

# HYPERCOVERINGS

## Contents

1. Introduction	1
2. Hypercoverings	2
3. Acyclicity	5
4. Čech cohomology and hypercoverings	8
5. Hypercoverings a la Verdier	10
6. Covering hypercoverings	12
7. Adding simplices	15
8. Homotopies	16
9. Cohomology and hypercoverings	17
10. Hypercoverings of spaces	21
11. Hypercoverings and weakly contractible objects	23
12. Other chapters	24
References	25

## 1. Introduction

Let  $\mathcal{C}$  be a site, see Sites, Definition 6.2. Let  $X$  be an object of  $\mathcal{C}$ . Given an abelian sheaf  $\mathcal{F}$  on  $\mathcal{C}$  we would like to compute its cohomology groups

$$H^i(X, \mathcal{F}).$$

According to our general definitions (Cohomology on Sites, Section 3) this cohomology group is computed by choosing an injective resolution  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \dots$  and setting

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^0) \rightarrow \Gamma(X, \mathcal{I}^1) \rightarrow \Gamma(X, \mathcal{I}^2) \rightarrow \dots)$$

The goal of this chapter is to show that we may also compute these cohomology groups without choosing an injective resolution (in the case that  $\mathcal{C}$  has fibre products). To do this we will use hypercoverings.

A hypercovering in a site is a generalization of a covering, see [AGV71, Exposé V, Sec. 7]. Given a hypercovering  $K$  of an object  $X$ , there is a Čech to cohomology spectral sequence expressing the cohomology of an abelian sheaf  $\mathcal{F}$  over  $X$  in terms of the cohomology of the sheaf over the components  $K_n$  of  $K$ . It turns out that there are always enough hypercoverings, so that taking the colimit over all hypercoverings, the spectral sequence degenerates and the cohomology of  $\mathcal{F}$  over  $X$  is computed by the colimit of the Čech cohomology groups.

A more general gadget one can consider is a simplicial augmentation where one has cohomological descent, see [AGV71, Exposé Vbis]. A nice manuscript on cohomological descent is the text by Brian Conrad, see <http://math.stanford.edu/~conrad/papers/hypercover.pdf>. We will come back to these issue in the chapter on simplicial spaces where we will show, for example, that proper hypercoverings of “locally compact” topological spaces are of cohomological descent (Simplicial Spaces, Section 6). Our method of attack will be to reduce this statement to the Čech to cohomology spectral sequence constructed in this chapter.

## 2. Hypercoverings

In order to start we make the following definition. The letters “SR” stand for Semi-Representable.

**Definition 2.1.** Let  $\mathcal{C}$  be a site. We denote  $\text{SR}(\mathcal{C})$  the category of *semi-representable objects* defined as follows

- (1) objects are families of objects  $\{U_i\}_{i \in I}$ , and
- (2) morphisms  $\{U_i\}_{i \in I} \rightarrow \{V_j\}_{j \in J}$  are given by a map  $\alpha : I \rightarrow J$  and for each  $i \in I$  a morphism  $f_i : U_i \rightarrow V_{\alpha(i)}$  of  $\mathcal{C}$ .

Let  $X \in \text{Ob}(\mathcal{C})$  be an object of  $\mathcal{C}$ . The category of *semi-representable objects over  $X$*  is the category  $\text{SR}(\mathcal{C}, X) = \text{SR}(\mathcal{C}/X)$ .

This definition differs from the one in [AGV71, Exposé V, Sec. 7], but it seems flexible enough to do all the required arguments. Note that this is a “big” category. We will later “bound” the size of the index sets  $I$  that we need for hypercoverings of  $X$ . We can then redefine  $\text{SR}(\mathcal{C}, X)$  to become a category. Let’s spell out the objects and morphisms  $\text{SR}(\mathcal{C}, X)$ :

- (1) objects are families of morphisms  $\{U_i \rightarrow X\}_{i \in I}$ , and
- (2) morphisms  $\{U_i \rightarrow X\}_{i \in I} \rightarrow \{V_j \rightarrow X\}_{j \in J}$  are given by a map  $\alpha : I \rightarrow J$  and for each  $i \in I$  a morphism  $f_i : U_i \rightarrow V_{\alpha(i)}$  over  $X$ .

There is a forgetful functor  $\text{SR}(\mathcal{C}, X) \rightarrow \text{SR}(\mathcal{C})$ .

**Definition 2.2.** Let  $\mathcal{C}$  be a site with fibre products. We denote  $F$  the functor which associates a presheaf to a semi-representable object. In a formula

$$\begin{aligned} F : \text{SR}(\mathcal{C}) &\longrightarrow \text{PSh}(\mathcal{C}) \\ \{U_i\}_{i \in I} &\longmapsto \coprod_{i \in I} h_{U_i} \end{aligned}$$

where  $h_U$  denotes the representable presheaf associated to the object  $U$ .

Given a morphism  $U \rightarrow X$  we obtain a morphism  $h_U \rightarrow h_X$  of representable presheaves. Thus we often think of  $F$  on  $\text{SR}(\mathcal{C}, X)$  as a functor into the category of presheaves of sets over  $h_X$ , namely  $\text{PSh}(\mathcal{C})/h_X$ . Here is a picture:

$$\begin{array}{ccc} \text{SR}(\mathcal{C}, X) & \xrightarrow{F} & \text{PSh}(\mathcal{C})/h_X \\ \downarrow & & \downarrow \\ \text{SR}(\mathcal{C}) & \xrightarrow{F} & \text{PSh}(\mathcal{C}) \end{array}$$

Next we discuss the existence of limits in the category of semi-representable objects.

**Lemma 2.3.** *Let  $\mathcal{C}$  be a site.*

- (1) *the category  $\text{SR}(\mathcal{C})$  has coproducts and  $F$  commutes with them,*

- (2) the functor  $F : SR(\mathcal{C}) \rightarrow PSh(\mathcal{C})$  commutes with limits,
- (3) if  $\mathcal{C}$  has fibre products, then  $SR(\mathcal{C})$  has fibre products,
- (4) if  $\mathcal{C}$  has products of pairs, then  $SR(\mathcal{C})$  has products of pairs,
- (5) if  $\mathcal{C}$  has equalizers, so does  $SR(\mathcal{C})$ , and
- (6) if  $\mathcal{C}$  has a final object, so does  $SR(\mathcal{C})$ .

Let  $X \in \text{Ob}(\mathcal{C})$ .

- (1) the category  $SR(\mathcal{C}, X)$  has coproducts and  $F$  commutes with them,
- (2) if  $\mathcal{C}$  has fibre products, then  $SR(\mathcal{C}, X)$  has finite limits and  $F : SR(\mathcal{C}, X) \rightarrow PSh(\mathcal{C})/h_X$  commutes with them.

**Proof.** Proof of the results on  $SR(\mathcal{C})$ . Proof of (1). The coproduct of  $\{U_i\}_{i \in I}$  and  $\{V_j\}_{j \in J}$  is  $\{U_i\}_{i \in I} \amalg \{V_j\}_{j \in J}$ , in other words, the family of objects whose index set is  $I \amalg J$  and for an element  $k \in I \amalg J$  gives  $U_i$  if  $k = i \in I$  and gives  $V_j$  if  $k = j \in J$ . Similarly for coproducts of families of objects. It is clear that  $F$  commutes with these.

Proof of (2). For  $U$  in  $\text{Ob}(\mathcal{C})$  consider the object  $\{U\}$  of  $SR(\mathcal{C})$ . It is clear that  $\text{Mor}_{SR(\mathcal{C})}(\{U\}, K) = F(K)(U)$  for  $K \in \text{Ob}(SR(\mathcal{C}))$ . Since limits of presheaves are computed at the level of sections (Sites, Section 4) we conclude that  $F$  commutes with limits.

Proof of (3). Suppose given a morphism  $(\alpha, f_i) : \{U_i\}_{i \in I} \rightarrow \{V_j\}_{j \in J}$  and a morphism  $(\beta, g_k) : \{W_k\}_{k \in K} \rightarrow \{V_j\}_{j \in J}$ . The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k\}_{(i,j,k) \in I \times J \times K \text{ such that } j=\alpha(i)=\beta(k)}$$

The fibre products exist if  $\mathcal{C}$  has fibre products.

Proof of (4). The product of  $\{U_i\}_{i \in I}$  and  $\{V_j\}_{j \in J}$  is  $\{U_i \times V_j\}_{i \in I, j \in J}$ . The products exist if  $\mathcal{C}$  has products.

Proof of (5). The equalizer of two maps  $(\alpha, f_i), (\alpha', f'_i) : \{U_i\}_{i \in I} \rightarrow \{V_j\}_{j \in J}$  is

$$\{\text{Eq}(f_i, f'_i : U_i \rightarrow V_{\alpha(i)})\}_{i \in I, \alpha(i)=\alpha'(i)}$$

The equalizers exist if  $\mathcal{C}$  has equalizers.

Proof of (6). If  $X$  is a final object of  $\mathcal{C}$ , then  $\{X\}$  is a final object of  $SR(\mathcal{C})$ .

Proof of the statements about  $SR(\mathcal{C}, X)$ . These follow from the results above applied to the category  $\mathcal{C}/X$  using that  $SR(\mathcal{C}/X) = SR(\mathcal{C}, X)$  and that  $PSh(\mathcal{C}/X) = PSh(\mathcal{C})/h_X$  (Sites, Lemma 24.4 applied to  $\mathcal{C}$  endowed with the chaotic topology). However we also argue directly as follows. It is clear that the coproduct of  $\{U_i \rightarrow X\}_{i \in I}$  and  $\{V_j \rightarrow X\}_{j \in J}$  is  $\{U_i \rightarrow X\}_{i \in I} \amalg \{V_j \rightarrow X\}_{j \in J}$  and similarly for coproducts of families of families of morphisms with target  $X$ . The object  $\{X \rightarrow X\}$  is a final object of  $SR(\mathcal{C}, X)$ . Suppose given a morphism  $(\alpha, f_i) : \{U_i \rightarrow X\}_{i \in I} \rightarrow \{V_j \rightarrow X\}_{j \in J}$  and a morphism  $(\beta, g_k) : \{W_k \rightarrow X\}_{k \in K} \rightarrow \{V_j \rightarrow X\}_{j \in J}$ . The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k \rightarrow X\}_{(i,j,k) \in I \times J \times K \text{ such that } j=\alpha(i)=\beta(k)}$$

The fibre products exist by the assumption that  $\mathcal{C}$  has fibre products. Thus  $SR(\mathcal{C}, X)$  has finite limits, see Categories, Lemma 18.4. We omit verifying the statements on the functor  $F$  in this case.  $\square$

**Definition 2.4.** Let  $\mathcal{C}$  be a site. Let  $f = (\alpha, f_i) : \{U_i\}_{i \in I} \rightarrow \{V_j\}_{j \in J}$  be a morphism in the category  $\text{SR}(\mathcal{C})$ . We say that  $f$  is a *covering* if for every  $j \in J$  the family of morphisms  $\{U_i \rightarrow V_j\}_{i \in I, \alpha(i)=j}$  is a covering for the site  $\mathcal{C}$ . Let  $X$  be an object of  $\mathcal{C}$ . A morphism  $K \rightarrow L$  in  $\text{SR}(\mathcal{C}, X)$  is a *covering* if its image in  $\text{SR}(\mathcal{C})$  is a covering.

**Lemma 2.5.** *Let  $\mathcal{C}$  be a site.*

- (1) *A composition of coverings in  $\text{SR}(\mathcal{C})$  is a covering.*
- (2) *If  $K \rightarrow L$  is a covering in  $\text{SR}(\mathcal{C})$  and  $L' \rightarrow L$  is a morphism, then  $L' \times_L K$  exists and  $L' \times_L K \rightarrow L'$  is a covering.*
- (3) *If  $\mathcal{C}$  has products of pairs, and  $A \rightarrow B$  and  $K \rightarrow L$  are coverings in  $\text{SR}(\mathcal{C})$ , then  $A \times K \rightarrow B \times L$  is a covering.*

*Let  $X \in \text{Ob}(\mathcal{C})$ . Then (1) and (2) holds for  $\text{SR}(\mathcal{C}, X)$  and (3) holds if  $\mathcal{C}$  has fibre products.*

**Proof.** Part (1) is immediate from the axioms of a site. Part (2) follows by the construction of fibre products in  $\text{SR}(\mathcal{C})$  in the proof of Lemma 2.3 and the requirement that the morphisms in a covering of  $\mathcal{C}$  are representable. Part (3) follows by thinking of  $A \times K \rightarrow B \times L$  as the composition  $A \times K \rightarrow B \times K \rightarrow B \times L$  and hence a composition of basechanges of coverings. The final statement follows because  $\text{SR}(\mathcal{C}, X) = \text{SR}(\mathcal{C}/X)$ .  $\square$

By Lemma 2.3 and Simplicial, Lemma 18.2 the coskeleton of a truncated simplicial object of  $\text{SR}(\mathcal{C}, X)$  exists if  $\mathcal{C}$  has fibre products. Hence the following definition makes sense.

**Definition 2.6.** Let  $\mathcal{C}$  be a site. Assume  $\mathcal{C}$  has fibre products. Let  $X \in \text{Ob}(\mathcal{C})$  be an object of  $\mathcal{C}$ . A *hypercovering* of  $X$  is a simplicial object  $K$  of  $\text{SR}(\mathcal{C}, X)$  such that

- (1) The object  $K_0$  is a covering of  $X$  for the site  $\mathcal{C}$ .
- (2) For every  $n \geq 0$  the canonical morphism

$$K_{n+1} \longrightarrow (\text{cosk}_n \text{sk}_n K)_{n+1}$$

is a covering in the sense defined above.

Condition (1) makes sense since each object of  $\text{SR}(\mathcal{C}, X)$  is after all a family of morphisms with target  $X$ . It could also be formulated as saying that the morphism of  $K_0$  to the final object of  $\text{SR}(\mathcal{C}, X)$  is a covering.

**Example 2.7.** Let  $\{U_i \rightarrow X\}_{i \in I}$  be a covering of the site  $\mathcal{C}$ . Set  $K_0 = \{U_i \rightarrow X\}_{i \in I}$ . Then  $K_0$  is a 0-truncated simplicial object of  $\text{SR}(\mathcal{C}, X)$ . Hence we may form

$$K = \text{cosk}_0 K_0.$$

Clearly  $K$  passes condition (1) of Definition 2.6. Since all the morphisms  $K_{n+1} \rightarrow (\text{cosk}_n \text{sk}_n K)_{n+1}$  are isomorphisms by Simplicial, Lemma 18.10 it also passes condition (2). Note that the terms  $K_n$  are the usual

$$K_n = \{U_{i_0} \times_X U_{i_1} \times_X \dots \times_X U_{i_n} \rightarrow X\}_{(i_0, i_1, \dots, i_n) \in I^{n+1}}$$

**Lemma 2.8.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X \in \text{Ob}(\mathcal{C})$  be an object of  $\mathcal{C}$ . The collection of all hypercoverings of  $X$  forms a set.*

**Proof.** Since  $\mathcal{C}$  is a site, the set of all coverings of  $X$  forms a set. Thus we see that the collection of possible  $K_0$  forms a set. Suppose we have shown that the collection of all possible  $K_0, \dots, K_n$  form a set. Then it is enough to show that given  $K_0, \dots, K_n$  the collection of all possible  $K_{n+1}$  forms a set. And this is clearly true since we have to choose  $K_{n+1}$  among all possible coverings of  $(\text{cosk}_n \text{sk}_n K)_{n+1}$ .  $\square$

**Remark 2.9.** The lemma does not just say that there is a cofinal system of choices of hypercoverings that is a set, but that really the hypercoverings form a set.

The category of presheaves on  $\mathcal{C}$  has finite (co)limits. Hence the functors  $\text{cosk}_n$  exists for presheaves of sets.

**Lemma 2.10.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X \in \text{Ob}(\mathcal{C})$  be an object of  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $X$ . Consider the simplicial object  $F(K)$  of  $PSh(\mathcal{C})$ , endowed with its augmentation to the constant simplicial presheaf  $h_X$ .*

- (1) *The morphism of presheaves  $F(K)_0 \rightarrow h_X$  becomes a surjection after sheafification.*
- (2) *The morphism*

$$(d_0^1, d_1^1) : F(K)_1 \longrightarrow F(K)_0 \times_{h_X} F(K)_0$$

*becomes a surjection after sheafification.*

- (3) *For every  $n \geq 1$  the morphism*

$$F(K)_{n+1} \longrightarrow (\text{cosk}_n \text{sk}_n F(K))_{n+1}$$

*turns into a surjection after sheafification.*

**Proof.** We will use the fact that if  $\{U_i \rightarrow U\}_{i \in I}$  is a covering of the site  $\mathcal{C}$ , then the morphism

$$\coprod_{i \in I} h_{U_i} \rightarrow h_U$$

becomes surjective after sheafification, see Sites, Lemma 13.4. Thus the first assertion follows immediately.

For the second assertion, note that according to Simplicial, Example 18.1 the simplicial object  $\text{cosk}_0 \text{sk}_0 K$  has terms  $K_0 \times \dots \times K_0$ . Thus according to the definition of a hypercovering we see that  $(d_0^1, d_1^1) : K_1 \rightarrow K_0 \times K_0$  is a covering. Hence (2) follows from the claim above and the fact that  $F$  transforms products into fibred products over  $h_X$ .

For the third, we claim that  $\text{cosk}_n \text{sk}_n F(K) = F(\text{cosk}_n \text{sk}_n K)$  for  $n \geq 1$ . To prove this, denote temporarily  $F'$  the functor  $\text{SR}(\mathcal{C}, X) \rightarrow PSh(\mathcal{C})/h_X$ . By Lemma 2.3 the functor  $F'$  commutes with finite limits. By our description of the  $\text{cosk}_n$  functor in Simplicial, Section 12 we see that  $\text{cosk}_n \text{sk}_n F'(K) = F'(\text{cosk}_n \text{sk}_n K)$ . Recall that the category used in the description of  $(\text{cosk}_n U)_m$  in Simplicial, Lemma 18.2 is the category  $(\Delta/[m])_{\leq n}^{opp}$ . It is an amusing exercise to show that  $(\Delta/[m])_{\leq n}$  is a connected category (see Categories, Definition 16.1) as soon as  $n \geq 1$ . Hence, Categories, Lemma 16.2 shows that  $\text{cosk}_n \text{sk}_n F'(K) = \text{cosk}_n \text{sk}_n F(K)$ . Whence the claim. Property (2) follows from this, because now we see that the morphism in (2) is the result of applying the functor  $F$  to a covering as in Definition 2.4, and the result follows from the first fact mentioned in this proof.  $\square$

### 3. Acyclicity

Let  $\mathcal{C}$  be a site. For a presheaf of sets  $\mathcal{F}$  we denote  $\mathbf{Z}_{\mathcal{F}}$  the presheaf of abelian groups defined by the rule

$$\mathbf{Z}_{\mathcal{F}}(U) = \text{free abelian group on } \mathcal{F}(U).$$

We will sometimes call this the *free abelian presheaf on  $\mathcal{F}$* . Of course the construction  $\mathcal{F} \mapsto \mathbf{Z}_{\mathcal{F}}$  is a functor and it is left adjoint to the forgetful functor  $PAb(\mathcal{C}) \rightarrow PSh(\mathcal{C})$ . Of course the sheafification  $\mathbf{Z}_{\mathcal{F}}^{\#}$  is a sheaf of abelian groups, and the functor  $\mathcal{F} \mapsto \mathbf{Z}_{\mathcal{F}}^{\#}$  is a left adjoint as well. We sometimes call  $\mathbf{Z}_{\mathcal{F}}^{\#}$  the *free abelian sheaf on  $\mathcal{F}$* .

For an object  $X$  of the site  $\mathcal{C}$  we denote  $\mathbf{Z}_X$  the free abelian presheaf on  $h_X$ , and we denote  $\mathbf{Z}_X^{\#}$  its sheafification.

**Definition 3.1.** Let  $\mathcal{C}$  be a site. Let  $K$  be a simplicial object of  $PSh(\mathcal{C})$ . By the above we get a simplicial object  $\mathbf{Z}_K^{\#}$  of  $Ab(\mathcal{C})$ . We can take its associated complex of abelian presheaves  $s(\mathbf{Z}_K^{\#})$ , see *Simplicial*, Section 22. The *homology of  $K$*  is the homology of the complex of abelian sheaves  $s(\mathbf{Z}_K^{\#})$ .

In other words, the  *$i$ th homology  $H_i(K)$  of  $K$*  is the sheaf of abelian groups  $H_i(K) = H_i(s(\mathbf{Z}_K^{\#}))$ . In this section we worry about the homology in case  $K$  is a hypercovering of an object  $X$  of  $\mathcal{C}$ .

**Lemma 3.2.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{F} \rightarrow \mathcal{G}$  be a morphism of presheaves of sets. Denote  $K$  the simplicial object of  $PSh(\mathcal{C})$  whose  $n$ th term is the  $(n+1)$ st fibre product of  $\mathcal{F}$  over  $\mathcal{G}$ , see *Simplicial*, Example 3.5. Then, if  $\mathcal{F} \rightarrow \mathcal{G}$  is surjective after sheafification, we have*

$$H_i(K) = \begin{cases} 0 & \text{if } i > 0 \\ \mathbf{Z}_{\mathcal{G}}^{\#} & \text{if } i = 0 \end{cases}$$

*The isomorphism in degree 0 is given by the morphism  $H_0(K) \rightarrow \mathbf{Z}_{\mathcal{G}}^{\#}$  coming from the map  $(\mathbf{Z}_K^{\#})_0 = \mathbf{Z}_{\mathcal{F}}^{\#} \rightarrow \mathbf{Z}_{\mathcal{G}}^{\#}$ .*

**Proof.** Let  $\mathcal{G}' \subset \mathcal{G}$  be the image of the morphism  $\mathcal{F} \rightarrow \mathcal{G}$ . Let  $U \in \text{Ob}(\mathcal{C})$ . Set  $A = \mathcal{F}(U)$  and  $B = \mathcal{G}'(U)$ . Then the simplicial set  $K(U)$  is equal to the simplicial set with  $n$ -simplices given by

$$A \times_B A \times_B \dots \times_B A \text{ (} n+1 \text{ factors)}.$$

By *Simplicial*, Lemma 31.3 the morphism  $K(U) \rightarrow B$  is a trivial Kan fibration. Thus it is a homotopy equivalence (*Simplicial*, Lemma 31.3). Hence applying the functor “free abelian group on” to this we deduce that

$$\mathbf{Z}_K(U) \longrightarrow \mathbf{Z}_B$$

is a homotopy equivalence. Note that  $s(\mathbf{Z}_B)$  is the complex

$$\dots \rightarrow \bigoplus_{b \in B} \mathbf{Z} \xrightarrow{0} \bigoplus_{b \in B} \mathbf{Z} \xrightarrow{1} \bigoplus_{b \in B} \mathbf{Z} \xrightarrow{0} \bigoplus_{b \in B} \mathbf{Z} \rightarrow 0$$

see *Simplicial*, Lemma 22.3. Thus we see that  $H_i(s(\mathbf{Z}_K(U))) = 0$  for  $i > 0$ , and  $H_0(s(\mathbf{Z}_K(U))) = \bigoplus_{b \in B} \mathbf{Z} = \bigoplus_{s \in \mathcal{G}'(U)} \mathbf{Z}$ . These identifications are compatible with restriction maps.

We conclude that  $H_i(s(\mathbf{Z}_K)) = 0$  for  $i > 0$  and  $H_0(s(\mathbf{Z}_K)) = \mathbf{Z}_{\mathcal{G}'}$ , where here we compute homology groups in  $PAb(\mathcal{C})$ . Since sheafification is an exact functor we deduce the result of the lemma. Namely, the exactness implies that  $H_0(s(\mathbf{Z}_K))^\# = H_0(s(\mathbf{Z}_K^\#))$ , and similarly for other indices.  $\square$

**Lemma 3.3.** *Let  $\mathcal{C}$  be a site. Let  $f : L \rightarrow K$  be a morphism of simplicial objects of  $PSh(\mathcal{C})$ . Let  $n \geq 0$  be an integer. Assume that*

- (1) *For  $i < n$  the morphism  $L_i \rightarrow K_i$  is an isomorphism.*
- (2) *The morphism  $L_n \rightarrow K_n$  is surjective after sheafification.*
- (3) *The canonical map  $L \rightarrow \text{cosk}_n \text{sk}_n L$  is an isomorphism.*
- (4) *The canonical map  $K \rightarrow \text{cosk}_n \text{sk}_n K$  is an isomorphism.*

*Then  $H_i(f) : H_i(L) \rightarrow H_i(K)$  is an isomorphism.*

**Proof.** This proof is exactly the same as the proof of Lemma 3.2 above. Namely, we first let  $K'_n \subset K_n$  be the sub presheaf which is the image of the map  $L_n \rightarrow K_n$ . Assumption (2) means that the sheafification of  $K'_n$  is equal to the sheafification of  $K_n$ . Moreover, since  $L_i = K_i$  for all  $i < n$  we see that get an  $n$ -truncated simplicial presheaf  $U$  by taking  $U_0 = L_0 = K_0, \dots, U_{n-1} = L_{n-1} = K_{n-1}, U_n = K'_n$ . Denote  $K' = \text{cosk}_n U$ , a simplicial presheaf. Because we can construct  $K'_m$  as a finite limit, and since sheafification is exact, we see that  $(K'_m)^\# = K_m$ . In other words,  $(K')^\# = K^\#$ . We conclude, by exactness of sheafification once more, that  $H_i(K) = H_i(K')$ . Thus it suffices to prove the lemma for the morphism  $L \rightarrow K'$ , in other words, we may assume that  $L_n \rightarrow K_n$  is a surjective morphism of *presheaves*!

In this case, for any object  $U$  of  $\mathcal{C}$  we see that the morphism of simplicial sets

$$L(U) \longrightarrow K(U)$$

satisfies all the assumptions of Simplicial, Lemma 31.1. Hence it is a trivial Kan fibration. In particular it is a homotopy equivalence (Simplicial, Lemma 29.8). Thus

$$\mathbf{Z}_L(U) \longrightarrow \mathbf{Z}_K(U)$$

is a homotopy equivalence too. This for all  $U$ . The result follows.  $\square$

**Lemma 3.4.** *Let  $\mathcal{C}$  be a site. Let  $K$  be a simplicial presheaf. Let  $\mathcal{G}$  be a presheaf. Let  $K \rightarrow \mathcal{G}$  be an augmentation of  $K$  towards  $\mathcal{G}$ . Assume that*

- (1) *The morphism of presheaves  $K_0 \rightarrow \mathcal{G}$  becomes a surjection after sheafification.*
- (2) *The morphism*

$$(d_0^1, d_1^1) : K_1 \longrightarrow K_0 \times_{\mathcal{G}} K_0$$

*becomes a surjection after sheafification.*

- (3) *For every  $n \geq 1$  the morphism*

$$K_{n+1} \longrightarrow (\text{cosk}_n \text{sk}_n K)_{n+1}$$

*turns into a surjection after sheafification.*

*Then  $H_i(K) = 0$  for  $i > 0$  and  $H_0(K) = \mathbf{Z}_{\mathcal{G}}^\#$ .*

**Proof.** Denote  $K^n = \text{cosk}_n \text{sk}_n K$  for  $n \geq 1$ . Define  $K^0$  as the simplicial object with terms  $(K^0)_n$  equal to the  $(n+1)$ -fold fibred product  $K_0 \times_{\mathcal{G}} \dots \times_{\mathcal{G}} K_0$ , see Simplicial, Example 3.5. We have morphisms

$$K \longrightarrow \dots \longrightarrow K^n \rightarrow K^{n-1} \rightarrow \dots \rightarrow K^1 \rightarrow K^0.$$

The morphisms  $K \rightarrow K^i, K^j \rightarrow K^i$  for  $j \geq i \geq 1$  come from the universal properties of the  $\text{cosk}_n$  functors. The morphism  $K^1 \rightarrow K^0$  is the canonical morphism from Simplicial, Remark 19.4. We also recall that  $K^0 \rightarrow \text{cosk}_1 \text{sk}_1 K^0$  is an isomorphism, see Simplicial, Lemma 19.3.

By Lemma 3.2 we see that  $H_i(K^0) = 0$  for  $i > 0$  and  $H_0(K^0) = \mathbf{Z}_G^\#$ .

Pick  $n \geq 1$ . Consider the morphism  $K^n \rightarrow K^{n-1}$ . It is an isomorphism on terms of degree  $< n$ . Note that  $K^n \rightarrow \text{cosk}_n \text{sk}_n K^n$  and  $K^{n-1} \rightarrow \text{cosk}_n \text{sk}_n K^{n-1}$  are isomorphisms. Note that  $(K^n)_n = K_n$  and that  $(K^{n-1})_n = (\text{cosk}_{n-1} \text{sk}_{n-1} K)_n$ . Hence by assumption, we have that  $(K^n)_n \rightarrow (K^{n-1})_n$  is a morphism of presheaves which becomes surjective after sheafification. By Lemma 3.3 we conclude that  $H_i(K^n) = H_i(K^{n-1})$ . Combined with the above this proves the lemma.  $\square$

**Lemma 3.5.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $X$ . The homology of the simplicial presheaf  $F(K)$  is 0 in degrees  $> 0$  and equal to  $\mathbf{Z}_X^\#$  in degree 0.*

**Proof.** Combine Lemmas 3.4 and 2.10.  $\square$

#### 4. Čech cohomology and hypercoverings

Let  $\mathcal{C}$  be a site. Consider a presheaf of abelian groups  $\mathcal{F}$  on the site  $\mathcal{C}$ . It defines a functor

$$\begin{aligned} \mathcal{F} : \text{SR}(\mathcal{C})^{\text{opp}} &\longrightarrow \text{Ab} \\ \{U_i\}_{i \in I} &\longmapsto \prod_{i \in I} \mathcal{F}(U_i) \end{aligned}$$

Thus a simplicial object  $K$  of  $\text{SR}(\mathcal{C})$  is turned into a cosimplicial object  $\mathcal{F}(K)$  of  $\text{Ab}$ . The cochain complex  $s(\mathcal{F})(K)$  associated to  $\mathcal{F}(K)$  (Simplicial, Section 24) is called the Čech complex of  $\mathcal{F}$  with respect to the simplicial object  $K$ . We set

$$\check{H}^i(K, \mathcal{F}) = H^i(s(\mathcal{F}(K))).$$

and we call it the  $i$ th Čech cohomology group of  $\mathcal{F}$  with respect to  $K$ . In this section we prove analogues of some of the results for Čech cohomology of open coverings proved in Cohomology, Sections 10, 11 and 12.

**Lemma 4.1.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $X$ . Let  $\mathcal{F}$  be a sheaf of abelian groups on  $\mathcal{C}$ . Then  $\check{H}^0(K, \mathcal{F}) = \mathcal{F}(X)$ .*

**Proof.** We have

$$\check{H}^0(K, \mathcal{F}) = \text{Ker}(\mathcal{F}(K_0) \longrightarrow \mathcal{F}(K_1))$$

Write  $K_0 = \{U_i \rightarrow X\}$ . It is a covering in the site  $\mathcal{C}$ . As well, we have that  $K_1 \rightarrow K_0 \times K_0$  is a covering in  $\text{SR}(\mathcal{C}, X)$ . Hence we may write  $K_1 = \coprod_{i_0, i_1 \in I} \{V_{i_0 i_1 j} \rightarrow X\}$  so that the morphism  $K_1 \rightarrow K_0 \times K_0$  is given by coverings  $\{V_{i_0 i_1 j} \rightarrow U_{i_0} \times_X U_{i_1}\}$  of the site  $\mathcal{C}$ . Thus we can further identify

$$\check{H}^0(K, \mathcal{F}) = \text{Ker}(\prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i_0 i_1 j} \mathcal{F}(V_{i_0 i_1 j}))$$

with obvious map. The sheaf property of  $\mathcal{F}$  implies that  $\check{H}^0(K, \mathcal{F}) = \mathcal{F}(X)$ .  $\square$

In fact this property characterizes the abelian sheaves among all abelian presheaves on  $\mathcal{C}$  of course. The analogue of Cohomology, Lemma 4.2 in this case is the following.



**Lemma 4.2.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $X$ . Let  $\mathcal{I}$  be an injective sheaf of abelian groups on  $\mathcal{C}$ . Then*

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} \mathcal{I}(X) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

**Proof.** Observe that for any object  $Z = \{U_i \rightarrow X\}$  of  $\text{SR}(\mathcal{C}, X)$  and any abelian sheaf  $\mathcal{F}$  on  $\mathcal{C}$  we have

$$\begin{aligned} \mathcal{F}(Z) &= \prod \mathcal{F}(U_i) \\ &= \prod \text{Mor}_{PSh(\mathcal{C})}(h_{U_i}, \mathcal{F}) \\ &= \text{Mor}_{PSh(\mathcal{C})}(F(Z), \mathcal{F}) \\ &= \text{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}_{F(Z)}, \mathcal{F}) \\ &= \text{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}_{F(Z)}^\#, \mathcal{F}) \end{aligned}$$

Thus we see, for any simplicial object  $K$  of  $\text{SR}(\mathcal{C}, X)$  that we have

$$(4.2.1) \quad s(\mathcal{F}(K)) = \text{Hom}_{Ab(\mathcal{C})}(s(\mathbf{Z}_{F(K)}^\#), \mathcal{F})$$

see Definition 3.1 for notation. The complex of sheaves  $s(\mathbf{Z}_{F(K)}^\#)$  is quasi-isomorphic to  $\mathbf{Z}_X^\#$  if  $K$  is a hypercovering, see Lemma 3.5. We conclude that if  $\mathcal{I}$  is an injective abelian sheaf, and  $K$  a hypercovering, then the complex  $s(\mathcal{I}(K))$  is acyclic except possibly in degree 0. In other words, we have

$$\check{H}^i(K, \mathcal{I}) = 0$$

for  $i > 0$ . Combined with Lemma 4.1 the lemma is proved.  $\square$

Next we come to the analogue of Cohomology on Sites, Lemma 11.6. Let  $\mathcal{C}$  be a site. Let  $\mathcal{F}$  be a sheaf of abelian groups on  $\mathcal{C}$ . Recall that  $\underline{H}^i(\mathcal{F})$  indicates the presheaf of abelian groups on  $\mathcal{C}$  which is defined by the rule  $\underline{H}^i(\mathcal{F}) : U \mapsto H^i(U, \mathcal{F})$ . We extend this to  $\text{SR}(\mathcal{C})$  as in the introduction to this section.

**Lemma 4.3.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $X$ . Let  $\mathcal{F}$  be a sheaf of abelian groups on  $\mathcal{C}$ . There is a map*

$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$$

*in  $D^+(Ab)$  functorial in  $\mathcal{F}$ , which induces natural transformations*

$$\check{H}^i(K, -) \longrightarrow H^i(X, -)$$

*as functors  $Ab(\mathcal{C}) \rightarrow Ab$ . Moreover, there is a spectral sequence  $(E_r, d_r)_{r \geq 0}$  with*

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

*converging to  $H^{p+q}(X, \mathcal{F})$ . This spectral sequence is functorial in  $\mathcal{F}$  and in the hypercovering  $K$ .*

**Proof.** We could prove this by the same method as employed in the corresponding lemma in the chapter on cohomology. Instead let us prove this by a double complex argument.

Choose an injective resolution  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  in the category of abelian sheaves on  $\mathcal{C}$ . Consider the double complex  $A^{\bullet, \bullet}$  with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential  $d_1^{p,q} : A^{p,q} \rightarrow A^{p+1,q}$  is the one coming from the differential  $\mathcal{I}^p \rightarrow \mathcal{I}^{p+1}$  and the differential  $d_2^{p,q} : A^{p,q} \rightarrow A^{p,q+1}$  is the one coming from the differential on the complex  $s(\mathcal{I}^p(K))$  associated to the cosimplicial abelian group  $\mathcal{I}^p(K)$  as explained above. As usual we denote  $sA^\bullet$  the simple complex associated to the double complex  $A^{\bullet,\bullet}$ . We will use the two spectral sequences  $({}'E_r, {}'d_r)$  and  $({}''E_r, {}''d_r)$  associated to this double complex, see Homology, Section 22.

By Lemma 4.2 the complexes  $s(\mathcal{I}^p(K))$  are acyclic in positive degrees and have  $H^0$  equal to  $\mathcal{I}^p(X)$ . Hence by Homology, Lemma 22.7 and its proof the spectral sequence  $({}'E_r, {}'d_r)$  degenerates, and the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow sA^\bullet$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that  $H^n(sA^\bullet) = H^n(X, \mathcal{F})$ .

The map  $s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$  of the lemma is the composition of the natural map  $s(\mathcal{F}(K)) \rightarrow sA^\bullet$  followed by the inverse of the displayed quasi-isomorphism above. This works because  $\mathcal{I}^\bullet(X)$  is a representative of  $R\Gamma(X, \mathcal{F})$ .

Consider the spectral sequence  $({}''E_r, {}''d_r)_{r \geq 0}$ . By Homology, Lemma 22.4 we see that

$${}''E_2^{p,q} = H_{II}^p(H_I^q(A^{\bullet,\bullet}))$$

In other words, we first take cohomology with respect to  $d_1$  which gives the groups  ${}''E_1^{p,q} = \underline{H}^p(\mathcal{F})(K_q)$ . Hence it is indeed the case (by the description of the differential  ${}''d_1$ ) that  ${}''E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$ . And by the other spectral sequence above we see that this one converges to  $H^n(X, \mathcal{F})$  as desired.

We omit the proof of the statements regarding the functoriality of the above constructions in the abelian sheaf  $\mathcal{F}$  and the hypercovering  $K$ .  $\square$

## 5. Hypercoverings a la Verdier

The astute reader will have noticed that all we need in order to get the Čech to cohomology spectral sequence for a hypercovering of an object  $X$ , is the conclusion of Lemma 2.10. Therefore the following definition makes sense.

**Definition 5.1.** Let  $\mathcal{C}$  be a site. Assume  $\mathcal{C}$  has equalizers and fibre products. Let  $\mathcal{G}$  be a presheaf of sets. A *hypercovering* of  $\mathcal{G}$  is a simplicial object  $K$  of  $\text{SR}(\mathcal{C})$  endowed with an augmentation  $F(K) \rightarrow \mathcal{G}$  such that

- (1)  $F(K_0) \rightarrow \mathcal{G}$  becomes surjective after sheafification,
- (2)  $F(K_1) \rightarrow F(K_0) \times_{\mathcal{G}} F(K_0)$  becomes surjective after sheafification, and
- (3)  $F(K_{n+1}) \rightarrow F((\text{cosk}_n \text{sk}_n K)_{n+1})$  for  $n \geq 1$  becomes surjective after sheafification.

We say that a simplicial object  $K$  of  $\text{SR}(\mathcal{C})$  is a *hypercovering* if  $K$  is a hypercovering of the final object  $*$  of  $\text{PSh}(\mathcal{C})$ .

The assumption that  $\mathcal{C}$  has fibre products and equalizers guarantees that  $\text{SR}(\mathcal{C})$  has fibre products and equalizers and  $F$  commutes with these (Lemma 2.3) which suffices to define the coskeleton functors used (see Simplicial, Remark 18.11 and Categories, Lemma 18.2). If  $\mathcal{C}$  is general, we can replace the condition (3) by the condition that  $F(K_{n+1}) \rightarrow ((\text{cosk}_n \text{sk}_n F(K))_{n+1})$  for  $n \geq 1$  becomes surjective after sheafification and the results of this section remain valid.

Let  $\mathcal{F}$  be an abelian sheaf on  $\mathcal{C}$ . In the previous section, we defined the Čech complex of  $\mathcal{F}$  with respect to a simplicial object  $K$  of  $\mathbf{SR}(\mathcal{C})$ . Next, given a presheaf  $\mathcal{G}$  we set

$$H^0(\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}^\#, \mathcal{F}) = H^0(\mathcal{G}^\#, \mathcal{F})$$

with notation as in Cohomology on Sites, Section 13). This is a left exact functor and its higher derived functors (briefly studied in Cohomology on Sites, Section 13) will be denoted  $H^i(\mathcal{G}, \mathcal{F})$ . We will show that given a hypercovering  $K$  of  $\mathcal{G}$ , there is a Čech to cohomology spectral sequence converging to the cohomology  $H^i(\mathcal{G}, \mathcal{F})$ . Note that if  $\mathcal{G} = *$ , then  $H^i(*, \mathcal{F}) = H^i(\mathcal{F})$  recovers the global cohomology of  $\mathcal{F}$ .

**Lemma 5.2.** *Let  $\mathcal{C}$  be a site with equalizers and fibre products. Let  $\mathcal{G}$  be a presheaf on  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $\mathcal{G}$ . Let  $\mathcal{F}$  be a sheaf of abelian groups on  $\mathcal{C}$ . Then  $\check{H}^0(K, \mathcal{F}) = H^0(\mathcal{G}, \mathcal{F})$ .*

**Proof.** This follows from the definition of  $H^0(\mathcal{G}, \mathcal{F})$  and the fact that

$$F(K_1) \rightrightarrows F(K_0) \longrightarrow \mathcal{G}$$

becomes an coequalizer diagram after sheafification.  $\square$

**Lemma 5.3.** *Let  $\mathcal{C}$  be a site with equalizers and fibre products. Let  $\mathcal{G}$  be a presheaf on  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $\mathcal{G}$ . Let  $\mathcal{I}$  be an injective sheaf of abelian groups on  $\mathcal{C}$ . Then*

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} H^0(\mathcal{G}, \mathcal{I}) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

**Proof.** By (4.2.1) we have

$$s(\mathcal{F}(K)) = \mathrm{Hom}_{Ab(\mathcal{C})}(s(\mathbf{Z}_{F(K)}^\#), \mathcal{F})$$

The complex  $s(\mathbf{Z}_{F(K)}^\#)$  is quasi-isomorphic to  $\mathbf{Z}_{\mathcal{G}}^\#$ , see Lemma 3.4. We conclude that if  $\mathcal{I}$  is an injective abelian sheaf, then the complex  $s(\mathcal{I}(K))$  is acyclic except possibly in degree 0. In other words, we have  $\check{H}^i(K, \mathcal{I}) = 0$  for  $i > 0$ . Combined with Lemma 5.2 the lemma is proved.  $\square$

**Lemma 5.4.** *Let  $\mathcal{C}$  be a site with equalizers and fibre products. Let  $\mathcal{G}$  be a presheaf on  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $\mathcal{G}$ . Let  $\mathcal{F}$  be a sheaf of abelian groups on  $\mathcal{C}$ . There is a map*

$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(\mathcal{G}, \mathcal{F})$$

*in  $D^+(Ab)$  functorial in  $\mathcal{F}$ , which induces a natural transformation*

$$\check{H}^i(K, -) \longrightarrow H^i(\mathcal{G}, -)$$

*of functors  $Ab(\mathcal{C}) \rightarrow Ab$ . Moreover, there is a spectral sequence  $(E_r, d_r)_{r \geq 0}$  with*

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

*converging to  $H^{p+q}(\mathcal{G}, \mathcal{F})$ . This spectral sequence is functorial in  $\mathcal{F}$  and in the hypercovering  $K$ .*

**Proof.** Choose an injective resolution  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  in the category of abelian sheaves on  $\mathcal{C}$ . Consider the double complex  $A^{\bullet, \bullet}$  with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential  $d_1^{p,q} : A^{p,q} \rightarrow A^{p+1,q}$  is the one coming from the differential  $\mathcal{I}^p \rightarrow \mathcal{I}^{p+1}$  and the differential  $d_2^{p,q} : A^{p,q} \rightarrow A^{p,q+1}$  is the one coming from the

differential on the complex  $s(\mathcal{I}^p(K))$  associated to the cosimplicial abelian group  $\mathcal{I}^p(K)$  as explained above. We will use the two spectral sequences  $({}'E_r, {}'d_r)$  and  $({}''E_r, {}''d_r)$  associated to this double complex, see Homology, Section 22.

By Lemma 5.3 the complexes  $s(\mathcal{I}^p(K))$  are acyclic in positive degrees and have  $H^0$  equal to  $H^0(\mathcal{G}, \mathcal{I}^p)$ . Hence by Homology, Lemma 22.7 and its proof the spectral sequence  $({}'E_r, {}'d_r)$  degenerates, and the natural map

$$H^0(\mathcal{G}, \mathcal{I}^\bullet) \longrightarrow \text{Tot}(A^{\bullet, \bullet})$$

is a quasi-isomorphism of complexes of abelian groups. The map  $s(\mathcal{F}(K)) \rightarrow R\Gamma(\mathcal{G}, \mathcal{F})$  of the lemma is the composition of the natural map  $s(\mathcal{F}(K)) \rightarrow \text{Tot}(A^{\bullet, \bullet})$  followed by the inverse of the displayed quasi-isomorphism above. This works because  $H^0(\mathcal{G}, \mathcal{I}^\bullet)$  is a representative of  $R\Gamma(\mathcal{G}, \mathcal{F})$ .

Consider the spectral sequence  $({}''E_r, {}''d_r)_{r \geq 0}$ . By Homology, Lemma 22.4 we see that

$${}''E_2^{p,q} = H_{II}^p(H_I^q(A^{\bullet, \bullet}))$$

In other words, we first take cohomology with respect to  $d_1$  which gives the groups  ${}''E_1^{p,q} = \underline{H}^p(\mathcal{F})(K_q)$ . Hence it is indeed the case (by the description of the differential  ${}''d_1$ ) that  ${}''E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$ . Since this spectral sequence converges to the cohomology of  $\text{Tot}(A^{\bullet, \bullet})$  the proof is finished.  $\square$

**Lemma 5.5.** *Let  $\mathcal{C}$  be a site with equalizers and fibre products. Let  $K$  be a hypercovering. Let  $\mathcal{F}$  be an abelian sheaf. There is a spectral sequence  $(E_r, d_r)_{r \geq 0}$  with*

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

*converging to the global cohomology groups  $H^{p+q}(\mathcal{F})$ .*

**Proof.** This is a special case of Lemma 5.4.  $\square$

## 6. Covering hypercoverings

Here are some ways to construct hypercoverings. We note that since the category  $\text{SR}(\mathcal{C}, X)$  has fibre products the category of simplicial objects of  $\text{SR}(\mathcal{C}, X)$  has fibre products as well, see Simplicial, Lemma 7.2.

**Lemma 6.1.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $K, L, M$  be simplicial objects of  $\text{SR}(\mathcal{C}, X)$ . Let  $a : K \rightarrow L$ ,  $b : M \rightarrow L$  be morphisms. Assume*

- (1)  *$K$  is a hypercovering of  $X$ ,*
- (2) *the morphism  $M_0 \rightarrow L_0$  is a covering, and*
- (3) *for all  $n \geq 0$  in the diagram*

$$\begin{array}{ccc}
 M_{n+1} & \xrightarrow{\quad} & (\text{cosk}_n \text{sk}_n M)_{n+1} \\
 \downarrow & \searrow \gamma & \uparrow \\
 & L_{n+1} \times (\text{cosk}_n \text{sk}_n L)_{n+1} & (\text{cosk}_n \text{sk}_n M)_{n+1} \\
 \downarrow & \swarrow & \downarrow \\
 L_{n+1} & \xrightarrow{\quad} & (\text{cosk}_n \text{sk}_n L)_{n+1}
 \end{array}$$

*the arrow  $\gamma$  is a covering.*

Then the fibre product  $K \times_L M$  is a hypercovering of  $X$ .

**Proof.** The morphism  $(K \times_L M)_0 = K_0 \times_{L_0} M_0 \rightarrow K_0$  is a base change of a covering by (2), hence a covering, see Lemma 2.5. And  $K_0 \rightarrow \{X \rightarrow X\}$  is a covering by (1). Thus  $(K \times_L M)_0 \rightarrow \{X \rightarrow X\}$  is a covering by Lemma 2.5. Hence  $K \times_L M$  satisfies the first condition of Definition 2.6.

We still have to check that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K \times_L M)_{n+1} \longrightarrow (\text{cosk}_n \text{sk}_n(K \times_L M))_{n+1}$$

is a covering for all  $n \geq 0$ . We abbreviate as follows:  $A = (\text{cosk}_n \text{sk}_n K)_{n+1}$ ,  $B = (\text{cosk}_n \text{sk}_n L)_{n+1}$ , and  $C = (\text{cosk}_n \text{sk}_n M)_{n+1}$ . The functor  $\text{cosk}_n \text{sk}_n$  commutes with fibre products, see Simplicial, Lemma 18.13. Thus the right hand side above is equal to  $A \times_B C$ . Consider the following commutative diagram

$$\begin{array}{ccccc} K_{n+1} \times_{L_{n+1}} M_{n+1} & \longrightarrow & M_{n+1} & & \\ \downarrow & & \downarrow & \searrow \gamma & \\ K_{n+1} & \longrightarrow & L_{n+1} & \longleftarrow & L_{n+1} \times_B C \longrightarrow C \\ & \searrow & & \swarrow & \downarrow \\ & & A & \longrightarrow & B \end{array}$$

This diagram shows that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1}$$

Now,  $K_{n+1} \times_B C \rightarrow A \times_B C$  is a base change of the covering  $K_{n+1} \rightarrow A$  via the morphism  $A \times_B C \rightarrow A$ , hence is a covering. By assumption (3) the morphism  $\gamma$  is a covering. Hence the morphism

$$(K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1} \longrightarrow K_{n+1} \times_B C$$

is a covering as a base change of a covering. The lemma follows as a composition of coverings is a covering.  $\square$

**Lemma 6.2.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . If  $K, L$  are hypercoverings of  $X$ , then  $K \times L$  is a hypercovering of  $X$ .*

**Proof.** You can either verify this directly, or use Lemma 6.1 above and check that  $L \rightarrow \{X \rightarrow X\}$  has property (3).  $\square$

Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Since the category  $\text{SR}(\mathcal{C}, X)$  has coproducts and finite limits, it is permissible to speak about the objects  $U \times K$  and  $\text{Hom}(U, K)$  for certain simplicial sets  $U$  (for example those with finitely many nondegenerate simplices) and any simplicial object  $K$  of  $\text{SR}(\mathcal{C}, X)$ . See Simplicial, Sections 13 and 16.

**Lemma 6.3.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $X$ . Let  $k \geq 0$  be an integer. Let  $u : Z \rightarrow K_k$  be a covering in  $\text{SR}(\mathcal{C}, X)$ . Then there exists a morphism of hypercoverings  $f : L \rightarrow K$  such that  $L_k \rightarrow K_k$  factors through  $u$ .*

**Proof.** Denote  $Y = K_k$ . There is a canonical morphism  $K \rightarrow \text{Hom}(\Delta[k], Y)$  corresponding to  $\text{id}_Y$  via *Simplicial*, Lemma 16.5. We will use the description of  $\text{Hom}(\Delta[k], Y)$  and  $\text{Hom}(\Delta[k], Z)$  given in that lemma. In particular there is a morphism  $\text{Hom}(\Delta[k], Y) \rightarrow \text{Hom}(\Delta[k], Z)$  which on degree  $n$  terms is the morphism

$$\prod_{\alpha:[k] \rightarrow [n]} Y \longrightarrow \prod_{\alpha:[k] \rightarrow [n]} Z.$$

Set

$$L = K \times_{\text{Hom}(\Delta[n], Y)} \text{Hom}(\Delta[n], Z).$$

The morphism  $L_k \rightarrow K_k$  sits in to a commutative diagram

$$\begin{array}{ccccc} L_k & \longrightarrow & \prod_{\alpha:[k] \rightarrow [n]} Y & \xrightarrow{\text{pr}_{\text{id}_{[k]}}} & Y \\ \downarrow & & \downarrow & & \downarrow \\ K_k & \longrightarrow & \prod_{\alpha:[k] \rightarrow [n]} Z & \xrightarrow{\text{pr}_{\text{id}_{[k]}}} & Z \end{array}$$

Since the composition of the two bottom arrows is the identity we conclude that we have the desired factorization.

We still have to show that  $L$  is a hypercovering of  $X$ . To see this we will use Lemma 6.1. Condition (1) is satisfied by assumption. For (2), the morphism

$$\text{Hom}(\Delta[k], Y)_0 \rightarrow \text{Hom}(\Delta[k], Z)_0$$

is a covering because it is a product of coverings, see Lemma 2.5. For (3) suppose first that  $n \geq 1$ . In this case by *Simplicial*, Lemma 20.12 we have  $\text{Hom}(\Delta[k], Y) = \text{cosk}_n \text{sk}_n \text{Hom}(\Delta[k], Y)$  and similarly for  $Z$ . Thus condition (3) for  $n > 0$  is clear. For  $n = 0$ , the diagram of condition (3) of Lemma 6.1 is, according to *Simplicial*, Lemma 20.13, the diagram

$$\begin{array}{ccc} \prod_{\alpha:[k] \rightarrow [1]} Z & \longrightarrow & Z \times Z \\ \downarrow & & \downarrow \\ \prod_{\alpha:[k] \rightarrow [1]} Y & \longrightarrow & Y \times Y \end{array}$$

with obvious horizontal arrows. Thus the morphism  $\gamma$  is the morphism

$$\prod_{\alpha:[k] \rightarrow [1]} Z \longrightarrow \prod_{\alpha:[k] \rightarrow [1] \text{ not onto}} Z \times \prod_{\alpha:[k] \rightarrow [1] \text{ onto}} Y$$

which is a product of coverings and hence a covering according to Lemma 6.1 once again.  $\square$

**Lemma 6.4.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $X$ . Let  $n \geq 0$  be an integer. Let  $u : \mathcal{F} \rightarrow F(K_n)$  be a morphism of presheaves which becomes surjective on sheafification. Then there exists a morphism of hypercoverings  $f : L \rightarrow K$  such that  $F(f_n) : F(L_n) \rightarrow F(K_n)$  factors through  $u$ .*

**Proof.** Write  $K_n = \{U_i \rightarrow X\}_{i \in I}$ . Thus the map  $u$  is a morphism of presheaves of sets  $u : \mathcal{F} \rightarrow \prod h_{u_i}$ . The assumption on  $u$  means that for every  $i \in I$  there exists a covering  $\{U_{ij} \rightarrow U_i\}_{j \in I_i}$  of the site  $\mathcal{C}$  and a morphism of presheaves  $t_{ij} : h_{U_{ij}} \rightarrow \mathcal{F}$  such that  $u \circ t_{ij}$  is the map  $h_{U_{ij}} \rightarrow h_{U_i}$  coming from the morphism  $U_{ij} \rightarrow U_i$ . Set  $J = \prod_{i \in I} I_i$ , and let  $\alpha : J \rightarrow I$  be the obvious map. For  $j \in J$  denote  $V_j = U_{\alpha(j)j}$ .

Set  $Z = \{V_j \rightarrow X\}_{j \in J}$ . Finally, consider the morphism  $u' : Z \rightarrow K_n$  given by  $\alpha : J \rightarrow I$  and the morphisms  $V_j = U_{\alpha(j)j} \rightarrow U_{\alpha(j)}$  above. Clearly, this is a covering in the category  $\text{SR}(\mathcal{C}, X)$ , and by construction  $F(u') : F(Z) \rightarrow F(K_n)$  factors through  $u$ . Thus the result follows from Lemma 6.3 above.  $\square$

## 7. Adding simplices

In this section we prove some technical lemmas which we will need later. Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . As we pointed out in Section 6 above, the objects  $U \times K$  and  $\text{Hom}(U, K)$  for certain simplicial sets  $U$  and any simplicial object  $K$  of  $\text{SR}(\mathcal{C}, X)$  are defined. See *Simplicial*, Sections 13 and 16.

**Lemma 7.1.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $X$ . Let  $U \subset V$  be simplicial sets, with  $U_n, V_n$  finite nonempty for all  $n$ . Assume that  $U$  has finitely many nondegenerate simplices. Suppose  $n \geq 0$  and  $x \in V_n$ ,  $x \notin U_n$  are such that*

- (1)  $V_i = U_i$  for  $i < n$ ,
- (2)  $V_n = U_n \cup \{x\}$ ,
- (3) any  $z \in V_j$ ,  $z \notin U_j$  for  $j > n$  is degenerate.

*Then the morphism*

$$\text{Hom}(V, K)_0 \longrightarrow \text{Hom}(U, K)_0$$

*of  $\text{SR}(\mathcal{C}, X)$  is a covering.*

**Proof.** If  $n = 0$ , then it follows easily that  $V = U \amalg \Delta[0]$  (see below). In this case  $\text{Hom}(V, K)_0 = \text{Hom}(U, K)_0 \times K_0$ . The result, in this case, then follows from Lemma 2.5.

Let  $a : \Delta[n] \rightarrow V$  be the morphism associated to  $x$  as in *Simplicial*, Lemma 11.3. Let us write  $\partial\Delta[n] = i_{(n-1)!}\text{sk}_{n-1}\Delta[n]$  for the  $(n-1)$ -skeleton of  $\Delta[n]$ . Let  $b : \partial\Delta[n] \rightarrow U$  be the restriction of  $a$  to the  $(n-1)$  skeleton of  $\Delta[n]$ . By *Simplicial*, Lemma 20.7 we have  $V = U \amalg_{\partial\Delta[n]} \Delta[n]$ . By *Simplicial*, Lemma 16.6 we get that

$$\begin{array}{ccc} \text{Hom}(V, K)_0 & \longrightarrow & \text{Hom}(U, K)_0 \\ \downarrow & & \downarrow \\ \text{Hom}(\Delta[n], K)_0 & \longrightarrow & \text{Hom}(\partial\Delta[n], K)_0 \end{array}$$

is a fibre product square. Thus it suffices to show that the bottom horizontal arrow is a covering. By *Simplicial*, Lemma 20.11 this arrow is identified with

$$K_n \rightarrow (\text{cosk}_{n-1}\text{sk}_{n-1}K)_n$$

and hence is a covering by definition of a hypercovering.  $\square$

**Lemma 7.2.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $K$  be a hypercovering of  $X$ . Let  $U \subset V$  be simplicial sets, with  $U_n, V_n$  finite nonempty for all  $n$ . Assume that  $U$  and  $V$  have finitely many nondegenerate simplices. Then the morphism*

$$\text{Hom}(V, K)_0 \longrightarrow \text{Hom}(U, K)_0$$

*of  $\text{SR}(\mathcal{C}, X)$  is a covering.*

**Proof.** By Lemma 7.1 above, it suffices to prove a simple lemma about inclusions of simplicial sets  $U \subset V$  as in the lemma. And this is exactly the result of *Simplicial*, Lemma 20.8.  $\square$

### 8. Homotopies

Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $L$  be a simplicial object of  $\text{SR}(\mathcal{C}, X)$ . According to *Simplicial*, Lemma 16.4 there exists an object  $\text{Hom}(\Delta[1], L)$  in the category  $\text{Simp}(\text{SR}(\mathcal{C}, X))$  which represents the functor

$$T \longmapsto \text{Mor}_{\text{Simp}(\text{SR}(\mathcal{C}, X))}(\Delta[1] \times T, L)$$

There is a canonical morphism

$$\text{Hom}(\Delta[1], L) \rightarrow L \times L$$

coming from  $e_i : \Delta[0] \rightarrow \Delta[1]$  and the identification  $\text{Hom}(\Delta[0], L) = L$ .

**Lemma 8.1.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $L$  be a simplicial object of  $\text{SR}(\mathcal{C}, X)$ . Let  $n \geq 0$ . Consider the commutative diagram*

$$(8.1.1) \quad \begin{array}{ccc} \text{Hom}(\Delta[1], L)_{n+1} & \longrightarrow & (\text{cosk}_n \text{sk}_n \text{Hom}(\Delta[1], L))_{n+1} \\ \downarrow & & \downarrow \\ (L \times L)_{n+1} & \longrightarrow & (\text{cosk}_n \text{sk}_n (L \times L))_{n+1} \end{array}$$

coming from the morphism defined above. We can identify the terms in this diagram as follows, where  $\partial\Delta[n+1] = i_{n!} \text{sk}_n \Delta[n+1]$  is the  $n$ -skeleton of the  $(n+1)$ -simplex:

$$\begin{aligned} \text{Hom}(\Delta[1], L)_{n+1} &= \text{Hom}(\Delta[1] \times \Delta[n+1], L)_0 \\ (\text{cosk}_n \text{sk}_n \text{Hom}(\Delta[1], L))_{n+1} &= \text{Hom}(\Delta[1] \times \partial\Delta[n+1], L)_0 \\ (L \times L)_{n+1} &= \text{Hom}((\Delta[n+1] \amalg \Delta[n+1]), L)_0 \\ (\text{cosk}_n \text{sk}_n (L \times L))_{n+1} &= \text{Hom}(\partial\Delta[n+1] \amalg \partial\Delta[n+1], L)_0 \end{aligned}$$

and the morphism between these objects of  $\text{SR}(\mathcal{C}, X)$  come from the commutative diagram of simplicial sets

$$(8.1.2) \quad \begin{array}{ccc} \Delta[1] \times \Delta[n+1] & \longleftarrow & \Delta[1] \times \partial\Delta[n+1] \\ \uparrow & & \uparrow \\ \Delta[n+1] \amalg \Delta[n+1] & \longleftarrow & \partial\Delta[n+1] \amalg \partial\Delta[n+1] \end{array}$$

Moreover the fibre product of the bottom arrow and the right arrow in (8.1.1) is equal to

$$\text{Hom}(U, L)_0$$

where  $U \subset \Delta[1] \times \Delta[n+1]$  is the smallest simplicial subset such that both  $\Delta[n+1] \amalg \Delta[n+1]$  and  $\Delta[1] \times \partial\Delta[n+1]$  map into it.

**Proof.** The first and third equalities are *Simplicial*, Lemma 16.4. The second and fourth follow from the cited lemma combined with *Simplicial*, Lemma 20.11. The last assertion follows from the fact that  $U$  is the push-out of the bottom and right arrow of the diagram (8.1.2), via *Simplicial*, Lemma 16.6. To see that  $U$  is equal to this push-out it suffices to see that the intersection of  $\Delta[n+1] \amalg \Delta[n+1]$  and  $\Delta[1] \times \partial\Delta[n+1]$  in  $\Delta[1] \times \Delta[n+1]$  is equal to  $\partial\Delta[n+1] \amalg \partial\Delta[n+1]$ . This we leave to the reader.  $\square$



**Lemma 8.2.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $K, L$  be hypercoverings of  $X$ . Let  $a, b : K \rightarrow L$  be morphisms of hypercoverings. There exists a morphism of hypercoverings  $c : K' \rightarrow K$  such that  $a \circ c$  is homotopic to  $b \circ c$ .*

**Proof.** Consider the following commutative diagram

$$\begin{array}{ccccc}
 K' & \xrightarrow{\text{def}} & K \times_{(L \times L)} \text{Hom}(\Delta[1], L) & \longrightarrow & \text{Hom}(\Delta[1], L) \\
 & \searrow c & \downarrow & & \downarrow \\
 & & K & \xrightarrow{(a, b)} & L \times L
 \end{array}$$

By the functorial property of  $\text{Hom}(\Delta[1], L)$  the composition of the horizontal morphisms corresponds to a morphism  $K' \Delta[1] \rightarrow L$  which defines a homotopy between  $c \circ a$  and  $c \circ b$ . Thus if we can show that  $K'$  is a hypercovering of  $X$ , then we obtain the lemma. To see this we will apply Lemma 6.1 to the pair of morphisms  $K \rightarrow L \times L$  and  $\text{Hom}(\Delta[1], L) \rightarrow L \times L$ . Condition (1) of Lemma 6.1 is satisfied. Condition (2) of Lemma 6.1 is true because  $\text{Hom}(\Delta[1], L)_0 = L_1$ , and the morphism  $(d_0^1, d_1^1) : L_1 \rightarrow L_0 \times L_0$  is a covering of  $\text{SR}(\mathcal{C}, X)$  by our assumption that  $L$  is a hypercovering. To prove condition (3) of Lemma 6.1 we use Lemma 8.1 above. According to this lemma the morphism  $\gamma$  of condition (3) of Lemma 6.1 is the morphism

$$\text{Hom}(\Delta[1] \times \Delta[n+1], L)_0 \longrightarrow \text{Hom}(U, L)_0$$

where  $U \subset \Delta[1] \times \Delta[n+1]$ . According to Lemma 7.2 this is a covering and hence the claim has been proven.  $\square$

**Remark 8.3.** Note that the crux of the proof is to use Lemma 7.2. This lemma is completely general and does not care about the exact shape of the simplicial sets (as long as they have only finitely many nondegenerate simplices). It seems altogether reasonable to expect a result of the following kind: Given any morphism  $a : K \times \partial\Delta[k] \rightarrow L$ , with  $K$  and  $L$  hypercoverings, there exists a morphism of hypercoverings  $c : K' \rightarrow K$  and a morphism  $g : K' \times \Delta[k] \rightarrow L$  such that  $g|_{K' \times \partial\Delta[k]} = a \circ (c \times \text{id}_{\partial\Delta[k]})$ . In other words, the category of hypercoverings is in a suitable sense contractible.

## 9. Cohomology and hypercoverings

Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $\mathcal{F}$  be a sheaf of abelian groups on  $\mathcal{C}$ . Let  $K, L$  be hypercoverings of  $X$ . If  $a, b : K \rightarrow L$  are homotopic maps, then  $\mathcal{F}(a), \mathcal{F}(b) : \mathcal{F}(K) \rightarrow \mathcal{F}(L)$  are homotopic maps, see *Simplicial*, Lemma 27.3. Hence have the same effect on cohomology groups of the associated cochain complexes, see *Simplicial*, Lemma 27.5. We are going to use this to define the colimit over all hypercoverings.

Let us temporarily denote  $\text{HC}(\mathcal{C}, X)$  the category of hypercoverings of  $X$ . We have seen that this is a category and not a “big” category, see Lemma 2.8. This will be the index category for our diagram, see *Categories*, Section 14 for notation. Consider the diagram

$$\check{H}^i(-, \mathcal{F}) : \text{HC}(\mathcal{C}, X) \longrightarrow \text{Ab}.$$

By Lemma 6.2 and Lemma 8.2, and the remark on homotopies above, this diagram is directed, see Categories, Definition 19.1. Thus the colimit

$$\check{H}_{\mathrm{HC}}^i(X, \mathcal{F}) = \operatorname{colim}_{K \in \mathrm{HC}(\mathcal{C}, X)} \check{H}^i(K, \mathcal{F})$$

has a particularly simple description (see location cited).

**Theorem 9.1.** *Let  $\mathcal{C}$  be a site with fibre products. Let  $X$  be an object of  $\mathcal{C}$ . Let  $i \geq 0$ . The functors*

$$\begin{aligned} \operatorname{Ab}(\mathcal{C}) &\longrightarrow \operatorname{Ab} \\ \mathcal{F} &\longmapsto H^i(X, \mathcal{F}) \\ \mathcal{F} &\longmapsto \check{H}_{\mathrm{HC}}^i(X, \mathcal{F}) \end{aligned}$$

*are canonically isomorphic.*

**Proof using spectral sequences.** Suppose that  $\xi \in H^p(X, \mathcal{F})$  for some  $p \geq 0$ . Let us show that  $\xi$  is in the image of the map  $\check{H}^p(X, \mathcal{F}) \rightarrow H^p(X, \mathcal{F})$  of Lemma 4.3 for some hypercovering  $K$  of  $X$ .

This is true if  $p = 0$  by Lemma 4.1. If  $p = 1$ , choose a Čech hypercovering  $K$  of  $X$  as in Example 2.7 starting with a covering  $K_0 = \{U_i \rightarrow X\}$  in the site  $\mathcal{C}$  such that  $\xi|_{U_i} = 0$ , see Cohomology on Sites, Lemma 8.3. It follows immediately from the spectral sequence in Lemma 4.3 that  $\xi$  comes from an element of  $\check{H}^1(K, \mathcal{F})$  in this case. In general, choose any hypercovering  $K$  of  $X$  such that  $\xi$  maps to zero in  $\underline{H}^p(\mathcal{F})(K_0)$  (using Example 2.7 and Cohomology on Sites, Lemma 8.3 again). By the spectral sequence of Lemma 4.3 the obstruction for  $\xi$  to come from an element of  $\check{H}^p(K, \mathcal{F})$  is a sequence of elements  $\xi_1, \dots, \xi_{p-1}$  with  $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathcal{F}))$  (more precisely the images of the  $\xi_q$  in certain subquotients of these groups).

We can inductively replace the hypercovering  $K$  by refinements such that the obstructions  $\xi_1, \dots, \xi_{p-1}$  restrict to zero (and not just the images in the subquotients – so no subtlety here). Indeed, suppose we have already managed to reach the situation where  $\xi_{q+1}, \dots, \xi_{p-1}$  are zero. Note that  $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathcal{F}))$  is the class of some element

$$\tilde{\xi}_q \in \underline{H}^q(\mathcal{F})(K_{p-q}) = \prod H^q(U_i, \mathcal{F})$$

if  $K_{p-q} = \{U_i \rightarrow X\}_{i \in I}$ . Let  $\xi_{q,i}$  be the component of  $\tilde{\xi}_q$  in  $H^q(U_i, \mathcal{F})$ . As  $q \geq 1$  we can use Cohomology on Sites, Lemma 8.3 yet again to choose coverings  $\{U_{i,j} \rightarrow U_i\}$  of the site such that each restriction  $\xi_{q,i}|_{U_{i,j}} = 0$ . Consider the object  $Z = \{U_{i,j} \rightarrow X\}$  of the category  $\mathrm{SR}(\mathcal{C}, X)$  and its obvious morphism  $u : Z \rightarrow K_{p-q}$ . It is clear that  $u$  is a covering, see Definition 2.4. By Lemma 6.3 there exists a morphism  $L \rightarrow K$  of hypercoverings of  $X$  such that  $L_{p-q} \rightarrow K_{p-q}$  factors through  $u$ . Then clearly the image of  $\xi_q$  in  $\underline{H}^q(\mathcal{F})(L_{p-q})$  is zero. Since the spectral sequence of Lemma 4.3 is functorial this means that after replacing  $K$  by  $L$  we reach the situation where  $\xi_q, \dots, \xi_{p-1}$  are all zero. Continuing like this we end up with a hypercovering where they are all zero and hence  $\xi$  is in the image of the map  $\check{H}^p(X, \mathcal{F}) \rightarrow H^p(X, \mathcal{F})$ .

Suppose that  $K$  is a hypercovering of  $X$ , that  $\xi \in \check{H}^p(K, \mathcal{F})$  and that the image of  $\xi$  under the map  $\check{H}^p(X, \mathcal{F}) \rightarrow H^p(X, \mathcal{F})$  of Lemma 4.3 is zero. To finish the proof of the theorem we have to show that there exists a morphism of hypercoverings  $L \rightarrow K$  such that  $\xi$  restricts to zero in  $\check{H}^p(L, \mathcal{F})$ . By the spectral sequence of Lemma 4.3 the vanishing of the image of  $\xi$  in  $H^p(X, \mathcal{F})$  means that there exist

elements  $\xi_1, \dots, \xi_{p-2}$  with  $\xi_q \in \check{H}^{p-1-q}(K, \underline{H}^q(\mathcal{F}))$  (more precisely the images of these in certain subquotients) such that the images  $d_{q+1}^{p-1-q, q} \xi_q$  (in the spectral sequence) add up to  $\xi$ . Hence by exactly the same mechanism as above we can find a morphism of hypercoverings  $L \rightarrow K$  such that the restrictions of the elements  $\xi_q$ ,  $q = 1, \dots, p-2$  in  $\check{H}^{p-1-q}(L, \underline{H}^q(\mathcal{F}))$  are zero. Then it follows that  $\xi$  is zero since the morphism  $L \rightarrow K$  induces a morphism of spectral sequences according to Lemma 4.3.  $\square$

**Proof without using spectral sequences.** We have seen the result for  $i = 0$ , see Lemma 4.1. We know that the functors  $H^i(X, -)$  form a universal  $\delta$ -functor, see Derived Categories, Lemma 20.4. In order to prove the theorem it suffices to show that the sequence of functors  $\check{H}_{HC}^i(X, -)$  forms a  $\delta$ -functor. Namely we know that Čech cohomology is zero on injective sheaves (Lemma 4.2) and then we can apply Homology, Lemma 11.4.

Let

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

be a short exact sequence of abelian sheaves on  $\mathcal{C}$ . Let  $\xi \in \check{H}_{HC}^p(X, \mathcal{H})$ . Choose a hypercovering  $K$  of  $X$  and an element  $\sigma \in \mathcal{H}(K_p)$  representing  $\xi$  in cohomology. There is a corresponding exact sequence of complexes

$$0 \rightarrow s(\mathcal{F}(K)) \rightarrow s(\mathcal{G}(K)) \rightarrow s(\mathcal{H}(K))$$

but we are not assured that there is a zero on the right also and this is the only thing that prevents us from defining  $\delta(\xi)$  by a simple application of the snake lemma. Recall that

$$\mathcal{H}(K_p) = \prod \mathcal{H}(U_i)$$

if  $K_p = \{U_i \rightarrow X\}$ . Let  $\sigma = \prod \sigma_i$  with  $\sigma_i \in \mathcal{H}(U_i)$ . Since  $\mathcal{G} \rightarrow \mathcal{H}$  is a surjection of sheaves we see that there exist coverings  $\{U_{i,j} \rightarrow U_i\}$  such that  $\sigma_i|_{U_{i,j}}$  is the image of some element  $\tau_{i,j} \in \mathcal{G}(U_{i,j})$ . Consider the object  $Z = \{U_{i,j} \rightarrow X\}$  of the category  $\text{SR}(\mathcal{C}, X)$  and its obvious morphism  $u : Z \rightarrow K_p$ . It is clear that  $u$  is a covering, see Definition 2.4. By Lemma 6.3 there exists a morphism  $L \rightarrow K$  of hypercoverings of  $X$  such that  $L_p \rightarrow K_p$  factors through  $u$ . After replacing  $K$  by  $L$  we may therefore assume that  $\sigma$  is the image of an element  $\tau \in \mathcal{G}(K_p)$ . Note that  $d(\sigma) = 0$ , but not necessarily  $d(\tau) = 0$ . Thus  $d(\tau) \in \mathcal{F}(K_{p+1})$  is a cocycle. In this situation we define  $\delta(\xi)$  as the class of the cocycle  $d(\tau)$  in  $\check{H}_{HC}^{p+1}(X, \mathcal{F})$ .

At this point there are several things to verify: (a)  $\delta(\xi)$  does not depend on the choice of  $\tau$ , (b)  $\delta(\xi)$  does not depend on the choice of the hypercovering  $L \rightarrow K$  such that  $\sigma$  lifts, and (c)  $\delta(\xi)$  does not depend on the initial hypercovering and  $\sigma$  chosen to represent  $\xi$ . We omit the verification of (a), (b), and (c); the independence of the choices of the hypercoverings really comes down to Lemmas 6.2 and 8.2. We also omit the verification that  $\delta$  is functorial with respect to morphisms of short exact sequences of abelian sheaves on  $\mathcal{C}$ .

Finally, we have to verify that with this definition of  $\delta$  our short exact sequence of abelian sheaves above leads to a long exact sequence of Čech cohomology groups. First we show that if  $\delta(\xi) = 0$  (with  $\xi$  as above) then  $\xi$  is the image of some element  $\xi' \in \check{H}_{HC}^p(X, \mathcal{G})$ . Namely, if  $\delta(\xi) = 0$ , then, with notation as above, we see that the class of  $d(\tau)$  is zero in  $\check{H}_{HC}^{p+1}(X, \mathcal{F})$ . Hence there exists a morphism of hypercoverings  $L \rightarrow K$  such that the restriction of  $d(\tau)$  to an element of  $\mathcal{F}(L_{p+1})$

is equal to  $d(v)$  for some  $v \in \mathcal{F}(L_p)$ . This implies that  $\tau|_{L_p} + v$  form a cocycle, and determine a class  $\xi' \in \check{H}^p(L, \mathcal{G})$  which maps to  $\xi$  as desired.

We omit the proof that if  $\xi' \in \check{H}_{HC}^{p+1}(X, \mathcal{F})$  maps to zero in  $\check{H}_{HC}^{p+1}(X, \mathcal{G})$ , then it is equal to  $\delta(\xi)$  for some  $\xi \in \check{H}_{HC}^p(X, \mathcal{H})$ .  $\square$

Next, we deduce Verdier's case of Theorem 9.1 by a sleight of hand.

**Proposition 9.2.** *Let  $\mathcal{C}$  be a site with fibre products and products of pairs. Let  $\mathcal{F}$  be an abelian sheaf on  $\mathcal{C}$ . Let  $i \geq 0$ . Then*

- (1) *for every  $\xi \in H^i(\mathcal{F})$  there exists a hypercovering  $K$  such that  $\xi$  is in the image of the canonical map  $\check{H}^i(K, \mathcal{F}) \rightarrow H^i(\mathcal{F})$ , and*
- (2) *if  $K, L$  are hypercoverings and  $\xi_K \in \check{H}^i(K, \mathcal{F})$ ,  $\xi_L \in \check{H}^i(L, \mathcal{F})$  are elements mapping to the same element of  $H^i(\mathcal{F})$ , then there exists a hypercovering  $M$  and morphisms  $M \rightarrow K$  and  $M \rightarrow L$  such that  $\xi_K$  and  $\xi_L$  map to the same element of  $\check{H}^i(M, \mathcal{F})$ .*

*In other words, modulo set theoretical issues, the cohomology groups of  $\mathcal{F}$  on  $\mathcal{C}$  are the colimit of the Čech cohomology groups of  $\mathcal{F}$  over all hypercoverings.*

**Proof.** This result is a trivial consequence of Theorem 9.1. Namely, we can artificially replace  $\mathcal{C}$  with a slightly bigger site  $\mathcal{C}'$  such that (I)  $\mathcal{C}'$  has a final object  $X$  and (II) hypercoverings in  $\mathcal{C}$  are more or less the same thing as hypercoverings of  $X$  in  $\mathcal{C}'$ . But due to the nature of things, there is quite a bit of bookkeeping to do.

Let us call a family of morphisms  $\{U_i \rightarrow U\}$  in  $\mathcal{C}$  with fixed target a *weak covering* if the sheafification of the map  $\prod_{i \in I} h_{U_i} \rightarrow h_U$  becomes surjective. We construct a new site  $\mathcal{C}'$  as follows

- (1) as a category set  $\text{Ob}(\mathcal{C}') = \text{Ob}(\mathcal{C}) \amalg \{X\}$  and add a unique morphism to  $X$  from every object of  $\mathcal{C}'$ ,
- (2)  $\mathcal{C}'$  has fibre products as fibre products and products of pairs exist in  $\mathcal{C}$ ,
- (3) coverings of  $\mathcal{C}'$  are weak coverings of  $\mathcal{C}$  together with those  $\{U_i \rightarrow X\}_{i \in I}$  such that either  $U_i = X$  for some  $i$ , or  $U_i \neq X$  for all  $i$  and the map  $\prod h_{U_i} \rightarrow *$  of presheaves on  $\mathcal{C}$  becomes surjective after sheafification on  $\mathcal{C}$ ,
- (4) we apply Sets, Lemma 11.1 to restrict the coverings to obtain our site  $\mathcal{C}'$ .

Then  $Sh(\mathcal{C}') = Sh(\mathcal{C})$  because the inclusion functor  $\mathcal{C} \rightarrow \mathcal{C}'$  is a special cocontinuous functor (see Sites, Definition 28.2). We omit the straightforward verifications.

Choose a covering  $\{U_i \rightarrow X\}$  of  $\mathcal{C}'$  such that  $U_i$  is an object of  $\mathcal{C}$  for all  $i$  (possible because  $\mathcal{C} \rightarrow \mathcal{C}'$  is special cocontinuous). Then  $K_0 = \{U_i \rightarrow X\}$  is a covering in the site  $\mathcal{C}'$  constructed above. We view  $K_0$  as an object of  $\text{SR}(\mathcal{C}', X)$  and we set  $K_{\text{init}} = \text{cosk}_0(K_0)$ . Then  $K_{\text{init}}$  is a hypercovering of  $X$ , see Example 2.7. Note that every  $K_{\text{init}, n}$  has the shape  $\{W_j \rightarrow X\}$  with  $W_j \in \text{Ob}(\mathcal{C})$ .

Proof of (1). Choose  $\xi \in H^i(\mathcal{F}) = H^i(X, \mathcal{F}')$  where  $\mathcal{F}'$  is the abelian sheaf on  $\mathcal{C}'$  corresponding to  $\mathcal{F}$  on  $\mathcal{C}$ . By Theorem 9.1 there exists a morphism of hypercoverings  $K' \rightarrow K_{\text{init}}$  of  $X$  in  $\mathcal{C}'$  such that  $\xi$  comes from an element of  $\check{H}^i(K', \mathcal{F})$ . Write  $K'_n = \{U_{n,j} \rightarrow X\}$ . Now since  $K'_n$  maps to  $K_{\text{init}, n}$  we see that  $U_{n,j}$  is an object of  $\mathcal{C}$ . Hence we can define a simplicial object  $K$  of  $\text{SR}(\mathcal{C})$  by setting  $K_n = \{U_{n,j}\}$ . Since coverings in  $\mathcal{C}'$  consisting of families of morphisms of  $\mathcal{C}$  are weak coverings, we see that  $K$  is a hypercovering in the sense of Definition 5.1. Finally, since  $\mathcal{F}'$  is the unique sheaf on  $\mathcal{C}'$  whose restriction to  $\mathcal{C}$  is equal to  $\mathcal{F}$  we see that the Čech

complexes  $s(\mathcal{F}(K))$  and  $s(\mathcal{F}'(K'))$  are identical and (1) follows. (Compatibility with map into cohomology groups omitted.)

Proof of (2). Let  $K$  and  $L$  be hypercoverings in  $\mathcal{C}$ . Let  $K'$  and  $L'$  be the simplicial objects of  $\mathrm{SR}(\mathcal{C}', X)$  gotten from  $K$  and  $L$  by the functor  $\mathrm{SR}(\mathcal{C}) \rightarrow \mathrm{SR}(\mathcal{C}', X)$ ,  $\{U_i\} \mapsto \{U_i \rightarrow X\}$ . As before we have equality of Čech complexes and hence we obtain  $\xi_{K'}$  and  $\xi_{L'}$  mapping to the same cohomology class of  $\mathcal{F}'$  over  $\mathcal{C}'$ . After possibly enlarging our choice of coverings in  $\mathcal{C}'$  (due to a set theoretical issue) we may assume that  $K'$  and  $L'$  are hypercoverings of  $X$  in  $\mathcal{C}'$ ; this is true by our definition of hypercoverings in Definition 5.1 and the fact that weak coverings in  $\mathcal{C}$  give coverings in  $\mathcal{C}'$ . By Theorem 9.1 there exists a hypercovering  $M'$  of  $X$  in  $\mathcal{C}'$  and morphisms  $M' \rightarrow K'$ ,  $M' \rightarrow L'$ , and  $M' \rightarrow K_{init}$  such that  $\xi_{K'}$  and  $\xi_{L'}$  restrict to the same element of  $\check{H}^i(M', \mathcal{F})$ . Unwinding this statement as above we find that (2) is true.  $\square$

## 10. Hypercoverings of spaces

The theory above is mildly interesting even in the case of topological spaces. In this case we can work out what a hypercovering is and see what the result actually says.

Let  $X$  be a topological space. Consider the site  $X_{Zar}$  of Sites, Example 6.4. Recall that an object of  $X_{Zar}$  is simply an open of  $X$  and that morphisms of  $X_{Zar}$  correspond simply to inclusions. So what is a hypercovering of  $X$  for the site  $X_{Zar}$ ?

Let us first unwind Definition 2.1. An object of  $\mathrm{SR}(X_{Zar}, X)$  is simply given by a set  $I$  and for each  $i \in I$  an open  $U_i \subset X$ . Let us denote this by  $\{U_i\}_{i \in I}$  since there can be no confusion about the morphism  $U_i \rightarrow X$ . A morphism  $\{U_i\}_{i \in I} \rightarrow \{V_j\}_{j \in J}$  between two such objects is given by a map of sets  $\alpha : I \rightarrow J$  such that  $U_i \subset V_{\alpha(i)}$  for all  $i \in I$ . When is such a morphism a covering? This is the case if and only if for every  $j \in J$  we have  $V_j = \bigcup_{i \in I, \alpha(i)=j} U_i$  (and is a covering in the site  $X_{Zar}$ ).

Using the above we get the following description of a hypercovering in the site  $X_{Zar}$ . A hypercovering of  $X$  in  $X_{Zar}$  is given by the following data

- (1) a simplicial set  $I$  (see Simplicial, Section 11), and
- (2) for each  $n \geq 0$  and every  $i \in I_n$  an open set  $U_i \subset X$ .

We will denote such a collection of data by the notation  $(I, \{U_i\})$ . In order for this to be a hypercovering of  $X$  we require the following properties

- for  $i \in I_n$  and  $0 \leq a \leq n$  we have  $U_i \subset U_{d_a^n(i)}$ ,
- for  $i \in I_n$  and  $0 \leq a \leq n$  we have  $U_i = U_{s_a^n(i)}$ ,
- we have

$$(10.0.1) \quad X = \bigcup_{i \in I_0} U_i,$$

- for every  $i_0, i_1 \in I_0$ , we have

$$(10.0.2) \quad U_{i_0} \cap U_{i_1} = \bigcup_{i \in I_1, d_0^1(i)=i_0, d_1^1(i)=i_1} U_i,$$

- for every  $n \geq 1$  and every  $(i_0, \dots, i_{n+1}) \in (I_n)^{n+2}$  such that  $d_{b-1}^n(i_a) = d_a^n(i_b)$  for all  $0 \leq a < b \leq n+1$  we have

$$(10.0.3) \quad U_{i_0} \cap \dots \cap U_{i_{n+1}} = \bigcup_{i \in I_{n+1}, d_a^{n+1}(i)=i_a, a=0, \dots, n+1} U_i,$$

- each of the open coverings (10.0.1), (10.0.2), and (10.0.3) is an element of  $\text{Cov}(X_{Zar})$  (this is a set theoretic condition, bounding the size of the index sets of the coverings).

Conditions (10.0.1) and (10.0.2) should be familiar from the chapter on sheaves on spaces for example, and condition (10.0.3) is the natural generalization.

**Remark 10.1.** One feature of this description is that if one of the multiple intersections  $U_{i_0} \cap \dots \cap U_{i_{n+1}}$  is empty then the covering on the right hand side may be the empty covering. Thus it is not automatically the case that the maps  $I_{n+1} \rightarrow (\text{cosk}_n \text{sk}_n I)_{n+1}$  are surjective. This means that the geometric realization of  $I$  may be an interesting (non-contractible) space.

In fact, let  $I'_n \subset I_n$  be the subset consisting of those simplices  $i \in I_n$  such that  $U_i \neq \emptyset$ . It is easy to see that  $I' \subset I$  is a subsimplicial set, and that  $(I', \{U_i\})$  is a hypercovering. Hence we can always refine a hypercovering to a hypercovering where none of the opens  $U_i$  is empty.

**Remark 10.2.** Let us repackage this information in yet another way. Namely, suppose that  $(I, \{U_i\})$  is a hypercovering of the topological space  $X$ . Given this data we can construct a simplicial topological space  $U_\bullet$  by setting

$$U_n = \coprod_{i \in I_n} U_i,$$

and where for given  $\varphi : [n] \rightarrow [m]$  we let morphisms  $U(\varphi) : U_n \rightarrow U_m$  be the morphism coming from the inclusions  $U_i \subset U_{\varphi(i)}$  for  $i \in I_n$ . This simplicial topological space comes with an augmentation  $\epsilon : U_\bullet \rightarrow X$ . With this morphism the simplicial space  $U_\bullet$  becomes a hypercovering of  $X$  along which one has cohomological descent in the sense of [AGV71, Exposé Vbis]. In other words,  $H^n(U_\bullet, \epsilon^* \mathcal{F}) = H^n(X, \mathcal{F})$ . (Insert future reference here to cohomology over simplicial spaces and cohomological descent formulated in those terms.) Suppose that  $\mathcal{F}$  is an abelian sheaf on  $X$ . In this case the spectral sequence of Lemma 4.3 becomes the spectral sequence with  $E_1$ -term

$$E_1^{p,q} = H^q(U_p, \epsilon_q^* \mathcal{F}) \Rightarrow H^{p+q}(U_\bullet, \epsilon^* \mathcal{F}) = H^{p+q}(X, \mathcal{F})$$

comparing the total cohomology of  $\epsilon^* \mathcal{F}$  to the cohomology groups of  $\mathcal{F}$  over the pieces of  $U_\bullet$ . (Insert future reference to this spectral sequence here.)

In topology we often want to find hypercoverings of  $X$  which have the property that all the  $U_i$  come from a given basis for the topology of  $X$  and that all the coverings (10.0.2) and (10.0.3) are from a given cofinal collection of coverings. Here are two example lemmas.

**Lemma 10.3.** *Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology of  $X$ . There exists a hypercovering  $(I, \{U_i\})$  of  $X$  such that each  $U_i$  is an element of  $\mathcal{B}$ .*

**Proof.** Let  $n \geq 0$ . Let us say that an  $n$ -truncated hypercovering of  $X$  is given by an  $n$ -truncated simplicial set  $I$  and for each  $i \in I_a$ ,  $0 \leq a \leq n$  an open  $U_i$  of  $X$  such that the conditions defining a hypercovering hold whenever they make sense. In other words we require the inclusion relations and covering conditions only when all simplices that occur in them are  $a$ -simplices with  $a \leq n$ . The lemma follows if we can prove that given a  $n$ -truncated hypercovering  $(I, \{U_i\})$  with all  $U_i \in \mathcal{B}$  we can extend it to an  $(n+1)$ -truncated hypercovering without adding any  $a$ -simplices

for  $a \leq n$ . This we do as follows. First we consider the  $(n+1)$ -truncated simplicial set  $I'$  defined by  $I' = \text{sk}_{n+1}(\text{cosk}_n I)$ . Recall that

$$I'_{n+1} = \left\{ (i_0, \dots, i_{n+1}) \in (I_n)^{n+2} \text{ such that } d_{b-1}^n(i_a) = d_a^n(i_b) \text{ for all } 0 \leq a < b \leq n+1 \right\}$$

If  $i' \in I'_{n+1}$  is degenerate, say  $i' = s_a^n(i)$  then we set  $U_{i'} = U_i$  (this is forced on us anyway by the second condition). We also set  $J_{i'} = \{i'\}$  in this case. If  $i' \in I'_{n+1}$  is nondegenerate, say  $i' = (i_0, \dots, i_{n+1})$ , then we choose a set  $J_{i'}$  and an open covering

$$(10.3.1) \quad U_{i_0} \cap \dots \cap U_{i_{n+1}} = \bigcup_{i \in J_{i'}} U_i,$$

with  $U_i \in \mathcal{B}$  for  $i \in J_{i'}$ . Set

$$I_{n+1} = \coprod_{i' \in I'_{n+1}} J_{i'}$$

There is a canonical map  $\pi : I_{n+1} \rightarrow I'_{n+1}$  which is a bijection over the set of degenerate simplices in  $I'_{n+1}$  by construction. For  $i \in I_{n+1}$  we define  $d_a^{n+1}(i) = d_a^{n+1}(\pi(i))$ . For  $i \in I_n$  we define  $s_a^n(i) \in I_{n+1}$  as the unique simplex lying over the degenerate simplex  $s_a^n(i) \in I'_{n+1}$ . We omit the verification that this defines an  $(n+1)$ -truncated hypercovering of  $X$ .  $\square$

**Lemma 10.4.** *Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology of  $X$ . Assume that*

- (1)  $X$  is quasi-compact,
- (2) each  $U \in \mathcal{B}$  is quasi-compact open, and
- (3) the intersection of any two quasi-compact opens in  $X$  is quasi-compact.

*Then there exists a hypercovering  $(I, \{U_i\})$  of  $X$  with the following properties*

- (1) each  $U_i$  is an element of the basis  $\mathcal{B}$ ,
- (2) each of the  $I_n$  is a finite set, and in particular
- (3) each of the coverings (10.0.1), (10.0.2), and (10.0.3) is finite.

**Proof.** This follows directly from the construction in the proof of Lemma 10.3 if we choose finite coverings by elements of  $\mathcal{B}$  in (10.3.1). Details omitted.  $\square$

## 11. Hypercoverings and weakly contractible objects

In this section we construct hypercoverings in the presence of enough weakly contractible objects (Sites, Definition 39.2). With our conventions this is particularly straightforward if every object has a covering by a single weakly contractible object.

**Lemma 11.1.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  be a subset. Assume*

- (1)  $\mathcal{C}$  has fibre products,
- (2) for all  $X \in \text{Ob}(\mathcal{C})$  there exists a covering  $\{U \rightarrow X\}$  with  $U \in \mathcal{B}$ ,
- (3) every element of  $\mathcal{B}$  is weakly contractible,
- (4) the topology is subcanonical.

*Then for every  $X$  there exists a hypercovering  $K$  of  $X$  such that each  $K_n = \{U_n \rightarrow X\}$  with  $U_n \in \mathcal{B}$ .*

**Proof.** We will construct  $K$  by induction. As a first approximation choose a covering  $\{U_0 \rightarrow X\}$  with  $U_0 \in \mathcal{B}$  and set  $K_0 = \{U_0 \rightarrow X\}$  and  $K = \text{cosk}_0 K_0$ , see Example 2.7. (This object will be denoted  $K^0$  in the final paragraph of the proof.)

Suppose for some  $n \geq 0$  we have constructed a hypercovering  $K$  such that  $K_k$  consists of a single object of  $\mathcal{B}$  mapping to  $X$  for  $0 \leq k \leq n$  and such that  $K = \text{cosk}_n \text{sk}_n K$ . In particular  $K_{n+1} = (\text{cosk}_n \text{sk}_n K)_{n+1}$  is a finite limit of  $K_k$  for  $k \leq n$ , see Simplicial, Section 12 (especially Lemmas 18.2 and 18.5). By the description of finite limits in  $\text{SR}(\mathcal{C}, X)$  (see proof of Lemma 2.3) we see that  $K_{n+1} = \{X_{n+1} \rightarrow X\}$  for some object  $X_{n+1}$  of  $\mathcal{C}$ . Choose a covering  $\{U_{n+1} \rightarrow X_{n+1}\}$  with  $U_{n+1} \in \mathcal{B}$ . Since  $U_n$  is weakly contractible, the topology is subcanonical, and  $\{U_{n+1} \rightarrow X_{n+1}\}$  is a covering, the morphisms  $s_i : U_n \rightarrow X_{n+1}$  lift to morphisms  $s'_i : U_n \rightarrow U_{n+1}$ . Set  $d'_j : U_{n+1} \rightarrow U_n$  equal to the composition of  $U_{n+1} \rightarrow X_{n+1}$  and  $d_j : X_{n+1} \rightarrow U_n$ . We obtain a truncated simplicial object  $K'$  of  $\text{SR}(\mathcal{C}, X)$  by setting  $K'_k = K_k$  for  $k \leq n$  and  $K'_{n+1} = U_{n+1}$  and morphisms  $d'_i = d_i$  and  $s'_i = s_i$  in degrees  $k \leq n-1$  and using the morphisms  $d'_j$  and  $s'_i$  in degree  $n$ . Extend this to a full simplicial object  $K'$  of  $\text{SR}(\mathcal{C}, X)$  using  $\text{cosk}_{n+1}$ . By functoriality of the coskeleton functors there is a morphism  $K' \rightarrow K$  of simplicial objects extending the given morphism of  $(n+1)$ -truncated simplicial objects.

It is immediately clear from the construction that the simplicial object  $K'$  so constructed is a hypercovering of  $X$ . Moreover, note that  $K' \rightarrow K$  is the identity morphism in degrees  $\leq n$ .

To finish the proof we take the inverse limit  $K = \lim K^n$  of the sequence of simplicial objects

$$\dots \rightarrow K^2 \rightarrow K^1 \rightarrow K^0$$

constructed above. Since we have stabilization in each degree it is clear that  $K$  agrees with  $K^n$  in degrees  $\leq n$  and therefore is a hypercovering of  $X$ .  $\square$

## 12. Other chapters

### Preliminaries

- |                          |                                  |
|--------------------------|----------------------------------|
| (1) Introduction         | (18) Modules on Sites            |
| (2) Conventions          | (19) Injectives                  |
| (3) Set Theory           | (20) Cohomology of Sheaves       |
| (4) Categories           | (21) Cohomology on Sites         |
| (5) Topology             | (22) Differential Graded Algebra |
| (6) Sheaves on Spaces    | (23) Divided Power Algebra       |
| (7) Sites and Sheaves    | (24) Hypercoverings              |
| (8) Stacks               |                                  |
| (9) Fields               | Schemes                          |
| (10) Commutative Algebra | (25) Schemes                     |
| (11) Brauer Groups       | (26) Constructions of Schemes    |
| (12) Homological Algebra | (27) Properties of Schemes       |
| (13) Derived Categories  | (28) Morphisms of Schemes        |
| (14) Simplicial Methods  | (29) Cohomology of Schemes       |
| (15) More on Algebra     | (30) Divisors                    |
| (16) Smoothing Ring Maps | (31) Limits of Schemes           |
| (17) Sheaves of Modules  | (32) Varieties                   |
|                          | (33) Topologies on Schemes       |
|                          | (34) Descent                     |



(35) Derived Categories of Schemes	(63) Quotients of Groupoids
(36) More on Morphisms	(64) Simplicial Spaces
(37) More on Flatness	(65) Formal Algebraic Spaces
(38) Groupoid Schemes	(66) Restricted Power Series
(39) More on Groupoid Schemes	(67) Resolution of Surfaces
(40) Étale Morphisms of Schemes	Deformation Theory
Topics in Scheme Theory	(68) Formal Deformation Theory
(41) Chow Homology	(69) Deformation Theory
(42) Adequate Modules	(70) The Cotangent Complex
(43) Dualizing Complexes	Algebraic Stacks
(44) Étale Cohomology	(71) Algebraic Stacks
(45) Crystalline Cohomology	(72) Examples of Stacks
(46) Pro-étale Cohomology	(73) Sheaves on Algebraic Stacks
Algebraic Spaces	(74) Criteria for Representability
(47) Algebraic Spaces	(75) Artin's Axioms
(48) Properties of Algebraic Spaces	(76) Quot and Hilbert Spaces
(49) Morphisms of Algebraic Spaces	(77) Properties of Algebraic Stacks
(50) Decent Algebraic Spaces	(78) Morphisms of Algebraic Stacks
(51) Cohomology of Algebraic Spaces	(79) Cohomology of Algebraic Stacks
(52) Limits of Algebraic Spaces	(80) Derived Categories of Stacks
(53) Divisors on Algebraic Spaces	(81) Introducing Algebraic Stacks
(54) Algebraic Spaces over Fields	Miscellany
(55) Topologies on Algebraic Spaces	(82) Examples
(56) Descent and Algebraic Spaces	(83) Exercises
(57) Derived Categories of Spaces	(84) Guide to Literature
(58) More on Morphisms of Spaces	(85) Desirables
(59) Pushouts of Algebraic Spaces	(86) Coding Style
(60) Groupoids in Algebraic Spaces	(87) Obsolete
(61) More on Groupoids in Spaces	(88) GNU Free Documentation License
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Topics in Geometry	

## References

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