OBSOLETE

Contents

1. Introduction

In this chapter we put some lemmas that have become "obsolete" (see [Mil17]).

2. Obsolete algebra lemmas

Lemma 2.1. *Let M be an R-module of finite presentation. For any surjection $\alpha : R^{\oplus n} \to M$ the kernel of α is a finite R-module.*

Proof. This is a special case of Algebra, Lemma 5.3. $\square$

The following technical lemma says that you can lift any sequence of relations from a fibre to the whole space of a ring map which is essentially of finite type, in a suitable sense.

Lemma 2.2. *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p}$. Assume $S_{\mathfrak{q}}$ is essentially of finite type over $R_{\mathfrak{p}}$. Assume given*

1. *an integer $n \geq 0$,*
2. *a prime $\mathfrak{a} \subset \kappa(\mathfrak{p})[x_1, \ldots, x_n]$,*
3. *a surjective $\kappa(\mathfrak{p})$-homomorphism*

$$\psi : (\kappa(\mathfrak{p})[x_1, \ldots, x_n])_{\mathfrak{a}} \longrightarrow S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}},$$

   and
4. *elements $\overline{f}_1, \ldots, \overline{f}_e$ in $Ker(\psi)$.*

Then there exist

- (1) *an integer $m \geq 0$,*
- (2) *and element $g \in S$, $g \notin \mathfrak{q}$,*
- (3) *a map*
$$\Psi : R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}] \longrightarrow S_g,$$
  and
- (4) *elements $f_1, \ldots, f_e, f_{e+1}, \ldots, f_{e+m}$ of $Ker(\Psi)$*

such that

- (1) *the following diagram commutes*

$$
\begin{array}{ccc}
R[x_1, \ldots, x_{n+m}] & \xrightarrow{\;x_{n+j} \mapsto 0\;} & (\kappa(\mathfrak{p})[x_1, \ldots, x_n])_{\mathfrak{a}} \;, \\
{\scriptstyle \Psi}\downarrow & & \downarrow{\scriptstyle \psi} \\
S_g & \xrightarrow{\hspace{4cm}} & S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}
\end{array}
$$

- (2) *the element f_i, $i \leq n$ maps to a unit times $\overline{f}_i$ in the local ring*
$$(\kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}])_{(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})},$$
- (3) *the element f_{e+j} maps to a unit times x_{n+j} in the same local ring, and*
- (4) *the induced map $R[x_1, \ldots, x_{n+m}]_{\mathfrak{b}} \to S_{\mathfrak{q}}$ is surjective, where $\mathfrak{b} = \Psi^{-1}(\mathfrak{q}S_g)$.*

Proof. We claim that it suffices to prove the lemma in case R and S are local with maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$. Namely, suppose we have constructed
$$\Psi' : R_{\mathfrak{p}}[x_1, \ldots, x_{n+m}] \longrightarrow S_{\mathfrak{q}}$$
and $f'_1, \ldots, f'_{e+m} \in R_{\mathfrak{p}}[x_1, \ldots, x_{n+m}]$ with all the required properties. Then there exists an element $f \in R$, $f \notin \mathfrak{p}$ such that each ff'_k comes from an element $f_k \in R[x_1, \ldots, x_{n+m}]$. Moreover, for a suitable $g \in S$, $g \notin \mathfrak{q}$ the elements $\Psi'(x_i)$ are the image of elements $y_i \in S_g$. Let Ψ be the R-algebra map defined by the rule $\Psi(x_i) = y_i$. Since $\Psi(f_i)$ is zero in the localization $S_{\mathfrak{q}}$ we may after possibly replacing g assume that $\Psi(f_i) = 0$. This proves the claim.

Thus we may assume R and S are local with maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$. Pick $y_1, \ldots, y_n \in S$ such that $y_i \bmod \mathfrak{p}S = \psi(x_i)$. Let $y_{n+1}, \ldots, y_{n+m} \in S$ be elements which generate an R-subalgebra of which S is the localization. These exist by the assumption that S is essentially of finite type over R. Since ψ is surjective we may write $y_{n+j} \bmod \mathfrak{p}S = \psi(h_j)$ for some $h_j \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]_{\mathfrak{a}}$. Write $h_j = g_j/d$, $g_j \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]$ for some common denominator $d \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]$, $d \notin \mathfrak{a}$. Choose lifts $G_j, D \in R[x_1, \ldots, x_n]$ of g_j and d. Set $y'_{n+j} = D(y_1, \ldots, y_n)y_{n+j} - G_j(y_1, \ldots, y_n)$. By construction $y'_{n+j} \in \mathfrak{p}S$. It is clear that $y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m}$ generate an R-subalgebra of S whose localization is S. We define
$$\Psi : R[x_1, \ldots, x_{n+m}] \to S$$
to be the map that sends x_i to y_i for $i = 1, \ldots, n$ and x_{n+j} to y'_{n+j} for $j = 1, \ldots, m$. Properties (1) and (4) are clear by construction. Moreover the ideal $\mathfrak{b}$ maps onto the ideal $(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})$ in the polynomial ring $\kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}]$.

Denote $J = \mathrm{Ker}(\Psi)$. We have a short exact sequence
$$0 \to J_{\mathfrak{b}} \to R[x_1, \ldots, x_{n+m}]_{\mathfrak{b}} \to S_{\mathfrak{q}} \to 0.$$

The surjectivity comes from our choice of $y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m}$ above. This implies that

$$J_{\mathfrak{b}}/\mathfrak{p}J_{\mathfrak{b}} \to \kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}]_{(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})} \to S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \to 0$$

is exact. By construction x_i maps to $\psi(x_i)$ and x_{n+j} maps to zero under the last map. Thus it is easy to choose f_i as in (2) and (3) of the lemma. $\square$

Remark 2.3 (Projective resolutions). Let R be a ring. For any set S we let $F(S)$ denote the free R-module on S. Then any left R-module has the following two step resolution

$$F(M \times M) \oplus F(R \times M) \to F(M) \to M \to 0.$$

The first map is given by the rule

$$[m_1, m_2] \oplus [r, m] \mapsto [m_1 + m_2] - [m_1] - [m_2] + [rm] - r[m].$$

Lemma 2.4. *Let $A \to B$ be a finite type, flat ring map with A an integral domain. Then B is a finitely presented A-algebra.*

Proof. Special case of More on Flatness, Proposition 12.9. $\square$

Lemma 2.5. *Let R be a domain with fraction field K. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over R. Let M be a finite S-module. Assume that M is flat over R. If for every subring $R \subset R' \subset K$, $R \neq R'$ the module $M \otimes_R R'$ is finitely presented over $S \otimes_R R'$, then M is finitely presented over S.*

Proof. This lemma is true because M is finitely presented even without the assumption that $M \otimes_R R'$ is finitely presented for every R' as in the statement of the lemma. This follows from More on Flatness, Proposition 12.9. Originally this lemma had an erroneous proof (thanks to Ofer Gabber for finding the gap) and was used in an alternative proof of the proposition cited. To reinstate this lemma, we need a correct argument in case R is a local normal domain using only results from the chapters on commutative algebra; please email stacks.project@gmail.com if you have an argument. $\square$

3. Lemmas related to ZMT

The lemmas in this section were originally used in the proof of the (algebraic version of) Zariski's Main Theorem, Algebra, Theorem 119.13.

Lemma 3.1. *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Set $u_n = \varphi(a_n)$, $u_{n-1} = u_n t + \varphi(a_{n-1})$, and so on till $u_1 = u_2 t + \varphi(a_1)$. Then all of $u_n, u_{n-1}, \ldots, u_1$ and $u_n t, u_{n-1}t, \ldots, u_1 t$ are integral over R, and the ideals $(\varphi(a_0), \ldots, \varphi(a_n))$ and $(u_n, \ldots, u_1)$ of S are equal.*

Proof. We prove this by induction on n. As $u_n = \varphi(a_n)$ we conclude from Algebra, Lemma 119.1 that $u_n t$ is integral over R. Of course $u_n = \varphi(a_n)$ is integral over R. Then $u_{n-1} = u_n t + \varphi(a_{n-1})$ is integral over R (see Algebra, Lemma 35.7) and we have

$$\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_{n-1})t^{n-1} + u_{n-1}t^{n-1} = 0.$$

Hence by the induction hypothesis applied to the map $S' \to S$ where S' is the integral closure of R in S and the displayed equation we see that $u_{n-1}, \ldots, u_1$ and $u_{n-1}t, \ldots, u_1 t$ are all in S' too. The statement on the ideals is immediate from the shape of the elements and the fact that $u_1 t + \varphi(a_0) = 0$. $\square$

Lemma 3.2. *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Let $J \subset S$ be an ideal such that for at least one i we have $\varphi(a_i) \notin J$. Then there exists a $u \in S$, $u \notin J$ such that both u and ut are integral over R.*

Proof. This is immediate from Lemma 3.1 since one of the elements u_i will not be in J. $\square$

The following two lemmas are a way of describing closed subschemes of $\mathbf{P}^1_R$ cut out by one (nondegenerate) equation.

Lemma 3.3. *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d. Assume that for every prime $\mathfrak{p}$ of R at least one coefficient of F is not in $\mathfrak{p}$. Let $S = R[X, Y]/(F)$ as a graded ring. Then for all $n \geq d$ the R-module S_n is finite locally free of rank d.*

Proof. The R-module S_n has a presentation

$$R[X, Y]_{n-d} \to R[X, Y]_n \to S_n \to 0.$$

Thus by Algebra, Lemma 76.3 it is enough to show that multiplication by F induces an injective map $\kappa(\mathfrak{p})[X, Y] \to \kappa(\mathfrak{p})[X, Y]$ for all primes $\mathfrak{p}$. This is clear from the assumption that F does not map to the zero polynomial mod $\mathfrak{p}$. The assertion on ranks is clear from this as well. $\square$

Lemma 3.4. *Let k be a field. Let $F, G \in k[X, Y]$ be homogeneous of degrees d, e. Assume F, G relatively prime. Then multiplication by G is injective on $S = k[X, Y]/(F)$.*

Proof. This is one way to define "relatively prime". If you have another definition, then you can show it is equivalent to this one. $\square$

Lemma 3.5. *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d. Let $S = R[X, Y]/(F)$ as a graded ring. Let $\mathfrak{p} \subset R$ be a prime such that some coefficient of F is not in $\mathfrak{p}$. There exists an $f \in R$ $f \notin \mathfrak{p}$, an integer e, and a $G \in R[X, Y]_e$ such that multiplication by G induces isomorphisms $(S_n)_f \to (S_{n+e})_f$ for all $n \geq d$.*

Proof. During the course of the proof we may replace R by R_f for $f \in R$, $f \notin \mathfrak{p}$ (finitely often). As a first step we do such a replacement such that some coefficient of F is invertible in R. In particular the modules S_n are now locally free of rank d for $n \geq d$ by Lemma 3.3. Pick any $G \in R[X, Y]_e$ such that the image of G in $\kappa(\mathfrak{p})[X, Y]$ is relatively prime to the image of $F(X, Y)$ (this is possible for some e). Apply Algebra, Lemma 76.3 to the map induced by multiplication by G from $S_d \to S_{d+e}$. By our choice of G and Lemma 3.4 we see $S_d \otimes \kappa(\mathfrak{p}) \to S_{d+e} \otimes \kappa(\mathfrak{p})$ is bijective. Thus, after replacing R by R_f for a suitable f we may assume that $G : S_d \to S_{d+e}$ is bijective. This in turn implies that the image of G in $\kappa(\mathfrak{p}')[X, Y]$ is relatively prime to the image of F for all primes $\mathfrak{p}'$ of R. And then by Algebra, Lemma 76.3 again we see that all the maps $G : S_d \to S_{d+e}$, $n \geq d$ are isomorphisms. $\square$

Remark 3.6. Let R be a ring. Suppose that we have $F \in R[X, Y]_d$ and $G \in R[X, Y]_e$ such that, setting $S = R[X, Y]/(F)$ we have (1) S_n is finite locally free of rank d for all $n \geq d$, and (2) multiplication by G defines isomorphisms $S_n \to S_{n+e}$ for all $n \geq d$. In this case we may define a finite, locally free R-algebra A as follows:

(1) as an R-module $A = S_{ed}$, and
(2) multiplication $A \times A \to A$ is given by the rule that $H_1 H_2 = H_3$ if and only if $G^d H_3 = H_1 H_2$ in S_{2ed}.

This makes sense because multiplication by G^d induces a bijective map $S_{de} \to S_{2de}$. It is easy to see that this defines a ring structure. Note the confusing fact that the element G^d defines the unit element of the ring A.

Lemma 3.7. *Let R be a ring, let $f \in R$. Suppose we have S, S' and the solid arrows forming the following commutative diagram of rings*

Assume that $R_f \to S'$ is finite. Then we can find a finite ring map $R \to S''$ and dotted arrows as in the diagram such that $S' = (S'')_f$.

Proof. Namely, suppose that S' is generated by x_i over R_f, $i = 1, \ldots, w$. Let $P_i(t) \in R_f[t]$ be a monic polynomial such that $P_i(x_i) = 0$. Say P_i has degree $d_i > 0$. Write $P_i(t) = t^{d_i} + \sum_{j < d_i} (a_{ij}/f^n) t^j$ for some uniform n. Also write the image of x_i in S_f as g_i/f^n for suitable $g_i \in S$. Then we know that the element $\xi_i = f^{n d_i} g_i^{d_i} + \sum_{j < d_i} f^{n(d_i - j)} a_{ij} g_i^j$ of S is killed by a power of f. Hence upon increasing n to n', which replaces g_i by $f^{n'-n} g_i$ we may assume $\xi_i = 0$. Then S' is generated by the elements $f^n x_i$, each of which is a zero of the monic polynomial $Q_i(t) = t^{d_i} + \sum_{j < d_i} f^{n(d_i - j)} a_{ij} t^j$ with coefficients in R. Also, by construction $Q_i(f^n g_i) = 0$ in S. Thus we get a finite R-algebra $S'' = R[z_1, \ldots, z_w]/(Q_1(z_1), \ldots, Q_w(z_w))$ which fits into a commutative diagram as above. The map $\alpha : S'' \to S$ maps z_i to $f^n g_i$ and the map $\beta : S'' \to S'$ maps z_i to $f^n x_i$. It may not yet be the case that β induces an isomorphism $(S'')_f \cong S'$. For the moment we only know that this map is surjective. The problem is that there could be elements $h/f^n \in (S'')_f$ which map to zero in S' but are not zero. In this case $\beta(h)$ is an element of S such that $f^N \beta(h) = 0$ for some N. Thus $f^N h$ is an element ot the ideal $J = \{h \in S'' \mid \alpha(h) = 0 \text{ and } \beta(h) = 0\}$ of S''. OK, and it is easy to see that S''/J does the job. $\square$

4. Formally smooth ring maps

Lemma 4.1. *Let R be a ring. Let S be a R-algebra. If S is of finite presentation and formally smooth over R then S is smooth over R.*

Proof. See Algebra, Proposition 133.13. $\square$

5. Simplicial methods

Lemma 5.1. *Assumptions and notation as in Simplicial, Lemma 31.1. There exists a section $g : U \to V$ to the morphism f and the composition $g \circ f$ is homotopy equivalent to the identity on V. In particular, the morphism f is a homotopy equivalence.*

Proof. Immediate from Simplicial, Lemmas 31.1 and 29.8. $\square$

6. Obsolete lemmas on schemes

Lemmas that seem superfluous.

Lemma 6.1. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $X \subset \mathbf{P}_R^n$ be a closed subscheme. Assume that $R = \Gamma(X, \mathcal{O}_X)$. Then the special fibre X_k is geometrically connected.*

Proof. This is a special case of More on Morphisms, Theorem 36.4. $\square$

Lemma 6.2. *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*
  (1) *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
  (2) *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
  (3) *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_* \mathcal{I}$.*
  (4) *There exists some coherent sheaf $\mathcal{G}$ on X such that*
      (a) *$Supp(\mathcal{G}) = Z_0$,*
      (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
      (c) *property $\mathcal{P}$ holds for $\mathcal{G}$.*
Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. The proof is a variant on the proof of Cohomology of Schemes, Lemma 12.5. In exactly the same manner as in that proof we see that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$.

Consider a coherent sheaf $\mathcal{G}$ as in (3). By Cohomology of Schemes, Lemma 12.2 there exists a sheaf of ideals $\mathcal{I}$ on Z_0 and a short exact sequence

$$0 \to ((Z_0 \to X)_* \mathcal{I})^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is strictly contained in Z_0. In particular $r > 0$ and $\mathcal{I}$ is nonzero because the support of $\mathcal{G}$ is equal to Z. Since $\mathcal{Q}$ has property $\mathcal{P}$ we conclude that also $((Z_0 \to X)_* \mathcal{I})^{\oplus r}$ has property $\mathcal{P}$. By (2) we deduce property $\mathcal{P}$ for $(Z_0 \to X)_* \mathcal{I}$. Slotting this into the proof of Cohomology of Schemes, Lemma 12.5 at the appropriate point gives the lemma. Some details omitted. $\square$

Lemma 6.3. *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*
  (1) *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
  (2) *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
  (3) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*
      (a) *$Supp(\mathcal{G}) = Z$,*
      (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
      (c) *property $\mathcal{P}$ holds for $\mathcal{G}$.*
Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. This follows from Lemma 6.2 in exactly the same way that Cohomology of Schemes, Lemma 12.6 follows from Cohomology of Schemes, Lemma 12.5. $\square$

7. Functor of quotients

Lemma 7.1. *Let $S = \mathrm{Spec}(R)$ be an affine scheme. Let X be an algebraic space over S. Let $q_i : \mathcal{F} \to \mathcal{Q}_i$, $i = 1, 2$ be surjective maps of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{Q}_1$ flat over S. Let $T \to S$ be a quasi-compact morphism of schemes such that there exists a factorization*

$$
\begin{array}{ccc}
 & \mathcal{F}_T & \\
{\scriptstyle q_{1,T}} \swarrow & & \searrow {\scriptstyle q_{2,T}} \\
\mathcal{Q}_{1,T} & \longleftarrow\cdots\cdots\cdots & \mathcal{Q}_{2,T}
\end{array}
$$

Then exists a closed subscheme $Z \subset S$ such that (a) $T \to S$ factors through Z and (b) $q_{1,Z}$ factors through $q_{2,Z}$. If $\mathrm{Ker}(q_2)$ is a finite type $\mathcal{O}_X$-module and X quasi-compact, then we can take $Z \to S$ of finite presentation.

Proof. Apply Quot, Lemma 7.5 to the map $\mathrm{Ker}(q_2) \to \mathcal{Q}_1$. $\square$

8. Spaces and fpqc coverings

The material here was made obsolete by Gabber's argument showing that algebraic spaces satisfy the sheaf condition with respect to fpqc coverings. Please visit Properties of Spaces, Section 14.

Lemma 8.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering of schemes over S. Then the map*

$$
\mathrm{Mor}_S(T, X) \longrightarrow \prod\nolimits_{i \in I} \mathrm{Mor}_S(T_i, X)
$$

is injective.

Proof. Immediate consequence of Properties of Spaces, Proposition 14.1. $\square$

Lemma 8.2. *Let S be a scheme. Let X be an algebraic space over S. Let $X = \bigcup_{j \in J} X_j$ be a Zariski covering, see Spaces, Definition 12.5. If each X_j satisfies the sheaf property for the fpqc topology then X satisfies the sheaf property for the fpqc topology.*

Proof. This is true because all algebraic spaces satisfy the sheaf property for the fpqc topology, see Properties of Spaces, Proposition 14.1. $\square$

Lemma 8.3. *Let S be a scheme. Let X be an algebraic space over S. If X is Zariski locally quasi-separated over S, then X satisfies the sheaf condition for the fpqc topology.*

Proof. Immediate consequence of the general Properties of Spaces, Proposition 14.1. $\square$

Remark 8.4. This remark used to discuss to what extend the original proof of Lemma 8.3 (of December 18, 2009) generalizes.

9. Very reasonable algebraic spaces

Material that is somewhat obsolete.

Lemma 9.1. *Let S be a scheme. Let X be a reasonable algebraic space over S. Then $|X|$ is Kolmogorov (see Topology, Definition 7.4).*

Proof. Follows from the definitions and Decent Spaces, Lemma 10.5. $\square$

In the rest of this section we make some remarks about very reasonable algebraic spaces. If there exists a scheme U and a surjective, étale, quasi-compact morphism $U \to X$, then X is very reasonable, see Decent Spaces, Lemma 4.7.

Lemma 9.2. *A scheme is very reasonable.*

Proof. This is true because the identity map is a quasi-compact, surjective étale morphism. $\square$

Lemma 9.3. *Let S be a scheme. Let X be an algebraic space over S. If there exists a Zariski open covering $X = \bigcup X_i$ such that each X_i is very reasonable, then X is very reasonable.*

Proof. This is case (ϵ) of Decent Spaces, Lemma 5.2. $\square$

Lemma 9.4. *An algebraic space which is Zariski locally quasi-separated is very reasonable. In particular any quasi-separated algebraic space is very reasonable.*

Proof. This is one of the implications of Decent Spaces, Lemma 5.1. $\square$

Lemma 9.5. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $Y \to X$ be a representable morphism. If X is very reasonable, so is Y.*

Proof. This is case (ϵ) of Decent Spaces, Lemma 5.3. $\square$

Remark 9.6. Very reasonable algebraic spaces form a strictly larger collection than Zariski locally quasi-separated algebraic spaces. Consider an algebraic space of the form $X = [U/G]$ (see Spaces, Definition 14.4) where G is a finite group acting without fixed points on a non-quasi-separated scheme U. Namely, in this case $U \times_X U = U \times G$ and clearly both projections to U are quasi-compact, hence X is very reasonable. On the other hand, the diagonal $U \times_X U \to U \times U$ is not quasi-compact, hence this algebraic space is not quasi-separated. Now, take U the infinite affine space over a field k of characteristic $\neq 2$ with zero doubled, see Schemes, Example 21.4. Let $0_1, 0_2$ be the two zeros of U. Let $G = \{+1, -1\}$, and let -1 act by -1 on all coordinates, and by switching 0_1 and 0_2. Then $[U/G]$ is very reasonable but not Zariski locally quasi-separated (details omitted).

Warning: The following lemma should be used with caution, as the schemes U_i in it are not necessarily separated or even quasi-separated.

Lemma 9.7. *Let S be a scheme. Let X be a very reasonable algebraic space over S. There exists a set of schemes U_i and morphisms $U_i \to X$ such that*

    (1) *each U_i is a quasi-compact scheme,*
    (2) *each $U_i \to X$ is étale,*
    (3) *both projections $U_i \times_X U_i \to U_i$ are quasi-compact, and*
    (4) *the morphism $\coprod U_i \to X$ is surjective (and étale).*

Proof. Decent Spaces, Definition 6.1 says that there exist $U_i \to X$ such that (2), (3) and (4) hold. Fix i, and set $R_i = U_i \times_X U_i$, and denote $s, t : R_i \to U_i$ the projections. For any affine open $W \subset U_i$ the open $W' = t(s^{-1}(W)) \subset U_i$ is a quasi-compact R_i-invariant open (see Groupoids, Lemma 17.2). Hence W' is a quasi-compact scheme, $W' \to X$ is étale, and $W' \times_X W' = s^{-1}(W') = t^{-1}(W')$ so both projections $W' \times_X W' \to W'$ are quasi-compact. This means the family of $W' \to X$, where $W \subset U_i$ runs through the members of affine open coverings of the U_i gives what we want. $\square$

10. Variants of cotangent complexes for schemes

This section gives an alternative construction of the cotangent complex of a morphism of schemes. This section is currently in the obsolete chapter as we can get by with the easier version discussed in Cotangent, Section 24 for applications.

Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{C}_{X/Y}$ be the category whose objects are commutative diagrams

(10.0.1)
$$\begin{array}{ccc} X \longleftarrow U \xrightarrow{\ i\ } A \\ \downarrow \qquad \downarrow \swarrow \\ Y \longleftarrow V \end{array}$$

of schemes where

(1) U is an open subscheme of X,
(2) V is an open subscheme of Y, and
(3) there exists an isomorphism $A = V \times \mathrm{Spec}(P)$ over V where P is a polynomial algebra over $\mathbf{Z}$ (on some set of variables).

In other words, A is an (infinite dimensional) affine space over V. Morphisms are given by commutative diagrams.

Notation. An object of $\mathcal{C}_{X/Y}$, i.e., a diagram (10.0.1), is often denoted $U \to A$ where it is understood that (a) U is an open subscheme of X, (b) $U \to A$ is a morphism over Y, (c) the image of the structure morphism $A \to Y$ is an open $V \subset Y$, and (d) $A \to V$ is an affine space. We'll write $U \to A/V$ to indicate $V \subset Y$ is the image of $A \to Y$. Recall that X_{Zar} denotes the small Zariski site X. There are forgetful functors

$$\mathcal{C}_{X/Y} \to X_{Zar}, \ (U \to A) \mapsto U \quad \text{and} \quad \mathcal{C}_{X/Y} \mapsto Y_{Zar}, \ (U \to A/V) \mapsto V.$$

Lemma 10.1. *Let $X \to Y$ be a morphism of schemes.*

(1) *The category $\mathcal{C}_{X/Y}$ is fibred over X_{Zar}.*
(2) *The category $\mathcal{C}_{X/Y}$ is fibred over Y_{Zar}.*
(3) *The category $\mathcal{C}_{X/Y}$ is fibred over the category of pairs (U, V) where $U \subset X$, $V \subset Y$ are open and $f(U) \subset V$.*

Proof. Ad (1). Given an object $U \to A$ of $\mathcal{C}_{X/Y}$ and a morphism $U' \to U$ of X_{Zar} consider the object $i' : U' \to A$ of $\mathcal{C}_{X/Y}$ where i' is the composition of i and $U' \to U$. The morphism $(U' \to A) \to (U \to A)$ of $\mathcal{C}_{X/Y}$ is strongly cartesian over X_{Zar}.

Ad (2). Given an object $U \to A/V$ and $V' \to V$ we can set $U' = U \cap f^{-1}(V')$ and $A' = V' \times_V A$ to obtain a strongly cartesian morphism $(U' \to A') \to (U \to A)$ over $V' \to V$.

Ad (3). Denote $(X/Y)_{Zar}$ the category in (3). Given $U \to A/V$ and a morphism $(U', V') \to (U, V)$ in $(X/Y)_{Zar}$ we can consider $A' = V' \times_V A$. Then the morphism $(U' \to A'/V') \to (U \to A/V)$ is strongly cartesian in $\mathcal{C}_{X/Y}$ over $(X/Y)_{Zar}$. $\square$

We obtain a topology τ_X on $\mathcal{C}_{X/Y}$ by using the topology inherited from X_{Zar} (see Stacks, Section 10). If not otherwise stated this is the topology on $\mathcal{C}_{X/Y}$ we will consider. To be precise, a family of morphisms $\{(U_i \to A_i) \to (U \to A)\}$ is a covering of $\mathcal{C}_{X/Y}$ if and only if

(1) $U = \bigcup U_i$, and
(2) $A_i = A$ for all i.

We obtain the same collection of sheaves if we allow $A_i \cong A$ in (2). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/Y}) \to Sh(X_{Zar})$.

The site $\mathcal{C}_{X/Y}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to A) \mapsto \mathcal{O}(A)$.
(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to A) \mapsto \mathcal{O}(U)$.
(3) The sheaf $\underline{\mathcal{O}}_Y$ given by the rule $(U \to A/V) \mapsto \mathcal{O}(V)$.

We obtain morphisms of ringed topoi

(10.1.1)
$$
\begin{array}{ccc}
(Sh(\mathcal{C}_{X/Y}), \underline{\mathcal{O}}_X) & \xrightarrow{\ i\ } & (Sh(\mathcal{C}_{X/Y}), \mathcal{O}) \\
{\scriptstyle \pi} \downarrow & & \\
(Sh(X_{Zar}), \mathcal{O}_X) & &
\end{array}
$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 28.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$.

Remark 10.2. We obtain a second topology τ_Y on $\mathcal{C}_{X/Y}$ by taking the topology inherited from Y_{Zar}. There is a third topology $\tau_{X \to Y}$ where a family of morphisms $\{(U_i \to A_i) \to (U \to A)\}$ is a covering if and only if $U = \bigcup U_i$, $V = \bigcup V_i$ and $A_i \cong V_i \times_V A$. This is the topology inherited from the topology on the site $(X/Y)_{Zar}$ whose underlying category is the category of pairs (U, V) as in Lemma 10.1 part (3). The coverings of $(X/Y)_{Zar}$ are families $\{(U_i, V_i) \to (U, V)\}$ such that $U = \bigcup U_i$ and $V = \bigcup V_i$. There are morphisms of topoi

$$
Sh(\mathcal{C}_{X/Y}) = Sh(\mathcal{C}_{X/Y}, \tau_X) \longleftarrow Sh(\mathcal{C}_{X/Y}, \tau_{X \to Y}) \longrightarrow Sh(\mathcal{C}_{X/Y}, \tau_Y)
$$

(recall that τ_X is our "default" topology). The pullback functors for these arrows are sheafification and pushforward is the identity on underlying presheaves. The diagram of topoi

$$
\begin{array}{ccccc}
Sh(X_{Zar}) & \xleftarrow{\ \pi\ } & Sh(\mathcal{C}_{X/Y}) & \longleftarrow & Sh(\mathcal{C}_{X/Y}, \tau_{X \to Y}) \\
{\scriptstyle f}\downarrow & & & & \downarrow \\
Sh(Y_{Zar}) & \longleftarrow & & & Sh(\mathcal{C}_{X/Y}, \tau_Y)
\end{array}
$$

is **not** commutative. Namely, the pullback of a nonzero abelian sheaf on Y is a nonzero abelian sheaf on $(\mathcal{C}_{X/Y}, \tau_{X \to Y})$, but we can certainly find examples where

such a sheaf pulls back to zero on X. Note that any presheaf $\mathcal{F}$ on Y_{Zar} gives a sheaf $\underline{\mathcal{F}}$ on $\mathcal{C}_{Y/X}$ by the rule which assigns to $(U \to A/V)$ the set $\mathcal{F}(V)$. Even if $\mathcal{F}$ happens to be a sheaf it isn't true in general that $\underline{\mathcal{F}} = \pi^{-1}f^{-1}\mathcal{F}$. This is related to the noncommutativity of the diagram above, as we can describe $\underline{\mathcal{F}}$ as the pushforward of the pullback of $\mathcal{F}$ to $Sh(\mathcal{C}_{X/Y}, \tau_{X \to Y})$ via the lower horizontal and right vertical arrows. An example is the sheaf $\underline{\mathcal{O}}_Y$. But what is true is that there is a map $\underline{\mathcal{F}} \to \pi^{-1}f^{-1}\mathcal{F}$ which is transformed (as we shall see later) into an isomorphism after applying $\pi_!$.

11. Deformations and obstructions of flat modules

In this section we sketch a construction of a deformation theory for the stack of coherent sheaves for any algebraic space X over a ring Λ. This material is obsolete due to the improved discussion in Quot, Section 6.

Our setup will be the following. We assume given

    (1) a ring Λ,
    (2) an algebraic space X over Λ,
    (3) a Λ-algebra A, set $X_A = X \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(A)$, and
    (4) a finitely presented $\mathcal{O}_{X_A}$-module $\mathcal{F}$ flat over A.

In this situation we will consider all possible surjections

$$0 \to I \to A' \to A \to 0$$

where A' is a Λ-algebra whose kernel I is an ideal of square zero in A'. Given A' we obtain a first order thickening $X_A \to X_{A'}$ of algebraic spaces over $\mathrm{Spec}(\Lambda)$. For each of these we consider the problem of lifting $\mathcal{F}$ to a finitely presented module $\mathcal{F}'$ on $X_{A'}$ flat over A'. We would like to replicate the results of Deformation Theory, Lemma 11.1 in this setting.

To be more precise let $Lift(\mathcal{F}, A')$ denote the category of pairs $(\mathcal{F}', \alpha)$ where $\mathcal{F}'$ is a finitely presented module on $X_{A'}$ flat over A' and $\alpha : \mathcal{F}'|_{X_A} \to \mathcal{F}$ is an isomorphism. Morphisms $(\mathcal{F}_1', \alpha_1) \to (\mathcal{F}_2', \alpha_2)$ are isomorphisms $\mathcal{F}_1' \to \mathcal{F}_2'$ which are compatible with α_1 and α_2. The set of isomorphism classes of $Lift(\mathcal{F}, A')$ is denoted $\mathrm{Lift}(\mathcal{F}, A')$.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X \otimes_\Lambda A$-modules on $X_{\acute{e}tale}$ flat over A. We introduce the category $Lift(\mathcal{G}, A')$ of pairs $(\mathcal{G}', \beta)$ where $\mathcal{G}'$ is a sheaf of $\mathcal{O}_X \otimes_\Lambda A'$-modules flat over A' and β is an isomorphism $\mathcal{G}' \otimes_{A'} A \to \mathcal{G}$.

Lemma 11.1. *Notation and assumptions as above. Let $p : X_A \to X$ denote the projection. Given A' denote $p' : X_{A'} \to X$ the projection. The functor p_*' induces an equivalence of categories between*

    (1) the category $Lift(\mathcal{F}, A')$, and
    (2) the category $Lift(p_*\mathcal{F}, A')$.

Proof. FIXME. $\square$

Let $\mathcal{H}$ be a sheaf of $\mathcal{O} \otimes_\Lambda A$-modules on $\mathcal{C}_{X/\Lambda}$ flat over A. We introduce the category $Lift_{\mathcal{O}}(\mathcal{H}, A')$ whose objects are pairs $(\mathcal{H}', \gamma)$ where $\mathcal{H}'$ is a sheaf of $\mathcal{O} \otimes_\Lambda A'$-modules flat over A' and $\gamma : \mathcal{H}' \otimes_A A' \to \mathcal{H}$ is an isomorphism of $\mathcal{O} \otimes_\Lambda A$-modules.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X \otimes_\Lambda A$-modules on $X_{\acute{e}tale}$ flat over A. Consider the morphisms i and π of Cotangent, Equation (26.1.1). Denote $\underline{\mathcal{G}} = \pi^{-1}(\mathcal{G})$. It is simply given

by the rule $(U \to \mathbf{A}) \mapsto \mathcal{G}(U)$ hence it is a sheaf of $\underline{\mathcal{O}}_X \otimes_\Lambda A$-modules. Denote $i_* \underline{\mathcal{G}}$ the same sheaf but viewed as a sheaf of $\mathcal{O} \otimes_\Lambda A$-modules.

Lemma 11.2. *Notation and assumptions as above. The functor $\pi_!$ induces an equivalence of categories between*

   (1) *the category $Lift_{\mathcal{O}}(i_* \underline{\mathcal{G}}, A')$, and*
   (2) *the category $Lift(\mathcal{G}, A')$.*

Proof. FIXME. $\square$

Lemma 11.3. *Notation and assumptions as in Lemma 11.2. Consider the object*

$$L = L(\Lambda, X, A, \mathcal{G}) = L\pi_!(Li^*(i_*(\underline{\mathcal{G}})))$$

of $D(\mathcal{O}_X \otimes_\Lambda A)$. Given a surjection $A' \to A$ of Λ-algebras with square zero kernel I we have

   (1) *The category $Lift(\mathcal{G}, A')$ is nonempty if and only if a certain class $\xi \in Ext^2_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$ is zero.*
   (2) *If $Lift(\mathcal{G}, A')$ is nonempty, then $Lift(\mathcal{G}, A')$ is principal homogeneous under $Ext^1_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$.*
   (3) *Given a lift $\mathcal{G}'$, the set of automorphisms of $\mathcal{G}'$ which pull back to $id_{\mathcal{G}}$ is canonically isomorphic to $Ext^0_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$.*

Proof. FIXME. $\square$

Finally, we put everything together as follows.

Proposition 11.4. *With Λ, X, A, $\mathcal{F}$ as above. There exists a canonical object $L = L(\Lambda, X, A, \mathcal{F})$ of $D(X_A)$ such that given a surjection $A' \to A$ of Λ-algebras with square zero kernel I we have*

   (1) *The category $Lift(\mathcal{F}, A')$ is nonempty if and only if a certain class $\xi \in Ext^2_{X_A}(L, \mathcal{F} \otimes_A I)$ is zero.*
   (2) *If $Lift(\mathcal{F}, A')$ is nonempty, then $Lift(\mathcal{F}, A')$ is principal homogeneous under $Ext^1_{X_A}(L, \mathcal{F} \otimes_A I)$.*
   (3) *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $id_{\mathcal{F}}$ is canonically isomorphic to $Ext^0_{X_A}(L, \mathcal{F} \otimes_A I)$.*

Proof. FIXME. $\square$

Lemma 11.5. *In the situation of Proposition 11.4, if $X \to \mathrm{Spec}(\Lambda)$ is locally of finite type and Λ is Noetherian, then L is pseudo-coherent.*

Proof. FIXME. $\square$

12. Modifications

Here are some obsolete results on the category of Restricted Power Series, Equation (13.0.1). Please visit Restricted Power Series, Section 13 for the current material.

Lemma 12.1. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. The category of Restricted Power Series, Equation (13.0.1) for A is equivalent to the category Restricted Power Series, Equation (13.0.1) for the henselization A^h of A.*

Proof. This is a special case of Restricted Power Series, Lemma 13.2. $\square$

13. Duplicate references

This section is a place where we collect duplicates.

Lemma 13.1. *Let S be a scheme. Let X be an algebraic space over S. The map* $\{\mathrm{Spec}(k) \to X \ monomorphism\} \to |X|$ *is injective.*

Proof. This is a duplicate of Properties of Spaces, Lemma 4.11. $\square$

Theorem 13.2. *Let $S = \mathrm{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. Then there is an equivalence of categories $Sh(S_{\acute{e}tale}) \to G\text{-Sets}, \ \mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$.*

Proof. This is a duplicate of Étale Cohomology, Theorem 57.3. $\square$

Remark 13.3. You got here because of a duplicate tag. Please see Formal Deformation Theory, Section 11 for the actual content.

14. Other chapters

References

[Mil17] George Abram Miller, *The Obsolete in Mathematics*, Amer. Math. Monthly **24** (1917), no. 10, 453–456.