RESOLUTION OF SURFACES

Contents

1. Introduction

This chapter discusses resolution of singularities of surfaces following Lipman [Lip78] and following the exposition in [Art86].

2. A trace map in positive characteristic

In this section p will be a prime number. Let R be an $\mathbf{F}_p$-algebra. Let M be an R-module and let $D : R \to M$ be a derivation. Given an $a \in R$ set $A = R[x]/(x^p - a)$. Define an R-linear map

$$\mathrm{Tr}_{x,D} : \Omega_{A/R} \longrightarrow M$$

by the rule

$$x^i \mathrm{d}x \longmapsto \begin{cases} 0 & \text{if} \quad 0 \leq i \leq p-2, \\ D(a) & \text{if} \quad i = p-1 \end{cases}$$

This makes sense as $\Omega_{A/R}$ is a free R-module with basis $x^i\mathrm{d}x$, $0 \leq i \leq p-1$. The following lemma implies that the trace map is well defined, i.e., independent of the choice of the coordinate x.

Lemma 2.1. *Let $\varphi : R[x]/(x^p - a) \to R[y]/(y^p - b)$ be an R-algebra homomorphism. Then $Tr_{x,D} = Tr_{y,D} \circ \varphi$.*

Proof. Say $\varphi(x) = \lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1}$ with $\lambda_i \in R$. The condition that mapping x to $\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1}y^{p-1}$ induces an R-algebra homomorphism $R[x]/(x^p - a) \to R[y]/(y^p - b)$ is equivalent to the condition that

$$a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$$

in the ring R. Consider the polynomial ring

$$R_{univ} = \mathbf{F}_p[b, \lambda_0, \ldots, \lambda_{p-1}]$$

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with the element $a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$ and with its universal derivation given by

$$D_{univ} = \mathrm{d} : R_{univ} \longrightarrow M_{univ} = \Omega_{R_{univ}/\mathbf{F}_p}$$

Consider the universal algebra map $\varphi_{univ} : R_{univ}[x]/(x^p - a) \to R_{univ}[y]/(y^p - b)$ given by mapping x to $\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1}$. We obtain a canonical maps

$$R_{univ} \longrightarrow R, \quad M_{univ} \longrightarrow M$$

compatible with derivations by sending b, λ_i to b, λ_i and sending $\mathrm{d}b, \mathrm{d}\lambda_i$ to $D(b), D(\lambda_i)$. By construction the maps

$$R_{univ}[x]/(x^p - a) \to R[x]/(x^p - a), \quad R_{univ}[y]/(y^p - b) \to R[y]/(y^p - b)$$

are compatible with the trace maps. Hence it suffices to prove the lemma for the map φ_{univ}. We will do this by evaluating $\mathrm{Tr}_{y,D}(\varphi(x)^i \mathrm{d}\varphi(x))$ for $i = 0, \ldots, p-1$.

The case $0 \leq i \leq p-2$. Expand

$$(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^i (\lambda_1 + 2\lambda_2 y + \ldots + (p-1)\lambda_{p-1} y^{p-2})$$

in the ring $R[y]/(y^p - b)$. We have to show that the coefficient of y^{p-1} is zero. For this it suffices to show that the expression above as a polynomial in y has vanishing coefficients in front of the powers y^{pk-1}. Then we write our polynomial as

$$\frac{\mathrm{d}}{(i+1)\mathrm{d}y}(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^{i+1}$$

and indeed the coefficients of y^{kp-1} are all zero.

The case $i = p-1$. Expand

$$(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^{p-1} (\lambda_1 + 2\lambda_2 y + \ldots + (p-1)\lambda_{p-1} y^{p-2})$$

in the ring $R[y]/(y^p - b)$. To finish the proof we have to show that the coefficient of y^{p-1} times $D(b)$ is $D(a)$. Here we use that R is S/pS where $S = \mathbf{Z}[b, \xi_j, \lambda_0, \ldots, \lambda_{p-1}, \xi_{ij}]$. Then the above, as a polynomial in y, is equal to

$$\frac{\mathrm{d}}{p\mathrm{d}y}(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^p$$

Since $\frac{\mathrm{d}}{\mathrm{d}y}(y^{pk}) = pky^{pk-1}$ it suffices to understand the coefficients of y^{pk} in the polynomial $(\lambda_0 + \lambda_1 y + \ldots + \lambda_{p-1} y^{p-1})^p$ modulo p. The sum of these terms gives

$$\lambda_0^p + \lambda_1^p y^p + \ldots + \lambda_{p-1}^p y^{p(p-1)} \bmod p$$

Whence we see that we obtain after applying the operator $\frac{\mathrm{d}}{p\mathrm{d}y}$ and after reducing modulo $y^p - b$ the value

$$\lambda_1^p + 2\lambda_2^p b + \ldots + (p-1)\lambda_{p-1} b^{p-2}$$

for the coefficient of y^{p-1} we wanted to compute. Now because $a = \lambda_0^p + \lambda_1^p b + \ldots + \lambda_{p-1}^p b^{p-1}$ in R we obtain that

$$D(a) = (\lambda_1^p + 2\lambda_2^p b + \ldots + (p-1)\lambda_{p-1}^p b^{p-2}) D(b)$$

in R. This proves that the coefficient of y^{p-1} is as desired. □

Lemma 2.2. *Let R be a Noetherian normal domain with fraction field K. Let $a \in K$ be an element such that there exists a derivation $D : R \to R$ with $D(a) \neq 0$. Then the integral closure of R in $L = K[x]/(x^p - a)$ is finite over R.*

Proof. After replacing x by fx and a by f^pa for some $f \in R$ we may assume $a \in R$. Hence also $D(a) \in R$. We will show by induction on $i \leq p-1$ that if

$$y = a_0 + a_1x + \ldots + a_ix^i, \quad a_j \in K$$

is integral over R, then $D(a)^ia_j \in R$. Thus the integral closure is contained in the finite R-module with basis $D(a)^{-p+1}x^j$, $j = 0, \ldots, p-1$. Since R is Noetherian this proves the lemma.

If $i = 0$, then $y = a_0$ is integral over R if and only if $a_0 \in R$ and the statement is true. Suppose the statement holds for some $i < p-1$ and suppose that

$$y = a_0 + a_1x + \ldots + a_{i+1}x^{i+1}, \quad a_j \in K$$

is integral over R. Then

$$y^p = a_0^p + a_1^pa + \ldots + a_{i+1}^pa^{i+1}$$

is an element of R (as it is in K and integral over R). Applying D we obtain

$$(a_1^p + 2a_2^pa + \ldots + (i+1)a_{i+1}^pa^i)D(a)$$

is in R. Hence it follows that

$$D(a)a_1 + 2D(a)a_2x + \ldots + (i+1)D(a)a_{i+1}x^i$$

is integral over R. By induction we find $D(a)^{i+1}a_j \in R$ for $j = 1, \ldots, i+1$. (Here we use that $1, \ldots, i+1$ are invertible.) Hence $D(a)^{i+1}a_0$ is also in R because it is the difference of y and $\sum_{j>0} D(a)^{i+1}a_jx^j$ which are integral over R (since x is integral over R as $a \in R$). $\square$

3. Modifications

Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. We set $S = \mathrm{Spec}(A)$ and $U = S \setminus \{\mathfrak{m}\}$. In this section we will consider the category

$$\left\{ f : X \longrightarrow S \;\middle|\; \begin{array}{c} X \text{ is an algebraic space} \\ f \text{ is a proper morphism} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array} \right\} \tag{3.0.1}$$

A morphism from X/S to X'/S will be a morphism of algebraic spaces $X \to X'$ compatible with the structure morphisms over S. In Restricted Power Series, Section 13 we have seen that this category only depends on the completion of A and we have proven some elementary properties of objects in this category. In this section we specifically study cases where $\dim(A) \leq 2$ or where the dimension of the closed fibre is at most 1.

Lemma 3.1. *Let $(A, \mathfrak{m}, \kappa)$ be a 2-dimensional Noetherian local domain such that $U = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ is a normal scheme. Then any modification $f : X \to S$ (as in Spaces over Fields, Definition 6.1) is a morphism as in (3.0.1).*

Proof. Let $f : X \to S$ be a modification. We have to show that $f^{-1}(U) \to U$ is an isomorphism. By Spaces over Fields, Lemma 6.2 there exists a nonempty open $V \subset S$ such that $f^{-1}(V) \to V$ is an isomorphism. Since X is integral we see that $f^{-1}(V)$ is dense in X. Note that every closed point u of U has codimension 1, i.e., that $\dim(\mathcal{O}_{U,u}) = 1$. Thus we may apply Spaces over Fields, Lemma 4.4 to see that $f^{-1}(U) \to U$ is finite. In particular $f^{-1}(U)$ is a scheme. Then $f^{-1}(U) \to U$ is an isomorphism, see Morphisms, Lemma 48.16. $\square$

Lemma 3.2. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $g : X \to Y$ be a morphism in the category (3.0.1). If the induced morphism $X_\kappa \to Y_\kappa$ of special fibres is a closed immersion, then g is a closed immersion.*

Proof. This is a special case of More on Morphisms of Spaces, Lemma 37.3. □

Lemma 3.3. *Let $(A, \mathfrak{m}, \kappa)$ be a complete Noetherian local ring. Let X be an algebraic space over* $\mathrm{Spec}(A)$*. If $X \to \mathrm{Spec}(A)$ is proper and $\dim(X_\kappa) \leq 1$, then X is a scheme projective over A.*

Proof. By Spaces over Fields, Lemma 7.5 the algebraic space X_κ is a scheme. Hence X_κ is a proper scheme of dimension ≤ 1 over κ. By Varieties, Lemma 23.4 we see that X_κ is H-projective over κ. Let $\mathcal{L}$ be an ample invertible sheaf on X_κ.

We are going to show that $\mathcal{L}$ lifts to a compatible system $\{\mathcal{L}_n\}$ of invertible sheaves on the nth infinitesimal neighbourhoods

$$X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}^n)$$

of $X_\kappa = X_1$. Recall that the étale sites of X_κ and all X_n are canonically equivalent, see More on Morphisms of Spaces, Lemma 8.6. In the rest of the proof we do not distinguish between sheaves on X_n and sheaves on X_m or X_κ. Suppose, given a lift $\mathcal{L}_n$ to X_n. We consider the exact sequence

$$1 \to (1 + \mathfrak{m}^n\mathcal{O}_X/\mathfrak{m}^{n+1}\mathcal{O}_X)^* \to \mathcal{O}_{X_{n+1}}^* \to \mathcal{O}_{X_n}^* \to 1$$

of sheaves on X_{n+1}. We have $(1+\mathfrak{m}^n\mathcal{O}_X/\mathfrak{m}^{n+1}\mathcal{O}_X)^* \cong \mathfrak{m}^n\mathcal{O}_X/\mathfrak{m}^{n+1}\mathcal{O}_X$ as abelian sheaves on X_{n+1}. The class of $\mathcal{L}_n$ in $H^1(X_n, \mathcal{O}_{X_n}^*)$ (see Cohomology on Sites, Lemma 7.1) can be lifted to an element of $H^1(X_{n+1}, \mathcal{O}_{X_{n+1}}^*)$ if and only if the obstruction in $H^2(X_{n+1}, \mathfrak{m}^n\mathcal{O}_X/\mathfrak{m}^{n+1}\mathcal{O}_X)$ is zero. Note that $\mathfrak{m}^n\mathcal{O}_X/\mathfrak{m}^{n+1}\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_{X_\kappa}$-module on X_κ. Hence its étale cohomology agrees with its cohomology on the scheme X_κ, see Descent, Proposition 7.10. However, as X_κ is a Noetherian scheme of dimension ≤ 1 this cohomology group vanishes (Cohomology, Proposition 21.6).

By Grothendieck's algebraization theorem (Cohomology of Schemes, Theorem 23.4) we find a projective morphism of schemes $Y \to \mathrm{Spec}(A)$ and a compatible system of isomorphisms $X_n \to Y_n$. (Here we use the assumption that A is complete.) By More on Morphisms of Spaces, Lemma 32.3 we see that $X \cong Y$ and the proof is complete. □

Lemma 3.4. *If $(A, \mathfrak{m}, \kappa)$ is a complete Noetherian local domain of dimension 2, then every modification of* $\mathrm{Spec}(A)$ *is projective over A.*

Proof. By Lemma 3.3 it suffices to show that the special fibre of any modification X of $\mathrm{Spec}(A)$ has dimension ≤ 1. Let $U \to X$ be an étale morphism with U affine. Since $X \to \mathrm{Spec}(A)$ is a modification (Spaces over Fields, Definition 6.1) we see that a dense open of U is étale over A. In particular, every generic point η of an irreducible component U' of U maps to the generic point of $\mathrm{Spec}(A)$ and $f.f.(A) \subset \kappa(\eta)$ is finite separable. If $u \in U'$ is a closed point lying over $\mathfrak{m} \in \mathrm{Spec}(A)$, then by the dimension formula we see that

$$\dim(\mathcal{O}_{U',u}) \leq \dim(A) = 2,$$

see Morphisms, Lemma 31.1. Since $\eta \notin U'_\kappa$, the dimension of U'_κ can be at most 1 as desired. □

4. Quadratic transformations

In this section we study what happens when we blow up a nonsingular point on a surface. We hesitate the formally define such a morphism as a *quadratic transformation* as on the one hand often other names are used and on the other hand the phrase "quadratic transformation" is sometimes used with a different meaning.

Lemma 4.1. *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. There is a closed immersion*

$$r : X \longrightarrow \mathbf{P}^1_S$$

over S such that $\mathcal{O}_X(1) = r^*\mathcal{O}_{\mathbf{P}^1_S}(1)$ and such that $r|_E : E \to \mathbf{P}^1_\kappa$ is an isomorphism.

Proof. As A is regular of dimension 2 we can write $\mathfrak{m} = (x, y)$. Then x and y placed in degree 1 generate the Rees algebra $\bigoplus_{n \geq 0} \mathfrak{m}^n$ over A. Recall that $X = \text{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n)$, see Divisors, Lemma 18.2. Thus the surjection

$$A[T_0, T_1] \longrightarrow \bigoplus\nolimits_{n \geq 0} \mathfrak{m}^n, \quad T_0 \mapsto x,\ T_1 \mapsto y$$

of graded A-algebras induces a closed immersion $r : X \to \mathbf{P}^1_S = \text{Proj}(A[T_0, T_1])$ such that $\mathcal{O}_X(1) = r^*\mathcal{O}_{\mathbf{P}^1_S}(1)$, see Constructions, Lemma 11.5. To prove the final statement note that

$$\left(\bigoplus\nolimits_{n \geq 0} \mathfrak{m}^n\right) \otimes_A \kappa = \bigoplus\nolimits_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \kappa[\overline{x}, \overline{y}]$$

a polynomial algebra, see Algebra, Lemma 102.1. This proves that the fibre of $X \to S$ over $\text{Spec}(\kappa)$ is equal to $\text{Proj}(\kappa[\overline{x}, \overline{y}]) = \mathbf{P}^1_\kappa$, see Constructions, Lemma 11.6. Recall that E is the closed subscheme of X defined by $\mathfrak{m}\mathcal{O}_X$, i.e., $E = X_\kappa$. By our choice of the morphism r we see that $r|_E$ in fact produces the identification of $E = X_\kappa$ with the special fibre of $\mathbf{P}^1_S \to S$. $\square$

Lemma 4.2. *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Then X is an irreducible regular scheme.*

Proof. Observe that X is integral by Divisors, Lemma 18.7 and Algebra, Lemma 102.2. To see X is regular it suffices to check that $\mathcal{O}_{X,x}$ is regular for closed points $x \in X$, see Properties, Lemma 9.2. Let $x \in X$ be a closed point. Since f is proper x maps to $\mathfrak{m}$, i.e., x is a point of the exceptional divisor E. Then E is an effective Cartier divisor and $E \cong \mathbf{P}^1_\kappa$. Thus if $f \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is a local equation for E, then $\mathcal{O}_{X,x}/(f) \cong \mathcal{O}_{\mathbf{P}^1_\kappa,x}$. Since $\mathbf{P}^1_\kappa$ is covered by two affine opens which are the spectrum of a polynomial ring over κ, we see that $\mathcal{O}_{\mathbf{P}^1_\kappa,x}$ is regular by Algebra, Lemma 110.1. We conclude by Algebra, Lemma 102.7. $\square$

Lemma 4.3. *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension 2. Let $f : X \to S = \text{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

1. *$H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$,*
2. *$H^1(X, \mathcal{O}_X(n)) = 0$ for $n \geq -1$,*
3. *$H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ or $\mathcal{F}(1)$ is globally generated,*
4. *$H^0(X, \mathcal{O}_X(n)) = \mathfrak{m}^{\max(0,n)}$,*
5. *$\text{length}_A H^1(X, \mathcal{O}_X(n)) = -n(-n-1)/2$ if $n < 0$.*

Proof. If $\mathfrak{m} = (x, y)$, then X is covered by the spectra of the affine blowup algebras $A[\frac{\mathfrak{m}}{x}]$ and $A[\frac{\mathfrak{m}}{y}]$ because x and y placed in degree 1 generate the Rees algebra $\bigoplus \mathfrak{m}^n$ over A. See Divisors, Lemma 18.2 and Constructions, Lemma 8.9. Since X is separated by Constructions, Lemma 8.8 we see that cohomology of quasi-coherent sheaves vanishes in degrees ≥ 2 by Cohomology of Schemes, Lemma 4.2.

Let $i : E \to X$ be the exceptional divisor, see Divisors, Definition 18.1. Recall that $\mathcal{O}_X(-E) = \mathcal{O}_X(1)$ is f-relatively ample, see Divisors, Lemma 18.4. Hence we know that $H^1(X, \mathcal{O}_X(-nE)) = 0$ for some $n > 0$, see Cohomology of Schemes, Lemma 15.4. Consider the filtration

$$\mathcal{O}_X(-nE) \subset \mathcal{O}_X(-(n-1)E) \subset \ldots \subset \mathcal{O}_X(-E) \subset \mathcal{O}_X \subset \mathcal{O}_X(E)$$

The succesive quotients are the sheaves

$$\mathcal{O}_X(-tE)/\mathcal{O}_X(-(t+1)E) = \mathcal{O}_X(t)/\mathcal{I}(t) = i_*\mathcal{O}_E(t)$$

where $\mathcal{I} = \mathcal{O}_X(-E)$ is the ideal sheaf of E. By Lemma 4.1 we have $E = \mathbf{P}^1_\kappa$ and $\mathcal{O}_E(1)$ indeed corresponds to the usual Serre twist of the structure sheaf on $\mathbf{P}^1$. Hence the cohomology of $\mathcal{O}_E(t)$ vanishes in degree 1 for $t \geq -1$, see Cohomology of Schemes, Lemma 8.1. Since this is equal to $H^1(X, i_*\mathcal{O}_E(t))$ (by Cohomology of Schemes, Lemma 2.4) we find that $H^1(X, \mathcal{O}_X(-(t+1)E)) \to H^1(X, \mathcal{O}_X(-tE))$ is surjective for $t \geq -1$. Hence

$$0 = H^1(X, \mathcal{O}_X(-nE)) \longrightarrow H^1(X, \mathcal{O}_X(-tE)) = H^1(X, \mathcal{O}_X(t))$$

is surjective for $t \geq -1$ which proves (2).

Let $\mathcal{F}$ be globally generated. This means there exists a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus\nolimits_{i \in I} \mathcal{O}_X \to \mathcal{F} \to 0$$

Note that $H^1(X, \bigoplus_{i \in I} \mathcal{O}_X) = \bigoplus_{i \in I} H^1(X, \mathcal{O}_X)$ by Cohomology, Lemma 20.1. By part (2) we have $H^1(X, \mathcal{O}_X) = 0$. If $\mathcal{F}(1)$ is globally generated, then we can find a surjection $\bigoplus_{i \in I} \mathcal{O}_X(-1) \to \mathcal{F}$ and argue in a similar fashion. In other words, part (3) follows from part (2).

For part (4) we note that for all n large enough we have $\Gamma(X, \mathcal{O}_X(n)) = \mathfrak{m}^n$, see Cohomology of Schemes, Lemma 15.3. If $n \geq 0$, then we can use the short exact sequence

$$0 \to \mathcal{O}_X(n) \to \mathcal{O}_X(n-1) \to i_*\mathcal{O}_E(n-1) \to 0$$

and the vanishing of H^1 for the sheaf on the left to get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}^{\max(0,n)} \arrow[r] \arrow[d] & \mathfrak{m}^{\max(0,n-1)} \arrow[r] \arrow[d] & \mathfrak{m}^{\max(0,n)}/\mathfrak{m}^{\max(0,n-1)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X, \mathcal{O}_X(n)) \arrow[r] & \Gamma(X, \mathcal{O}_X(n-1)) \arrow[r] & \Gamma(E, \mathcal{O}_E(n-1)) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. In fact, the rows are exact also for $n < 0$ because in this case the groups on the right are zero. In the proof of Lemma 4.1 we have seen that the right vertical arrow is an isomorphism (details omitted). Hence if the left vertical arrow is an isomorphism, so is the middle one. In this way we see that (4) holds by descending induction on n.

Finally, we prove (5) by descending induction on n and the sequences

$$0 \to \mathcal{O}_X(n) \to \mathcal{O}_X(n-1) \to i_*\mathcal{O}_E(n-1) \to 0$$

Namely, for $n \geq -1$ we already know $H^1(X, \mathcal{O}_X(n)) = 0$. Since

$$H^1(X, i_*\mathcal{O}_E(-2)) = H^1(E, \mathcal{O}_E(-2)) = H^1(\mathbf{P}^1_\kappa, \mathcal{O}(-2)) \cong \kappa$$

by Cohomology of Schemes, Lemma 8.1 which has length 1 as an A-module, we conclude from the long exact cohomology sequence that (5) holds for $n = -2$. And so on and so forth. $\square$

Lemma 4.4. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2. Let $f : X \to S = \mathrm{Spec}(A)$ be the blowing up of A in $\mathfrak{m}$. Let $\mathfrak{m}^n \subset I \subset \mathfrak{m}$ be an ideal. Let $d \geq 0$ be the largest integer such that*

$$I\mathcal{O}_X \subset \mathcal{O}_X(-dE)$$

where E is the exceptional divisor. Set $\mathcal{I}' = I\mathcal{O}_X(dE) \subset \mathcal{O}_X$. Then $d > 0$, the sheaf $\mathcal{O}_X/\mathcal{I}'$ is supported in finitely many closed points $x_1, \ldots, x_r$ of X, and

$$\begin{aligned} \mathit{length}_A(A/I) &> \mathit{length}_A\Gamma(X, \mathcal{O}_X/\mathcal{I}') \\ &\geq \sum\nolimits_{i=1,\ldots,r} \mathit{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i}) \end{aligned}$$

Proof. Since $I \subset \mathfrak{m}$ we see that every element of I vanishes on E. Thus we see that $d \geq 1$. On the other hand, since $\mathfrak{m}^n \subset I$ we see that $d \leq n$. Consider the short exact sequence

$$0 \to I\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_X/I\mathcal{O}_X \to 0$$

Since $I\mathcal{O}_X$ is globally generated, we see that $H^1(X, I\mathcal{O}_X) = 0$ by Lemma 4.3. Hence we obtain a surjection $A/I \to \Gamma(X, \mathcal{O}_X/I\mathcal{O}_X)$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(-dE)/I\mathcal{O}_X \to \mathcal{O}_X/I\mathcal{O}_X \to \mathcal{O}_X/\mathcal{O}_X(-dE) \to 0$$

By Divisors, Lemma 9.24 we see that $\mathcal{O}_X(-dE)/I\mathcal{O}_X$ is supported in finitely many closed points of X. In particular, this coherent sheaf has vanishing higher cohomology groups (detail omitted). Thus in the following diagram

$$\begin{tikzcd}
 & & A/I \arrow[d] & & \\
0 \arrow[r] & \Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) \arrow[r] & \Gamma(X, \mathcal{O}_X/I\mathcal{O}_X) \arrow[r] & \Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-dE)) \arrow[r] & 0
\end{tikzcd}$$

the bottom row is exact and the vertical arrow surjective. We have

$$\mathrm{length}_A\Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) < \mathrm{length}_A(A/I)$$

since $\Gamma(X, \mathcal{O}_X/\mathcal{O}_X(-dE))$ is nonzero. Namely, the image of $1 \in \Gamma(X, \mathcal{O}_X)$ is nonzero as $d > 0$.

To finish the proof we translate the results above into the statements of the lemma. Since $\mathcal{O}_X(dE)$ is invertible we have

$$\mathcal{O}_X/\mathcal{I}' = \mathcal{O}_X(-dE)/I\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(dE).$$

Thus $\mathcal{O}_X/\mathcal{I}'$ and $\mathcal{O}_X(-dE)/I\mathcal{O}_X$ are supported in the same set of finitely many closed points, say $x_1, \ldots, x_r \in E \subset X$. Moreover we obtain

$$\Gamma(X, \mathcal{O}_X(-dE)/I\mathcal{O}_X) = \bigoplus \mathcal{O}_X(-dE)_{x_i}/I\mathcal{O}_{X,x_i} \cong \bigoplus \mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i} = \Gamma(X, \mathcal{O}_X/\mathcal{I}')$$

because an invertible module over a local ring is trivial. Thus we obtain the strict inequality. We also get the second because

$$\mathrm{length}_A(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i}) \geq \mathrm{length}_{\mathcal{O}_{X,x_i}}(\mathcal{O}_{X,x_i}/\mathcal{I}'_{x_i})$$

as is immediate from the definition of length. □

5. Quadratic transformations of spaces

Using the result above we can prove that blowups in points dominate any modification of a regular 2 dimensional algebraic space.

Let X be a decent algebraic space over some base scheme S. Let $x \in |X|$ be a closed point. By Decent Spaces, Lemma 12.5 we can represent x by a closed immersion $i : \mathrm{Spec}(k) \to X$. Then the *blowing up of X at x* means the blowing up of X in the closed subspace $Z = i(\mathrm{Spec}(k)) \subset X$.

Lemma 5.1. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $T \subset |X|$ be a finite set of closed points x such that (1) X is regular at x and (2) the local ring of X at x has dimension 2. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals such that $\mathcal{O}_X/\mathcal{I}$ is supported on T. Then there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf.

Proof. Say $T = \{x_1, \ldots, x_r\}$. Pick an étale morphism $U \to X$ where U is a scheme with points $u_i \in U$ lying over x_i. By Decent Spaces, Lemma 10.3 the points u_i are closed points. After shrinking U we may assume these are the only points of U mapping to T. The local rings $\mathcal{O}_{U,u_i}$ are regular local of dimension 2, see Properties of Spaces, Definitions 23.2 and 20.2. Let $I_i \subset \mathcal{O}_{U,u_i}$ be the stalk of $\mathcal{I}|_U$ at u_i. Set

$$n_i = \mathrm{length}_{\mathcal{O}_{U,u_i}}(\mathcal{O}_{U,u_i}/I_i)$$

This is finite as $\mathcal{O}_X/\mathcal{I}$ is supported on T and hence $\mathcal{O}_{U,u_i}/I_i$ has support equal to $\{\mathfrak{m}_{u_i}\}$ (see Algebra, Lemma 61.3). We are going to use induction on $\sum n_i$. If $n_i = 0$ for all i, then $\mathcal{I} = \mathcal{O}_X$ and we are done.

Suppose $n_i > 0$. Let $X' \to X$ be the blowing up of X in x_i (see discussion above the lemma). Since $U \to X$ is étale and u_i is the unique point of U lying over x we see that $U' = U \times_X X'$ is the blowup of U in u_i, see Divisors on Spaces, Lemma 6.3. Since $\mathrm{Spec}(\mathcal{O}_{U,u_i}) \to U$ is flat we see that $U' \times_U \mathrm{Spec}(\mathcal{O}_{U,u_i})$ is the blowup of the ring $\mathcal{O}_{U,u_i}$ in the maximal ideal. Hence both squares in the commutative diagram

$$\begin{tikzcd}
\mathrm{Proj}(\bigoplus_{d \geq 0} \mathfrak{m}_{u_i}^d) \arrow[r] \arrow[d] & U' \arrow[r] \arrow[d] & X' \arrow[d] \\
\mathrm{Spec}(\mathcal{O}_{U,u_i}) \arrow[r] & U \arrow[r] & X
\end{tikzcd}$$

are cartesian. Let $E \subset X'$, $E' \subset U'$, $E'' \subset \mathrm{Proj}(\bigoplus_{d \geq 0} \mathfrak{m}_{u_i}^d)$ be the exceptional divisors. Let $d \geq 1$ be the integer found in Lemma 4.4 for the ideal $\mathcal{I}_i \subset \mathcal{O}_{U,u_i}$. Since the horizontal arrows in the diagram are flat, since $E'' \to E$ is surjective, and since E'' is the pullback of E, we see that

$$\mathcal{I}\mathcal{O}_{X'} \subset \mathcal{O}_{X'}(-dE)$$

(some details omitted). Set $\mathcal{I}' = \mathcal{I}\mathcal{O}_{X'}(dE) \subset \mathcal{O}_{X'}$. Then we see that $\mathcal{O}_{X'}/\mathcal{I}'$ is supported in finitely many closed points $T' \subset |X'|$ because this holds over $X \setminus \{x_i\}$ and for the pullback to $\text{Proj}(\bigoplus_{d \geq 0} \mathfrak{m}_{u_i}^d)$. The final assertion of Lemma 4.4 tells us that the sum of the lengths of the stalks $\mathcal{O}_{U',u'}/\mathcal{I}'\mathcal{O}_{U',u'}$ for u' lying over u_i is $< n_i$. Hence the sum of the lengths has decreased.

By induction hypothesis, there exists a sequence

$$X_n' \to \ldots \to X_1' \to X'$$

of blowups at closed points lying over T' such that $\mathcal{I}'\mathcal{O}_{X_n'}$ is invertible. Since $\mathcal{I}'\mathcal{O}_{X'}(-dE) = \mathcal{I}\mathcal{O}_{X'}$, we see that $\mathcal{I}\mathcal{O}_{X_n'} = \mathcal{I}'\mathcal{O}_{X_n'}(-d(f')^{-1}E)$ where $f' : X_n' \to X'$ is the composition. Note that $(f')^{-1}E$ is an effective Cartier divisor by Divisors on Spaces, Lemma 6.8. Thus we are done by Divisors on Spaces, Lemma 2.7. $\square$

Lemma 5.2. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $T \subset |X|$ be a finite set of closed points x such that (1) X is regular at x and (2) the local ring of X at x has dimension 2. Let $f : Y \to X$ be a proper morphism of algebraic spaces which is an isomorphism over $U = X \setminus T$. Then there exists a sequence*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T and a factorization $X_n \to Y \to X$ of the composition.

Proof. By More on Morphisms of Spaces, Lemma 28.3 there exists a U-admissible blowup $X' \to X$ which dominates $Y \to X$. Hence we may assume there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{O}_X/\mathcal{I}$ is supported on T and such that Y is the blowing up of X in $\mathcal{I}$. By Lemma 5.1 there exists a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

where $X_{i+1} \to X_i$ is the blowing up of X_i at a closed point x_i lying above a point of T such that $\mathcal{I}\mathcal{O}_{X_n}$ is an invertible ideal sheaf. By the universal property of blowing up (Divisors on Spaces, Lemma 6.5) we find the desired factorization. $\square$

6. Examples

Some examples related to the results earlier in this chapter.

Example 6.1. Let k be a field. The ring $A = k[x, y, z]/(x^r + y^s + z^t)$ is a UFD for r, s, t pairwise coprime integers. Namely, since $x^r + y^s + z^t$ is irreducible A is a domain. The element z is a prime element, i.e., generates a prime ideal in A. On the other hand, if $r = 1 + ers$ for some e, then

$$A[1/z] \cong k[x', y', 1/z]$$

where $x' = x/z^{es}$, $y' = y/z^{et}$ and $z = (x')^r + (y')^s$. Thus $A[1/z]$ is a localization of a polynomial ring and hence a UFD. It follows from an argument of Nagata that A is a UFD. See Algebra, Lemma 116.7. A similar argument can be given if r is not congruent to 1 modulo rs.

Example 6.2. The ring $A = \mathbf{C}[[x, y, z]]/(x^r + y^s + z^t)$ is not a UFD when $r < s < t$ are pairwise coprime integers and not equal to $2, 3, 5$. For example consider the special case $A = \mathbf{C}[[x, y, z]]/(x^2 + y^5 + z^7)$. Consider the maps

$$\psi_\zeta : \mathbf{C}[[x, y, z]]/(x^2 + y^5 + z^7) \to \mathbf{C}[[t]]$$

given by

$$x \mapsto t^7, \quad y \mapsto t^3, \quad z \mapsto -\zeta t^2(1+t)^{1/7}$$

where ζ is a 7th root of unity. The kernel $\mathfrak{p}_\zeta$ of ψ_ζ is a height one prime, hence if A is a UFD, then it is principal, say given by $f_\zeta \in \mathbf{C}[[x,y,z]]$. Note that $V(x^3 - y^7) = \bigcup V(\mathfrak{p}_\zeta)$ and $A/(x^3 - y^7)$ is reduced away from the closed point. Hence, still assuming A is a UFD, we would obtain

$$\prod\nolimits_\zeta f_\zeta = u(x^3 - y^7) + a(x^2 + y^5 + z^7) \quad \text{in} \quad \mathbf{C}[[x,y,z]]$$

for some unit $u \in \mathbf{C}[[x,y,z]]$ and some element $a \in \mathbf{C}[[x,y,z]]$. After scaling by a constant we may assume $u(0,0,0) = 1$. Note that the left hand side vanishes to order 7. Hence $a = -x \bmod \mathfrak{m}^2$. But then we get a term xy^5 on the right hand side which does not occur on the left hand side. A contradiction.

Example 6.3. There exists an excellent 2-dimensional Noetherian local ring and a modification $X \to S = \mathrm{Spec}(A)$ which is not a scheme. We sketch a construction. Let X be a normal surface over $\mathbf{C}$ with a unique singular point $x \in X$. Assume that there exists a resolution $\pi : X' \to X$ such that the exceptional fibre $C = \pi^{-1}(x)_{red}$ is a smooth projective curve. Furthermore, assume there exists a point $c \in C$ such that if $\mathcal{O}_C(nc)$ is in the image of $\mathrm{Pic}(X') \to \mathrm{Pic}(C)$, then $n = 0$. Then we let $X'' \to X'$ be the blowing up in the nonsingular point c. Let $C' \subset X''$ be the strict transform of C and let $E \subset X''$ be the exceptional fibre. By Artin's results ([Art70]; use for example [Mum61] to see that the normal bundle of C' is negative) we can blow down the curve C' in X'' to obtain an algebraic space X'''. Picture

$$\begin{tikzcd} & X'' \arrow[dl] \arrow[dr] & \\ X' \arrow[dr] & & X''' \arrow[dl] \\ & X & \end{tikzcd}$$

We claim that X''' is not a scheme. This provides us with our example because X''' is a scheme if and only if the base change of X''' to $A = \mathcal{O}_{X,x}$ is a scheme (details omitted). If X''' where a scheme, then the image of C' in X''' would have an affine neighbourhood. The complement of this neighbourhood would be an effective Cartier divisor on X''' (because X''' is nonsingular apart from 1 point). This effective Cartier divisor would correspond to an effective Cartier divisor on X'' meeting E and avoiding C'. Taking the image in X' we obtain an effective Cartier divisor meeting C (set theoretically) in c. This is impossible as no multiple of c is the restriction of a Cartier divisor by assumption.

To finish we have to find such a singular surface X. We can just take X to be the affine surface given by

$$x^3 + y^3 + z^3 + x^4 + y^4 + z^4 = 0$$

in $\mathbf{A}^3_{\mathbf{C}} = \mathrm{Spec}(\mathbf{C}[x,y,z])$ and singular point $(0,0,0)$. Then $(0,0,0)$ is the only singular point. Blowing up X in the maximal ideal corresponding to $(0,0,0)$ we find three charts each isomorphic to the smooth affine surface

$$1 + s^3 + t^3 + x(1 + s^4 + t^4) = 0$$

which is nonsingular with exceptional divisor C given by $x = 0$. The reader will recognize C as an elliptic curve. Finally, the surface X is rational as projection from $(0, 0, 0)$ shows, or because in the equation for the blow up we can solve for x. Finally, the Picard group of a nonsingular rational surface is countable, whereas the Picard group of an elliptic curve over the complex numbers is uncountable. Hence we can find a closed point c as indicated.

7. Other chapters

References

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