DERIVED CATEGORIES OF SPACES

Contents

1. Introduction

In this chapter we discuss derived categories of modules on algebraic spaces. There do not seem to be good introductory references addressing this topic; it is covered in the literature by referring to papers dealing with derived categories of modules on algebraic stacks, for example see [Ols07].

2. Conventions

If $\mathcal{A}$ is an abelian category and M is an object of $\mathcal{A}$ then we also denote M the object of $K(\mathcal{A})$ and/or $D(\mathcal{A})$ corresponding to the complex which has M in degree 0 and is zero in all other degrees.

If we have a ring A, then $K(A)$ denotes the homotopy category of complexes of A-modules and $D(A)$ the associated derived category. Similarly, if we have a ringed

This is a chapter of the Stacks Project, version 714e994, compiled on Oct 28, 2014.

space $(X, \mathcal{O}_X)$ the symbol $K(\mathcal{O}_X)$ denotes the homotopy category of complexes of $\mathcal{O}_X$-modules and $D(\mathcal{O}_X)$ the associated derived category.

3. Generalities

In this section we put some general results on cohomology of unbounded complexes of modules on algebraic spaces.

Lemma 3.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Given an étale morphism $V \to Y$, set $U = V \times_Y X$ and denote $g : U \to V$ the projection morphism. Then $(Rf_*E)|_V = Rg_*(E|_U)$ for E in $D(\mathcal{O}_X)$.*

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_U) = g_*(\mathcal{I}^\bullet|_U)$ by Cohomology on Sites, Lemma 20.1. Hence the result follows from Properties of Spaces, Lemma 24.2. □

Definition 3.2. Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D(\mathcal{O}_X)$. Let $T \subset |X|$ be a closed subset. We say E is *supported on* T if the cohomology sheaves $H^i(E)$ are supported on T.

4. Derived category of quasi-coherent modules on the small étale site

Let X be a scheme. In this section we show that $D_{QCoh}(\mathcal{O}_X)$ can be defined in terms of the small étale site $X_{étale}$ of X. Denote $\mathcal{O}_{étale}$ the structure sheaf on $X_{étale}$. Consider the morphism of ringed sites

$$\epsilon : (X_{étale}, \mathcal{O}_{étale}) \longrightarrow (X_{Zar}, \mathcal{O}_X). \tag{4.0.1}$$

denoted $\text{id}_{small,étale,Zar}$ in Descent, Lemma 7.5.

Lemma 4.1. *The morphism ϵ of (4.0.1) is a flat morphism of ringed sites. In particular the functor $\epsilon^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_{étale})$ is exact. Moreover, if $\epsilon^*\mathcal{F} = 0$, then $\mathcal{F} = 0$.*

Proof. The second assertion follows from the first by Modules on Sites, Lemma 30.2. To prove the first assertion we have to show that $\mathcal{O}_{étale}$ is a flat $\epsilon^{-1}\mathcal{O}_X$-module. To do this it suffices to check $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ is flat for any geometric point $\overline{x}$ of X, see Modules on Sites, Lemma 38.2, Sites, Lemma 33.1, and Étale Cohomology, Remarks 29.11. By Étale Cohomology, Lemma 33.1 we see that $\mathcal{O}_{étale,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X,x}$. Thus $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ is faithfully flat by More on Algebra, Lemma 34.1. The final statement follows also: if $\epsilon^*\mathcal{F} = 0$, then

$$0 = \epsilon^*\mathcal{F}_{\overline{x}} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{étale}$$

(Modules on Sites, Lemma 35.4) for all geometric points $\overline{x}$. By faithful flatness of $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ we conclude $\mathcal{F}_x = 0$ for all $x \in X$. □

Let X be a scheme. Notation as in (4.0.1). Recall that $\epsilon^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_{étale})$ is an equivalence by Descent, Proposition 7.11 and Remark 7.6. Moreover, $QCoh(\mathcal{O}_{étale})$ forms a Serre subcategory of $Mod(\mathcal{O}_{étale})$ by Descent, Lemma 7.13. Hence we can let $D_{QCoh}(\mathcal{O}_{étale})$ be the triangulated subcategory of $D(\mathcal{O}_{étale})$ whose objects are the complexes with quasi-coherent cohomology sheaves, see Derived Categories, Section 13. The functor ϵ^* is exact (Lemma 4.1) hence induces $\epsilon^* : D(\mathcal{O}_X) \to D(\mathcal{O}_{étale})$ and since pullbacks of quasi-coherent modules are quasi-coherent also $\epsilon^* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{étale})$.

Lemma 4.2. *Let X be a scheme. The functor $\epsilon^* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{étale})$ defined above is an equivalence.*

Proof. We will prove this by showing the functor $R\epsilon_* : D(\mathcal{O}_{étale}) \to D(\mathcal{O}_X)$ induces a quasi-inverse.

Every quasi-coherent $\mathcal{O}_{étale}$-module $\mathcal{H}$ is of the form $\epsilon^*\mathcal{F}$ for some quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, see Descent, Proposition 7.11. Since $\mathcal{F} = \epsilon_*\mathcal{H}$ in this case (as ϵ_* is the restriction to $X_{Zar} \subset X_{étale}$) we conclude that the adjunction map $\epsilon^*\epsilon_*\mathcal{H} \to \mathcal{H}$ is an isomorphism for all quasi-coherent $\mathcal{O}_{étale}$-modules $\mathcal{H}$.

Let E be an object of $D_{QCoh}(\mathcal{O}_{étale})$ and denote $\mathcal{H}^i = H^i(E)$ its ith cohomology sheaf. Let $\mathcal{B}$ be the set of affine objects of $X_{étale}$. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Descent, Proposition 7.10 and Cohomology of Schemes, Lemma 2.2. According to Cohomology on Sites, Lemma 22.3 this implies E is represented by a K-injective complex $\mathcal{I}^\bullet$ and $\mathcal{I}^\bullet = \lim \mathcal{I}_n^\bullet$ where each $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives, the maps in the system $\ldots \to \mathcal{I}_2^\bullet \to \mathcal{I}_1^\bullet$ are termwise split surjections, and each $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}E$. In particular,

$$R\epsilon_*E = \epsilon_*\mathcal{I}^\bullet = \lim \epsilon_*\mathcal{I}_n^\bullet$$

For every $U \in \mathcal{B}$ we have

$$H^m(\mathcal{I}_n^\bullet(U)) = \begin{cases} \mathcal{H}^m(U) & \text{if} \quad m \geq -n \\ 0 & \text{if} \quad m < n \end{cases}$$

by the vanishing of $H^p(U, \mathcal{H}^i)$ for $p > 0$, the spectral sequence Derived Categories, Lemma 21.3, and the fact that $\tau_{\geq -n}E \cong \mathcal{I}_n^\bullet$. Hence we can apply Homology, Lemma 27.7 to the sequence of complexes

$$\lim_n \mathcal{I}_n^{m-2}(U) \to \lim_n \mathcal{I}_n^{m-1}(U) \to \lim_n \mathcal{I}_n^{m}(U) \to \lim_n \mathcal{I}_n^{m+1}(U)$$

to conclude that $H^m(\mathcal{I}^\bullet(U)) = \mathcal{H}^m(U)$ for $U \in \mathcal{B}$. Since ϵ_* is restriction to X_{Zar} we see, on applying the above to $U \subset X$ affine open, that $H^m(\epsilon_*\mathcal{I}^\bullet) = \epsilon_*\mathcal{H}^m$. Thus $R\epsilon_*$ indeed gives rise to a functor

$$R\epsilon_* : D_{QCoh}(\mathcal{O}_{étale}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

For our object E of $D_{QCoh}(\mathcal{O}_{étale})$ above the adjunction map $\epsilon^*R\epsilon_*E \to E$ is an isomorphism as we've seen that the cohomology sheaves of $R\epsilon_*E$ are $\epsilon_*\mathcal{H}^m$ and we have $\epsilon^*\epsilon_*\mathcal{H}^m = \mathcal{H}^m$ (see above). For $F \in D_{QCoh}(\mathcal{O}_X)$ the adjunction map $F \to R\epsilon_*\epsilon^*F$ is an isomorphism for the same reason, i.e., because the cohomology sheaves of $R\epsilon_*\epsilon^*F$ are isomorphic to $\epsilon_*H^m(\epsilon^*F) = \epsilon_*\epsilon^*H^m(F) = H^m(F)$. □

5. Derived category of quasi-coherent modules

Let S be a scheme. Lemma 4.2 shows that the category $D_{QCoh}(\mathcal{O}_S)$ can be defined in terms of complexes of $\mathcal{O}_S$-modules on the scheme S or by complexes of $\mathcal{O}$-modules on the small étale site of S. Hence the following definition is compatible with the definition in the case of schemes.

Definition 5.1. Let S be a scheme. Let X be an algebraic space over S. The *derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves* is denoted $D_{QCoh}(\mathcal{O}_X)$.

This makes sense by Properties of Spaces, Lemma 27.7 and Derived Categories, Section 13. Thus we obtain a canonical functor

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X) \tag{5.1.1}$$

see Derived Categories, Equation (13.1.1).

Observe that a flat morphism $f : Y \to X$ of algebraic spaces induces an exact functor $f^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$, see Morphisms of Spaces, Lemma 28.9 and Modules on Sites, Lemma 30.2. In particular $Lf^* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$ is computed on any representative complex (Derived Categories, Lemma 17.8). We will write $Lf^* = f^*$ when f is flat and we have $H^i(f^*E) = f^*H^i(E)$ for E in $D(\mathcal{O}_X)$ in this case. We will use this often when f is étale. Of course in the étale case the pullback functor is just the restriction to $Y_{étale}$, see Properties of Spaces, Equation (24.1.1).

Lemma 5.2. *Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent*

1. *E is in $D_{QCoh}(\mathcal{O}_X)$,*
2. *for every étale morphism $\varphi : U \to X$ where U is an affine scheme φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$,*
3. *for every étale morphism $\varphi : U \to X$ where U is a scheme φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$,*
4. *there exists a surjective étale morphism $\varphi : U \to X$ where U is a scheme such that φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$, and*
5. *there exists a surjective étale morphism of algebraic spaces $f : Y \to X$ such that Lf^*E is an object of $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. This follows immediately from the discussion preceding the lemma and Properties of Spaces, Lemma 27.6. ☐

Lemma 5.3. *Let S be a scheme. Let X be an algebraic space over S. Then $D_{QCoh}(\mathcal{O}_X)$ has direct sums.*

Proof. By Injectives, Lemma 13.4 the derived category $D(\mathcal{O}_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Properties of Spaces, Lemma 27.7. ☐

Lemma 5.4. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The functor Lf^* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. Choose a diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where U and V are schemes, the vertical arrows are étale, and a is surjective. Since $a^* \circ Lf^* = Lh^* \circ b^*$ the result follows from Lemma 5.2 and the case of schemes which is Derived Categories of Schemes, Lemma 3.6. ☐

Lemma 5.5. *Let S be a scheme. Let X be an algebraic space over S. For objects K, L of $D_{QCoh}(\mathcal{O}_X)$ the derived tensor product $K \otimes^{\mathbf{L}} L$ is in $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Let $\varphi : U \to X$ be a surjective étale morphism from a scheme U. Since $\varphi^*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = \varphi^*K \otimes^{\mathbf{L}}_{\mathcal{O}_U} \varphi^*L$ we see from Lemma 5.2 that this follows from the case of schemes which is Derived Categories of Schemes, Lemma 3.7. $\square$

The following lemma will help us to "compute" a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

Lemma 5.6. *Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then there exists an inverse system $\mathcal{I}_n^\bullet$ of complexes of $\mathcal{O}_X$-modules such that*

1. *$\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$ represents E,*
2. *$\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,*
3. *$\mathcal{I}^\bullet \to \mathcal{I}_n^\bullet$ induces an identification $\tau_{\geq -n}E \to \mathcal{I}_n^\bullet$ in $D(\mathcal{O}_X)$,*
4. *the transition maps $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections, and*
5. *$\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules.*

Moreover, E is the derived limit of the inverse system of its canonical truncations $\tau_{\geq -n}E$.

Proof. Denote $\mathcal{H}^i = H^i(E)$ the ith cohomology sheaf of E. Let $\mathcal{B}$ be the set of affine objects of $X_{étale}$. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$ as U is an affine scheme. See discussion in Cohomology of Spaces, Section 3 and Cohomology of Schemes, Lemma 2.2. Thus the lemma follows from Cohomology on Sites, Lemmas 22.3 and 22.4. $\square$

Lemma 5.7. *Let S be a scheme. Let X be an algebraic space over S. Let $F : Mod(\mathcal{O}_X) \to Ab$ be a functor and $N \geq 0$ an integer. Assume that*

1. *F is left exact,*
2. *F commutes with countable direct products,*
3. *$R^pF(\mathcal{F}) = 0$ for all $p \geq N$ and $\mathcal{F}$ quasi-coherent.*

Then for $E \in D_{QCoh}(\mathcal{O}_X)$ the maps $R^pF(E) \to R^pF(\tau_{\geq p-N+1}E)$ are isomorphisms.

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. By shifting the complex we see it suffices to prove the assertion for $p = 0$. Choose $\mathcal{I}^\bullet = \lim \mathcal{I}_n^\bullet$ as in Lemma 5.6. As $\mathcal{I}^\bullet$ is K-injective $RF(E)$ is represented by $F(\mathcal{I}^\bullet)$. As F commutes with countable direct products, and since the maps $\mathcal{I}_n^m \to \mathcal{I}_{n-1}^m$ are split surjections, we get $F(\mathcal{I}^\bullet) = \lim F(\mathcal{I}_n^\bullet)$. The cohomology of

$$F(\mathcal{I}_n^{-2}) \to F(\mathcal{I}_n^{-1}) \to F(\mathcal{I}_n^0) \to F(\mathcal{I}_n^1) \tag{5.7.1}$$

in degree 0, resp. -1 is equal to $R^0F(\tau_{\geq -n}E)$, resp. $R^{-1}F(\tau_{\geq -n}E)$ because $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives representing $\tau_{\geq -n}E$. We have a distinguished triangle

$$H^{-n}(E)[n] \to \tau_{\geq -n}E \to \tau_{\geq -n+1}E \to H^{-n}(E)[n+1]$$

(Derived Categories, Remark 12.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$R^pF(H^{-n}(E)[n]) = R^{p+n}F(H^{-n}(E)) = 0$$

for $p + n \geq N$ and

$$R^pF(H^{-n}(E)[n+1]) = R^{p+n+1}F(H^{-n}(E)) = 0$$

for $p + n + 1 \geq N$. We conclude that

$$R^pF(\tau_{\geq -n}E) \to R^pF(\tau_{\geq -n+1}E)$$

is an isomorphism for all $n \gg p$ and an isomorphism for $n \geq N$ for $p = 0$. Thus Homology, Lemma 27.7 applies to the system of sequences (5.7.1) and we conclude that $R^0F(E) = \lim R^0F(\tau_{\geq -n}E)$. By the above the system $R^0F(\tau_{\geq -n}E)$ is constant starting with $n = N - 1$ as desired. $\square$

6. Total direct image

The following lemma is the analogue of Cohomology of Spaces, Lemma 7.1.

Lemma 6.1. *Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S.*

1. *The functor Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*
2. *If Y is quasi-compact, there exists an integer $N = N(X, Y, f)$ such that for an object E of $D_{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m > N$.*
3. *In fact, if Y is quasi-compact we can find $N = N(X, Y, f)$ such that for every morphism of algebraic spaces $Y' \to Y$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \to Y'$ is the base change of f.*

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that Rf_*E has quasi-coherent cohomology sheaves. This question is local on Y, hence we may assume Y is quasi-compact. Pick $N = N(X, Y, f)$ as in Cohomology of Spaces, Lemma 7.1. Thus $R^pf_*\mathcal{F} = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $p \geq N$. In particular, for any affine object V of $Y_{étale}$ we have $H^p(V \times_Y X, \mathcal{F}) = 0$ for $p \geq N$, see Cohomology of Spaces, Lemma 3.3.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Choose $\mathcal{I}^\bullet = \lim \mathcal{I}_n^\bullet$ as in Lemma 5.6. As $\mathcal{I}^\bullet$ is K-injective Rf_*E is represented by $f_*\mathcal{I}^\bullet = \lim f_*\mathcal{I}_n^\bullet$. Let V be an affine object of $Y_{étale}$. The cohomology $H^m(f_*\mathcal{I}_n^\bullet(V))$ of

$$f_*\mathcal{I}_n^{m-1}(V) \to f_*\mathcal{I}_n^m(V) \to f_*\mathcal{I}_n^{m+1}(V)$$

is equal to $H^m(V \times_Y X, \tau_{\geq -n}E)$ because $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives representing $\tau_{\geq -n}E$. We have a distinguished triangle

$$H^{-n}(E)[n] \to \tau_{\geq -n}E \to \tau_{\geq -n+1}E \to H^{-n}(E)[n+1]$$

in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have $H^m(V \times_Y X, H^{-n}(E)[n]) = 0$ for $n + m \geq N$ by our choice of N. Similarly, $H^m(V \times_Y X, H^{-n}(E)[n+1]) = 0$ for $n + m + 1 \geq N$. We conclude that

$$H^m(f_*\mathcal{I}_n^\bullet(V)) \to H^m(f_*\mathcal{I}_{n-1}^\bullet(V))$$

is an isomorphism for all $n \geq N - m$. Thus Cohomology on Sites, Lemma 22.1 applies to show that the mth cohomology sheaf of $\lim f_*\mathcal{I}_n^\bullet$ agrees with the mth cohomology sheaf of $f_*\mathcal{I}_n^\bullet$ for $n \geq N - m$. Since these cohomology sheaves are quasi-coherent by Cohomology of Spaces, Lemma 3.2 we get (1).

Finally, we show that (2) and (3) hold with our choice of N. Namely, the stabilization proven above gives that $H^m(Rf_*E)$ is equal to $H^m(Rf_*(\tau_{\geq -n}E))$ for all n

large enough which means we can work with objects in $D^+(\mathcal{O}_X)$ in order to prove (2) and (3). In this case we can for example use the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

(Derived Categories, Lemma 21.3) and the vanishing of $R^p f_* H^q(E)$ for $p \geq N$ to conclude. Some details omitted. $\square$

Lemma 6.2. *Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ commutes with direct sums.*

Proof. Let E_i be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_* E_i \longrightarrow Rf_* E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer N as in Lemma 6.1. Then $R^0 f_* E = R^0 f_* \tau_{\geq -N} E$ and $R^0 f_* E_i = R^0 f_* \tau_{\geq -N} E_i$ by the lemma cited. Observe that $\tau_{\geq -N} E = \bigoplus \tau_{\geq -N} E_i$. Thus we may assume all of the E_i have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E \quad \text{and} \quad R^p f_* H^q(E_i) \Rightarrow R^{p+q} f_* E_i$$

(Derived Categories, Lemma 21.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Spaces, Lemma 4.2. $\square$

Remark 6.3. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y over S. Let $f_0 : X_0 \to Y_0$ be a morphism of schemes representing f (awkward but temporary notation). Then we claim the diagrams

$$\begin{tikzcd} D_{QCoh}(\mathcal{O}_{X_0}) \arrow[r, equal, "\text{Lemma 4.2}"'] \arrow[d, "Rf_{0,*}"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D_{QCoh}(\mathcal{O}_{Y_0}) \arrow[r, equal, "\text{Lemma 4.2}"] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

(Lemma 6.1 and Derived Categories of Schemes, Lemma 4.1) and

$$\begin{tikzcd} D_{QCoh}(\mathcal{O}_{X_0}) \arrow[r, equal, "\text{Lemma 4.2}"'] & D_{QCoh}(\mathcal{O}_X) \\ D_{QCoh}(\mathcal{O}_{Y_0}) \arrow[r, equal, "\text{Lemma 4.2}"] \arrow[u, "Lf_0^*"] & D_{QCoh}(\mathcal{O}_Y) \arrow[u, "Lf^*"'] \end{tikzcd}$$

(Lemma 5.4 and Derived Categories of Schemes, Lemma 3.6) are commutative. The result for Lf^* and Lf_0^* follows as the equivalences $D_{QCoh}(\mathcal{O}_{X_0}) \to D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_{Y_0}) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 4.2 come from pulling back by the (flat) morphisms of ringed sites $\epsilon : X_{\acute{e}tale} \to X_{0,Zar}$ and $\epsilon : Y_{\acute{e}tale} \to Y_{0,Zar}$ and the diagram of ringed sites

$$\begin{tikzcd} X_{0,Zar} \arrow[d, "f_0"'] & X_{\acute{e}tale} \arrow[l, "\epsilon"'] \arrow[d, "f"] \\ Y_{0,Zar} & Y_{\acute{e}tale} \arrow[l, "\epsilon"] \end{tikzcd}$$

is commutative (details omitted). In fact the commutativity of the first diagram also follows as the proof of Lemma 4.2 shows that the functor $R\epsilon_*$ gives the equivalences $D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{X_0})$ and $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{Y_0})$.

Lemma 6.4. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms.*

Proof. The statement means that a morphism $\alpha : E \to F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_*\alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is étale local on Y. Hence we may assume Y and therefore X is affine. In this case the problem reduces to the case of schemes (Derived Categories of Schemes, Lemma 4.3) via Lemma 4.2 and Remark 6.3. □

Lemma 6.5. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. For E in $D_{QCoh}(\mathcal{O}_Y)$ we have $Rf_*Lf^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{O}_X$.*

Proof. Since f is affine the map $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$ is an isomorphism (Cohomology of Spaces, Lemma 7.2). There is a canonical map $E \otimes^{\mathbf{L}} f_*\mathcal{O}_X = E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X \to Rf_*Lf^*E$ adjoint to the map

$$Lf^*(E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X) = Lf^*E \otimes^{\mathbf{L}} Lf^*Rf_*\mathcal{O}_X \longrightarrow Lf^*E \otimes^{\mathbf{L}} \mathcal{O}_X = Lf^*E$$

coming from $1 : Lf^*E \to Lf^*E$ and the canonical map $Lf^*Rf_*\mathcal{O}_X \to \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on Y. Hence we may assume Y and therefore X is affine. In this case the problem reduces to the case of schemes (Derived Categories of Schemes, Lemma 4.4) via Lemma 4.2 and Remark 6.3. □

7. Derived category of coherent modules

Let S be a scheme. Let X be a locally Noetherian algebraic space over S. In this case the category $Coh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules is a weak Serre subcategory, see Homology, Section 9 and Cohomology of Spaces, Lemma 11.3. Denote

$$D_{Coh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 13. Thus we obtain a canonical functor

$$D(Coh(\mathcal{O}_X)) \longrightarrow D_{Coh}(\mathcal{O}_X) \tag{7.0.1}$$

see Derived Categories, Equation (13.1.1).

Lemma 7.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y is Noetherian. Let E be an object of $D^b_{Coh}(\mathcal{O}_X)$ such that the scheme theoretic support of $H^i(E)$ is proper over Y for all i. Then Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_Y)$.*

Proof. Consider the spectral sequence

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E$$

see Derived Categories, Lemma 21.3. By assumption and Cohomology of Spaces, Remark 19.3 the sheaves $R^pf_*H^q(E)$ are coherent. Hence $R^{p+q}f_*E$ is coherent, i.e., $E \in D_{Coh}(\mathcal{O}_S)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Spaces, Lemma 7.1 or from Lemma 6.1. □

8. Induction principle

In this section we discuss an induction principle for algebraic spaces analogues to what is Cohomology of Schemes, Lemma 8.3 for schemes. To formulate it we introduce the notion of an *elementary distinguished square*; this terminology is borrowed from [MV99]. The principle as formulated here is implicit in the paper [GR71] by Raynaud and Gruson. A related principle for algebraic stacks is [Ryd10, Theorem D] by David Rydh.

Definition 8.1. Let S be a scheme. A commutative diagram

$$\begin{tikzcd} U \times_W V \arrow[r] \arrow[d] & V \arrow[d, "f"] \\ U \arrow[r, "j"] & W \end{tikzcd}$$

of algebraic spaces over S is called an *elementary distinguished square* if

(1) U is an open subspace of W and j is the inclusion morphism,
(2) f is étale, and
(3) setting $T = W \setminus U$ (with reduced induced subspace structure) the morphism $f^{-1}(T) \to T$ is an isomorphism.

We will indicate this by saying: "Let $(U \subset W, f : V \to W)$ be an elementary distinguished square."

Note that if $(U \subset W, f : V \to W)$ is an elementary distinguished square, then we have $W = U \cup f(V)$. Thus $\{U \to W, V \to W\}$ is an étale covering of W. It turns out that these étale coverings have nice properties and that in some sense there are "enough" of them.

Lemma 8.2. *Let S be a scheme. Let $(U \subset W, f : V \to W)$ be an elementary distinguished square of algebraic spaces over S.*

(1) *If $V' \subset V$ and $U \subset U' \subset W$ are open subspaces and $W' = U' \cup f(V')$ then $(U' \subset W', f|_{V'} : V' \to W')$ is an elementary distinguished square.*
(2) *If $p : W' \to W$ is a morphism of algebraic spaces, then $(p^{-1}(U) \subset W', V \times_W W' \to W')$ is an elementary distinguished square.*

Proof. Omitted. □

Lemma 8.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let P be a property of the quasi-compact and quasi-separated objects of $X_{spaces,étale}$. Assume that*

(1) *P holds for every affine object of $X_{spaces,étale}$,*
(2) *for every elementary distinguished square $(U \subset W, f : V \to W)$ such that*
    (a) *W is a quasi-compact and quasi-separated object of $X_{spaces,étale}$,*
    (b) *U is quasi-compact,*
    (c) *V is affine, and*
    (d) *P holds for U, V, and $U \times_W V$,*

    then P holds for W.

Then P holds for every quasi-compact and quasi-separated object of $X_{spaces,étale}$ and in particular for X.

Proof. We first claim that P holds for every representable quasi-compact and quasi-separated object of $X_{spaces,étale}$. Namely, suppose that $U \to X$ is étale and U is a quasi-compact and quasi-separated scheme. By assumption (1) property P holds for every affine open of U. Moreover, if $W, V \subset U$ are quasi-compact open with V affine and P holds for W, V, and $W \cap V$, then P holds for $W \cup V$ by (2) (as the pair $(W \subset W \cup V, V \to W \cup V)$ is an elementary distinguished square). Thus P holds for U by the induction principle for schemes, see Cohomology of Schemes, Lemma 4.1.

To finish the proof it suffices to prove P holds for X (because we can simply replace X by any quasi-compact and quasi-separated object of $X_{spaces,étale}$ we want to prove the result for). We will use the filtration

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and the morphisms $f_p : V_p \to U_p$ of Decent Spaces, Lemma 8.5. We will prove that P holds for U_p by descending induction on p. Note that P holds for U_{n+1} by (1) as an empty algebraic space is affine. Assume P holds for U_{p+1}. Note that $(U_{p+1} \subset U_p, f_p : V_p \to U_p)$ is an elementary distinguished square, but (2) may not apply as V_p may not be affine. However, as V_p is a quasi-compact scheme we may choose a finite affine open covering $V_p = V_{p,1} \cup \ldots \cup V_{p,m}$. Set $W_{p,0} = U_{p+1}$ and

$$W_{p,i} = U_{p+1} \cup f_p(V_{p,1} \cup \ldots \cup V_{p,i})$$

for $i = 1, \ldots, m$. These are quasi-compact open subspaces of X. Then we have

$$U_{p+1} = W_{p,0} \subset W_{p,1} \subset \ldots \subset W_{p,m} = U_p$$

and the pairs

$$(W_{p,0} \subset W_{p,1}, f_p|_{V_{p,1}}), (W_{p,1} \subset W_{p,2}, f_p|_{V_{p,2}}), \ldots, (W_{p,m-1} \subset W_{p,m}, f_p|_{V_{p,m}})$$

are elementary distinguished squares by Lemma 8.2. Note that P holds for each $V_{p,1}$ (as affine schemes) and for $W_{p,i} \times_{W_{p,i+1}} V_{p,i+1}$ as this is a quasi-compact open of $V_{p,i+1}$ and hence P holds for it by the first paragraph of this proof. Thus (2) applies to each of these and we inductively conclude P holds for $W_{p,1}, \ldots, W_{p,m} = U_p$. □

Lemma 8.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{B} \subset \text{Ob}(X_{spaces,étale})$. Let P be a property of the elements of $\mathcal{B}$. Assume that*

1. *every $W \in \mathcal{B}$ is quasi-compact and quasi-separated,*
2. *if $W \in \mathcal{B}$ and $U \subset W$ is quasi-compact open, then $U \in \mathcal{B}$,*
3. *if $V \in \text{Ob}(X_{spaces,étale})$ is affine, then (a) $V \in \mathcal{B}$ and (b) P holds for V,*
4. *for every elementary distinguished square $(U \subset W, f : V \to W)$ such that*
   (a) *$W \in \mathcal{B}$,*
   (b) *U is quasi-compact,*
   (c) *V is affine, and*
   (d) *P holds for U, V, and $U \times_W V$,*

   then P holds for W.

Then P holds for every $W \in \mathcal{B}$.

Proof. This is proved in exactly the same manner as the proof of Lemma 8.3. (We remark that (4)(d) makes sense as $U \times_W V$ is a quasi-compact open of V hence an element of $\mathcal{B}$ by conditions (2) and (3).) □

Remark 8.5. How to choose the collection $\mathcal{B}$ in Lemma 8.4? Here are some examples:

1. If X is quasi-compact and separated, then we can choose $\mathcal{B}$ to be the set of quasi-compact and separated objects of $X_{spaces,étale}$. Then $X \in \mathcal{B}$ and $\mathcal{B}$ satisfies (1), (2), and (3)(a). With this choice of $\mathcal{B}$ Lemma 8.4 reproduces Lemma 8.3.
2. If X is quasi-compact with affine diagonal, then we can choose $\mathcal{B}$ to be the set of objects of $X_{spaces,étale}$ which are quasi-compact and have affine diagonal. Again $X \in \mathcal{B}$ and $\mathcal{B}$ satisfies (1), (2), and (3)(a).
3. If X is quasi-compact and quasi-separated, then the smallest subset $\mathcal{B}$ which contains X and satisfies (1), (2), and (3)(a) is given by the rule $W \in \mathcal{B}$ if and only if either W is a quasi-compact open subspace of X, or W is a quasi-compact open of an affine object of $X_{spaces,étale}$.

Here is a variant where we extend the truth from an open to larger opens.

Lemma 8.6. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $W \subset X$ be a quasi-compact open subspace. Let P be a property of quasi-compact open subspaces of X. Assume that*

1. *P holds for W, and*
2. *for every elementary distinguished square $(W_1 \subset W_2, f : V \to W_2)$ where such that*
   - (a) *W_1, W_2 are quasi-compact open subspaces of X,*
   - (b) *$W \subset W_1$,*
   - (c) *V is affine, and*
   - (d) *P holds for W_1,*

   then P holds for W_2.

Then P holds for X.

Proof. We can deduce this from Lemma 8.4, but instead we will give a direct argument by eplicitly redoing the proof of Lemma 8.3. We will use the filtration

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and the morphisms $f_p : V_p \to U_p$ of Decent Spaces, Lemma 8.5. We will prove that P holds for $W_p = W \cup U_p$ by descending induction on p. This will finish the proof as $W_1 = X$. Note that P holds for $W_{n+1} = W \cap U_{n+1} = W$ by (1). Assume P holds for W_{p+1}. Observe that $W_p \setminus W_{p+1}$ (with reduced induced subspace structure) is a closed subspace of $U_p \setminus U_{p+1}$. Since $(U_{p+1} \subset U_p, f_p : V_p \to U_p)$ is an elementary distinguished square, the same is true for $(W_{p+1} \subset W_p, f_p : V_p \to W_p)$. However (2) may not apply as V_p may not be affine. However, as V_p is a quasi-compact scheme we may choose a finite affine open covering $V_p = V_{p,1} \cup \ldots \cup V_{p,m}$. Set $W_{p,0} = W_{p+1}$ and

$$W_{p,i} = W_{p+1} \cup f_p(V_{p,1} \cup \ldots \cup V_{p,i})$$

for $i = 1, \ldots, m$. These are quasi-compact open subspaces of X containing W. Then we have

$$W_{p+1} = W_{p,0} \subset W_{p,1} \subset \ldots \subset W_{p,m} = W_p$$

and the pairs

$$(W_{p,0} \subset W_{p,1}, f_p|_{V_{p,1}}), (W_{p,1} \subset W_{p,2}, f_p|_{V_{p,2}}), \ldots, (W_{p,m-1} \subset W_{p,m}, f_p|_{V_{p,m}})$$

are elementary distinguished squares by Lemma 8.2. Now (2) applies to each of these and we inductively conclude P holds for $W_{p,1}, \ldots, W_{p,m} = W_p$. ☐

9. Mayer-Vietoris

In this section we prove that an elementary distinguished triangle gives rise to various Mayer-Vietoris sequences.

Let S be a scheme. Let $U \to X$ be an étale morphism of algebraic spaces over S. In Properties of Spaces, Section 25 it was shown that $U_{spaces,étale} = X_{spaces,étale}/U$ compatible with structure sheaves. Hence in this situation we often think of the morphism $j_U : U \to X$ as a localization morphism (see Modules on Sites, Definition 19.1). In particular we think of pullback j_U^* as restriction to U and we often denote it by $|_U$; this is compatible with Properties of Spaces, Equation (24.1.1). In particular we see that

$$(\mathcal{F}|_U)_{\overline{u}} = \mathcal{F}_{\overline{x}} \tag{9.0.1}$$

if $\overline{u}$ is a geometric point of U and $\overline{x}$ the image of $\overline{u}$ in X. Moreover, restriction has an exact left adjoint $j_{U!}$, see Modules on Sites, Lemmas 19.2 and 19.3. Finally, recall that if $\mathcal{G}$ is an $\mathcal{O}_X$-module, then

$$(j_{U!}\mathcal{G})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathcal{G}_{\overline{u}} \tag{9.0.2}$$

for any geometric point $\overline{x} : \mathrm{Spec}(k) \to X$ where the direct sum is over those morphism $\overline{u} : \mathrm{Spec}(k) \to U$ such that $j_U \circ \overline{u} = \overline{x}$, see Modules on Sites, Lemma 37.1 and Properties of Spaces, Lemma 16.13.

Lemma 9.1. *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S.*

(1) *For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have a short exact sequence*
$$0 \to j_{U \times_X V!}\mathcal{F}|_{U \times_X V} \to j_{U!}\mathcal{F}|_U \oplus j_{V!}\mathcal{F}|_V \to \mathcal{F} \to 0$$
(2) *For an object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*
$$j_{U \times_X V!}E|_{U \times_X V} \to j_{U!}E|_U \oplus j_{V!}E|_V \to E \to j_{U \times_X V!}E|_{U \times_X V}[1]$$
in $D(\mathcal{O}_X)$.

Proof. To show the sequence of (1) is exact we may check on stalks at geometric points by Properties of Spaces, Theorem 16.12. Let $\overline{x}$ be a geometric point of X. By Equations (9.0.1) and (9.0.2) taking stalks at $\overline{x}$ we obtain the sequence

$$0 \to \bigoplus\nolimits_{(\overline{u},\overline{v})} \mathcal{F}_{\overline{x}} \to \bigoplus\nolimits_{\overline{u}} \mathcal{F}_{\overline{x}} \oplus \bigoplus\nolimits_{\overline{v}} \mathcal{F}_{\overline{x}} \to \mathcal{F}_{\overline{x}} \to 0$$

This sequence is exact because for every $\overline{x}$ there either is exactly one $\overline{u}$ mapping to $\overline{x}$, or there is no $\overline{u}$ and exactly one $\overline{v}$ mapping to $\overline{x}$.

Proof of (2). We have seen in Cohomology on Sites, Section 20 that the restriction functors and the extension by zero functors on derived categories are computed by just applying the functor to any complex. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules representing E. The distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes of $\mathcal{O}_X$-modules

$$0 \to j_{U \times_X V!}\mathcal{E}^\bullet|_{U \times_X V} \to j_{U!}\mathcal{E}^\bullet|_U \oplus j_{V!}\mathcal{E}^\bullet|_V \to \mathcal{E}^\bullet \to 0$$

which is short exact by (1). ☐

Lemma 9.2. *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S.*

(1) *For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have a short exact sequence*

$$0 \to \mathcal{F} \to j_{U,*}\mathcal{F}|_U \oplus j_{V,*}\mathcal{F}|_V \to j_{U \times_X V,*}\mathcal{F}|_{U \times_X V} \to 0$$

(2) *For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U \times_X V,*}E|_{U \times_X V} \to E[1]$$

in $D(\mathcal{O}_X)$.

Proof. Let W be an object of $X_{étale}$. We claim the sequence

$$0 \to \mathcal{F}(W) \to \mathcal{F}(W \times_X U) \oplus \mathcal{F}(W \times_X V) \to \mathcal{F}(W \times_X U \times_X V)$$

is exact and that an element of the last group can locally on W be lifted to the middle one. By Lemma 8.2 the pair $(W \times_X U \subset W, V \times_X W \to W)$ is an elementary distinguished square. Thus we may assume $W = X$ and it suffices to prove the same thing for

$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \times_X V)$$

We have seen that

$$0 \to j_{U \times_X V!}\mathcal{O}_{U \times_X V} \to j_{U!}\mathcal{O}_U \oplus j_{V!}\mathcal{O}_V \to \mathcal{O}_X \to 0$$

is a exact sequence of $\mathcal{O}_X$-modules in Lemma 9.1 and applying the right exact functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{F})$ gives the sequence above. This also means that the obstruction to lifting $s \in \mathcal{F}(U \times_X V)$ to an element of $\mathcal{F}(U) \oplus \mathcal{F}(V)$ lies in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = H^1(X, \mathcal{F})$. By locality of cohomology (Cohomology on Sites, Lemma 8.3) this obstruction vanishes étale locally on X and the proof of (1) is complete.

Proof of (2). Choose a K-injective complex $\mathcal{I}^\bullet$ representing E whose terms $\mathcal{I}^n$ are injective objects of $Mod(\mathcal{O}_X)$, see Injectives, Theorem 12.6. Then $\mathcal{I}^\bullet|U$ is a K-injective complex (Cohomology on Sites, Lemma 20.1). Hence $Rj_{U,*}E|_U$ is represented by $j_{U,*}\mathcal{I}^\bullet|_U$. Similarly for V and $U \times_X V$. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U \times_X V,*}\mathcal{I}^\bullet|_{U \times_X V} \to 0.$$

This sequence is exact by (1). $\square$

Lemma 9.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $(U \subset X, V \to X)$ be an elementary distinguished square. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \times_X V} : U \times_X V \to Y$ the restrictions. For every object E of $D(\mathcal{O}_X)$ there exists a distinguished triangle*

$$Rf_*E \to Ra_*(E|_U) \oplus Rb_*(E|_V) \to Rc_*(E|_{U \times_X V}) \to Rf_*E[1]$$

in $D(\mathcal{O}_Y)$. This triangle is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $Mod(\mathcal{O}_X)$ for all n, see Injectives, Theorem 12.6. Then Rf_*E is computed by $f_*\mathcal{I}^\bullet$. Similarly for U, V, and $U \cap V$ by Cohomology on Sites, Lemma 20.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 12 and especially Lemma 12.1) to the short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \times_X V} \to 0.$$

To see this is a short exact sequence of complexes we argue as follows. Pick an injective object $\mathcal{I}$ of $Mod(\mathcal{O}_X)$. Apply f_* to the short exact sequence

$$0 \to \mathcal{I} \to j_{U,*}\mathcal{I}|_U \oplus j_{V,*}\mathcal{I}|_V \to j_{U \times_X V,*}\mathcal{I}|_{U \times_X V} \to 0$$

of Lemma 9.2 and use that $R^1 f_* \mathcal{I} = 0$ to get a short exact sequence

$$0 \to f_*\mathcal{I} \to f_*j_{U,*}\mathcal{I}|_U \oplus f_*j_{V,*}\mathcal{I}|_V \to f_*j_{U \times_X V,*}\mathcal{I}|_{U \times_X V} \to 0$$

The proof is finished by observing that $a_* = f_* j_{U,*}$ and similarly for b_* and c_*. $\square$

Lemma 9.4. *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S. For objects E, F of $D(\mathcal{O}_X)$ we have a Mayer-Vietoris sequence*

$$\begin{tikzcd} & & \ldots \arrow[r] & Ext^{-1}(E_{U \times_X V}, F_{U \times_X V}) \arrow[dlll] \\ \mathrm{Hom}(E, F) \arrow[r] & \mathrm{Hom}(E_U, F_U) \oplus \mathrm{Hom}(E_V, F_V) \arrow[rr] & & \mathrm{Hom}(E_{U \times_X V}, F_{U \times_X V}) \end{tikzcd}$$

where the subscripts denote restrictions to the relevant opens and the Hom*'s are taken in the relevant derived categories.*

Proof. Use the distinguished triangle of Lemma 9.1 to obtain a long exact sequence of Hom's (from Derived Categories, Lemma 4.2) and use that $\mathrm{Hom}(j_{U!}E|_U, F) = \mathrm{Hom}(E|_U, F|_U)$ by Cohomology on Sites, Lemma 20.2. $\square$

Lemma 9.5. *Let S be a scheme. Let $j : U \to X$ be a étale morphism of algebraic spaces over S. Given an étale morphism $V \to Y$, set $W = V \times_X U$ and denote $j_W : W \to V$ the projection morphism. Then $(j_!E)|_V = j_{W!}(E|_W)$ for E in $D(\mathcal{O}_U)$.*

Proof. This is true because $(j_!\mathcal{F})|_V = j_{W!}(\mathcal{F}|_W)$ for an $\mathcal{O}_X$-module $\mathcal{F}$ as follows immediately from the construction of the functors $j_!$ and $j_{W!}$, see Modules on Sites, Lemma 19.2. $\square$

Lemma 9.6. *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Set $T = |X| \setminus |U|$.*

(1) *If E is an object of $D(\mathcal{O}_X)$ supported on T, then (a) $E \to Rj_*(E|_V)$ and (b) $j_!(E|_V) \to E$ are isomorphisms.*
(2) *If F is an object of $D(\mathcal{O}_V)$ supported on $j^{-1}T$, then (a) $F \to (j_!F)|_V$, (b) $(Rj_*F)|_V \to F$, and (c) $j_!F \to Rj_*F$ are isomorphisms.*

Proof. Let E be an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T. Then we see that $E|_U = 0$ and $E|_{U \times_X V} = 0$ as T doesn't meet U and $j^{-1}T$ doesn't meet $U \times_X V$. Thus (1)(a) follows from Lemma 9.2. In exactly the same way (1)(b) follows from Lemma 9.1.

Let F be an object of $D(\mathcal{O}_V)$ whose cohomology sheaves are supported on $j^{-1}T$. By Lemma 3.1 we have $(Rj_*F)|_U = Rj_{W,*}(F|_W) = 0$ because $F|_W = 0$ by our assumption. Similarly $(j_!F)|_U = j_{W!}(F|_W) = 0$ by Lemma 9.5. Thus $j_!F$ and Rj_*F are supported on T and $(j_!F)|_V$ and $(Rj_*F)|_V$ are supported on $j^{-1}(T)$.

To check that the maps (2)(a), (b), (c) are isomorphisms in the derived category, it suffices to check that these map induce isomorphisms on stalks of cohomology sheaves at geometric points of T and $j^{-1}(T)$ by Properties of Spaces, Theorem 16.12. This we may do after replacing X by V, U by $U \times_X V$, V by $V \times_X V$ and F by $F|_{V \times_X V}$ (restriction via first projection), see Lemmas 3.1, 9.5, and 8.2. Since $V \times_X V \to V$ has a section this reduces (2) to the case that $j : V \to X$ has a section.

Assume j has a section $\sigma : X \to V$. Set $V' = \sigma(X)$. This is an open subspace of V. Set $U' = j^{-1}(U)$. This is another open subspace of V. Then $(U' \subset V, V' \to V)$ is an elementary distinguished square. Observe that $F|_{U'} = 0$ and $F|_{V' \cap U'} = 0$ because F is supported on $j^{-1}(T)$. Denote $j' : V' \to V$ the open immersion and $j_{V'} : V' \to X$ the composition $V' \to V \to X$ which is the inverse of σ. Set $F' = \sigma^*F$. The distinguished triangles of Lemmas 9.1 and 9.2 show that $F = j'_!(F|_{V'})$ and $F = Rj'_*(F|_{V'})$. It follows that $j_!F = j_!j'_!(F|_{V'}) = j_{V'!}F = F'$ because $j_{V'} : V' \to X$ is an isomorphism and the inverse of σ. Similarly, $Rj_*F = Rj_*Rj'_*F = Rj_{V',*}F = F'$. This proves (2)(c). To prove (2)(a) and (2)(b) it suffices to show that $F = F'|_V$. This is clear because both F and $F'|_V$ restrict to zero on U' and $U' \cap V'$ and the same object on V'. □

We can glue complexes!

Lemma 9.7. *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S. Suppose given*

1. *an object E of $D(\mathcal{O}_X)$,*
2. *a morphism $a : A \to E|_U$ of $D(\mathcal{O}_U)$,*
3. *a morphism $b : B \to E|_V$ of $D(\mathcal{O}_V)$,*
4. *an isomorphism $c : A|_{U \times_X V} \to B|_{U \times_X V}$*

such that

$$a|_{U \times_X V} = b|_{U \times_X V} \circ c.$$

Then there exists a morphism $F \to E$ in $D(\mathcal{O}_X)$ whose restriction to U is isomorphic to a and whose restriction to V is isomorphic to b.

Proof. Denote j_U, j_V, $j_{U \times_X V}$ the corresponding morphisms towards X. Choose a distinguished triangle

$$F \to Rj_{U,*}A \oplus Rj_{V,*}B \to Rj_{U \times_X V,*}(B|_{U \times_X V}) \to F[1]$$

Here the map $Rj_{V,*}B \to Rj_{U \times_X V,*}(B|_{U \times_X V})$ is the obvious one. The map $Rj_{U,*}A \to Rj_{U \times_X V,*}(B|_{U \times_X V})$ is the composition of $Rj_{U,*}A \to Rj_{U \times_X V,*}(A|_{U \times_X V})$ with $Rj_{U \times_X V,*}c$. Restricting to U we obtain

$$F|_U \to A \oplus (Rj_{V,*}B)|_U \to (Rj_{U \times_X V,*}(B|_{U \times_X V}))|_U \to F|_U[1]$$

Denote $j : U \times_X V \to U$. Compatibility of restriction and total direct image (Lemma 3.1) shows that both $(Rj_{V,*}B)|_U$ and $(Rj_{U \times_X V,*}(B|_{U \times_X V}))|_U$ are canonically isomorphic to $Rj_*(B|_{U \times_X V})$. Hence the second arrow of the last displayed equation has a section, and we conclude that the morphism $F|_U \to A$ is an isomorphism.

To see that the morphism $F|_V \to B$ is an isomorphism we will use a trick. Namely, choose a distinguished triangle

$$F|_V \to B \to B' \to F[1]|_V$$

in $D(\mathcal{O}_V)$. Since $F|_U \to A$ is an isomorphism, and since we have the isomorphism $c : A|_{U\times_X V} \to B|_{U\times_X V}$ the restriction of $F|_V \to B$ is an isomorphism over $U \times_X V$. Thus B' is supported on $j_V^{-1}(T)$ where $T = |X| \setminus |U|$. On the other hand, there is a morphism of distinguished triangles

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & Rj_{U,*}F|_U \oplus Rj_{V,*}F|_V \arrow[r] \arrow[d] & Rj_{U\times_X V,*}F|_{U\times_X V} \arrow[r] \arrow[d] & F[1] \arrow[d] \\
F \arrow[r] & Rj_{U,*}A \oplus Rj_{V,*}B \arrow[r] & Rj_{U\times_X V,*}(B|_{U\times_X V}) \arrow[r] & F[1]
\end{tikzcd}$$

The all of the vertical maps in this diagram are isomorphisms, except for the map $Rj_{V,*}F|_V \to Rj_{V,*}B$, hence that is an isomorphism too (Derived Categories, Lemma 4.3). This implies that $Rj_{V,*}B' = 0$. Hence $B' = 0$ by Lemma 9.6.

The existence of the morphism $F \to E$ follows from the Mayer-Vietoris sequence for Hom, see Lemma 9.4. □

10. The coherator

Let S be a scheme. Let X be an algebraic space over S. The *coherator* is a functor

$$Q_X : \mathit{Mod}(\mathcal{O}_X) \longrightarrow \mathit{QCoh}(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor $\mathit{QCoh}(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$. It exists for any algebraic space X and moreover the adjunction mapping $Q_X(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every quasi-coherent module $\mathcal{F}$, see Properties of Spaces, Proposition 30.2. Since Q_X is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(\mathit{QCoh}(\mathcal{O}_X)).$$

As this functor is constructed by applying Q_X to a K-injective replacement we see that RQ_X is a right adjoint to the canonical functor $D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathcal{O}_X)$.

Lemma 10.1. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then f_* defines a derived functor $f_* : D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathit{QCoh}(\mathcal{O}_Y))$. This functor has the property that*

$$\begin{tikzcd}
D(\mathit{QCoh}(\mathcal{O}_X)) \arrow[r] \arrow[d, "f_*"'] & D_{\mathit{QCoh}}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\
D(\mathit{QCoh}(\mathcal{O}_Y)) \arrow[r] & D_{\mathit{QCoh}}(\mathcal{O}_Y)
\end{tikzcd}$$

commutes.

Proof. The functor $f_* : \mathit{QCoh}(\mathcal{O}_X) \to \mathit{QCoh}(\mathcal{O}_Y)$ is exact, see Cohomology of Spaces, Lemma 7.2. Hence f_* defines a derived functor $f_* : D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathit{QCoh}(\mathcal{O}_Y))$ by simply applying f_* to any representative complex, see Derived Categories, Lemma 17.8. For any complex of $\mathcal{O}_X$-modules $\mathcal{F}^\bullet$ there is a canonical map $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$. To finish the proof we show this is a quasi-isomorphism when $\mathcal{F}^\bullet$ is a complex with each $\mathcal{F}^n$ quasi-coherent. The statement is étale local on Y hence we may assume Y affine. As an affine morphism is representable we reduce to the case of schemes by the compatibility of Remark 6.3. The case of schemes is Derived Categories of Schemes, Lemma 6.1. □

Lemma 10.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

1. *f is quasi-compact, quasi-separated, and flat, and*
2. *denoting*
$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$
the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ the diagram
$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$
commutes.

Then $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.

Proof. Since f is quasi-compact and quasi-separated, we see that f_* preserve quasi-coherence, see Morphisms of Spaces, Lemma 11.2. Recall that $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties of Spaces, Proposition 30.2). Hence any K in $D(QCoh(\mathcal{O}_X))$ can be represented by a K-injective complex $\mathcal{I}^\bullet$ of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 12.6. Then we can define $\Phi(K) = f_*\mathcal{I}^\bullet$.

Since f is flat, the functor f^* is exact. Hence f^* defines $f^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ and also $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$. The functor $f^* = Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ is left adjoint to $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$, see Cohomology on Sites, Lemma 19.1. Similarly, the functor $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$ is left adjoint to $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by Derived Categories, Lemma 28.4.

Let A be an object of $D(QCoh(\mathcal{O}_Y))$ and E an object of $D(\mathcal{O}_X)$. Then
$$\begin{aligned} \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_*E)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(A, Rf_*E) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*A, E) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(f^*A, RQ_X(E)) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E))) \end{aligned}$$
This implies what we want. □

Lemma 10.3. *Let S be a scheme. Let X be an affine algebraic space over S. Set $A = \Gamma(X, \mathcal{O}_X)$. Then*

1. *$Q_X : Mod(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$ is the functor which sends $\mathcal{F}$ to the quasi-coherent $\mathcal{O}_X$-module associated to the A-module $\Gamma(X, \mathcal{F})$,*
2. *$RQ_X : D(\mathcal{O}_X) \to D(QCoh(\mathcal{O}_X))$ is the functor which sends E to the complex of quasi-coherent $\mathcal{O}_X$-modules associated to the object $R\Gamma(X, E)$ of $D(A)$,*
3. *restricted to $D_{QCoh}(\mathcal{O}_X)$ the functor RQ_X defines a quasi-inverse to (5.1.1).*

Proof. Let $X_0 = \mathrm{Spec}(A)$ be the affine scheme representing X. Recall that there is a morphism of ringed sites $\epsilon : X_{\acute{e}tale} \to X_{0,Zar}$ which induces equivalences
$$QCoh(\mathcal{O}_X) \underset{\epsilon^*}{\overset{\epsilon_*}{\rightleftarrows}} QCoh(\mathcal{O}_{X_0}),$$
see Lemma 4.2. Hence we see that $Q_X = \epsilon^* \circ Q_{X_0} \circ \epsilon_*$ by uniqueness of adjoint functors. Hence (1) follows from the description of Q_{X_0} in Derived Categories of

Schemes, Lemma 6.3 and the fact that $\Gamma(X_0, \epsilon_*\mathcal{F}) = \Gamma(X, \mathcal{F})$. Part (2) follows from (1) and the fact that the functor from A-modules to quasi-coherent $\mathcal{O}_X$-modules is exact. The third assertion now follows from the result for schemes (Derived Categories of Schemes, Lemma 6.3) and Lemma 4.2. $\square$

Proposition 10.4. *Let S be a scheme. Let X be a quasi-compact algebraic space over S with affine diagonal. Then the functor (5.1.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. We first use the induction principle to prove i_X is fully faithful. Let $\mathcal{B} \subset \text{Ob}(X_{spaces,\acute{e}tale})$ be the set of objects which are quasi-compact and have affine diagonal. For $U \in \mathcal{B}$ let $P(U)$ = "the functor $i_U : D(QCoh(\mathcal{O}_U)) \to D_{QCoh}(\mathcal{O}_U)$ is fully faithful". By Remark 8.5 conditions (1), (2), and (3)(a) of Lemma 8.4 hold and we are left with proving (3)(b) and (4). Condition (3)(b) holds by Lemma 10.3.

Let $(U \subset W, V \to W)$ be an elementary distinguished square with V affine. Assume that P holds for U, V, and $U \times_W V$. We have to show that P holds for W. We may replace X by W, i.e., we may assume $W = X$ (we do this just to simplify the notation).

Suppose that A, B are objects of $D(QCoh(\mathcal{O}_X))$. We want to show that

$$\text{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B) \longrightarrow \text{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B))$$

is bijective. Let $T = |X| \setminus |U|$.

Assume first $i_X(B)$ is supported on T. In this case the map

$$i_X(B) \to Rj_{V,*}(i_X(B)|_V) = Rj_{V,*}(i_V(B|_V))$$

is a quasi-isomorphism (Lemma 9.6). The morphism $V \to X$ is affine as V is affine and X has affine diagonal (Morphisms of Spaces, Lemma 20.11). Thus we have an object $j_{V,*}(B|_V)$ in $QCoh(\mathcal{O}_X)$ and an isomorphism $i_X(j_{V,*}(B|_V)) \to Rj_{V,*}(i_V(B|_V))$ in $D(\mathcal{O}_X)$ (Lemma 10.1). Moreover, $j_{V,*}$ and $-|_V$ are adjoint functors on the derived categories of quasi-coherent modules, see proof Lemma 10.2. The adjunction map $B \to j_{V,*}(B|_V)$ becomes an isomorphism after applying i_X, whence is an isomorphism in $D(QCoh(\mathcal{O}_X))$. Hence

$$\begin{aligned}
\text{Mor}_{D(QCoh(\mathcal{O}_X))}(A, B) &= \text{Mor}_{D(QCoh(\mathcal{O}_X))}(A, j_{V,*}(B|_V)) \\
&= \text{Mor}_{D(QCoh(\mathcal{O}_V))}(A|_V, B|_V) \\
&= \text{Mor}_{D(\mathcal{O}_V)}(i_V(A|_V), i_V(B|_V)) \\
&= \text{Mor}_{D(\mathcal{O}_X)}(i_X(A), Rj_{V,*}(i_V(B|_V))) \\
&= \text{Mor}_{D(\mathcal{O}_X)}(i_X(A), i_X(B))
\end{aligned}$$

as desired.

In general, choose any complex $\mathcal{B}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules representing B. Next, choose any quasi-isomorphism $s : \mathcal{B}^\bullet|_U \to \mathcal{C}^\bullet$ of complexes of quasi-coherent modules on U. As $j_U : U \to X$ is quasi-compact and quasi-separated the functor $j_{U,*}$ transforms quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 11.2). Thus there is a canonical map $\mathcal{B}^\bullet \to j_{U,*}(\mathcal{B}^\bullet|_U) \to j_{U,*}\mathcal{C}^\bullet$

of complexes of quasi-coherent modules on X. Set $B'' = j_{U,*}\mathcal{C}^\bullet$ in $D(QCoh(\mathcal{O}_X))$ and choose a distinguished triangle

$$B \to B'' \to B' \to B^\bullet[1]$$

in $D(QCoh(\mathcal{O}_X))$. Since the first arrow of the triangle restricts to an isomorphism over U we see that B' is supported on T. Hence in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'[-1]) \arrow[r] \arrow[d] & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')[-1]) \arrow[d] \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B) \arrow[r] \arrow[d] & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B)) \arrow[d] \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'') \arrow[r] \arrow[d] & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B'')) \arrow[d] \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B') \arrow[r] & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B'))
\end{tikzcd}$$

we have exact columns and the top and bottom horizontal arrows are bijective. Finally, choose a complex $\mathcal{A}^\bullet$ of quasi-coherent modules representing A.

Let $\alpha : i_X(A) \to i_X(B)$ be a morphism between in $D(\mathcal{O}_X)$. The restriction $\alpha|_U$ comes from a morphism in $D(QCoh(\mathcal{O}_U))$ by assumption. Hence there exists a choice of $s : \mathcal{B}^\bullet|_U \to \mathcal{C}^\bullet$ as above such that $\alpha|_U$ is represented by an actual map of complexes $\mathcal{A}^\bullet|_U \to \mathcal{C}^\bullet$. This corresponds to a map of complexes $\mathcal{A} \to j_{U,*}\mathcal{C}^\bullet$. In other words, the image of α in $\mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B''))$ comes from an element of $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'')$. A diagram chase then shows that α comes from a morphism $A \to B$ in $D(QCoh(\mathcal{O}_X))$. Finally, suppose that $a : A \to B$ is a morphism of $D(QCoh(\mathcal{O}_X))$ which becomes zero in $D(\mathcal{O}_X)$. After choosing $\mathcal{B}^\bullet$ suitably, we may assume a is represented by a morphism of complexes $a^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$. Since P holds for U the restriction $a^\bullet|_U$ is zero in $D(QCoh(\mathcal{O}_U))$. Thus we can choose s such that $s \circ a^\bullet|_U : \mathcal{A}^\bullet|_U \to \mathcal{C}^\bullet$ is homotopic to zero. Applying the functor $j_{U,*}$ we conclude that $\mathcal{A}^\bullet \to j_{U,*}\mathcal{C}^\bullet$ is homotopic to zero. Thus a maps to zero in $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'')$. Thus we may assume that a is the image of an element of $b \in \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'[-1])$. The image of b in $\mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')[-1])$ comes from a $\gamma \in \mathrm{Hom}_{D(\mathcal{O}_X)}(A, B''[-1])$ (as a maps to zero in the group on the right). Since we've seen above the horizontal arrows are surjective, we see that γ comes from a c in $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B''[-1])$ which implies $a = 0$ as desired.

Since i_X is fully faithful with right adjoint RQ_X we see that $RQ_X \circ i_X = \mathrm{id}$ (Categories, Lemma 24.3). To finish the proof we show that for any E in $D_{QCoh}(\mathcal{O}_X)$ the map $i_X(RQ_X(E)) \to E$ is an isomorphism. Choose a distinguished triangle

$$i_X(RQ_X(E)) \to E \to E' \to i_X(RQ_X(E))[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. A formal argument using the above shows that $i_X(RQ_X(E')) = 0$. Thus it suffices to prove that for $E \in D_{QCoh}(\mathcal{O}_X)$ the condition $i_X(RQ_X(E)) = 0$ implies that $E = 0$. Consider an étale morphism $j : V \to X$ with V affine. By Lemmas 10.3, 10.1, and 10.2 we have

$$Rj_*(E|_V) = Rj_*(i_V(RQ_V(E|_V))) = i_X(j_*(RQ_V(E|_V))) = i_X(RQ_X(Rj_*(E|_V)))$$

Choose a distinguished triangle

$$E \to Rj_*(E|_V) \to E' \to E[1]$$

Apply RQ_X to get a distinguished triangle

$$0 \to RQ_X(Rj_*(E|_V)) \to RQ_X(E') \to 0[1]$$

in other words the map in the middle is an isomorphism. Combined with the string of equalities above we find that our first distinghuished triangle becomes a distinguished triangle

$$E \to i_X(RQ_X(E')) \to E' \to E[1]$$

where the middle morphism is the adjunction map. However, the composition $E \to E'$ is zero, hence $E \to i_X(RQ_X(E'))$ is zero by adjunction! Since this morphism is isomorphic to the morphism $E \to Rj_*(E|_V)$ adjoint to $\text{id} : E|_V \to E|_V$ we conclude that $E|_V$ is zero. Since this holds for all affine V étale over X we conlude E is zero as desired. $\square$

Remark 10.5. Analyzing the proof of Proposition 10.4 we see that we have shown the following. Let X be a quasi-compact and quasi-separated scheme. Suppose that for every étale morphism $j : V \to X$ with V affine the right derived functor

$$\Phi : D(\mathit{QCoh}(\mathcal{O}_U)) \to D(\mathit{QCoh}(\mathcal{O}_X))$$

of the left exact functor $j_* : \mathit{QCoh}(\mathcal{O}_V) \to \mathit{QCoh}(\mathcal{O}_X)$ fits into a commutative diagram

$$\begin{tikzcd}
D(\mathit{QCoh}(\mathcal{O}_V)) \arrow[r, "i_V"'] \arrow[d, "\Phi"'] & D_{\mathit{QCoh}}(\mathcal{O}_V) \arrow[d, "Rj_*"] \\
D(\mathit{QCoh}(\mathcal{O}_X)) \arrow[r, "i_X"] & D_{\mathit{QCoh}}(\mathcal{O}_X)
\end{tikzcd}$$

Then the functor (5.1.1)

$$D(\mathit{QCoh}(\mathcal{O}_X)) \longrightarrow D_{\mathit{QCoh}}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

11. The coherator for Noetherian spaces

We need a little bit more about injective modules to treat the case of a Noetherian algebraic space.

Lemma 11.1. *Let S be a Noetherian affine scheme. Every injective object of $\mathit{QCoh}(\mathcal{O}_S)$ is a filtered colimit $\mathrm{colim}_i \mathcal{F}_i$ of quasi-coherent sheaves of the form*

$$\mathcal{F}_i = (Z_i \to S)_* \mathcal{G}_i$$

where Z_i is the spectrum of an Artinian ring and $\mathcal{G}_i$ is a coherent module on Z_i.

Proof. Let $S = \mathrm{Spec}(A)$. Let $\mathcal{J}$ be an injective object of $\mathit{QCoh}(\mathcal{O}_S)$. Since $\mathit{QCoh}(\mathcal{O}_S)$ is equivalent to the category of A-modules we see that $\mathcal{J}$ is equal to $\widetilde{J}$ for some injective A-module J. By Dualizing Complexes, Proposition 5.9 we can write $J = \bigoplus E_\alpha$ with E_α indecomposable and therefore isomorphic to the injective hull of a reside field at a point. Thus (because finite disjoint unions of Artinian schemes are Artinian) we may assume that J is the injective hull of $\kappa(\mathfrak{p})$ for some prime $\mathfrak{p}$ of A. Then $J = \bigcup J[\mathfrak{p}^n]$ where $J[\mathfrak{p}^n]$ is the injective hull of

$\kappa(\mathfrak{p})$ over $A_{/}\mathfrak{p}^n A_\mathfrak{p}$, see Dualizing Complexes, Lemma 7.3. Thus $\widetilde{J}$ is the colimit of the sheaves $(Z_n \to X)_*\mathcal{G}_n$ where $Z_n = \text{Spec}(A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p})$ and $\mathfrak{G}_n$ the coherent sheaf associated to the finite $A_{/}\mathfrak{p}^n A_\mathfrak{p}$-module $J[\mathfrak{p}^n]$. Finiteness follows from Dualizing Complexes, Lemma 6.1. □

Lemma 11.2. *Let S be an affine scheme. Let X be a Noetherian algebraic space over S. Every injective object of $QCoh(\mathcal{O}_X)$ is a direct summand of a filtered colimit* $\text{colim}_i \mathcal{F}_i$ *of quasi-coherent sheaves of the form*

$$\mathcal{F}_i = (Z_i \to X)_*\mathcal{G}_i$$

where Z_i is the spectrum of an Artinian ring and $\mathcal{G}_i$ is a coherent module on Z_i.

Proof. Choose an affine scheme U and a surjective étale morphism $j : U \to X$ (Properties of Spaces, Lemma 6.3). Then U is a Noetherian affine scheme. Choose an injective object $\mathcal{J}'$ of $QCoh(\mathcal{O}_U)$ such that there exists an injection $\mathcal{J}|_U \to \mathcal{J}'$. Then

$$\mathcal{J} \to j_*\mathcal{J}'$$

is an injective morphism in $QCoh(\mathcal{O}_X)$, hence identifies $\mathcal{J}$ as a direct summand of $j_*\mathcal{J}'$. Thus the result follows from the corresponding result for $\mathcal{J}'$ proved in Lemma 11.1. □

Lemma 11.3. *Let S be a scheme. Let $f : X \to Y$ be a flat, quasi-compact, and quasi-separated morphism of algebraic spaces over S. If $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$, then $f_*\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_Y)$.*

Proof. Since f is quasi-compact and quasi-separated, the functor f_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Morphisms of Spaces, Lemma 11.2). The functor f^* is a left adjoint to f_* which transforms injections into injections. Hence the result follows from Homology, Lemma 25.1 □

Lemma 11.4. *Let S be a scheme. Let X be a Noetherian algebraic space over S. If $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$, then*

1. *$H^p(U, \mathcal{J}|_U) = 0$ for $p > 0$ and for every quasi-compact and quasi-separated algebraic space U étale over X,*
2. *for any morphism $f : X \to Y$ of algebraic spaces over S with Y quasi-separated we have $R^p f_*\mathcal{J} = 0$ for $p > 0$.*

Proof. Proof of (1). Write $\mathcal{J}$ as a direct summand of $\text{colim}\, \mathcal{F}_i$ with $\mathcal{F}_i = (Z_i \to X)_*\mathcal{G}_i$ as in Lemma 11.2. It is clear that it suffices to prove the vanishing for $\text{colim}\, \mathcal{F}_i$. Since pullback commutes with colimits and since U is quasi-compact and quasi-separated, it suffices to prove $H^p(U, \mathcal{F}_i|_U) = 0$ for $p > 0$, see Cohomology of Spaces, Lemma 4.1. Observe that $Z_i \to X$ is an affine morphism, see Morphisms of Spaces, Lemma 20.12. Thus

$$\mathcal{F}_i|_U = (Z_i \times_X U \to U)_*\mathcal{G}'_i = R(Z_i \times_X U \to U)_*\mathcal{G}'_i$$

where $\mathcal{G}'_i$ is the pullback of $\mathcal{G}_i$ to $Z_i \times_X U$, see Cohomology of Spaces, Lemma 10.2. Since $Z_i \times_X U$ is affine we conlude that $\mathcal{G}'_i$ has no higher cohomology on $Z_i \times_X U$. By the Leray spectral sequence we conclude the same thing is true for $\mathcal{F}_i|_U$ (Cohomology on Sites, Lemma 14.6).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \to Y$ be an étale morphism with V affine. Then $V \times_Y X \to X$ is an étale morphism and $V \times_Y X$ is a quasi-compact and quasi-separated algebraic space étale over X (details omitted). Hence $H^p(V \times_Y X, \mathcal{J})$ is zero by part (1). Since $R^p f_* \mathcal{J}$ is the sheaf associated to the presheaf $V \mapsto H^p(V \times_Y X, \mathcal{J})$ the result is proved. $\square$

Lemma 11.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of Noetherian algebraic spaces over S. Then f_* on quasi-coherent sheaves has a right derived extension $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ such that the diagram*

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

commutes.

Proof. Since X and Y are Noetherian the morphism is quasi-compact and quasi-separated (see Morphisms of Spaces, Lemma 8.9). Thus f_* preserve quasi-coherence, see Morphisms of Spaces, Lemma 11.2. Next, Let K be an object of $D(QCoh(\mathcal{O}_X))$. Since $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties of Spaces, Proposition 30.2), we can represent K by a K-injective complex $\mathcal{I}^\bullet$ such that each $\mathcal{I}^n$ is an injective object of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 12.6. Thus we see that the functor Φ is defined by setting

$$\Phi(K) = f_* \mathcal{I}^\bullet$$

where the right hand side is viewed as an object of $D(QCoh(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$f_* \mathcal{I}^\bullet \longrightarrow Rf_* \mathcal{I}^\bullet$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this it suffices to prove the map induces an isomorphism on cohomology sheaves. Pick any $m \in \mathbf{Z}$. Let $N = N(X, Y, f)$ be as in Lemma 6.1. Consider the short exact sequence

$$0 \to \sigma_{\geq m-N-1} \mathcal{I}^\bullet \to \mathcal{I}^\bullet \to \sigma_{\leq m-N-2} \mathcal{I}^\bullet \to 0$$

of complexes of quasi-coherent sheaves on X. By Lemma 6.1 we see that the cohomology sheaves of $Rf_* \sigma_{\leq m-N-2} \mathcal{I}^\bullet$ are zero in degrees $\geq m - 1$. Thus we see that $R^m f_* \mathcal{I}^\bullet$ is isomorphic to $R^m f_* \sigma_{\geq m-N-1} \mathcal{I}^\bullet$. In other words, we may assume that $\mathcal{I}^\bullet$ is a bounded below complex of injective objects of $QCoh(\mathcal{O}_X)$. This case follows from Leray's acyclicity lemma (Derived Categories, Lemma 17.7) with required vanishing because of Lemma 11.4. $\square$

Proposition 11.6. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Then the functor (5.1.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. This follows using the exact same argument as in the proof of Proposition 10.4 using Lemma 11.5. See discussion in Remark 10.5. $\square$

12. Pseudo-coherent and perfect complexes

In this section we study the general notions defined in Cohomology on Sites, Sections 33, 34, 35, and 36 for the étale site of an algebraic space. In particular we match this with what happens for schemes.

First we compare the notion of a pseudo-coherent complex on a scheme and on its associated small étale site.

Lemma 12.1. *Let X be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is of finite type as an $\mathcal{O}_X$-module, and*
(2) *$\epsilon^*\mathcal{F}$ is of finite type as an $\mathcal{O}_{étale}$-module on the small étale site of X.*

Here ϵ is as in (4.0.1).

Proof. The implication (1) $\Rightarrow$ (2) is a general fact, see Modules on Sites, Lemma 23.4. Assume (2). By assumption there exists an étale covering $\{f_i : X_i \to X\}$ such that $\epsilon^*\mathcal{F}|_{(X_i)_{étale}}$ is generated by finitely many sections. Let $x \in X$. We will show that $\mathcal{F}$ is generated by finitely many sections in a neighbourhood of x. Say x is in the image of $X_i \to X$ and denote $X' = X_i$. Let $s_1, \ldots, s_n \in \Gamma(X', \epsilon^*\mathcal{F}|_{X'_{étale}})$ be generating sections. As $\epsilon^*\mathcal{F} = \epsilon^{-1}\mathcal{F} \otimes_{\epsilon^{-1}\mathcal{O}_X} \mathcal{O}_{étale}$ we can find an étale morphism $X'' \to X'$ such that x is in the image of $X' \to X$ and such that $s_i|_{X''} = \sum s_{ij} \otimes a_{ij}$ for some sections $s_{ij} \in \epsilon^{-1}\mathcal{F}(X'')$ and $a_{ij} \in \mathcal{O}_{étale}(X'')$. Denote $U \subset X$ the image of $X'' \to X$. This is an open subscheme as $f'' : X'' \to X$ is étale (Morphisms, Lemma 37.13). After possibly shrinking X'' more we may assume s_{ij} come from elements $t_{ij} \in \mathcal{F}(U)$ as follows from the construction of the inverse image functor ϵ^{-1}. Now we claim that t_{ij} generate $\mathcal{F}|_U$ which finishes the proof of the lemma. Namely, the corresponding map $\mathcal{O}_U^{\oplus N} \to \mathcal{F}|_U$ has the property that its pullback by f'' to X'' is surjective. Since $f'' : X'' \to U$ is a surjective flat morphism of schemes, this implies that $\mathcal{O}_U^{\oplus N} \to \mathcal{F}|_U$ is surjective by looking at stalks and using that $\mathcal{O}_{U,f''(z)} \to \mathcal{O}_{X'',z}$ is faithfully flat for all $z \in X''$. □

In the situation above the morphism of sites ϵ is flat hence defines a pullback on complexes of modules.

Lemma 12.2. *Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent*

(1) *E is m-pseudo-coherent, and*
(2) *ϵ^*E is m-pseudo-coherent on the small étale site of X.*

Here ϵ is as in (4.0.1).

Proof. The implication (1) $\Rightarrow$ (2) is a general fact, see Cohomology on Sites, Lemma 34.3. Assume ϵ^*E is m-pseudo-coherent. We will use without further mention that ϵ^* is an exact functor and that therefore

$$\epsilon^*H^i(E) = H^i(\epsilon^*E).$$

To show that E is m-pseudo-coherent we may work locally on X, hence we may assume that X is quasi-compact (for example affine). Since X is quasi-compact every étale covering $\{U_i \to X\}$ has a finite refinement. Thus we see that ϵ^*E is an object of $D^-(\mathcal{O}_{étale})$, see comments following Cohomology on Sites, Definition 34.1. By Lemma 4.1 it follows that E is an object of $D^-(\mathcal{O}_X)$.

Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(E)$ is nonzero; then n is also the largest integer such that $H^n(\epsilon^*E)$ is nonzero. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is clearly true. If $n \geq m$, then $H^n(\epsilon^*E)$ is a finite $\mathcal{O}_{étale}$-module, see Cohomology on Sites, Lemma 34.7. Hence $H^n(E)$ is a finite $\mathcal{O}_X$-module, see Lemma 12.1. After replacing X by the members of an open covering, we may assume there exists a surjection $\mathcal{O}_X^{\oplus t} \to H^n(E)$. We may locally on X lift this to a map of complexes $\alpha : \mathcal{O}_X^{\oplus t}[-n] \to E$ (details omitted). Choose a distinguished triangle

$$\mathcal{O}_X^{\oplus t}[-n] \to E \to C \to \mathcal{O}_X^{\oplus t}[-n+1]$$

Then C has vanishing cohomology in degrees $\geq n$. On the other hand, the complex ϵ^*C is m-pseudo-coherent, see Cohomology on Sites, Lemma 34.4. Hence by induction we see that C is m-pseudo-coherent. Applying Cohomology on Sites, Lemma 34.4 once more we conclude. □

Lemma 12.3. *Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Then*

(1) *E has tor amplitude in $[a, b]$ if and only if ϵ^*E has tor amplitude in $[a, b]$.*
(2) *E has finite tor dimension if and only if ϵ^*E has finite tor dimension.*

Here ϵ is as in (4.0.1).

Proof. The easy implication follows from the general result contained in Cohomology on Sites, Lemma 35.4 (and the fact that the small étale site of X has enough points, see Étale Cohomology, Remarks 29.11). For the converse, assume that ϵ^*E has tor amplitude in $[a, b]$ Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. As ϵ is a flat morphism of ringed sites (Lemma 4.1) we have

$$\epsilon^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = \epsilon^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{étale}} \epsilon^*\mathcal{F}$$

Thus the (assumed) vanishing of cohomology sheaves on the right hand side implies the desired vanishing of the cohomology sheaves of $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ via Lemma 4.1. □

Lemma 12.4. *Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Then E is a perfect object of $D(\mathcal{O}_X)$ if and only if ϵ^*E is a perfect object of $D(\mathcal{O}_{étale})$. Here ϵ is as in (4.0.1).*

Proof. The easy implication follows from the general result contained in Cohomology on Sites, Lemma 36.5 (and the fact that the small étale site of X has enough points, see Étale Cohomology, Remarks 29.11). For the converse, we can use the equivalence of Cohomology on Sites, Lemma 36.4 and the corresponding results for pseudo-coherent and complexes of finite tor dimension, namely Lemmas 12.2 and 12.3. Some details omitted. □

Lemma 12.5. *Let S be a scheme. Let X be an algebraic space over S. If E is an m-pseudo-coherent object of $D(\mathcal{O}_X)$, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$. If E is pseudo-coherent, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Locally $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ with $\mathcal{E}^\bullet$ strictly perfect. The sheaves $\mathcal{E}^i$ are direct summands of finite free modules, hence quasi-coherent. The lemma follows. □

Lemma 12.6. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. For $m \in \mathbf{Z}$ the following are equivalent*

(1) *$H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and*

(2) *E is m-pseudo-coherent.*

In particular, E is pseudo-coherent if and only if E is an object of $D^-_{Coh}(\mathcal{O}_X)$.

Proof. As X is quasi-compact we can find an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 6.3). Observe that U is Noetherian. Note that E is m-pseudo-coherent if and only if $E|_U$ is m-pseudo-coherent (follows from the definition or from Cohomology on Sites, Lemma 34.2). Similarly, $H^i(E)$ is coherent if and only if $H^i(E)|_U = H^i(E|_U)$ is coherent (see Cohomology of Spaces, Lemma 11.2). Thus we may assume that X is representable.

If X is representable by a scheme X_0 then (Lemma 4.2) we can write $E = \epsilon^* E_0$ where E_0 is an object of $D_{QCoh}(\mathcal{O}_{X_0})$ and $\epsilon : X_{\acute{e}tale} \to (X_0)_{Zar}$ is as in (4.0.1). In this case E is m-pseudo-coherent if and only if E_0 is by Lemma 12.2. Similarly, $H^i(E_0)$ is of finite type (i.e., coherent) if and only if $H^i(E)$ is by Lemma 12.1. Finally, $H^i(E_0) = 0$ if and only if $H^i(E) = 0$ by Lemma 4.1. Thus we reduce to the case of schemes which is Derived Categories of Schemes, Lemma 9.4. □

Lemma 12.7. *Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent*

(1) *E has tor amplitude in $[a, b]$, and*
(2) *for all $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for $i \notin [a, b]$.*

Proof. It is clear that (1) implies (2). Assume (2). Let $j : U \to X$ be an étale morphism with U affine. As X is quasi-separated $j : U \to X$ is quasi-compact and separated, hence j_* transforms quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 11.2). Thus the functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{G})$ for $i \notin [a, b]$ for all quasi-coherent $\mathcal{O}_U$-modules $\mathcal{G}$. Since it suffices to prove that $E|_U$ has tor amplitude in $[a, b]$ we reduce to the case where X is representable.

If X is representable by a scheme X_0 then (Lemma 4.2) we can write $E = \epsilon^* E_0$ where E_0 is an object of $D_{QCoh}(\mathcal{O}_{X_0})$ and $\epsilon : X_{\acute{e}tale} \to (X_0)_{Zar}$ is as in (4.0.1). For every quasi-coherent module $\mathcal{F}_0$ on X_0 the module $\epsilon^* \mathcal{F}_0$ is quasi-coherent on X and

$$H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \epsilon^* \mathcal{F}_0) = \epsilon^* H^i(E_0 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_0}} \mathcal{F}_0)$$

as ϵ is flat (Lemma 4.1). Moreover, the vanishing of these sheaves for $i \notin [a, b]$ implies the same thing for $H^i(E_0 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_0}} \mathcal{F}_0)$ by the same lemma. Thus we've reduced the problem to the case of schemes which is treated in Derived Categories of Schemes, Lemma 9.6. □

Lemma 12.8. *Let X be a scheme. Let E, F be objects of $D(\mathcal{O}_X)$. Assume either*

(1) *E is pseudo-coherent and F lies in $D^+(\mathcal{O}_X)$, or*
(2) *E is perfect and F arbitrary,*

then there is a canonical isomorphism

$$\epsilon^* R\mathcal{H}om(E, F) \longrightarrow R\mathcal{H}om(\epsilon^* E, \epsilon^* F)$$

Here ϵ is as in (4.0.1).

Proof. Recall that ϵ is flat (Lemma 4.1) and hence $\epsilon^* = L\epsilon^*$. There is a canonical map from left to right by Cohomology on Sites, Remark 26.9. To see this is an isomorphism we can work locally, i.e., we may assume X is an affine scheme.

In case (1) we can represent E by a bounded above complex $\mathcal{E}^\bullet$ of finite free $\mathcal{O}_X$-modules, see Derived Categories of Schemes, Lemma 11.2. We may also represent F by a bounded below complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules. Applying Cohomology, Lemma 35.10 we see that $R\mathcal{H}om(E, F)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{F}^q)$$

Applying Cohomology on Sites, Lemma 33.10 we see that $R\mathcal{H}om(\epsilon^*E, \epsilon^*F)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_{\acute{e}tale}}(\epsilon^*\mathcal{E}^p, \epsilon^*\mathcal{F}^q)$$

Thus the statement of the lemma boils down to the true fact that the canonical map

$$\epsilon^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{\acute{e}tale}}(\epsilon^*\mathcal{E}, \epsilon^*\mathcal{F})$$

is an isomorphism for any $\mathcal{O}_X$-module $\mathcal{F}$ and finite free $\mathcal{O}_X$-module $\mathcal{E}$.

In case (2) we can represent E by a strictly perfect complex $\mathcal{E}^\bullet$ of $\mathcal{O}_X$-modules, use Derived Categories of Schemes, Lemmas 3.4 and 9.7 and the fact that a perfect complex of modules is represented by a finite complex of finite projective modules. Thus we can do the exact same proof as above, replacing the reference to Cohomology, Lemma 35.10 by a reference to Cohomology, Lemma 35.9. $\square$

Lemma 12.9. *Let S be a scheme. Let X be an algebraic space over S. Let L, K be objects of $D(\mathcal{O}_X)$. If either*

1. *L in $D^+_{QCoh}(\mathcal{O}_X)$ and K is pseudo-coherent,*
2. *L in $D_{QCoh}(\mathcal{O}_X)$ and K is perfect,*

then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$.

Proof. This follows from the analogue for schemes (Derived Categories of Schemes, Lemma 9.8) via the criterion of Lemma 5.2, the criterion of Lemmas 12.2 and 12.4, and the result of Lemma 12.8. $\square$

13. Approximation by perfect complexes

In this section we continue the discussion started in Derived Categories of Schemes, Section 12.

Definition 13.1. Let S be a scheme. Let X be an algebraic space over S. Consider triples (T, E, m) where

1. $T \subset |X|$ is a closed subset,
2. E is an object of $D_{QCoh}(\mathcal{O}_X)$, and
3. $m \in \mathbf{Z}$.

We say *approximation holds for the triple* (T, E, m) if there exists a perfect object P of $D(\mathcal{O}_X)$ supported on T and a map $\alpha : P \to E$ which induces isomorphisms $H^i(P) \to H^i(E)$ for $i > m$ and a surjection $H^m(P) \to H^m(E)$.

Approximation cannot hold for every triple. Please read the remarks following Derived Categories of Schemes, Definition 12.1 to see why.

Definition 13.2. Let S be a scheme. Let X be an algebraic space over S. We say *approximation by perfect complexes holds* on X if for any closed subset $T \subset |X|$ such that the morphism $X \setminus T \to X$ is quasi-compact there exists an integer r such that for every triple (T, E, m) as in Definition 13.1 with

(1) E is $(m-r)$-pseudo-coherent, and
(2) $H^i(E)$ is supported on T for $i \geq m - r$

approximation holds.

Lemma 13.3. *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic space over S. Let E be a perfect object of $D(\mathcal{O}_V)$ supported on $j^{-1}(T)$ where $T = |X| \setminus |U|$. Then Rj_*E is a perfect object of $D(\mathcal{O}_X)$.*

Proof. Being perfect is local on $X_{étale}$. Thus it suffices to check that Rj_*E is perfect when restricted to U and V. We have $Rj_*E|_V = E$ by Lemma 9.6 which is perfect. We have $Rj_*E|_U = 0$ because $E|_{V \setminus j^{-1}(T)} = 0$ (use Lemma 3.1). $\square$

Lemma 13.4. *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Let T be a closed subset of $|X| \setminus |U|$ and let (T, E, m) be a triple as in Definition 13.1. If*

(1) *approximation holds for $(j^{-1}T, E|_V, m)$, and*
(2) *the sheaves $H^i(E)$ for $i \geq m$ are supported on T,*

then approximation holds for (T, E, m).

Proof. Let $P \to E|_V$ be an approximation of the triple $(j^{-1}T, E|_V, m)$ over V. Then Rj_*P is a perfect object of $D(\mathcal{O}_X)$ by Lemma 13.3. On the other hand, $Rj_*P = j_!P$ by Lemma 9.6. We see that $j_!P$ is supported on T for example by (9.0.2). Hence we obtain an approximation $Rj_*P = j_!P \to j_!(E|_V) \to E$. $\square$

Lemma 13.5. *Let S be a scheme. Let X be an algebraic space over S which is representable by an affine scheme. Then approximation holds for every triple (T, E, m) as in Definition 13.1 such that there exists an integer $r \geq 0$ with*

(1) *E is m-pseudo-coherent,*
(2) *$H^i(E)$ is supported on T for $i \geq m - r + 1$,*
(3) *$X \setminus T$ is the union of r affine opens.*

In particular, approximation by perfect complexes holds for affine schemes.

Proof. Let X_0 be an affine scheme representing X. Let $T_0 \subset X_0$ by the closed subset corresponding to T. Let $\epsilon : X_{étale} \to X_{0,Zar}$ be the morphism (4.0.1). We may write $E = \epsilon^*E_0$ for some object E_0 of $D_{QCoh}(\mathcal{O}_{X_0})$, see Lemma 4.2. Then E_0 is m-pseudo-coherent, see Lemma 12.2. Comparing stalks of cohomology sheaves (see proof of Lemma 4.1) we see that $H^i(E_0)$ is supported on T_0 for $i \geq m - r + 1$. By Derived Categories of Schemes, Lemma 12.4 there exists an approximation $P_0 \to E_0$ of (T_0, E_0, m). By Lemma 12.4 we see that $P = \epsilon^*P_0$ is a perfect object of $D(\mathcal{O}_X)$. Pulling back we obtain an approximation $P = \epsilon^*P_0 \to \epsilon^*E_0 = E$ as desired. $\square$

Lemma 13.6. *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Assume U quasi-compact, V affine, and $U \times_X V$ quasi-compact. If approximation by perfect complexes holds on U, then approximation by perfect complexes holds on X.*

Proof. Let $T \subset |X|$ be a closed subset with $X \setminus T \to X$ quasi-compact. Let r_U be the integer of Definition 13.2 adapted to the pair $(U, T \cap |U|)$. Set $T' = T \setminus |U|$. Endow T' with the induced reduced subspace structure. Since $|T'|$ is contained in $|X| \setminus |U|$ we see that $j^{-1}(T') \to T'$ is an isomorphism. Moreover, $V \setminus j^{-1}(T')$ is quasi-compact as it is the fibre product of $U \times_X V$ with $X \setminus T$ over X and we've assumed $U \times_X V$ quasi-compact and $X \setminus T \to X$ quasi-compact. Let r' be the number of affines needed to cover $V \setminus j^{-1}(T')$. We claim that $r = \max(r_U, r')$ works for the pair (X, T).

To see this choose a triple (T, E, m) such that E is $(m-r)$-pseudo-coherent and $H^i(E)$ is supported on T for $i \geq m - r$. Let t be the largest integer such that $H^t(E)|_U$ is nonzero. (Such an integer exists as U is quasi-compact and $E|_U$ is $(m-r)$-pseudo-coherent.) We will prove that E can be approximated by induction on t.

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on T' for $i \geq m - r'$. Hence Lemma 13.5 guarantees the existence of an approximation $P \to E|_V$ of $(T', E|_V, m)$ on V. Applying Lemma 13.4 we see that (T', E, m) can be approximated. Such an approximation is also an approximation of (T, E, m).

Induction step. Choose an approximation $P \to E|_U$ of $(T \cap |U|, E|_U, m)$. This in particular gives a surjection $H^t(P) \to H^t(E|_U)$. In the rest of the proof we will use the equivalence of Lemma 4.2 (and the compatibilities of Remark 6.3) for the representable algebraic spaces V and $U \times_X V$. We will also use the fact that $(m-r)$-pseudo-coherence, resp. perfectness on the Zariski site and étale site agree, see Lemmas 12.2 and 12.4. Thus we can use the results of Derived Categories of Schemes, Section 11 for the open immersion $U \times_X V \subset V$. In this way Derived Categories of Schemes, Lemma 11.8 implies there exists a perfect object Q in $D(\mathcal{O}_V)$ supported on $j^{-1}(T)$ and an isomorphism $Q|_{U \times_X V} \to (P \oplus P[1])|_{U \times_X V}$. By Derived Categories of Schemes, Lemma 11.5 we can replace Q by $Q \otimes^{\mathbf{L}} I$ and assume that the map

$$Q|_{U \times_X V} \longrightarrow (P \oplus P[1])|_{U \times_X V} \longrightarrow P|_{U \times_X V} \longrightarrow E|_{U \times_X V}$$

lifts to $Q \to E|_V$. By Lemma 9.7 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T and the map $H^t(R) \to H^t(E)$ is surjective on restriction to U. Choose a distinguised triangle

$$R \to E \to E' \to R[1]$$

Then E' is $(m-r)$-pseudo-coherent (Cohomology on Sites, Lemma 34.4), $H^i(E')|_U = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r$. By induction we find an approximation $R' \to E'$ of (T, E', m). Fit the composition $R' \to E' \to R[1]$ into a distringuished triangle $R \to R'' \to R' \to R[1]$ and extend the morphisms $R' \to E'$ and $R[1] \to R[1]$ into a morphism of distinguished triangles

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R'' \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d] & R[1] \arrow[d] \\
R \arrow[r] & E \arrow[r] & E' \arrow[r] & R[1]
\end{tikzcd}$$

using TR3. Then R'' is a perfect complex (Cohomology on Sites, Lemma 36.6) supported on T. An easy diagram chase shows that $R'' \to E$ is the desired approximation. □

Theorem 13.7. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Then approximation by perfect complexes holds on X.*

Proof. This follows from the induction principle of Lemma 8.3 and Lemmas 13.6 and 13.5. □

14. Generating derived categories

This section is the analogue of Derived Categories of Schemes, Section 13. However, we first prove the following lemma which is the analogue of Derived Categories of Schemes, Lemma 11.9.

Lemma 14.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $W \subset X$ be a quasi-compact open. Let $T \subset |X|$ be a closed subset such that $X \setminus T \to X$ is a quasi-compact morphism. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\alpha : P \to E|_W$ be a map where P is a perfect object of $D(\mathcal{O}_W)$ supported on $T \cap W$. Then there exists a map $\beta : R \to E$ where R is a perfect object of $D(\mathcal{O}_X)$ supported on T such that P is a direct summand of $R|_W$ in $D(\mathcal{O}_W)$ compatible α and $\beta|_W$.*

Proof. We will use the induction principle of Lemma 8.6 to prove this. Thus we immediately reduce to the case where we have an elementary distinguished square $(W \subset X, f : V \to X)$ with V affine and $P \to E|_W$ as in the statement of the lemma. In the rest of the proof we will use Lemma 4.2 (and the compatibilities of Remark 6.3) for the representable algebraic spaces V and $W \times_X V$. We will also use the fact that perfectness on the Zariski site and étale site agree, see Lemma 12.4.

By Derived Categories of Schemes, Lemma 11.8 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $f^{-1}T$ and an isomorphism $Q|_{W \times_X V} \to (P \oplus P[1])|_{W \times_X V}$. By Derived Categories of Schemes, Lemma 11.5 we can replace Q by $Q \otimes^{\mathbf{L}} I$ (still supported on $f^{-1}T$) and assume that the map

$$Q|_{W \times_X V} \to (P \oplus P[1])|_{W \times V} \longrightarrow P|_{W \times_X V} \longrightarrow E|_{W \times_X V}$$

lifts to $Q \to E|_V$. By Lemma 9.7 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_W$ is isomorphic to $P \oplus P[1] \to E|_W$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T as desired. □

Remark 14.2. The proof of Lemma 14.1 shows that

$$R|_W = P \oplus P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_W$ equals that of P. By repeating the construction for the map $P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m] \to R|_W$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_W$ and P above any given degree.

Lemma 14.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let W be a quasi-compact open subspace of X. Let P be a perfect object of $D(\mathcal{O}_W)$. Then P is a direct summand of the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Special case of Lemma 14.1. □

Theorem 14.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. The category $D_{QCoh}(\mathcal{O}_X)$ can be generated by a single perfect object. More precisely, there exists a perfect object P of $D(\mathcal{O}_X)$ such that for $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

1. *$E = 0$, and*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$ for all $n \in \mathbf{Z}$.*

Proof. We will prove this using the induction principle of Lemma 8.3

If X is affine, then $\mathcal{O}_X$ is a perfect generator. This follows from Lemma 4.2 and Derived Categories of Schemes, Lemma 3.4.

Assume that $(U \subset X, f : V \to X)$ is an elementary distinguished square with U quasi-compact such that the theorem holds for U and V is an affine scheme. Let P be a perfect object of $D(\mathcal{O}_U)$ which is a generator for $D_{QCoh}(\mathcal{O}_U)$. Using Lemma 14.3 we may choose a perfect object Q of $D(\mathcal{O}_X)$ whose restriction to U is a direct sum one of whose summands is P. Say $V = \mathrm{Spec}(A)$. Let $Z \subset V$ be the reduced closed subscheme which is the inverse image of $X \setminus U$ and maps isomorphically to it (see Definition 8.1). This is a retrocompact closed subset of V. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on Z, the pushforward $K' = Rf_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 13.3 and 9.6). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Lemma 9.6 we have $K' = f_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Derived Categories of Schemes, Lemma 13.2 (using also Lemma 4.2) the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \times_X V \to V)_*E|_{U \times_X V}$. This implies that $E = R(U \to X)_*E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. □

The following result is an strengthening of Theorem 14.4 proved using exactly the same methods. Let $T \subset |X|$ be a closed subset where X is an algebraic space. Let's denote $D_T(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on T.

Lemma 14.5. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that $|X| \setminus T$ is quasi-compact. With notation as above, the category $D_{QCoh,T}(\mathcal{O}_X)$ is generated by a single perfect object.*

Proof. We will prove this using the induction principle of Lemma 8.3. The property is true for representable quasi-compact and quasi-separated objects of the site $X_{spaces,étale}$ by Derived Categories of Schemes, Lemma 13.5.

Assume that $(U \subset X, f : V \to X)$ is an elementary distinguished square such that the lemma holds for U and V is affine. To finish the proof we have to show that the result holds for X. Let P be a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$ which is a generator for $D_{QCoh, T\cap U}(\mathcal{O}_U)$. Using Lemma 14.1 we may choose a perfect object Q of $D(\mathcal{O}_X)$ supported on T whose restriction to U is a direct sum one of whose summands is P. Write $V = \mathrm{Spec}(B)$. Let $Z = X \setminus U$. Then $f^{-1}Z$ is a closed subset of V such that $V \setminus f^{-1}Z$ is quasi-compact. As X is quasi-separated, it follows that $f^{-1}Z \cap f^{-1}T = f^{-1}(Z \cap T)$ is a closed subset of V such that $W = V \setminus f^{-1}(Z \cap T)$ is quasi-compact. Thus we can choose $g_1, \ldots, g_s \in B$ such that $f^{-1}(Z \cap T) = V(g_1, \ldots, g_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $g_1, \ldots, g_s$ over B. Note that since K is supported on $f^{-1}(Z \cap T) \subset V$ closed, the pushforward $K' = R(V \to X)_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 13.3 and 9.6). We claim that $Q \oplus K'$ is a generator for $D_{QCoh,T}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh,T}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Lemma 9.6 we have $K' = R(V \to X)_! K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Derived Categories of Schemes, Lemma 13.2 we have $E|_V = Rj_*E|_W$ where $j : W \to V$ is the inclusion. Picture

$$\begin{tikzcd}
W \arrow[r, "j"] & V & Z \cap T \arrow[l] \arrow[d] \\
V \setminus f^{-1}Z \arrow[u, "j'"] \arrow[ur, "j''"'] & & Z \arrow[ul]
\end{tikzcd}$$

Since E is supported on T we see that $E|_W$ is supported on $f^{-1}T \cap W = f^{-1}T \cap (V \setminus f^{-1}Z)$ which is closed in W. We conclude that

$$E|_V = Rj_*(E|_W) = Rj_*(Rj'_*(E|_{U\cap V})) = Rj''_*(E|_{U\cap V})$$

Here the second equality is part (1) of Cohomology, Lemma 30.9 which applies because V is a scheme and E has quasi-coherent cohomology sheaves hence pushforward along the quasi-compact open immersion j' agrees with pushforward on the underlying schemes, see Remark 6.3. This implies that $E = R(U \to X)_*E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. $\square$

15. Compact and perfect objects

This section is the analogue of Derived Categories of Schemes, Section 14.

Proposition 15.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.*

Proof. By Cohomology on Sites, Lemma 39.1 the perfect objects even define compact objects of $D(\mathcal{O}_X)$. Conversely, let K be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that K is perfect, it suffices to show that $K|_U$ is perfect for every affine scheme U étale over X, see Cohomology on Sites, Lemma 36.2. Observe that $j : U \to X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 6.2. Thus the adjointness of restriction to U and Rj_* implies that $K|_U$ is a perfect object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that X is affine, in particular a quasi-compact and quasi-separated scheme. Via Lemma 4.2 and 12.4 we reduce to the case of schemes, i.e., to Derived Categories of Schemes, Proposition 14.1. □

The following result is a strengthening of Proposition 15.1. Let $T \subset |X|$ be a closed subset where X is an algebraic space. As before $D_T(\mathcal{O}_X)$ denotes the the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on T. Since taking direct sums commutes with taking cohomology sheaves, it follows that $D_T(\mathcal{O}_X)$ has direct sums and that they are equal to direct sums in $D(\mathcal{O}_X)$.

Lemma 15.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that $|X| \setminus T$ is quasi-compact. An object of $D_{QCoh,T}(\mathcal{O}_X)$ is compact if and only if it is perfect as an object of $D(\mathcal{O}_X)$.*

Proof. We observe that $D_{QCoh,T}(\mathcal{O}_X)$ is a triangulated category with direct sums by the remark preceding the lemma. By Cohomology on Sites, Lemma 39.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$ hence a fortiori of any subcategory preserved under taking direct sums. For the converse we will use there exists a generator $E \in D_{QCoh,T}(\mathcal{O}_X)$ which is a perfect complex of $\mathcal{O}_X$-modules, see Lemma 14.5. Hence by the above, E is compact. Then it follows from Derived Categories, Proposition 34.6 that E is a classical generator of the full subcategory of compact objects of $D_{QCoh,T}(\mathcal{O}_X)$. Thus any compact object can be constructed out of E by a finite sequence of operations consisting of (a) taking shifts, (b) taking finite direct sums, (c) taking cones, and (d) taking direct summands. Each of these operations preserves the property of being perfect and the result follows. □

The following lemma is an application of the ideas that go into the proof of the preceding lemma.

Lemma 15.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $T \subset |X|$ be a closed subset such that the complement $U \subset X$ is quasi-compact. Let $\alpha : P \to E$ be a morphism of $D_{QCoh}(\mathcal{O}_X)$ with either*

1. *P is perfect and E supported on T, or*
2. *P pseudo-coherent, E supported on T, and E bounded below.*

Then there exists a perfect complex of $\mathcal{O}_X$-modules I and a map $I \to \mathcal{O}_X[0]$ such that $I \otimes^{\mathbf{L}} P \to E$ is zero and such that $I|_U \to \mathcal{O}_U[0]$ is an isomorphism.

Proof. Set $\mathcal{D} = D_{QCoh,T}(\mathcal{O}_X)$. In both cases the complex $K = R\mathcal{H}om(P, E)$ is an object of $\mathcal{D}$. See Lemma 12.9 for quasi-coherence. It is clear that K is supported on T as formation of $R\mathcal{H}om$ commutes with restriction to opens. The map α defines an element of $H^0(K) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[0], K)$. Then it suffices to prove the result for the map $\alpha : \mathcal{O}_X[0] \to K$.

Let $E \in \mathcal{D}$ be a perfect generator, see Lemma 14.5. Write

$$K = \mathrm{hocolim} K_n$$

as in Derived Categories, Lemma 34.3 using the generator E. Since the functor $\mathcal{D} \to D(\mathcal{O}_X)$ commutes with direct sums, we see that $K = \text{hocolim} K_n$ also in $D(\mathcal{O}_X)$. Since $\mathcal{O}_X$ is a compact object of $D(\mathcal{O}_X)$ we find an n and a morphism $\alpha_n : \mathcal{O}_X \to K_n$ which gives rise to α. By Derived Categories, Lemma 34.4 applied to the morphism $\mathcal{O}_X[0] \to K_n$ in the ambient category $D(\mathcal{O}_X)$ we see that α_n factors as $\mathcal{O}_X[0] \to Q \to K_n$ where Q is an object of $\langle E \rangle$. We conclude that Q is a perfect complex supported on T.

Choose a distinguished triangle

$$I \to \mathcal{O}_X[0] \to Q \to I[1]$$

By construction I is perfect, the map $I \to \mathcal{O}_X[0]$ restricts to an isomorphism over U, and the composition $I \to K$ is zero as α factors through Q. This proves the lemma. □

16. Derived categories as module categories

The section is the analogue of Derived Categories of Schemes, Section 15.

Lemma 16.1. *Let S be a scheme. Let X be an algebraic space over S. Let $K^\bullet$ be a complex of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Let $(E, d) = \text{Hom}_{Comp^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor*

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 25.3 has image contained in $D_{QCoh}(\mathcal{O}_X)$.

Proof. Let P be a differential graded E-module with property P. Let $F_\bullet$ be a filtration on P as in Differential Graded Algebra, Section 13. Then we have

$$P \otimes_E K^\bullet = \text{hocolim } F_i P \otimes_E K^\bullet$$

Each of the F_iP has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. □

The following lemma can be strengthened (there is a uniformity in the vanishing over all L with nonzero cohomology sheaves only in a fixed range).

Lemma 16.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let K, L be objects of $D(\mathcal{O}_X)$ with K perfect and L in $D^b_{QCoh}(\mathcal{O}_X)$. Then $Ext^n_{D(\mathcal{O}_X)}(K, L)$ is nonzero for only a finite number of n.*

Proof. Since K is perfect we have

$$\text{Ext}^i_{D(\mathcal{O}_X)}(K, L) = H^i(X, K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

where $K^\wedge$ is the "dual" perfect complex to K, see Cohomology on Sites, Lemma 36.9. Note that $P = K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(X)$ by Lemmas 5.5 and 12.5 (to see that a perfect complex has quasi-coherent cohomology sheaves). On the other hand, the spectral sequence

$$E_1^{p,q} = H^p(K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(L)) \Rightarrow H^{p+q}(K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = H^{p+q}(P),$$

the boundedness of L, and the finite tor amplitude of $K^\wedge$ show that P has only finitely many nonzero cohomology sheaves. It follows that $H^n(X, P) = 0$ for $n \ll 0$. But also $H^n(X, P) = 0$ for $n \gg 0$ by Cohomology of Spaces, Lemma 6.3 and the spectral sequence expressing $H^n(X, P^\bullet)$ in terms of $H^p(X, H^q(P^\bullet))$ using that the cohomology sheaves of P are quasi-coherent. □

The following is the analogue of Derived Categories of Schemes, Theorem 15.3.

Theorem 16.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Then there exist a differential graded algebra (E, d) with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{QCoh}(\mathcal{O}_X)$ is equivalent to $D(E, d)$.*

Proof. Let $K^\bullet$ be a K-injective complex of $\mathcal{O}$-modules which is perfect and generates $D_{QCoh}(\mathcal{O}_X)$. Such a thing exists by Theorem 14.4 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, \text{d}) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where $\text{Comp}^{dg}(\mathcal{O}_X)$ is the differential graded category of complexes of $\mathcal{O}$-modules. Please see Differential Graded Algebra, Section 25. Since $K^\bullet$ is K-injective we have

$$H^n(E) = \text{Ext}^n_{D(\mathcal{O}_X)}(K^\bullet, K^\bullet) \tag{16.3.1}$$

for all $n \in \mathbf{Z}$. Only a finite number of these Exts are nonzero by Lemma 16.2. Consider the functor

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, \text{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 25.3. Since $K^\bullet$ is perfect, it defines a compact object of $D(\mathcal{O}_X)$, see Proposition 15.1. Combined with (16.3.1) the functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 25.5. It has a right adjoint

$$R\,\text{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, \text{d})$$

by Differential Graded Algebra, Lemmas 25.4 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 16.1 that we obtain

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, \text{d}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of $K^\bullet$ as a generator of $D_{QCoh}(\mathcal{O}_X)$ the kernel of the adjoint restricted to $D_{QCoh}(\mathcal{O}_X)$ is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 7.2. □

17. Cohomology and base change, IV

This section is the analogue of Derived Categories of Schemes, Section 16.

Lemma 17.1. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. For E in $D_{QCoh}(\mathcal{O}_X)$ and K in $D_{QCoh}(\mathcal{O}_Y)$ we have*

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

Proof. Without any assumptions there is a map $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \to Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$. Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K) = Lf^*(Rf_*(E)) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$. See Cohomology on Sites, Lemmas 18.4 and 19.1. To check it is an isomorphism we may work étale locally on Y. Hence we reduce to the case that Y is an affine scheme.

Suppose that $K = \bigoplus K_i$ is a direct sum of some complexes $K_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each K_i, then it holds for K. Namely, the functors Lf^*

and $\otimes^{\mathbf{L}}$ preserve direct sums by construction and Rf_* commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 6.2. Moreover, suppose that $K \to L \to M \to K[1]$ is a distinguished triangle in $D_{QCoh}(Y)$. Then if the statement of the lemma holds for two of K, L, M, then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume Y affine, say $Y = \mathrm{Spec}(A)$. The functor $\widetilde{\ } : D(A) \to D_{QCoh}(\mathcal{O}_Y)$ is an equivalence by Lemma 4.2 and Derived Categories of Schemes, Lemma 3.4. Let T be the property for $K \in D(A)$ that the statement of the lemma holds for $\widetilde{K}$. The discussion above and More on Algebra, Remark 45.11 shows that it suffices to prove T holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. □

Definition 17.2. Let S be a scheme. Let B be an algebraic space over S. Let X, Y be algebraic spaces over B. We say X and Y are *Tor independent over B* if and only if for every commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "\overline{x}"] \arrow[d, "\overline{y}"] \arrow[rd, "\overline{b}"] & X \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

of geometric points the rings $\mathcal{O}_{X,\overline{x}}$ and $\mathcal{O}_{Y,\overline{y}}$ are Tor independent over $\mathcal{O}_{B,\overline{b}}$ (see More on Algebra, Definition 47.1).

The following lemma shows in particular that this definition agrees with our definition in the case of representable algebraic spaces.

Lemma 17.3. *Let S be a scheme. Let B be an algebraic space over S. Let X, Y be algebraic spaces over B. The following are equivalent*

(1) *X and Y are Tor independent over B,*
(2) *for every commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & W \arrow[d] & V \arrow[l] \arrow[d] \\ X \arrow[r] & B & Y \arrow[l] \end{tikzcd}$$

with étale vertical arrows U and V are Tor independent over W,
(3) *for some commutative diagram as in (2) with (a) $W \to B$ étale surjective, (b) $U \to X \times_B W$ étale surjective, (c) $V \to Y \times_B W$ étale surjective, the spaces U and V are Tor independent over W, and*
(4) *for some commutative diagram as in (3) with U, V, W schemes, the schemes U and V are Tor independent over W in the sense of Derived Categories of Schemes, Definition 16.2.*

Proof. For an étale morphism $\varphi : U \to X$ of algebraic spaces and geometric point $\overline{u}$ the map of local rings $\mathcal{O}_{X,\varphi(\overline{u})} \to \mathcal{O}_{U,\overline{u}}$ is an isomorphism. Hence the equivalence of (1) and (2) follows. So does the implication (1) $\Rightarrow$ (3). Assume (3) and pick a diagram of geometric points as in Definition 17.2. The assumptions imply that we can first lift $\overline{b}$ to a geometric point $\overline{w}$ of W, then lift the geometric point $(\overline{x}, \overline{b})$ to a geometric point $\overline{u}$ of U, and finally lift the geometric point $(\overline{y}, \overline{b})$ to a geometric point $\overline{v}$ of V. Use Properties of Spaces, Lemma 16.4 to find the lifts. Using the remark on local rings above we conclude that the condition of the definition is satisfied for the given diagram.

Having made these initial points, it is clear that (4) comes down to the statement that Definition 17.2 agrees with Derived Categories of Schemes, Definition 16.2 when X, Y, and B are schemes.

Let $\overline{x}, \overline{b}, \overline{y}$ be as in Definition 17.2 lying over the points x, y, b. Recall that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{X,x}^{sh}$ (Properties of Spaces, Lemma 19.1) and similarly for the other two. By Algebra, Lemma 145.28 we see that $\mathcal{O}_{X,\overline{x}}$ is a strict henselization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,\overline{b}}$. In particular, the ring map

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,\overline{b}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$

is flat (More on Algebra, Lemma 34.1). By More on Algebra, Lemma 47.3 we see that

$$\text{Tor}_i^{\mathcal{O}_{B,b}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y}} (\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{B,\overline{b}}} \mathcal{O}_{Y,\overline{y}}) = \text{Tor}_i^{\mathcal{O}_{B,\overline{b}}}(\mathcal{O}_{X,\overline{x}}, \mathcal{O}_{Y,\overline{y}})$$

Hence it follows that if X and Y are Tor independent over B as schemes, then X and Y are Tor independent as algebraic spaces over B.

For the converse, we may assume X, Y, and B are affine. Observe that the ring map

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{B,\overline{b}}} \mathcal{O}_{Y,\overline{y}}$$

is flat by the observations given above. Moreover, the image of the map on spectra includes all primes $\mathfrak{s} \subset \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y}$ lying over $\mathfrak{m}_x$ and $\mathfrak{m}_y$. Hence from this and the displayed formula of Tor's above we see that if X and Y are Tor independent over B as algebraic spaces, then

$$\text{Tor}_i^{\mathcal{O}_{B,b}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y})_{\mathfrak{s}} = 0$$

for all $i > 0$ and all $\mathfrak{s}$ as above. By More on Algebra, Lemma 47.4 applied to the ring maps $\Gamma(B, \mathcal{O}_B) \to \Gamma(X, \mathcal{O}_X)$ and $\Gamma(B, \mathcal{O}_B) \to \Gamma(X, \mathcal{O}_X)$ this implies that X and Y are Tor independent over B. $\square$

Lemma 17.4. *Let S be a scheme. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Consider the base change diagram*

$$\begin{tikzcd} X' \arrow[r, "h"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

If X and Y' are Tor independent over Y, then for all $E \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf'_*Lh^*E = Lg^*Rf_*E$.

Proof. For any object E of $D(\mathcal{O}_X)$ we can use Cohomology on Sites, Remark 19.2 to get a canonical base change map $Lg^*Rf_*E \to Rf'_*Lh^*E$. To check this is an isomorphism we may work étale locally on Y'. Hence we may assume $g : Y' \to Y$ is a morphism of affine schemes. In particular, g is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*Lh^*E = Rf_*(Rh_*Lh^*E)$$

is an isomorphism, see Lemma 6.4 (and use Lemmas 5.4, 5.5, and 6.1 to see that the objects Rf'_*Lh^*E and Lg^*Rf_*E have quasi-coherent cohomology sheaves). Note that h is affine as well (Morphisms of Spaces, Lemma 20.5). By Lemma 6.5 the map becomes a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g_*\mathcal{O}_{Y'} \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} h_*\mathcal{O}_{X'})$$

Observe that $h_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{Y'}$. Thus by Lemma 17.1 it suffices to prove that $Lf^*g_*\mathcal{O}_{Y'} = f^*g_*\mathcal{O}_{Y'}$. This follows from our assumption that X and Y' are Tor independent over Y. Namely, to check it we may work étale locally on X, hence we may also assume X is affine. Say $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$ and $Y' = \mathrm{Spec}(R')$. Our assumption implies that A and R' are Tor independent over R (see Lemma 17.3 and More on Algebra, Lemma 47.4), i.e., $\mathrm{Tor}_i^R(A, R') = 0$ for $i > 0$. In other words $A \otimes^{\mathbf{L}}_R R' = A \otimes_R R'$ which exactly means that $Lf^*g_*\mathcal{O}_{Y'} = f^*g_*\mathcal{O}_{Y'}$. $\square$

The following two lemmas remain true if we replace $\mathcal{G}$ with a bounded complex of quasi-coherent $\mathcal{O}_X$-modules each flat over S.

Lemma 17.5. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module flat over Y. Then formation of*

$$Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : Y' \to Y$ be a morphism of algebraic spaces and consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "h"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in other words $X' = Y' \times_Y X$. Set $E' = Lh^*E$ and $\mathcal{G}' = h^*\mathcal{G}$ (here we do **not** use the derived pullback). The lemma asserts that we have

$$Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) = Rf'_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{G}')$$

To prove this, note that in Cohomology on Sites, Remark 19.2 we have constructed an arrow

$$Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) \longrightarrow R(f')_*(Lh^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})) = R(f')_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} lh^*\mathcal{G}))$$

which we can compose with the map $Lh^*\mathcal{G} \to h^*\mathcal{G}$ to get a canonical map

$$Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) \longrightarrow Rf'_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{G}')$$

To check this map is an isomorphism we may work étale locally on Y'. Hence we may assume $g : Y' \to Y$ is a morphism of affine schemes. In this case, we will use the induction principle to prove this map is always an isomorphism for any quasi-compact and quasi-separated algebraic space X over Y (Lemma 8.3).

If X is a scheme (for example affine), then the result holds. Namekly, E comes from an object of the derived category of the underlying scheme by Lemma 4.2. Furthermore, the constructions Rf_* (derived pushforward) and Lg^* (derived pullback) are (in the current situation) compatible with pulling back from the Zariski site (Remark 6.3). Thus in this case the result follows from the case of schemes which is Derived Categories of Schemes, Lemma 16.4.

The induction step. Let $(U \subset X, f : V \to X)$ be an elementary distinguished square with U, V, $U \times_X V$ quasi-compact such that the result holds for the restriction of E and $\mathcal{G}$ to U, V, and $U \times_X V$. Denote $a = f|_U$, $b = f|_V$ and $c = f|_{U \times_X V}$. Let $a' : U' \to Y'$, $b' : V' \to Y'$ and $c' : U' \times_{X'} V' \to Y'$ be the base changes of a, b, and c. Note that formation of $R\mathcal{H}om$ commutes with restriction (Cohomology on Sites, Lemma 26.3). Set $H = E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}$ and $H' = E' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{G}'$. Using the distinguished triangles from relative Mayer-Vietoris (Lemma 9.3) we obtain a commutative diagram

$$\begin{tikzcd}
Lg^*Rf_*H \arrow[r] \arrow[d] & Rf'_*H' \arrow[d] \\
Lg^*Ra_*H|_U \oplus Lg^*Rb_*H|_V \arrow[r] \arrow[d] & Ra'_*H'|_{U'} \oplus Rb'_*H'|_{V'} \arrow[d] \\
Lg^*Rc_*H|_{U \times_X V} \arrow[r] \arrow[d] & Rc'_*H'|_{U' \times_{X'} V'} \arrow[d] \\
Lg^*Rf_*H[1] \arrow[r] & Rf'_*H'[1]
\end{tikzcd}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 4.3) and the proof of the lemma is finished. $\square$

Lemma 17.6. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module flat over Y. Then formation of*

$$Rf_*R\mathcal{H}om(E, \mathcal{G})$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : Y' \to Y$ be a morphism of algebraic spaces and consider the base change diagram

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

in other words $X' = Y' \times_Y X$. Set $E' = Lh^*E$ and $\mathcal{G}' = h^*\mathcal{G}$ (here we do **not** use the derived pullback). The lemma asserts that we have

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) = Rf'_*R\mathcal{H}om(E', \mathcal{G}')$$

To prove this, note that in Cohomology on Sites, Remark 26.10 we have constructed an arrow

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) \longrightarrow R(f')_*R\mathcal{H}om(Lh^*E, Lh^*\mathcal{G})$$

which we can compose with the map $Lh^*\mathcal{G} \to h^*\mathcal{G}$ to get a canonical map

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) \to Rf'_*R\mathcal{H}om(E', \mathcal{G}')$$

With these preliminaries out of the way, we deduce the result from Lemma 17.5. Namely, since E is a perfect complex there exists a dual perfect complex E_{dual}, see Cohomology on Sites, Lemma 36.9, such that $R\mathcal{H}om(E, \mathcal{G}) = E_{dual} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}$. We omit the verification that the base change map of Lemma 17.5 for E_{dual} agrees with the base change map for E constructed above. $\square$

18. Producing perfect complexes

The following lemma is our main technical tool for producing perfect complexes. Later versions of this result will reduce to this by Noetherian approximation.

Lemma 18.1. *Let S be a scheme. Let Y be a Noetherian algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ such that*

1. *$E \in D^b_{Coh}(\mathcal{O}_X)$,*
2. *the scheme theoretic support of $H^i(E)$ is proper over Y for all i,*
3. *E has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_Y)$.*

Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$.

Proof. By Lemma 7.1 we see that Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_Y)$. Hence Rf_*E is pseudo-coherent (Lemma 12.6). Hence it suffices to show that Rf_*E has finite tor dimension, see Cohomology on Sites, Lemma 36.4. By Lemma 12.7 it suffices to check that $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F}$ has universally bounded cohomology for all quasi-coherent sheaves $\mathcal{F}$ on Y. Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset X$ be the union of the supports of $H^i(E)$ for all i. Then T is proper over Y by assumptions (1) and (2). In particular there exists a quasi-compact open subspace $X' \subset X$ containing T. Setting $f' = f|_{X'}$ we have $Rf_*(E) = Rf'_*(E|_{X'})$ because E restricts to zero on $X \setminus T$. Thus we may replace X by X' and assume f is quasi-compact. We have assumed f is quasi-separated. Thus

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F} = Rf_*\left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}\right) = Rf_*\left(E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}\right)$$

by Lemma 17.1 and Cohomology on Sites, Lemma 18.5. By assumption (3) the complex $E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$ has cohomology sheaves in a given finite range, say $[a, b]$. Then Rf_* of it has cohomology in the range $[a, \infty)$ and we win. $\square$

19. Computing Ext groups and base change

The results in this section will be used to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems.

Lemma 19.1. *Let S be a scheme. Let B be a Noetherian algebraic space over S. Let $f : X \to B$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module flat over B with scheme theoretic support proper over B. Then $K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$ is a perfect object of $D(\mathcal{O}_B)$ and there are functorial isomorphisms*

$$H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \longrightarrow H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof).

Proof. We have

$$\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F} = \mathcal{G} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_B} f^{-1}\mathcal{F} = \mathcal{G} \otimes_{f^{-1}\mathcal{O}_B} f^{-1}\mathcal{F} = \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}$$

the first equality by Cohomology on Sites, Lemma 18.5, the second as $\mathcal{G}$ is a flat $f^{-1}\mathcal{O}_B$-module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})) &= H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}) \\ &= H^i(B, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})) \\ &= H^i(B, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \\ &= H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology on Sites, Remark 14.4), and the third equality by Lemma 17.1. The object K is perfect by Lemma 18.1. We check the lemma applies. Locally E is isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules. Hence locally $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}$ is isomorphic to a finite complex whose terms are finite direct sums of copies of $\mathcal{G}$. This immediately implies the hypotheses on the cohomology sheaves $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$. The hypothesis on finite tor dimension follows as $\mathcal{G}$ is flat over $f^{-1}\mathcal{O}_B$.

The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{tikzcd} H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3) \arrow[r] \arrow[d, "\delta"'] & H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3)) \arrow[d, "\delta"] \\ H^{i+1}(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1) \arrow[r] & H^{i+1}(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1)) \end{tikzcd}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

This sequence is exact because $\mathcal{G}$ is flat over B. We omit the verification of the commutativity of the displayed diagram. $\square$

Lemma 19.2. *Let S be a scheme. Let B be a Noetherian algebraic space over S. Let $f : X \to B$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module flat over B with scheme theoretic support proper over B. Then*

$$K = Rf_* R\mathcal{H}om(E, \mathcal{G})$$

is a perfect object of $D(\mathcal{O}_B)$ and there are functorial isomorphisms

$$H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \longrightarrow Ext^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof).

Proof. Since E is a perfect complex there exists a dual perfect complex E_{dual}, see Cohomology on Sites, Lemma 36.9. Observe that $R\mathcal{H}om(E, \mathcal{G}) = E_{dual} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}$ and that

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) = H^i(X, E_{dual} \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

by construction of E_{dual}. Thus the perfectness of K and the isomorphisms follow from the corresponding results of Lemma 19.1 applied to E_{dual} and $\mathcal{G}$.

The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{tikzcd} H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3) \arrow[r] \arrow[d, "\delta"'] & \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3) \arrow[d, "\delta"] \\ H^{i+1}(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1) \end{tikzcd}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

This sequence is exact because $\mathcal{G}$ is flat over B. We omit the verification of the commutativity of the displayed diagram. $\square$

Lemma 19.3. *Let S be a scheme. Let B be a Noetherian algebraic space over S. Let $f : X \to B$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ and $\mathcal{G}$ an $\mathcal{O}_X$-module. Assume*

1. *$E \in D^-_{Coh}(\mathcal{O}_X)$, and*
2. *$\mathcal{G}$ is a coherent $\mathcal{O}_X$-module flat over B with scheme theoretic support proper over B.*

Then for every $m \in \mathbf{Z}$ there exists a perfect object K of $D(\mathcal{O}_B)$ and functorial maps

$$\alpha^i_{\mathcal{F}} : \mathit{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof) such that $\alpha^i_{\mathcal{F}}$ is an isomorphism for $i \leq m$.

Proof. We may replace X by a quasi-compact open neighbourhood of the support of $\mathcal{G}$, hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, -m-1)$ (possible by Theorem 13.7). Then the induced map

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(P, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\mathrm{Ext}^i_{\mathcal{O}_X}(C, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \quad \text{resp.} \quad \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(C, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

where C is the cone of $P \to E$. Since C has vanishing cohomology sheaves in degrees $\geq -m-1$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 27.3. This reduces us to the case that E is a perfect complex which is Lemma 19.2.

The statement on boundaries is explained in the proof of Lemma 19.2. $\square$

20. Limits and derived categories

In this section we collect some results about the derived category of an algebraic space which is the limit of an inverse system of algebraic spaces. More precisely, we will work in the following setting.

Situation 20.1. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We denote $f_i : X \to X_i$ the projection. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

Lemma 20.2. *In Situation 20.1. Let E_0 and K_0 be objects of $D(\mathcal{O}_{X_0})$. Set $E_i = Lf_{i0}^*E_0$ and $K_i = Lf_{i0}^*K_0$ for $i \geq 0$ and set $E = Lf_0^*E_0$ and $K = Lf_0^*K_0$. Then the map*

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{X_i})}(E_i, K_i) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(E, K)$$

is an isomorphism if either

(1) *E_0 is perfect and $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$, or*
(2) *E_0 is pseudo-coherent and $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has finite tor dimension.*

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the canonical map

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for each U_0 by the induction principle of Lemma 8.3. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Lemma 9.4. Thus it suffices to prove the lemma when X_0 is affine.

If X_0 is affine, then the result follows from the case of schemes, see Derived Categories of Schemes, Lemma 19.2. To see this use the equivalence of Lemma 4.2 and use the translation of properties explained in Lemmas 12.2, 12.3, and 12.4. $\square$

Lemma 20.3. *In Situation 20.1 the category of perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{X_i})$.*

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the functor

$$\mathrm{colim}_{i \geq 0} D_{perf}(\mathcal{O}_{U_i}) \longrightarrow D_{perf}(\mathcal{O}_U)$$

is an equivalence where $_{perf}$ indicates the full subcategory of perfect objects and where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for every U_0 by the induction principle of Lemma 8.3. First, we observe that we already know the functor is fully faithful by Lemma 20.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have an elementary distinguished square $(U_0 \subset X_0, V_0 \to X_0)$ and that P holds for U_0, V_0, and $U_0 \times_{X_0} V_0$. Let E be a perfect object of $D(\mathcal{O}_X)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on U_i and $E_{V,i}$ perfect on V_i whose pullback to U and V are isomorphic to $E|_U$ and $E|_V$. Denote

$$a : E_{U,i} \to (R(X \to X_i)_*E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (R(X \to X_i)_*E)|_{V_i}$$

the maps adjoint to the isomorphisms $L(U \to U_i)^*E_{U,i} \to E|_U$ and $L(V \to V_i)^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \times_{X_i} V_i} \to E_{V,i}|_{U_i \times_{X_i} V_i}$ which pulls back to the identifications

$$L(U \to U_i)^*E_{U,i}|_{U \times_X V} \to E|_{U \times_X V} \to L(V \to V_i)^*E_{V,i}|_{U \times_X V}.$$

Apply Lemma 9.7 to get an object E_i on X_i and a map $d : E_i \to R(X \to X_i)_*E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is perfect and that d is adjoint to an isomorphism $L(X \to X_i)^*E_i \to E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when X_0 is affine. This follows from the case of schemes, see Derived Categories of Schemes, Lemma 19.3. To see this use the equivalence of Lemma 4.2 and use the translation of Lemma 12.4. □

21. Cohomology and base change, V

A final section on cohomology and base change continueing the discussion of Sections 17 and 18. An easy to grok special case is given in Remark 21.2.

Lemma 21.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of finite presentation between algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module, flat over Y, with support proper over Y. Then*

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$$

is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 17.5. Thus it suffices to show that K is a perfect object. If Y is Noetherian, then this follows from Lemma 19.1. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on Y, hence we may assume Y is affine. Say $Y = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits of Spaces, Lemma 7.1 there exists an i and an algebraic space X_i of finite presentation over R_i whose base change to R is X. By Limits of Spaces, Lemma 7.2 we may assume after increasing i, that there exists a finitely presented $\mathcal{O}_{X_i}$-module $\mathcal{G}_i$ whose pullback to X is $\mathcal{G}$. After increasing i we may assume $\mathcal{G}_i$ is flat over R_i, see Limits of Spaces, Lemma 6.11. After increasing i we may assume the support of $\mathcal{G}_i$ is proper over R_i, see Limits of Spaces, Lemma 12.3. Finally, by Lemma 12.4 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 19.1 to $X_i \to \mathrm{Spec}(R_i)$, E_i, $\mathcal{G}_i$ and using the base change property already shown we obtain the result. □

Remark 21.2. Let R be a ring. Let X be an algebraic space of finite presentation over R. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module flat over R with scheme theoretic support proper over R. By Lemma 21.1 there exists a finite complex of finite projective R-modules $M^\bullet$ such that we have

$$R\Gamma(X_{R'}, \mathcal{G}_{R'}) = M^\bullet \otimes_R R'$$

functorially in the R-algebra R'.

Lemma 21.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of finite presentation between algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module, flat over Y, with support proper over Y. Then*

$$K = Rf_*R\mathcal{H}om(E, \mathcal{G})$$

is a perfect object of $D(\mathcal{O}_Y)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 17.6. Thus it suffices to show that K is a perfect object. If Y is Noetherian, then this follows from Lemma 19.2. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on Y, hence we may assume Y is affine. Say $Y = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits of Spaces, Lemma 7.1 there exists an i and an algebraic space X_i of finite presentation over R_i whose base change to R is X. By Limits of Spaces, Lemma 7.2 we may assume after increasing i, that there exists a finitely presented $\mathcal{O}_{X_i}$-module $\mathcal{G}_i$ whose pullback to X is $\mathcal{G}$. After increasing i we may assume $\mathcal{G}_i$ is flat over R_i, see Limits of Spaces, Lemma 6.11. After increasing i we may assume the support of $\mathcal{G}_i$ is proper over R_i, see Limits of Spaces, Lemma 12.3. Finally, by Lemma 12.4 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 19.2 to $X_i \to \mathrm{Spec}(R_i)$, E_i, $\mathcal{G}_i$ and using the base change property already shown we obtain the result. □

22. Other chapters

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