

CRITERIA FOR REPRESENTABILITY

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1. Introduction

The purpose of this chapter is to find criteria guaranteeing that a stack in groupoids over the category of schemes with the fppf topology is an algebraic stack. Historically, this often involved proving that certain functors were representable, see Grothendieck's lectures [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d]. This explains the title of this chapter. Another important source of this material comes from the work of Artin, see [Art69b], [Art70], [Art73], [Art71b], [Art71a], [Art69a], [Art69c], and [Art74].

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 1 for an explanation.

2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 2.

3. What we already know

The analogue of this chapter for algebraic spaces is the chapter entitled “Bootstrap”, see Bootstrap, Section 1. That chapter already contains some representability results. Moreover, some of the preliminary material treated there we already have worked out in the chapter on algebraic stacks. Here is a list:

- (1) We discuss morphisms of presheaves representable by algebraic spaces in Bootstrap, Section 3. In Algebraic Stacks, Section 9 we discuss the notion of a 1-morphism of categories fibred in groupoids being representable by algebraic spaces.
- (2) We discuss properties of morphisms of presheaves representable by algebraic spaces in Bootstrap, Section 4. In Algebraic Stacks, Section 10 we discuss the notion of a 1-morphism of categories fibred in groupoids being representable by algebraic spaces.
- (3) We proved that if F is a sheaf whose diagonal is representable by algebraic spaces and which has an étale covering by an algebraic space, then F is an algebraic space, see Bootstrap, Theorem 6.1. (This is a weak version of the result in the next item on the list.)
- (4) We proved that if F is a sheaf and if there exists an algebraic space U and a morphism $U \rightarrow F$ which is representable by algebraic spaces, surjective, flat, and locally of finite presentation, then F is an algebraic space, see Bootstrap, Theorem 10.1.
- (5) We have also proved the “smooth” analogue of (4) for algebraic stacks: If \mathcal{X} is a stack in groupoids over $(Sch/S)_{fppf}$ and if there exists a stack in groupoids \mathcal{U} over $(Sch/S)_{fppf}$ which is representable by an algebraic space and a 1-morphism $u : \mathcal{U} \rightarrow \mathcal{X}$ which is representable by algebraic spaces, surjective, and smooth then \mathcal{X} is an algebraic stack, see Algebraic Stacks, Lemma 15.3.

Our first task now is to prove the analogue of (4) for algebraic stacks in general; it is Theorem 16.1.

4. Morphisms of stacks in groupoids

This section is preliminary and should be skipped on a first reading.

Lemma 4.1. *Let $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X} \rightarrow \mathcal{Z}$ and $\mathcal{Y} \rightarrow \mathcal{Z}$ are representable by algebraic spaces and étale so is $\mathcal{X} \rightarrow \mathcal{Y}$.*

Proof. Let \mathcal{U} be a representable category fibred in groupoids over S . Let $f : \mathcal{U} \rightarrow \mathcal{Y}$ be a 1-morphism. We have to show that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U}$ is representable by an algebraic space and étale over \mathcal{U} . Consider the composition $h : \mathcal{U} \rightarrow \mathcal{Z}$. Then

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{U} \longrightarrow \mathcal{Y} \times_{\mathcal{Z}} \mathcal{U}$$

is a 1-morphism between categories fibres in groupoids which are both representable by algebraic spaces and both étale over \mathcal{U} . Hence by Properties of Spaces, Lemma 13.6 this is represented by an étale morphism of algebraic spaces. Finally, we obtain

the result we want as the morphism f induces a morphism $\mathcal{U} \rightarrow \mathcal{Y} \times_{\mathcal{Z}} \mathcal{U}$ and we have

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} = (\mathcal{X} \times_{\mathcal{Z}} \mathcal{U}) \times_{(\mathcal{Y} \times_{\mathcal{Z}} \mathcal{U})} \mathcal{U}.$$

□

Lemma 4.2. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be stacks in groupoids over $(Sch/S)_{fppf}$. Suppose that $\mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{Z} \rightarrow \mathcal{Y}$ are 1-morphisms. If*

- (1) \mathcal{Y}, \mathcal{Z} are representable by algebraic spaces Y, Z over S ,
- (2) the associated morphism of algebraic spaces $Y \rightarrow Z$ is surjective, flat and locally of finite presentation, and
- (3) $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X}$ is a stack in setoids,

then \mathcal{X} is a stack in setoids.

Proof. This is a special case of Stacks, Lemma 6.10. □

The following lemma is the analogue of Algebraic Stacks, Lemma 15.3 and will be superseded by the stronger Theorem 16.1.

Lemma 4.3. *Let S be a scheme. Let $u : \mathcal{U} \rightarrow \mathcal{X}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

- (1) \mathcal{U} is representable by an algebraic space, and
- (2) u is representable by algebraic spaces, surjective, flat and locally of finite presentation,

then $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ representable by algebraic spaces.

Proof. Given two schemes T_1, T_2 over S denote $\mathcal{T}_i = (Sch/T_i)_{fppf}$ the associated representable fibre categories. Suppose given 1-morphisms $f_i : \mathcal{T}_i \rightarrow \mathcal{X}$. According to Algebraic Stacks, Lemma 10.11 it suffices to prove that the 2-fibered product $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is representable by an algebraic space. By Stacks, Lemma 6.8 this is in any case a stack in setoids. Thus $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ corresponds to some sheaf F on $(Sch/S)_{fppf}$, see Stacks, Lemma 6.3. Let U be the algebraic space which represents U . By assumption

$$\mathcal{T}'_i = \mathcal{U} \times_{u, \mathcal{X}, f_i} \mathcal{T}_i$$

is representable by an algebraic space T'_i over S . Hence $\mathcal{T}'_1 \times_{\mathcal{U}} \mathcal{T}'_2$ is representable by the algebraic space $T'_1 \times_U T'_2$. Consider the commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 & \longrightarrow & \mathcal{T}_1 \\
 & \nearrow & \downarrow & & \downarrow \\
 \mathcal{T}'_1 \times_{\mathcal{U}} \mathcal{T}'_2 & \longrightarrow & \mathcal{T}'_1 & & \mathcal{X} \\
 \downarrow & & \downarrow & & \downarrow \\
 & \nearrow & \mathcal{T}_2 & \longrightarrow & \mathcal{X} \\
 \mathcal{T}'_2 & \longrightarrow & \mathcal{U} & & \mathcal{X}
 \end{array}$$

In this diagram the bottom square, the right square, the back square, and the front square are 2-fibre products. A formal argument then shows that $\mathcal{T}'_1 \times_{\mathcal{U}} \mathcal{T}'_2 \rightarrow$

$\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is the “base change” of $\mathcal{U} \rightarrow \mathcal{X}$, more precisely the diagram

$$\begin{array}{ccc} \mathcal{T}'_1 \times_{\mathcal{U}} \mathcal{T}'_2 & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 & \longrightarrow & \mathcal{X} \end{array}$$

is a 2-fibre square. Hence $\mathcal{T}'_1 \times_{\mathcal{U}} \mathcal{T}'_2 \rightarrow \mathcal{U}$ is representable by algebraic spaces, flat, locally of finite presentation and surjective, see Algebraic Stacks, Lemmas 9.6, 9.7, 10.4, and 10.6. Therefore \mathcal{U} is an algebraic space by Bootstrap, Theorem 10.1 and we win. \square

Lemma 4.4. *Let \mathcal{X} be a category fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent*

- (1) $\Delta_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,
- (2) for every 1-morphism $\mathcal{V} \rightarrow \mathcal{X} \times \mathcal{X}$ with \mathcal{V} representable (by a scheme) and the fibre product $\mathcal{Y} = \mathcal{X} \times_{\Delta_{\mathcal{X} \times \mathcal{X}}} \mathcal{V}$ has diagonal representable by algebraic spaces.

Proof. Although this is a bit of a brain twister, it is completely formal. Namely, recall that $\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X} = \mathcal{I}_{\mathcal{X}}$ is the inertia of \mathcal{X} and that $\Delta_{\mathcal{X}}$ is the identity section of $\mathcal{I}_{\mathcal{X}}$, see Categories, Section 32. Thus condition (1) says the following: Given a scheme V , an object x of \mathcal{X} over V , and a morphism $\alpha : x \rightarrow x$ of \mathcal{X}_V the condition “ $\alpha = \text{id}_x$ ” defines an algebraic space over V . (In other words, there exists a monomorphism of algebraic spaces $W \rightarrow V$ such that a morphism of schemes $f : T \rightarrow V$ factors through W if and only if $f^* \alpha = \text{id}_{f^*x}$.)

On the other hand, let V be a scheme and let x, y be objects of \mathcal{X} over V . Then (x, y) define a morphism $\mathcal{V} = (Sch/V)_{fppf} \rightarrow \mathcal{X} \times \mathcal{X}$. Next, let $h : V' \rightarrow V$ be a morphism of schemes and let $\alpha : h^*x \rightarrow h^*y$ and $\beta : h^*x \rightarrow h^*y$ be morphisms of $\mathcal{X}_{V'}$. Then (α, β) define a morphism $\mathcal{V}' = (Sch/V)_{fppf} \rightarrow \mathcal{V} \times \mathcal{V}$. Condition (2) now says that (with any choices as above) the condition “ $\alpha = \beta$ ” defines an algebraic space over V .

To see the equivalence, given (α, β) as in (2) we see that (1) implies that “ $\alpha^{-1} \circ \beta = \text{id}_{h^*x}$ ” defines an algebraic space. The implication (2) \Rightarrow (1) follows by taking $h = \text{id}_V$ and $\beta = \text{id}_x$. \square

5. Limit preserving on objects

Let S be a scheme. Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is *limit preserving on objects* if the following condition holds: Given any data consisting of

- (1) an affine scheme $U = \lim_{i \in I} U_i$ which is written as the directed limit of affine schemes U_i over S ,
- (2) an object y_i of \mathcal{Y} over U_i for some i ,
- (3) an object x of \mathcal{X} over U , and
- (4) an isomorphism $\gamma : p(x) \rightarrow y_i|_U$,

then there exists an $i' \geq i$, an object $x_{i'}$ of \mathcal{X} over $U_{i'}$, an isomorphism $\beta : x_{i'}|_U \rightarrow x$, and an isomorphism $\gamma_{i'} : p(x_{i'}) \rightarrow y_i|_{U_{i'}}$ such that

$$(5.0.1) \quad \begin{array}{ccc} p(x_{i'}|_U) & \xrightarrow{\gamma_{i'}|_U} & (y_i|_{U_{i'}})|_U \\ p(\beta) \downarrow & & \parallel \\ p(x) & \xrightarrow{\gamma} & y_i|_U \end{array}$$

commutes. In this situation we say that “ $(i', x_{i'}, \beta, \gamma_{i'})$ is a *solution* to the problem posed by our data (1), (2), (3), (4)”. The motivation for this definition comes from Limits of Spaces, Lemma 3.2.

Lemma 5.1. *Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Z} \rightarrow \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \rightarrow \mathcal{Y}$ is limit preserving on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Z}$ of p by q .*

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S , let z_i be an object of \mathcal{Z} over U_i for some i , let w be an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U , and let $\delta : p'(w) \rightarrow z_i|_U$ be an isomorphism. We may write $w = (U, x, z, \alpha)$ for some object x of \mathcal{X} over U and object z of \mathcal{Z} over U and isomorphism $\alpha : p(x) \rightarrow q(z)$. Note that $p'(w) = z$ hence $\delta : z \rightarrow z_i|_U$. Set $y_i = q(z_i)$ and $\gamma = q(\delta) \circ \alpha : p(x) \rightarrow y_i|_U$. As p is limit preserving on objects there exists an $i' \geq i$ and an object $x_{i'}$ of \mathcal{X} over $U_{i'}$ as well as isomorphisms $\beta : x_{i'}|_U \rightarrow x$ and $\gamma_{i'} : p(x_{i'}) \rightarrow y_i|_{U_{i'}}$ such that (5.0.1) commutes. Then we consider the object $w_{i'} = (U_{i'}, x_{i'}, z_i|_{U_{i'}}, \gamma_{i'})$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over $U_{i'}$ and define isomorphisms

$$w_{i'}|_U = (U, x_{i'}|_U, z_i|_U, \gamma_{i'}|_U) \xrightarrow{(\beta, \delta^{-1})} (U, x, z, \alpha) = w$$

and

$$p'(w_{i'}) = z_i|_{U_{i'}} \xrightarrow{\text{id}} z_i|_{U_{i'}}.$$

These combine to give a solution to the problem. \square

Lemma 5.2. *Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Y} \rightarrow \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are limit preserving on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S , let z_i be an object of \mathcal{Z} over U_i for some i , let x be an object of \mathcal{X} over U , and let $\gamma : q(p(x)) \rightarrow z_i|_U$ be an isomorphism. As q is limit preserving on objects there exist an $i' \geq i$, an object $y_{i'}$ of \mathcal{Y} over $U_{i'}$, an isomorphism $\beta : y_{i'}|_U \rightarrow p(x)$, and an isomorphism $\gamma_{i'} : q(y_{i'}) \rightarrow z_i|_{U_{i'}}$ such that (5.0.1) is commutative. As p is limit preserving on objects there exist an $i'' \geq i'$, an object $x_{i''}$ of \mathcal{X} over $U_{i''}$, an isomorphism $\beta' : x_{i''}|_U \rightarrow x$, and an isomorphism $\gamma'_{i''} : p(x_{i''}) \rightarrow y_{i'}|_{U_{i''}}$ such that (5.0.1) is commutative. The solution is to take $x_{i''}$ over $U_{i''}$ with isomorphism

$$q(p(x_{i''})) \xrightarrow{q(\gamma'_{i''})} q(y_{i'}|_{U_{i''}}) \xrightarrow{\gamma_{i'}|_{U_{i''}}} z_i|_{U_{i''}}$$

and isomorphism $\beta' : x_{i''}|_U \rightarrow x$. We omit the verification that (5.0.1) is commutative. \square

Lemma 5.3. *Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

- (1) p is limit preserving on objects, and
- (2) p is locally of finite presentation (see *Algebraic Stacks*, Definition 10.1).

Proof. Assume (2). Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S , let y_i be an object of \mathcal{Y} over U_i for some i , let x be an object of \mathcal{X} over U , and let $\gamma : p(x) \rightarrow y_i|_U$ be an isomorphism. Let X_{y_i} denote an algebraic space over U_i representing the 2-fibre product

$$(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, p} \mathcal{X}.$$

Note that $\xi = (U, U \rightarrow U_i, x, \gamma^{-1})$ defines an object of this 2-fibre product over U . Via the 2-Yoneda lemma ξ corresponds to a morphism $f_\xi : U \rightarrow X_{y_i}$ over U_i . By Limits of Spaces, Proposition 3.9 there exists an $i' \geq i$ and a morphism $f_{i'} : U_{i'} \rightarrow X_{y_i}$ such that f_ξ is the composition of $f_{i'}$ and the projection morphism $U \rightarrow U_{i'}$. Also, the 2-Yoneda lemma tells us that $f_{i'}$ corresponds to an object $\xi_{i'} = (U_{i'}, U_{i'} \rightarrow U_i, x_{i'}, \alpha)$ of the displayed 2-fibre product over $U_{i'}$ whose restriction to U recovers ξ . In particular we obtain an isomorphism $\gamma : x_{i'}|_U \rightarrow x$. Note that $\alpha : y_i|_{U_{i'}} \rightarrow p(x_{i'})$. Hence we see that taking $x_{i'}$, the isomorphism $\gamma : x_{i'}|_U \rightarrow x$, and the isomorphism $\beta = \alpha^{-1} : p(x_{i'}) \rightarrow y_i|_{U_{i'}}$ is a solution to the problem.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \rightarrow \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y, \mathcal{Y}, p} \mathcal{X}$. We have to show that $X_y \rightarrow T$ is locally of finite presentation. To do this we may use Limits of Spaces, Proposition 3.9 in the form described in Limits of Spaces, Remark 3.10. Hence it suffices to show that given an affine scheme $U = \lim_{i \in I} U_i$ written as the directed limit of affine schemes over T , then $X_y(U) = \text{colim}_i X_y(U_i)$. Pick any $i \in I$ and set $y_i = y|_{U_i}$. Also denote i' an element of I which is bigger than or equal to i . By the 2-Yoneda lemma morphisms $U \rightarrow X_y$ over T correspond bijectively to isomorphism classes of pairs (x, α) where x is an object of \mathcal{X} over U and $\alpha : y|_U \rightarrow p(x)$ is an isomorphism. Of course giving α is, up to an inverse, the same thing as giving an isomorphism $\gamma : p(x) \rightarrow y_i|_U$. Similarly for morphisms $U_{i'} \rightarrow X_y$ over T . Hence (1) guarantees that

$$X_y(U) = \text{colim}_{i' \geq i} X_y(U_{i'})$$

in this situation and we win. \square

Lemma 5.4. *Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume p is representable by algebraic spaces and an open immersion. Then p is limit preserving on objects.*

Proof. This follows from Lemma 5.3 and (via the general principle *Algebraic Stacks*, Lemma 10.9) from the fact that an open immersion of algebraic spaces is locally of finite presentation, see *Morphisms of Spaces*, Lemma 27.11. \square

Let S be a scheme. In the following lemma we need the notion of the *size* of an algebraic space X over S . Namely, given a cardinal κ we will say X has $\text{size}(X) \leq \kappa$ if and only if there exists a scheme U with $\text{size}(U) \leq \kappa$ (see *Sets*, Section 9) and a surjective étale morphism $U \rightarrow X$.

Lemma 5.5. *Let S be a scheme. Let $\kappa = \text{size}(T)$ for some $T \in \text{Ob}((Sch/S)_{fppf})$. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ such that*

- (1) $\mathcal{Y} \rightarrow (Sch/S)_{fppf}$ is limit preserving on objects,

- (2) for an affine scheme V locally of finite presentation over S and $y \in \text{Ob}(\mathcal{Y}_V)$ the fibre product $(\text{Sch}/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$ is representable by an algebraic space of size $\leq \kappa^1$,
- (3) \mathcal{X} and \mathcal{Y} are stacks for the Zariski topology.

Then f is representable by algebraic spaces.

Proof. Let V be a scheme over S and $y \in \mathcal{Y}_V$. We have to prove $(\text{Sch}/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$ is representable by an algebraic space.

Case I: V is affine and maps into an affine open $\text{Spec}(\Lambda) \subset S$. Then we can write $V = \lim V_i$ with each V_i affine and of finite presentation over $\text{Spec}(\Lambda)$, see Algebra, Lemma 123.1. Then y comes from an object y_i over V_i for some i by assumption (1). By assumption (3) the fibre product $(\text{Sch}/V_i)_{fppf} \times_{y_i, \mathcal{Y}} \mathcal{X}$ is representable by an algebraic space Z_i . Then $(\text{Sch}/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$ is representable by $Z \times_{V_i} V$.

Case II: V is general. Choose an affine open covering $V = \bigcup_{i \in I} V_i$ such that each V_i maps into an affine open of S . We first claim that $\mathcal{Z} = (\text{Sch}/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$ is a stack in setoids for the Zariski topology. Namely, it is a stack in groupoids for the Zariski topology by Stacks, Lemma 5.6. Then suppose that z is an object of \mathcal{Z} over a scheme T . Denote $g : T \rightarrow V$ the morphism corresponding to the projection of z in $(\text{Sch}/V)_{fppf}$. Consider the Zariski sheaf $I = \text{Isom}_{\mathcal{Z}}(z, z)$. By Case I we see that $I|_{g^{-1}(V_i)} = *$ (the singleton sheaf). Hence $\mathcal{I} = *$. Thus \mathcal{Z} is fibred in setoids. To finish the proof we have to show that the Zariski sheaf $Z : T \mapsto \text{Ob}(\mathcal{Z}_T)/\cong$ is an algebraic space, see Algebraic Stacks, Lemma 8.2. There is a map $p : Z \rightarrow V$ (transformation of functors) and by Case I we know that $Z_i = p^{-1}(V_i)$ is an algebraic space. The morphisms $Z_i \rightarrow Z$ are representable by open immersions and $\coprod Z_i \rightarrow Z$ is surjective (in the Zariski topology). Hence Z is a sheaf for the fppf topology by Bootstrap, Lemma 3.11. Thus Spaces, Lemma 8.4 applies and we conclude that Z is an algebraic space². \square

Lemma 5.6. *Let S be a scheme. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(\text{Sch}/S)_{fppf}$. Let \mathcal{P} be a property of morphisms of algebraic spaces as in Algebraic Stacks, Definition 10.1. If*

- (1) f is representable by algebraic spaces,
- (2) $\mathcal{Y} \rightarrow (\text{Sch}/S)_{fppf}$ is limit preserving on objects,
- (3) for an affine scheme V locally of finite presentation over S and $y \in \mathcal{Y}_V$ the resulting morphism of algebraic spaces $f_y : F_y \rightarrow V$, see Algebraic Stacks, Equation (9.1.1), has property \mathcal{P} .

Then f has property \mathcal{P} .

Proof. Let V be a scheme over S and $y \in \mathcal{Y}_V$. We have to show that $F_y \rightarrow V$ has property \mathcal{P} . Since \mathcal{P} is fppf local on the base we may assume that V is an affine scheme which maps into an affine open $\text{Spec}(\Lambda) \subset S$. Thus we can write $V = \lim V_i$ with each V_i affine and of finite presentation over $\text{Spec}(\Lambda)$, see Algebra, Lemma 123.1. Then y comes from an object y_i over V_i for some i by assumption

¹The condition on size can be dropped by those ignoring set theoretic issues.

²To see that the set theoretic condition of that lemma is satisfied we argue as follows: First choose the open covering such that $|I| \leq \text{size}(V)$. Next, choose schemes U_i of size $\leq \max(\kappa, \text{size}(V))$ and surjective étale morphisms $U_i \rightarrow Z_i$; we can do this by assumption (2) and Sets, Lemma 9.6 (details omitted). Then Sets, Lemma 9.9 implies that $\coprod U_i$ is an object of $(\text{Sch}/S)_{fppf}$. Hence $\coprod Z_i$ is an algebraic space by Spaces, Lemma 8.3.

(2). By assumption (3) the morphism $F_{y_i} \rightarrow V_i$ has property \mathcal{P} . As \mathcal{P} is stable under arbitrary base change and since $F_y = F_{y_i} \times_{V_i} V$ we conclude that $F_y \rightarrow V$ has property \mathcal{P} as desired. \square

6. Formally smooth on objects

Let S be a scheme. Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is *formally smooth on objects* if the following condition holds: Given any data consisting of

- (1) a first order thickening $U \subset U'$ of affine schemes over S ,
- (2) an object y' of \mathcal{Y} over U' ,
- (3) an object x of \mathcal{X} over U , and
- (4) an isomorphism $\gamma : p(x) \rightarrow y'|_U$,

then there exists an object x' of \mathcal{X} over U' with an isomorphism $\beta : x'|_U \rightarrow x$ and an isomorphism $\gamma' : p(x') \rightarrow y'$ such that

$$(6.0.1) \quad \begin{array}{ccc} p(x'|_U) & \xrightarrow{\gamma'|_U} & y'|_U \\ p(\beta) \downarrow & & \parallel \\ p(x) & \xrightarrow{\gamma} & y'|_U \end{array}$$

commutes. In this situation we say that “ (x', β, γ') is a *solution* to the problem posed by our data (1), (2), (3), (4)”.

Lemma 6.1. *Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Z} \rightarrow \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \rightarrow \mathcal{Y}$ is formally smooth on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Z}$ of p by q .*

Proof. This is formal. Let $U \subset U'$ be a first order thickening of affine schemes over S , let z' be an object of \mathcal{Z} over U' , let w be an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U , and let $\delta : p'(w) \rightarrow z'|_U$ be an isomorphism. We may write $w = (U, x, z, \alpha)$ for some object x of \mathcal{X} over U and object z of \mathcal{Z} over U and isomorphism $\alpha : p(x) \rightarrow q(z)$. Note that $p'(w) = z$ hence $\delta : z \rightarrow z|_U$. Set $y' = q(z')$ and $\gamma = q(\delta) \circ \alpha : p(x) \rightarrow y'|_U$. As p is formally smooth on objects there exists an object x' of \mathcal{X} over U' as well as isomorphisms $\beta : x'|_U \rightarrow x$ and $\gamma' : p(x') \rightarrow y'$ such that (6.0.1) commutes. Then we consider the object $w' = (U', x', z', \gamma')$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U' and define isomorphisms

$$w'|_U = (U, x'|_U, z'|_U, \gamma'|_U) \xrightarrow{(\beta, \delta^{-1})} (U, x, z, \alpha) = w$$

and

$$p'(w') = z' \xrightarrow{\text{id}} z'.$$

These combine to give a solution to the problem. \square

Lemma 6.2. *Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Y} \rightarrow \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are formally smooth on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let $U \subset U'$ be a first order thickening of affine schemes over S , let z' be an object of \mathcal{Z} over U' , let x be an object of \mathcal{X} over U , and let $\gamma : q(p(x)) \rightarrow z'|_U$ be an isomorphism. As q is formally smooth on objects there exist an object y' of \mathcal{Y} over U' , an isomorphism $\beta : y'|_U \rightarrow p(x)$, and an isomorphism

$\gamma' : q(y') \rightarrow z'$ such that (6.0.1) is commutative. As p is formally smooth on objects there exist an object x' of \mathcal{X} over U' , an isomorphism $\beta' : x'|_U \rightarrow x$, and an isomorphism $\gamma'' : p(x') \rightarrow y'$ such that (6.0.1) is commutative. The solution is to take x' over U' with isomorphism

$$q(p(x')) \xrightarrow{q(\gamma'')} q(y') \xrightarrow{\gamma'} z'$$

and isomorphism $\beta' : x'|_U \rightarrow x$. We omit the verification that (6.0.1) is commutative. \square

Note that the class of formally smooth morphisms of algebraic spaces is stable under arbitrary base change and local on the target in the fpqc topology, see More on Morphisms of Spaces, Lemma 16.3 and 16.10. Hence condition (2) in the lemma below makes sense.

Lemma 6.3. *Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

- (1) p is formally smooth on objects, and
- (2) p is formally smooth (see Algebraic Stacks, Definition 10.1).

Proof. Assume (2). Let $U \subset U'$ be a first order thickening of affine schemes over S , let y' be an object of \mathcal{Y} over U' , let x be an object of \mathcal{X} over U , and let $\gamma : p(x) \rightarrow y'|_U$ be an isomorphism. Let $X_{y'}$ denote an algebraic space over U' representing the 2-fibre product

$$(Sch/U')_{fppf} \times_{y', \mathcal{Y}, p} \mathcal{X}.$$

Note that $\xi = (U, U \rightarrow U', x, \gamma^{-1})$ defines an object of this 2-fibre product over U . Via the 2-Yoneda lemma ξ corresponds to a morphism $f_\xi : U \rightarrow X_{y'}$ over U' . As $X_{y'} \rightarrow U'$ is formally smooth by assumption there exists a morphism $f' : U' \rightarrow X_{y'}$ such that f_ξ is the composition of f' and the morphism $U \rightarrow U'$. Also, the 2-Yoneda lemma tells us that f' corresponds to an object $\xi' = (U', U' \rightarrow U', x', \alpha)$ of the displayed 2-fibre product over U' whose restriction to U recovers ξ . In particular we obtain an isomorphism $\gamma : x'|_U \rightarrow x$. Note that $\alpha : y' \rightarrow p(x')$. Hence we see that taking x' , the isomorphism $\gamma : x'|_U \rightarrow x$, and the isomorphism $\beta = \alpha^{-1} : p(x') \rightarrow y'$ is a solution to the problem.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \rightarrow \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y, \mathcal{Y}, p} \mathcal{X}$. We have to show that $X_y \rightarrow T$ is formally smooth. Hence it suffices to show that given a first order thickening $U \subset U'$ of affine schemes over T , then $X_y(U') \rightarrow X_y(U)$ is surjective (morphisms in the category of algebraic spaces over T). Set $y' = y|_{U'}$. By the 2-Yoneda lemma morphisms $U \rightarrow X_y$ over T correspond bijectively to isomorphism classes of pairs (x, α) where x is an object of \mathcal{X} over U and $\alpha : y|_U \rightarrow p(x)$ is an isomorphism. Of course giving α is, up to an inverse, the same thing as giving an isomorphism $\gamma : p(x) \rightarrow y|_U$. Similarly for morphisms $U' \rightarrow X_y$ over T . Hence (1) guarantees the surjectivity of $X_y(U') \rightarrow X_y(U)$ in this situation and we win. \square

7. Surjective on objects

Let S be a scheme. Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is *surjective on objects* if the following condition holds: Given any data consisting of

- (1) a field k over S , and
- (2) an object y of \mathcal{Y} over $\text{Spec}(k)$,

then there exists an extension $k \subset K$ of fields over S , an object x of \mathcal{X} over $\text{Spec}(K)$ such that $p(x) \cong y|_{\text{Spec}(K)}$.

Lemma 7.1. *Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Z} \rightarrow \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \rightarrow \mathcal{Y}$ is surjective on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Z}$ of p by q .*

Proof. This is formal. Let z be an object of \mathcal{Z} over a field k . As p is surjective on objects there exists an extension $k \subset K$ and an object x of \mathcal{X} over K and an isomorphism $\alpha : p(x) \rightarrow q(z)|_{\text{Spec}(K)}$. Then $w = (\text{Spec}(K), x, z|_{\text{Spec}(K)}, \alpha)$ is an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over K with $p'(w) = z|_{\text{Spec}(K)}$. \square

Lemma 7.2. *Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ and $q : \mathcal{Y} \rightarrow \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are surjective on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let z be an object of \mathcal{Z} over a field k . As q is surjective on objects there exists a field extension $k \subset K$ and an object y of \mathcal{Y} over K such that $q(y) \cong z|_{\text{Spec}(K)}$. As p is surjective on objects there exists a field extension $K \subset L$ and an object x of \mathcal{X} over L such that $p(x) \cong y|_{\text{Spec}(L)}$. Then the field extension $k \subset L$ and the object x of \mathcal{X} over L satisfy $q(p(x)) \cong z|_{\text{Spec}(L)}$ as desired. \square

Lemma 7.3. *Let $p : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

- (1) p is surjective on objects, and
- (2) p is surjective (see *Algebraic Stacks, Definition 10.1*).

Proof. Assume (2). Let k be a field and let y be an object of \mathcal{Y} over k . Let X_y denote an algebraic space over k representing the 2-fibre product

$$(Sch/\text{Spec}(k))_{fppf} \times_{y, \mathcal{Y}, p} \mathcal{X}.$$

As we've assumed that p is surjective we see that X_y is not empty. Hence we can find a field extension $k \subset K$ and a K -valued point x of X_y . Via the 2-Yoneda lemma this corresponds to an object x of \mathcal{X} over K together with an isomorphism $p(x) \cong y|_{\text{Spec}(K)}$ and we see that (1) holds.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \rightarrow \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y, \mathcal{Y}, p} \mathcal{X}$. We have to show that $X_y \rightarrow T$ is surjective. By Morphisms of Spaces, Definition 5.2 we have to show that $|X_y| \rightarrow |T|$ is surjective. This means exactly that given a field k over T and a morphism $t : \text{Spec}(k) \rightarrow T$ there exists a field extension $k \subset K$

and a morphism $x : \mathrm{Spec}(K) \rightarrow X_y$ such that

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & X_y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{t} & T \end{array}$$

commutes. By the 2-Yoneda lemma this means exactly that we have to find $k \subset K$ and an object x of \mathcal{X} over K such that $p(x) \cong t^*y|_{\mathrm{Spec}(K)}$. Hence (1) guarantees that this is the case and we win. \square

8. Algebraic morphisms

The following notion is occasionally useful.

Definition 8.1. Let S be a scheme. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(\mathrm{Sch}/S)_{fppf}$. We say that F is *algebraic* if for every scheme T and every object ξ of \mathcal{Y} over T the 2-fibre product

$$(\mathrm{Sch}/T)_{fppf} \times_{\xi, \mathcal{Y}} \mathcal{X}$$

is an algebraic stack over S .

With this terminology in place we have the following result that generalizes Algebraic Stacks, Lemma 15.4.

Lemma 8.2. *Let S be a scheme. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(\mathrm{Sch}/S)_{fppf}$. If*

- (1) \mathcal{Y} is an algebraic stack, and
- (2) F is algebraic (see above),

then \mathcal{X} is an algebraic stack.

Proof. By assumption (1) there exists a scheme T and an object ξ of \mathcal{Y} over T such that the corresponding 1-morphism $\xi : (\mathrm{Sch}/T)_{fppf} \rightarrow \mathcal{Y}$ is smooth and surjective. Then $\mathcal{U} = (\mathrm{Sch}/T)_{fppf} \times_{\xi, \mathcal{Y}} \mathcal{X}$ is an algebraic stack by assumption (2). Choose a scheme U and a surjective smooth 1-morphism $(\mathrm{Sch}/U)_{fppf} \rightarrow \mathcal{U}$. The projection $\mathcal{U} \rightarrow \mathcal{X}$ is, as the base change of the morphism $\xi : (\mathrm{Sch}/T)_{fppf} \rightarrow \mathcal{Y}$, surjective and smooth, see Algebraic Stacks, Lemma 10.6. Then the composition $(\mathrm{Sch}/U)_{fppf} \rightarrow \mathcal{U} \rightarrow \mathcal{X}$ is surjective and smooth as a composition of surjective and smooth morphisms, see Algebraic Stacks, Lemma 10.5. Hence \mathcal{X} is an algebraic stack by Algebraic Stacks, Lemma 15.3. \square

Lemma 8.3. *Let S be a scheme. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(\mathrm{Sch}/S)_{fppf}$. If \mathcal{X} is an algebraic stack and $\Delta : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces, then F is algebraic.*

Proof. Choose a representable stack in groupoids \mathcal{U} and a surjective smooth 1-morphism $\mathcal{U} \rightarrow \mathcal{X}$. Let T be a scheme and let ξ be an object of \mathcal{Y} over T . The morphism of 2-fibre products

$$(\mathrm{Sch}/T)_{fppf} \times_{\xi, \mathcal{Y}} \mathcal{U} \longrightarrow (\mathrm{Sch}/T)_{fppf} \times_{\xi, \mathcal{Y}} \mathcal{X}$$

is representable by algebraic spaces, surjective, and smooth as a base change of $\mathcal{U} \rightarrow \mathcal{X}$, see Algebraic Stacks, Lemmas 9.7 and 10.6. By our condition on the diagonal of \mathcal{Y} we see that the source of this morphism is representable by an algebraic space, see

Algebraic Stacks, Lemma 10.11. Hence the target is an algebraic stack by Algebraic Stacks, Lemma 15.3. \square

9. Spaces of sections

Given morphisms $W \rightarrow Z \rightarrow U$ we can consider the functor that associates to a scheme U' over U the set of sections $\sigma : Z_{U'} \rightarrow W_{U'}$ of the base change $W_{U'} \rightarrow Z_{U'}$ of the morphism $W \rightarrow Z$. In this section we prove some preliminary lemmas on this functor.

Lemma 9.1. *Let $Z \rightarrow U$ be a finite morphism of schemes. Let W be an algebraic space and let $W \rightarrow Z$ be a surjective étale morphism. Then there exists a surjective étale morphism $U' \rightarrow U$ and a section*

$$\sigma : Z_{U'} \rightarrow W_{U'}$$

of the morphism $W_{U'} \rightarrow Z_{U'}$.

Proof. We may choose a separated scheme W' and a surjective étale morphism $W' \rightarrow W$. Hence after replacing W by W' we may assume that W is a separated scheme. Write $f : W \rightarrow Z$ and $\pi : Z \rightarrow U$. Note that $f \circ \pi : W \rightarrow U$ is separated as W is separated (see Schemes, Lemma 21.14). Let $u \in U$ be a point. Clearly it suffices to find an étale neighbourhood (U', u') of (U, u) such that a section σ exists over U' . Let z_1, \dots, z_r be the points of Z lying above u . For each i choose a point $w_i \in W$ which maps to z_i . We may pick an étale neighbourhood $(U', u') \rightarrow (U, u)$ such that the conclusions of More on Morphisms, Lemma 30.5 hold for both $Z \rightarrow U$ and the points z_1, \dots, z_r and $W \rightarrow U$ and the points w_1, \dots, w_r . Hence, after replacing (U, u) by (U', u') and relabeling, we may assume that all the field extensions $\kappa(u) \subset \kappa(z_i)$ and $\kappa(u) \subset \kappa(w_i)$ are purely inseparable, and moreover that there exist disjoint union decompositions

$$Z = V_1 \amalg \dots \amalg V_r \amalg A, \quad W = W_1 \amalg \dots \amalg W_r \amalg B$$

by open and closed subschemes with $z_i \in V_i$, $w_i \in W_i$ and $V_i \rightarrow U$, $W_i \rightarrow U$ finite. After replacing U by $U \setminus \pi(A)$ we may assume that $A = \emptyset$, i.e., $Z = V_1 \amalg \dots \amalg V_r$. After replacing W_i by $W_i \cap f^{-1}(V_i)$ and B by $B \cup \bigcup W_i \cap f^{-1}(Z \setminus V_i)$ we may assume that f maps W_i into V_i . Then $f_i = f|_{W_i} : W_i \rightarrow V_i$ is a morphism of schemes finite over U , hence finite (see Morphisms, Lemma 44.12). It is also étale (by assumption), $f_i^{-1}(\{z_i\}) = w_i$, and induces an isomorphism of residue fields $\kappa(z_i) = \kappa(w_i)$ (because both are purely inseparable extensions of $\kappa(u)$ and $\kappa(z_i) \subset \kappa(w_i)$ is separable as f is étale). Hence by Étale Morphisms, Lemma 14.2 we see that f_i is an isomorphism in a neighbourhood V'_i of z_i . Since $\pi : Z \rightarrow U$ is closed, after shrinking U , we may assume that $W_i \rightarrow V_i$ is an isomorphism. This proves the lemma. \square

Lemma 9.2. *Let $Z \rightarrow U$ be a finite locally free morphism of schemes. Let W be an algebraic space and let $W \rightarrow Z$ be an étale morphism. Then the functor*

$$F : (\text{Sch}/U)_{fppf}^{\text{opp}} \longrightarrow \text{Sets},$$

defined by the rule

$$U' \longmapsto F(U') = \{\sigma : Z_{U'} \rightarrow W_{U'} \text{ section of } W_{U'} \rightarrow Z_{U'}\}$$

is an algebraic space and the morphism $F \rightarrow U$ is étale.

Proof. Assume first that $W \rightarrow Z$ is also separated. Let U' be a scheme over U and let $\sigma \in F(U')$. By Morphisms of Spaces, Lemma 4.7 the morphism σ is a closed immersion. Moreover, σ is étale by Properties of Spaces, Lemma 13.6. Hence σ is also an open immersion, see Morphisms of Spaces, Lemma 45.2. In other words, $Z_\sigma = \sigma(Z_{U'}) \subset W_{U'}$ is an open subspace such that the morphism $Z_\sigma \rightarrow Z_{U'}$ is an isomorphism. In particular, the morphism $Z_\sigma \rightarrow U'$ is finite. Hence we obtain a transformation of functors

$$F \longrightarrow (W/U)_{fin}, \quad \sigma \longmapsto (U' \rightarrow U, Z_\sigma)$$

where $(W/U)_{fin}$ is the finite part of the morphism $W \rightarrow U$ introduced in More on Groupoids in Spaces, Section 9. It is clear that this transformation of functors is injective (since we can recover σ from Z_σ as the inverse of the isomorphism $Z_\sigma \rightarrow Z_{U'}$). By More on Groupoids in Spaces, Proposition 9.11 we know that $(W/U)_{fin}$ is an algebraic space étale over U . Hence to finish the proof in this case it suffices to show that $F \rightarrow (W/U)_{fin}$ is representable and an open immersion. To see this suppose that we are given a morphism of schemes $U' \rightarrow U$ and an open subspace $Z' \subset W_{U'}$ such that $Z' \rightarrow U'$ is finite. Then it suffices to show that there exists an open subscheme $U'' \subset U'$ such that a morphism $T \rightarrow U'$ factors through U'' if and only if $Z' \times_{U'} T$ maps isomorphically to $Z \times_{U'} T$. This follows from More on Morphisms of Spaces, Lemma 37.6 (here we use that $Z \rightarrow B$ is flat and locally of finite presentation as well as finite). Hence we have proved the lemma in case $W \rightarrow Z$ is separated as well as étale.

In the general case we choose a separated scheme W' and a surjective étale morphism $W' \rightarrow W$. Note that the morphisms $W' \rightarrow W$ and $W \rightarrow Z$ are separated as their source is separated. Denote F' the functor associated to $W' \rightarrow Z \rightarrow U$ as in the lemma. In the first paragraph of the proof we showed that F' is representable by an algebraic space étale over U . By Lemma 9.1 the map of functors $F' \rightarrow F$ is surjective for the étale topology on Sch/U . Moreover, if U' and $\sigma : Z_{U'} \rightarrow W_{U'}$ define a point $\xi \in F(U')$, then the fibre product

$$F'' = F' \times_{F, \xi} U'$$

is the functor on Sch/U' associated to the morphisms

$$W'_{U'} \times_{W_{U'}, \sigma} Z_{U'} \rightarrow Z_{U'} \rightarrow U'.$$

Since the first morphism is separated as a base change of a separated morphism, we see that F'' is an algebraic space étale over U' by the result of the first paragraph. It follows that $F' \rightarrow F$ is a surjective étale transformation of functors, which is representable by algebraic spaces. Hence F is an algebraic space by Bootstrap, Theorem 10.1. Since $F' \rightarrow F$ is an étale surjective morphism of algebraic spaces it follows that $F \rightarrow U$ is étale because $F' \rightarrow U$ is étale. \square

10. Relative morphisms

Let S be a scheme. Let $Z \rightarrow B$ and $X \rightarrow B$ be morphisms of algebraic spaces over S . Given a scheme T we can consider pairs (a, b) where $a : T \rightarrow B$ is a morphism

and $b : T \times_{a,B} Z \rightarrow T \times_{a,B} X$ is a morphism over T . Picture

$$(10.0.1) \quad \begin{array}{ccccc} T \times_{a,B} Z & \xrightarrow{b} & T \times_{a,B} X & & Z & & X \\ & \searrow & \swarrow & & \searrow & & \swarrow \\ & & T & \xrightarrow{a} & B & & \end{array}$$

Of course, we can also think of b as a morphism $b : T \times_{a,B} Z \rightarrow X$ such that

$$\begin{array}{ccccc} T \times_{a,B} Z & \xrightarrow{\quad} & Z & \xrightarrow{b} & X \\ \downarrow & & \searrow & & \swarrow \\ T & \xrightarrow{a} & B & & \end{array}$$

commutes. In this situation we can define a functor

$$(10.0.2) \quad \text{Mor}_B(Z, X) : (\text{Sch}/S)^{\text{opp}} \rightarrow \text{Sets}, \quad T \mapsto \{(a, b) \text{ as above}\}$$

Sometimes we think of this as a functor defined on the category of schemes over B , in which case we drop a from the notation.

Lemma 10.1. *Let S be a scheme. Let $Z \rightarrow B$ and $X \rightarrow B$ be morphisms of algebraic spaces over S . Then*

- (1) $\text{Mor}_B(Z, X)$ is a sheaf on $(\text{Sch}/S)_{\text{fppf}}$.
- (2) If T is an algebraic space over S , then there is a canonical bijection

$$\text{Mor}_{\text{Sh}((\text{Sch}/S)_{\text{fppf}})}(T, \text{Mor}_B(Z, X)) = \{(a, b) \text{ as in (10.0.1)}\}$$

Proof. Let T be an algebraic space over S . Let $\{T_i \rightarrow T\}$ be an fppf covering of T (as in Topologies on Spaces, Section 4). Suppose that $(a_i, b_i) \in \text{Mor}_B(Z, X)(T_i)$ such that $(a_i, b_i)|_{T_i \times_T T_j} = (a_j, b_j)|_{T_i \times_T T_j}$ for all i, j . Then by Descent on Spaces, Lemma 6.2 there exists a unique morphism $a : T \rightarrow B$ such that a_i is the composition of $T_i \rightarrow T$ and a . Then $\{T_i \times_{a_i, B} Z \rightarrow T \times_{a, B} Z\}$ is an fppf covering too and the same lemma implies there exists a unique morphism $b : T \times_{a, B} Z \rightarrow T \times_{a, B} X$ such that b_i is the composition of $T_i \times_{a_i, B} Z \rightarrow T \times_{a, B} Z$ and b . Hence $(a, b) \in \text{Mor}_B(Z, X)(T)$ restricts to (a_i, b_i) over T_i for all i .

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S . In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say ‘‘pair’’ we mean a pair (a, b) fitting into (10.0.1).

Let $v : T \rightarrow \text{Mor}_B(Z, X)$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \rightarrow T$. Then $v(p) \in \text{Mor}_B(Z, X)(U)$ corresponds to a pair (a_U, b_U) over U . Let $R = U \times_T U$ with projections $t, s : R \rightarrow U$. As v is a transformation of functors we see that the pullbacks of (a_U, b_U) by s and t agree. Hence, since $\{U \rightarrow T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, b) over T .

Conversely, let (a, b) be a pair over T . Let $U \rightarrow T$, $R = U \times_T U$, and $t, s : R \rightarrow U$ be as above. Then the restriction $(a, b)|_U$ gives rise to a transformation of functors $v : h_U \rightarrow \text{Mor}_B(Z, X)$ by the Yoneda lemma (Categories, Lemma 3.5). As the two pullbacks $s^*(a, b)|_U$ and $t^*(a, b)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \rightarrow h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 9.1

and since $Mor_B(Z, X)$ is an fppf sheaf by (1) we conclude that v factors through a map $T \rightarrow Mor_B(Z, X)$.

We omit the verification that the two constructions above are mutually inverse. \square

Lemma 10.2. *Let S be a scheme. Let $Z \rightarrow B$, $X \rightarrow B$, and $B' \rightarrow B$ be morphisms of algebraic spaces over S . Set $Z' = B' \times_B Z$ and $X' = B' \times_B X$. Then*

$$Mor_{B'}(Z', X') = B' \times_B Mor_B(Z, X)$$

in $Sh((Sch/S)_{fppf})$.

Proof. The equality as functors follows immediately from the definitions. The equality as sheaves follows from this because both sides are sheaves according to Lemma 10.1 and the fact that a fibre product of sheaves is the same as the corresponding fibre product of pre-sheaves (i.e., functors). \square

Lemma 10.3. *Let S be a scheme. Let $Z \rightarrow B$ and $X' \rightarrow X \rightarrow B$ be morphisms of algebraic spaces over S . Assume*

- (1) $X' \rightarrow X$ is étale, and
- (2) $Z \rightarrow B$ is finite locally free.

Then $Mor_B(Z, X') \rightarrow Mor_B(Z, X)$ is representable by algebraic spaces and étale. If $X' \rightarrow X$ is also surjective, then $Mor_B(Z, X') \rightarrow Mor_B(Z, X)$ is surjective.

Proof. Let U be a scheme and let $\xi = (a, b)$ be an element of $Mor_B(Z, X)(U)$. We have to prove that the functor

$$h_U \times_{\xi, Mor_B(Z, X)} Mor_B(Z, X')$$

is representable by an algebraic space étale over U . Set $Z_U = U \times_{a, B} Z$ and $W = Z_U \times_{b, X} X'$. Then $W \rightarrow Z_U \rightarrow U$ is as in Lemma 9.2 and the sheaf F defined there is identified with the fibre product displayed above. Hence the first assertion of the lemma. The second assertion follows from this and Lemma 9.1 which guarantees that $F \rightarrow U$ is surjective in the situation above. \square

Lemma 10.4. *Let $Z \rightarrow B$ and $X \rightarrow B$ be morphisms of affine schemes. Assume $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$ -module. Then $Mor_B(Z, X)$ is representable by an affine scheme over B .*

Proof. Write $B = \text{Spec}(R)$. Choose a basis $\{e_1, \dots, e_m\}$ for $\Gamma(Z, \mathcal{O}_Z)$. Finally, choose a presentation

$$\Gamma(X, \mathcal{O}_X) = R[\{x_i\}_{i \in I}] / (\{f_k\}_{k \in K}).$$

We will denote \bar{x}_i the image of x_i in this quotient. Write

$$P = R[\{a_{ij}\}_{i \in I, 1 \leq j \leq m}].$$

Consider the R -algebra map

$$\Psi : R[\{x_i\}_{i \in I}] \longrightarrow P \otimes_R \Gamma(Z, \mathcal{O}_Z), \quad x_i \longmapsto \sum_j a_{ij} \otimes e_j.$$

Write $\Psi(f_k) = \sum c_{kj} \otimes e_j$ with $c_{kj} \in P$. Finally, denote $J \subset P$ the ideal generated by the elements c_{kj} , $k \in K$, $1 \leq j \leq m$. We claim that $W = \text{Spec}(P/J)$ represents the functor $Mor_B(Z, X)$.

First, note that by construction P/J is an R -algebra, hence a morphism $a_{univ} : W \rightarrow B$. Second, by construction the map Ψ factors through $\Gamma(X, \mathcal{O}_X)$, hence we obtain an P/J -algebra homomorphism

$$P/J \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow P/J \otimes_R \Gamma(Z, \mathcal{O}_Z)$$

which determines a morphism $b_{univ} : W \times_{a_{univ}, B} Z \rightarrow W \times_{a_{univ}, B} X$. By the Yoneda lemma the pair (a_{univ}, b_{univ}) determines a transformation of functors $W \rightarrow \text{Mor}_B(Z, X)$ which we claim is an isomorphism. To show that it is an isomorphism it suffices to show that it induces a bijection of sets $W(T) \rightarrow \text{Mor}_B(Z, X)(T)$ over any affine scheme T .

Suppose $T = \text{Spec}(R')$ is an affine scheme and $(a, b) \in \text{Mor}_B(Z, X)(T)$, then a defines an R -algebra structure on R' and b defines an R' -algebra map

$$b^\sharp : R' \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow R' \otimes_R \Gamma(Z, \mathcal{O}_Z).$$

In particular we can write $b^\sharp(1 \otimes \bar{x}_i) = \sum \alpha_{ij} \otimes e_j$ for some $\alpha_{ij} \in R'$. This corresponds to an R -algebra map $P \rightarrow R'$ determined by the rule $a_{ij} \mapsto \alpha_{ij}$. This map factors through the quotient P/J by the construction of the ideal J to give a map $P/J \rightarrow R'$. This in turn corresponds to a morphism $T \rightarrow W$ such that (a, b) is the pullback of (a_{univ}, b_{univ}) . Some details omitted. \square

Proposition 10.5. *Let S be a scheme. Let $Z \rightarrow B$ and $X \rightarrow B$ be morphisms of algebraic spaces over S . If $Z \rightarrow B$ is finite locally free then $\text{Mor}_B(Z, X)$ is an algebraic space.*

Proof. Choose a scheme $B' = \coprod B'_i$ which is a disjoint union of affine schemes B'_i and an étale surjective morphism $B' \rightarrow B$. We may also assume that $B'_i \times_B Z$ is the spectrum of a ring which is finite free as a $\Gamma(B'_i, \mathcal{O}_{B'_i})$ -module. By Lemma 10.2 and Spaces, Lemma 5.5 the morphism $\text{Mor}_{B'}(Z', X') \rightarrow \text{Mor}_B(Z, X)$ is surjective étale. Hence by Bootstrap, Theorem 10.1 it suffices to prove the proposition when $B = B'$ is a disjoint union of affine schemes B'_i so that each $B'_i \times_B Z$ is finite free over B'_i . Then it actually suffices to prove the result for the restriction to each B'_i . Thus we may assume that B is affine and that $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$ -module.

Choose a scheme X' which is a disjoint union of affine schemes and a surjective étale morphism $X' \rightarrow X$. By Lemma 10.3 the morphism $\text{Mor}_B(Z, X') \rightarrow \text{Mor}_B(Z, X)$ is representable by algebraic spaces, étale, and surjective. Hence by Bootstrap, Theorem 10.1 it suffices to prove the proposition when X is a disjoint union of affine schemes. This reduces us to the case discussed in the next paragraph.

Assume $X = \coprod_{i \in I} X_i$ is a disjoint union of affine schemes, B is affine, and that $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$ -module. For any finite subset $E \subset I$ set

$$F_E = \text{Mor}_B(Z, \coprod_{i \in E} X_i).$$

By Lemma 10.4 we see that F_E is an algebraic space. Consider the morphism

$$\coprod_{E \subset I \text{ finite}} F_E \longrightarrow \text{Mor}_B(Z, X)$$

Each of the morphisms $F_E \rightarrow \text{Mor}_B(Z, X)$ is an open immersion, because it is simply the locus parametrizing pairs (a, b) where b maps into the open subscheme $\coprod_{i \in E} X_i$ of X . Moreover, if T is quasi-compact, then for any pair (a, b) the image

of b is contained in $\coprod_{i \in E} X_i$ for some $E \subset I$ finite. Hence the displayed arrow is in fact an open covering and we win³ by Spaces, Lemma 8.4. \square

11. Restriction of scalars

Suppose $X \rightarrow Z \rightarrow B$ are morphisms of algebraic spaces over S . Given a scheme T we can consider pairs (a, b) where $a : T \rightarrow B$ is a morphism and $b : T \times_{a, B} Z \rightarrow X$ is a morphism over Z . Picture

$$(11.0.1) \quad \begin{array}{ccc} & & X \\ & \nearrow b & \downarrow \\ T \times_{a, B} Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ T & \xrightarrow{a} & B \end{array}$$

In this situation we can define a functor

$$(11.0.2) \quad \text{Res}_{Z/B}(X) : (\text{Sch}/S)^{\text{opp}} \longrightarrow \text{Sets}, \quad T \longmapsto \{(a, b) \text{ as above}\}$$

Sometimes we think of this as a functor defined on the category of schemes over B , in which case we drop a from the notation.

Lemma 11.1. *Let S be a scheme. Let $X \rightarrow Z \rightarrow B$ be morphisms of algebraic spaces over S . Then*

- (1) $\text{Res}_{Z/B}(X)$ is a sheaf on $(\text{Sch}/S)_{\text{fppf}}$.
- (2) If T is an algebraic space over S , then there is a canonical bijection

$$\text{Mor}_{\text{Sh}((\text{Sch}/S)_{\text{fppf}})}(T, \text{Res}_{Z/B}(X)) = \{(a, b) \text{ as in (11.0.1)}\}$$

Proof. Let T be an algebraic space over S . Let $\{T_i \rightarrow T\}$ be an fppf covering of T (as in Topologies on Spaces, Section 4). Suppose that $(a_i, b_i) \in \text{Res}_{Z/B}(X)(T_i)$ such that $(a_i, b_i)|_{T_i \times_T T_j} = (a_j, b_j)|_{T_i \times_T T_j}$ for all i, j . Then by Descent on Spaces, Lemma 6.2 there exists a unique morphism $a : T \rightarrow B$ such that a_i is the composition of $T_i \rightarrow T$ and a . Then $\{T_i \times_{a_i, B} Z \rightarrow T \times_{a, B} Z\}$ is an fppf covering too and the same lemma implies there exists a unique morphism $b : T \times_{a, B} Z \rightarrow X$ such that b_i is the composition of $T_i \times_{a_i, B} Z \rightarrow T \times_{a, B} Z$ and b . Hence $(a, b) \in \text{Res}_{Z/B}(X)(T)$ restricts to (a_i, b_i) over T_i for all i .

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S . In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say ‘‘pair’’ we mean a pair (a, b) fitting into (11.0.1).

Let $v : T \rightarrow \text{Res}_{Z/B}(X)$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \rightarrow T$. Then $v(p) \in \text{Res}_{Z/B}(X)(U)$ corresponds to a pair (a_U, b_U) over U . Let $R = U \times_T U$ with projections $t, s : R \rightarrow U$. As v is a transformation of functors we see that the pullbacks of (a_U, b_U) by s and t agree. Hence, since $\{U \rightarrow T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, b) over T .

³Modulo some set theoretic arguments. Namely, we have to show that $\coprod F_E$ is an algebraic space. This follows because $|I| \leq \text{size}(X)$ and $\text{size}(F_E) \leq \text{size}(X)$ as follows from the explicit description of F_E in the proof of Lemma 10.4. Some details omitted.

Conversely, let (a, b) be a pair over T . Let $U \rightarrow T$, $R = U \times_T U$, and $t, s : R \rightarrow U$ be as above. Then the restriction $(a, b)|_U$ gives rise to a transformation of functors $v : h_U \rightarrow \text{Res}_{Z/B}(X)$ by the Yoneda lemma (Categories, Lemma 3.5). As the two pullbacks $s^*(a, b)|_U$ and $t^*(a, b)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \rightarrow h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 9.1 and since $\text{Res}_{Z/B}(X)$ is an fppf sheaf by (1) we conclude that v factors through a map $T \rightarrow \text{Res}_{Z/B}(X)$.

We omit the verification that the two constructions above are mutually inverse. \square

Of course the sheaf $\text{Res}_{Z/B}(X)$ comes with a natural transformation of functors $\text{Res}_{Z/B}(X) \rightarrow B$. We will use this without further mention in the following.

Lemma 11.2. *Let S be a scheme. Let $X \rightarrow Z \rightarrow B$ and $B' \rightarrow B$ be morphisms of algebraic spaces over S . Set $Z' = B' \times_B Z$ and $X' = B' \times_B X$. Then*

$$\text{Res}_{Z'/B'}(X') = B' \times_B \text{Res}_{Z/B}(X)$$

in $\text{Sh}((\text{Sch}/S)_{\text{fppf}})$.

Proof. The equality as functors follows immediately from the definitions. The equality as sheaves follows from this because both sides are sheaves according to Lemma 11.1 and the fact that a fibre product of sheaves is the same as the corresponding fibre product of pre-sheaves (i.e., functors). \square

Lemma 11.3. *Let S be a scheme. Let $X' \rightarrow X \rightarrow Z \rightarrow B$ be morphisms of algebraic spaces over S . Assume*

- (1) $X' \rightarrow X$ is étale, and
- (2) $Z \rightarrow B$ is finite locally free.

Then $\text{Res}_{Z/B}(X') \rightarrow \text{Res}_{Z/B}(X)$ is representable by algebraic spaces and étale. If $X' \rightarrow X$ is also surjective, then $\text{Res}_{Z/B}(X') \rightarrow \text{Res}_{Z/B}(X)$ is surjective.

Proof. Let U be a scheme and let $\xi = (a, b)$ be an element of $\text{Res}_{Z/B}(X)(U)$. We have to prove that the functor

$$h_U \times_{\xi, \text{Res}_{Z/B}(X)} \text{Res}_{Z/B}(X')$$

is representable by an algebraic space étale over U . Set $Z_U = U \times_{a,B} Z$ and $W = Z_U \times_{b,X} X'$. Then $W \rightarrow Z_U \rightarrow U$ is as in Lemma 9.2 and the sheaf F defined there is identified with the fibre product displayed above. Hence the first assertion of the lemma. The second assertion follows from this and Lemma 9.1 which guarantees that $F \rightarrow U$ is surjective in the situation above. \square

At this point we can use the lemmas above to prove that $\text{Res}_{Z/B}(X)$ is an algebraic space whenever $Z \rightarrow B$ is finite locally free in almost exactly the same way as in the proof that $\text{Mor}_B(Z, X)$ is an algebraic spaces, see Proposition 10.5. Instead we will directly deduce this result from the following lemma and the fact that $\text{Mor}_B(Z, X)$ is an algebraic space.

Lemma 11.4. *Let S be a scheme. Let $X \rightarrow Z \rightarrow B$ be morphisms of algebraic spaces over S . The following diagram*

$$\begin{array}{ccc} \text{Mor}_B(Z, X) & \longrightarrow & \text{Mor}_B(Z, Z) \\ \uparrow & & \uparrow \text{id}_Z \\ \text{Res}_{Z/B}(X) & \longrightarrow & B \end{array}$$

is a cartesian diagram of sheaves on $(\text{Sch}/S)_{\text{fppf}}$.

Proof. Omitted. Hint: Exercise in the functorial point of view in algebraic geometry. \square

Proposition 11.5. *Let S be a scheme. Let $X \rightarrow Z \rightarrow B$ be morphisms of algebraic spaces over S . If $Z \rightarrow B$ is finite locally free then $\text{Res}_{Z/B}(X)$ is an algebraic space.*

Proof. By Proposition 10.5 the functors $\text{Mor}_B(Z, X)$ and $\text{Mor}_B(Z, Z)$ are algebraic spaces. Hence this follows from the cartesian diagram of Lemma 11.4 and the fact that fibre products of algebraic spaces exist and are given by the fibre product in the underlying category of sheaves of sets (see Spaces, Lemma 7.2). \square

12. Finite Hilbert stacks

In this section we prove some results concerning the finite Hilbert stacks $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ introduced in Examples of Stacks, Section 17.

Lemma 12.1. *Consider a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{G} & \mathcal{X} \\ F' \downarrow & & \downarrow F \\ \mathcal{Y}' & \xrightarrow{H} & \mathcal{Y} \end{array}$$

of stacks in groupoids over $(\text{Sch}/S)_{\text{fppf}}$ with a given 2-isomorphism $\gamma : H \circ F' \rightarrow F \circ G$. In this situation we obtain a canonical 1-morphism $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \rightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. This morphism is compatible with the forgetful 1-morphisms of Examples of Stacks, Equation (17.2.1).

Proof. We map the object (U, Z, y', x', α') to the object $(U, Z, H(y'), G(x'), \gamma \star \text{id}_H \star \alpha')$ where \star denotes horizontal composition of 2-morphisms, see Categories, Definition 26.1. To a morphism $(f, g, b, a) : (U_1, Z_1, y'_1, x'_1, \alpha'_1) \rightarrow (U_2, Z_2, y'_2, x'_2, \alpha'_2)$ we assign $(f, g, H(b), G(a))$. We omit the verification that this defines a functor between categories over $(\text{Sch}/S)_{\text{fppf}}$. \square

Lemma 12.2. *In the situation of Lemma 12.1 assume that the given square is 2-cartesian. Then the diagram*

$$\begin{array}{ccc} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') & \longrightarrow & \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

is 2-cartesian.

Proof. We get a 2-commutative diagram by Lemma 12.1 and hence we get a 1-morphism (i.e., a functor)

$$\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \longrightarrow \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

We indicate why this functor is essentially surjective. Namely, an object of the category on the right hand side is given by a scheme U over S , an object y' of \mathcal{Y}'_U , an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U and an isomorphism $H(y') \rightarrow y$ in \mathcal{Y}_U . The assumption means exactly that there exists an object x' of \mathcal{X}'_Z such that there exist isomorphisms $G(x') \cong x$ and $\alpha' : y'|_Z \rightarrow F'(x')$ compatible with α . Then we see that (U, Z, y', x', α') is an object of $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}')$ over U . Details omitted. \square

Lemma 12.3. *In the situation of Lemma 12.1 assume*

- (1) $\mathcal{Y}' = \mathcal{Y}$ and $H = id_{\mathcal{Y}}$,
- (2) G is representable by algebraic spaces and étale.

Then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \rightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. If G is also surjective, then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \rightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is surjective.

Proof. Let U be a scheme and let $\xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U . We have to prove that the 2-fibre product

$$(12.3.1) \quad (Sch/U)_{fppf} \times_{\xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y})$$

is representable by an algebraic space étale over U . An object of this over U' corresponds to an object x' in the fibre category of \mathcal{X}' over $Z_{U'}$ such that $G(x') \cong x|_{Z_{U'}}$. By assumption the 2-fibre product

$$(Sch/Z)_{fppf} \times_{x, \mathcal{X}} \mathcal{X}'$$

is representable by an algebraic space W such that the projection $W \rightarrow Z$ is étale. Then (12.3.1) is representable by the algebraic space F parametrizing sections of $W \rightarrow Z$ over U introduced in Lemma 9.2. Since $F \rightarrow U$ is étale we conclude that $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \rightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. Finally, if $\mathcal{X}' \rightarrow \mathcal{X}$ is surjective also, then $W \rightarrow Z$ is surjective, and hence $F \rightarrow U$ is surjective by Lemma 9.1. Thus in this case $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \rightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is also surjective. \square

Lemma 12.4. *In the situation of Lemma 12.1. Assume that G, H are representable by algebraic spaces and étale. Then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \rightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. If also H is surjective and the induced functor $\mathcal{X}' \rightarrow \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is surjective, then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \rightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is surjective.*

Proof. Set $\mathcal{X}'' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 4.1 the 1-morphism $\mathcal{X}' \rightarrow \mathcal{X}''$ is representable by algebraic spaces and étale (in particular the condition in the second statement of the lemma that $\mathcal{X}' \rightarrow \mathcal{X}''$ be surjective makes sense). We obtain a 2-commutative diagram

$$\begin{array}{ccccc} \mathcal{X}' & \longrightarrow & \mathcal{X}'' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

It follows from Lemma 12.2 that $\mathcal{H}_d(\mathcal{X}''/\mathcal{Y}')$ is the base change of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ by $\mathcal{Y}' \rightarrow \mathcal{Y}$. In particular we see that $\mathcal{H}_d(\mathcal{X}''/\mathcal{Y}') \rightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale, see Algebraic Stacks, Lemma 10.6. Moreover, it is also surjective if H is. Hence if we can show that the result holds for the left square

in the diagram, then we're done. In this way we reduce to the case where $\mathcal{Y}' = \mathcal{Y}$ which is the content of Lemma 12.3. \square

Lemma 12.5. *Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that $\Delta : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces. Then*

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}$$

see *Examples of Stacks, Equation (17.2.1)* is representable by algebraic spaces.

Proof. Let U be a scheme and let $\xi = (U, Z, p, x, 1)$ be an object of $\mathcal{H}_d(\mathcal{X}) = \mathcal{H}_d(\mathcal{X}/S)$ over U . Here p is just the structure morphism of U . The fifth component 1 exists and is unique since everything is over S . Also, let y be an object of \mathcal{Y} over U . We have to show the 2-fibre product

$$(12.5.1) \quad (Sch/U)_{fppf} \times_{\xi \times y, \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable by an algebraic space. To explain why this is so we introduce

$$I = \text{Isom}_{\mathcal{Y}}(y|_Z, F(x))$$

which is an algebraic space over Z by assumption. Let $a : U' \rightarrow U$ be a scheme over U . What does it mean to give an object of the fibre category of (12.5.1) over U' ? Well, it means that we have an object $\xi' = (U', Z', y', x', \alpha')$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U' and isomorphisms $(U', Z', p', x', 1) \cong (U, Z, p, x, 1)|_{U'}$ and $y' \cong y|_{U'}$. Thus ξ' is isomorphic to $(U', U' \times_{a,U} Z, a^*y, x|_{U' \times_{a,U} Z}, \alpha)$ for some morphism

$$\alpha : a^*y|_{U' \times_{a,U} Z} \longrightarrow F(x|_{U' \times_{a,U} Z})$$

in the fibre category of \mathcal{Y} over $U' \times_{a,U} Z$. Hence we can view α as a morphism $b : U' \times_{a,U} Z \rightarrow I$. In this way we see that (12.5.1) is representable by $\text{Res}_{Z/U}(I)$ which is an algebraic space by Proposition 11.5. \square

The following lemma is a (partial) generalization of Lemma 12.3.

Lemma 12.6. *Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ and $G : \mathcal{X}' \rightarrow \mathcal{X}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If G is representable by algebraic spaces, then the 1-morphism*

$$\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable by algebraic spaces.

Proof. Let U be a scheme and let $\xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U . We have to prove that the 2-fibre product

$$(12.6.1) \quad (Sch/U)_{fppf} \times_{\xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y})$$

is representable by an algebraic space étale over U . An object of this over $a : U' \rightarrow U$ corresponds to an object x' of \mathcal{X}' over $U' \times_{a,U} Z$ such that $G(x') \cong x|_{U' \times_{a,U} Z}$. By assumption the 2-fibre product

$$(Sch/Z)_{fppf} \times_{x, \mathcal{X}} \mathcal{X}'$$

is representable by an algebraic space X over Z . It follows that (12.6.1) is representable by $\text{Res}_{Z/U}(X)$, which is an algebraic space by Proposition 11.5. \square

Lemma 12.7. *Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces and locally of finite presentation. Then*

$$p : \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \rightarrow \mathcal{Y}$$

is limit preserving on objects.

Proof. This means we have to show the following: Given

- (1) an affine scheme $U = \lim_i U_i$ which is written as the directed limit of affine schemes U_i over S ,
- (2) an object y_i of \mathcal{Y} over U_i for some i , and
- (3) an object $\Xi = (U, Z, y, x, \alpha)$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U such that $y = y_i|_U$,

then there exists an $i' \geq i$ and an object $\Xi_{i'} = (U_{i'}, Z_{i'}, y_{i'}, x_{i'}, \alpha_{i'})$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over $U_{i'}$ with $\Xi_{i'}|_U = \Xi$ and $y_{i'} = y_i|_{U_{i'}}$. Namely, the last two equalities will take care of the commutativity of (5.0.1).

Let $X_{y_i} \rightarrow U_i$ be an algebraic space representing the 2-fibre product

$$(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X}.$$

Note that $X_{y_i} \rightarrow U_i$ is locally of finite presentation by our assumption on F . Write Ξ . It is clear that $\xi = (Z, Z \rightarrow U_i, x, \alpha)$ is an object of the 2-fibre product displayed above, hence ξ gives rise to a morphism $f_\xi : Z \rightarrow X_{y_i}$ of algebraic spaces over U_i (since X_{y_i} is the functor of isomorphisms classes of objects of $(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X}$, see Algebraic Stacks, Lemma 8.2). By Limits, Lemmas 9.1 and 7.7 there exists an $i' \geq i$ and a finite locally free morphism $Z_{i'} \rightarrow U_{i'}$ of degree d whose base change to U is Z . By Limits of Spaces, Proposition 3.9 we may, after replacing i' by a bigger index, assume there exists a morphism $f_{i'} : Z_{i'} \rightarrow X_{y_i}$ such that

$$\begin{array}{ccccc} & & f_\xi & & \\ & & \curvearrowright & & \\ Z & \longrightarrow & Z_{i'} & \xrightarrow{f_{i'}} & X_{y_i} \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & U_{i'} & \longrightarrow & U_i \end{array}$$

is commutative. We set $\Xi_{i'} = (U_{i'}, Z_{i'}, y_{i'}, x_{i'}, \alpha_{i'})$ where

- (1) $y_{i'}$ is the object of \mathcal{Y} over $U_{i'}$ which is the pullback of y_i to $U_{i'}$,
- (2) $x_{i'}$ is the object of \mathcal{X} over $Z_{i'}$ corresponding via the 2-Yoneda lemma to the 1-morphism

$$(Sch/Z_{i'})_{fppf} \rightarrow \mathcal{S}_{X_{y_i}} \rightarrow (Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X} \rightarrow \mathcal{X}$$

where the middle arrow is the equivalence which defines X_{y_i} (notation as in Algebraic Stacks, Sections 8 and 7).

- (3) $\alpha_{i'} : y_{i'}|_{Z_{i'}} \rightarrow F(x_{i'})$ is the isomorphism coming from the 2-commutativity of the diagram

$$\begin{array}{ccccc} (Sch/Z_{i'})_{fppf} & \longrightarrow & (Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X} & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow F \\ & & (Sch/U_{i'})_{fppf} & \longrightarrow & \mathcal{Y} \end{array}$$

Recall that $f_\xi : Z \rightarrow X_{y_i}$ was the morphism corresponding to the object $\xi = (Z, Z \rightarrow U_i, x, \alpha)$ of $(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X}$ over Z . By construction $f_{i'}$ is the morphism corresponding to the object $\xi_{i'} = (Z_{i'}, Z_{i'} \rightarrow U_i, x_{i'}, \alpha_{i'})$. As $f_\xi = f_{i'} \circ (Z \rightarrow Z_{i'})$ we see that the object $\xi_{i'} = (Z_{i'}, Z_{i'} \rightarrow U_i, x_{i'}, \alpha_{i'})$ pulls back to ξ over Z . Thus $x_{i'}$ pulls back to x and $\alpha_{i'}$ pulls back to α . This means that $\Xi_{i'}$ pulls back to Ξ over U and we win. \square

13. The finite Hilbert stack of a point

Let $d \geq 1$ be an integer. In Examples of Stacks, Definition 17.2 we defined a stack in groupoids \mathcal{H}_d . In this section we prove that \mathcal{H}_d is an algebraic stack. We will throughout assume that $S = \text{Spec}(\mathbf{Z})$. The general case will follow from this by base change. Recall that the fibre category of \mathcal{H}_d over a scheme T is the category of finite locally free morphisms $\pi : Z \rightarrow T$ of degree d . Instead of classifying these directly we first study the quasi-coherent sheaves of algebras $\pi_* \mathcal{O}_Z$.

Let R be a ring. Let us temporarily make the following definition: A *free d -dimensional algebra over R* is given by a commutative R -algebra structure m on $R^{\oplus d}$ such that $e_1 = (1, 0, \dots, 0)$ is a unit⁴. We think of m as an R -linear map

$$m : R^{\oplus d} \otimes_R R^{\oplus d} \longrightarrow R^{\oplus d}$$

such that $m(e_1, x) = m(x, e_1) = x$ and such that m defines a commutative and associative ring structure. If we write $m(e_i, e_j) = \sum a_{ij}^k e_k$ then we see this boils down to the conditions

$$\begin{cases} \sum_l a_{ij}^l a_{lk}^m = \sum_l a_{il}^m a_{jk}^l & \forall i, j, k, m \\ a_{ij}^k = a_{ji}^k & \forall i, j, k \\ a_{i1}^j = \delta_{ij} & \forall i, j \end{cases}$$

where δ_{ij} is the Kronecker δ -function. OK, so let's define

$$R_{univ} = \mathbf{Z}[a_{ij}^k]/J$$

where the ideal J is the ideal generated by the relations displayed above. Denote

$$m_{univ} : R_{univ}^{\oplus d} \otimes_{R_{univ}} R_{univ}^{\oplus d} \longrightarrow R_{univ}^{\oplus d}$$

the free d -dimensional algebra m over R_{univ} whose structure constants are the classes of a_{ij}^k modulo J . Then it is clear that given any free d -dimensional algebra m over a ring R there exists a unique \mathbf{Z} -algebra homomorphism $\psi : R_{univ} \rightarrow R$ such that $\psi_* m_{univ} = m$ (this means that m is what you get by applying the base change functor $-\otimes_{R_{univ}} R$ to m_{univ}). In other words, setting $X = \text{Spec}(R_{univ})$ we obtain a canonical identification

$$X(T) = \{\text{free } d\text{-dimensional algebras } m \text{ over } R\}$$

for varying $T = \text{Spec}(R)$. By Zariski localization we obtain the following seemingly more general identification

$$(13.0.1) \quad X(T) = \{\text{free } d\text{-dimensional algebras } m \text{ over } \Gamma(T, \mathcal{O}_T)\}$$

for any scheme T .

⁴It may be better to think of this as a pair consisting of a multiplication map $m : R^{\oplus d} \otimes_R R^{\oplus d} \rightarrow R^{\oplus d}$ and a ring map $\psi : R \rightarrow R^{\oplus d}$ satisfying a bunch of axioms.

Next we talk a little bit about *isomorphisms of free d -dimensional R -algebras*. Namely, suppose that m, m' are two free d -dimensional algebras over a ring R . An *isomorphism from m to m'* is given by an invertible R -linear map

$$\varphi : R^{\oplus d} \longrightarrow R^{\oplus d}$$

such that $\varphi(e_1) = e_1$ and such that

$$m \circ \varphi \otimes \varphi = \varphi \circ m'.$$

Note that we can compose these so that the collection of free d -dimensional algebras over R becomes a category. In this way we obtain a functor

$$(13.0.2) \quad FA_d : Sch_{fppf}^{opp} \longrightarrow \text{Groupoids}$$

from the category of schemes to groupoids: to a scheme T we associate the set of free d -dimensional algebras over $\Gamma(T, \mathcal{O}_T)$ endowed with the structure of a category using the notion of isomorphisms just defined.

The above suggests we consider the functor G in groups which associates to any scheme T the group

$$G(T) = \{g \in \text{GL}_d(\Gamma(T, \mathcal{O}_T)) \mid g(e_1) = e_1\}$$

It is clear that $G \subset \text{GL}_d$ (see Groupoids, Example 5.4) is the closed subgroup scheme cut out by the equations $x_{11} = 1$ and $x_{i1} = 0$ for $i > 1$. Hence G is a smooth affine group scheme over $\text{Spec}(\mathbf{Z})$. Consider the action

$$a : G \times_{\text{Spec}(\mathbf{Z})} X \longrightarrow X$$

which associates to a T -valued point (g, m) with $T = \text{Spec}(R)$ on the left hand side the free d -dimensional algebra over R given by

$$a(g, m) = g^{-1} \circ m \circ g \otimes g.$$

Note that this means that g defines an isomorphism $m \rightarrow a(g, m)$ of d -dimensional free R -algebras. We omit the verification that a indeed defines an action of the group scheme G on the scheme X .

Lemma 13.1. *The functor in groupoids FA_d defined in (13.0.2) is isomorphic (!) to the functor in groupoids which associates to a scheme T the category with*

- (1) *set of objects is $X(T)$,*
- (2) *set of morphisms is $G(T) \times X(T)$,*
- (3) *$s : G(T) \times X(T) \rightarrow X(T)$ is the projection map,*
- (4) *$t : G(T) \times X(T) \rightarrow X(T)$ is $a(T)$, and*
- (5) *composition $G(T) \times X(T) \times_{s, X(T), t} G(T) \times X(T) \rightarrow G(T) \times X(T)$ is given by $((g, m), (g', m')) \mapsto (gg', m')$.*

Proof. We have seen the rule on objects in (13.0.1). We have also seen above that $g \in G(T)$ can be viewed as a morphism from m to $a(g, m)$ for any free d -dimensional algebra m . Conversely, any morphism $m \rightarrow m'$ is given by an invertible linear map φ which corresponds to an element $g \in G(T)$ such that $m' = a(g, m)$. \square

In fact the groupoid $(X, G \times X, s, t, c)$ described in the lemma above is the groupoid associated to the action $a : G \times X \rightarrow X$ as defined in Groupoids, Lemma 14.1. Since G is smooth over $\text{Spec}(\mathbf{Z})$ we see that the two morphisms $s, t : G \times X \rightarrow X$ are smooth: by symmetry it suffices to prove that one of them is, and s is the base change of $G \rightarrow \text{Spec}(\mathbf{Z})$. Hence $(G \times X, X, s, t, c)$ is a smooth groupoid scheme,

and the quotient stack $[X/G]$ is an algebraic stack by Algebraic Stacks, Theorem 17.3.

Proposition 13.2. *The stack \mathcal{H}_d is equivalent to the quotient stack $[X/G]$ described above. In particular \mathcal{H}_d is an algebraic stack.*

Proof. Note that by Groupoids in Spaces, Definition 19.1 the quotient stack $[X/G]$ is the stackification of the category fibred in groupoids associated to the “presheaf in groupoids” which associates to a scheme T the groupoid

$$(X(T), G(T) \times X(T), s, t, c).$$

Since this “presheaf in groupoids” is isomorphic to FA_d by Lemma 13.1 it suffices to prove that the \mathcal{H}_d is the stackification of (the category fibred in groupoids associated to the “presheaf in groupoids”) FA_d . To do this we first define a functor

$$\text{Spec} : FA_d \longrightarrow \mathcal{H}_d$$

Recall that the fibre category of \mathcal{H}_d over a scheme T is the category of finite locally free morphisms $Z \rightarrow T$ of degree d . Thus given a scheme T and a free d -dimensional $\Gamma(T, \mathcal{O}_T)$ -algebra m we may assign to this the object

$$Z = \underline{\text{Spec}}_T(\mathcal{A})$$

of $\mathcal{H}_{d,T}$ where $\mathcal{A} = \mathcal{O}_T^{\oplus d}$ endowed with a \mathcal{O}_T -algebra structure via m . Moreover, if m' is a second such free d -dimensional $\Gamma(T, \mathcal{O}_T)$ -algebra and if $\varphi : m \rightarrow m'$ is an isomorphism of these, then the induced \mathcal{O}_T -linear map $\varphi : \mathcal{O}_T^{\oplus d} \rightarrow \mathcal{O}_T^{\oplus d}$ induces an isomorphism

$$\varphi : \mathcal{A}' \longrightarrow \mathcal{A}$$

of quasi-coherent \mathcal{O}_T -algebras. Hence

$$\underline{\text{Spec}}_T(\varphi) : \underline{\text{Spec}}_T(\mathcal{A}) \longrightarrow \underline{\text{Spec}}_T(\mathcal{A}')$$

is a morphism in the fibre category $\mathcal{H}_{d,T}$. We omit the verification that this construction is compatible with base change so we get indeed a functor $\text{Spec} : FA_d \rightarrow \mathcal{H}_d$ as claimed above.

To show that $\text{Spec} : FA_d \rightarrow \mathcal{H}_d$ induces an equivalence between the stackification of FA_d and \mathcal{H}_d it suffices to check that

- (1) $\text{Isom}(m, m') = \text{Isom}(\text{Spec}(m), \text{Spec}(m'))$ for any $m, m' \in FA_d(T)$.
- (2) for any scheme T and any object $Z \rightarrow T$ of $\mathcal{H}_{d,T}$ there exists a covering $\{T_i \rightarrow T\}$ such that $Z|_{T_i}$ is isomorphic to $\text{Spec}(m)$ for some $m \in FA_d(T_i)$, and

see Stacks, Lemma 9.1. The first statement follows from the observation that any isomorphism

$$\underline{\text{Spec}}_T(\mathcal{A}) \longrightarrow \underline{\text{Spec}}_T(\mathcal{A}')$$

is necessarily given by a global invertible matrix g when $\mathcal{A} = \mathcal{A}' = \mathcal{O}_T^{\oplus d}$ as modules. To prove the second statement let $\pi : Z \rightarrow T$ be a finite locally free morphism of degree d . Then \mathcal{A} is a locally free sheaf \mathcal{O}_T -modules of rank d . Consider the element $1 \in \Gamma(T, \mathcal{A})$. This element is nonzero in $\mathcal{A} \otimes_{\mathcal{O}_{T,t}} \kappa(t)$ for every $t \in T$ since the scheme $Z_t = \text{Spec}(\mathcal{A} \otimes_{\mathcal{O}_{T,t}} \kappa(t))$ is nonempty being of degree $d > 0$ over $\kappa(t)$. Thus $1 : \mathcal{O}_T \rightarrow \mathcal{A}$ can locally be used as the first basis element (for example you can use Algebra, Lemma 76.3 parts (1) and (2) to see this). Thus, after localizing on T we may assume that there exists an isomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{O}_T^{\oplus d}$ such that $1 \in \Gamma(\mathcal{A})$

corresponds to the first basis element. In this situation the multiplication map $\mathcal{A} \otimes_{\mathcal{O}_T} \mathcal{A} \rightarrow \mathcal{A}$ translates via φ into a free d -dimensional algebra m over $\Gamma(T, \mathcal{O}_T)$. This finishes the proof. \square

14. Finite Hilbert stacks of spaces

The finite Hilbert stack of an algebraic space is an algebraic stack.

Lemma 14.1. *Let S be a scheme. Let X be an algebraic space over S . Then $\mathcal{H}_d(X)$ is an algebraic stack.*

Proof. The 1-morphism

$$\mathcal{H}_d(X) \longrightarrow \mathcal{H}_d$$

is representable by algebraic spaces according to Lemma 12.6. The stack \mathcal{H}_d is an algebraic stack according to Proposition 13.2. Hence $\mathcal{H}_d(X)$ is an algebraic stack by Algebraic Stacks, Lemma 15.4. \square

This lemma allows us to bootstrap.

Lemma 14.2. *Let S be a scheme. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$ such that*

- (1) \mathcal{X} is representable by an algebraic space, and
- (2) F is representable by algebraic spaces, surjective, flat, and locally of finite presentation.

Then $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is an algebraic stack.

Proof. Choose a representable stack in groupoids \mathcal{U} over S and a 1-morphism $f : \mathcal{U} \rightarrow \mathcal{H}_d(\mathcal{X})$ which is representable by algebraic spaces, smooth, and surjective. This is possible because $\mathcal{H}_d(\mathcal{X})$ is an algebraic stack by Lemma 14.1. Consider the 2-fibre product

$$\mathcal{W} = \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{H}_d(\mathcal{X}), f} \mathcal{U}.$$

Since \mathcal{U} is representable (in particular a stack in setoids) it follows from Examples of Stacks, Lemma 17.3 and Stacks, Lemma 6.7 that \mathcal{W} is a stack in setoids. The 1-morphism $\mathcal{W} \rightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces, smooth, and surjective as a base change of the morphism f (see Algebraic Stacks, Lemmas 9.7 and 10.6). Thus, if we can show that \mathcal{W} is representable by an algebraic space, then the lemma follows from Algebraic Stacks, Lemma 15.3.

The diagonal of \mathcal{Y} is representable by algebraic spaces according to Lemma 4.3. We may apply Lemma 12.5 to see that the 1-morphism

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}$$

is representable by algebraic spaces. Consider the 2-fibre product

$$\mathcal{V} = \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{(\mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}), f \times F} (\mathcal{U} \times \mathcal{X}).$$

The projection morphism $\mathcal{V} \rightarrow \mathcal{U} \times \mathcal{X}$ is representable by algebraic spaces as a base change of the last displayed morphism. Hence \mathcal{V} is an algebraic space (see

Bootstrap, Lemma 3.6 or Algebraic Stacks, Lemma 9.8). The 1-morphism $\mathcal{V} \rightarrow \mathcal{U}$ fits into the following 2-cartesian diagram

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow F \\ \mathcal{W} & \longrightarrow & \mathcal{Y} \end{array}$$

because

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{(\mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}), f \times F} (\mathcal{U} \times \mathcal{X}) = (\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{H}_d(\mathcal{X}), f} \mathcal{U}) \times_{\mathcal{Y}, F} \mathcal{X}.$$

Hence $\mathcal{V} \rightarrow \mathcal{W}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation as a base change of F . It follows that the same thing is true for the corresponding sheaves of sets associated to \mathcal{V} and \mathcal{W} , see Algebraic Stacks, Lemma 10.4. Thus we conclude that the sheaf associated to \mathcal{W} is an algebraic space by Bootstrap, Theorem 10.1. \square

15. LCI locus in the Hilbert stack

Please consult Examples of Stacks, Section 17 for notation. Fix a 1-morphism $F : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that F is representable by algebraic spaces. Fix $d \geq 1$. Consider an object (U, Z, y, x, α) of \mathcal{H}_d . There is an induced 1-morphism

$$(Sch/Z)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$$

(by the universal property of 2-fibre products) which is representable by a morphism of algebraic spaces over U . Namely, since F is representable by algebraic spaces, we may choose an algebraic space X_y over U which represents the 2-fibre product $(Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$. Since $\alpha : y|_Z \rightarrow F(x)$ is an isomorphism we see that $\xi = (Z, Z \rightarrow U, x, \alpha)$ is an object of the 2-fibre product $(Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$ over Z . Hence ξ gives rise to a morphism $x_\alpha : Z \rightarrow X_y$ of algebraic spaces over U as X_y is the functor of isomorphisms classes of objects of $(Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$, see Algebraic Stacks, Lemma 8.2. Here is a picture

$$(15.0.1) \quad \begin{array}{ccccc} Z & \xrightarrow{x_\alpha} & X_y & & (Sch/Z)_{fppf} \xrightarrow{x, \alpha} (Sch/U)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X} \longrightarrow \mathcal{X} \\ & \searrow & \downarrow & & \downarrow \\ & & U & & (Sch/U)_{fppf} \xrightarrow{y} \mathcal{Y} \\ & & & & \downarrow F \end{array}$$

We remark that if $(f, g, b, a) : (U, Z, y, x, \alpha) \rightarrow (U', Z', y', x', \alpha')$ is a morphism between objects of \mathcal{H}_d , then the morphism $x'_{\alpha'} : Z' \rightarrow X'_{y'}$ is the base change of the morphism x_α by the morphism $g : U' \rightarrow U$ (details omitted).

Now assume moreover that F is flat and locally of finite presentation. In this situation we define a full subcategory

$$\mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y}) \subset \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

consisting of those objects (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ such that the corresponding morphism $x_\alpha : Z \rightarrow X_y$ is unramified and a local complete intersection morphism (see Morphisms of Spaces, Definition 35.1 and More on Morphisms of Spaces, Definition 36.1 for definitions).

Lemma 15.1. *Let S be a scheme. Fix a 1-morphism $F : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, and locally of finite presentation. Then $\mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y})$ is a stack in groupoids and the inclusion functor*

$$\mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y}) \rightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable and an open immersion.

Proof. Let $\Xi = (U, Z, y, x, \alpha)$ be an object of \mathcal{H}_d . It follows from the remark following (15.0.1) that the pullback of Ξ by $U' \rightarrow U$ belongs to $\mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y})$ if and only if the base change of x_α is unramified and a local complete intersection morphism. Note that $Z \rightarrow U$ is finite locally free (hence flat, locally of finite presentation and universally closed) and that $X_y \rightarrow U$ is flat and locally of finite presentation by our assumption on F . Then More on Morphisms of Spaces, Lemmas 37.1 and 37.7 imply exists an open subscheme $W \subset U$ such that a morphism $U' \rightarrow U$ factors through W if and only if the base change of x_α via $U' \rightarrow U$ is unramified and a local complete intersection morphism. This implies that

$$(Sch/U)_{fppf} \times_{\Xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y})$$

is representable by W . Hence the final statement of the lemma holds. The first statement (that $\mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y})$ is a stack in groupoids) follows from this an Algebraic Stacks, Lemma 15.5. \square

Local complete intersection morphisms are “locally unobstructed”. This holds in much greater generality than the special case that we need in this chapter here.

Lemma 15.2. *Let $U \subset U'$ be a first order thickening of affine schemes. Let X' be an algebraic space flat over U' . Set $X = U \times_{U'} X'$. Let $Z \rightarrow U$ be finite locally free of degree d . Finally, let $f : Z \rightarrow X$ be unramified and a local complete intersection morphism. Then there exists a commutative diagram*

$$\begin{array}{ccc} (Z \subset Z') & \xrightarrow{(f, f')} & (X \subset X') \\ & \searrow & \swarrow \\ & (U \subset U') & \end{array}$$

of algebraic spaces over U' such that $Z' \rightarrow U'$ is finite locally free of degree d and $Z = U \times_{U'} Z'$.

Proof. By More on Morphisms of Spaces, Lemma 36.8 the conormal sheaf $\mathcal{C}_{Z/X}$ of the unramified morphism $Z \rightarrow X$ is a finite locally free \mathcal{O}_Z -module and by More on Morphisms of Spaces, Lemma 36.9 we have an exact sequence

$$0 \rightarrow i^* \mathcal{C}_{X/X'} \rightarrow \mathcal{C}_{Z/X'} \rightarrow \mathcal{C}_{Z/X} \rightarrow 0$$

of conormal sheaves. Since Z is affine this sequence is split. Choose a splitting

$$\mathcal{C}_{Z/X'} = i^* \mathcal{C}_{X/X'} \oplus \mathcal{C}_{Z/X}$$

Let $Z \subset Z''$ be the universal first order thickening of Z over X' (see More on Morphisms of Spaces, Section 12). Denote $\mathcal{I} \subset \mathcal{O}_{Z''}$ the quasi-coherent sheaf of ideals corresponding to $Z \subset Z''$. By definition we have $\mathcal{C}_{Z/X'}$ is \mathcal{I} viewed as a sheaf on Z . Hence the splitting above determines a splitting

$$\mathcal{I} = i^* \mathcal{C}_{X/X'} \oplus \mathcal{C}_{Z/X}$$

Let $Z' \subset Z''$ be the closed subscheme cut out by $\mathcal{C}_{Z/X} \subset \mathcal{I}$ viewed as a quasi-coherent sheaf of ideals on Z'' . It is clear that Z' is a first order thickening of Z and that we obtain a commutative diagram of first order thickenings as in the statement of the lemma.

Since $X' \rightarrow U'$ is flat and since $X = U \times_{U'} X'$ we see that $\mathcal{C}_{X/X'}$ is the pullback of $\mathcal{C}_{U/U'}$ to X , see More on Morphisms of Spaces, Lemma 15.1. Note that by construction $\mathcal{C}_{Z/Z'} = i^* \mathcal{C}_{X/X'}$ hence we conclude that $\mathcal{C}_{Z/Z'}$ is isomorphic to the pullback of $\mathcal{C}_{U/U'}$ to Z . Applying More on Morphisms of Spaces, Lemma 15.1 once again (or its analogue for schemes, see More on Morphisms, Lemma 8.1) we conclude that $Z' \rightarrow U'$ is flat and that $Z = U \times_{U'} Z'$. Finally, More on Morphisms, Lemma 8.3 shows that $Z' \rightarrow U'$ is finite locally free of degree d . \square

Lemma 15.3. *Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, and locally of finite presentation. Then*

$$p : \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \rightarrow \mathcal{Y}$$

is formally smooth on objects.

Proof. We have to show the following: Given

- (1) an object (U, Z, y, x, α) of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ over an affine scheme U ,
- (2) a first order thickening $U \subset U'$, and
- (3) an object y' of \mathcal{Y} over U' such that $y'|_U = y$,

then there exists an object $(U', Z', y', x', \alpha')$ of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ over U' with $Z = U \times_{U'} Z'$, with $x = x'|_Z$, and with $\alpha = \alpha'|_U$. Namely, the last two equalities will take care of the commutativity of (6.0.1).

Consider the morphism $x_\alpha : Z \rightarrow X_y$ constructed in Equation (15.0.1). Denote similarly $X'_{y'}$ the algebraic space over U' representing the 2-fibre product $(Sch/U')_{fppf} \times_{y', \mathcal{Y}, F} \mathcal{X}$. By assumption the morphism $X'_{y'} \rightarrow U'$ is flat (and locally of finite presentation). As $y'|_U = y$ we see that $X_y = U \times_{U'} X'_{y'}$. Hence we may apply Lemma 15.2 to find $Z' \rightarrow U'$ finite locally free of degree d with $Z = U \times_{U'} Z'$ and with $Z' \rightarrow X'_{y'}$ extending x_α . By construction the morphism $Z' \rightarrow X'_{y'}$ corresponds to a pair (x', α') . It is clear that $(U', Z', y', x', \alpha')$ is an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U' with $Z = U \times_{U'} Z'$, with $x = x'|_Z$, and with $\alpha = \alpha'|_U$. As we've seen in Lemma 15.1 that $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \subset \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is an "open substack" it follows that $(U', Z', y', x', \alpha')$ is an object of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ as desired. \square

Lemma 15.4. *Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, surjective, and locally of finite presentation. Then*

$$\coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{Y}$$

is surjective on objects.

Proof. It suffices to prove the following: For any field k and object y of \mathcal{Y} over $\text{Spec}(k)$ there exists an integer $d \geq 1$ and an object (U, Z, y, x, α) of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ with $U = \text{Spec}(k)$. Namely, in this case we see that p is surjective on objects in the strong sense that an extension of the field is not needed.

Denote X_y the algebraic space over $U = \text{Spec}(k)$ representing the 2-fibre product $(Sch/U')_{fppf} \times_{y', \mathcal{Y}, F} \mathcal{X}$. By assumption the morphism $X_y \rightarrow \text{Spec}(k)$ is surjective

and locally of finite presentation (and flat). In particular X_y is nonempty. Choose a nonempty affine scheme V and an étale morphism $V \rightarrow X_y$. Note that $V \rightarrow \text{Spec}(k)$ is (flat), surjective, and locally of finite presentation (by Morphisms of Spaces, Definition 27.1). Pick a closed point $v \in V$ where $V \rightarrow \text{Spec}(k)$ is Cohen-Macaulay (i.e., V is Cohen-Macaulay at v), see More on Morphisms, Lemma 17.5. Applying More on Morphisms, Lemma 18.4 we find a regular immersion $Z \rightarrow V$ with $Z = \{v\}$. This implies $Z \rightarrow V$ is a closed immersion. Moreover, it follows that $Z \rightarrow \text{Spec}(k)$ is finite (for example by Algebra, Lemma 118.1). Hence $Z \rightarrow \text{Spec}(k)$ is finite locally free of some degree d . Now $Z \rightarrow X_y$ is unramified as the composition of an closed immersion followed by an étale morphism (see Morphisms of Spaces, Lemmas 35.3, 36.10, and 35.8). Finally, $Z \rightarrow X_y$ is a local complete intersection morphism as a composition of a regular immersion of schemes and an étale morphism of algebraic spaces (see More on Morphisms, Lemma 42.9 and Morphisms of Spaces, Lemmas 36.6 and 34.8 and More on Morphisms of Spaces, Lemmas 36.6 and 36.5). The morphism $Z \rightarrow X_y$ corresponds to an object x of \mathcal{X} over Z together with an isomorphism $\alpha : y|_Z \rightarrow F(x)$. We obtain an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. By what was said above about the morphism $Z \rightarrow X_y$ we see that it actually is an object of the subcategory $\mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y})$ and we win. \square

16. Bootstrapping algebraic stacks

The following theorem is one of the main results of this chapter.

Theorem 16.1. *Let S be a scheme. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

- (1) \mathcal{X} is representable by an algebraic space, and
- (2) F is representable by algebraic spaces, surjective, flat and locally of finite presentation,

then \mathcal{Y} is an algebraic stack.

Proof. By Lemma 4.3 we see that the diagonal of \mathcal{Y} is representable by algebraic spaces. Hence we only need to verify the existence of a 1-morphism $f : \mathcal{V} \rightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$ with \mathcal{V} representable and f surjective and smooth. By Lemma 14.2 we know that

$$\coprod_{d \geq 1} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is an algebraic stack. It follows from Lemma 15.1 and Algebraic Stacks, Lemma 15.5 that

$$\coprod_{d \geq 1} \mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y})$$

is an algebraic stack as well. Choose a representable stack in groupoids \mathcal{V} over $(Sch/S)_{fppf}$ and a surjective and smooth 1-morphism

$$\mathcal{V} \longrightarrow \coprod_{d \geq 1} \mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y}).$$

We claim that the composition

$$\mathcal{V} \longrightarrow \coprod_{d \geq 1} \mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{Y}$$

is smooth and surjective which finishes the proof of the theorem. In fact, the smoothness will be a consequence of Lemmas 12.7 and 15.3 and the surjectivity a consequence of Lemma 15.4. We spell out the details in the following paragraph.

By construction $\mathcal{V} \rightarrow \coprod_{d \geq 1} \mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces, surjective, and smooth (and hence also locally of finite presentation and formally smooth by the general principle Algebraic Stacks, Lemma 10.9 and More on Morphisms of Spaces, Lemma 16.6). Applying Lemmas 5.3, 6.3, and 7.3 we see that $\mathcal{V} \rightarrow \coprod_{d \geq 1} \mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y})$ is limit preserving on objects, formally smooth on objects, and surjective on objects. The 1-morphism $\coprod_{d \geq 1} \mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y}) \rightarrow \mathcal{Y}$ is

- (1) limit preserving on objects: this is Lemma 12.7 for $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \rightarrow \mathcal{Y}$ and we combine it with Lemmas 15.1, 5.4, and 5.2 to get it for $\mathcal{H}_{d, lci}(\mathcal{X}/\mathcal{Y}) \rightarrow \mathcal{Y}$,
- (2) formally smooth on objects by Lemma 15.3, and
- (3) surjective on objects by Lemma 15.4.

Using Lemmas 5.2, 6.2, and 7.2 we conclude that the composition $\mathcal{V} \rightarrow \mathcal{Y}$ is limit preserving on objects, formally smooth on objects, and surjective on objects. Using Lemmas 5.3, 6.3, and 7.3 we see that $\mathcal{V} \rightarrow \mathcal{Y}$ is locally of finite presentation, formally smooth, and surjective. Finally, using (via the general principle Algebraic Stacks, Lemma 10.9) the infinitesimal lifting criterion (More on Morphisms of Spaces, Lemma 16.6) we see that $\mathcal{V} \rightarrow \mathcal{Y}$ is smooth and we win. \square

17. Applications

Our first task is to show that the quotient stack $[U/R]$ associated to a “flat and locally finitely presented groupoid” is an algebraic stack. See Groupoids in Spaces, Definition 19.1 for the definition of the quotient stack. The following lemma is preliminary and is the analogue of Algebraic Stacks, Lemma 17.2.

Lemma 17.1. *Let S be a scheme contained in Sch_{fppf} . Let (U, R, s, t, c) be a groupoid in algebraic spaces over S . Assume s, t are flat and locally of finite presentation. Then the morphism $\mathcal{S}_U \rightarrow [U/R]$ is flat, locally of finite presentation, and surjective.*

Proof. Let T be a scheme and let $x : (Sch/T)_{fppf} \rightarrow [U/R]$ be a 1-morphism. We have to show that the projection

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} \longrightarrow (Sch/T)_{fppf}$$

is surjective and smooth. We already know that the left hand side is representable by an algebraic space F , see Algebraic Stacks, Lemmas 17.1 and 10.11. Hence we have to show the corresponding morphism $F \rightarrow T$ of algebraic spaces is surjective, locally of finite presentation, and flat. Since we are working with properties of morphisms of algebraic spaces which are local on the target in the fppf topology we may check this fppf locally on T . By construction, there exists an fppf covering $\{T_i \rightarrow T\}$ of T such that $x|_{(Sch/T_i)_{fppf}}$ comes from a morphism $x_i : T_i \rightarrow U$. (Note that $F \times_T T_i$ represents the 2-fibre product $\mathcal{S}_U \times_{[U/R]} (Sch/T_i)_{fppf}$ so everything is compatible with the base change via $T_i \rightarrow T$.) Hence we may assume that x comes from $x : T \rightarrow U$. In this case we see that

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} = (\mathcal{S}_U \times_{[U/R]} \mathcal{S}_U) \times_{\mathcal{S}_U} (Sch/T)_{fppf} = \mathcal{S}_R \times_{\mathcal{S}_U} (Sch/T)_{fppf}$$

The first equality by Categories, Lemma 29.10 and the second equality by Groupoids in Spaces, Lemma 21.2. Clearly the last 2-fibre product is represented by the algebraic space $F = R \times_{s, U, x} T$ and the projection $R \times_{s, U, x} T \rightarrow T$ is flat and locally of finite presentation as the base change of the flat locally finitely presented

morphism of algebraic spaces $s : R \rightarrow U$. It is also surjective as s has a section (namely the identity $e : U \rightarrow R$ of the groupoid). This proves the lemma. \square

Here is the first main result of this section.

Theorem 17.2. *Let S be a scheme contained in Sch_{fppf} . Let (U, R, s, t, c) be a groupoid in algebraic spaces over S . Assume s, t are flat and locally of finite presentation. Then the quotient stack $[U/R]$ is an algebraic stack over S .*

Proof. We check the two conditions of Theorem 16.1 for the morphism

$$(Sch/U)_{fppf} \longrightarrow [U/R].$$

The first is trivial (as U is an algebraic space). The second is Lemma 17.1. \square

18. When is a quotient stack algebraic?

In Groupoids in Spaces, Section 19 we have defined the quotient stack $[U/R]$ associated to a groupoid (U, R, s, t, c) in algebraic spaces. Note that $[U/R]$ is a stack in groupoids whose diagonal is representable by algebraic spaces (see Bootstrap, Lemma 11.5 and Algebraic Stacks, Lemma 10.11) and such that there exists an algebraic space U and a 1-morphism $(Sch/U)_{fppf} \rightarrow [U/R]$ which is an “fppf surjection” in the sense that it induces a map on presheaves of isomorphism classes of objects which becomes surjective after sheafification. However, it is not the case that $[U/R]$ is an algebraic stack in general. This is not a contradiction with Theorem 16.1 as the 1-morphism $(Sch/U)_{fppf} \rightarrow [U/R]$ is not representable by algebraic spaces in general, and if it is it may not be flat and locally of finite presentation.

The easiest way to make examples of non-algebraic quotient stacks is to look at quotients of the form $[S/G]$ where S is a scheme and G is a group scheme over S acting trivially on S . Namely, we will see below (Lemma 18.3) that if $[S/G]$ is algebraic, then $G \rightarrow S$ has to be flat and locally of finite presentation. An explicit example can be found in Examples, Section 44.

Lemma 18.1. *Let S be a scheme and let B be an algebraic space over S . Let (U, R, s, t, c) be a groupoid in algebraic spaces over B . The quotient stack $[U/R]$ is an algebraic stack if and only if there exists a morphism of algebraic spaces $g : U' \rightarrow U$ such that*

- (1) *the composition $U' \times_{g, U, t} R \rightarrow R \xrightarrow{s} U$ is a surjection of sheaves, and*
- (2) *the morphisms $s', t' : R' \rightarrow U'$ are flat and locally of finite presentation where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via g .*

Proof. First, assume that $g : U' \rightarrow U$ satisfies (1) and (2). Property (1) implies that $[U'/R'] \rightarrow [U/R]$ is an equivalence, see Groupoids in Spaces, Lemma 24.2. By Theorem 17.2 the quotient stack $[U'/R']$ is an algebraic stack. Hence $[U/R]$ is an algebraic stack too, see Algebraic Stacks, Lemma 12.4.

Conversely, assume that $[U/R]$ is an algebraic stack. We may choose a scheme W and a surjective smooth 1-morphism

$$f : (Sch/W)_{fppf} \longrightarrow [U/R].$$

By the 2-Yoneda lemma (Algebraic Stacks, Section 5) this corresponds to an object ξ of $[U/R]$ over W . By the description of $[U/R]$ in Groupoids in Spaces, Lemma 23.1 we can find a surjective, flat, locally finitely presented morphism $b : U' \rightarrow W$

of schemes such that $\xi' = b^*\xi$ corresponds to a morphism $g : U' \rightarrow U$. Note that the 1-morphism

$$f' : (Sch/U')_{fppf} \longrightarrow [U/R],$$

corresponding to ξ' is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Lemma 10.5. Hence $(Sch/U')_{fppf} \times_{[U/R]} (Sch/U')_{fppf}$ which is represented by the algebraic space

$$Isom_{[U/R]}(\mathrm{pr}_0^*\xi', \mathrm{pr}_1^*\xi') = (U' \times_S U') \times_{(g \circ \mathrm{pr}_0, g \circ \mathrm{pr}_1), U \times_S U} R = R'$$

(see Groupoids in Spaces, Lemma 21.1 for the first equality; the second is the definition of restriction) is flat and locally of finite presentation over U' via both s' and t' (by base change, see Algebraic Stacks, Lemma 10.6). By this description of R' and by Algebraic Stacks, Lemma 16.1 we obtain a canonical fully faithful 1-morphism $[U'/R'] \rightarrow [U/R]$. This 1-morphism is essentially surjective because f' is flat, locally of finite presentation, and surjective (see Stacks, Lemma 4.8); another way to prove this is to use Algebraic Stacks, Remark 16.3. Finally, we can use Groupoids in Spaces, Lemma 24.2 to conclude that the composition $U' \times_{g,U,t} R \rightarrow R \xrightarrow{s} U$ is a surjection of sheaves. \square

Lemma 18.2. *Let S be a scheme and let B be an algebraic space over S . Let G be a group algebraic space over B . Let X be an algebraic space over B and let $a : G \times_B X \rightarrow X$ be an action of G on X over B . The quotient stack $[X/G]$ is an algebraic stack if and only if there exists a morphism of algebraic spaces $\varphi : X' \rightarrow X$ such that*

- (1) $G \times_B X' \rightarrow X$, $(g, x') \mapsto a(g, \varphi(x'))$ is a surjection of sheaves, and
- (2) the two projections $X'' \rightarrow X'$ of the algebraic space X'' given by the rule

$$T \longmapsto \{(x'_1, g, x'_2) \in (X' \times_B G \times_B X')(T) \mid \varphi(x'_1) = a(g, \varphi(x'_2))\}$$

are flat and locally of finite presentation.

Proof. This lemma is a special case of Lemma 18.1. Namely, the quotient stack $[X/G]$ is by Groupoids in Spaces, Definition 19.1 equal to the quotient stack $[X/G \times_B X]$ of the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ associated to the group action in Groupoids in Spaces, Lemma 14.1. There is one small observation that is needed to get condition (1). Namely, the morphism $s : G \times_B X \rightarrow X$ is the second projection and the morphism $t : G \times_B X \rightarrow X$ is the action morphism a . Hence the morphism $h : U' \times_{g,U,t} R \rightarrow R \xrightarrow{s} U$ from Lemma 18.1 corresponds to the morphism

$$X' \times_{\varphi, X, a} (G \times_B X) \xrightarrow{\mathrm{pr}_1} X$$

in the current setting. However, because of the symmetry given by the inverse of G this morphism is isomorphic to the morphism

$$(G \times_B X) \times_{\mathrm{pr}_1, X, \varphi} X' \xrightarrow{a} X$$

of the statement of the lemma. Details omitted. \square

Lemma 18.3. *Let S be a scheme and let B be an algebraic space over S . Let G be a group algebraic space over B . Endow B with the trivial action of G . Then the quotient stack $[B/G]$ is an algebraic stack if and only if G is flat and locally of finite presentation over B .*

Proof. If G is flat and locally of finite presentation over B , then $[B/G]$ is an algebraic stack by Theorem 17.2.

Conversely, assume that $[B/G]$ is an algebraic stack. By Lemma 18.2 and because the action is trivial, we see there exists an algebraic space B' and a morphism $B' \rightarrow B$ such that (1) $B' \rightarrow B$ is a surjection of sheaves and (2) the projections

$$B' \times_B G \times_B B' \rightarrow B'$$

are flat and locally of finite presentation. Note that the base change $B' \times_B G \times_B B' \rightarrow G \times_B B'$ of $B' \rightarrow B$ is a surjection of sheaves also. Thus it follows from Descent on Spaces, Lemma 7.1 that the projection $G \times_B B' \rightarrow B'$ is flat and locally of finite presentation. By (1) we can find an fppf covering $\{B_i \rightarrow B\}$ such that $B_i \rightarrow B$ factors through $B' \rightarrow B$. Hence $G \times_B B_i \rightarrow B_i$ is flat and locally of finite presentation by base change. By Descent on Spaces, Lemmas 10.11 and 10.8 we conclude that $G \rightarrow B$ is flat and locally of finite presentation. \square

19. Algebraic stacks in the étale topology

Let S be a scheme. Instead of working with stacks in groupoids over the big fppf site $(Sch/S)_{fppf}$ we could work with stacks in groupoids over the big étale site $(Sch/S)_{\acute{e}tale}$. All of the material in Algebraic Stacks, Sections 4, 5, 6, 7, 8, 9, 10, and 11 makes sense for categories fibred in groupoids over $(Sch/S)_{\acute{e}tale}$. Thus we get a second notion of an algebraic stack by working in the étale topology. This notion is (a priori) weaker than the notion introduced in Algebraic Stacks, Definition 12.1 since a stack in the fppf topology is certainly a stack in the étale topology. However, the notions are equivalent as is shown by the following lemma.

Lemma 19.1. *Denote the common underlying category of Sch_{fppf} and $Sch_{\acute{e}tale}$ by Sch_α (see Sheaves on Stacks, Section 4 and Topologies, Remark 9.1). Let S be an object of Sch_α . Let*

$$p : \mathcal{X} \rightarrow Sch_\alpha/S$$

be a category fibred in groupoids with the following properties:

- (1) \mathcal{X} is a stack in groupoids over $(Sch/S)_{\acute{e}tale}$,
- (2) the diagonal $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces⁵, and
- (3) there exists $U \in \text{Ob}(Sch_\alpha/S)$ and a 1-morphism $(Sch/U)_{\acute{e}tale} \rightarrow \mathcal{X}$ which is surjective and smooth.

Then \mathcal{X} is an algebraic stack in the sense of Algebraic Stacks, Definition 12.1.

Proof. Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Stacks, Definition 12.1 are independent of the topology. This is true because these properties involve only the notion of a 2-fibre product of categories fibred in groupoids, 1- and 2-morphisms of categories fibred in groupoids, the notion of a 1-morphism of categories fibred in groupoids representable by algebraic spaces, and what it means for such a 1-morphism to be surjective and smooth. Thus all we have to prove is that an étale stack in groupoids \mathcal{X} with properties (2) and (3) is also an fppf stack in groupoids.

⁵Here we can either mean sheaves in the étale topology whose diagonal is representable and which have an étale surjective covering by a scheme or algebraic spaces as defined in Algebraic Spaces, Definition 6.1. Namely, by Bootstrap, Lemma 12.1 there is no difference.

Using (2) let R be an algebraic space representing

$$(Sch_\alpha/U) \times_{\mathcal{X}} (Sch_\alpha/U)$$

By (3) the projections $s, t : R \rightarrow U$ are smooth. Exactly as in the proof of Algebraic Stacks, Lemma 16.1 there exists a groupoid in spaces (U, R, s, t, c) and a canonical fully faithful 1-morphism $[U/R]_{\acute{e}tale} \rightarrow \mathcal{X}$ where $[U/R]_{\acute{e}tale}$ is the étale stackification of presheaf in groupoids

$$T \longmapsto (U(T), R(T), s(T), t(T), c(T))$$

Claim: If $V \rightarrow T$ is a surjective smooth morphism from an algebraic space V to a scheme T , then there exists an étale covering $\{T_i \rightarrow T\}$ refining the covering $\{V \rightarrow T\}$. This follows from More on Morphisms, Lemma 28.7 or the more general Sheaves on Stacks, Lemma 18.10. Using the claim and arguing exactly as in Algebraic Stacks, Lemma 16.2 it follows that $[U/R]_{\acute{e}tale} \rightarrow \mathcal{X}$ is an equivalence.

Next, let $[U/R]$ denote the quotient stack in the fppf topology which is an algebraic stack by Algebraic Stacks, Theorem 17.3. Thus we have 1-morphisms

$$U \rightarrow [U/R]_{\acute{e}tale} \rightarrow [U/R].$$

Both $U \rightarrow [U/R]_{\acute{e}tale} \cong \mathcal{X}$ and $U \rightarrow [U/R]$ are surjective and smooth (the first by assumption and the second by the theorem) and in both cases the fibre product $U \times_{\mathcal{X}} U$ and $U \times_{[U/R]} U$ is representable by R . Hence the 1-morphism $[U/R]_{\acute{e}tale} \rightarrow [U/R]$ is fully faithful (since morphisms in the quotient stacks are given by morphisms into R , see Groupoids in Spaces, Section 23).

Finally, for any scheme T and morphism $t : T \rightarrow [U/R]$ the fibre product $V = T \times_{[U/R]} U$ is an algebraic space surjective and smooth over T . By the claim above there exists an étale covering $\{T_i \rightarrow T\}_{i \in I}$ and morphisms $T_i \rightarrow V$ over T . This proves that the object t of $[U/R]$ over T comes étale locally from U . We conclude that $[U/R]_{\acute{e}tale} \rightarrow [U/R]$ is an equivalence of stacks in groupoids over $(Sch/S)_{\acute{e}tale}$ by Stacks, Lemma 4.8. This concludes the proof. \square

20. Other chapters

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(2) Conventions	(19) Injectives
(3) Set Theory	(20) Cohomology of Sheaves
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