

# DIVIDED POWER ALGEBRA

## Contents

1. Introduction	1
2. Divided powers	1
3. Divided power rings	4
4. Extending divided powers	6
5. Divided power polynomial algebras	8
6. Tate resolutions	10
7. Application to complete intersections	15
8. Local complete intersection rings	17
9. Local complete intersection maps	21
10. Other chapters	24
References	25

## 1. Introduction

In this chapter we talk about divided power algebras and what you can do with them. A reference is the book [Ber74].

## 2. Divided powers

In this section we collect some results on divided power rings. We will use the convention  $0! = 1$  (as empty products should give 1).

**Definition 2.1.** Let  $A$  be a ring. Let  $I$  be an ideal of  $A$ . A collection of maps  $\gamma_n : I \rightarrow I$ ,  $n > 0$  is called a *divided power structure* on  $I$  if for all  $n \geq 0$ ,  $m > 0$ ,  $x, y \in I$ , and  $a \in A$  we have

- (1)  $\gamma_1(x) = x$ , we also set  $\gamma_0(x) = 1$ ,
- (2)  $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$ ,
- (3)  $\gamma_n(ax) = a^n\gamma_n(x)$ ,
- (4)  $\gamma_n(x+y) = \sum_{i=0, \dots, n} \gamma_i(x)\gamma_{n-i}(y)$ ,
- (5)  $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$ .

Note that the rational numbers  $\frac{(n+m)!}{n!m!}$  and  $\frac{(nm)!}{n!(m!)^n}$  occurring in the definition are in fact integers; the first is the number of ways to choose  $n$  out of  $n+m$  and the second counts the number of ways to divide a group of  $nm$  objects into  $n$  groups of  $m$ . We make some remarks about the definition which show that  $\gamma_n(x)$  is a replacement for  $x^n/n!$  in  $I$ .

**Lemma 2.2.** *Let  $A$  be a ring. Let  $I$  be an ideal of  $A$ .*

- (1) *If  $\gamma$  is a divided power structure on  $I$ , then  $n!\gamma_n(x) = x^n$  for  $n \geq 1$ ,  $x \in I$ .*

Assume  $A$  is torsion free as a  $\mathbf{Z}$ -module.

(2) A divided power structure on  $I$ , if it exists, is unique.

(3) If  $\gamma_n : I \rightarrow I$  are maps then

$$\gamma \text{ is a divided power structure} \Leftrightarrow n!\gamma_n(x) = x^n \quad \forall x \in I, n \geq 1.$$

(4) The ideal  $I$  has a divided power structure if and only if there exists a set of generators  $x_i$  of  $I$  as an ideal such that for all  $n \geq 1$  we have  $x_i^n \in (n!)I$ .

**Proof.** Proof of (1). If  $\gamma$  is a divided power structure, then condition (2) implies that  $n\gamma_n(x) = \gamma_1(x)\gamma_{n-1}(x)$ . Hence by induction and condition (1) we get  $n!\gamma_n(x) = x^n$ .

Assume  $A$  is torsion free as a  $\mathbf{Z}$ -module. Proof of (2). This is clear from (1).

Proof of (3). Assume that  $n!\gamma_n(x) = x^n$  for all  $x \in I$  and  $n \geq 1$ . Since  $A \subset A \otimes_{\mathbf{Z}} \mathbf{Q}$  it suffices to prove (1) – (5) in case  $A$  is a  $\mathbf{Q}$ -algebra. In this case  $\gamma_n(x) = x^n/n!$  and it is straightforward to verify (1) – (5), for example (4) corresponds to the binomial formula

$$(x + y)^n = \sum \frac{n!}{i!(n-i)!} x^i y^{n-i}$$

We encourage the reader to do the verifications to make sure that we have the coefficients correct.

Proof of (4). Assume we have generators  $x_i$  of  $I$  as an ideal such that  $x_i^n \in (n!)I$  for all  $n \geq 1$ . We claim that for all  $x \in I$  we have  $x^n \in (n!)I$ . If the claim holds then we can set  $\gamma_n(x) = x^n/n!$  which is a divided power structure by (3). To prove the claim we note that it holds for  $x = ax_i$ . Hence we see that the claim holds for a set of generators of  $I$  as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove the claim for  $x + y$  if it holds for  $x$  and  $y$ . This follows immediately from the binomial theorem.  $\square$

**Example 2.3.** Let  $p$  be a prime number. Let  $A$  be a ring such that every integer  $n$  not divisible by  $p$  is invertible, i.e.,  $A$  is a  $\mathbf{Z}_{(p)}$ -algebra. Then  $I = pA$  has a canonical divided power structure. Namely, given  $x = pa \in A$  we set

$$\gamma_n(x) = \frac{p^n}{n!} a^n$$

The reader verifies immediately that  $p^n/n! \in \mathbf{Z}_{(p)}$  so that the definition makes sense. It is a straightforward exercise to verify that conditions (1) – (5) of Definition 2.1 are satisfied. Alternatively, it is clear that the definition works for  $A_0 = \mathbf{Z}_{(p)}$  and then the result follows from Lemma 4.2.

**Lemma 2.4.** Let  $A$  be a ring. Let  $I$  be an ideal of  $A$ . Let  $\gamma_n : I \rightarrow I$ ,  $n \geq 1$  be a sequence of maps. Assume

(a) (1), (3), and (4) of Definition 2.1 hold for all  $x, y \in I$ , and

(b) properties (2) and (5) hold for  $x$  in set of generators of  $I$  as an ideal.

Then  $\gamma$  is a divided power structure on  $I$ .

**Proof.** The numbers (1), (2), (3), (4), (5) in this proof refer to the conditions listed in Definition 2.1. Applying (3) we see that if (2) and (5) hold for  $x$  then (2) and (5) hold for  $ax$  for all  $a \in A$ . Hence we see (b) implies (2) and (5) hold for a set of generators of  $I$  as an abelian group. Hence, by induction of the length of an

expression in terms of these it suffices to prove that, given  $x, y \in I$  such that (2) and (5) hold for  $x$  and  $y$ , then (2) and (5) hold for  $x + y$ .

Proof of (2) for  $x + y$ . By (4) we have

$$\gamma_n(x + y)\gamma_m(x + y) = \sum_{i+j=n, k+l=m} \gamma_i(x)\gamma_k(x)\gamma_j(y)\gamma_l(y)$$

Using (2) for  $x$  and  $y$  this equals

$$\sum \frac{(i+k)!}{i!k!} \frac{(j+l)!}{j!l!} \gamma_{i+k}(x)\gamma_{j+l}(y)$$

Comparing this with the expansion

$$\gamma_{n+m}(x + y) = \sum \gamma_a(x)\gamma_b(y)$$

we see that we have to prove that given  $a + b = n + m$  we have

$$\sum_{i+k=a, j+l=b, i+j=n, k+l=m} \frac{(i+k)!}{i!k!} \frac{(j+l)!}{j!l!} = \frac{(n+m)!}{n!m!}.$$

Instead of arguing this directly, we note that the result is true for the ideal  $I = (x, y)$  in the polynomial ring  $\mathbf{Q}[x, y]$  because  $\gamma_n(f) = f^n/n!$ ,  $f \in I$  defines a divided power structure on  $I$ . Hence the equality of rational numbers above is true.

Proof of (5) for  $x + y$  given that (1) – (4) hold and that (5) holds for  $x$  and  $y$ . We will again reduce the proof to an equality of rational numbers. Namely, using (4) we can write  $\gamma_n(\gamma_m(x + y)) = \gamma_n(\sum \gamma_i(x)\gamma_j(y))$ . Using (4) we can write  $\gamma_n(\gamma_m(x + y))$  as a sum of terms which are products of factors of the form  $\gamma_k(\gamma_i(x)\gamma_j(y))$ . If  $i > 0$  then

$$\begin{aligned} \gamma_k(\gamma_i(x)\gamma_j(y)) &= \gamma_j(y)^k \gamma_k(\gamma_i(x)) \\ &= \frac{(ki)!}{k!(i!)^k} \gamma_j(y)^k \gamma_{ki}(x) \\ &= \frac{(ki)!}{k!(i!)^k} \frac{(kj)!}{k!(j!)^k} \gamma_{ik}(x) \gamma_{kj}(y) \end{aligned}$$

using (3) in the first equality, (5) for  $x$  in the second, and (2) exactly  $k$  times in the third. Using (5) for  $y$  we see the same equality holds when  $i = 0$ . Continuing like this using all axioms but (5) we see that we can write

$$\gamma_n(\gamma_m(x + y)) = \sum_{i+j=nm} c_{ij} \gamma_i(x)\gamma_j(y)$$

for certain universal constants  $c_{ij} \in \mathbf{Z}$ . Again the fact that the equality is valid in the polynomial ring  $\mathbf{Q}[x, y]$  implies that the coefficients  $c_{ij}$  are all equal to  $(nm)!/n!(m!)^n$  as desired.  $\square$

**Lemma 2.5.** *Let  $A$  be a ring with two ideals  $I, J \subset A$ . Let  $\gamma$  be a divided power structure on  $I$  and let  $\delta$  be a divided power structure on  $J$ . Then*

- (1)  $\gamma$  and  $\delta$  agree on  $IJ$ ,
- (2) if  $\gamma$  and  $\delta$  agree on  $I \cap J$  then they are the restriction of a unique divided power structure  $\epsilon$  on  $I + J$ .

**Proof.** Let  $x \in I$  and  $y \in J$ . Then

$$\gamma_n(xy) = y^n \gamma_n(x) = n! \delta_n(y) \gamma_n(x) = \delta_n(y) x^n = \delta_n(xy).$$

Hence  $\gamma$  and  $\delta$  agree on a set of (additive) generators of  $IJ$ . By property (4) of Definition 2.1 it follows that they agree on all of  $IJ$ .

Let  $z \in I + J$ . Write  $z = x + y$  with  $x \in I$  and  $y \in J$ . Then we set

$$\epsilon_n(z) = \sum \gamma_i(x) \delta_{n-i}(y)$$

To see that this is well defined, suppose that  $z = x' + y'$  is another representation with  $x' \in I$  and  $y' \in J$ . Then  $w = x - x' = y' - y \in I \cap J$ . Hence

$$\begin{aligned} \sum_{i+j=n} \gamma_i(x) \delta_j(y) &= \sum_{i+j=n} \gamma_i(x' + w) \delta_j(y) \\ &= \sum_{i'+l+j=n} \gamma_{i'}(x') \gamma_l(w) \delta_j(y) \\ &= \sum_{i'+l+j=n} \gamma_{i'}(x') \delta_l(w) \delta_j(y) \\ &= \sum_{i'+j'=n} \gamma_{i'}(x') \delta_{j'}(y + w) \\ &= \sum_{i'+j'=n} \gamma_{i'}(x') \delta_{j'}(y') \end{aligned}$$

as desired. Next, we prove conditions (1) – (5) of Definition 2.1. Properties (1) and (3) are clear. To see (4), suppose that  $z = x + y$  and  $z' = x' + y'$  with  $x, x' \in I$  and  $y, y' \in J$  and compute

$$\begin{aligned} \epsilon_n(z + z') &= \sum_{a+b=n} \gamma_i(x + x') \delta_j(y + y') \\ &= \sum_{i+i'+j+j'=n} \gamma_i(x) \gamma_{i'}(x') \delta_j(y) \delta_{j'}(y') \\ &= \sum_{k=0, \dots, n} \sum_{i+j=k} \gamma_i(x) \delta_j(y) \sum_{i'+j'=n-k} \gamma_{i'}(x') \delta_{j'}(y') \\ &= \sum_{k=0, \dots, n} \epsilon_k(z) \epsilon_{n-k}(z') \end{aligned}$$

as desired. Now we see that it suffices to prove (2) and (5) for elements of  $I$  or  $J$ , see Lemma 2.4. This is clear because  $\gamma$  and  $\delta$  are divided power structures.  $\square$

**Lemma 2.6.** *Let  $p$  be a prime number. Let  $A$  be a ring, let  $I \subset A$  be an ideal, and let  $\gamma$  be a divided power structure on  $I$ . Assume  $p$  is nilpotent in  $A/I$ . Then  $I$  is locally nilpotent if and only if  $p$  is nilpotent in  $A$ .*

**Proof.** If  $p^N = 0$  in  $A$ , then for  $x \in I$  we have  $x^{p^N} = (pN)! \gamma_N(x) = 0$  because  $(pN)!$  is divisible by  $p^N$ . Conversely, assume  $I$  is locally nilpotent. We've also assumed that  $p$  is nilpotent in  $A/I$ , hence  $p^r \in I$  for some  $r$ , hence  $p^r$  nilpotent, hence  $p$  nilpotent.  $\square$

### 3. Divided power rings

There is a category of divided power rings. Here is the definition.

**Definition 3.1.** A *divided power ring* is a triple  $(A, I, \gamma)$  where  $A$  is a ring,  $I \subset A$  is an ideal, and  $\gamma = (\gamma_n)_{n \geq 1}$  is a divided power structure on  $I$ . A *homomorphism of divided power rings*  $\varphi : (A, I, \gamma) \rightarrow (B, J, \delta)$  is a ring homomorphism  $\varphi : A \rightarrow B$  such that  $\varphi(I) \subset J$  and such that  $\delta_n(\varphi(x)) = \varphi(\gamma_n(x))$  for all  $x \in I$ .

We sometimes say “let  $(B, J, \delta)$  be a divided power algebra over  $(A, I, \gamma)$ ” to indicate that  $(B, J, \delta)$  is a divided power ring which comes equipped with a homomorphism of divided power rings  $(A, I, \gamma) \rightarrow (B, J, \delta)$ .

**Lemma 3.2.** *The category of divided power rings has all limits and they agree with limits in the category of rings.*

**Proof.** The empty limit is the zero ring (that’s weird but we need it). The product of a collection of divided power rings  $(A_t, I_t, \gamma_t)$ ,  $t \in T$  is given by  $(\prod A_t, \prod I_t, \gamma)$  where  $\gamma_n((x_t)) = (\gamma_{t,n}(x_t))$ . The equalizer of  $\alpha, \beta : (A, I, \gamma) \rightarrow (B, J, \delta)$  is just  $C = \{a \in A \mid \alpha(a) = \beta(a)\}$  with ideal  $C \cap I$  and induced divided powers. It follows that all limits exist, see Categories, Lemma 14.10.  $\square$

The following lemma illustrates a very general category theoretic phenomenon in the case of divided power algebras.

**Lemma 3.3.** *Let  $\mathcal{C}$  be the category of divided power rings. Let  $F : \mathcal{C} \rightarrow \text{Sets}$  be a functor. Assume that*

- (1) *there exists a cardinal  $\kappa$  such that for every  $f \in F(A, I, \gamma)$  there exists a morphism  $(A', I', \gamma') \rightarrow (A, I, \gamma)$  of  $\mathcal{C}$  such that  $f$  is the image of  $f' \in F(A', I', \gamma')$  and  $|A'| \leq \kappa$ , and*
- (2)  *$F$  commutes with limits.*

*Then  $F$  is representable, i.e., there exists an object  $(B, J, \delta)$  of  $\mathcal{C}$  such that*

$$F(A, I, \gamma) = \text{Hom}_{\mathcal{C}}((B, J, \delta), (A, I, \gamma))$$

*functorially in  $(A, I, \gamma)$ .*

**Proof.** This is a special case of Categories, Lemma 40.1.  $\square$

**Lemma 3.4.** *The category of divided power rings has all colimits.*

**Proof.** The empty colimit is  $\mathbf{Z}$  with divided power ideal  $(0)$ . Let’s discuss general colimits. Let  $\mathcal{C}$  be a category and let  $c \mapsto (A_c, I_c, \gamma_c)$  be a diagram. Consider the functor

$$F(B, J, \delta) = \lim_{c \in \mathcal{C}} \text{Hom}((A_c, I_c, \gamma_c), (B, J, \delta))$$

Note that any  $f = (f_c)_{c \in \mathcal{C}} \in F(B, J, \delta)$  has the property that all the images  $f_c(A_c)$  generate a subring  $B'$  of  $B$  of bounded cardinality  $\kappa$  and that all the images  $f_c(I_c)$  generate a divided power sub ideal  $J'$  of  $B'$ . And we get a factorization of  $f$  as a  $f'$  in  $F(B')$  followed by the inclusion  $B' \rightarrow B$ . Also,  $F$  commutes with limits. Hence we may apply Lemma 3.3 to see that  $F$  is representable and we win.  $\square$

**Remark 3.5.** The forgetful functor  $(A, I, \gamma) \mapsto A$  does not commute with colimits. For example, let

$$\begin{array}{ccc} (B, J, \delta) & \longrightarrow & (B'', J'', \delta'') \\ \uparrow & & \uparrow \\ (A, I, \gamma) & \longrightarrow & (B', J', \delta') \end{array}$$

be a pushout in the category of divided power rings. Then in general the map  $B \otimes_A B' \rightarrow B''$  isn’t an isomorphism. (It is always surjective.) An explicit example is given by  $(A, I, \gamma) = (\mathbf{Z}, (0), \emptyset)$ ,  $(B, J, \delta) = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta)$ , and  $(B', J', \delta') = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta')$  where  $\delta_2(2) = 2$  and  $\delta'_2(2) = 0$  and all higher divided powers

equal to zero. Then  $(B'', J'', \delta'') = (\mathbf{F}_2, (0), \emptyset)$  which doesn't agree with the tensor product. However, note that it is always true that

$$B''/J'' = B/J \otimes_{A/I} B'/J'$$

as can be seen from the universal property of the pushout by considering maps into divided power algebras of the form  $(C, (0), \emptyset)$ .

#### 4. Extending divided powers

Here is the definition.

**Definition 4.1.** Given a divided power ring  $(A, I, \gamma)$  and a ring map  $A \rightarrow B$  we say  $\gamma$  *extends* to  $B$  if there exists a divided power structure  $\bar{\gamma}$  on  $IB$  such that  $(A, I, \gamma) \rightarrow (B, IB, \bar{\gamma})$  is a homomorphism of divided power rings.

**Lemma 4.2.** *Let  $(A, I, \gamma)$  be a divided power ring. Let  $A \rightarrow B$  be a ring map. If  $\gamma$  extends to  $B$  then it extends uniquely. Assume (at least) one of the following conditions holds*

- (1)  $IB = 0$ ,
- (2)  $I$  is principal, or
- (3)  $A \rightarrow B$  is flat.

*Then  $\gamma$  extends to  $B$ .*

**Proof.** Any element of  $IB$  can be written as a finite sum  $\sum b_i x_i$  with  $b_i \in B$  and  $x_i \in I$ . If  $\gamma$  extends to  $\bar{\gamma}$  on  $IB$  then  $\bar{\gamma}_n(x_i) = \gamma_n(x_i)$ . Thus conditions (3) and (4) imply that

$$\bar{\gamma}_n(\sum b_i x_i) = \sum_{n_1 + \dots + n_t = n} \prod_{i=1}^t b_i^{n_i} \gamma_{n_i}(x_i)$$

Thus we see that  $\bar{\gamma}$  is unique if it exists.

If  $IB = 0$  then setting  $\bar{\gamma}_n(0) = 0$  works. If  $I = (x)$  then we define  $\bar{\gamma}_n(bx) = b^n \gamma_n(x)$ . This is well defined: if  $b'x = bx$ , i.e.,  $(b - b')x = 0$  then

$$\begin{aligned} b^n \gamma_n(x) - (b')^n \gamma_n(x) &= (b^n - (b')^n) \gamma_n(x) \\ &= (b^{n-1} + \dots + (b')^{n-1})(b - b') \gamma_n(x) = 0 \end{aligned}$$

because  $\gamma_n(x)$  is divisible by  $x$  and hence annihilated by  $b - b'$ . Next, we prove conditions (1) – (5) of Definition 2.1. Parts (1), (2), (3), (5) are obvious from the construction. For (4) suppose that  $y, z \in IB$ , say  $y = bx$  and  $z = cx$ . Then  $y + z = (b + c)x$  hence

$$\begin{aligned} \bar{\gamma}_n(y + z) &= (b + c)^n \gamma_n(x) \\ &= \sum \frac{n!}{i!(n-i)!} b^i c^{n-i} \gamma_n(x) \\ &= \sum b^i c^{n-i} \gamma_i(x) \gamma_{n-i}(x) \\ &= \sum \bar{\gamma}_i(y) \bar{\gamma}_{n-i}(z) \end{aligned}$$

as desired.

Assume  $A \rightarrow B$  is flat. Suppose that  $b_1, \dots, b_r \in B$  and  $x_1, \dots, x_r \in I$ . Then

$$\bar{\gamma}_n(\sum b_i x_i) = \sum b_1^{e_1} \dots b_r^{e_r} \gamma_{e_1}(x_1) \dots \gamma_{e_r}(x_r)$$

where the sum is over  $e_1 + \dots + e_r = n$  if  $\bar{\gamma}_n$  exists. Next suppose that we have  $c_1, \dots, c_s \in B$  and  $a_{ij} \in A$  such that  $b_i = \sum a_{ij}c_j$ . Setting  $y_j = \sum a_{ij}x_i$  we claim that

$$\sum b_1^{e_1} \dots b_r^{e_r} \gamma_{e_1}(x_1) \dots \gamma_{e_r}(x_r) = \sum c_1^{d_1} \dots c_s^{d_s} \gamma_{d_1}(y_1) \dots \gamma_{d_s}(y_s)$$

in  $B$  where on the right hand side we are summing over  $d_1 + \dots + d_s = n$ . Namely, using the axioms of a divided power structure we can expand both sides into a sum with coefficients in  $\mathbf{Z}[a_{ij}]$  of terms of the form  $c_1^{d_1} \dots c_s^{d_s} \gamma_{e_1}(x_1) \dots \gamma_{e_r}(x_r)$ . To see that the coefficients agree we note that the result is true in  $\mathbf{Q}[x_1, \dots, x_r, c_1, \dots, c_s, a_{ij}]$  with  $\gamma$  the unique divided power structure on  $(x_1, \dots, x_r)$ . By Lazard's theorem (Algebra, Theorem 78.4) we can write  $B$  as a directed colimit of finite free  $A$ -modules. In particular, if  $z \in IB$  is written as  $z = \sum x_i b_i$  and  $z = \sum x'_i b'_i$ , then we can find  $c_1, \dots, c_s \in B$  and  $a_{ij}, a'_{ij} \in A$  such that  $b_i = \sum a_{ij}c_j$  and  $b'_i = \sum a'_{ij}c_j$  such that  $y_j = \sum x_i a_{ij} = \sum x'_i a'_{ij}$ . Hence the procedure above gives a well defined map  $\bar{\gamma}_n$  on  $IB$ . By construction  $\bar{\gamma}$  satisfies conditions (1), (3), and (4). Moreover, for  $x \in I$  we have  $\bar{\gamma}_n(x) = \gamma_n(x)$ . Hence it follows from Lemma 2.4 that  $\bar{\gamma}$  is a divided power structure on  $IB$ .  $\square$

**Lemma 4.3.** *Let  $(A, I, \gamma)$  be a divided power ring.*

- (1) *If  $\varphi : (A, I, \gamma) \rightarrow (B, J, \delta)$  is a homomorphism of divided power rings, then  $\text{Ker}(\varphi) \cap I$  is preserved by  $\gamma_n$  for all  $n \geq 1$ .*
- (2) *Let  $\mathfrak{a} \subset A$  be an ideal and set  $I' = I \cap \mathfrak{a}$ . The following are equivalent*
  - (a)  *$I'$  is preserved by  $\gamma_n$  for all  $n > 0$ ,*
  - (b)  *$\gamma$  extends to  $A/\mathfrak{a}$ , and*
  - (c) *there exist a set of generators  $x_i$  of  $I'$  as an ideal such that  $\gamma_n(x_i) \in I'$  for all  $n > 0$ .*

**Proof.** Proof of (1). This is clear. Assume (2)(a). Define  $\bar{\gamma}_n(x \bmod I') = \gamma_n(x) \bmod I'$  for  $x \in I$ . This is well defined since  $\gamma_n(x+y) = \gamma_n(x) \bmod I'$  for  $y \in I'$  by Definition 2.1 (4) and the fact that  $\gamma_j(y) \in I'$  by assumption. It is clear that  $\bar{\gamma}$  is a divided power structure as  $\gamma$  is one. Hence (2)(b) holds. Also, (2)(b) implies (2)(a) by part (1). It is clear that (2)(a) implies (2)(c). Assume (2)(c). Note that  $\gamma_n(x) = a^n \gamma_n(x_i) \in I'$  for  $x = ax_i$ . Hence we see that  $\gamma_n(x) \in I'$  for a set of generators of  $I'$  as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove  $\forall n : \gamma_n(x+y) \in I'$  if  $\forall n : \gamma_n(x), \gamma_n(y) \in I'$ . This follows immediately from the fourth axiom of a divided power structure.  $\square$

**Lemma 4.4.** *Let  $(A, I, \gamma)$  be a divided power ring. Let  $E \subset I$  be a subset. Then the smallest ideal  $J \subset I$  preserved by  $\gamma$  and containing all  $f \in E$  is the ideal  $J$  generated by  $\gamma_n(f)$ ,  $n \geq 1$ ,  $f \in E$ .*

**Proof.** Follows immediately from Lemma 4.3.  $\square$

**Lemma 4.5.** *Let  $(A, I, \gamma)$  be a divided power ring. Let  $p$  be a prime. If  $p$  is nilpotent in  $A/I$ , then*

- (1) *the  $p$ -adic completion  $A^\wedge = \lim_e A/p^e A$  surjects onto  $A/I$ ,*
- (2) *the kernel of this map is the  $p$ -adic completion  $I^\wedge$  of  $I$ , and*
- (3) *each  $\gamma_n$  is continuous for the  $p$ -adic topology and extends to  $\gamma_n^\wedge : I^\wedge \rightarrow I^\wedge$  defining a divided power structure on  $I^\wedge$ .*

*If moreover  $A$  is a  $\mathbf{Z}_{(p)}$ -algebra, then*

- (4) for  $e$  large enough the ideal  $p^e A \subset I$  is preserved by the divided power structure  $\gamma$  and

$$(A^\wedge, I^\wedge, \gamma^\wedge) = \lim_e (A/p^e A, I/p^e A, \bar{\gamma})$$

in the category of divided power rings.

**Proof.** Let  $t \geq 1$  be an integer such that  $p^t A/I = 0$ , i.e.,  $p^t A \subset I$ . The map  $A^\wedge \rightarrow A/I$  is the composition  $A^\wedge \rightarrow A/p^t A \rightarrow A/I$  which is surjective (for example by Algebra, Lemma 93.1). As  $p^e I \subset p^e A \cap I \subset p^{e-t} I$  for  $e \geq t$  we see that the kernel of the composition  $A^\wedge \rightarrow A/I$  is the  $p$ -adic completion of  $I$ . The map  $\gamma_n$  is continuous because

$$\gamma_n(x + p^e y) = \sum_{i+j=n} p^{j_e} \gamma_i(x) \gamma_j(y) = \gamma_n(x) \bmod p^e I$$

by the axioms of a divided power structure. It is clear that the axioms for divided power structures are inherited by the maps  $\gamma_n^\wedge$  from the maps  $\gamma_n$ . Finally, to see the last statement say  $e > t$ . Then  $p^e A \subset I$  and  $\gamma_1(p^e A) \subset p^e A$  and for  $n > 1$  we have

$$\gamma_n(p^e a) = p^n \gamma_n(p^{e-1} a) = \frac{p^n}{n!} p^{n(e-1)} a^n \in p^e A$$

as  $p^n/n! \in \mathbf{Z}_{(p)}$  and as  $n \geq 2$  and  $e \geq 2$  so  $n(e-1) \geq e$ . This proves that  $\gamma$  extends to  $A/p^e A$ , see Lemma 4.3. The statement on limits is clear from the construction of limits in the proof of Lemma 3.2.  $\square$

## 5. Divided power polynomial algebras

A very useful example is the *divided power polynomial algebra*. Let  $A$  be a ring. Let  $t \geq 1$ . We will denote  $A\langle x_1, \dots, x_t \rangle$  the following  $A$ -algebra: As an  $A$ -module we set

$$A\langle x_1, \dots, x_t \rangle = \bigoplus_{n_1, \dots, n_t \geq 0} A x_1^{[n_1]} \dots x_t^{[n_t]}$$

with multiplication given by

$$x_i^{[n]} x_i^{[m]} = \frac{(n+m)!}{n!m!} x_i^{[n+m]}.$$

We also set  $x_i = x_i^{[1]}$ . Note that  $1 = x_1^{[0]} \dots x_t^{[0]}$ . There is a similar construction which gives the divided power polynomial algebra in infinitely many variables. There is an canonical  $A$ -algebra map  $A\langle x_1, \dots, x_t \rangle \rightarrow A$  sending  $x_i^{[n]}$  to zero for  $n > 0$ . The kernel of this map is denoted  $A\langle x_1, \dots, x_t \rangle_+$ .

**Lemma 5.1.** *Let  $(A, I, \gamma)$  be a divided power ring. There exists a unique divided power structure  $\delta$  on*

$$J = IA\langle x_1, \dots, x_t \rangle + A\langle x_1, \dots, x_t \rangle_+$$

such that

- (1)  $\delta_n(x_i) = x_i^{[n]}$ , and
- (2)  $(A, I, \gamma) \rightarrow (A\langle x_1, \dots, x_t \rangle, J, \delta)$  is a homomorphism of divided power rings.

Moreover,  $(A\langle x_1, \dots, x_t \rangle, J, \delta)$  has the following universal property: A homomorphism of divided power rings  $\varphi : (A\langle x \rangle, J, \delta) \rightarrow (C, K, \epsilon)$  is the same thing as a homomorphism of divided power rings  $A \rightarrow C$  and elements  $k_1, \dots, k_t \in K$ .

**Proof.** We will prove the lemma in case of a divided power polynomial algebra in one variable. The result for the general case can be argued in exactly the same way, or by noting that  $A\langle x_1, \dots, x_t \rangle$  is isomorphic to the ring obtained by adjoining the divided power variables  $x_1, \dots, x_t$  one by one.

Let  $A\langle x \rangle_+$  be the ideal generated by  $x, x^{[2]}, x^{[3]}, \dots$ . Note that  $J = IA\langle x \rangle + A\langle x \rangle_+$  and that

$$IA\langle x \rangle \cap A\langle x \rangle_+ = IA\langle x \rangle \cdot A\langle x \rangle_+$$

Hence by Lemma 2.5 it suffices to show that there exist divided power structures on the ideals  $IA\langle x \rangle$  and  $A\langle x \rangle_+$ . The existence of the first follows from Lemma 4.2 as  $A \rightarrow A\langle x \rangle$  is flat. For the second, note that if  $A$  is torsion free, then we can apply Lemma 2.2 (4) to see that  $\delta$  exists. Namely, choosing as generators the elements  $x^{[m]}$  we see that  $(x^{[m]})^n = \frac{(nm)!}{(m!)^n} x^{[nm]}$  and  $n!$  divides the integer  $\frac{(nm)!}{(m!)^n}$ . In general write  $A = R/\mathfrak{a}$  for some torsion free ring  $R$  (e.g., a polynomial ring over  $\mathbf{Z}$ ). The kernel of  $R\langle x \rangle \rightarrow A\langle x \rangle$  is  $\bigoplus \mathfrak{a}x^{[m]}$ . Applying criterion (2)(c) of Lemma 4.3 we see that the divided power structure on  $R\langle x \rangle_+$  extends to  $A\langle x \rangle$  as desired.

Proof of the universal property. Given a homomorphism  $\varphi : A \rightarrow C$  of divided power rings and  $k_1, \dots, k_t \in K$  we consider

$$A\langle x_1, \dots, x_t \rangle \rightarrow C, \quad x_1^{[n_1]} \dots x_t^{[n_t]} \mapsto \epsilon_{n_1}(k_1) \dots \epsilon_{n_t}(k_t)$$

using  $\varphi$  on coefficients. The only thing to check is that this is an  $A$ -algebra homomorphism (details omitted). The inverse construction is clear.  $\square$

**Remark 5.2.** Let  $(A, I, \gamma)$  be a divided power ring. There is a variant of Lemma 5.1 for infinitely many variables. First note that if  $s < t$  then there is a canonical map

$$A\langle x_1, \dots, x_s \rangle \rightarrow A\langle x_1, \dots, x_t \rangle$$

Hence if  $W$  is any set, then we set

$$A\langle x_w, w \in W \rangle = \operatorname{colim}_{E \subset W} A\langle x_e, e \in E \rangle$$

(colimit over  $E$  finite subset of  $W$ ) with transition maps as above. By the definition of a colimit we see that the universal mapping property of  $A\langle x_w, w \in W \rangle$  is completely analogous to the mapping property stated in Lemma 5.1.

The following lemma can be found in [BO83].

**Lemma 5.3.** *Let  $p$  be a prime number. Let  $A$  be a ring such that every integer  $n$  not divisible by  $p$  is invertible, i.e.,  $A$  is a  $\mathbf{Z}_{(p)}$ -algebra. Let  $I \subset A$  be an ideal. Two divided power structures  $\gamma, \gamma'$  on  $I$  are equal if and only if  $\gamma_p = \gamma'_p$ . Moreover, given a map  $\delta : I \rightarrow I$  such that*

- (1)  $p!\delta(x) = x^p$  for all  $x \in I$ ,
- (2)  $\delta(ax) = a^p\delta(x)$  for all  $a \in A, x \in I$ , and
- (3)  $\delta(x+y) = \delta(x) + \sum_{i+j=p, i, j \geq 1} \frac{1}{i!j!} x^i y^j + \delta(y)$  for all  $x, y \in I$ ,

then there exists a unique divided power structure  $\gamma$  on  $I$  such that  $\gamma_p = \delta$ .

**Proof.** If  $n$  is not divisible by  $p$ , then  $\gamma_n(x) = cx\gamma_{n-1}(x)$  where  $c$  is a unit in  $\mathbf{Z}_{(p)}$ . Moreover,

$$\gamma_{pm}(x) = c\gamma_m(\gamma_p(x))$$

where  $c$  is a unit in  $\mathbf{Z}_{(p)}$ . Thus the first assertion is clear. For the second assertion, we can, working backwards, use these equalities to define all  $\gamma_n$ . More precisely, if  $n = a_0 + a_1p + \dots + a_ep^e$  with  $a_i \in \{0, \dots, p-1\}$  then we set

$$\gamma_n(x) = c_n x^{a_0} \delta(x)^{a_1} \dots \delta^e(x)^{a_e}$$

for  $c_n \in \mathbf{Z}_{(p)}$  defined by

$$c_n = (p!)^{a_1 + a_2(1+p) + \dots + a_e(1 + \dots + p^{e-1})} / n!.$$

Now we have to show the axioms (1) – (5) of a divided power structure, see Definition 2.1. We observe that (1) and (3) are immediate. Verification of (2) and (5) is by a direct calculation which we omit. Let  $x, y \in I$ . We claim there is a ring map

$$\varphi : \mathbf{Z}_{(p)}\langle u, v \rangle \longrightarrow A$$

which maps  $u^{[n]}$  to  $\gamma_n(x)$  and  $v^{[n]}$  to  $\gamma_n(y)$ . By construction of  $\mathbf{Z}_{(p)}\langle u, v \rangle$  this means we have to check that

$$\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!} \gamma_{n+m}(x)$$

in  $A$  and similarly for  $y$ . This is true because (2) holds for  $\gamma$ . Let  $\epsilon$  denote the divided power structure on the ideal  $\mathbf{Z}_{(p)}\langle u, v \rangle_+$  of  $\mathbf{Z}_{(p)}\langle u, v \rangle$ . Next, we claim that  $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$  for  $f \in \mathbf{Z}_{(p)}\langle u, v \rangle_+$  and all  $n$ . This is clear for  $n = 0, 1, \dots, p-1$ . For  $n = p$  it suffices to prove it for a set of generators of the ideal  $\mathbf{Z}_{(p)}\langle u, v \rangle_+$  because both  $\epsilon_p$  and  $\gamma_p = \delta$  satisfy properties (1) and (3) of the lemma. Hence it suffices to prove that  $\gamma_p(\gamma_n(x)) = \frac{(pn)!}{p!(n!)^p} \gamma_{pn}(x)$  and similarly for  $y$ , which follows as (5) holds for  $\gamma$ . Now, if  $n = a_0 + a_1p + \dots + a_ep^e$  is an arbitrary integer written in  $p$ -adic expansion as above, then

$$\epsilon_n(f) = c_n f^{a_0} \gamma_p(f)^{a_1} \dots \gamma_p^e(f)^{a_e}$$

because  $\epsilon$  is a divided power structure. Hence we see that  $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$  holds for all  $n$ . Applying this for  $f = u + v$  we see that axiom (4) for  $\gamma$  follows from the fact that  $\epsilon$  is a divided power structure.  $\square$

## 6. Tate resolutions

In this section we briefly discuss the resolutions constructed in [Tat57] which combine divided power structures with differential graded algebras. In this section we will use *homological notation* for differential graded algebras. Our differential graded algebras will sit in nonnegative homological degrees. Thus our differential graded algebras  $(A, d)$  will be given as chain complexes

$$\dots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0 \rightarrow 0 \rightarrow \dots$$

endowed with a multiplication.

Let  $R$  be a ring. In this section we will often consider graded  $R$ -algebras  $A = \bigoplus_{d \geq 0} A_d$  whose components are zero in negative degrees. We will set  $A_+ = \bigoplus_{d > 0} A_d$ . We will write  $A_{\text{even}} = \bigoplus_{d \geq 0} A_{2d}$  and  $A_{\text{odd}} = \bigoplus_{d \geq 0} A_{2d+1}$ . Recall that  $A$  is graded commutative if  $xy = (-1)^{\deg(x)\deg(y)}yx$  for homogeneous elements  $x, y$ . Recall that  $A$  is strictly graded commutative if in addition  $x^2 = 0$  for homogeneous elements  $x$  of odd degree. Finally, to understand the following definition, keep in mind that  $\gamma_n(x) = x^n/n!$  if  $A$  is a  $\mathbf{Q}$ -algebra.

**Definition 6.1.** Let  $R$  be a ring. Let  $A = \bigoplus_{d \geq 0} A_d$  be a graded  $R$ -algebra which is strictly graded commutative. A collection of maps  $\gamma_n : A_{even,+} \rightarrow A_{even,+}$  defined for all  $n > 0$  is called a *divided power structure* on  $A$  if we have

- (1)  $\gamma_n(x) \in A_{2nd}$  if  $x \in A_{2d}$ ,
- (2)  $\gamma_1(x) = x$  for any  $x$ , we also set  $\gamma_0(x) = 1$ ,
- (3)  $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!} \gamma_{n+m}(x)$ ,
- (4)  $\gamma_n(xy) = x^n \gamma_n(y)$  for all  $x \in A_{even}$  and  $y \in A_{even,+}$ ,
- (5)  $\gamma_n(xy) = 0$  if  $x, y \in A_{odd}$  homogeneous and  $n > 1$
- (6) if  $x, y \in A_{even,+}$  then  $\gamma_n(x+y) = \sum_{i=0, \dots, n} \gamma_i(x)\gamma_{n-i}(y)$ ,
- (7)  $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n} \gamma_{nm}(x)$  for  $x \in A_{even,+}$ .

Observe that conditions (2), (3), (4), (6), and (7) imply that  $\gamma$  is a ‘‘usual’’ divided power structure on the ideal  $A_{even,+}$  of the (commutative) ring  $A_{even}$ , see Sections 2, 3, 4, and 5. In particular, we have  $\gamma_n(x) = n!x^n$  for all  $x \in A_{even,+}$ . Condition (1) states that  $\gamma$  is compatible with grading and condition (5) tells us  $\gamma_n$  for  $n > 1$  vanishes on products of homogeneous elements of odd degree. But note that it may happen that

$$\gamma_2(z_1 z_2 + z_3 z_4) = z_1 z_2 z_3 z_4$$

is nonzero if  $z_1, z_2, z_3, z_4$  are homogeneous elements of odd degree.

**Example 6.2** (Adjoining odd variable). Let  $R$  be a ring. Let  $(A, \gamma)$  be a strictly graded commutative graded  $R$ -algebra endowed with a divided power structure as in the definition above. Let  $d > 0$  be an odd integer. In this setting we can adjoin a variable  $T$  of degree  $d$  to  $A$ . Namely, set

$$A\langle T \rangle = A \oplus AT$$

with grading given by  $A\langle T \rangle_m = A_m \oplus A_{m-d}T$ . We claim there is a unique divided power structure on  $A\langle T \rangle$  compatible with the given divided power structure on  $A$ . Namely, we set

$$\gamma_n(x + yT) = \gamma_n(x) + \gamma_{n-1}(x)yT$$

for  $x \in A_{even,+}$  and  $y \in A_{odd}$ .

**Example 6.3** (Adjoining even variable). Let  $R$  be a ring. Let  $(A, \gamma)$  be a strictly graded commutative graded  $R$ -algebra endowed with a divided power structure as in the definition above. Let  $d > 0$  be an even integer. In this setting we can adjoin a variable  $T$  of degree  $d$  to  $A$ . Namely, set

$$A\langle T \rangle = A \oplus AT \oplus AT^{(2)} \oplus AT^{(3)} \oplus \dots$$

with multiplication given by

$$T^{(n)}T^{(m)} = \frac{(n+m)!}{n!m!} T^{(n+m)}$$

and with grading given by

$$A\langle T \rangle_m = A_m \oplus A_{m-d}T \oplus A_{m-2d}T^{(2)} \oplus \dots$$

We claim there is a unique divided power structure on  $A\langle T \rangle$  compatible with the given divided power structure on  $A$  such that  $\gamma_n(T^{(i)}) = T^{(ni)}$ . To define the divided power structure we first set

$$\gamma_n \left( \sum_{i>0} x_i T^{(i)} \right) = \sum \prod_{n=\sum e_i} x_i^{e_i} T^{(ie_i)}$$

if  $x_i$  is in  $A_{\text{even}}$ . If  $x_0 \in A_{\text{even},+}$  then we take

$$\gamma_n \left( \sum_{i \geq 0} x_i T^{(i)} \right) = \sum_{a+b=n} \gamma_a(x_0) \gamma_b \left( \sum_{i > 0} x_i T^{(i)} \right)$$

where  $\gamma_b$  is as defined above.

At this point we tie in the definition of divided power structures with differentials. To understand the definition note that  $d(x^n/n!) = d(x)x^{n-1}/(n-1)!$  if  $A$  is a  $\mathbf{Q}$ -algebra and  $x \in A_{\text{even},+}$ .

**Definition 6.4.** Let  $R$  be a ring. Let  $A = \bigoplus_{d \geq 0} A_d$  be a differential graded  $R$ -algebra which is strictly graded commutative. A divided power structure  $\gamma$  on  $A$  is compatible with the differential graded structure if  $d(\gamma_n(x)) = d(x)\gamma_{n-1}(x)$  for all  $x \in A_{\text{even},+}$ .

Warning: Let  $(A, d, \gamma)$  be as in Definition 6.4. It may not be true that  $\gamma_n(x)$  is a boundary, if  $x$  is a boundary. Thus  $\gamma$  in general does not induce a divided power structure on the homology algebra  $H(A)$ . In some papers the authors put an additional compatibility condition in order to insure this is the case, but we elect not to do so.

**Lemma 6.5.** Let  $(A, d, \gamma)$  and  $(B, d, \gamma)$  be as in Definition 6.4. Let  $f : A \rightarrow B$  be a map of differential graded algebras compatible with divided power structures. Assume

- (1)  $H_k(A) = 0$  for  $k > 0$ , and
- (2)  $f$  is surjective.

Then  $\gamma$  induces a divided power structure on the graded  $R$ -algebra  $H(B)$ .

**Proof.** Suppose that  $x$  and  $x'$  are homogeneous of the same degree  $2d$  and define the same cohomology class in  $H(B)$ . Say  $x' - x = d(w)$ . Choose a lift  $y \in A_{2d}$  of  $x$  and a lift  $z \in A_{2d+1}$  of  $w$ . Then  $y' = y + d(z)$  is a lift of  $x'$ . Hence

$$\gamma_n(y') = \sum \gamma_i(y) \gamma_{n-i}(d(z)) = \gamma_n(y) + \sum_{i < n} \gamma_i(y) \gamma_{n-i}(d(z))$$

Since  $A$  is acyclic in positive degrees and since  $d(\gamma_j(d(z))) = 0$  for all  $j$  we can write this as

$$\gamma_n(y') = \gamma_n(y) + \sum_{i < n} \gamma_i(y) d(z_i)$$

for some  $z_i$  in  $A$ . Moreover, for  $0 < i < n$  we have

$$d(\gamma_i(y)z_i) = d(\gamma_i(y))z_i + \gamma_i(y)d(z_i) = d(y)\gamma_{i-1}(y)z_i + \gamma_i(y)d(z_i)$$

and the first term maps to zero in  $B$  as  $d(y)$  maps to zero in  $B$ . Hence  $\gamma_n(x')$  and  $\gamma_n(x)$  map to the same element of  $H(B)$ . Thus we obtain a well defined map  $\gamma_n : H_{2d}(B) \rightarrow H_{2nd}(B)$  for all  $d > 0$  and  $n > 0$ . We omit the verification that this defines a divided power structure on  $H(B)$ .  $\square$

**Lemma 6.6.** Let  $(A, d, \gamma)$  is as in Definition 6.4. Let  $R \rightarrow R'$  be a ring map. Then  $d$  and  $\gamma$  induce similar structures on  $A' = A \otimes_R R'$  such that  $(A, d, \gamma)$  is as in Definition 6.4.

**Proof.** Observe that  $A'_{\text{even}} = A_{\text{even}} \otimes_R R'$  and  $A'_{\text{even},+} = A_{\text{even},+} \otimes_R R'$ . Hence we are trying to show that the divided powers  $\gamma$  extend to  $A'_{\text{even}}$  (terminology as in Definition 4.1). Once we have shown  $\gamma$  extends it follows easily that this extension has all the desired properties.

Choose a polynomial  $R$ -algebra  $P$  and a surjection of  $R$ -algebras  $P \rightarrow R'$ . The ring map  $A_{\text{even}} \rightarrow A_{\text{even}} \otimes_R P$  is flat, hence the divided powers  $\gamma$  extend to  $A_{\text{even}} \otimes_R P$  uniquely by Lemma 4.2. Let  $J = \text{Ker}(P \rightarrow R')$ . To show that  $\gamma$  extends to  $A \otimes_R R'$  it suffices to show that  $I' = \text{Ker}(A_{\text{even},+} \otimes_R P \rightarrow A_{\text{even},+} \otimes_R R')$  is generated by elements  $z$  such that  $\gamma_n(z) \in I'$  for all  $n > 0$ . This is clear as  $I'$  is generated by elements of the form  $x \otimes f$  with  $x \in A_{\text{even},+}$  and  $f \in \text{Ker}(P \rightarrow R')$ .  $\square$

**Lemma 6.7.** *Let  $(A, d, \gamma)$  be as in Definition 6.4. Let  $d \geq 1$  be an integer. Let  $A\langle T \rangle$  be the graded divided power polynomial algebra on  $T$  with  $\deg(T) = d$  constructed in Example 6.2 or 6.3. Let  $f \in A_{d-1}$  be an element with  $d(f) = 0$ . There exists a unique differential  $d$  on  $A\langle T \rangle$  such that  $d(T) = f$  and such that  $d$  is compatible with the divided power structure on  $A\langle T \rangle$ .*

**Proof.** This is proved by a direct computation which is omitted.  $\square$

Here is the construction of Tate.

**Lemma 6.8.** *Assume that  $R$  is a Noetherian ring and  $R \rightarrow S$  a ring map of finite type. There exists a factorization*

$$R \rightarrow A \rightarrow S$$

with the following properties

- (1)  $(A, d, \gamma)$  is as in Definition 6.4,
- (2)  $A \rightarrow S$  is a quasi-isomorphism (if we endow  $S$  with the zero differential),
- (3)  $A$  is a graded divided power polynomial algebra over  $R$  with finitely many variables in each degree.

The last condition means that  $A$  is constructed out of  $R$  by successively adjoining variables  $T$  as in Examples 6.2 and 6.3.

**Proof.** Start of the construction. Let  $A(0) = R[x_1, \dots, x_n]$  be a (usual) polynomial ring and let  $A(0) \rightarrow S$  be a surjection. As grading we take  $A(0)_0 = A(0)$  and  $A(0)_d = 0$  for  $d \neq 0$ . Thus  $d = 0$  and  $\gamma_n, n > 0$  is zero as well.

Choose generators  $f_1, \dots, f_m \in R[x_1, \dots, x_m]$  for the kernel of the given map  $A(0) = R[x_1, \dots, x_m] \rightarrow S$ . We apply Examples 6.2  $m$  times to get

$$A(1) = A(0)\langle T_1, \dots, T_m \rangle$$

with  $\deg(T_i) = 1$  as a graded divided power polynomial algebra. We set  $d(T_i) = f_i$ . Since  $A(1)$  is a divided power polynomial algebra over  $A(0)$  and since  $d(f_i) = 0$  this extends uniquely to a differential on  $A(1)$  by Lemma 6.7.

Induction hypothesis: Assume we are given factorizations

$$R \rightarrow A(0) \rightarrow A(1) \rightarrow \dots \rightarrow A(m) \rightarrow S$$

where  $A(0)$  and  $A(1)$  are as above and each  $R \rightarrow A(m') \rightarrow S$  for  $2 \leq m' \leq m$  satisfies properties (1) and (3) of the statement of the lemma and (2) replaced by the condition that  $H_i(A(m')) \rightarrow H_i(S)$  is an isomorphism for  $m' > i \geq 0$ . The base case is  $m = 1$ .

Induction step. Assume we have  $R \rightarrow A(m) \rightarrow S$  as in the induction hypothesis. Consider the group  $H_m(A(m))$ . This is a module over  $H_0(A(m)) = S$ . In fact, it is

a subquotient of  $A(m)_m$  which is a finite type module over  $A(m)_0 = R[x_1, \dots, x_n]$ . Thus we can pick finitely many elements

$$e_1, \dots, e_t \in \text{Ker}(d : A(m)_m \rightarrow A(m)_{m-1})$$

which map to generators of this module. Applying Example 6.3 or 6.2  $t$  times we get

$$A(m+1) = A(m)\langle T_1, \dots, T_t \rangle$$

with  $\deg(T_i) = m+1$  as a graded divided power algebra. We set  $d(T_i) = e_i$ . Since  $A(1)$  is a divided power polynomial algebra over  $A(0)$  and since  $d(e_i) = 0$  this extends uniquely to a differential on  $A(m+1)$  compatible with the divided power structure. Since we've added only material in degree  $m+1$  and higher we see that  $H_i(A(m+1)) = H_i(A(m))$  for  $i < m$ . Moreover, it is clear that  $H_m(A(m+1)) = 0$  by construction.

To finish the proof we observe that we have shown there exists a sequence of maps

$$R \rightarrow A(0) \rightarrow A(1) \rightarrow \dots \rightarrow A(m) \rightarrow A(m+1) \rightarrow \dots \rightarrow S$$

and to finish the proof we set  $A = \text{colim } A(m)$ .  $\square$

**Lemma 6.9.** *Let  $R$  be a ring. Suppose that  $(A, d, \gamma)$  and  $(B, d, \gamma)$  are as in Definition 6.4. Let  $\bar{\varphi} : H_0(A) \rightarrow H_0(B)$  be an  $R$ -algebra map. Assume*

- (1)  *$A$  is a graded divided power polynomial algebra over  $R$  with finitely many variables in each degree,*
- (2)  *$H_k(B) = 0$  for  $k > 0$ .*

*Then there exists a map  $\varphi : A \rightarrow B$  of differential graded  $R$ -algebras compatible with divided powers lifting  $\bar{\varphi}$ .*

**Proof.** Since  $A$  is obtained from  $R$  by adjoining divided power variables, we obtain filtrations  $R \subset A(0) \subset A(1) \subset \dots$  such that  $A(m+1)$  is obtained from  $A(m)$  by adjoining finitely many divided power variables of degree  $m+1$ . Then  $A(0) \rightarrow S$  is a surjection from a (usual) polynomial algebra over  $R$  onto  $S$ . Thus we can lift  $\bar{\varphi}$  to an  $R$ -algebra map  $\varphi(0) : A(0) \rightarrow B(0)$ .

Write  $A(1) = A(0)\langle T_1, \dots, T_m \rangle$  for some divided power variables  $T_j$  of degree 1. Let  $f_j \in B_0$  be  $f_j = \varphi(0)(d(T_j))$ . Observe that  $f_j$  maps to zero in  $H_0(B)$  as  $dT_j$  maps to zero in  $H_0(A)$ . Thus we can find  $b_j \in B_1$  with  $d(b_j) = f_j$ . By the universal property of divided power polynomial algebras we find a lift  $\varphi(1) : A(1) \rightarrow B$  of  $\varphi(0)$  mapping  $T_j$  to  $f_j$ .

Having constructed  $\varphi(m)$  for some  $m \geq 1$  we can construct  $\varphi(m+1) : A(m+1) \rightarrow B$  in exactly the same manner. We omit the details.  $\square$

**Lemma 6.10.** *Let  $R$  be a Noetherian ring. Let  $R \rightarrow S$  and  $R \rightarrow T$  be finite type ring maps. There exists a canonical structure of a divided power graded  $R$ -algebra on*

$$\text{Tor}_*^R(S, T)$$

**Proof.** Choose a factorization  $R \rightarrow A \rightarrow S$  as above. Since  $A \rightarrow S$  is a quasi-isomorphism and since  $A_d$  is a free  $R$ -module, we see that the differential graded algebra  $B = A \otimes_R T$  computes the tor groups displayed in the lemma. Choose a surjection  $R[y_1, \dots, y_k] \rightarrow T$ . Then we see that  $B$  is a quotient of the differential graded algebra  $A[y_1, \dots, y_k]$  whose homology sits in degree 0 (it is equal to

$S[y_1, \dots, y_k]$ ). By Lemma 6.6 the differential graded algebras  $B$  and  $A[y_1, \dots, y_k]$  have divided power structures compatible with the differentials. Hence we obtain our divided power structure on  $H(B)$  by Lemma 6.5.

The divided power algebra structure constructed in this way is independent of the choice of  $A$ . Namely, if  $A'$  is a second choice, then Lemma 6.9 implies there is a map  $A \rightarrow A'$  preserving all structure and the augmentations towards  $S$ . Then the induced map  $B = A \otimes_R T \rightarrow A' \otimes_R T' = B'$  is likewise and is a quasi-isomorphism. The induced isomorphism of Tor algebras is therefore compatible with all multiplication and divided powers.  $\square$

### 7. Application to complete intersections

Let  $R$  be a ring. Let  $(A, d, \gamma)$  be as in Definition 6.4. A *derivation* of degree 2 is an  $R$ -linear map  $\theta : A \rightarrow A$  with the following properties

- (1)  $\theta(A_d) \subset A_{d-2}$ ,
- (2)  $\theta(xy) = \theta(x)y + x\theta(y)$ ,
- (3)  $\theta$  commutes with  $d$ ,
- (4)  $\theta(\gamma_n(x)) = \theta(x)\gamma_{n-1}(x)$  for all  $x \in A_{2d}$  all  $d$ .

In the following lemma we construct a derivation.

**Lemma 7.1.** *Let  $R$  be a ring. Let  $(A, d, \gamma)$  be as in Definition 6.4. Let  $R' \rightarrow R$  be a surjection of rings whose kernel has square zero and is generated by one element  $f$ . If  $A$  is a graded divided power polynomial algebra over  $R$  with finitely many variables in each degree, then we obtain a derivation  $\theta : A/IA \rightarrow A/IA$  where  $I$  is the annihilator of  $f$  in  $R$ .*

**Proof.** Since  $A$  is a divided power polynomial algebra, we can find a divided power polynomial algebra  $A'$  over  $R'$  such that  $A = A' \otimes_R R'$ . Moreover, we can lift  $d$  to an  $R$ -linear operator  $d$  on  $A'$  such that

- (1)  $d(xy) = d(x)y + (-1)^{\deg(x)}xd(y)$  for  $x, y \in A'$  homogeneous, and
- (2)  $d(\gamma_n(x)) = d(x)\gamma_{n-1}(x)$  for  $x \in A'_{\text{even},+}$ .

We omit the details (hint: proceed one variable at the time). However, it may not be the case that  $d^2$  is zero on  $A'$ . It is clear that  $d^2$  maps  $A'$  into  $fA' \cong A/IA$ . Hence  $d^2$  annihilates  $fA'$  and factors as a map  $A \rightarrow A/IA$ . Since  $d^2$  is  $R$ -linear we obtain our map  $\theta : A/IA \rightarrow A/IA$ . The verification of the properties of a derivation is immediate.  $\square$

**Lemma 7.2.** *Assumption and notation as in Lemma 7.1. Suppose  $S = H_0(A)$  is isomorphic to  $R[x_1, \dots, x_n]/(f_1, \dots, f_m)$  for some  $n, m$ , and  $f_j \in R[x_1, \dots, x_n]$ . Moreover, suppose given a relation*

$$\sum r_j f_j = 0$$

with  $r_j \in R[x_1, \dots, x_n]$ . Choose  $r'_j, f'_j \in R'[x_1, \dots, x_n]$  lifting  $r_j, f_j$ . Write  $\sum r'_j f'_j = gf$  for some  $g \in R/I[x_1, \dots, x_n]$ . If  $H_1(A) = 0$  and all the coefficients of each  $r_j$  are in  $I$ , then there exists an element  $\xi \in H_2(A/IA)$  such that  $\theta(\xi) = g$  in  $S/IS$ .

**Proof.** Let  $A(0) \subset A(1) \subset A(2) \subset \dots$  be the filtration of  $A$  such that  $A(m)$  is gotten from  $A(m-1)$  by adjoining divided power variables of degree  $m$ . Then  $A(0)$

is a polynomial algebra over  $R$  equipped with an  $R$ -algebra surjection  $A(0) \rightarrow S$ . Thus we can choose a map

$$\varphi : R[x_1, \dots, x_n] \rightarrow A(0)$$

lifting the augmentations to  $S$ . Next,  $A(1) = A(0)\langle T_1, \dots, T_t \rangle$  for some divided power variables  $T_i$  of degree 1. Since  $H_0(A) = S$  we can pick  $\xi_j \in \sum A(0)T_i$  with  $d(\xi_j) = \varphi(f_j)$ . Then

$$d\left(\sum \varphi(r_j)\xi_j\right) = \sum \varphi(r_j)\varphi(f_j) = \sum \varphi(r_j f_j) = 0$$

Since  $H_1(A) = 0$  we can pick  $\xi \in A_2$  with  $d(\xi) = \sum \varphi(r_j)\xi_j$ . If the coefficients of  $r_j$  are in  $I$ , then the same is true for  $\varphi(r_j)$ . In this case  $d(\xi)$  dies in  $A_1/IA_1$  and hence  $\xi$  defines a class in  $H_2(A/IA)$ .

The construction of  $\theta$  in the proof of Lemma 7.1 proceeds by successively lifting  $A(i)$  to  $A'(i)$  and lifting the differential  $d$ . We lift  $\varphi$  to  $\varphi' : R'[x_1, \dots, x_n] \rightarrow A'(0)$ . Next, we have  $A'(1) = A'(0)\langle T_1, \dots, T_t \rangle$ . Moreover, we can lift  $\xi_j$  to  $\xi'_j \in \sum A'(0)T_i$ . Then  $d(\xi'_j) = \varphi'(f'_j) + f a_j$  for some  $a_j \in A'(0)$ . Consider a lift  $\xi' \in A'_2$  of  $\xi$ . Then we know that

$$d(\xi') = \sum \varphi'(r'_j)\xi'_j + \sum f b_i T_i$$

for some  $b_i \in A(0)$ . Applying  $d$  again we find

$$\theta(\xi) = \sum \varphi'(r'_j)\varphi'(f'_j) + \sum f \varphi'(r'_j)a_j + \sum f b_i d(T_i)$$

The first term gives us what we want. The second term is zero because the coefficients of  $r_j$  are in  $I$  and hence are annihilated by  $f$ . The third term maps to zero in  $H_0$  because  $d(T_i)$  maps to zero.  $\square$

The method of proof of the following lemma is apparantly due to Gulliksen.

**Lemma 7.3.** *Let  $R' \rightarrow R$  be a surjection of Noetherian rings whose kernel has square zero and is generated by one element  $f$ . Let  $S = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$ . Let  $\sum r_j f_j = 0$  be a relation in  $R[x_1, \dots, x_n]$ . Assume that*

- (1) *each  $r_j$  has coefficients in the annihilator  $I$  of  $f$  in  $R$ ,*
- (2) *for some lifts  $r'_j, f'_j \in R'[x_1, \dots, x_n]$  we have  $\sum r'_j f'_j = gf$  where  $g$  is not nilpotent in  $S$ .*

*Then  $S$  does not have finite tor dimension over  $R$  (i.e.,  $S$  is not a perfect  $R$ -algebra).*

**Proof.** Choose a Tate resolution  $R \rightarrow A \rightarrow S$  as in Lemma 6.8. Let  $\xi \in H_2(A/IA)$  and  $\theta : A/IA \rightarrow A/IA$  be the element and derivation found in Lemmas 7.1 and 7.2. Observe that

$$\theta^n(\gamma_n(\xi)) = g^n$$

Hence if  $g$  is not nilpotent, then  $\xi^n$  is nonzero in  $H_{2n}(A/IA)$  for all  $n > 0$ . Since  $H_{2n}(A/IA) = \text{Tor}_{2n}^R(S, R/I)$  we conclude.  $\square$

The following result can be found in [Rod88].

**Lemma 7.4.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring. Let  $I \subset J \subset A$  be proper ideals. If  $A/J$  has finite tor dimension over  $A/I$ , then  $I/\mathfrak{m}I \rightarrow J/\mathfrak{m}J$  is injective.*

**Proof.** Let  $f \in I$  be an element mapping to a nonzero element of  $I/\mathfrak{m}I$  which is mapped to zero in  $J/\mathfrak{m}J$ . We can choose an ideal  $I'$  with  $\mathfrak{m}I \subset I' \subset I$  such that  $I/I'$  is generated by the image of  $f$ . Set  $R = A/I$  and  $R' = A/I'$ . Let  $J = (a_1, \dots, a_m)$  for some  $a_j \in A$ . Then  $f = \sum b_j a_j$  for some  $b_j \in \mathfrak{m}$ . Let  $r_j, f_j \in R$  resp.  $r'_j, f'_j \in R'$  be the image of  $b_j, a_j$ . Then we see we are in the situation of Lemma 7.3 (with the ideal  $I$  of that lemma equal to  $\mathfrak{m}_R$ ) and the lemma is proved.  $\square$

**Lemma 7.5.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring. Let  $I \subset J \subset A$  be proper ideals. Assume*

- (1)  $A/J$  has finite tor dimension over  $A/I$ , and
- (2)  $J$  is generated by a regular sequence.

*Then  $I$  is generated by a regular sequence and  $J/I$  is generated by a regular sequence.*

**Proof.** By Lemma 7.4 we see that  $I/\mathfrak{m}I \rightarrow J/\mathfrak{m}J$  is injective. Thus we can find  $s \leq r$  and a minimal system of generators  $f_1, \dots, f_r$  of  $J$  such that  $f_1, \dots, f_s$  are in  $I$  and form a minimal system of generators of  $I$ . The lemma follows as any minimal system of generators of  $J$  is a regular sequence by More on Algebra, Lemmas 21.14 and 21.6.  $\square$

**Lemma 7.6.** *Let  $R \rightarrow S$  be a local ring map of Noetherian local rings. Let  $I \subset R$  and  $J \subset S$  be ideals with  $IS \subset J$ . If  $R \rightarrow S$  is flat and  $S/\mathfrak{m}_R S$  is regular, then the following are equivalent*

- (1)  $J$  is generated by a regular sequence and  $S/J$  has finite tor dimension as a module over  $R/I$ ,
- (2)  $J$  is generated by a regular sequence and  $\text{Tor}_p^{R/I}(S/J, R/\mathfrak{m}_R)$  is nonzero for only finitely many  $p$ ,
- (3)  $I$  is generated by a regular sequence and  $J/IS$  is generated by a regular sequence in  $S/IS$ .

**Proof.** If (3) holds, then  $J$  is generated by a regular sequence, see for example More on Algebra, Lemmas 21.12 and 21.6. Moreover, if (3) holds, then  $S/J = (S/I)/(J/I)$  has finite projective dimension over  $S/IS$  because the Koszul complex will be a finite free resolution of  $S/J$  over  $S/IS$ . Since  $R/I \rightarrow S/IS$  is flat, it then follows that  $S/J$  has finite tor dimension over  $R/I$  by More on Algebra, Lemma 51.9. Thus (3) implies (1).

The implication (1)  $\Rightarrow$  (2) is trivial. Assume (2). By More on Algebra, Lemma 56.20 we find that  $S/J$  has finite tor dimension over  $S/IS$ . Thus we can apply Lemma 7.5 to conclude that  $IS$  and  $J/IS$  are generated by regular sequences. Let  $f_1, \dots, f_r \in I$  be a minimal system of generators of  $I$ . Since  $R \rightarrow S$  is flat, we see that  $f_1, \dots, f_r$  form a minimal system of generators for  $IS$  in  $S$ . Thus  $f_1, \dots, f_r \in R$  is a sequence of elements whose images in  $S$  form a regular sequence by More on Algebra, Lemmas 21.14 and 21.6. Thus  $f_1, \dots, f_r$  is a regular sequence in  $R$  by Algebra, Lemma 67.7.  $\square$

## 8. Local complete intersection rings

Let  $(A, \mathfrak{m})$  be a Noetherian complete local ring. By the Cohen structure theorem (see Algebra, Theorem 149.8) we can write  $A$  as the quotient of a regular Noetherian complete local ring  $R$ . Let us say that  $A$  is a *complete intersection* if there exists some surjection  $R \rightarrow A$  with  $R$  a regular local ring such that the kernel is generated

by a regular sequence. The following lemma shows this notion is independent of the choice of the surjection.

**Lemma 8.1.** *Let  $(A, \mathfrak{m})$  be a Noetherian complete local ring. The following are equivalent*

- (1) *for every surjection of local rings  $R \rightarrow A$  with  $R$  a regular local ring, the kernel of  $R \rightarrow A$  is generated by a regular sequence, and*
- (2) *for some surjection of local rings  $R \rightarrow A$  with  $R$  a regular local ring, the kernel of  $R \rightarrow A$  is generated by a regular sequence.*

**Proof.** Let  $k$  be the residue field of  $A$ . If the characteristic of  $k$  is  $p > 0$ , then we denote  $\Lambda$  a Cohen ring (Algebra, Definition 149.5) with residue field  $k$  (Algebra, Lemma 149.6). If the characteristic of  $k$  is 0 we set  $\Lambda = k$ . Recall that  $\Lambda[[x_1, \dots, x_n]]$  for any  $n$  is formally smooth over  $\mathbf{Z}$ , resp.  $\mathbf{Q}$  in the  $\mathfrak{m}$ -adic topology, see More on Algebra, Lemma 28.1. Fix a surjection  $\Lambda[[x_1, \dots, x_n]] \rightarrow A$  as in the Cohen structure theorem (Algebra, Theorem 149.8).

Let  $R \rightarrow A$  be a surjection from a regular local ring  $R$ . Let  $f_1, \dots, f_r$  be a minimal sequence of generators of  $\text{Ker}(R \rightarrow A)$ . We will use without further mention that an ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence. Observe that  $f_1, \dots, f_r$  is a regular sequence in  $R$  if and only if  $f_1, \dots, f_r$  is a regular sequence in the completion  $R^\wedge$  by Algebra, Lemmas 67.7 and 93.3. Moreover, we have

$$R^\wedge / (f_1, \dots, f_r)R^\wedge = (R / (f_1, \dots, f_r))^\wedge = A^\wedge = A$$

because  $A$  is  $\mathfrak{m}_A$ -adically complete (first equality by Algebra, Lemma 93.2). Finally, the ring  $R^\wedge$  is regular since  $R$  is regular (More on Algebra, Lemma 32.4). Hence we may assume  $R$  is complete.

If  $R$  is complete we can choose a map  $\Lambda[[x_1, \dots, x_n]] \rightarrow R$  lifting the given map  $\Lambda[[x_1, \dots, x_n]] \rightarrow A$ , see More on Algebra, Lemma 27.5. By adding some more variables  $y_1, \dots, y_m$  mapping to generators of the kernel of  $R \rightarrow A$  we may assume that  $\Lambda[[x_1, \dots, x_n, y_1, \dots, y_m]] \rightarrow R$  is surjective (some details omitted). Then we can consider the commutative diagram

$$\begin{array}{ccc} \Lambda[[x_1, \dots, x_n, y_1, \dots, y_m]] & \longrightarrow & R \\ \downarrow & & \downarrow \\ \Lambda[[x_1, \dots, x_n]] & \longrightarrow & A \end{array}$$

By Algebra, Lemma 130.6 we see that the condition for  $R \rightarrow A$  is equivalent to the condition for the fixed chosen map  $\Lambda[[x_1, \dots, x_n]] \rightarrow A$ . This finishes the proof of the lemma.  $\square$

The following two lemmas are sanity checks on the definition given above.

**Lemma 8.2.** *Let  $R$  be a regular ring. Let  $\mathfrak{p} \subset R$  be a prime. Let  $f_1, \dots, f_r \in \mathfrak{p}$  be a regular sequence. Then the completion of*

$$A = (R / (f_1, \dots, f_r))_{\mathfrak{p}} = R_{\mathfrak{p}} / (f_1, \dots, f_r)R_{\mathfrak{p}}$$

*is a complete intersection in the sense defined above.*

**Proof.** The completion of  $A$  is equal to  $A^\wedge = R_{\mathfrak{p}}^\wedge / (f_1, \dots, f_r)R_{\mathfrak{p}}^\wedge$  because completion for finite modules over the Noetherian ring  $R_{\mathfrak{p}}$  is exact (Algebra, Lemma 93.2). The image of the sequence  $f_1, \dots, f_r$  in  $R_{\mathfrak{p}}$  is a regular sequence by Algebra, Lemmas 93.3 and 67.7. Moreover,  $R_{\mathfrak{p}}^\wedge$  is a regular local ring by More on Algebra, Lemma 32.4. Hence the result holds by our definition of complete intersection for complete local rings.  $\square$

The following lemma is the analogue of Algebra, Lemma 130.4.

**Lemma 8.3.** *Let  $R$  be a regular ring. Let  $\mathfrak{p} \subset R$  be a prime. Let  $I \subset \mathfrak{p}$  be an ideal. Set  $A = (R/I)_{\mathfrak{p}} = R_{\mathfrak{p}}/I_{\mathfrak{p}}$ . The following are equivalent*

- (1) *the completion of  $A$  is a complete intersection in the sense above,*
- (2)  *$I_{\mathfrak{p}} \subset R_{\mathfrak{p}}$  is generated by a regular sequence,*
- (3) *the module  $(I/I^2)_{\mathfrak{p}}$  can be generated by  $\dim(R_{\mathfrak{p}}) - \dim(A)$  elements,*
- (4) *add more here.*

**Proof.** We may and do replace  $R$  by its localization at  $\mathfrak{p}$ . Then  $\mathfrak{p} = \mathfrak{m}$  is the maximal ideal of  $R$  and  $A = R/I$ . Let  $f_1, \dots, f_r \in I$  be a minimal sequence of generators. The completion of  $A$  is equal to  $A^\wedge = R^\wedge / (f_1, \dots, f_r)R^\wedge$  because completion for finite modules over the Noetherian ring  $R_{\mathfrak{p}}$  is exact (Algebra, Lemma 93.2).

If (1) holds, then the image of the sequence  $f_1, \dots, f_r$  in  $R^\wedge$  is a regular sequence by assumption. Hence it is a regular sequence in  $R$  by Algebra, Lemmas 93.3 and 67.7. Thus (1) implies (2).

Assume (3) holds. Set  $c = \dim(R) - \dim(A)$  and let  $f_1, \dots, f_c \in I$  map to generators of  $I/I^2$ . by Nakayama's lemme (Algebra, Lemma 19.1) we see that  $I = (f_1, \dots, f_c)$ . Since  $R$  is regular and hence Cohen-Macaulay (Algebra, Proposition 99.5) we see that  $f_1, \dots, f_c$  is a regular sequence by Algebra, Proposition 99.5. Thus (3) implies (2). Finally, (2) implies (1) by Lemma 8.2.  $\square$

The following result is due to Avramov, see [Avr75].

**Proposition 8.4.** *Let  $A \rightarrow B$  be a flat local homomorphism of Noetherian local rings. Then the following are equivalent*

- (1)  *$B^\wedge$  is a complete intersection,*
- (2)  *$A^\wedge$  and  $(B/\mathfrak{m}_A B)^\wedge$  are complete intersections.*

**Proof.** Consider the diagram

$$\begin{array}{ccc} B & \longrightarrow & B^\wedge \\ \uparrow & & \uparrow \\ A & \longrightarrow & A^\wedge \end{array}$$

Since the horizontal maps are faithfully flat (Algebra, Lemma 93.4) we conclude that the right vertical arrow is flat (for example by Algebra, Lemma 95.14). Moreover, we have  $(B/\mathfrak{m}_A B)^\wedge = B^\wedge / \mathfrak{m}_{A^\wedge} B^\wedge$  by Algebra, Lemma 93.2. Thus we may assume  $A$  and  $B$  are complete local Noetherian rings.

Assume  $A$  and  $B$  are complete local Noetherian rings. Choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in More on Algebra, Lemma 28.3. Let  $I = \text{Ker}(R \rightarrow A)$  and  $J = \text{Ker}(S \rightarrow B)$ . Note that since  $R/I = A \rightarrow B = S/J$  is flat the map  $J/I \otimes_R R/\mathfrak{m}_R \rightarrow J/J \cap \mathfrak{m}_R S$  is an isomorphism. Hence a minimal system of generators of  $J/I$  maps to a minimal system of generators of  $\text{Ker}(S/\mathfrak{m}_R S \rightarrow B/\mathfrak{m}_A B)$ . Finally,  $S/\mathfrak{m}_R S$  is a regular local ring.

Assume (1) holds, i.e.,  $J$  is generated by a regular sequence. Since  $A = R/I \rightarrow B = S/J$  is flat we see Lemma 7.6 applies and we deduce that  $I$  and  $J/I$  are generated by regular sequences. We have  $\dim(B) = \dim(A) + \dim(B/\mathfrak{m}_A B)$  and  $\dim(S/IS) = \dim(A) + \dim(S/\mathfrak{m}_R S)$  (Algebra, Lemma 108.7). Thus  $J/I$  is generated by

$$\dim(S/J) - \dim(S/IS) = \dim(S/\mathfrak{m}_R S) - \dim(B/\mathfrak{m}_A B)$$

elements (Algebra, Lemma 59.11). It follows that  $\text{Ker}(S/\mathfrak{m}_R S \rightarrow B/\mathfrak{m}_A B)$  is generated by the same number of elements (see above). Hence  $\text{Ker}(S/\mathfrak{m}_R S \rightarrow B/\mathfrak{m}_A B)$  is generated by a regular sequence, see for example Lemma 8.3. In this way we see that (2) holds.

If (2) holds, then  $I$  and  $J/J \cap \mathfrak{m}_R S$  are generated by regular sequences. Lifting these generators (see above), using flatness of  $R/I \rightarrow S/IS$ , and using Grothendieck's lemma (Algebra, Lemma 95.3) we find that  $J/I$  is generated by a regular sequence in  $S/IS$ . Thus Lemma 7.6 tells us that  $J$  is generated by a regular sequence, whence (1) holds.  $\square$

**Definition 8.5.** Let  $A$  be a Noetherian ring.

- (1) If  $A$  is local, then we say  $A$  is a *complete intersection* if its completion is a complete intersection in the sense above.
- (2) In general we say  $A$  is a *local complete intersection* if all of its local rings are complete intersections.

We will check below that this does not conflict with the terminology introduced in Algebra, Definitions 130.1 and 130.5. But first, we show this “makes sense” by showing that if  $A$  is a Noetherian local complete intersection, then  $A$  is a local complete intersection, i.e., all of its local rings are complete intersections.

**Lemma 8.6.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring. Let  $\mathfrak{p} \subset A$  be a prime ideal. If  $A$  is a complete intersection, then  $A_{\mathfrak{p}}$  is a complete intersection too.*

**Proof.** Choose a prime  $\mathfrak{q}$  of  $A^\wedge$  lying over  $\mathfrak{p}$  (this is possible as  $A \rightarrow A^\wedge$  is faithfully flat by Algebra, Lemma 93.4). Then  $A_{\mathfrak{p}} \rightarrow (A^\wedge)_{\mathfrak{q}}$  is a flat local ring homomorphism. Thus by Proposition 8.4 we see that  $A_{\mathfrak{p}}$  is a complete intersection if and only if  $(A^\wedge)_{\mathfrak{q}}$  is a complete intersection. Thus it suffices to prove the lemma in case  $A$  is complete (this is the key step of the proof).

Assume  $A$  is complete. By definition we may write  $A = R/(f_1, \dots, f_r)$  for some regular sequence  $f_1, \dots, f_r$  in a regular local ring  $R$ . Let  $\mathfrak{q} \subset R$  be the prime corresponding to  $\mathfrak{p}$ . Observe that  $f_1, \dots, f_r \in \mathfrak{q}$  and that  $A_{\mathfrak{p}} = R_{\mathfrak{q}}/(f_1, \dots, f_r)R_{\mathfrak{q}}$ . Hence  $A_{\mathfrak{p}}$  is a complete intersection by Lemma 8.2.  $\square$

**Lemma 8.7.** *Let  $A$  be a Noetherian ring. Then  $A$  is a local complete intersection if and only if  $A_{\mathfrak{m}}$  is a complete intersection for every maximal ideal  $\mathfrak{m}$  of  $A$ .*

**Proof.** This follows immediately from Lemma 8.6 and the definitions.  $\square$

**Lemma 8.8.** *Let  $S$  be a finite type algebra over a field  $k$ .*

- (1) *for a prime  $\mathfrak{q} \subset S$  the local ring  $S_{\mathfrak{q}}$  is a complete intersection in the sense of Algebra, Definition 130.5 if and only if  $S_{\mathfrak{q}}$  is a complete intersection in the sense of Definition 8.5, and*
- (2)  *$S$  is a local complete intersection in the sense of Algebra, Definition 130.1 if and only if  $S$  is a local complete intersection in the sense of Definition 8.5.*

**Proof.** Proof of (1). Let  $k[x_1, \dots, x_n] \rightarrow S$  be a surjection. Let  $\mathfrak{p} \subset k[x_1, \dots, x_n]$  be the prime ideal corresponding to  $\mathfrak{q}$ . Let  $I \subset k[x_1, \dots, x_n]$  be the kernel of our surjection. Note that  $k[x_1, \dots, x_n]_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}}$  is surjective with kernel  $I_{\mathfrak{p}}$ . Observe that  $k[x_1, \dots, x_n]$  is a regular ring by Algebra, Proposition 110.2. Hence the equivalence of the two notions in (1) follows by combining Lemma 8.3 with Algebra, Lemma 130.7.

Having proved (1) the equivalence in (2) follows from the definition and Algebra, Lemma 130.9.  $\square$

**Lemma 8.9.** *Let  $A \rightarrow B$  be a flat local homomorphism of Noetherian local rings. Then the following are equivalent*

- (1)  *$B$  is a complete intersection,*
- (2)  *$A$  and  $B/\mathfrak{m}_A B$  are complete intersections.*

**Proof.** Now that the definition makes sense this is a trivial reformulation of the (nontrivial) Proposition 8.4.  $\square$

### 9. Local complete intersection maps

Let  $A \rightarrow B$  be a local homomorphism of Noetherian complete local rings. A consequence of the Cohen structure theorem is that we can find a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ & \searrow & \uparrow \\ & & A \end{array}$$

of Noetherian complete local rings with  $S \rightarrow B$  surjective,  $A \rightarrow S$  flat, and  $S/\mathfrak{m}_A S$  a regular local ring. This follows from More on Algebra, Lemma 28.3. Let us (temporarily) say  $A \rightarrow S \rightarrow B$  is a *good factorization* of  $A \rightarrow B$  if  $S$  is a Noetherian local ring,  $A \rightarrow S \rightarrow B$  are local ring maps,  $S \rightarrow B$  surjective,  $A \rightarrow S$  flat, and  $S/\mathfrak{m}_A S$  regular. Let us say that  $A \rightarrow B$  is a *complete intersection homomorphism* if there exists some good factorization  $A \rightarrow S \rightarrow B$  such that the kernel of  $S \rightarrow B$  is generated by a regular sequence. The following lemma shows this notion is independent of the choice of the diagram.

**Lemma 9.1.** *Let  $A \rightarrow B$  be a local homomorphism of Noetherian complete local rings. The following are equivalent*

- (1) for some good factorization  $A \rightarrow S \rightarrow B$  the kernel of  $S \rightarrow B$  is generated by a regular sequence, and
- (2) for every good factorization  $A \rightarrow S \rightarrow B$  the kernel of  $S \rightarrow B$  is generated by a regular sequence.

**Proof.** Let  $A \rightarrow S \rightarrow B$  be a good factorization. As  $B$  is complete we obtain a factorization  $A \rightarrow S^\wedge \rightarrow B$  where  $S^\wedge$  is the completion of  $S$ . Note that this is also a good factorization: The ring map  $S \rightarrow S^\wedge$  is flat (Algebra, Lemma 93.3), hence  $A \rightarrow S^\wedge$  is flat. The ring  $S^\wedge/\mathfrak{m}_A S^\wedge = (S/\mathfrak{m}_A S)^\wedge$  is regular since  $S/\mathfrak{m}_A S$  is regular (More on Algebra, Lemma 32.4). Let  $f_1, \dots, f_r$  be a minimal sequence of generators of  $\text{Ker}(S \rightarrow B)$ . We will use without further mention that an ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence. Observe that  $f_1, \dots, f_r$  is a regular sequence in  $S$  if and only if  $f_1, \dots, f_r$  is a regular sequence in the completion  $S^\wedge$  by Algebra, Lemma 67.7. Moreover, we have

$$S^\wedge/(f_1, \dots, f_r)R^\wedge = (S/(f_1, \dots, f_r))^\wedge = B^\wedge = B$$

because  $B$  is  $\mathfrak{m}_B$ -adically complete (first equality by Algebra, Lemma 93.2). Thus the kernel of  $S \rightarrow B$  is generated by a regular sequence if and only if the kernel of  $S^\wedge \rightarrow B$  is generated by a regular sequence. Hence it suffices to consider good factorizations where  $S$  is complete.

Assume we have two factorizations  $A \rightarrow S \rightarrow B$  and  $A \rightarrow S' \rightarrow B$  with  $S$  and  $S'$  complete. By More on Algebra, Lemma 28.4 the ring  $S \times_B S'$  is a Noetherian complete local ring. Hence, using More on Algebra, Lemma 28.3 we can choose a good factorization  $A \rightarrow S'' \rightarrow S \times_B S'$  with  $S''$  complete. Thus it suffices to show: If  $A \rightarrow S' \rightarrow S \rightarrow B$  are comparable good factorizations, then  $\text{Ker}(S \rightarrow B)$  is generated by a regular sequence if and only if  $\text{Ker}(S' \rightarrow B)$  is generated by a regular sequence.

Let  $A \rightarrow S' \rightarrow S \rightarrow B$  be comparable good factorizations. First, since  $S'/\mathfrak{m}_R S' \rightarrow S/\mathfrak{m}_R S$  is a surjection of regular local rings, the kernel is generated by a regular sequence  $\bar{x}_1, \dots, \bar{x}_c \in \mathfrak{m}_{S'}/\mathfrak{m}_R S'$  which can be extended to a regular system of parameters for the regular local ring  $S'/\mathfrak{m}_R S'$ , see (Algebra, Lemma 102.4). Set  $I = \text{Ker}(S' \rightarrow S)$ . By flatness of  $S$  over  $R$  we have

$$I/\mathfrak{m}_R I = \text{Ker}(S'/\mathfrak{m}_R S' \rightarrow S/\mathfrak{m}_R S) = (\bar{x}_1, \dots, \bar{x}_c).$$

Choose lifts  $x_1, \dots, x_c \in I$ . These lifts form a regular sequence generating  $I$  as  $S'$  is flat over  $R$ , see Algebra, Lemma 95.3.

We conclude that if also  $\text{Ker}(S \rightarrow B)$  is generated by a regular sequence, then so is  $\text{Ker}(S' \rightarrow B)$ , see More on Algebra, Lemmas 21.12 and 21.6.

Conversely, assume that  $J = \text{Ker}(S' \rightarrow B)$  is generated by a regular sequence. Because the generators  $x_1, \dots, x_c$  of  $I$  map to linearly independent elements of  $\mathfrak{m}_{S'}/\mathfrak{m}_R^2 S'$ , we see that  $I/\mathfrak{m}_{S'} I \rightarrow J/\mathfrak{m}_{S'} J$  is injective. Hence there exists a minimal system of generators  $x_1, \dots, x_c, y_1, \dots, y_d$  for  $J$ . Then  $x_1, \dots, x_c, y_1, \dots, y_d$  is a regular sequence and it follows that the images of  $y_1, \dots, y_d$  in  $S$  form a regular sequence generating  $\text{Ker}(S \rightarrow B)$ . This finishes the proof of the lemma.  $\square$

In the following proposition observe that the condition on vanishing of Tor's applies in particular if  $B$  has finite tor dimension over  $A$  and thus in particular if  $B$  is flat over  $A$ .

**Proposition 9.2.** *Let  $A \rightarrow B$  be a local homomorphism of Noetherian local rings. Then the following are equivalent*

- (1)  $B$  is a complete intersection and  $\text{Tor}_p^A(B, A/\mathfrak{m}_A)$  is nonzero for only finitely many  $p$ ,
- (2)  $A$  is a complete intersection and  $A^\wedge \rightarrow B^\wedge$  is a complete intersection homomorphism in the sense defined above.

**Proof.** Let  $F_\bullet \rightarrow A/\mathfrak{m}_A$  be a resolution by finite free  $A$ -modules. Observe that  $\text{Tor}_p^A(B, A/\mathfrak{m}_A)$  is the  $p$ th homology of the complex  $F_\bullet \otimes_A B$ . Let  $F_\bullet^\wedge = F_\bullet \otimes_A A^\wedge$  be the completion. Then  $F_\bullet^\wedge$  is a resolution of  $A^\wedge/\mathfrak{m}_{A^\wedge}$  by finite free  $A^\wedge$ -modules (as  $A \rightarrow A^\wedge$  is flat and completion on finite modules is exact, see Algebra, Lemmas 93.2 and 93.3). It follows that

$$F_\bullet^\wedge \otimes_{A^\wedge} B^\wedge = F_\bullet \otimes_A B \otimes_B B^\wedge$$

By flatness of  $B \rightarrow B^\wedge$  we conclude that

$$\text{Tor}_p^{A^\wedge}(B^\wedge, A^\wedge/\mathfrak{m}_{A^\wedge}) = \text{Tor}_p^A(B, A/\mathfrak{m}_A) \otimes_B B^\wedge$$

In this way we see that the condition in (1) on the local ring map  $A \rightarrow B$  is equivalent to the same condition for the local ring map  $A^\wedge \rightarrow B^\wedge$ . Thus we may assume  $A$  and  $B$  are complete local Noetherian rings (since the other conditions are formulated in terms of the completions in any case).

Assume  $A$  and  $B$  are complete local Noetherian rings. Choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in More on Algebra, Lemma 28.3. Let  $I = \text{Ker}(R \rightarrow A)$  and  $J = \text{Ker}(S \rightarrow B)$ . The proposition now follows from Lemma 7.6.  $\square$

**Remark 9.3.** It appears difficult to define an good notion of “local complete intersection homomorphisms” for maps between general Noetherian rings. The reason is that, for a local Noetherian ring  $A$ , the fibres of  $A \rightarrow A^\wedge$  are not local complete intersection rings. Thus, if  $A \rightarrow B$  is a local homomorphism of local Noetherian rings, and the map of completions  $A^\wedge \rightarrow B^\wedge$  is a complete intersection homomorphism in the sense defined above, then  $(A_{\mathfrak{p}})^\wedge \rightarrow (B_{\mathfrak{q}})^\wedge$  is in general **not** a complete intersection homomorphism in the sense defined above. A solution can be had by working exclusively with excellent Noetherian rings. More generally, one could work with those Noetherian rings whose formal fibres are complete intersections, see [Rod87]. We will develop this theory here if we ever need it.

To finish of this section we compare the notion defined above with the notion introduced in More on Algebra, Section 8.

**Lemma 9.4.** *Consider a commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & B \\ & \searrow & \uparrow \\ & & A \end{array}$$

of Noetherian local rings with  $S \rightarrow B$  surjective,  $A \rightarrow S$  flat, and  $S/\mathfrak{m}_A S$  a regular local ring. The following are equivalent

- (1)  $\text{Ker}(S \rightarrow B)$  is generated by a regular sequence, and
- (2)  $A^\wedge \rightarrow B^\wedge$  is a complete intersection homomorphism as defined above.

**Proof.** Omitted. Hint: the proof is identical to the argument given in the first paragraph of the proof of Lemma 9.1.  $\square$

**Lemma 9.5.** *Let  $A$  be a Noetherian ring. Let  $A \rightarrow B$  be a finite type ring map. The following are equivalent*

- (1)  $A \rightarrow B$  is a local complete intersection in the sense of More on Algebra, Definition 23.2,
- (2) for every prime  $\mathfrak{q} \subset B$  and with  $\mathfrak{p} = A \cap \mathfrak{q}$  the ring map  $(A_{\mathfrak{p}})^\wedge \rightarrow (B_{\mathfrak{q}})^\wedge$  is a complete intersection homomorphism in the sense defined above.

**Proof.** Choose a surjection  $R = A[x_1, \dots, x_n] \rightarrow B$ . Observe that  $A \rightarrow R$  is flat with regular fibres. Let  $I$  be the kernel of  $R \rightarrow B$ . Assume (2). Then we see that  $I$  is locally generated by a regular sequence by Lemma 9.4 and Algebra, Lemma 67.8. In other words, (1) holds. Conversely, assume (1). Then after localizing on  $R$  and  $B$  we can assume that  $I$  is generated by a Koszul regular sequence. By More on Algebra, Lemma 21.6 we find that  $I$  is locally generated by a regular sequence. Hence (2) hold by Lemma 9.4. Some details omitted.  $\square$

**Lemma 9.6.** *Let  $A$  be a Noetherian ring. Let  $A \rightarrow B$  be a finite type ring map such that the image of  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  contains all closed points of  $\text{Spec}(A)$ . Then the following are equivalent*

- (1)  $B$  is a complete intersection and  $A \rightarrow B$  has finite tor dimension,
- (2)  $A$  is a complete intersection and  $A \rightarrow B$  is a complete intersection in the sense of More on Algebra, Definition 23.2.

**Proof.** This is a reformulation of Proposition 9.2 via Lemma 9.5. We omit the details.  $\square$

## 10. Other chapters

Preliminaries	(15) More on Algebra
	(16) Smoothing Ring Maps
(1) Introduction	(17) Sheaves of Modules
(2) Conventions	(18) Modules on Sites
(3) Set Theory	(19) Injectives
(4) Categories	(20) Cohomology of Sheaves
(5) Topology	(21) Cohomology on Sites
(6) Sheaves on Spaces	(22) Differential Graded Algebra
(7) Sites and Sheaves	(23) Divided Power Algebra
(8) Stacks	(24) Hypercoverings
(9) Fields	
(10) Commutative Algebra	Schemes
(11) Brauer Groups	(25) Schemes
(12) Homological Algebra	(26) Constructions of Schemes
(13) Derived Categories	(27) Properties of Schemes
(14) Simplicial Methods	(28) Morphisms of Schemes

(29) Cohomology of Schemes	(61) More on Groupoids in Spaces
(30) Divisors	(62) Bootstrap
(31) Limits of Schemes	Topics in Geometry
(32) Varieties	(63) Quotients of Groupoids
(33) Topologies on Schemes	(64) Simplicial Spaces
(34) Descent	(65) Formal Algebraic Spaces
(35) Derived Categories of Schemes	(66) Restricted Power Series
(36) More on Morphisms	(67) Resolution of Surfaces
(37) More on Flatness	Deformation Theory
(38) Groupoid Schemes	(68) Formal Deformation Theory
(39) More on Groupoid Schemes	(69) Deformation Theory
(40) Étale Morphisms of Schemes	(70) The Cotangent Complex
Topics in Scheme Theory	Algebraic Stacks
(41) Chow Homology	(71) Algebraic Stacks
(42) Adequate Modules	(72) Examples of Stacks
(43) Dualizing Complexes	(73) Sheaves on Algebraic Stacks
(44) Étale Cohomology	(74) Criteria for Representability
(45) Crystalline Cohomology	(75) Artin's Axioms
(46) Pro-étale Cohomology	(76) Quot and Hilbert Spaces
Algebraic Spaces	(77) Properties of Algebraic Stacks
(47) Algebraic Spaces	(78) Morphisms of Algebraic Stacks
(48) Properties of Algebraic Spaces	(79) Cohomology of Algebraic Stacks
(49) Morphisms of Algebraic Spaces	(80) Derived Categories of Stacks
(50) Decent Algebraic Spaces	(81) Introducing Algebraic Stacks
(51) Cohomology of Algebraic Spaces	Miscellany
(52) Limits of Algebraic Spaces	(82) Examples
(53) Divisors on Algebraic Spaces	(83) Exercises
(54) Algebraic Spaces over Fields	(84) Guide to Literature
(55) Topologies on Algebraic Spaces	(85) Desirables
(56) Descent and Algebraic Spaces	(86) Coding Style
(57) Derived Categories of Spaces	(87) Obsolete
(58) More on Morphisms of Spaces	(88) GNU Free Documentation License
(59) Pushouts of Algebraic Spaces	(89) Auto Generated Index
(60) Groupoids in Algebraic Spaces	

### References

- [Avr75] Luchezar L. Avramov, *Flat morphisms of complete intersections*, Dokl. Akad. Nauk SSSR **225** (1975), no. 1, 11–14.
- [Ber74] Pierre Berthelot, *Cohomologie cristalline des schémas de caractéristique  $p > 0$* , Lecture Notes in Mathematics, Vol. 407, Springer-Verlag, Berlin, 1974.
- [BO83] Pierre Berthelot and Arthur Ogus, *F-isocrystals and de Rham cohomology. I*, Invent. Math. **72** (1983), no. 2, 159–199.
- [Rod87] Antonio G. Rodicio, *Local rings whose formal fibres are complete intersections*, Extracta Math. **2** (1987), no. 2, 71–73.
- [Rod88] ———, *On a result of Avramov*, Manuscripta Math. **62** (1988), no. 2, 181–185.
- [Tat57] John Tate, *Homology of Noetherian rings and local rings*, Illinois J. Math. **1** (1957), 14–27.