0'
\end{array}
$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and Y. By analogy with Topologies, Lemma 3.16 we define

$$f_{\mathrm{cris}} : (X/S)_{\mathrm{cris}} \longrightarrow (Y/S')_{\mathrm{cris}}$$

by the formula $f_{\mathrm{cris}} = \pi_Y \circ f_{\mathrm{CRIS}} \circ i_X$ where i_X and π_Y are as in Lemma 9.2 for X and Y and where f_{CRIS} is as in Remark 8.5.

Remark 9.4 (Comparison with Zariski site). In Situation 7.5. The functor (9.1.1) is continuous, cocontinuous, and commutes with products and fibred products. Hence we obtain a morphism of topoi

$$u_{X/S} : (X/S)_{\mathrm{cris}} \longrightarrow Sh(X_{Zar})$$

relating the small crystalline topos of X/S with the small Zariski topos of X. See Sites, Section 20.

Lemma 9.5. *In Situation 7.5. Let $X' \subset X$ and $S' \subset S$ be open subschemes such that X' maps into S'. Then there is a fully faithful functor $\mathrm{Cris}(X'/S') \to \mathrm{Cris}(X/S)$ which gives rise to a morphism of topoi fitting into the commutative diagram*

$$
\begin{array}{ccc}
(X'/S')_{cris} & \longrightarrow & (X/S)_{cris} \\
{\scriptstyle u_{X'/S'}}\downarrow & & \downarrow{\scriptstyle u_{X/S}} \\
Sh(X'_{Zar}) & \longrightarrow & Sh(X_{Zar})
\end{array}
$$

Moreover, this diagram is an example of localization of morphisms of topoi as in Sites, Lemma 30.1.

Proof. The fully faithful functor comes from thinking of objects of $\mathrm{Cris}(X'/S')$ as divided power thickenings (U, T, δ) of X where $U \to X$ factors through $X' \subset X$ (since then automatically $T \to S$ will factor through S'). This functor is clearly cocontinuous hence we obtain a morphism of topoi as indicated. Let $h_{X'} \in Sh(X_{Zar})$ be the representable sheaf associated to X' viewed as an object of X_{Zar}. It is clear that $Sh(X'_{Zar})$ is the localization $Sh(X_{Zar})/h_{X'}$. On the other hand, the category $\mathrm{Cris}(X/S)/u_{X/S}^{-1}h_{X'}$ (see Sites, Lemma 29.3) is canonically identified with $\mathrm{Cris}(X'/S')$ by the functor above. This finishes the proof. $\square$

Remark 9.6 (Structure morphism)**.** In Situation 7.5. Consider the closed subscheme $S_0 = V(\mathcal{I}) \subset S$. If we assume that p is locally nilpotent on S_0 (which is always the case in practice) then we obtain a situation as in Definition 8.1 with S_0 instead of X. Hence we get a site $\mathrm{Cris}(S_0/S)$. If $f : X \to S_0$ is the structure morphism of X over S, then we get a commutative diagram of ringed topoi

$$
\begin{array}{ccc}
(X/S)_{\mathrm{cris}} & \xrightarrow{\ f_{\mathrm{cris}}\ } & (S_0/S)_{\mathrm{cris}} \\
{\scriptstyle u_{X/S}}\downarrow & & \downarrow{\scriptstyle u_{S_0/S}} \\
Sh(X_{Zar}) & \xrightarrow{\ f_{small}\ } & Sh(S_{0,Zar}) \\
& & \searrow \\
& & \qquad Sh(S_{Zar})
\end{array}
$$

see Remark 9.3. We think of the composition $(X/S)_{\mathrm{cris}} \to Sh(S_{Zar})$ as the structure morphism of the crystalline site. Even if p is not locally nilpotent on S_0 the structure morphism

$$
\tau_{X/S} : (X/S)_{\mathrm{cris}} \longrightarrow Sh(S_{Zar})
$$

is defined as we can take the lower route through the diagram above.

Remark 9.7 (Compatibilities). The morphisms defined above satisfy numerous compatibilities. For example, in the situation of Remark 9.3 we obtain a commutative diagram of ringed topoi

$$
\begin{array}{ccc}
(X/S)_{\mathrm{cris}} & \longrightarrow & (Y/S')_{\mathrm{cris}} \\
\downarrow & & \downarrow \\
Sh((Sch/S)_{Zar}) & \longrightarrow & Sh((Sch/S')_{Zar})
\end{array}
$$

where the vertical arrows are the structure morphisms.

10. Sheaves on the crystalline site

Notation and assumptions as in Situation 7.5. In order to discuss the small and big crystalline sites of X/S simultaneously in this section we let

$$
\mathcal{C} = \mathrm{CRIS}(X/S) \quad \text{or} \quad \mathcal{C} = \mathrm{Cris}(X/S).
$$

A sheaf $\mathcal{F}$ on $\mathcal{C}$ gives rise to a *restriction* $\mathcal{F}_T$ for every object (U, T, δ) of $\mathcal{C}$. Namely, $\mathcal{F}_T$ is the Zariski sheaf on the scheme T defined by the rule

$$
\mathcal{F}_T(W) = \mathcal{F}(U \cap W, W, \delta|_W)
$$

for $W \subset T$ is open. Moreover, if $f : T \to T'$ is a morphism between objects (U, T, δ) and (U', T', δ') of $\mathcal{C}$, then there is a canonical *comparison* map

(10.0.1) $$ c_f : f^{-1}\mathcal{F}_{T'} \longrightarrow \mathcal{F}_T. $$

Namely, if $W' \subset T'$ is open then f induces a morphism

$$
f|_{f^{-1}W'} : (U \cap f^{-1}(W'), f^{-1}W', \delta|_{f^{-1}W'}) \longrightarrow (U' \cap W', W', \delta|_{W'})
$$

of $\mathcal{C}$, hence we can use the restriction mapping $(f|_{f^{-1}W'})^*$ of $\mathcal{F}$ to define a map $\mathcal{F}_{T'}(W') \to \mathcal{F}_T(f^{-1}W')$. These maps are clearly compatible with further restriction, hence define an f-map from $\mathcal{F}_{T'}$ to $\mathcal{F}_T$ (see Sheaves, Section 21 and especially Sheaves, Definition 21.7). Thus a map c_f as in (10.0.1). Note that if f is an open immersion, then c_f is an isomorphism, because in that case $\mathcal{F}_T$ is just the restriction of $\mathcal{F}_{T'}$ to T.

Conversely, given Zariski sheaves $\mathcal{F}_T$ for every object (U, T, δ) of $\mathcal{C}$ and comparison maps c_f as above which (a) are isomorphisms for open immersions, and (b) satisfy a suitable cocycle condition, we obtain a sheaf on $\mathcal{C}$. This is proved exactly as in Topologies, Lemma 3.18.

The *structure sheaf* on $\mathcal{C}$ is the sheaf $\mathcal{O}_{X/S}$ defined by the rule

$$
\mathcal{O}_{X/S} : (U, T, \delta) \longmapsto \Gamma(T, \mathcal{O}_T)
$$

This is a sheaf by the definition of coverings in $\mathcal{C}$. Suppose that $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X/S}$-modules. In this case the comparison mappings (10.0.1) define a comparison map

(10.0.2) $$ c_f : f^*\mathcal{F}_T \longrightarrow \mathcal{F}_{T'} $$

of $\mathcal{O}_T$-modules.

Another type of example comes by starting with a sheaf $\mathcal{G}$ on $(Sch/X)_{Zar}$ or X_{Zar} (depending on whether $\mathcal{C} = \mathrm{CRIS}(X/S)$ or $\mathcal{C} = \mathrm{Cris}(X/S)$). Then $\underline{\mathcal{G}}$ defined by the rule

$$
\underline{\mathcal{G}} : (U, T, \delta) \longmapsto \mathcal{G}(U)
$$

is a sheaf on $\mathcal{C}$. In particular, if we take $\mathcal{G} = \mathbf{G}_a = \mathcal{O}_X$, then we obtain

$$\underline{\mathbf{G}_a} : (U, T, \delta) \longmapsto \Gamma(U, \mathcal{O}_U)$$

There is a surjective map of sheaves $\mathcal{O}_{X/S} \to \underline{\mathbf{G}_a}$ defined by the canonical maps $\Gamma(T, \mathcal{O}_T) \to \Gamma(U, \mathcal{O}_U)$ for objects (U, T, δ). The kernel of this map is denoted $\mathcal{J}_{X/S}$, hence a short exact sequence

$$0 \to \mathcal{J}_{X/S} \to \mathcal{O}_{X/S} \to \underline{\mathbf{G}_a} \to 0$$

Note that $\mathcal{J}_{X/S}$ comes equipped with a canonical divided power structure. After all, for each object (U, T, δ) the third component δ *is* a divided power structure on the kernel of $\mathcal{O}_T \to \mathcal{O}_U$. Hence the (big) crystalline topos is a divided power topos.

11. Crystals in modules

It turns out that a crystal is a very general gadget. However, the definition may be a bit hard to parse, so we first give the definition in the case of modules on the crystalline sites.

Definition 11.1. In Situation 7.5. Let $\mathcal{C} = \mathrm{CRIS}(X/S)$ or $\mathcal{C} = \mathrm{Cris}(X/S)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\mathcal{C}$.

(1) We say $\mathcal{F}$ is *locally quasi-coherent* if for every object (U, T, δ) of $\mathcal{C}$ the restriction $\mathcal{F}_T$ is a quasi-coherent $\mathcal{O}_T$-module.
(2) We say $\mathcal{F}$ is *quasi-coherent* if it is quasi-coherent in the sense of Modules on Sites, Definition 23.1.
(3) We say $\mathcal{F}$ is a *crystal in $\mathcal{O}_{X/S}$-modules* if all the comparison maps (10.0.2) are isomorphisms.

It turns out that we can relate these notions as follows.

Lemma 11.2. *With notation $X/S, \mathcal{I}, \gamma, \mathcal{C}, \mathcal{F}$ as in Definition 11.1. The following are equivalent*

(1) *$\mathcal{F}$ is quasi-coherent, and*
(2) *$\mathcal{F}$ is locally quasi-coherent and a crystal in $\mathcal{O}_{X/S}$-modules.*

Proof. Assume (1). Let $f : (U', T', \delta') \to (U, T, \delta)$ be an object of $\mathcal{C}$. We have to prove (a) $\mathcal{F}_T$ is a quasi-coherent $\mathcal{O}_T$-module and (b) $c_f : f^* \mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism. The assumption means that we can find a covering $\{(T_i, U_i, \delta_i) \to (T, U, \delta)\}$ and for each i the restriction of $\mathcal{F}$ to $\mathcal{C}/(T_i, U_i, \delta_i)$ has a global presentation. Since it suffices to prove (a) and (b) Zariski locally, we may replace $f : (T', U', \delta') \to (T, U, \delta)$ by the base change to (T_i, U_i, δ_i) and assume that $\mathcal{F}$ restricted to $\mathcal{C}/(T, U, \delta)$ has a global presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_{X/S}|_{\mathcal{C}/(U, T, \delta)} \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_{X/S}|_{\mathcal{C}/(U, T, \delta)} \longrightarrow \mathcal{F}|_{\mathcal{C}/(U, T, \delta)} \longrightarrow 0$$

It is clear that this gives a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_T \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_T \longrightarrow \mathcal{F}_T \longrightarrow 0$$

and hence (a) holds. Moreover, the presentation restricts to T' to give a similar presentation of $\mathcal{F}_{T'}$, whence (b) holds.

Assume (2). Let (U, T, δ) be an object of $\mathcal{C}$. We have to find a covering of (U, T, δ) such that $\mathcal{F}$ has a global presentation when we restrict to the localization of $\mathcal{C}$ at

the members of the covering. Thus we may assume that T is affine. In this case we can choose a presentation

$$\bigoplus_{j \in J} \mathcal{O}_T \longrightarrow \bigoplus_{i \in I} \mathcal{O}_T \longrightarrow \mathcal{F}_T \longrightarrow 0$$

as $\mathcal{F}_T$ is assumed to be a quasi-coherent $\mathcal{O}_T$-module. Then by the crystal property of $\mathcal{F}$ we see that this pulls back to a presentation of $\mathcal{F}_{T'}$ for any morphism $f : (U', T', \delta') \to (U, T, \delta)$ of $\mathcal{C}$. Thus the desired presentation of $\mathcal{F}|_{\mathcal{C}/(U,T,\delta)}$. $\qquad \square$

Definition 11.3. If $\mathcal{F}$ satisfies the equivalent conditions of Lemma 11.2, then we say that $\mathcal{F}$ is a *crystal in quasi-coherent modules*. We say that $\mathcal{F}$ is a *crystal in finite locally free modules* if, in addition, $\mathcal{F}$ is finite locally free.

Of course, as Lemma 11.2 shows, this notation is somewhat heavy since a quasi-coherent module is always a crystal. But it is standard terminology in the literature.

Remark 11.4. To formulate the general notion of a crystal we use the language of stacks and strongly cartesian morphisms, see Stacks, Definition 4.1 and Categories, Definition 31.1. In Situation 7.5 let $p : \mathcal{C} \to \mathrm{Cris}(X/S)$ be a stack. A *crystal in objects of $\mathcal{C}$ on X relative to S* is a *cartesian section* $\sigma : \mathrm{Cris}(X/S) \to \mathcal{C}$, i.e., a functor σ such that $p \circ \sigma = \mathrm{id}$ and such that $\sigma(f)$ is strongly cartesian for all morphisms f of $\mathrm{Cris}(X/S)$. Similarly for the big crystalline site.

12. Sheaf of differentials

In this section we will stick with the (small) crystalline site as it seems more natural. We globalize Definition 6.1 as follows.

Definition 12.1. In Situation 7.5 let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$. An *S-derivation* $D : \mathcal{O}_{X/S} \to \mathcal{F}$ is a map of sheaves such that for every object (U, T, δ) of $\mathrm{Cris}(X/S)$ the map

$$D : \Gamma(T, \mathcal{O}_T) \longrightarrow \Gamma(T, \mathcal{F})$$

is a divided power $\Gamma(V, \mathcal{O}_V)$-derivation where $V \subset S$ is any open such that $T \to S$ factors through V.

This means that D is additive, satisfies the Leibniz rule, annihilates functions coming from S, and satisfies $D(f^{[n]}) = f^{[n-1]}D(f)$ for a local section f of the divided power ideal $\mathcal{J}_{X/S}$. This is a special case of a very general notion which we now describe.

Please compare the following discussion with Modules on Sites, Section 32. Let $\mathcal{C}$ be a site, let $\mathcal{A} \to \mathcal{B}$ be a map of sheaves of rings on $\mathcal{C}$, let $\mathcal{J} \subset \mathcal{B}$ be a sheaf of ideals, let δ be a divided power structure on $\mathcal{J}$, and let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. Then there is a notion of a *divided power $\mathcal{A}$-derivation* $D : \mathcal{B} \to \mathcal{F}$. This means that D is $\mathcal{A}$-linear, satisfies the Leibniz rule, and satisfies $D(\delta_n(x)) = \delta_{n-1}(x)D(x)$ for local sections x of $\mathcal{J}$. In this situation there exists a *universal divided power $\mathcal{A}$-derivation*

$$d_{\mathcal{B}/\mathcal{A},\delta} : \mathcal{B} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A},\delta}$$

Moreover, $d_{\mathcal{B}/\mathcal{A},\delta}$ is the composition

$$\mathcal{B} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A}} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A},\delta}$$

where the first map is the universal derivation constructed in the proof of Modules on Sites, Lemma 32.2 and the second arrow is the quotient by the submodule generated by the local sections $d_{\mathcal{B}/\mathcal{A}}(\delta_n(x)) - \delta_{n-1}(x)d_{\mathcal{B}/\mathcal{A}}(x)$.

We translate this into a relative notion as follows. Suppose $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ is a morphism of ringed topoi, $\mathcal{J} \subset \mathcal{O}$ a sheaf of ideals, δ a divided power structure on $\mathcal{J}$, and $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules. In this situation we say $D : \mathcal{O} \to \mathcal{F}$ is a divided power $\mathcal{O}'$-derivation if D is a divided power $f^{-1}\mathcal{O}'$-derivation as defined above. Moreover, we write

$$\Omega_{\mathcal{O}/\mathcal{O}',\delta} = \Omega_{\mathcal{O}/f^{-1}\mathcal{O}',\delta}$$

which is the receptacle of the universal divided power $\mathcal{O}'$-derivation.

Applying this to the structure morphism

$$(X/S)_{\mathrm{Cris}} \longrightarrow Sh(S_{Zar})$$

(see Remark 9.6) we recover the notion of Definition 12.1 above. In particular, there is a universal divided power derivation

$$d_{X/S} : \mathcal{O}_{X/S} \to \Omega_{X/S}$$

Note that we omit from the notation the decoration indicating the module of differentials is compatible with divided powers (it seems unlikely anybody would ever consider the usual module of differentials of the structure sheaf on the crystalline site).

Lemma 12.2. *Let $(T, \mathcal{J}, \delta)$ be a divided power scheme. Let $T \to S$ be a morphism of schemes. The quotient $\Omega_{T/S} \to \Omega_{T/S,\delta}$ described above is a quasi-coherent $\mathcal{O}_T$-module. For $W \subset T$ affine open mapping into $V \subset S$ affine open we have*

$$\Gamma(W, \Omega_{T/S,\delta}) = \Omega_{\Gamma(W,\mathcal{O})/\Gamma(V,\mathcal{O}_V),\delta}$$

where the right hand side is as constructed in Section 6.

Proof. Omitted. $\square$

Lemma 12.3. *In Situation 7.5. For (U, T, δ) in $Cris(X/S)$ the restriction $(\Omega_{X/S})_T$ to T is $\Omega_{T/S,\delta}$ and the restriction $d_{X/S}|_T$ is equal to $d_{T/S,\delta}$.*

Proof. Omitted. $\square$

Lemma 12.4. *In Situation 7.5. For any affine object (U, T, δ) of $Cris(X/S)$ mapping into an affine open $V \subset S$ we have*

$$\Gamma((U, T, \delta), \Omega_{X/S}) = \Omega_{\Gamma(T,\mathcal{O})/\Gamma(V,\mathcal{O}_V),\delta}$$

where the right hand side is as constructed in Section 6.

Proof. Combine Lemmas 12.2 and 12.3. $\square$

Lemma 12.5. *In Situation 7.5. Let (U, T, δ) be an object of $Cris(X/S)$. Let*

$$(U(1), T(1), \delta(1)) = (U, T, \delta) \times (U, T, \delta)$$

in $Cris(X/S)$. Let $\mathcal{K} \subset \mathcal{O}_{T(1)}$ be the quasi-coherent sheaf of ideals corresponding to the closed immersion $\Delta : T \to T(1)$. Then $\mathcal{K} \subset \mathcal{J}_{T(1)}$ is preserved by the divided structure on $\mathcal{J}_{T(1)}$ and we have

$$(\Omega_{X/S})_T = \mathcal{K}/\mathcal{K}^{[2]}$$

Proof. Note that $U = U(1)$ as $U \to X$ is an open immersion and as (9.1.1) commutes with products. Hence we see that $\mathcal{K} \subset \mathcal{J}_{T(1)}$. Given this fact the lemma follows by working affine locally on T and using Lemmas 12.4 and 6.5. $\qquad\square$

It turns out that $\Omega_{X/S}$ is not a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. But it does satisfy two closely related properties (compare with Lemma 11.2).

Lemma 12.6. *In Situation 7.5. The sheaf of differentials $\Omega_{X/S}$ has the following two properties:*

(1) *$\Omega_{X/S}$ is locally quasi-coherent, and*
(2) *for any morphism $(U, T, \delta) \to (U', T', \delta')$ of $\mathrm{Cris}(X/S)$ where $f : T \to T'$ is a closed immersion the map $c_f : f^*(\Omega_{X/S})_{T'} \to (\Omega_{X/S})_T$ is surjective.*

Proof. Part (1) follows from a combination of Lemmas 12.2 and 12.3. Part (2) follows from the fact that $(\Omega_{X/S})_T = \Omega_{T/S,\delta}$ is a quotient of $\Omega_{T/S}$ and that $f^*\Omega_{T'/S} \to \Omega_{T/S}$ is surjective. $\qquad\square$

13. Two universal thickenings

The constructions in this section will help us define a connection on a crystal in modules on the crystalline site. In some sense the constructions here are the "sheafified, universal" versions of the constructions in Section 3.

Remark 13.1. In Situation 7.5. Let (U, T, δ) be an object of $\mathrm{Cris}(X/S)$. Write $\Omega_{T/S,\delta} = (\Omega_{X/S})_T$, see Lemma 12.3. We explicitly describe a first order thickening T' of T. Namely, set

$$\mathcal{O}_{T'} = \mathcal{O}_T \oplus \Omega_{T/S,\delta}$$

with algebra structure such that $\Omega_{T/S,\delta}$ is an ideal of square zero. Let $\mathcal{J} \subset \mathcal{O}_T$ be the ideal sheaf of the closed immersion $U \to T$. Set $\mathcal{J}' = \mathcal{J} \oplus \Omega_{T/S,\delta}$. Define a divided power structure on $\mathcal{J}'$ by setting

$$\delta'_n(f, \omega) = (\delta_n(f), \delta_{n-1}(f)\omega),$$

see Lemma 3.1. There are two ring maps

$$p_0, p_1 : \mathcal{O}_T \to \mathcal{O}_{T'}$$

The first is given by $f \mapsto (f, 0)$ and the second by $f \mapsto (f, \mathrm{d}_{T/S,\delta}f)$. Note that both are compatible with the divided power structures on $\mathcal{J}$ and $\mathcal{J}'$ and so is the quotient map $\mathcal{O}_{T'} \to \mathcal{O}_T$. Thus we get an object (U, T', δ') of $\mathrm{Cris}(X/S)$ and a commutative diagram

$$\begin{array}{ccc} & T & \\ \mathrm{id} \swarrow & \downarrow i & \searrow \mathrm{id} \\ T \xleftarrow{\;p_0\;} & T' & \xrightarrow{\;p_1\;} T \end{array}$$

of $\mathrm{Cris}(X/S)$ such that i is a first order thickening whose ideal sheaf is identified with $\Omega_{T/S,\delta}$ and such that $p_1^* - p_0^* : \mathcal{O}_T \to \mathcal{O}_{T'}$ is identified with the universal derivation $\mathrm{d}_{T/S,\delta}$ composed with the inclusion $\Omega_{T/S,\delta} \to \mathcal{O}_{T'}$.

Remark 13.2. In Situation 7.5. Let (U, T, δ) be an object of $\mathrm{Cris}(X/S)$. Write $\Omega_{T/S,\delta} = (\Omega_{X/S})_T$, see Lemma 12.3. We also write $\Omega^2_{T/S,\delta}$ for its second exterior power. We explicitly describe a second order thickening T'' of T. Namely, set

$$\mathcal{O}_{T''} = \mathcal{O}_T \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta} \oplus \Omega^2_{T/S,\delta}$$

with algebra structure defined in the following way

$$(f, \omega_1, \omega_2, \eta) \cdot (f', \omega_1', \omega_2', \eta') = (ff', f\omega_1' + f'\omega_1, f\omega_2' + f'\omega_2, f\eta' + f'\eta + \omega_1 \wedge \omega_2' + \omega_1' \wedge \omega_2).$$

Let $\mathcal{J} \subset \mathcal{O}_T$ be the ideal sheaf of the closed immersion $U \to T$. Let $\mathcal{J}''$ be the inverse image of $\mathcal{J}$ under the projection $\mathcal{O}_{T''} \to \mathcal{O}_T$. Define a divided power structure on $\mathcal{J}''$ by setting

$$\delta_n''(f, \omega_1, \omega_2, \eta) = (\delta_n(f), \delta_{n-1}(f)\omega_1, \delta_{n-1}(f)\omega_2, \delta_{n-1}(f)\eta + \delta_{n-2}(f)\omega_1 \wedge \omega_2)$$

see Lemma 3.2. There are three ring maps $q_0, q_1, q_2 : \mathcal{O}_T \to \mathcal{O}_{T''}$ given by

$$q_0(f) = (f, 0, 0, 0),$$
$$q_1(f) = (f, \mathrm{d}f, 0, 0),$$
$$q_2(f) = (f, \mathrm{d}f, \mathrm{d}f, 0)$$

where $\mathrm{d} = \mathrm{d}_{T/S, \delta}$. Note that all three are compatible with the divided power structures on $\mathcal{J}$ and $\mathcal{J}''$. There are three ring maps $q_{01}, q_{12}, q_{02} : \mathcal{O}_{T'} \to \mathcal{O}_{T''}$ where $\mathcal{O}_{T'}$ is as in Remark 13.1. Namely, set

$$q_{01}(f, \omega) = (f, \omega, 0, 0),$$
$$q_{12}(f, \omega) = (f, \mathrm{d}f, \omega, \mathrm{d}\omega),$$
$$q_{02}(f, \omega) = (f, \omega, \omega, 0)$$

These are also compatible with the given divided power structures. Let's do the verifications for q_{12}: Note that q_{12} is a ring homomorphism as

$$
\begin{aligned}
q_{12}(f, \omega)q_{12}(g, \eta) &= (f, \mathrm{d}f, \omega, \mathrm{d}\omega)(g, \mathrm{d}g, \eta, \mathrm{d}\eta) \\
&= (fg, f\mathrm{d}g + g\mathrm{d}f, f\eta + g\omega, f\mathrm{d}\eta + g\mathrm{d}\omega + \mathrm{d}f \wedge \eta + \mathrm{d}g \wedge \omega) \\
&= q_{12}(fg, f\eta + g\omega) = q_{12}((f, \omega)(g, \eta))
\end{aligned}
$$

Note that q_{12} is compatible with divided powers because

$$
\begin{aligned}
\delta_n''(q_{12}(f, \omega)) &= \delta_n''((f, \mathrm{d}f, \omega, \mathrm{d}\omega)) \\
&= (\delta_n(f), \delta_{n-1}(f)\mathrm{d}f, \delta_{n-1}(f)\omega, \delta_{n-1}(f)\mathrm{d}\omega + \delta_{n-2}(f)\mathrm{d}(f) \wedge \omega) \\
&= q_{12}((\delta_n(f), \delta_{n-1}(f)\omega)) = q_{12}(\delta_n'(f, \omega))
\end{aligned}
$$

The verifications for q_{01} and q_{02} are easier. Note that $q_0 = q_{01} \circ p_0$, $q_1 = q_{01} \circ p_1$, $q_1 = q_{12} \circ p_0$, $q_2 = q_{12} \circ p_1$, $q_0 = q_{02} \circ p_0$, and $q_2 = q_{02} \circ p_1$. Thus (U, T'', δ'') is an object of $\mathrm{Cris}(X/S)$ and we get morphisms

$$T'' \underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}} T' \underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}} T$$

of $\mathrm{Cris}(X/S)$ satisfying the relations described above. In applications we will use $q_i : T'' \to T$ and $q_{ij} : T'' \to T'$ to denote the morphisms associated to the ring maps described above.

14. The de Rham complex

In Situation 7.5. Working on the (small) crystalline site, we define $\Omega_{X/S}^i = \wedge_{\mathcal{O}_{X/S}}^i \Omega_{X/S}$ for $i \geq 0$. The universal S-derivation $\mathrm{d}_{X/S}$ gives rise to the *de Rham complex*

$$\mathcal{O}_{X/S} \to \Omega_{X/S}^1 \to \Omega_{X/S}^2 \to \dots$$

on $\mathrm{Cris}(X/S)$, see Lemma 12.4 and Remark 6.9.

15. Connections

In Situation 7.5. Given an $\mathcal{O}_{X/S}$-module $\mathcal{F}$ on $\mathrm{Cris}(X/S)$ a *connection* is a map of abelian sheaves

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega_{X/S}$$

such that $\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f$ for local sections s, f of $\mathcal{F}$ and $\mathcal{O}_{X/S}$. Given a connection there are canonical maps $\nabla : \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{i+1}_{X/S}$ defined by the rule $\nabla(s \otimes \omega) = \nabla(s) \wedge \omega + s \otimes \mathrm{d}\omega$ as in Remark 6.10. We say the connection is *integrable* if $\nabla \circ \nabla = 0$. If ∇ is integrable we obtain the *de Rham complex*

$$\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^2_{X/S} \to \dots$$

on $\mathrm{Cris}(X/S)$. It turns out that any crystal in $\mathcal{O}_{X/S}$-modules comes equipped with a canonical integrable connection.

Lemma 15.1. *In Situation 7.5. Let $\mathcal{F}$ be a crystal in $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$. Then $\mathcal{F}$ comes equipped with a canonical integrable connection.*

Proof. Say (U, T, δ) is an object of $\mathrm{Cris}(X/S)$. Let (U, T', δ') be the infinitesimal thickening of T by $(\Omega_{X/S})_T = \Omega_{T/S,\delta}$ constructed in Remark 13.1. It comes with projections $p_0, p_1 : T' \to T$ and a diagonal $i : T \to T(1)$. By assumption we get isomorphisms

$$p_0^* \mathcal{F}_T \xrightarrow{c_0} \mathcal{F}_{T'} \xleftarrow{c_1} p_1^* \mathcal{F}_T$$

of $\mathcal{O}_{T'}$-modules. Pulling $c = c_1^{-1} \circ c_0$ back to T by i we obtain the identity map of $\mathcal{F}_T$. Hence if $s \in \Gamma(T, \mathcal{F}_T)$ then $\nabla(s) = p_1^* s - c(p_0^* s)$ is a section of $p_1^* \mathcal{F}_T$ which vanishes on pulling back by Δ. Hence $\nabla(s)$ is a section of

$$\mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S,\delta}$$

because this is the kernel of $p_1^* \mathcal{F}_T \to \mathcal{F}_T$ as $\Omega_{T/S,\delta}$ is the kernel of $\mathcal{O}_{T'} \to \mathcal{O}_T$ by construction.

The collection of maps

$$\nabla : \Gamma(T, \mathcal{F}_T) \to \Gamma(T, \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S,\delta})$$

so obtained is functorial in T because the construction of T' is functorial in T. Hence we obtain a connection.

To show that the connection is integrable we consider the object (U, T'', δ'') constructed in Remark 13.2. Because $\mathcal{F}$ is a sheaf we see that

is a commutative map of $\mathcal{O}_{T''}$-modules. For $s \in \Gamma(T, \mathcal{F}_T)$ we have $c(p_0^* s) = p_1^* s - \nabla(s)$. Write $\nabla(s) = \sum p_1^* s_i \cdot \omega_i$ where s_i is a local section of $\mathcal{F}_T$ and ω_i is a local section of $\Omega_{T/S,\delta}$. We think of ω_i as a local section of the structure sheaf of $\mathcal{O}_{T'}$ and hence we write product instead of tensor product. On the one hand

$$q_{12}^* c \circ q_{01}^* c(q_0^* s) = q_{12}^* c(q_1^* s - \sum q_1^* s_i \cdot q_{01}^* \omega_i)$$

$$= q_2^* s - \sum q_2^* s_i \cdot q_{12}^* \omega_i - \sum q_2^* s_i \cdot q_{01}^* \omega_i + \sum q_{12}^* \nabla(s_i) \cdot q_{01}^* \omega_i$$

and on the other hand

$$q_{02}^* c(q_0^* s) = q_2^* s - \sum q_2^* s_i \cdot q_{02}^* \omega_i.$$

From the formulae of Remark 13.2 we see that $q_{01}^* \omega_i + q_{12}^* \omega_i - q_{02}^* \omega_i = \mathrm{d}\omega_i$. Hence the difference of the two expressions above is

$$\sum q_2^* s_i \cdot \mathrm{d}\omega_i - \sum q_{12}^* \nabla(s_i) \cdot q_{01}^* \omega_i$$

Note that $q_{12}^* \omega \cdot q_{01}^* \omega' = \omega' \wedge \omega = -\omega \wedge \omega'$ by the definition of the multiplication on $\mathcal{O}_{T''}$. Thus the expression above is $\nabla^2(s)$ viewed as a section of the subsheaf $\mathcal{F}_T \otimes \Omega^2_{T/S,\delta}$ of $q_2^* \mathcal{F}$. Hence we get the integrability condition. $\qquad\square$

16. Cosimplicial algebra

This section should be moved somewhere else. A *cosimplicial ring* is a cosimplicial object in the category of rings. Given a ring R, a *cosimplicial R-algebra* is a cosimplicial object in the category of R-algebras. A *cosimplicial ideal* in a cosimplicial ring A_* is given by an ideal $I_n \subset A_n$ for all n such that $A(f)(I_n) \subset I_m$ for all $f : [n] \to [m]$ in Δ.

Let A_* be a cosimplicial ring. Let $\mathcal{C}$ be the category of pairs (A, M) where A is a ring and M is a module over A. A morphism $(A, M) \to (A', M')$ consists of a ring map $A \to A'$ and an A-module map $M \to M'$ where M' is viewed as an A-module via $A \to A'$ and the A'-module structure on M'. Having said this we can define a *cosimplicial module M_* over A_** as a cosimplicial object (A_*, M_*) of $\mathcal{C}$ whose first entry is equal to A_*. A *homomorphism $\varphi_* : M_* \to N_*$ of cosimplicial modules over A_** is a morphism $(A_*, M_*) \to (A_*, N_*)$ of cosimplicial objects in $\mathcal{C}$ whose first component is 1_{A_*}.

A *homotopy* between homomorphisms $\varphi_*, \psi_* : M_* \to N_*$ of cosimplicial modules over A_* is a homotopy between the associated maps $(A_*, M_*) \to (A_*, N_*)$ whose first component is the trivial homotopy (dual to Simplicial, Example 25.3). We spell out what this means. Such a homotopy is a homotopy

$$h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], N_*)$$

between φ_* and ψ_* as homomorphisms of cosimplicial abelian groups such that for each n the map $h_n : M_n \to \prod_{\alpha \in \Delta[1]_n} N_n$ is A_n-linear. The following lemma is a version of Simplicial, Lemma 27.3 for cosimplicial modules.

Lemma 16.1. *Let A_* be a cosimplicial ring. Let $\varphi_*, \psi_* : K_* \to M_*$ be homomorphisms of cosimplicial A_*-modules.*

(1) *If φ_* and ψ_* are homotopic, then*

$$\varphi_* \otimes 1, \psi_* \otimes 1 : K_* \otimes_{A_*} L_* \longrightarrow M_* \otimes_{A_*} L_*$$

are homotopic for any cosimplicial A_*-module L_*.

(2) *If φ_* and ψ_* are homotopic, then*

$$\wedge^i(\varphi_*), \wedge^i(\psi_*) : \wedge^i(K_*) \longrightarrow \wedge^i(M_*)$$

are homotopic.

(3) If φ_* and ψ_* are homotopic, and $A_* \to B_*$ is a homomorphism of cosimplicial rings, then

$$\varphi_* \otimes 1, \psi_* \otimes 1 : K_* \otimes_{A_*} B_* \longrightarrow M_* \otimes_{A_*} B_*$$

are homotopic as homomorphisms of cosimplicial B_*-modules.

(4) If $I_* \subset A_*$ is a cosimplicial ideal, then the induced maps

$$\varphi_*^\wedge, \psi_*^\wedge : K_*^\wedge \longrightarrow M_*^\wedge$$

between completions are homotopic.

(5) Add more here as needed, for example symmetric powers.

Proof. Let $h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], N_*)$ be the given homotopy. In degree n we have

$$h_n = (h_{n,\alpha}) : K_n \longrightarrow \prod_{\alpha \in \Delta[1]_n} K_n$$

see Simplicial, Section 27. In order for a collection of $h_{n,\alpha}$ to form a homotopy, it is necessary and sufficient if for every $f : [n] \to [m]$ we have

$$h_{m,\alpha} \circ M_*(f) = N_*(f) \circ h_{n,\alpha \circ f}$$

see Simplicial, Equation (27.1.1). We also should have that $\psi_n = h_{n,0:[n]\to[1]}$ and $\varphi_n = h_{n,1:[n]\to[1]}$.

In each of the cases of the lemma we can produce the corresponding maps. Case (1). We can use the homotopy $h \otimes 1$ defined in degree n by setting

$$(h \otimes 1)_{n,\alpha} = h_{n,\alpha} \otimes 1_{L_n} : K_n \otimes_{A_n} L_n \longrightarrow M_n \otimes_{A_n} L_n.$$

Case (2). We can use the homotopy $\wedge^i h$ defined in degree n by setting

$$\wedge^i(h)_{n,\alpha} = \wedge^i(h_{n,\alpha}) : \wedge_{A_n}(K_n) \longrightarrow \wedge^i_{A_n}(M_n).$$

Case (3). We can use the homotopy $h \otimes 1$ defined in degree n by setting

$$(h \otimes 1)_{n,\alpha} = h_{n,\alpha} \otimes 1 : K_n \otimes_{A_n} B_n \longrightarrow M_n \otimes_{A_n} B_n.$$

Case (4). We can use the homotopy $h^\wedge$ defined in degree n by setting

$$(h^\wedge)_{n,\alpha} = h_{n,\alpha}^\wedge : K_n^\wedge \longrightarrow M_n^\wedge.$$

This works because each $h_{n,\alpha}$ is A_n-linear. $\qquad\square$

17. Crystals in quasi-coherent modules

In Situation 5.1. Set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. We are going to classify crystals in quasi-coherent modules on $\mathrm{Cris}(X/S)$. Before we do so we fix some notation.

Choose a polynomial ring $P = A[x_i]$ over A and a surjection $P \to C$ of A-algebras with kernel $J = \mathrm{Ker}(P \to C)$. Set

$$(17.0.1) \qquad\qquad D = \lim_e D_{P,\gamma}(J)/p^e D_{P,\gamma}(J)$$

for the p-adically completed divided power envelope. This ring comes with a divided power ideal $\bar{J}$ and divided power structure $\bar{\gamma}$, see Lemma 5.5. Set $D_e = D/p^e D$ and denote $\bar{J}_e$ the image of $\bar{J}$ in D_e. We will use the short hand

$$(17.0.2) \qquad\qquad \Omega_D = \lim_e \Omega_{D_e/A,\bar{\gamma}} = \lim_e \Omega_{D/A,\bar{\gamma}}/p^e \Omega_{D/A,\bar{\gamma}}$$

for the p-adic completion of the module of divided power differentials, see Lemma 6.12. It is also the p-adic completion of $\Omega_{D_{P,\gamma}(J)/A,\bar{\gamma}}$ which is free on $\mathrm{d}x_i$, see Lemma

6.6. Hence any element of Ω_D can be written uniquely as a sum $\sum f_i \mathrm{d}x_i$ with for all e only finitely many f_i not in $p^e D$. Moreover, the maps $\mathrm{d}_{D_e/A,\bar{\gamma}} : D_e \to \Omega_{D_e/A,\bar{\gamma}}$ fit together to define a divided power A-derivation

$$(17.0.3) \qquad\qquad\qquad \mathrm{d} : D \longrightarrow \Omega_D$$

on p-adic completions.

We will also need the "products $\mathrm{Spec}(D(n))$ of $\mathrm{Spec}(D)$", see Proposition 21.1 and its proof for an explanation. Formally these are defined as follows. For $n \geq 0$ let $J(n) = \mathrm{Ker}(P \otimes_A \ldots \otimes_A P \to C)$ where the tensor product has $n+1$ factors. We set

$$(17.0.4) \qquad D(n) = \lim_e D_{P \otimes_A \ldots \otimes_A P, \gamma}(J(n))/p^e D_{P \otimes_A \ldots \otimes_A P, \gamma}(J(n))$$

equal to the p-adic completion of the divided power envelope. We denote $\bar{J}(n)$ its divided power ideal and $\bar{\gamma}(n)$ its divided powers. We also introduce $D(n)_e = D(n)/p^e D(n)$ as well as the p-adically completed module of differentials

$$(17.0.5) \qquad \Omega_{D(n)} = \lim_e \Omega_{D(n)_e/A,\bar{\gamma}} = \lim_e \Omega_{D(n)/A,\bar{\gamma}}/p^e \Omega_{D(n)/A,\bar{\gamma}}$$

and derivation

$$(17.0.6) \qquad\qquad\qquad \mathrm{d} : D(n) \longrightarrow \Omega_{D(n)}$$

Of course we have $D = D(0)$. Note that the rings $D(0), D(1), D(2), \ldots$ form a cosimplicial object in the category of divided power rings.

Lemma 17.1. *Let D and $D(n)$ be as in (17.0.1) and (17.0.4). The coprojection $P \to P \otimes_A \ldots \otimes_A P$, $f \mapsto f \otimes 1 \otimes \ldots \otimes 1$ induces an isomorphism*

$$(17.1.1) \qquad\qquad D(n) = \lim_e D\langle \xi_i(j) \rangle / p^e D\langle \xi_i(j) \rangle$$

of algebras over D with

$$\xi_i(j) = x_i \otimes 1 \otimes \ldots \otimes 1 - 1 \otimes \ldots \otimes 1 \otimes x_i \otimes 1 \otimes \ldots \otimes 1$$

for $j = 1, \ldots, n$.

Proof. We have

$$P \otimes_A \ldots \otimes_A P = P[\xi_i(j)]$$

and $J(n)$ is generated by J and the elements $\xi_i(j)$. Hence the lemma follows from Lemma 2.5. $\square$

Lemma 17.2. *Let D and $D(n)$ be as in (17.0.1) and (17.0.4). Then $(D, \bar{J}, \bar{\gamma})$ and $(D(n), \bar{J}(n), \bar{\gamma}(n))$ are objects of $\mathrm{Cris}^\wedge(C/A)$, see Remark 5.4, and*

$$D(n) = \coprod_{j=0,\ldots,n} D$$

in $\mathrm{Cris}^\wedge(C/A)$.

Proof. The first assertion is clear. For the second, if $(B \to C, \delta)$ is an object of $\mathrm{Cris}^\wedge(C/A)$, then we have

$$\mathrm{Mor}_{\mathrm{Cris}^\wedge(C/A)}(D, B) = \mathrm{Hom}_A((P, J), (B, \mathrm{Ker}(B \to C)))$$

and similarly for $D(n)$ replacing (P, J) by $(P \otimes_A \ldots \otimes_A P, J(n))$. The property on coproducts follows as $P \otimes_A \ldots \otimes_A P$ is a coproduct. $\square$

In the lemma below we will consider pairs (M, ∇) satisfying the following conditions

    (1) M is a p-adically complete D-module,

(2) $\nabla : M \to M \otimes_D^\wedge \Omega_D$ is a connection, i.e., $\nabla(fm) = m \otimes \mathrm{d}f + f\nabla(m)$,

(3) ∇ is integrable (see Remark 6.10), and

(4) ∇ is *topologically quasi-nilpotent*: If we write $\nabla(m) = \sum \theta_i(m)\mathrm{d}x_i$ for some operators $\theta_i : M \to M$, then for any $m \in M$ there are only finitely many pairs (i,k) such that $\theta_i^k(m) \notin pM$.

The operators θ_i are sometimes denoted $\nabla_{\partial/\partial x_i}$ in the literature. In the following lemma we construct a functor from crystals in quasi-coherent modules on $\mathrm{Cris}(X/S)$ to the category of such pairs. We will show this functor is an equivalence in Proposition 17.4.

Lemma 17.3. *In the situation above there is a functor*

$$
\begin{array}{c}
\textit{crystals in quasi-coherent} \\
\mathcal{O}_{X/S}\textit{-modules on } \mathrm{Cris}(X/S)
\end{array}
\longrightarrow
\begin{array}{c}
\textit{pairs } (M, \nabla) \textit{ satisfying} \\
\textit{(1), (2), (3), and (4)}
\end{array}
$$

Proof. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on X/S. Set $T_e = \mathrm{Spec}(D_e)$ so that $(X, T_e, \bar\gamma)$ is an object of $\mathrm{Cris}(X/S)$ for $e \gg 0$. We have morphisms

$$(X, T_e, \bar\gamma) \to (X, T_{e+1}, \bar\gamma) \to \ldots$$

which are closed immersions. We set

$$M = \lim_e \Gamma((X, T_e, \bar\gamma), \mathcal{F}) = \lim_e \Gamma(T_e, \mathcal{F}_{T_e}) = \lim_e M_e$$

Note that since $\mathcal{F}$ is locally quasi-coherent we have $\mathcal{F}_{T_e} = \widetilde{M_e}$. Since $\mathcal{F}$ is a crystal we have $M_e = M_{e+1}/p^e M_{e+1}$. Hence we see that $M_e = M/p^e M$ and that M is p-adically complete.

By Lemma 15.1 we know that $\mathcal{F}$ comes endowed with a canonical integrable connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_{X/S}$. If we evaluate this connection on the objects T_e constructed above we obtain a canonical integrable connection

$$\nabla : M \longrightarrow M \otimes_D^\wedge \Omega_D$$

To see that this is topologically nilpotent we work out what this means.

Now we can do the same procedure for the rings $D(n)$. This produces a p-adically complete $D(n)$-module $M(n)$. Again using the crystal property of $\mathcal{F}$ we obtain isomorphisms

$$M \otimes_{D,p_0}^\wedge D(1) \to M(1) \leftarrow M \otimes_{D,p_1}^\wedge D(1)$$

compare with the proof of Lemma 15.1. Denote c the composition from left to right. Pick $m \in M$. Write $\xi_i = x_i \otimes 1 - 1 \otimes x_i$. Using (17.1.1) we can write uniquely

$$c(m \otimes 1) = \sum_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

for some $\theta_K(m) \in M$ where the sum is over multi-indices $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$. Set $\theta_i = \theta_K$ where K has a 1 in the ith spot and zeros elsewhere. We have

$$\nabla(m) = \sum \theta_i(m)\mathrm{d}x_i.$$

as can be seen by comparing with the definition of ∇. Namely, the defining equation is $p_1^* m = \nabla(m) - c(p_0^* m)$ in Lemma 15.1 but the sign works out because in the stacks project we consistently use $\mathrm{d}f = p_1(f) - p_0(f)$ modulo the ideal of the diagonal squared, and hence $\xi_i = x_i \otimes 1 - 1 \otimes x_i$ maps to $-\mathrm{d}x_i$ modulo the ideal of the diagonal squared.

Denote $q_i : D \to D(2)$ and $q_{ij} : D(1) \to D(2)$ the coprojections corresponding to the indices i, j. As in the last paragraph of the proof of Lemma 15.1 we see that

$$q_{02}^* c = q_{12}^* c \circ q_{01}^* c.$$

This means that

$$\sum\nolimits_{K''} \theta_{K''}(m) \otimes \prod \zeta_i''^{[k_i'']} = \sum\nolimits_{K',K} \theta_{K'}(\theta_K(m)) \otimes \prod \zeta_i'^{[k_i']} \prod \zeta_i^{[k_i]}$$

in $M \otimes_{D,q_2}^{\wedge} D(2)$ where

$$\zeta_i = x_i \otimes 1 \otimes 1 - 1 \otimes x_i \otimes 1,$$
$$\zeta_i' = 1 \otimes x_i \otimes 1 - 1 \otimes 1 \otimes x_i,$$
$$\zeta_i'' = x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_i.$$

In particular $\zeta_i'' = \zeta_i + \zeta_i'$ and we have that $D(2)$ is the p-adic completion of the divided power polynomial ring in ζ_i, ζ_i' over $q_2(D)$, see Lemma 17.1. Comparing coefficients in the expression above it follows immediately that $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ (this provides an alternative proof of the integrability of ∇) and that

$$\theta_K(m) = (\prod \theta_i^{k_i})(m).$$

In particular, as the sum expressing $c(m \otimes 1)$ above has to converge p-adically we conclude that for each i and each $m \in M$ only a finite number of $\theta_i^k(m)$ are allowed to be nonzero modulo p. $\qquad\square$

Proposition 17.4. *The functor*

$$\begin{array}{c} \text{crystals in quasi-coherent} \\ \mathcal{O}_{X/S}\text{-modules on } \mathrm{Cris}(X/S) \end{array} \longrightarrow \begin{array}{c} \text{pairs } (M, \nabla) \text{ satisfying} \\ \text{(1), (2), (3), and (4)} \end{array}$$

of Lemma 17.3 is an equivalence of categories.

Proof. Let (M, ∇) be given. We are going to construct a crystal in quasi-coherent modules $\mathcal{F}$. Write $\nabla(m) = \sum \theta_i(m) dx_i$. Then $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ and we can set $\theta_K(m) = (\prod \theta_i^{k_i})(m)$ for any multi-index $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$.

Let (U, T, δ) be any object of $\mathrm{Cris}(X/S)$ with T affine. Say $T = \mathrm{Spec}(B)$ and the ideal of $U \to T$ is $J_B \subset B$. By Lemma 5.6 there exists an integer e and a morphism

$$f : (U, T, \delta) \longrightarrow (X, T_e, \bar\gamma)$$

where $T_e = \mathrm{Spec}(D_e)$ as in the proof of Lemma 17.3. Choose such an e and f; denote $f : D \to B$ also the corresponding divided power A-algebra map. We will set $\mathcal{F}_T$ equal to the quasi-coherent sheaf of $\mathcal{O}_T$-modules associated to the B-module

$$M \otimes_{D,f} B.$$

However, we have to show that this is independent of the choice of f. Suppose that $g : D \to B$ is a second such morphism. Since f and g are morphisms in $\mathrm{Cris}(X/S)$ we see that the image of $f - g : D \to B$ is contained in the divided power ideal J_B. Write $\xi_i = f(x_i) - g(x_i) \in J_B$. By analogy with the proof of Lemma 17.3 we define an isomorphism

$$c_{f,g} : M \otimes_{D,f} B \longrightarrow M \otimes_{D,g} B$$

by the formula

$$m \otimes 1 \longmapsto \sum\nolimits_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

which makes sense by our remarks above and the fact that ∇ is topologically quasi-nilpotent (so the sum is finite!). A computation shows that

$$c_{g,h} \circ c_{f,g} = c_{f,h}$$

if given a third morphism $h : (U, T, \delta) \longrightarrow (X, T_e, \bar{\gamma})$. It is also true that $c_{f,f} = 1$. Hence these maps are all isomorphisms and we see that the module $\mathcal{F}_T$ is independent of the choice of f.

If $a : (U', T', \delta') \to (U, T, \delta)$ is a morphism of affine objects of $\mathrm{Cris}(X/S)$, then choosing $f' = f \circ a$ it is clear that there exists a canonical isomorphism $a^* \mathcal{F}_T \to \mathcal{F}_{T'}$. We omit the verification that this map is independent of the choice of f. Using these maps as the restriction maps it is clear that we obtain a crystal in quasi-coherent modules on the full subcategory of $\mathrm{Cris}(X/S)$ consisting of affine objects. We omit the proof that this extends to a crystal on all of $\mathrm{Cris}(X/S)$. We also omit the proof that this procedure is a functor and that it is quasi-inverse to the functor constructed in Lemma 17.3. $\qquad\square$

Lemma 17.5. *In Situation 5.1. Let $A \to P' \to C$ be ring maps with $A \to P'$ smooth and $P' \to C$ surjective with kernel J'. Let D' be the p-adic completion of $D_{P',\gamma}(J')$. There are homomorphisms of divided power A-algebras*

$$a : D \longrightarrow D', \quad b : D' \longrightarrow D$$

compatible with the maps $D \to C$ and $D' \to C$ such that $a \circ b = \mathrm{id}_{D'}$. These maps induce an equivalence of categories of pairs (M, ∇) satisfying (1), (2), (3), and (4) over D and pairs (M', ∇') satisfying (1), (2), (3), and (4) over D'. In particular, the equivalence of categories of Proposition 17.4 also holds for the corresponding functor towards pairs over D'.

Proof. We can pick the map $P = A[x_i] \to C$ such that it factors through a surjection of A-algebras $P \to P'$ (we may have to increase the number of variables in P to do this). Hence we obtain a surjective map $a : D \to D'$ by functoriality of divided power envelopes and completion. Pick e large enough so that D_e is a divided power thickening of C over A. Then $D_e \to C$ is a surjection whose kernel is locally nilpotent, see Divided Power Algebra, Lemma 2.6. Setting $D'_e = D'/p^e D'$ we see that the kernel of $D_e \to D'_e$ is locally nilpotent. Hence by Algebra, Lemma 133.16 we can find a lift $\beta_e : P' \to D_e$ of the map $P' \to D'_e$. Note that $D_{e+i+1} \to D_{e+i} \times_{D'_{e+i}} D'_{e+i+1}$ is surjective with square zero kernel for any $i \geq 0$ because $p^{e+i}D \to p^{e+i}D'$ is surjective. Applying the usual lifting property (Algebra, Proposition 133.13) successively to the diagrams

$$
\begin{array}{ccc}
P' & \longrightarrow & D_{e+i} \times_{D'_{e+i}} D'_{e+i+1} \\
\uparrow & & \uparrow \\
A & \longrightarrow & D_{e+i+1}
\end{array}
$$

we see that we can find an A-algebra map $\beta : P' \to D$ whose composition with a is the given map $P' \to D'$. By the universal property of the divided power envelope we obtain a map $D_{P',\gamma}(J') \to D$. As D is p-adically complete we obtain $b : D' \to D$ such that $a \circ b = \mathrm{id}_{D'}$.

Consider the base change functor

$$(M, \nabla) \longmapsto (M \otimes_D^\wedge D', \nabla')$$

from pairs for D to pairs for D', see Remark 6.11. Similarly, we have the base change functor corresponding to the divided power homomorphism $D' \to D$. To finish the proof of the lemma we have to show that the base change for the compositions $b \circ a : D \to D$ and $a \circ b : D' \to D'$ are isomorphic to the identity functor. This is clear for the second as $a \circ b = \mathrm{id}_{D'}$. To prove it for the first, we use the functorial isomorphism

$$c_{\mathrm{id}_D, b \circ a} : M \otimes_{D, \mathrm{id}_D} D \longrightarrow M \otimes_{D, b \circ a} D$$

of the proof of Proposition 17.4. The only thing to prove is that these maps are horizontal, which we omit.

The last statement of the proof now follows. $\square$

Remark 17.6. The equivalence of Proposition 17.4 holds if we start with a surjection $P \to C$ where P/A satisfies the strong lifting property of Algebra, Lemma 133.16. To prove this we can argue as in the proof of Lemma 17.5. (Details will be added here if we ever need this.) Presumably there is also a direct proof of this result, but the advantage of using polynomial rings is that the rings $D(n)$ are p-adic completions of divided power polynomial rings and the algebra is simplified.

18. General remarks on cohomology

In this section we do a bit of work to translate the cohomology of modules on the cristalline site of an affine scheme into an algebraic question.

Lemma 18.1. *In Situation 7.5. Let $\mathcal{F}$ be a locally quasi-coherent $\mathcal{O}_{X/S}$-module on $Cris(X/S)$. Then we have*

$$H^p((U, T, \delta), \mathcal{F}) = 0$$

for all $p > 0$ and all (U, T, δ) with T or U affine.

Proof. As $U \to T$ is a thickening we see that U is affine if and only if T is affine, see Limits, Lemma 10.1. Having said this, let us apply Cohomology on Sites, Lemma 11.9 to the collection $\mathcal{B}$ of affine objects (U, T, δ) and the collection Cov of affine open coverings $\mathcal{U} = \{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$. The Čech complex $\check{C}^*(\mathcal{U}, \mathcal{F})$ for such a covering is simply the Cech complex of the quasi-coherent $\mathcal{O}_T$-module $\mathcal{F}_T$ (here we are using the assumption that $\mathcal{F}$ is locally quasi-coherent) with respect to the affine open covering $\{T_i \to T\}$ of the affine scheme T. Hence the Čech cohomology is zero by Cohomology of Schemes, Lemma 2.5 and 2.2. Thus the hypothesis of Cohomology on Sites, Lemma 11.9 are satisfied and we win. $\square$

Lemma 18.2. *In Situation 7.5. Assume moreover X and S are affine schemes. Consider the full subcategory $\mathcal{C} \subset Cris(X/S)$ consisting of divided power thickenings (X, T, δ) endowed with the chaotic topology (see Sites, Example 6.6). For any locally quasi-coherent $\mathcal{O}_{X/S}$-module $\mathcal{F}$ we have*

$$R\Gamma(\mathcal{C}, \mathcal{F}|_{\mathcal{C}}) = R\Gamma(Cris(X/S), \mathcal{F})$$

Proof. We will use without further mention that $\mathcal{C}$ and $Cris(X/S)$ have products and fibre products, see Lemma 8.2. Note that the inclusion functor $u : \mathcal{C} \to$

$\mathrm{Cris}(X/S)$ is fully faithful, continuous and commutes with products and fibre products. We claim it defines a morphism of ringed sites

$$f : (\mathrm{Cris}(X/S), \mathcal{O}_{X/S}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}_{X/S}|_{\mathcal{C}})$$

To see this we will use Sites, Lemma 15.5. Note that $\mathcal{C}$ has fibre products and u commutes with them so the categories $\mathcal{I}^u_{(U,T,\delta)}$ are disjoint unions of directed categories (by Sites, Lemma 5.1 and Categories, Lemma 19.7). Hence it suffices to show that $\mathcal{I}^u_{(U,T,\delta)}$ is connected. Nonempty follows from Lemma 5.6 and connectedness follows from the fact that $\mathcal{C}$ has products and that u commutes with them (compare with the proof of Sites, Lemma 5.2).

Note that $f_*\mathcal{F} = \mathcal{F}|_{\mathcal{C}}$. Hence the lemma follows if $R^p f_*\mathcal{F} = 0$ for $p > 0$, see Cohomology on Sites, Lemma 14.6. By Cohomology on Sites, Lemma 8.4 it suffices to show that $H^p((X,T,\delta), \mathcal{F}) = 0$ for all (X,T,δ). This follows from Lemma 18.1. $\qquad\square$

Lemma 18.3. *In Situation 5.1. Set $\mathcal{C} = (Cris(C/A))^{opp}$ and $\mathcal{C}^{\wedge} = (Cris^{\wedge}(C/A))^{opp}$ endowed with the chaotic topology, see Remark 5.4 for notation. There is a morphism of topoi*

$$g : Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{C}^{\wedge})$$

such that if $\mathcal{F}$ is a sheaf of abelian groups on $\mathcal{C}$, then

$$R^p g_*\mathcal{F}(B \to C, \delta) = \begin{cases} \lim_e \mathcal{F}(B_e \to C, \delta) & \text{if } p = 0 \\ R^1 \lim_e \mathcal{F}(B_e \to C, \delta) & \text{if } p = 1 \\ 0 & \text{else} \end{cases}$$

where $B_e = B/p^e B$ for $e \gg 0$.

Proof. Any functor between categories defines a morphism between chaotic topoi in the same direction, for example because such a functor can be considered as a cocontinuous functor between sites, see Sites, Section 20. Proof of the description of $g_*\mathcal{F}$ is omitted. Note that in the statement we take $(B_e \to C, \delta)$ is an object of $\mathrm{Cris}(C/A)$ only for e large enough. Let $\mathcal{I}$ be an injective abelian sheaf on $\mathcal{C}$. Then the transition maps

$$\mathcal{I}(B_e \to C, \delta) \leftarrow \mathcal{I}(B_{e+1} \to C, \delta)$$

are surjective as the morphisms

$$(B_e \to C, \delta) \longrightarrow (B_{e+1} \to C, \delta)$$

are monomorphisms in the category $\mathcal{C}$. Hence for an injective abelian sheaf both sides of the displayed formula of the lemma agree. Taking an injective resolution of $\mathcal{F}$ one easily obtains the result (sheaves are presheaves, so exactness is measured on the level of groups of sections over objects). $\qquad\square$

Lemma 18.4. *Let $\mathcal{C}$ be a category endowed with the chaotic topology. Let X be an object of $\mathcal{C}$ such that every object of $\mathcal{C}$ has a morphism towards X. Assume that $\mathcal{C}$ has products. Then for every abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ the total cohomology $R\Gamma(\mathcal{C}, \mathcal{F})$ is represented by the complex*

$$\mathcal{F}(X) \to \mathcal{F}(X \times X) \to \mathcal{F}(X \times X \times X) \to \dots$$

associated to the cosimplicial abelian group $[n] \mapsto \mathcal{F}(X^n)$.

Proof. Note that $H^q(X^p, \mathcal{F}) = 0$ for all $q > 0$ as sheaves are presheaves on $\mathcal{C}$. The assumption on X is that $h_X \to *$ is surjective. Using that $H^q(X, \mathcal{F}) = H^p(h_X, \mathcal{F})$ and $H^p(\mathcal{C}, \mathcal{F}) = H^p(*, \mathcal{F})$ we see that our statement is a special case of Cohomology on Sites, Lemma 13.2. $\qquad\square$

19. Cosimplicial preparations

In this section we compare crystalline cohomology with de Rham cohomology. We follow [BdJ11].

Example 19.1. Suppose that A_* is any cosimplicial ring. Consider the cosimplicial module M_* defined by the rule

$$M_n = \bigoplus\nolimits_{i=0,\ldots,n} A_n e_i$$

For a map $f : [n] \to [m]$ define $M_*(f) : M_n \to M_m$ to be the unique $A_*(f)$-linear map which maps e_i to $e_{f(i)}$. We claim the identity on M_* is homotopic to 0. Namely, a homotopy is given by a map of cosimplicial modules

$$h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], M_*)$$

see Section 16. For $j \in \{0, \ldots, n+1\}$ we let $\alpha_j^n : [n] \to [1]$ be the map defined by $\alpha_j^n(i) = 0 \Leftrightarrow i < j$. Then $\Delta[1]_n = \{\alpha_0^n, \ldots, \alpha_{n+1}^n\}$ and correspondingly $\mathrm{Hom}(\Delta[1], M_*)_n = \prod_{j=0,\ldots,n+1} M_n$, see Simplicial, Sections 25 and 27. Instead of using this product representation, we think of an element in $\mathrm{Hom}(\Delta[1], M_*)_n$ as a function $\Delta[1]_n \to M_n$. Using this notation, we define h in degree n by the rule

$$h_n(e_i)(\alpha_j^n) = \begin{cases} e_i & \text{if} \quad i < j \\ 0 & \text{else} \end{cases}$$

We first check h is a morphism of cosimplicial modules. Namely, for $f : [n] \to [m]$ we will show that

(19.1.1) $$h_m \circ M_*(f) = \mathrm{Hom}(\Delta[1], M_*)(f) \circ h_n$$

The left hand side of (19.1.1) evaluated at e_i and then in turn evaluated at α_j^m is

$$h_m(e_{f(i)})(\alpha_j^m) = \begin{cases} e_{f(i)} & \text{if} \quad f(i) < j \\ 0 & \text{else} \end{cases}$$

Note that $\alpha_j^m \circ f = \alpha_{j'}^n$ where $0 \le j' \le n+1$ is the unique index such that $f(i) < j$ if and only if $i < j'$. Thus the right hand side of (19.1.1) evaluated at e_i and then in turn evaluated at α_j^m is

$$M_*(f)(h_n(e_i)(\alpha_j^m \circ f) = M_*(f)(h_n(e_i)(\alpha_{j'}^n)) = \begin{cases} e_{f(i)} & \text{if} \quad i < j' \\ 0 & \text{else} \end{cases}$$

It follows from our description of j' that the two answers are equal. Hence h is a map of cosimplicial modules. Let $0 : \Delta[0] \to \Delta[1]$ and $1 : \Delta[0] \to \Delta[1]$ be the obvious maps, and denote $ev_0, ev_1 : \mathrm{Hom}(\Delta[1], M_*) \to M_*$ the corresponding evaluation maps. The reader verifies readily that the the compositions

$$ev_0 \circ h, ev_1 \circ h : M_* \longrightarrow M_*$$

are 0 and 1 respectively, whence h is the desired homotopy between 0 and 1.

Lemma 19.2. *With notation as in (17.0.5) the complex*

$$\Omega_{D(0)} \to \Omega_{D(1)} \to \Omega_{D(2)} \to \dots$$

is homotopic to zero as a $D(*)$-cosimplicial module.

Proof. We are going to use the principle of Simplicial, Lemma 27.3 and more specifically Lemma 16.1 which tells us that homotopic maps between (co)simplicial objects are transformed by any functor into homotopic maps. The complex of the lemma is equal to the p-adic completion of the base change of the cosimplicial module

$$M_* = \left(\Omega_{P/A} \to \Omega_{P\otimes_A P/A} \to \Omega_{P\otimes_A P\otimes_A P/A} \to \dots \right)$$

via the cosimplicial ring map $P \otimes_A \dots \otimes_A P \to D(n)$. This follows from Lemma 6.6, see comments following (17.0.2). Hence it suffices to show that the cosimplicial module M_* is homotopic to zero (uses base change and p-adic completion). We can even assume $A = \mathbf{Z}$ and $P = \mathbf{Z}[\{x_i\}_{i\in I}]$ as we can use base change with $\mathbf{Z} \to A$. In this case $P^{\otimes n+1}$ is the polynomial algebra on the elements

$$x_i(e) = 1 \otimes \dots \otimes x_i \otimes \dots \otimes 1$$

with x_i in the eth slot. The modules of the complex are free on the generators $\mathrm{d}x_i(e)$. Note that if $f : [n] \to [m]$ is a map then we see that

$$M_*(f)(\mathrm{d}x_i(e)) = \mathrm{d}x_i(f(e))$$

Hence we see that M_* is a direct sum over I of copies of the module studied in Example 19.1 and we win. $\square$

Lemma 19.3. *With notation as in (17.0.4) and (17.0.5), given any cosimplicial module M_* over $D(*)$ and $i > 0$ the cosimplicial module*

$$M_0 \otimes^\wedge_{D(0)} \Omega^i_{D(0)} \to M_1 \otimes^\wedge_{D(1)} \Omega^i_{D(1)} \to M_2 \otimes^\wedge_{D(2)} \Omega^i_{D(2)} \to \dots$$

is homotopic to zero, where $\Omega^i_{D(n)}$ is the p-adic completion of the ith exterior power of $\Omega_{D(n)}$.

Proof. By Lemma 19.2 the endomorphisms 0 and 1 of $\Omega_{D(*)}$ are homotopic. If we apply the functor $\wedge^i$ we see that the same is true for the cosimplicial module $\wedge^i \Omega_{D(*)}$, see Lemma 16.1. Another application of the same lemma shows the p-adic completion $\Omega^i_{D(*)}$ is homotopy equivalent to zero. Tensoring with M_* we see that $M_* \otimes_{D(*)} \Omega^i_{D(*)}$ is homotopic to zero, see Lemma 16.1 again. A final application of the p-adic completion functor finishes the proof. $\square$

20. Divided power Poincaré lemma

Just the simplest possible version.

Lemma 20.1. *Let A be a ring. Let $P = A\langle x_i \rangle$ be a divided power polynomial ring over A. For any A-module M the complex*

$$0 \to M \to M \otimes_A P \to M \otimes_A \Omega^1_{P/A,\delta} \to M \otimes_A \Omega^2_{P/A,\delta} \to \dots$$

is exact. Let D be the p-adic completion of P. Let Ω^i_D be the p-adic completion of the ith exterior power of $\Omega_{D/A,\delta}$. For any p-adically complete A-module M the complex

$$0 \to M \to M \otimes^\wedge_A D \to M \otimes^\wedge_A \Omega^1_D \to M \otimes^\wedge_A \Omega^2_D \to \dots$$

is exact.

Proof. It suffices to show that the complex

$$E : (0 \to A \to P \to \Omega^1_{P/A,\delta} \to \Omega^2_{P/A,\delta} \to \ldots)$$

is homotopy equivalent to zero as a complex of A-modules. For every multi-index $K = (k_i)$ we can consider the subcomplex $E(K)$ which in degree j consists of

$$\bigoplus\nolimits_{I=\{i_1,\ldots,i_j\}\subset \mathrm{Supp}(K)} A \prod\nolimits_{i\notin I} x_i^{[k_i]} \prod\nolimits_{i\in I} x_i^{[k_i-1]} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_j}$$

Since $E = \bigoplus E(K)$ we see that it suffices to prove each of the complexes $E(K)$ is homotopic to zero. If $K = 0$, then $E(K) : (A \to A)$ is homotopic to zero. If K has nonempty (finite) support S, then the complex $E(K)$ is isomorphic to the complex

$$0 \to A \to \bigoplus\nolimits_{s\in S} A \to \wedge^2(\bigoplus\nolimits_{s\in S} A) \to \ldots \to \wedge^{\#S}(\bigoplus\nolimits_{s\in S} A) \to 0$$

which is homotopic to zero, for example by More on Algebra, Lemma 20.5. $\square$

An alternative (more direct) approach to the following lemma is explained in Example 25.2.

Lemma 20.2. *Let A be a ring. Let (B, J, δ) be a divided power ring. Let $P = B\langle x_i\rangle$ be a divided power polynomial ring over B with divided power ideal $J = IP + B\langle x_i\rangle_+$ as usual. Let M be a B-module endowed with an integrable connection $\nabla : M \to M \otimes_B \Omega^1_{B/A,\delta}$. Then the map of de Rham complexes*

$$M \otimes_B \Omega^*_{B/A,\delta} \longrightarrow M \otimes_P \Omega^*_{P/A,\delta}$$

is a quasi-isomorphism. Let D, resp. D' be the p-adic completion of B, resp. P and let Ω^i_D, resp. $\Omega^i_{D'}$ be the p-adic completion of $\Omega^i_{B/A,\delta}$, resp. $\Omega^i_{P/A,\delta}$. Let M be a p-adically complete D-module endowed with an integral connection $\nabla : M \to M \otimes^\wedge_D \Omega^1_D$. Then the map of de Rham complexes

$$M \otimes^\wedge_D \Omega^*_D \longrightarrow M \otimes^\wedge_D \Omega^*_{D'}$$

is a quasi-isomorphism.

Proof. Consider the decreasing filtration F^* on $\Omega^*_{B/A,\delta}$ given by the subcomplexes $F^i(\Omega^*_{B/A,\delta}) = \sigma_{\geq i}\Omega^*_{B/A,\delta}$. See Homology, Section 13. This induces a decreasing filtration F^* on $\Omega^*_{P/A,\delta}$ by setting

$$F^i(\Omega^*_{P/A,\delta}) = F^i(\Omega^*_{B/A,\delta}) \wedge \Omega^*_{P/A,\delta}.$$

We have a split short exact sequence

$$0 \to \Omega^1_{B/A,\delta} \otimes_B P \to \Omega^1_{P/A,\delta} \to \Omega^1_{P/B,\delta} \to 0$$

and the last module is free on $\mathrm{d}x_i$. It follows from this that $F^i(\Omega^*_{P/A,\delta}) \to \Omega^*_{P/A,\delta}$ is a termwise split injection and that

$$\mathrm{gr}^i_F(\Omega^*_{B/A,\delta}) = \Omega^i_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta}$$

as complexes. Thus we can define a filtration F^* on $M \otimes_B \Omega^*_{B/A,\delta}$ by setting

$$F^i(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes_B F^i(\Omega^*_{P/A,\delta})$$

and we have

$$\mathrm{gr}^i_F(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes_B \Omega^i_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta}$$

as complexes. By Lemma 20.1 each of these complexes is quasi-isomorphic to $M \otimes_B \Omega^i_{B/A,\delta}$ placed in degree 0. Hence we see that the first displayed map of

the lemma is a morphism of filtered complexes which induces a quasi-isomorphism on graded pieces. This implies that it is a quasi-isomorphism, for example by the spectral sequence associated to a filtered complex, see Homology, Section 21.

The proof of the second quasi-isomorphism is exactly the same. $\qquad\square$

21. Cohomology in the affine case

Let's go back to the situation studied in Section 17. We start with (A, I, γ) and $A/I \to C$ and set $X = \operatorname{Spec}(C)$ and $S = \operatorname{Spec}(A)$. Then we choose a polynomial ring P over A and a surjection $P \to C$ with kernel J. We obtain D and $D(n)$ see (17.0.1) and (17.0.4). Set $T(n)_e = \operatorname{Spec}(D(n)/p^e D(n))$ so that $(X, T(n)_e, \delta(n))$ is an object of $\operatorname{Cris}(X/S)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules and set

$$M(n) = \lim_e \Gamma((X, T(n)_e, \delta(n)), \mathcal{F})$$

for $n = 0, 1, 2, 3, \ldots$. This forms a cosimplicial module over the cosimplicial ring $D(0), D(1), D(2), \ldots$.

Proposition 21.1. *With notations as above assume that*

1. *$\mathcal{F}$ is locally quasi-coherent, and*
2. *for any morphism $(U, T, \delta) \to (U', T', \delta')$ of $\operatorname{Cris}(X/S)$ where $f : T \to T'$ is a closed immersion the map $c_f : f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is surjective.*

Then the complex

$$M(0) \to M(1) \to M(2) \to \ldots$$

computes $R\Gamma(\operatorname{Cris}(X/S), \mathcal{F})$.

Proof. Using assumption (1) and Lemma 18.2 we see that $R\Gamma(\operatorname{Cris}(X/S), \mathcal{F})$ is isomorphic to $R\Gamma(\mathcal{C}, \mathcal{F})$. Note that the categories $\mathcal{C}$ used in Lemmas 18.2 and 18.3 agree. Let $f : T \to T'$ be a closed immersion as in (2). Surjectivity of $c_f : f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is equivalent to surjectivity of $\mathcal{F}_{T'} \to f_*\mathcal{F}_T$. Hence, if $\mathcal{F}$ satisfies (1) and (2), then we obtain a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{F}_{T'} \to f_*\mathcal{F}_T \to 0$$

of quasi-coherent $\mathcal{O}_{T'}$-modules on T', see Schemes, Section 24 and in particular Lemma 24.1. Thus, if T' is affine, then we conclude that the restriction map $\mathcal{F}(U', T', \delta') \to \mathcal{F}(U, T, \delta)$ is surjective by the vanishing of $H^1(T', \mathcal{K})$, see Cohomology of Schemes, Lemma 2.2. Hence the transition maps of the inverse systems in Lemma 18.3 are surjective. We conclude that that $R^p g_*(\mathcal{F}|_\mathcal{C}) = 0$ for all $p \geq 1$ where g is as in Lemma 18.3. The object D of the category $\mathcal{C}^\wedge$ satisfies the assumption of Lemma 18.4 by Lemma 5.7 with

$$D \times \ldots \times D = D(n)$$

in $\mathcal{C}$ because $D(n)$ is the $n+1$-fold coproduct of D in $\operatorname{Cris}^\wedge(C/A)$, see Lemma 17.2. Thus we win. $\qquad\square$

Lemma 21.2. *Assumptions and notation as in Proposition 21.1. Then*

$$H^j(\operatorname{Cris}(X/S), \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0$$

for all $i > 0$ and all $j \geq 0$.

Proof. Using Lemma 12.6 it follows that $\mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}$ also satisfies assumptions (1) and (2) of Proposition 21.1. Write $M(n)_e = \Gamma((X, T(n)_e, \delta(n)), \mathcal{F})$ so that $M(n) = \lim_e M(n)_e$. Then

$$\lim_e \Gamma((X, T(n)_e, \delta(n)), \mathcal{H}) = \lim_e M(n)_e \otimes_{D(n)_e} \Omega_{D(n)}/p^e \Omega_{D(n)}$$
$$= \lim_e M(n)_e \otimes_{D(n)} \Omega_{D(n)}$$

By Lemma 19.3 the cosimplicial modules

$$M(0)_e \otimes_{D(0)} \Omega^i_{D(0)} \to M(1)_e \otimes_{D(1)} \Omega^i_{D(1)} \to M(2)_e \otimes_{D(2)} \Omega^i_{D(2)} \to \cdots$$

are homotopic to zero. Because the transition maps $M(n)_{e+1} \to M(n)_e$ are surjective, we see that the inverse limit of the associated complexes are acyclic[5]. Hence the vanishing of cohomology of $\mathcal{H}$ by Proposition 21.1. $\qquad\square$

Proposition 21.3. *Assumptions as in Proposition 21.1 but now assume that $\mathcal{F}$ is a crystal in quasi-coherent modules. Let (M, ∇) be the corresponding module with connection over D, see Proposition 17.4. Then the complex*

$$M \otimes_D^\wedge \Omega^*_D$$

computes $R\Gamma(Cris(X/S), \mathcal{F})$.

Proof. We will prove this using the two spectral sequences associated to the double complex $K^{*,*}$ with terms

$$K^{a,b} = M \otimes_D^\wedge \Omega^a_{D(b)}$$

What do we know so far? Well, Lemma 19.3 tells us that each column $K^{a,*}$, $a > 0$ is acyclic. Proposition 21.1 tells us that the first column $K^{0,*}$ is quasi-isomorphic to $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$. Hence the first spectral sequence associated to the double complex shows that there is a canonical quasi-isomorphism of $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ with $\mathrm{Tot}(K^{*,*})$.

Next, let's consider the rows $K^{*,b}$. By Lemma 17.1 each of the $b+1$ maps $D \to D(b)$ presents $D(b)$ as the p-adic completion of a divided power polynomial algebra over D. Hence Lemma 20.2 shows that the map

$$M \otimes_D^\wedge \Omega^*_D \longrightarrow M \otimes_{D(b)}^\wedge \Omega^*_{D(b)} = K^{*,b}$$

is a quasi-isomorphism. Note that each of these maps defines the *same* map on cohomology (and even the same map in the derived category) as the inverse is given by the co-diagonal map $D(b) \to D$ (corresponding to the multiplication map $P \otimes_A \ldots \otimes_A P \to P$). Hence if we look at the E_1 page of the second spectral sequence we obtain

$$E_1^{a,b} = H^a(M \otimes_D^\wedge \Omega^*_D)$$

with differentials

$$E_1^{a,0} \xrightarrow{0} E_1^{a,1} \xrightarrow{1} E_1^{a,2} \xrightarrow{0} E_1^{a,3} \xrightarrow{1} \cdots$$

as each of these is the alternation sum of the given identifications $H^a(M \otimes_D^\wedge \Omega^*_D) = E_1^{a,0} = E_1^{a,1} = \cdots$. Thus we see that the E_2 page is equal $H^a(M \otimes_D^\wedge \Omega^*_D)$ on the first row and zero elsewhere. It follows that the identification of $M \otimes_D^\wedge \Omega^*_D$ with the first row induces a quasi-isomorphism of $M \otimes_D^\wedge \Omega^*_D$ with $\mathrm{Tot}(K^{*,*})$. $\qquad\square$

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[5]Actually, they are even homotopic to zero as the homotopies fit together, but we don't need this. The reason for this roundabout argument is that the limit $\lim_e M(n)_e \otimes_{D(n)} \Omega^i_{D(n)}$ isn't the p-adic completion of $M(n) \otimes_{D(n)} \Omega^i_{D(n)}$ as with the assumptions of the lemma we don't know that $M(n)_e = M(n)_{e+1}/p^e M(n)_{e+1}$. If $\mathcal{F}$ is a crystal then this does hold.

Lemma 21.4. *Assumptions as in Proposition 21.3. Let $A \to P' \to C$ be ring maps with $A \to P'$ smooth and $P' \to C$ surjective with kernel J'. Let D' be the p-adic completion of $D_{P',\gamma}(J')$. Let (M', ∇') be the pair over D' corresponding to $\mathcal{F}$, see Lemma 17.5. Then the complex*

$$M' \otimes_{D'}^{\wedge} \Omega_{D'}^*$$

computes $R\Gamma(Cris(X/S), \mathcal{F})$.

Proof. Choose $a : D \to D'$ and $b : D' \to D$ as in Lemma 17.5. Note that the base change $M = M' \otimes_{D',b} D$ with its connection ∇ corresponds to $\mathcal{F}$. Hence we know that $M \otimes_D^{\wedge} \Omega_D^*$ computes the crystalline cohomology of $\mathcal{F}$, see Proposition 21.3. Hence it suffices to show that the base change maps (induced by a and b)

$$M' \otimes_{D'}^{\wedge} \Omega_{D'}^* \longrightarrow M \otimes_D^{\wedge} \Omega_D^* \quad \text{and} \quad M \otimes_D^{\wedge} \Omega_D^* \longrightarrow M' \otimes_{D'}^{\wedge} \Omega_{D'}^*$$

are quasi-isomorphisms. Since $a \circ b = \text{id}_{D'}$ we see that the composition one way around is the identity on the complex $M' \otimes_{D'}^{\wedge} \Omega_{D'}^*$. Hence it suffices to show that the map

$$M \otimes_D^{\wedge} \Omega_D^* \longrightarrow M \otimes_D^{\wedge} \Omega_D^*$$

induced by $b \circ a : D \to D$ is a quasi-isomorphism. (Note that we have the same complex on both sides as $M = M' \otimes_{D',b} D$, hence $M \otimes_{D,b \circ a}^{\wedge} D = M' \otimes_{D',b \circ a \circ b}^{\wedge} D = M' \otimes_{D',b}^{\wedge} D = M$.) In fact, we claim that for any divided power A-algebra homomorphism $\rho : D \to D$ compatible with the augmentation to C the induced map $M \otimes_D^{\wedge} \Omega_D^* \to M \otimes_{D,\rho}^{\wedge} \Omega_D^*$ is a quasi-isomorphism.

Write $\rho(x_i) = x_i + z_i$. The elements z_i are in the divided power ideal of D because ρ is compatible with the augmentation to C. Hence we can factor the map ρ as a composition

$$D \xrightarrow{\sigma} D\langle \xi_i \rangle^{\wedge} \xrightarrow{\tau} D$$

where the first map is given by $x_i \mapsto x_i + \xi_i$ and the second map is the divided power D-algebra map which maps ξ_i to z_i. (This uses the universal properties of polynomial algebra, divided power polynomial algebras, divided power envelopes, and p-adic completion.) Note that there exists an *automorphism* α of $D\langle \xi_i \rangle^{\wedge}$ with $\alpha(x_i) = x_i - \xi_i$ and $\alpha(\xi_i) = \xi_i$. Applying Lemma 20.2 to $\alpha \circ \sigma$ (which maps x_i to x_i) and using that α is an isomorphism we conclude that σ induces a quasi-isomorphism of $M \otimes_D^{\wedge} \Omega_D^*$ with $M \otimes_{D,\sigma}^{\wedge} \Omega_{D\langle x_i \rangle^{\wedge}}^*$. On the other hand the map τ has as a left inverse the map $D \to D\langle x_i \rangle^{\wedge}$, $x_i \mapsto x_i$ and we conclude (using Lemma 20.2 once more) that τ induces a quasi-isomorphism of $M \otimes_{D,\sigma}^{\wedge} \Omega_{D\langle x_i \rangle^{\wedge}}^*$ with $M \otimes_{D,\tau \circ \sigma}^{\wedge} \Omega_D^*$. Composing these two quasi-isomorphisms we obtain that ρ induces a quasi-isomorphism $M \otimes_D^{\wedge} \Omega_D^* \to M \otimes_{D,\rho}^{\wedge} \Omega_D^*$ as desired. $\square$

22. Two counter examples

Before we turn to some of the successes of crystalline cohomology, let us give two examples which explain why crystalline cohomology does not work very well if the schemes in question are either not proper over the base, or singular. The first example can be found in [BO83].

Example 22.1. Let $A = \mathbf{Z}_p$ with divided power ideal (p) endowed with its unique divided powers γ. Let $C = \mathbf{F}_p[x, y]/(x^2, xy, y^2)$. We choose the presentation

$$C = P/J = \mathbf{Z}_p[x, y]/(x^2, xy, y^2, p)$$

Let $D = D_{P,\gamma}(J)^\wedge$ with divided power ideal $(\bar{J}, \bar{\gamma})$ as in Section 17. We will denote x, y also the images of x and y in D. Consider the element

$$\tau = \bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2) - \bar{\gamma}_p(xy)^2 \in D$$

We note that $p\tau = 0$ as

$$p!\bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2) = x^{2p}\bar{\gamma}_p(y^2) = \bar{\gamma}_p(x^2 y^2) = x^p y^p \bar{\gamma}_p(xy) = p!\bar{\gamma}_p(xy)^2$$

in D. We also note that $\mathrm{d}\tau = 0$ in Ω_D as

$$\begin{aligned}
\mathrm{d}(\bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2)) &= \bar{\gamma}_{p-1}(x^2)\bar{\gamma}_p(y^2)\mathrm{d}x^2 + \bar{\gamma}_p(x^2)\bar{\gamma}_{p-1}(y^2)\mathrm{d}y^2 \\
&= 2x\bar{\gamma}_{p-1}(x^2)\bar{\gamma}_p(y^2)\mathrm{d}x + 2y\bar{\gamma}_p(x^2)\bar{\gamma}_{p-1}(y^2)\mathrm{d}y \\
&= 2/(p-1)!(x^{2p-1}\bar{\gamma}_p(y^2)\mathrm{d}x + y^{2p-1}\bar{\gamma}_p(x^2)\mathrm{d}y) \\
&= 2/(p-1)!(x^{p-1}\bar{\gamma}_p(xy^2)\mathrm{d}x + y^{p-1}\bar{\gamma}_p(x^2 y)\mathrm{d}y) \\
&= 2/(p-1)!(x^{p-1}y^p\bar{\gamma}_p(xy)\mathrm{d}x + x^p y^{p-1}\bar{\gamma}_p(xy)\mathrm{d}y) \\
&= 2\bar{\gamma}_{p-1}(xy)\bar{\gamma}_p(xy)(y\mathrm{d}x + x\mathrm{d}y) \\
&= \mathrm{d}(\bar{\gamma}_p(xy)^2)
\end{aligned}$$

Finally, we claim that $\tau \neq 0$ in D. To see this it suffices to produce an object $(B \to \mathbf{F}_p[x, y]/(x^2, xy, y^2), \delta)$ of $\mathrm{Cris}(C/S)$ such that τ does not map to zero in B. To do this take

$$B = \mathbf{F}_p[x, y, u, v]/(x^3, x^2 y, xy^2, y^3, xu, yu, xv, yv, u^2, v^2)$$

with the obvious surjection to C. Let $K = \mathrm{Ker}(B \to C)$ and consider the map

$$\delta_p : K \longrightarrow K, \quad ay^2 + bxy + cy^2 + du + ev + fuv \longmapsto a^p u + c^p v$$

One checks this satisfies the assumptions (1), (2), (3) of Divided Power Algebra, Lemma 5.3 and hence defines a divided power structure. Moreover, we see that τ maps to uv which is not zero in B. Set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. We draw the following conclusions

(1) $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ has p-torsion, and
(2) pulling back by frobenius $F^* : H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S}) \to H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is not injective.

Namely, τ defines a nonzero torsion element of $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ by Proposition 21.3. Similarly, $F^*(\tau) = \sigma(\tau)$ where $\sigma : D \to D$ is the map induced by any lift of Frobenius on P. If we choose $\sigma(x) = x^p$ and $\sigma(y) = y^p$, then an easy computation shows that $F^*(\tau) = 0$.

The next example shows that even for affine n-space crystalline cohomology does not give the correct thing.

Example 22.2. Let $A = \mathbf{Z}_p$ with divided power ideal (p) endowed with its unique divided powers γ. Let $C = \mathbf{F}_p[x_1, \ldots, x_r]$. We choose the presentation

$$C = P/J = P/pP \quad \text{with} \quad P = \mathbf{Z}_p[x_1, \ldots, x_r]$$

Note that pP has divided powers by Divided Power Algebra, Lemma 4.2. Hence setting $D = P^\wedge$ with divided power ideal (p) we obtain a situation as in Section 17. We conclude that $R\Gamma(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is represented by the complex

$$D \to \Omega_D^1 \to \Omega_D^2 \to \ldots \to \Omega_D^r$$

see Proposition 21.3. Assuming $r > 0$ we conclude the following

(1) The cristalline cohomology of the cristalline structure sheaf of $X = \mathbf{A}^r_{\mathbf{F}_p}$ over $S = \operatorname{Spec}(\mathbf{Z}_p)$ is zero except in degrees $0, \ldots, r$.
(2) We have $H^0(\operatorname{Cris}(X/S), \mathcal{O}_{X/S}) = \mathbf{Z}_p$.
(3) The cohomology group $H^r(\operatorname{Cris}(X/S), \mathcal{O}_{X/S})$ is infinite and is not a torsion abelian group.
(4) The cohomology group $H^r(\operatorname{Cris}(X/S), \mathcal{O}_{X/S})$ is not separated for the p-adic topology.

While the first two statements are reasonable, parts (3) and (4) are disconcerting! The truth of these statements follows immediately from working out what the complex displayed above looks like. Let's just do this in case $r = 1$. Then we are just looking at the two term complex of p-adically complete modules

$$\mathrm{d} : D = \left(\bigoplus\nolimits_{n \geq 0} \mathbf{Z}_p x^n \right)^\wedge \longrightarrow \Omega^1_D = \left(\bigoplus\nolimits_{n \geq 1} \mathbf{Z}_p x^{n-1} \mathrm{d}x \right)^\wedge$$

The map is given by $\operatorname{diag}(0, 1, 2, 3, 4, \ldots)$ except that the first summand is missing on the right hand side. Now it is clear that $\bigoplus_{n>0} \mathbf{Z}_p/n\mathbf{Z}_p$ is a subgroup of the cokernel, hence the cokernel is infinite. In fact, the element

$$\omega = \sum\nolimits_{e>0} p^e x^{p^{2e}-1} \mathrm{d}x$$

is clearly not a torsion element of the cokernel. But it gets worse. Namely, consider the element

$$\eta = \sum\nolimits_{e>0} p^e x^{p^e-1} \mathrm{d}x$$

For every $t > 0$ the element η is congruent to $\sum_{e>t} p^e x^{p^e-1} \mathrm{d}x$ modulo the image of d which is divisible by p^t. But η is not in the image of d because it would have to be the image of $a + \sum_{e>0} x^{p^e}$ for some $a \in \mathbf{Z}_p$ which is not an element of the left hand side. In fact, $p^N \eta$ is similarly not in the image of d for any integer N. This implies that η "generates" a copy of $\mathbf{Q}_p$ inside of $H^1_{\operatorname{cris}}(\mathbf{A}^1_{\mathbf{F}_p}/\operatorname{Spec}(\mathbf{Z}_p))$.

23. Applications

In this section we collect some applications of the material in the previous sections.

Proposition 23.1. *In Situation 7.5. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on $\operatorname{Cris}(X/S)$. The truncation map of complexes*

$$(\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^2_{X/S} \to \ldots) \longrightarrow \mathcal{F}[0],$$

while not a quasi-isomorphism, becomes a quasi-isomorphism after applying $Ru_{X/S,*}$. In fact, for any $i > 0$, we have

$$Ru_{X/S,*}(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0.$$

Proof. By Lemma 15.1 we get a de Rham complex as indicated in the lemma. We abbreviate $\mathcal{H} = \mathcal{F} \otimes \Omega^i_{X/S}$. Let $X' \subset X$ be an affine open subscheme which maps into an affine open subscheme $S' \subset S$. Then

$$(Ru_{X/S,*}\mathcal{H})|_{X'_{Zar}} = Ru_{X'/S',*}(\mathcal{H}|_{\operatorname{Cris}(X'/S')}),$$

see Lemma 9.5. Thus Lemma 21.2 shows that $Ru_{X/S,*}\mathcal{H}$ is a complex of sheaves on X_{Zar} whose cohomology on any affine open is trivial. As X has a basis for its topology consisting of affine opens this implies that $Ru_{X/S,*}\mathcal{H}$ is quasi-isomorphic to zero. $\square$

Remark 23.2. The proof of Proposition 23.1 shows that the conclusion

$$Ru_{X/S,*}(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0$$

for $i > 0$ is true for any $\mathcal{O}_{X/S}$-module $\mathcal{F}$ which satisfies conditions (1) and (2) of Proposition 21.1. This applies to the following non-crystals: $\Omega^i_{X/S}$ for all i, and any sheaf of the form $\underline{\mathcal{F}}$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. In particular, it applies to the sheaf $\underline{\mathcal{O}_X} = \mathbf{G}_a$. But note that we need something like Lemma 15.1 to produce a de Rham complex which requires $\mathcal{F}$ to be a crystal. Hence (currently) the collection of sheaves of modules for which the full statement of Proposition 23.1 holds is exactly the category of crystals in quasi-coherent modules.

In Situation 7.5. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on $\mathrm{Cris}(X/S)$. Let (U, T, δ) be an object of $\mathrm{Cris}(X/S)$. Proposition 23.1 allows us to construct a canonical map

(23.2.1) $$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \longrightarrow R\Gamma(T, \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^*_{T/S,\delta})$$

Namely, we have $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F} \otimes \Omega^*_{X/S})$, we can restrict global cohomology classes to T, and $\Omega_{X/S}$ restricts to $\Omega_{T/S,\delta}$ by Lemma 12.3.

24. Some further results

In this section we mention some results whose proof is missing. We will formulate these as a series of remarks and we will convert them into actual lemmas and propositions only when we add detailed proofs.

Remark 24.1 (Higher direct images). Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Let

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X' \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S'_0 \end{array}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and X'. Let $\mathcal{F}$ be an $\mathcal{O}_{X/S}$-module on $\mathrm{Cris}(X/S)$. Then $Rf_{\mathrm{cris},*}\mathcal{F}$ can be computed as follows.

Given an object (U', T', δ') of $\mathrm{Cris}(X'/S')$ set $U = X \times_{X'} U' = f^{-1}(U')$ (an open subscheme of X). Denote (T_0, T, δ) the divided power scheme over S such that

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

is cartesian in the category of divided power schemes, see Lemma 7.4. There is an induced morphism $U \to T_0$ and we obtain a morphism $(U/T)_{\mathrm{cris}} \to (X/S)_{\mathrm{cris}}$, see Remark 9.3. Let $\mathcal{F}_U$ be the pullback of $\mathcal{F}$. Let $\tau_{U/T} : (U/T)_{\mathrm{cris}} \to T_{Zar}$ be the structure morphism. Then we have

(24.1.1) $$(Rf_{\mathrm{cris},*}\mathcal{F})_{T'} = R(T \to T')_* \left(R\tau_{U/T,*}\mathcal{F}_U \right)$$

where the left hand side is the restriction (see Section 10).

Hints: First, show that $\mathrm{Cris}(U/T)$ is the localization (in the sense of Sites, Lemma 29.3) of $\mathrm{Cris}(X/S)$ at the sheaf of sets $f_{\mathrm{cris}}^{-1}h_{(U',T',\delta')}$. Next, reduce the statement to the case where $\mathcal{F}$ is an injective module and pushforward of modules using that the pullback of an injective $\mathcal{O}_{X/S}$-module is an injective $\mathcal{O}_{U/T}$-module on $\mathrm{Cris}(U/T)$. Finally, check the result holds for plain pushforward.

Remark 24.2 (Mayer-Vietoris). In the situation of Remark 24.1 suppose we have an open covering $X = X' \cup X''$. Denote $X''' = X' \cap X''$. Let f', f'', and f'' be the restriction of f to X', X'', and X'''. Moreover, Let $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{F}'''$ be the restriction of $\mathcal{F}$ to the crystalline sites of X', X'', and X'''. Then there exists a distinguished triangle

$$Rf_{\mathrm{cris},*}\mathcal{F} \longrightarrow Rf'_{\mathrm{cris},*}\mathcal{F}' \oplus Rf''_{\mathrm{cris},*}\mathcal{F}'' \longrightarrow Rf'''_{\mathrm{cris},*}\mathcal{F}''' \longrightarrow Rf_{\mathrm{cris},*}\mathcal{F}[1]$$

in $D(\mathcal{O}_{X'/S'})$.

Hints: This is a formal consequence of the fact that the subcategories $\mathrm{Cris}(X'/S)$, $\mathrm{Cris}(X''/S)$, $\mathrm{Cris}(X'''/S)$ correspond to open subobjects of the final sheaf on $\mathrm{Cris}(X/S)$ and that the last is the intersection of the first two.

Remark 24.3 (Čech complex). Let p be a prime number. Let (A, I, γ) be a divided power ring with A a $\mathbf{Z}_{(p)}$-algebra. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a separated[6] scheme over S_0 such that p is locally nilpotent on X. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules.

Choose an affine open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ of X. Write $U_\lambda = \mathrm{Spec}(C_\lambda)$. Choose a polynomial algebra P_λ over A and a surjection $P_\lambda \to C_\lambda$. Having fixed these choices we can construct a Čech complex which computes $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$.

Given $n \geq 0$ and $\lambda_0, \ldots, \lambda_n \in \Lambda$ write $U_{\lambda_0 \ldots \lambda_n} = U_{\lambda_0} \cap \ldots \cap U_{\lambda_n}$. This is an affine scheme by assumption. Write $U_{\lambda_0 \ldots \lambda_n} = \mathrm{Spec}(C_{\lambda_0 \ldots \lambda_n})$. Set

$$P_{\lambda_0 \ldots \lambda_n} = P_{\lambda_0} \otimes_A \ldots \otimes_A P_{\lambda_n}$$

which comes with a canonical surjection onto $C_{\lambda_0 \ldots \lambda_n}$. Denote the kernel $J_{\lambda_0 \ldots \lambda_n}$ and set $D_{\lambda_0 \ldots \lambda_n}$ the p-adically completed divided power envelope of $J_{\lambda_0 \ldots \lambda_n}$ in $P_{\lambda_0 \ldots \lambda_n}$ relative to γ. Let $M_{\lambda_0 \ldots \lambda_n}$ be the $P_{\lambda_0 \ldots \lambda_n}$-module corresponding to the restriction of $\mathcal{F}$ to $\mathrm{Cris}(U_{\lambda_0 \ldots \lambda_n}/S)$ via Proposition 17.4. By construction we obtain a cosimplicial divided power ring $D(*)$ having in degree n the ring

$$D(n) = \prod_{\lambda_0 \ldots \lambda_n} D_{\lambda_0 \ldots \lambda_n}$$

(use that divided power envelopes are functorial and the trivial cosimplicial structure on the ring $P(*)$ defined similarly). Since $M_{\lambda_0 \ldots \lambda_n}$ is the "value" of $\mathcal{F}$ on the objects $\mathrm{Spec}(D_{\lambda_0 \ldots \lambda_n})$ we see that $M(*)$ defined by the rule

$$M(n) = \prod_{\lambda_0 \ldots \lambda_n} M_{\lambda_0 \ldots \lambda_n}$$

forms a cosimplicial $D(*)$-module. Now we claim that we have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = s(M(*))$$

Here $s(-)$ denotes the cochain complex associated to a cosimplicial module (see Simplicial, Section 24).

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[6]This assumption is not strictly necessary, as using hypercoverings the construction of the remark can be extended to the general case.

Hints: The proof of this is similar to the proof of Proposition 21.1 (in particular the result holds for any module satisfying the assumptions of that proposition).

Remark 24.4 (Alternating Čech complex). Let p be a prime number. Let (A, I, γ) be a divided power ring with A a $\mathbf{Z}_{(p)}$-algebra. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a separated quasi-compact scheme over S_0 such that p is locally nilpotent on X. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules.

Choose a finite affine open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ of X and a total ordering on Λ. Write $U_\lambda = \mathrm{Spec}(C_\lambda)$. Choose a polynomial algebra P_λ over A and a surjection $P_\lambda \to C_\lambda$. Having fixed these choices we can construct an alternating Čech complex which computes $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$.

We are going to use the notation introduced in Remark 24.3. Denote $\Omega_{\lambda_0 \ldots \lambda_n}$ the p-adically completed module of differentials of $D_{\lambda_0 \ldots \lambda_n}$ over A compatible with the divided power structure. Let ∇ be the integrable connection on $M_{\lambda_0 \ldots \lambda_n}$ coming from Proposition 17.4. Consider the double complex $M^{\bullet, \bullet}$ with terms

$$M^{n,m} = \bigoplus_{\lambda_0 < \ldots < \lambda_n} M_{\lambda_0 \ldots \lambda_n} \otimes^\wedge_{D_{\lambda_0 \ldots \lambda_n}} \Omega^m_{D_{\lambda_0 \ldots \lambda_n}}.$$

For the differential d_1 (increasing n) we use the usual Čech differential and for the differential d_2 we use the connection, i.e., the differential of the de Rham complex. We claim that

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = \mathrm{Tot}(M^{\bullet, \bullet})$$

Here $\mathrm{Tot}(-)$ denotes the total complex associated to a double complex, see Homology, Definition 22.3.

Hints: We have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^\bullet_{X/S})$$

by Proposition 23.1. The right hand side of the formula is simply the alternating Čech complex for the covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ (which induces an open covering of the final sheaf of $\mathrm{Cris}(X/S)$) and the complex $\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^\bullet_{X/S}$, see Proposition 21.3. Now the result follows from a general result in cohomology on sites, namely that the alternating Čech complex computes the cohomology provided it gives the correct answer on all the pieces (insert future reference here).

Remark 24.5 (Quasi-coherence). In the situation of Remark 24.1 assume that $S \to S'$ is quasi-compact and quasi-separated and that $X \to S_0$ is quasi-compact and quasi-separated. Then for a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ the sheaves $R^i f_{\mathrm{cris}, *} \mathcal{F}$ are locally quasi-coherent.

Hints: We have to show that the restrictions to T' are quasi-coherent $\mathcal{O}_{T'}$-modules, where (U', T', δ') is any object of $\mathrm{Cris}(X'/S')$. It suffices to do this when T' is affine. We use the formula (24.1.1), the fact that $T \to T'$ is quasi-compact and quasi-separated (as T is affine over the base change of T' by $S \to S'$), and Cohomology of Schemes, Lemma 4.4 to see that it suffices to show that the sheaves $R^i \tau_{U/T, *} \mathcal{F}_U$ are quasi-coherent. Note that $U \to T_0$ is also quasi-compact and quasi-separated, see Schemes, Lemmas 21.15 and 21.15.

This reduces us to proving that $R^i \tau_{X/S, *} \mathcal{F}$ is quasi-coherent on S in the case that p locally nilpotent on S. Here $\tau_{X/S}$ is the structure morphism, see Remark 9.6. We may work locally on S, hence we may assume S affine (see Lemma 9.5). Induction

on the number of affines covering X and Mayer-Vietoris (Remark 24.2) reduces the question to the case where X is also affine (as in the proof of Cohomology of Schemes, Lemma 4.4). Say $X = \operatorname{Spec}(C)$ and $S = \operatorname{Spec}(A)$ so that (A, I, γ) and $A \to C$ are as in Situation 5.1. Choose a polynomial algebra P over A and a surjection $P \to C$ as in Section 17. Let (M, ∇) be the module corresponding to $\mathcal{F}$, see Proposition 17.4. Applying Proposition 21.3 we see that $R\Gamma(\operatorname{Cris}(X/S), \mathcal{F})$ is represented by $M \otimes_D \Omega_D^*$. Note that completion isn't necessary as p is nilpotent in A! We have to show that this is compatible with taking principal opens in $S = \operatorname{Spec}(A)$. Suppose that $g \in A$. Then we conclude that similarly $R\Gamma(\operatorname{Cris}(X_g/S_g), \mathcal{F})$ is computed by $M_g \otimes_{D_g} \Omega_{D_g}^*$ (again this uses that p-adic completion isn't necessary). Hence we conclude because localization is an exact functor on A-modules.

Remark 24.6 (Boundedness)**.** In the situation of Remark 24.1 assume that $S \to S'$ is quasi-compact and quasi-separated and that $X \to S_0$ is of finite type and quasi-separated. Then there exists an integer i_0 such that for any crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ we have $R^i f_{\operatorname{cris},*} \mathcal{F} = 0$ for all $i > i_0$.

Hints: Arguing as in Remark 24.5 (using Cohomology of Schemes, Lemma 4.4) we reduce to proving that $H^i(\operatorname{Cris}(X/S), \mathcal{F}) = 0$ for $i \gg 0$ in the situation of Proposition 21.3 when C is a finite type algebra over A. This is clear as we can choose a finite polynomial algebra and we see that $\Omega_D^i = 0$ for $i \gg 0$.

Remark 24.7 (Specific boundedness)**.** In Situation 7.5 let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Assume that S_0 has a unique point and that $X \to S_0$ is of finite presentation.

(1) If $\dim X = d$ and X/S_0 has embedding dimension e, then $H^i(\operatorname{Cris}(X/S), \mathcal{F}) = 0$ for $i > d + e$.
(2) If X is separated and can be covered by q affines, and X/S_0 has embedding dimension e, then $H^i(\operatorname{Cris}(X/S), \mathcal{F}) = 0$ for $i > q + e$.

Hints: In case (1) we can use that

$$H^i(\operatorname{Cris}(X/S), \mathcal{F}) = H^i(X_{Zar}, Ru_{X/S,*} \mathcal{F})$$

and that $Ru_{X/S,*} \mathcal{F}$ is locally calculated by a de Rham complex constructed using an embedding of X into a smooth scheme of dimension e over S (see Lemma 21.4). These de Rham complexes are zero in all degrees $> e$. Hence (1) follows from Cohomology, Proposition 21.6. In case (2) we use the alternating Čech complex (see Remark 24.4) to reduce to the case X affine. In the affine case we prove the result using the de Rham complex associated to an embedding of X into a smooth scheme of dimension e over S (it takes some work to construct such a thing).

Remark 24.8 (Base change map)**.** In the situation of Remark 24.1 assume $S = \operatorname{Spec}(A)$ and $S' = \operatorname{Spec}(A')$ are affine. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'/S'}$-module. Let $\mathcal{F}$ be the pullback of $\mathcal{F}'$. Then there is a canonical base change map

$$L(S' \to S)^* R\tau_{X'/S',*} \mathcal{F}' \longrightarrow R\tau_{X/S,*} \mathcal{F}$$

where $\tau_{X/S}$ and $\tau_{X'/S'}$ are the structure morphisms, see Remark 9.6. On global sections this gives a base change map

(24.8.1) $$R\Gamma(\operatorname{Cris}(X'/S'), \mathcal{F}') \otimes_{A'}^{\mathbf{L}} A \longrightarrow R\Gamma(\operatorname{Cris}(X/S), \mathcal{F})$$

in $D(A)$.

Hint: Compose the very general base change map of Cohomology on Sites, Remark 19.2 with the canonical map $Lf_{\text{cris}}^* \mathcal{F}' \to f_{\text{cris}}^* \mathcal{F}' = \mathcal{F}$.

Remark 24.9 (Base change isomorphism)**.** The map (24.8.1) is an isomorphism provided all of the following conditions are satisfied:

1. p is nilpotent in A',
2. $\mathcal{F}'$ is a crystal in quasi-coherent $\mathcal{O}_{X'/S'}$-modules,
3. $X' \to S_0'$ is a quasi-compact, quasi-separated morphism,
4. $X = X' \times_{S_0'} S_0$,
5. $\mathcal{F}'$ is a flat $\mathcal{O}_{X'/S'}$-module,
6. $X' \to S_0'$ is a local complete intersection morphism (see More on Morphisms, Definition 42.2; this holds for example if $X' \to S_0'$ is syntomic or smooth),
7. X' and S_0 are Tor independent over S_0' (see More on Algebra, Definition 47.1; this holds for example if either $S_0 \to S_0'$ or $X' \to S_0'$ is flat).

Hints: Condition (1) means that in the arguments below p-adic completion does nothing and can be ignored. Using condition (3) and Mayer Vietoris (see Remark 24.2) this reduces to the case where X' is affine. In fact by condition (6), after shrinking further, we can assume that $X' = \text{Spec}(C')$ and we are given a presentation $C' = A'/I'[x_1, \ldots, x_n]/(\bar{f}_1', \ldots, \bar{f}_c')$ where $\bar{f}_1', \ldots, \bar{f}_c'$ is a Koszul-regular sequence in A'/I'. (This means that smooth locally $\bar{f}_1', \ldots, \bar{f}_c'$ forms a regular sequence, see More on Algebra, Lemma 21.17.) We choose a lift of $\bar{f}_i'$ to an element $f_i' \in A'[x_1, \ldots, x_n]$. By (4) we see that $X = \text{Spec}(C)$ with $C = A/I[x_1, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_c)$ where $f_i \in A[x_1, \ldots, x_n]$ is the image of f_i'. By property (7) we see that $\bar{f}_1, \ldots, \bar{f}_c$ is a Koszul-regular sequence in $A/I[x_1, \ldots, x_n]$. The divided power envelope of $I'A'[x_1, \ldots, x_n] + (f_1', \ldots, f_c')$ in $A'[x_1, \ldots, x_n]$ relative to γ' is

$$D' = A'[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c \rangle / (\xi_i - f_i')$$

see Lemma 2.4. Then you check that $\xi_1 - f_1', \ldots, \xi_n - f_n'$ is a Koszul-regular sequence in the ring $A'[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c \rangle$. Similarly the divided power envelope of $IA[x_1, \ldots, x_n] + (f_1, \ldots, f_c)$ in $A[x_1, \ldots, x_n]$ relative to γ is

$$D = A[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c \rangle / (\xi_i - f_i)$$

and $\xi_1 - f_1, \ldots, \xi_n - f_n$ is a Koszul-regular sequence in the ring $A[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c \rangle$. It follows that $D' \otimes_{A'}^{\mathbf{L}} A = D$. Condition (2) implies $\mathcal{F}'$ corresponds to a pair (M', ∇) consisting of a D'-module with connection, see Proposition 17.4. Then $M = M' \otimes_{D'} D$ corresponds to the pullback $\mathcal{F}$. By assumption (5) we see that M' is a flat D'-module, hence

$$M = M' \otimes_{D'} D = M' \otimes_{D'} D' \otimes_{A'}^{\mathbf{L}} A = M' \otimes_{A'}^{\mathbf{L}} A$$

Since the modules of differentials $\Omega_{D'}$ and Ω_D (as defined in Section 17) are free D'-modules on the same generators we see that

$$M \otimes_D \Omega_D^\bullet = M' \otimes_{D'} \Omega_{D'}^\bullet \otimes_{D'} D = M' \otimes_{D'} \Omega_{D'}^\bullet \otimes_{A'}^{\mathbf{L}} A$$

which proves what we want by Proposition 21.3.

Remark 24.10 (Rlim)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A an algebra over $\mathbf{Z}_{(p)}$ with p nilpotent in A/I. Set $S = \text{Spec}(A)$ and $S_0 = \text{Spec}(A/I)$. Let X be a scheme over S_0 with p locally nilpotent on X. Let $\mathcal{F}$

be any $\mathcal{O}_{X/S}$-module. For $e \gg 0$ we have $(p^e) \subset I$ is preserved by γ, see Divided Power Algebra, Lemma 4.5. Set $S_e = \mathrm{Spec}(A/p^eA)$ for $e \gg 0$. Then $\mathrm{Cris}(X/S_e)$ is a full subcategory of $\mathrm{Cris}(X/S)$ and we denote $\mathcal{F}_e$ the restriction of $\mathcal{F}$ to $\mathrm{Cris}(X/S_e)$. Then

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\lim_e R\Gamma(\mathrm{Cris}(X/S_e), \mathcal{F}_e)$$

Hints: Suffices to prove this for $\mathcal{F}$ injective. In this case the sheaves $\mathcal{F}_e$ are injective modules too, the transition maps $\Gamma(\mathcal{F}_{e+1}) \to \Gamma(\mathcal{F}_e)$ are surjective, and we have $\Gamma(\mathcal{F}) = \lim_e \Gamma(\mathcal{F}_e)$ because any object of $\mathrm{Cris}(X/S)$ is locally an object of one of the categories $\mathrm{Cris}(X/S_e)$ by definition of $\mathrm{Cris}(X/S)$.

Remark 24.11 (Comparison)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with p nilpotent in A. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let Y be a smooth scheme over S and set $X = Y \times_S S_0$. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Then

(1) γ extends to a divided power structure on the ideal of X in Y so that (X, Y, γ) is an object of $\mathrm{Cris}(X/S)$,
(2) the restriction $\mathcal{F}_Y$ (see Section 10) comes endowed with a canonical integrable connection $\nabla : \mathcal{F}_Y \to \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$, and
(3) we have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S})$$

in $D(A)$.

Hints: See Divided Power Algebra, Lemma 4.2 for (1). See Lemma 15.1 for (2). For Part (3) note that there is a map, see (23.2.1). This map is an isomorphism when X is affine, see Lemma 21.4. This shows that $Ru_{X/S,*}\mathcal{F}$ and $\mathcal{F}_Y \otimes \Omega^\bullet_{Y/S}$ are quasi-isomorphic as complexes on $Y_{Zar} = X_{Zar}$. Since $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(X_{Zar}, Ru_{X/S,*}\mathcal{F})$ the result follows.

Remark 24.12 (Perfectness)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with p nilpotent in A. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a proper smooth scheme over S_0. Let $\mathcal{F}$ be a crystal in finite locally free quasi-coherent $\mathcal{O}_{X/S}$-modules. Then $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is a perfect object of $D(A)$.

Hints: By Remark 24.9 we have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \otimes^{\mathbf{L}}_A A/I \cong R\Gamma(\mathrm{Cris}(X/S_0), \mathcal{F}|_{\mathrm{Cris}(X/S_0)})$$

By Remark 24.11 we have

$$R\Gamma(\mathrm{Cris}(X/S_0), \mathcal{F}|_{\mathrm{Cris}(X/S_0)}) = R\Gamma(X, \mathcal{F}_X \otimes \Omega^\bullet_{X/S_0})$$

Using the stupid filtration on the de Rham complex we see that the last displayed complex is perfect in $D(A/I)$ as soon as the complexes

$$R\Gamma(X, \mathcal{F}_X \otimes \Omega^q_{X/S_0})$$

are perfect complexes in $D(A/I)$, see More on Algebra, Lemma 56.4. This is true by standard arguments in coherent cohomology using that $\mathcal{F}_X \otimes \Omega^q_{X/S_0}$ is a finite locally free sheaf and $X \to S_0$ is proper and flat (insert future reference here). Applying More on Algebra, Lemma 57.4 we see that

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \otimes^{\mathbf{L}}_A A/I^n$$

is a perfect object of $D(A/I^n)$ for all n. This isn't quite enough unless A is Noetherian. Namely, even though I is locally nilpotent by our assumption that p is nilpotent, see Divided Power Algebra, Lemma 2.6, we cannot conclude that $I^n = 0$ for some n. A counter example is $\mathbf{F}_p\langle x\rangle$. To prove it in general when $\mathcal{F} = \mathcal{O}_{X/S}$ the argument of `http://math.columbia.edu/~dejong/wordpress/?p=2227` works. When the coefficients $\mathcal{F}$ are non-trivial the argument of [Fal99] seems to be as follows. Reduce to the case $pA = 0$ by More on Algebra, Lemma 57.4. In this case the Frobenius map $A \to A$, $a \mapsto a^p$ factors as $A \to A/I \xrightarrow{\varphi} A$ (as $x^p = 0$ for $x \in I$). Set $X^{(1)} = X \otimes_{A/I,\varphi} A$. The absolute Frobenius morphism of X factors through a morphism $F_X : X \to X^{(1)}$ (a kind of relative Frobenius). Affine locally if $X = \mathrm{Spec}(C)$ then $X^{(1)} = \mathrm{Spec}(C \otimes_{A/I,\varphi} A)$ and F_X corresponds to $C \otimes_{A/I,\varphi} A \to C$, $c \otimes a \mapsto c^p a$. This defines morphisms of ringed topoi

$$(X/S)_{\mathrm{cris}} \xrightarrow{(F_X)_{\mathrm{cris}}} (X^{(1)}/S)_{\mathrm{cris}} \xrightarrow{u_{X^{(1)}/S}} Sh(X_{Zar}^{(1)})$$

whose composition is denoted Frob_X. One then shows that $R\mathrm{Frob}_{X,*}\mathcal{F}$ is representable by a perfect complex of $\mathcal{O}_{X^{(1)}}$-modules(!) by a local calculation.

Remark 24.13 (Complete perfectness). Let p be a prime number. Let (A, I, γ) be a divided power ring with A a p-adically complete ring and p nilpotent in A/I. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a proper smooth scheme over S_0. Let $\mathcal{F}$ be a crystal in finite locally free quasi-coherent $\mathcal{O}_{X/S}$-modules. Then $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is a perfect object of $D(A)$.

Hints: We know that $K = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is the derived limit $K = R\lim K_e$ of the cohomologies over $A/p^e A$, see Remark 24.10. Each K_e is a perfect complex of $D(A/p^e A)$ by Remark 24.12. Since A is p-adically complete the result follows from More on Algebra, Lemma 65.3.

Remark 24.14 (Complete comparison). Let p be a prime number. Let (A, I, γ) be a divided power ring with A a Noetherian p-adically complete ring and p nilpotent in A/I. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let Y be a proper smooth scheme over S and set $X = Y \times_S S_0$. Let $\mathcal{F}$ be a finite type crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Then

(1) there exists a coherent $\mathcal{O}_Y$-module $\mathcal{F}_Y$ endowed with integrable connection

$$\nabla : \mathcal{F}_Y \longrightarrow \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$$

such that $\mathcal{F}_Y/p^e \mathcal{F}_Y$ is the module with connection over $A/p^e A$ found in Remark 24.11, and

(2) we have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^{\bullet})$$

in $D(A)$.

Hints: The existence of $\mathcal{F}_Y$ is Grothendieck's existence theorem (insert future reference here). The isomorphism of cohomologies follows as both sides are computed as $R\lim$ of the versions modulo p^e (see Remark 24.10 for the left hand side; use the theorem on formal functions, see Cohomology of Schemes, Theorem 18.5 for the right hand side). Each of the versions modulo p^e are isomorphic by Remark 24.11.

25. Pulling back along purely inseparable maps

By an α_p-cover we mean a morphism of the form

$$X' = \mathrm{Spec}(C[z]/(z^p - c)) \longrightarrow \mathrm{Spec}(C) = X$$

where C is an $\mathbf{F}_p$-algebra and $c \in C$. Equivalently, X' is an α_p-torsor over X. An *iterated α_p-cover*[7] is a morphism of schemes in characteristic p which is locally on the target a composition of finitely many α_p-covers. In this section we prove that pullback along such a morphism induces a quasi-isomorphism on crystalline cohomology after inverting the prime p. In fact, we prove a precise version of this result. We beging with a preliminary lemma whose formulation needs some notation.

Assume we have a ring map $B \to B'$ and quotients $\Omega_B \to \Omega$ and $\Omega_{B'} \to \Omega'$ satisfying the assumptions of Remark 6.11. Thus (6.11.1) provides a canonical map of complexes

$$c_M^\bullet : M \otimes_B \Omega^\bullet \longrightarrow M \otimes_B (\Omega')^\bullet$$

for all B-modules M endowed with integrable connection $\nabla : M \to M \otimes_B \Omega_B$.

Suppose we have $a \in B$, $z \in B'$, and a map $\theta : B' \to B'$ satisfying the following assumptions

(1) $\mathrm{d}(a) = 0$,
(2) $\Omega' = B' \otimes_B \Omega \oplus B'\mathrm{d}z$; we write $\mathrm{d}(f) = \mathrm{d}_1(f) + \partial_z(f)\mathrm{d}z$ with $\mathrm{d}_1(f) \in B' \otimes \Omega$ and $\partial_z(f) \in N'$ for all $f \in B'$,
(3) $\theta : B' \to B'$ is B-linear,
(4) $\partial_z \circ \theta = a$,
(5) $B \to B'$ is universally injective (and hence $\Omega \to \Omega'$ is injective),
(6) $af - \theta(\partial_z(f)) \in B$ for all $f \in B'$,
(7) $(\theta \otimes 1)(\mathrm{d}_1(f)) - \mathrm{d}_1(\theta(f)) \in \Omega$ for all $f \in B'$ where $\theta \otimes 1 : B' \otimes \Omega \to B' \otimes \Omega$

These conditions are not logically independent. For example, assumption (4) implies that $\partial_z(af - \theta(\partial_z(f))) = 0$. Hence if the image of $B \to B'$ is the collection of elements annihilated by ∂_z, then (6) follows. A similar argument can be made for condition (7).

Lemma 25.1. *In the situation above there exists a map of complexes*

$$e_M^\bullet : M \otimes_B (\Omega')^\bullet \longrightarrow M \otimes_B \Omega^\bullet$$

such that $c_M^\bullet \circ e_M^\bullet$ and $e_M^\bullet \circ c_M^\bullet$ are homotopic to multiplication by a.

Proof. In this proof all tensor products are over B. Assumption (2) implies that

$$M \otimes (\Omega')^i = (B' \otimes M \otimes \Omega^i) \oplus (B'\mathrm{d}z \otimes M \otimes \Omega^{i-1})$$

for all $i \geq 0$. A collection of additive generators for $M \otimes (\Omega')^i$ is formed by elements of the form $f\omega$ and elements of the form $f\mathrm{d}z \wedge \eta$ where $f \in B'$, $\omega \in M \otimes \Omega^i$, and $\eta \in M \otimes \Omega^{i-1}$.

For $f \in B'$ we write

$$\epsilon(f) = af - \theta(\partial_z(f)) \quad \text{and} \quad \epsilon'(f) = (\theta \otimes 1)(\mathrm{d}_1(f)) - \mathrm{d}_1(\theta(f))$$

---

[7]This is nonstandard notation.

so that $\epsilon(f) \in B$ and $\epsilon'(f) \in \Omega$ by assumptions (6) and (7). We define $e_M^\bullet$ by the rules $e_M^i(f\omega) = \epsilon(f)\omega$ and $e_M^i(f\mathrm{d}z \wedge \eta) = \epsilon'(f) \wedge \eta$. We will see below that the collection of maps e_M^i is a map of complexes.

We define

$$h^i : M \otimes_B (\Omega')^i \longrightarrow M \otimes_B (\Omega')^{i-1}$$

by the rules $h^i(f\omega) = 0$ and $h^i(f\mathrm{d}z \wedge \eta) = \theta(f)\eta$ for elements as above. We claim that

$$\mathrm{d} \circ h + h \circ \mathrm{d} = a - c_M^\bullet \circ e_M^\bullet$$

Note that multiplication by a is a map of complexes by (1). Hence, since $c_M^\bullet$ is an injective map of complexes by assumption (5), we conclude that $e_M^\bullet$ is a map of complexes. To prove the claim we compute

$$
\begin{aligned}
(\mathrm{d} \circ h + h \circ \mathrm{d})(f\omega) &= h\,(\mathrm{d}(f) \wedge \omega + f\nabla(\omega)) \\
&= \theta(\partial_z(f))\omega \\
&= af\omega - \epsilon(f)\omega \\
&= af\omega - c_M^i(e_M^i(f\omega))
\end{aligned}
$$

The second equality because $\mathrm{d}z$ does not occur in $\nabla(\omega)$ and the third equality by assumption (6). Similarly, we have

$$
\begin{aligned}
(\mathrm{d} \circ h + h \circ \mathrm{d})(f\mathrm{d}z \wedge \eta) &= \mathrm{d}(\theta(f)\eta) + h\,(\mathrm{d}(f) \wedge \mathrm{d}z \wedge \eta - f\mathrm{d}z \wedge \nabla(\eta)) \\
&= \mathrm{d}(\theta(f)) \wedge \eta + \theta(f)\nabla(\eta) - (\theta \otimes 1)(\mathrm{d}_1(f)) \wedge \eta - \theta(f)\nabla(\eta) \\
&= \mathrm{d}_1(\theta(f)) \wedge \eta + \partial_z(\theta(f))\mathrm{d}z \wedge \eta - (\theta \otimes 1)(\mathrm{d}_1(f)) \wedge \eta \\
&= af\mathrm{d}z \wedge \eta - \epsilon'(f) \wedge \eta \\
&= af\mathrm{d}z \wedge \eta - c_M^i(e_M^i(f\mathrm{d}z \wedge \eta))
\end{aligned}
$$

The second equality because $\mathrm{d}(f) \wedge \mathrm{d}z \wedge \eta = -\mathrm{d}z \wedge \mathrm{d}_1(f) \wedge \eta$. The fourth equality by assumption (4). On the other hand it is immediate from the definitions that $e_M^i(c_M^i(\omega)) = \epsilon(1)\omega = a\omega$. This proves the lemma. $\qquad \square$

Example 25.2. A standard example of the situation above occurs when $B' = B\langle z \rangle$ is the divided power polynomial ring over a divided power ring (B, J, δ) with divided powers δ' on $J' = B'_+ + JB' \subset B'$. Namely, we take $\Omega = \Omega_{B,\delta}$ and $\Omega' = \Omega_{B',\delta'}$. In this case we can take $a = 1$ and

$$\theta\left(\sum b_m z^{[m]}\right) = \sum b_m z^{[m+1]}$$

Note that

$$f - \theta(\partial_z(f)) = f(0)$$

equals the constant term. It follows that in this case Lemma 25.1 recovers the crystalline Poincaré lemma (Lemma 20.2).

Lemma 25.3. *In Situation 5.1. Assume D and Ω_D are as in (17.0.1) and (17.0.2). Let $\lambda \in D$. Let D' be the p-adic completion of*

$$D[z]\langle \xi \rangle / (\xi - (z^p - \lambda))$$

and let $\Omega_{D'}$ be the p-adic completion of the module of divided power differentials of D' over A. For any pair (M, ∇) over D satisfying (1), (2), (3), and (4) the canonical map of complexes (6.11.1)

$$c_M^\bullet : M \otimes_D^\wedge \Omega_D^\bullet \longrightarrow M \otimes_D^\wedge \Omega_{D'}^\bullet$$

has the following property: There exists a map $e_M^\bullet$ in the opposite direction such that both $c_M^\bullet \circ e_M^\bullet$ and $e_M^\bullet \circ c_M^\bullet$ are homotopic to multiplication by p.

Proof. We will prove this using Lemma 25.1 with $a = p$. Thus we have to find $\theta : D' \to D'$ and prove (1), (2), (3), (4), (5), (6), (7). We first collect some information about the rings D and D' and the modules Ω_D and $\Omega_{D'}$.

Writing

$$D[z]\langle\xi\rangle/(\xi - (z^p - \lambda)) = D\langle\xi\rangle[z]/(z^p - \xi - \lambda)$$

we see that D' is the p-adic completion of the free D-module

$$\bigoplus_{i=0,\ldots,p-1} \bigoplus_{n\geq 0} z^i \xi^{[n]} D$$

where $\xi^{[0]} = 1$. It follows that $D \to D'$ has a continuous D-linear section, in particular $D \to D'$ is universally injective, i.e., (5) holds. We think of D' as a divided power algebra over A with divided power ideal $\overline{J}' = \overline{J}D' + (\xi)$. Then D' is also the p-adic completion of the divided power envelope of the ideal generated by $z^p - \lambda$ in D, see Lemma 2.4. Hence

$$\Omega_{D'} = \Omega_D \otimes_D^\wedge D' \oplus D'\mathrm{d}z$$

by Lemma 6.6. This proves (2). Note that (1) is obvious.

At this point we construct θ. (We wrote a PARI/gp script theta.gp verifying some of the formulas in this proof which can be found in the scripts subdirectory of the stacks project.) Before we do so we compute the derivative of the elements $z^i \xi^{[n]}$. We have $\mathrm{d}z^i = iz^{i-1}\mathrm{d}z$. For $n \geq 1$ we have

$$\mathrm{d}\xi^{[n]} = \xi^{[n-1]}\mathrm{d}\xi = -\xi^{[n-1]}\mathrm{d}\lambda + pz^{p-1}\xi^{[n-1]}\mathrm{d}z$$

because $\xi = z^p - \lambda$. For $0 < i < p$ and $n \geq 1$ we have

$$\begin{aligned}
\mathrm{d}(z^i \xi^{[n]}) &= iz^{i-1}\xi^{[n]}\mathrm{d}z + z^i\xi^{[n-1]}\mathrm{d}\xi \\
&= iz^{i-1}\xi^{[n]}\mathrm{d}z + z^i\xi^{[n-1]}\mathrm{d}(z^p - \lambda) \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + (iz^{i-1}\xi^{[n]} + pz^{i+p-1}\xi^{[n-1]})\mathrm{d}z \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + (iz^{i-1}\xi^{[n]} + pz^{i-1}(\xi + \lambda)\xi^{[n-1]})\mathrm{d}z \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + ((i + pn)z^{i-1}\xi^{[n]} + p\lambda z^{i-1}\xi^{[n-1]})\mathrm{d}z
\end{aligned}$$

the last equality because $\xi\xi^{[n-1]} = n\xi^{[n]}$. Thus we see that

$$\partial_z(z^i) = iz^{i-1}$$
$$\partial_z(\xi^{[n]}) = pz^{p-1}\xi^{[n-1]}$$
$$\partial_z(z^i\xi^{[n]}) = (i + pn)z^{i-1}\xi^{[n]} + p\lambda z^{i-1}\xi^{[n-1]}$$

Motivated by these formulas we define θ by the rules

$$\begin{array}{rcll}
\theta(z^j) &=& p\frac{z^{j+1}}{j+1} & j = 0,\ldots p-1, \\
\theta(z^{p-1}\xi^{[m]}) &=& \xi^{[m+1]} & m \geq 1, \\
\theta(z^j\xi^{[m]}) &=& \frac{pz^{j+1}\xi^{[m]} - \theta(p\lambda z^j\xi^{[m-1]})}{(j+1+pm)} & 0 \leq j < p-1, m \geq 1
\end{array}$$

where in the last line we use induction on m to define our choice of θ. Working this out we get (for $0 \leq j < p-1$ and $1 \leq m$)

$$\theta(z^j\xi^{[m]}) = \frac{pz^{j+1}\xi^{[m]}}{(j+1+pm)} - \frac{p^2\lambda z^{j+1}\xi^{[m-1]}}{(j+1+pm)(j+1+p(m-1))} + \ldots + \frac{(-1)^m p^{m+1}\lambda^m z^{j+1}}{(j+1+pm)\ldots(j+1)}$$

although we will not use this expression below. It is clear that θ extends uniquely to a p-adically continuous D-linear map on D'. By construction we have (3) and (4). It remains to prove (6) and (7).

Proof of (6) and (7). As θ is D-linear and continuous it suffices to prove that $p - \theta \circ \partial_z$, resp. $(\theta \otimes 1) \circ d_1 - d_1 \circ \theta$ gives an element of D, resp. Ω_D when evaluated on the elements $z^i \xi^{[n]}$[8]. Set $D_0 = \mathbf{Z}_{(p)}[\lambda]$ and $D_0' = \mathbf{Z}_{(p)}[z, \lambda]\langle \xi \rangle / (\xi - z^p + \lambda)$. Observe that each of the expressions above is an element of D_0' or $\Omega_{D_0'}$. Hence it suffices to prove the result in the case of $D_0 \to D_0'$. Note that D_0 and D_0' are torsion free rings and that $D_0 \otimes \mathbf{Q} = \mathbf{Q}[\lambda]$ and $D_0' \otimes \mathbf{Q} = \mathbf{Q}[z, \lambda]$. Hence $D_0 \subset D_0'$ is the subring of elements annihilated by ∂_z and (6) follows from (4), see the discussion directly preceding Lemma 25.1. Similarly, we have $d_1(f) = \partial_\lambda(f) d\lambda$ hence

$$((\theta \otimes 1) \circ d_1 - d_1 \circ \theta)(f) = (\theta(\partial_\lambda(f)) - \partial_\lambda(\theta(f))) \, d\lambda$$

Applying ∂_z to the coefficient we obtain

$$\begin{aligned}
\partial_z \left(\theta(\partial_\lambda(f)) - \partial_\lambda(\theta(f)) \right) &= p\partial_\lambda(f) - \partial_z(\partial_\lambda(\theta(f))) \\
&= p\partial_\lambda(f) - \partial_\lambda(\partial_z(\theta(f))) \\
&= p\partial_\lambda(f) - \partial_\lambda(pf) = 0
\end{aligned}$$

whence the coefficient does not depend on z as desired. This finishes the proof of the lemma. $\qquad\square$

Note that an iterated α_p-cover $X' \to X$ (as defined in the introduction to this section) is finite locally free. Hence if X is connected the degree of $X' \to X$ is constant and is a power of p.

Lemma 25.4. *Let p be a prime number. Let $(S, \mathcal{I}, \gamma)$ be a divided power scheme over $\mathbf{Z}_{(p)}$ with $p \in \mathcal{I}$. We set $S_0 = V(\mathcal{I}) \subset S$. Let $f : X' \to X$ be an iterated α_p-cover of schemes over S_0 with constant degree q. Let $\mathcal{F}$ be any crystal in quasi-coherent sheaves on X and set $\mathcal{F}' = f_{cris}^* \mathcal{F}$. In the distinguished triangle*

$$Ru_{X/S,*}\mathcal{F} \longrightarrow f_* Ru_{X'/S,*}\mathcal{F}' \longrightarrow E \longrightarrow Ru_{X/S,*}\mathcal{F}[1]$$

the object E has cohomology sheaves annihilated by q.

Proof. Note that $X' \to X$ is a homeomorphism hence we can identify the under-lying topological spaces of X and X'. The question is clearly local on X, hence we may assume X, X', and S affine and $X' \to X$ given as a composition

$$X' = X_n \to X_{n-1} \to X_{n-2} \to \ldots \to X_0 = X$$

where each morphism $X_{i+1} \to X_i$ is an α_p-cover. Denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. It suffices to prove that each of the maps

$$R\Gamma(\mathrm{Cris}(X_i/S), \mathcal{F}_i) \longrightarrow R\Gamma(\mathrm{Cris}(X_{i+1}/S), \mathcal{F}_{i+1})$$

fits into a triangle whose third member has cohomology groups annihilated by p. (This uses axiom TR4 for the triangulated category $D(X)$. Details omitted.)

Hence we may assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(C)$, $X' = \mathrm{Spec}(C')$ and $C' = C[z]/(z^p - c)$ for some $c \in C$. Choose a polynomial algebra P over A and a surjection $P \to C$. Let D be the p-adically completed divided power envelope of

---

[8]This can be done by direct computation: It turns out that $p - \theta \circ \partial_z$ evaluated on $z^i \xi^{[n]}$ gives zero except for 1 which is mapped to p and ξ which is mapped to $-p\lambda$. It turns out that $(\theta \otimes 1) \circ d_1 - d_1 \circ \theta$ evaluated on $z^i \xi^{[n]}$ gives zero except for $z^{p-1}\xi$ which is mapped to $-\lambda$.

Ker$(P \to C)$ in P as in (17.0.1). Set $P' = P[z]$ with surjection $P' \to C'$ mapping z to the class of z in C'. Choose a lift $\lambda \in D$ of $c \in C$. Then we see that the p-adically completed divided power envelope D' of Ker$(P' \to C')$ in P' is isomorphic to the p-adic completion of $D[z]\langle\xi\rangle/(\xi - (z^p - \lambda))$, see Lemma 25.3 and its proof. Thus we see that the result follows from this lemma by the computation of cohomology of crystals in quasi-coherent modules in Proposition 21.3. $\qquad\square$

The bound in the following lemma is probably not optimal.

Lemma 25.5. *With notations and assumptions as in Lemma 25.4 the map*

$$f^* : H^i(Cris(X/S), \mathcal{F}) \longrightarrow H^i(Cris(X'/S), \mathcal{F}')$$

has kernel and cokernel annihilated by q^{i+1}.

Proof. This follows from the fact that E has nonzero cohomology sheaves in degrees -1 and up, so that the spectral sequence $H^a(\mathcal{H}^b(E)) \Rightarrow H^{a+b}(E)$ converges. This combined with the long exact cohomology sequence associated to a distinguished triangle gives the bound. $\qquad\square$

In Situation 7.5 assume that $p \in \mathcal{I}$. Set

$$X^{(1)} = X \times_{S_0, F_{S_0}} S_0.$$

Denote $F_{X/S_0} : X \to X^{(1)}$ the relative Frobenius morphism.

Lemma 25.6. *In the situation above, assume that $X \to S_0$ is smooth of relative dimension d. Then F_{X/S_0} is an iterated α_p-cover of degree p^d. Hence Lemmas 25.4 and 25.5 apply to this situation. In particular, for any crystal in quasi-coherent modules $\mathcal{G}$ on $Cris(X^{(1)}/S)$ the map*

$$F_{X/S_0}^* : H^i(Cris(X^{(1)}/S), \mathcal{G}) \longrightarrow H^i(Cris(X/S), F_{X/S_0, cris}^*\mathcal{G})$$

has kernel and cokernel annihilated by $p^{d(i+1)}$.

Proof. It suffices to prove the first statement. To see this we may assume that X is étale over $\mathbf{A}_{S_0}^d$, see Morphisms, Lemma 37.20. Denote $\varphi : X \to \mathbf{A}_{S_0}^d$ this étale morphism. In this case the relative Frobenius of X/S_0 fits into a diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X^{(1)} \\
\downarrow & & \downarrow \\
\mathbf{A}_{S_0}^d & \longrightarrow & \mathbf{A}_{S_0}^d
\end{array}
$$

where the lower horizontal arrow is the relative frobenius morphism of $\mathbf{A}_{S_0}^d$ over S_0. This is the morphism which raises all the coordinates to the pth power, hence it is an iterated α_p-cover. The proof is finished by observing that the diagram is a fibre square, see the proof of Étale Cohomology, Theorem 80.4. $\qquad\square$

26. Frobenius action on crystalline cohomology

In this section we prove that Frobenius pullback induces a quasi-isomorphism on crystalline cohomology after inverting the prime p. But in order to even formulate this we need to work in a special situation.

Situation 26.1. In Situation 7.5 assume the following

(1) $S = \mathrm{Spec}(A)$ for some divided power ring (A, I, γ) with $p \in I$,
(2) there is given a homomorphism of divided power rings $\sigma : A \to A$ such that
$\sigma(x) = x^p \bmod pA$ for all $x \in A$.

In Situation 26.1 the morphism $\mathrm{Spec}(\sigma) : S \to S$ is a lift of the absolute Frobenius $F_{S_0} : S_0 \to S_0$ and since the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;F_X\;} & X \\
\downarrow & & \downarrow \\
S_0 & \xrightarrow{\;F_{S_0}\;} & S_0
\end{array}
$$

is commutative where $F_X : X \to X$ is the absolute Frobenius morphism of X. Thus we obtain a morphism of crystalline topoi

$$(F_X)_{\mathrm{cris}} : (X/S)_{\mathrm{cris}} \longrightarrow (X/S)_{\mathrm{cris}}$$

see Remark 9.3. Here is the terminology concerning F-crystals following the notation of Saavedra, see [SR72].

Definition 26.2. In Situation 26.1 an *F-crystal on X/S (relative to σ)* is a pair $(\mathcal{E}, F_{\mathcal{E}})$ given by a crystal in finite locally free $\mathcal{O}_{X/S}$-modules $\mathcal{E}$ together with a map

$$F_{\mathcal{E}} : (F_X)^*_{\mathrm{cris}} \mathcal{E} \longrightarrow \mathcal{E}$$

An F-crystal is called *nondegenerate* if there exists an integer $i \geq 0$ a map $V : \mathcal{E} \to (F_X)^*_{\mathrm{cris}} \mathcal{E}$ such that $V \circ F_{\mathcal{E}} = p^i \mathrm{id}$.

Remark 26.3. Let $(\mathcal{E}, F)$ be an F-crystal as in Definition 26.2. In the literature the nondegeneracy condition is often part of the definition of an F-crystal. Moreover, often it is also assumed that $F \circ V = p^n \mathrm{id}$. What is needed for the result below is that there exists an integer $j \geq 0$ such that $\mathrm{Ker}(F)$ and $\mathrm{Coker}(F)$ are killed by p^j. If the rank of $\mathcal{E}$ is bounded (for example if X is quasi-compact), then both of these conditions follow from the nondegeneracy condition as formulated in the definition. Namely, suppose R is a ring, $r \geq 1$ is an integer and $K, L \in \mathrm{Mat}(r \times r, R)$ are matrices with $KL = p^i 1_{r \times r}$. Then $\det(K) \det(L) = p^{ri}$. Let L' be the adjugate matrix of L, i.e., $L'L = LL' = \det(L)$. Set $K' = p^{ri} K$ and $j = ri + i$. Then we have $K'L = p^j 1_{r \times r}$ as $KL = p^i$ and

$$LK' = LK \det(L) \det(M) = LKLL' \det(M) = Lp^i L' \det(M) = p^j 1_{r \times r}$$

It follows that if V is as in Definition 26.2 then setting $V' = p^N V$ where $N > i \cdot \mathrm{rank}(\mathcal{E})$ we get $V' \circ F = p^{N+i}$ and $F \circ V' = p^{N+i}$.

Theorem 26.4. *In Situation 26.1 let $(\mathcal{E}, F_{\mathcal{E}})$ be a nondegenerate F-crystal. Assume A is a p-adically complete Noetherian ring and that $X \to S_0$ is proper smooth. Then the canonical map*

$$F_{\mathcal{E}} \circ (F_X)^*_{\mathrm{cris}} : R\Gamma(Cris(X/S), \mathcal{E}) \otimes^{\mathbf{L}}_{A,\sigma} A \longrightarrow R\Gamma(Cris(X/S), \mathcal{E})$$

becomes an isomorphism after inverting p.

Proof. We first write the arrow as a composition of three arrows. Namely, set

$$X^{(1)} = X \times_{S_0, F_{S_0}} S_0$$

and denote $F_{X/S_0} : X \to X^{(1)}$ the relative Frobenius morphism. Denote $\mathcal{E}^{(1)}$ the base change of $\mathcal{E}$ by $\mathrm{Spec}(\sigma)$, in other words the pullback of $\mathcal{E}$ to $\mathrm{Cris}(X^{(1)}/S)$ by the morphism of crystalline topoi associated to the commutative diagram

$$
\begin{array}{ccc}
X^{(1)} & \longrightarrow & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{\mathrm{Spec}(\sigma)} & S
\end{array}
$$

Then we have the base change map

$(26.4.1)$ $\qquad R\Gamma(\mathrm{Cris}(X/S), \mathcal{E}) \otimes^{\mathbf{L}}_{A,\sigma} A \longrightarrow R\Gamma(\mathrm{Cris}(X^{(1)}/S), \mathcal{E}^{(1)})$

see Remark 24.8. Note that the composition of $F_{X/S_0} : X \to X^{(1)}$ with the projection $X^{(1)} \to X$ is the absolute Frobenius morphism F_X. Hence we see that $F_{X/S_0}^* \mathcal{E}^{(1)} = (F_X)^*_{\mathrm{cris}} \mathcal{E}$. Thus pullback by F_{X/S_0} is a map

$(26.4.2)$ $\qquad F_{X/S_0}^* : R\Gamma(\mathrm{Cris}(X^{(1)}/S), \mathcal{E}^{(1)}) \longrightarrow R\Gamma(\mathrm{Cris}(X/S), (F_X)^*_{\mathrm{cris}} \mathcal{E})$

Finally we can use $F_{\mathcal{E}}$ to get a map

$(26.4.3)$ $\qquad R\Gamma(\mathrm{Cris}(X/S), (F_X)^*_{\mathrm{cris}} \mathcal{E}) \longrightarrow R\Gamma(\mathrm{Cris}(X/S), \mathcal{E})$

The map of the theorem is the composition of the three maps (26.4.1), (26.4.2), and (26.4.3) above. The first is a quasi-isomorphism modulo all powers of p by Remark 24.9. Hence it is a quasi-isomorphism since the complexes involved are perfect in $D(A)$ see Remark 24.13. The third map is a quasi-isomorphism after inverting p simply because $F_{\mathcal{E}}$ has an inverse up to a power of p, see Remark 26.3. Finally, the second is an isomorphism after inverting p by Lemma 25.6. $\square$

27. Other chapters

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