SIMPLICIAL SPACES

Contents

1. Introduction

This chapter develops some theory concerning simplicial topological spaces, simplicial ringed spaces, simplicial schemes, and simplicial algebraic spaces. The theory of simplicial spaces sometimes allows one to prove local to global principles which appear difficult to prove in other ways. Some example applications can be found in the papers [Fal03], [Kie72], and [Del74].

We assume throughout that the reader is familiar with the basic concepts and results of the chapter Simplical Methods, see Simplicial, Section 1. In particular, we continue to write X and not $X_\bullet$ for a simplicial object.

2. Simplicial topological spaces

A *simplicial space* is a simplicial object in the category of topological spaces where morphisms are continuous maps of topological spaces. (We will use "simplicial algebraic space" to refer to simplicial objects in the category of algebraic spaces.) We may picture a simplicial space X as follows

$$
\begin{tikzcd}
X_2 \arrow[r, shift left=3] \arrow[r, leftarrow, shift left=1] \arrow[r, shift right=1] \arrow[r, leftarrow, shift right=3] \arrow[r, shift right=5] & X_1 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & X_0
\end{tikzcd}
$$

Here there are two morphisms $d^1_0, d^1_1 : X_1 \to X_0$ and a single morphism $s^0_0 : X_0 \to X_1$, etc. It is important to keep in mind that $d^n_i : X_n \to X_{n-1}$ should be thought of as a "projection forgetting the ith coordinate" and $s^n_j : X_n \to X_{n+1}$ as the diagonal map repeating the jth coordinate.

This is a chapter of the Stacks Project, version 714e994, compiled on Oct 28, 2014.

Let X be a simplicial space. We associate a site X_{Zar}[1] to X as follows.

(1) An object of X_{Zar} is an open U of X_n for some n,
(2) a morphism $U \to V$ of X_{Zar} is given by a $\varphi : [m] \to [n]$ where n, m are such that $U \subset X_n$, $V \subset X_m$ and φ is such that $X(\varphi)(U) \subset V$, and
(3) a covering $\{U_i \to U\}$ in X_{Zar} means that $U, U_i \subset X_n$ are open, the maps $U_i \to U$ are given by $\mathrm{id} : [n] \to [n]$, and $U = \bigcup U_i$.

Note that in particular, if $U \to V$ is a morphism of X_{Zar} give by φ, then $X(\varphi) : X_n \to X_m$ does in fact induce a continuous map $U \to V$ of topological spaces.
It is clear that the above is a special case of a construction that associates to any diagram of topological spaces a site. We formulate the obligatory lemma.

Lemma 2.1. *Let X be a simplicial space. Then X_{Zar} as defined above is a site.*

Proof. Omitted. □

Let X be a simplicial space. Let $\mathcal{F}$ be a sheaf on X_{Zar}. It is clear from the definition of coverings, that the restriction of $\mathcal{F}$ to the opens of X_n defines a sheaf $\mathcal{F}_n$ on the topological space X_n. For every $\varphi : [m] \to [n]$ the restriction maps of $\mathcal{F}$ for pairs $U \subset X_n$, $V \subset X_m$ with $X(\varphi)(U) \subset V$, define an $X(\varphi)$-map $\mathcal{F}(\varphi) : \mathcal{F}_m \to \mathcal{F}_n$, see Sheaves, Definition 21.7. Moreover, given $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ we have

$$\mathcal{F}(\psi) \circ \mathcal{F}(\varphi) = \mathcal{F}(\varphi \circ \psi)$$

(LHS uses composition of f-maps, see Sheaves, Definition 21.9). Clearly, the converse is true as well: if we have a system $(\{\mathcal{F}_n\}_{n \geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi \in \mathrm{Arrows}(\Delta)})$ as above, satisfying the displayed equalities, then we obtain a sheaf on X_{Zar}.

Lemma 2.2. *Let X be a simplicial space. There is an equivalence of categories between*

(1) *$Sh(X_{Zar})$, and*
(2) *category of systems $(\mathcal{F}_n, \mathcal{F}(\varphi))$ described above.*

Proof. See discussion above. □

Lemma 2.3. *Let $f : Y \to X$ be a morphism of simplicial spaces. Then the functor $u : X_{Zar} \to Y_{Zar}$ which associates to the open $U \subset X_n$ the open $f_n^{-1}(U) \subset Y_n$ defines a morphism of sites $f_{Zar} : Y_{Zar} \to X_{Zar}$.*

Proof. It is clear that u is a continuous functor. Hence we obtain functors $f_{Zar,*} = u^s$ and $f_{Zar}^{-1} = u_s$, see Sites, Section 15. To see that we obtain a morphism of sites we have to show that u^s is exact. We will use Sites, Lemma 15.5 to see this. Let $V \subset Y_n$ be an open subset. The category $\mathcal{I}_V^u$ (see Sites, Section 5) consists of pairs (U, φ) where $\varphi : [m] \to [n]$ and $U \subset X_m$ open such that $Y(\varphi)(V) \subset f_m^{-1}(U)$. Moreover, a morphism $(U, \varphi) \to (U', \varphi')$ is given by a $\psi : [m'] \to [m]$ such that $X(\psi)(U) \subset U'$ and $\varphi \circ \psi = \varphi'$. It is our task to show that $\mathcal{I}_V^u$ is cofiltered.

We verify the conditions of Categories, Definition 20.1. Condition (1) holds because $(X_n, \mathrm{id}_{[n]})$ is an object. Let (U, φ) be an object. The condition $Y(\varphi)(V) \subset f_m^{-1}(U)$ is equivalent to $V \subset f_n^{-1}(X(\varphi)^{-1}(U))$. Hence we obtain a morphism $(X(\varphi)^{-1}(U), \mathrm{id}_{[n]}) \to (U, \varphi)$ given by setting $\psi = \varphi$. Moreover, given a pair of objects of the form $(U, \mathrm{id}_{[n]})$ and $(U', \mathrm{id}_{[n]})$ we see there exists an object, namely $(U \cap U', \mathrm{id}_{[n]})$, which maps to both of them. Thus condition (2) holds. To verify

[1] This notation is similar to the notation in Sites, Example 6.4 and Topologies, Definition 3.7.

condition (3) suppose given two morphisms $a, a' : (U, \varphi) \to (U', \varphi')$ given by $\psi, \psi' : [m'] \to [m]$. Then precomposing with the morphism $(X(\varphi)^{-1}(U), \mathrm{id}_{[n]}) \to (U, \varphi)$ given by φ equalizes a, a' because $\varphi \circ \psi = \varphi' = \varphi \circ \psi'$. This finishes the proof. $\square$

Lemma 2.4. *Let $f : Y \to X$ be a morphism of simplicial spaces. In terms of the description of sheaves in Lemma 2.2 the morphism f_{Zar} of Lemma 2.3 can be described as follows.*

(1) *If $\mathcal{G}$ is a sheaf on Y, then $(f_{Zar,*}\mathcal{G})_n = f_{n,*}\mathcal{G}_n$.*
(2) *If $\mathcal{F}$ is a sheaf on X, then $(f_{Zar}^{-1}\mathcal{F})_n = f_n^{-1}\mathcal{F}_n$.*

Proof. The first part is immediate from the definitions. For the second part, note that in the proof of Lemma 2.3 we have shown that for a $V \subset Y_n$ open the category $(\mathcal{I}_V^u)^{opp}$ contains as a cofinal subcategory the category of opens $U \subset X_n$ with $f_n^{-1}(U) \supset V$ and morphisms given by inclusions. Hence we see that the restriction of $u_p\mathcal{F}$ to opens of Y_n is the presheaf $f_{n,p}\mathcal{F}_n$ as defined in Sheaves, Lemma 21.3. Since $f_{Zar}^{-1}\mathcal{F} = u_s\mathcal{F}$ is the sheafification of $u_p\mathcal{F}$ and since sheafification uses only coverings and since coverings in Y_{Zar} use only inclusions between opens on the same Y_n, the result follows from the fact that $f_n^{-1}\mathcal{F}_n$ is (correspondingly) the sheafification of $f_{n,p}\mathcal{F}_n$, see Sheaves, Section 21. $\square$

Let X be a topological space. In Sites, Example 6.4 we denoted X_{Zar} the site consisting of opens of X with inclusions as morphisms and coverings given by open coverings. We identify the topos $Sh(X_{Zar})$ with the category of sheaves on X.

Lemma 2.5. *Let X be a simplicial space. The functor $X_{n,Zar} \to X_{Zar}$, $U \mapsto U$ is continuous and cocontinuous. The associated morphism of topoi $g : Sh(X_n) \to Sh(X_{Zar})$ satisfies*

(1) *g^{-1} associates to the sheaf $\mathcal{F}$ on X the sheaf $\mathcal{F}_n$ on X_n,*
(2) *g^{-1} has a left adjoint $g_!^{Sh}$ which commutes with finite connected limits,*
(3) *$g^{-1} : Ab(X_{Zar}) \to Ab(X_n)$ has a left adjoint $g_! : Ab(X_n) \to Ab(X_{Zar})$ which is exact.*

Proof. Besides the properties of our functor mentioned in the statement, the category $X_{n,Zar}$ has fibre products and equalizers and the functor commutes with them (beware that X_{Zar} does not have all fibre products). Hence the lemma follows from the discussion in Sites, Sections 19 and 20 and Modules on Sites, Section 16. More precisely, Sites, Lemmas 20.1, 20.5, and 20.6 and Modules on Sites, Lemmas 16.2 and 16.3. $\square$

Lemma 2.6. *Let X be a simplicial space. If $\mathcal{I}$ is an injective abelian sheaf on X_{Zar}, then $\mathcal{I}_n$ is an injective abelian sheaf on X_n.*

Proof. This follows from Homology, Lemma 25.1 and Lemma 2.5. $\square$

Lemma 2.7. *Let $f : Y \to X$ be a morphism of simplicial spaces. Then*

$$\begin{tikzcd} Sh(Y_n) \arrow[r, "f_n"'] \arrow[d] & Sh(X_n) \arrow[d] \\ Sh(Y_{Zar}) \arrow[r, "f_{Zar}"] & Sh(X_{Zar}) \end{tikzcd}$$

is a commutative diagram of topoi.

Proof. Direct from the description of pullback functors in Lemmas 2.4 and 2.5. □

Let X be a topological space. Denote $X_\bullet$ the constant simplicial topological space with value X. By Lemma 2.2 a sheaf on $X_{\bullet,Zar}$ is the same thing as a cosimplicial object in the category of sheaves on X.

Lemma 2.8. *Let X be a topological space. Let $X_\bullet$ be the constant simplical topological space with value X. The functor*

$$X_{\bullet,Zar} \longrightarrow X_{Zar}, \quad U \longmapsto U$$

is continuous and cocontinuous and defines a morphism of topoi $g : Sh(X_{\bullet,Zar}) \to Sh(X)$ as well as a left adjoint $g_!$ to g^{-1}. We have

1. *g^{-1} associates to a sheaf on X the constant cosimplicial sheaf on X,*
2. *$g_!$ associates to a sheaf $\mathcal{F}$ on $X_{\bullet,Zar}$ the sheaf $\mathcal{F}_0$, and*
3. *g_* associates to a sheaf $\mathcal{F}$ on $X_{\bullet,Zar}$ the equalizer of the two maps $\mathcal{F}_0 \to \mathcal{F}_1$.*

Proof. The statements about the functor are straightforward to verify. The existence of g and $g_!$ follow from Sites, Lemmas 20.1 and 20.5. The description of g^{-1} is immediate from Sites, Lemma 20.5. The description of g_* and $g_!$ follows as the functors given are right and left adjoint to g^{-1}. □

Lemma 2.9. *Let Y be a simplicial space and X a topological space. Let $a : Y \to X$ be an augmentation (Simplicial, Definition 19.1). There is a canonical morphism of topoi*

$$a : Sh(Y_{Zar}) \to Sh(X)$$

which comes from composing the morphism $a_{Zar} : Sh(Y_{Zar}) \to Sh(X_{\bullet,Zar})$ of Lemma 2.3 with the morphism $g : Sh(X_{\bullet,Zar}) \to Sh(X)$ of Lemma 2.8.

Proof. This lemma proves itself. □

Lemma 2.10. *Let X be a simplicial topological space. The complex of abelian presheaves on X_{Zar}*

$$\ldots \to \mathbf{Z}_{X_2} \to \mathbf{Z}_{X_1} \to \mathbf{Z}_{X_0}$$

with boundary $\sum(-1)^i d_i^n$ is a resolution of the constant presheaf $\mathbf{Z}$.

Proof. Let $U \subset X_m$ be an object of X_{Zar}. Then the value of the complex above on U is the complex of abelian groups

$$\ldots \to \mathbf{Z}[\mathrm{Mor}_\Delta([2],[m])] \to \mathbf{Z}[\mathrm{Mor}_\Delta([1],[m])] \to \mathbf{Z}[\mathrm{Mor}_\Delta([0],[m])]$$

In other words, this is the complex associated to the free abelian group on the simplicial set $\Delta[m]$, see Simplicial, Example 11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 25.7, and since "taking free abelian groups" is a functor, we see that the complex above is homotopy equivalent to the free abelian group on $\Delta[0]$ (Simplicial, Remark 25.4 and Lemma 26.2). This complex is acyclic in positive degrees and equal to $\mathbf{Z}$ in degree 0. □

Lemma 2.11. *Let X be a simplicial topological space. Let $\mathcal{F}$ be an abelian sheaf on X. There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_p)$$

converging to $H^{p+q}(X_{Zar}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Consider the double complex with terms

$$A^{p,q} = \mathcal{I}^q(X_p)$$

and first differential given by the alternating sum along the maps d_i^{p+1}-maps $\mathcal{I}_p^q \to \mathcal{I}_{p+1}^q$, see Lemma 2.2. Note that

$$A^{p,q} = \Gamma(X_p, \mathcal{I}_p^q) = \mathrm{Mor}_{PSh}(h_{X_p}, \mathcal{I}^q) = \mathrm{Mor}_{PAb}(\mathbf{Z}_{X_p}, \mathcal{I}^q)$$

Hence it follows from Lemma 2.10 and Cohomology on Sites, Lemma 11.1 that the rows of the double complex are exact in positive degrees and evaluate to $\Gamma(X_{Zar}, \mathcal{I}^q)$ in degree 0. On the other hand, since restriction is exact (Lemma 2.5) the map

$$\mathcal{F}_p \to \mathcal{I}_p^\bullet$$

is a resolution. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on X_p (Lemma 2.6). Hence the cohomology of the columns computes the groups $H^q(X_p, \mathcal{F}_p)$. We conclude by applying Homology, Lemmas 22.6 and 22.7. □

3. Simplicial sites and topoi

It seems natural to define a *simplicial site* as a simplicial object in the (big) category whose objects are sites and whose morphisms are morphisms of sites. See Sites, Definitions 6.2 and 15.1 with composition of morphisms as in Sites, Lemma 15.3. But here are some variants one might want to consider: (a) we could work with cocontinuous functors (see Sites, Sections 19 and 20) between sites instead, (b) we could work in a suitable 2-category of sites where one introduces the notion of a 2-morphism between morphisms of sites, (c) we could work in a 2-category constructed out of cocontinuous functors. Instead of picking one of these variants as a definition we will simply develop theory as needed.

Certainly a *simplicial topos* should probably be defined as a pseudo-functor from Δ^{opp} into the 2-category of topoi. See Categories, Definition 27.5 and Sites, Section 16 and 35. We will try to avoid working with such a beast if possible.

Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. This means that for every morphism $\varphi : [m] \to [n]$ of Δ we have a morphism of sites $f_\varphi : \mathcal{C}_n \to \mathcal{C}_m$. This morphism is given by a continuous functor in the oppsite direction which we will denote $u_\varphi : \mathcal{C}_m \to \mathcal{C}_n$.

Lemma 3.1. *Let $\mathcal{C}$ be a simplicial object in the category of sites. With notation as above we construct a site $\mathcal{C}_{site}$ as follows.*

(1) *An object of $\mathcal{C}_{site}$ is an object U of $\mathcal{C}_n$ for some n,*
(2) *a morphism $(\varphi, f) : U \to V$ of $\mathcal{C}_{site}$ is given by a map $\varphi : [m] \to [n]$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$, $V \in \mathrm{Ob}(\mathcal{C}_m)$ and a morphism $f : U \to u_\varphi(V)$ of $\mathcal{C}_n$, and*
(3) *a covering $\{(id, f_i) : U_i \to U\}$ in $\mathcal{C}_{site}$ is given by an n and a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}_n$.*

Proof. Composition of $(\varphi, f) : U \to V$ with $(\psi, g) : V \to W$ is given by $(\varphi \circ \psi, u_\varphi(g) \circ f)$. This uses that $u_\varphi \circ u_\psi = u_{\varphi \circ \psi}$.

Let $\{(\mathrm{id}, f_i) : U_i \to U\}$ be a covering as in (3) and let $(\varphi, g) : W \to U$ be a morphism with $W \in \mathrm{Ob}(\mathcal{C}_m)$. We claim that

$$W \times_{(\varphi,g),U,(\mathrm{id},f_i)} U_i = W \times_{g,u_\varphi(U),u_\varphi(f_i)} u_\varphi(U_i)$$

in the category $\mathcal{C}_{site}$. This makes sense as by our definition of morphisms of sites, the required fibre products in $\mathcal{C}_m$ exist since u_φ transforms coverings into coverings. The same reasoning implies the claim (details omitted). Thus we see that the collection of coverings is stable under base change. The other axioms of a site are immediate. $\square$

Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors. This means that for every morphism $\varphi : [m] \to [n]$ of Δ we have a cocontinuous functor denoted $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$.

Lemma 3.2. *Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors. With notation as above, assume the functors $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ have property P of Sites, Remark 19.5. Then we can construct a site $\mathcal{C}_{site}$ as follows.*

1. *An object of $\mathcal{C}_{site}$ is an object U of $\mathcal{C}_n$ for some n,*
2. *a morphism $(\varphi, f) : U \to V$ of $\mathcal{C}_{site}$ is given by a map $\varphi : [m] \to [n]$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$, $V \in \mathrm{Ob}(\mathcal{C}_m)$ and a morphism $f : u_\varphi(U) \to V$ of $\mathcal{C}_m$, and*
3. *a covering $\{(id, f_i) : U_i \to U\}$ in $\mathcal{C}_{site}$ is given by an n and a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}_n$.*

Proof. Composition of $(\varphi, f) : U \to V$ with $(\psi, g) : V \to W$ is given by $(\varphi \circ \psi, g \circ u_\psi(f))$. This uses that $u_\psi \circ u_\varphi = u_{\varphi\circ\psi}$.

Let $\{(\mathrm{id}, f_i) : U_i \to U\}$ be a covering as in (3) and let $(\varphi, g) : W \to U$ be a morphism with $W \in \mathrm{Ob}(\mathcal{C}_m)$. We claim that

$$W \times_{(\varphi,g),U,(\mathrm{id},f_i)} U_i = W \times_{g,U,f_i} U_i$$

in the category $\mathcal{C}_{site}$ where the right hand side is the object of $\mathcal{C}_m$ defined in Sites, Remark 19.5 which exists by property P. Compatibility of this type of fibre product with compositions of functors implies the claim (details omitted). Since the family $\{W \times_{g,U,f_i} U_i \to W\}$ is a covering of $\mathcal{C}_m$ by property P we see that the collection of coverings is stable under base change. The other axioms of a site are immediate. $\square$

Situation 3.3. Here we have one of the following two cases

(A) $\mathcal{C}$ is a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. For every morphism $\varphi : [m] \to [n]$ of Δ we have a morphism of sites $f_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ given by a continuous functor $u_\varphi : \mathcal{C}_m \to \mathcal{C}_n$.

(B) $\mathcal{C}$ is a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors having property P of Sites, Remark 19.5. For every morphism $\varphi : [m] \to [n]$ of Δ we have a cocontinuous functor $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ which induces a morphism of topoi $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$.

As usual we will denote f_φ^{-1} and $f_{\varphi,*}$ the pullback and pushforward. We let $\mathcal{C}_{site}$ denote the site defined in Lemma 3.1 (case A) or Lemma 3.2 (case B).

Let $\mathcal{C}$ be as in Situation 3.3. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}_{site}$. It is clear from the definition of coverings, that the restriction of $\mathcal{F}$ to the objects of $\mathcal{C}_n$ defines a sheaf $\mathcal{F}_n$ on the site $\mathcal{C}_n$. For every $\varphi : [m] \to [n]$ the restriction maps of $\mathcal{F}$ along the morphisms $(\varphi, f) : U \to V$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$ and $V \in \mathrm{Ob}(\mathcal{C}_m)$ define an element $\mathcal{F}(\varphi)$ of

$$\mathrm{Mor}_{Sh(\mathcal{C}_m)}(\mathcal{F}_m, f_{\varphi,*}\mathcal{F}_n) = \mathrm{Mor}_{Sh(\mathcal{C}_n)}(f_\varphi^{-1}\mathcal{F}_m, \mathcal{F}_n)$$

Moreover, given $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ we have

$$f_\varphi^{-1}\mathcal{F}(\psi) \circ \mathcal{F}(\varphi) = \mathcal{F}(\varphi \circ \psi)$$

Clearly, the converse is true as well: if we have a system $(\{\mathcal{F}_n\}_{n\geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi \in \text{Arrows}(\Delta)})$ as above, satisfying the displayed equalities, then we obtain a sheaf on $\mathcal{C}_{site}$.

Lemma 3.4. *In Situation 3.3 there is an equivalence of categories between*

(1) *$Sh(\mathcal{C}_{site})$, and*
(2) *category of systems $(\mathcal{F}_n, \mathcal{F}(\varphi))$ described above.*

In particular, the topos $Sh(\mathcal{C}_{site})$ only depends on the topoi $Sh(\mathcal{C}_n)$ and the morphisms of topoi f_φ.

Proof. See discussion above. □

Lemma 3.5. *In Situation 3.3 the functor $\mathcal{C}_n \to \mathcal{C}_{site}$, $U \mapsto U$ is continuous and cocontinuous. The associated morphism of topoi $g : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{site})$ satisfies*

(1) *g^{-1} associates to the sheaf $\mathcal{F}$ on $\mathcal{C}_{site}$ the sheaf $\mathcal{F}_n$ on $\mathcal{C}_n$,*
(2) *g^{-1} has a left adjoint $g_!^{Sh}$ which commutes with finite connected limits, and*
(3) *$g^{-1} : Ab(\mathcal{C}_{site}) \to Ab(\mathcal{C}_n)$ has a left adjoint $g_! : Ab(\mathcal{C}_n) \to Ab(\mathcal{C}_{site})$ which is exact.*

Proof. It is clear that functor $\mathcal{C}_n \to \mathcal{C}_{site}$ is continuous and cocontinuous. Hence part (1) and the existence of $g_!^{Sh}$ and $g_!$ follows from Sites, Lemmas 20.1 and 20.5 and Modules on Sites, Lemmas 16.2 and 16.4.

Next, we come to the exactness properties of $g_!^{Sh}$ and $g_!$. Perhaps the most straightforward way to prove this is to give a formula for these functors. If $\mathcal{G}$ is a sheaf on $\mathcal{C}_n$, then we claim $\mathcal{H} = g_!^{Sh}\mathcal{G}$ is the sheaf on $\mathcal{C}_{site}$ whose degree m part is the sheaf

$$\mathcal{H}_m = \coprod\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$$

Given a map $\psi : [m] \to [m']$ the map $\mathcal{H}(\psi) : f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is given on components by the identifications

$$f_\psi^{-1} f_\varphi^{-1}\mathcal{G} \to f_{\psi\circ\varphi}^{-1}\mathcal{G}$$

Observe that given a map $a : \mathcal{H} \to \mathcal{F}$ of sheaves on $\mathcal{C}_{site}$ we obtain a map $\mathcal{G} \to \mathcal{F}_n$ corresponding to the restriction of a_n to the component $\mathcal{G}$ in $\mathcal{H}_n$. Conversely, given $b : \mathcal{G} \to \mathcal{H}_n$ we can define $a : \mathcal{H} \to \mathcal{F}$ by letting a_m be the map which on components

$$f_\varphi^{-1}\mathcal{G} \to \mathcal{F}_m$$

uses the maps adjoint to $\mathcal{F}(\varphi) \circ f_\varphi^{-1} b$. We omit the arguments showing these two constructions give mutually inverse maps

$$\text{Mor}_{Sh(\mathcal{C}_n)}(\mathcal{G}, \mathcal{F}_n) = \text{Mor}_{Sh(\mathcal{C}_{site})}(\mathcal{H}, \mathcal{F})$$

thus verifying the claim above. If $\mathcal{G}$ is an abelian sheaf on $\mathcal{C}_n$, then $g_!\mathcal{G}$ is the abelian sheaf on $\mathcal{C}_{site}$ whose degree m part is the sheaf

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$$

with transition maps defined exactly as above. By definition of the site $\mathcal{C}_{site}$ we see that these functors have the desired exactness properties and we conclude. □

Lemma 3.6. *In Situation 3.3. If $\mathcal{I}$ is an injective abelian sheaf on $\mathcal{C}_{site}$, then $\mathcal{I}_n$ is an injective abelian sheaf on $\mathcal{C}_n$.*

Proof. This follows from Homology, Lemma 25.1 and Lemma 3.5. □

Let $\mathcal{C}$ be as in Situation 3.3. In statement of the following lemmas we will let $g_n : \mathcal{C}_n \to \mathcal{C}_{site}$ be the functor of Lemma 3.5. If $\varphi : [m] \to [n]$ is a morphism of Δ, then the diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C}_n) \arrow[rr, "f_\varphi"'] \arrow[dr, "g_n"'] & & Sh(\mathcal{C}_m) \arrow[dl, "g_m"] \\ & Sh(\mathcal{C}_{site}) & \end{tikzcd}$$

is not commutative, but there is a 2-morphism $g_n \to g_m \circ f_\varphi$ coming from the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$. See Sites, Section 35.

Lemma 3.7. *In Situation 3.3 and with notation as above there is a complex*

$$\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}$$

of abelian sheaves on $\mathcal{C}_{site}$ which forms a resolution of the constant sheaf with value $\mathbf{Z}$ on $\mathcal{C}_{site}$.

Proof. We will use the description of the functors $g_{n!}$ in the proof of Lemma 3.5 without further mention. As maps of the complex we take $\sum (-1)^i d^n_i$ where $d^n_i : g_{n!}\mathbf{Z} \to g_{n-1!}\mathbf{Z}$ is the adjoint to the map $\mathbf{Z} \to \bigoplus_{[n-1]\to[n]} \mathbf{Z} = g_n^{-1}g_{n-1!}\mathbf{Z}$ corresponding to the factor labeled with $\delta^n_i : [n-1] \to [n]$. Then g_m^{-1} applied to the complex gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha \in \mathrm{Mor}_\Delta([2],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha \in \mathrm{Mor}_\Delta([1],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha \in \mathrm{Mor}_\Delta([0],[m])} \mathbf{Z}$$

on $\mathcal{C}_m$. In other words, this is the complex associated to the free abelian sheaf on the simplicial set $\Delta[m]$, see Simplicial, Example 11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 25.7, and since "taking free abelian sheaf on" is a functor, we see that the complex above is homotopy equivalent to the free abelian sheaf on $\Delta[0]$ (Simplicial, Remark 25.4 and Lemma 26.2). This complex is acyclic in positive degrees and equal to $\mathbf{Z}$ in degree 0. □

Lemma 3.8. *In Situation 3.3. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{site}$. There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_1^{p,q} = H^q(\mathcal{C}_p, \mathcal{F}_p)$$

converging to $H^{p+q}(\mathcal{C}_{site}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Consider the double complex with terms

$$A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}^q_p)$$

and first differential given by the alternating sum along the maps d^{p+1}_i-maps $\mathcal{I}^q_p \to \mathcal{I}^q_{p+1}$, see Lemma 3.4. Note that

$$A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}^q_p) = \mathrm{Mor}_{Ab(\mathcal{C}_{site})}(g_{p!}\mathbf{Z}, \mathcal{I}^q)$$

Hence it follows from Lemma 3.7 that the rows of the double complex are exact in positive degrees and evaluate to $\Gamma(\mathcal{C}_{site}, \mathcal{I}^q)$ in degree 0. On the other hand, since restriction is exact (Lemma 3.5) the map

$$\mathcal{F}_p \to \mathcal{I}^\bullet_p$$

is a resolution. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on $\mathcal{C}_p$ (Lemma 3.6). Hence the cohomology of the columns computes the groups $H^q(\mathcal{C}_p, \mathcal{F}_p)$. We conclude by applying Homology, Lemmas 22.6 and 22.7. □

4. Simplicial semi-representable objects

Let $\mathcal{C}$ be a site. Recall that $\text{SR}(\mathcal{C})$ denotes the category of semi-representable objects of $\mathcal{C}$. See Hypercoverings, Definition 2.1. For an object $K = \{U_i\}_{i\in I}$ of $\text{SR}(\mathcal{C})$ we will use the notation

$$\mathcal{C}/K = \coprod\nolimits_{i\in I} \mathcal{C}/U_i$$

and we will call it the *localization of* $\mathcal{C}$ *at* K. There is a natural structure of a site on this category, with coverings inherited from the localizations $\mathcal{C}/U_i$ (and whence from $\mathcal{C}$). If $f : K \to L$ is a morphism of $\text{SR}(\mathcal{C})$, then we obtain a cocontinuous functor

$$f : \mathcal{C}/K \longrightarrow \mathcal{C}/L$$

by applying the construction of Sites, Lemma 24.7 to the components. More precisely, if $f = (\alpha, f_i)$ where $K = \{U_i\}_{i\in I}$, $L = \{V_j\}_{j\in J}$, $\alpha : I \to J$, and $f_i : U_i \to V_{\alpha(i)}$ then f maps the component $\mathcal{C}/U_i$ into the component $\mathcal{C}/V_{\alpha(i)}$ via the construction of the aforementioned lemma.

Let K be a simplicial object of $\text{SR}(\mathcal{C})$. By the construction above we obtain a simplicial object $n \mapsto \mathcal{C}/K_n$ in the category whose objects are sites and whose morphisms are cocontinuous functors of sites. Since these localization functors satisfy the assumption of Lemma 3.2 by Sites, Remark 24.10 we obtain a site $(\mathcal{C}/K)_{site}$.

We can describe this site explicitly as follows. Say $K_n = \{U_{n,i}\}_{i\in I_n}$ and that for $\varphi : [m] \to [n]$ the morphism $K(\varphi) : K_n \to K_m$ is given by $a(\varphi) : I_n \to I_m$ and $f_{\varphi,i} : U_{n,i} \to U_{m,a(\varphi)(i)}$ for $i \in I_n$. Then we have

(1) an object of $\mathcal{C}/K$ corresponds to an object $(U/U_{n,i})$ of $\mathcal{C}/U_{n,i}$ for some n and some $i \in I_n$,
(2) a morphism between U and V is a pair (φ, f) where $\varphi : [m] \to [n]$ with $U/U_{n,i}$ and $V/U_{m,a(\varphi)(i)}$ and $f : U \to V$ is a morphism of $\mathcal{C}$ such that

$$\begin{tikzcd} U \arrow[r, "f"'] \arrow[d] & V \arrow[d] \\ U_{n,i} \arrow[r, "f_{\varphi,i}"] & U_{m,a(\varphi)(i)} \end{tikzcd}$$

is commutative, and
(3) a covering $\{(\text{id}, f_j) : U_j \to U\}$ is given by an n and $i \in I_n$ and objects $U/U_{n,i}$, $U_j/U_{n,i}$ such that $\{f_j : U_j \to U\}$ is a covering of $\mathcal{C}$.

Lemma 4.1. *Let $\mathcal{C}$ be a site. Let K be a simplicial object of $SR(\mathcal{C})$. If $\mathcal{C}$ has fibre products, then $\mathcal{C}/K$ can also be viewed as a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. The construction of Lemma 3.1 then produces the same site as the construction above.*

Proof. Given a morphism of objects $U \to V$ of $\mathcal{C}$ the localization morphism $j : \mathcal{C}/U \to \mathcal{C}/U$ is a left adjoint to the base change functor $\mathcal{C}/V \to \mathcal{C}/U$. The base change functor is continuous and induces the same morphism of topoi as j. See

Sites, Lemma 26.3. Argueing as above we can use this to define a morphism of sites $\mathcal{C}/A \to \mathcal{C}/B$ given any morphism $A \to B$ of $\text{SR}(\mathcal{C})$. Applying this to the morphisms of the simplicial object K we obtain simplicial object $(\mathcal{C}/K)'$ in the category of sites with morphisms of sites. Let $(\mathcal{C}/K)'_{site}$ be the site constructed in Lemma 3.1. Since the base change functors are adjoint to the localization functors, we find that $(\mathcal{C}/K)'_{site}$ is the same as the category $(\mathcal{C}/K)_{site}$. Equality of the sets of coverings is immediate from the definitions. □

Let $\mathcal{C}$ be a site. Let $L = \{V_i\}$ be an object of $\text{SR}(\mathcal{C})$. There is a continuous and cocontinuous localization functor $j : \mathcal{C}/K \to \mathcal{C}$ which is the product of the localization functors $\mathcal{C}/V_i \to \mathcal{C}$. We obtain functors j^{-1}, j_*, $j_!^{Sh}$, and $j_!$ exactly as in Sites, Section 24 and Modules on Sites, Section 19. Given a simplicial object K of $\text{SR}(\mathcal{C})$ we obtain a family of localization functors $j_n : \mathcal{C}/K_n \to \mathcal{C}$.

Lemma 4.2. *Let $\mathcal{C}$ be a site. Let K be a simplicial object of $SR(\mathcal{C})$. The forgetful functor $(\mathcal{C}/K)_{site} \to \mathcal{C}$ is continuous and cocontinuous and induces a morphism of topoi*

$$g : Sh((\mathcal{C}/K)_{site}) \longrightarrow Sh(\mathcal{C})$$

as well as functors $g_!^{Sh}$ and $g_!$ left adjoint to g^{-1} on sheaves of sets and abelian groups with the following properties:

(1) *the functor g^{-1} associates to a sheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf on $(\mathcal{C}/K)_{site}$ wich in degree n is equal to $j_n^{-1}\mathcal{F}$,*
(2) *the functor g_* associates to a sheaf $\mathcal{G}$ on $(\mathcal{C}/K)_{site}$ the equalizer of the two maps $j_{0,*}\mathcal{G}_0 \to j_{1,*}\mathcal{G}_1$,*

Proof. The functor is continuous and cocontinuous by our choice of coverings and our description of (certain) fibre products in $(\mathcal{C}/K)_{site}$ in the proof of Lemma 3.2. Details omitted. Thus we obtain a morphism of topoi and functors $g_!^{Sh}$ and $g_!$, see Sites, Section 20 and Modules on Sites, Section 16. The description of g^{-1} is immediate from the definition as the compostion $\mathcal{C}/K_n \to \mathcal{C}/K \to \mathcal{C}$ is the localization morphism j_n.

Proof of (2). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$ and let $\mathcal{G}$ be a sheaf on $(\mathcal{C}/K)_{site}$. A map $a : g^{-1}\mathcal{F} \to \mathcal{G}$ corresponds to a system of maps $a_n : j_n^{-1}\mathcal{F} \to \mathcal{G}_n$ on $\mathcal{C}/K_n$ by Lemma 3.4. Taking $n = 0$ we get a map $j_0^{-1}\mathcal{F} \to \mathcal{G}_0$ which is adjoint to a map $a_0 : \mathcal{F} \to j_{0,*}\mathcal{G}_0$. Since a_0 is compatible with a_1 via the two maps $j_{0,*}\mathcal{G}_0 \to j_{1,*}\mathcal{G}_1$ we see that a_0 maps into the equalizer. Conversely, given a map $a_0 : \mathcal{F} \to j_{0,*}\mathcal{G}_0$ into the equalizer we can pick, for any n, one of the maps $j_{0,*}\mathcal{G}_0 \to j_{n,*}\mathcal{G}_n$ and compose to get a well defined map $a_n : \mathcal{F} \to j_{n,*}\mathcal{G}_n$. These fit together to define a map of sheaves $g^{-1}\mathcal{F} \to \mathcal{G}$. □

Lemma 4.3. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf of sets on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$, see Hypercoverings, Definition 5.1. Then we have a canonical isomorphism*

$$R\Gamma(\mathcal{G}, E) = R\Gamma((\mathcal{C}/K)_{site}, g^{-1}E)$$

for $E \in D^+(\mathcal{C})$. If K is a hypercovering, then $R\Gamma(E) = R\Gamma((\mathcal{C}/K)_{site}, g^{-1}E)$.

Proof. First, let $\mathcal{I}$ be an injective abelian sheaf on $\mathcal{C}$. Then the spectral sequence of Lemma 3.8 for the sheaf $g^{-1}\mathcal{I}$ degenerates as $(g^{-1}\mathcal{I})_p$ is the restriction of $\mathcal{I}$ to $\mathcal{C}/K_p$ which is injective by Cohomology on Sites, Lemma 8.1 (extended in the obvious manner to localization at semi-representable objects of $\mathcal{C}$). Thus we see that the complex

$$\mathcal{I}(K_0) \to \mathcal{I}(K_1) \to \mathcal{I}(K_2) \to \ldots$$

computes $R\Gamma((\mathcal{C}/K)_{site}, g^{-1}\mathcal{I})$. This is exactly the Čech complex of $\mathcal{I}$ with respect to the simplicial object K of $\text{SR}(\mathcal{C})$ as defined in Hypercoverings, Section 4. Thus Hypercoverings, Lemma 5.3 shows that this complex computes $R\Gamma(\mathcal{G}, \mathcal{I})$ (which has zero higher cohomology groups as $\mathcal{I}$ is injective). In other words, we have $H^0(\mathcal{G}, \mathcal{I}) = H^0((\mathcal{C}/K)_{site}, \mathcal{I})$ and $H^p(\mathcal{G}, \mathcal{I}) = H^p((\mathcal{C}/K)_{site}, \mathcal{I}) = 0$ for all $p > 0$.

The lemma now follows formally. Namely, let $A \in D^+(\mathcal{C})$ be arbitrary. We can represent A by a bounded below complex $\mathcal{I}^\bullet$ of injective abelian sheaves. By Leray's acyclicity lemma (Derived Categories, Lemma 17.7) $R\Gamma((\mathcal{C}/K)_{site}, A)$ is computed by the complex $\Gamma((\mathcal{C}/K)_{site}, g^{-1}\mathcal{I}^\bullet)$ and $R\Gamma(\mathcal{G}, A)$ is computed by $\Gamma(\mathcal{G}, \mathcal{I}^\bullet)$. Since these complexes are the same we obtain the conclusion.

The final statement refers to the special case where $\mathcal{G} = *$ is the final object in the category of presheaves on $\mathcal{C}$. □

Lemma 4.4. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X, see Hypercoverings, Definition 2.6. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/K)_{site}, g^{-1}E)$$

for $E \in D^+(\mathcal{C})$.

Proof. If $\mathcal{C}$ also has equalizers, then this is a special case of Lemma 4.3 because a hypercovering of X is a hypercovering of h_X by Hypercoverings, Lemma 2.10. This also uses that $H^q(h_X, \mathcal{F}) = H^q(h_X^\#, \mathcal{F}) = H^q(X, \mathcal{F})$, see discussion in Hypercoverings, Section 5 and Cohomology on Sites, Section 13. In general (when $\mathcal{C}$ does not have equalizers) one proves this using *exactly* the same argument as in the proof of Lemma 4.3 but substituting Hypercoverings, Lemma 4.2 for Hypercoverings, Lemma 5.3. □

5. Hypercovering in a site

In the previous section we worked out, in great generality, how hypercoverings give rise to simplicial sites and how cohomology of (say) constant sheaves on this site computes the cohomology of the object the hypercovering is augmented towards. In this section we explain what this means in a special case.

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$ and let $X_\bullet$ be a simplicial object of $\mathcal{C}$. Assume we have an augmentation

$$a : X_\bullet \to X$$

The discussion above turns this into a morphism of topoi

$$g : (\mathcal{C}/X_\bullet)_{site} \longrightarrow \mathcal{C}/X$$

Here an object of the site $(\mathcal{C}/X_\bullet)_{site}$ is given by a U/X_n and a morphism $(\varphi, f) : U/X_n \to V/X_m$ is given by a morphism $\varphi : [m] \to [n]$ in Δ and a morphism

$f : U \to V$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{\varphi} & X_m \end{array}$$

is commutative. The morphism of topoi g is given by the cocontinuous functor $U/X_n \mapsto U/X$. That's all folks!

Thus we may translate some of the results above to this setting. For example, let us say that the augmentation is a *hypercovering* if the following hold

(1) $\{X_0 \to X\}$ is a covering of $\mathcal{C}$,
(2) $\{X_1 \to X_0 \times_X X_0\}$ is a covering of $\mathcal{C}$,
(3) $\{X_{n+1} \to (\text{cosk}_n \text{sk}_n X_\bullet)_{n+1}\}$ is a covering of $\mathcal{C}$ for $n \geq 1$.

The category $\mathcal{C}/X$ has all finite limits, hence the coskeleta used in the formulation above exist.

Lemma 5.1. *In the situation above assume that $X_\bullet$ is a hypercovering of X. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/X_\bullet)_{site}, g^{-1}E)$$

for $E \in D^+(\mathcal{C}/X)$.

Proof. This is a special case of Lemma 4.4. $\square$

6. Proper hypercoverings in topology

Let's work in the category LC of Hausdorff and locally quasi-compact topological spaces and continuous maps, see Cohomology on Sites, Section 23. Let X be an object of LC and let $X_\bullet$ be a simplicial object of LC. Assume we have an augmentation

$$a : X_\bullet \to X$$

We say that $X_\bullet$ is a *proper hypercovering* of X if

(1) $X_0 \to X$ is a proper surjective map,
(2) $X_1 \to X_0 \times_X X_0$ is a proper surjective map,
(3) $X_{n+1} \to (\text{cosk}_n \text{sk}_n X_\bullet)_{n+1}$ is a proper surjective map for $n \geq 1$.

The category LC has all finite limits, hence the coskeleta used in the formulation above exist.

$$\boxed{\text{Principle: Proper hypercoverings can be used to compute cohomology.}}$$

A key idea behind the proof of the principle is to find a topology on LC which is stronger than the usual one such that (A) a surjective proper map defines a covering, and (B) cohomology of usual sheaves with respect to this stronger topology agrees with the usual cohomology. Properties (A) and (B) hold for the qc topology, see Cohomology on Sites, Section 23. Once we have (A) and (B) we deduce the principle via a combination of the spectral sequences of Hypercoverings, Lemma 4.3 and Lemma 2.11. The following lemma is just a first step.

Lemma 6.1. *In the situation above, let $\mathcal{F}$ be an abelian sheaf on X. Let $\mathcal{F}_n$ be the pullback to X_n. If $X_\bullet$ is a proper hypercovering of X, then there exists a canonical spectral sequence*

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_p)$$

converging to $H^{p+q}(X, \mathcal{F})$.

Proof. By Cohomology on Sites, Lemma 23.6 we have

$$H^*(X, \mathcal{F}) = H^*(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F}).$$

Since a proper surjective map defines a qc covering (Cohomology on Sites, Lemma 23.7) we see that $X_\bullet \to X$ is a hypercovering in the site LC_{qc} as in Section 5. Thus we have

$$R\Gamma(X, \mathcal{F}) = R\Gamma(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F}) = R\Gamma((LC/X_\bullet)_{site}, g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F})$$

by Lemma 5.1. By Lemma 3.8 there is a spectral sequence with

$$E_1^{p,q} = H^q(LC_{qc}/X_p, (g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F})_p)$$

converging to the cohomology of $g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F}$. Finally, the restriction $(g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F})_p$ is just the restriction to LC_{qc}/X_p of $\epsilon^{-1}\pi^{-1}\mathcal{F}$ which by Cohomology on Sites, Lemma 23.5 is the pullback of $\mathcal{F}_p$ to LC_{qc}/X_p. By Cohomology on Sites, Lemma 23.6 again we conclude that

$$H^q(LC_{qc}/X_p, (g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F})_p) = H^q(X_p, \mathcal{F}_p)$$

and the proof is finished. □

Lemma 6.2. *In the situation above, let $\mathcal{F}$ be an abelian sheaf on X. Let $\mathcal{F}_\bullet$ be the pullback of $\mathcal{F}$ via $a : X_\bullet \to X$. If $X_\bullet$ is a proper hypercovering of X, then*

$$H^*(X, \mathcal{F}) = H^*((X_\bullet)_{Zar}, \mathcal{F}_\bullet)$$

Proof. Consider the continuous functor

$$(X_\bullet)_{Zar} \longrightarrow (LC_{qc}/X_\bullet)_{site}, \quad U \longmapsto U$$

We obtain a commutative diagram of topoi

$$\begin{tikzcd} Sh((LC_{qc}/X_\bullet)_{site}) \arrow[r] \arrow[d, "g"'] & Sh((X_\bullet)_{Zar}) \arrow[d, "g"'] \\ Sh(LC_{qc}/X) \arrow[r, "\pi\circ\epsilon"] & Sh(X_{Zar}) \end{tikzcd}$$

Thus our sheaf $\mathcal{F}$ gives rise to a compatible collection of abelian sheaves in each topos. In the proof of Lemma 6.1 we have seen that the sheaf $\mathcal{F}$ has the same cohomology as the sheaf $\epsilon^{-1}\pi^{-1}\mathcal{F}$ and $g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F}$. On the other hand, the terms of the spectral sequence of Lemma 2.11 for $\mathcal{F}_\bullet$ are the same as those in the statement and proof of Lemma 6.1. A simple argument with spectral sequences then shows that the map

$$R\Gamma((X_\bullet)_{Zar}, \mathcal{F}_\bullet) \longrightarrow R\Gamma((LC_{qc}/X_\bullet)_{site}, g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F})$$

is an isomorphism. Some details omitted. □

Lemma 6.3. *In the situation above, assume $a : X_\bullet \to X$ gives a proper hypercovering of X. Then for all $K \in D^+(X)$*

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism where $a : Sh((X_\bullet)_{Zar}) \to Sh(X)$ is as in Lemma 2.9.

Proof. Observe that for any abelian sheaf $\mathcal{F}$ on X the sheaf $R^q a_*(a^{-1}\mathcal{F})$ is the sheaf associated to the presheaf

$$U \mapsto H^q((U_\bullet)_{Zar}, a^{-1}\mathcal{F}) = H^q(U, \mathcal{F})$$

where $U_\bullet = a^{-1}(U)$. The last equality holds by Lemma 6.2. Thus $R^q a_*(a^{-1}\mathcal{F})$ is zero for $q > 0$ and equal to $\mathcal{F}$ for $q = 0$. This proves the result in case K consists of a single abelian sheaf in a single degree. The general case follows from this immediately. $\square$

7. Simplicial schemes

A *simplicial scheme* is a simplicial object in the category of schemes, see Simplicial, Definition 3.1. Recall that a simplicial scheme looks like

$$\begin{tikzcd} X_2 \arrow[r, shift left=3] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=3, leftarrow] \arrow[r, shift right=5] & X_1 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & X_0 \end{tikzcd}$$

Here there are two morphisms $d^1_0, d^1_1 : X_1 \to X_0$ and a single morphism $s^0_0 : X_0 \to X_1$, etc. It is important to keep in mind that $d^n_i : X_n \to X_{n-1}$ should be thought of as a "projection forgetting the ith coordinate" and $s^n_j : X_n \to X_{n+1}$ as the diagonal map repeating the jth coordinate.

8. Descent in terms of simplicial schemes

Cartesian morphisms are defined as follows.

Definition 8.1. Let $a : Y \to X$ be a morphism of simplicial schemes. We say a is *cartesian*, or that Y *is cartesian over* X, if for every morphism $\varphi : [n] \to [m]$ of Δ the corresponding diagram

$$\begin{tikzcd} Y_m \arrow[r, "a"'] \arrow[d, "Y(\varphi)"'] & X_m \arrow[d, "X(\varphi)"] \\ Y_n \arrow[r, "a"] & X_n \end{tikzcd}$$

is a fibre square in the category of schemes.

Cartesian morphisms are related to descent data. First we prove a general lemma describing the category of cartesian simplicial schemes over a fixed simplicial scheme. In this lemma we denote $f^* : Sch/X \to Sch/Y$ the base change functor associated to a morphism of schemes $Y \to X$.

Lemma 8.2. *Let X be a simplicial scheme. The category of simplicial schemes cartesian over X is equivalent to the category of pairs (V, φ) where V is a scheme over X_0 and*

$$\varphi : V \times_{X_0, d^1_1} X_1 \longrightarrow X_1 \times_{d^1_0, X_0} V$$

is an isomorphism over X_1 such that $(s^0_0)^*\varphi = id_V$ and such that

$$(d^2_1)^*\varphi = (d^2_0)^*\varphi \circ (d^2_2)^*\varphi$$

as morphisms of schemes over X_2.

Proof. The statement of the displayed equality makes sense because $d^1_1 \circ d^2_2 = d^1_1 \circ d^2_1$, $d^1_1 \circ d^2_0 = d^1_0 \circ d^2_2$, and $d^1_0 \circ d^2_0 = d^1_0 \circ d^2_1$ as morphisms $X_2 \to X_0$, see Simplicial, Remark 3.3 hence we can picture these maps as follows

$$\begin{tikzcd}
 & X_2 \times_{d^1_1 \circ d^2_0, X_0} V \arrow[r, "(d^2_0)^*\varphi"'] & X_2 \times_{d^1_0 \circ d^2_0, X_0} V \arrow[dr, equal] & \\
X_2 \times_{d^1_0 \circ d^2_2, X_0} V \arrow[ur, equal] & & & X_2 \times_{d^1_0 \circ d^2_1, X_0} V \\
 & X_2 \times_{d^1_1 \circ d^2_2, X_0} V \arrow[ul, "(d^2_2)^*\varphi"] \arrow[r, equal] & X_2 \times_{d^1_1 \circ d^2_1, X_0} V \arrow[ur, "(d^2_1)^*\varphi"'] &
\end{tikzcd}$$

and the condition signifies the diagram is commutative. It is clear that given a simplicial scheme Y cartesian over X we can set $V = Y_0$ and φ equal to the composition

$$V \times_{X_0, d^1_1} X_1 = Y_0 \times_{X_0, d^1_1} X_1 = Y_1 = X_1 \times_{X_0, d^1_0} Y_0 = X_1 \times_{X_0, d^1_0} V$$

of identifications given by the cartesian structure. To prove this functor is an equivalence we construct a quasi-inverse. The construction of the quasi-inverse is analogous to the construction discussed in Descent, Section 3 from which we borrow the notation $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and $\tau^n_{ij} : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Namely, given a pair (V, φ) as in the lemma we set $Y_n = X_n \times_{X(\tau^n_n), X_0} V$. Then given $\beta : [n] \to [m]$ we define $V(\beta) : Y_m \to Y_n$ as the pullback by $X(\tau^m_{\beta(n)m})$ of the map φ postcomposed by the projection $X_m \times_{X(\beta), X_n} Y_n \to Y_n$. This makes sense because

$$X_m \times_{X(\tau^m_{\beta(n)m}), X_1} X_1 \times_{d^1_1, X_0} V = X_m \times_{X(\tau^m_m), X_0} V = Y_m$$

and

$$X_m \times_{X(\tau^m_{\beta(n)m}), X_1} X_1 \times_{d^1_0, X_0} V = X_m \times_{X(\tau^m_{\beta(n)}), X_0} V = X_m \times_{X(\beta), X_n} Y_n.$$

We omit the verification that the commutativity of the displayed diagram above implies the maps compose correctly. We also omit the verification that the two functors are quasi-inverse to each other. ☐

Definition 8.3. Let $f : X \to S$ be a morphism of schemes. The *simplicial scheme associated to* f, denoted $(X/S)_\bullet$, is the functor $\Delta^{opp} \to Sch$, $[n] \mapsto X \times_S \ldots \times_S X$ described in Simplicial, Example 3.5.

Thus $(X/S)_n$ is the $(n+1)$-fold fibre product of X over S. The morphism $d^1_0 : X \times_S X \to X$ is the map $(x_0, x_1) \mapsto x_1$ and the morphism d^1_1 is the other projection. The morphism s^0_0 is the diagonal morphism $X \to X \times_S X$.

Lemma 8.4. *Let $f : X \to S$ be a morphism of schemes. Let $\pi : Y \to (X/S)_\bullet$ be a cartesian morphism of simplicial schemes. Set $V = Y_0$ considered as a scheme over X. The morphisms $d^1_0, d^1_1 : Y_1 \to Y_0$ and the morphism $\pi_1 : Y_1 \to X \times_S X$ induce isomorphisms*

$$V \times_S X \xleftarrow{(d^1_1, pr_1 \circ \pi_1)} Y_1 \xrightarrow{(pr_0 \circ \pi_1, d^1_0)} X \times_S V.$$

Denote $\varphi : V \times_S X \to X \times_S V$ the resulting isomorphism. Then the pair (V, φ) is a descent datum relative to $X \to S$.

Proof. This is a special case of (part of) Lemma 8.2 as the displayed equation of that lemma is equivalent to the cocycle condition of Descent, Definition 30.1. ☐

Lemma 8.5. *Let $f : X \to S$ be a morphism of schemes. The construction*

$$\begin{matrix}\text{category of cartesian} \\ \text{schemes over } (X/S)_\bullet\end{matrix} \longrightarrow \begin{matrix}\text{category of descent data} \\ \text{relative to } X/S\end{matrix}$$

of Lemma 8.4 is an equivalence of categories.

Proof. The functor from left to right is given in Lemma 8.4. Hence this is a special case of Lemma 8.2. ☐

We may reinterpret the pullback of Descent, Lemma 30.6 as follows. Suppose given a morphism of simplicial schemes $f : X' \to X$ and a cartesian morphism of simplicial schemes $Y \to X$. Then the fibre product (viewed as a "pullback")

$$f^*Y = Y \times_X X'$$

of simplicial schemes is a simplicial scheme cartesian over X'. Suppose given a commutative diagram of morphisms of schemes

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

This gives rise to a morphism of simplicial schemes

$$f_\bullet : (X'/S')_\bullet \longrightarrow (X/S)_\bullet.$$

We claim that the "pullback" $f_\bullet^*$ along the morphism $f_\bullet : (X'/S')_\bullet \to (X/S)_\bullet$ corresponds via Lemma 8.5 with the pullback defined in terms of descent data in the aforementioned Descent, Lemma 30.6.

9. Quasi-coherent modules on simplicial schemes

In the following definition we make use of the description of sheaves on a simplicial space given in Lemma 2.2.

Definition 9.1. Let S be a scheme. Let U be a simplicial scheme over S.

(1) A *quasi-coherent sheaf* on U is given by a sheaf of $\mathcal{O}_U$-modules $\mathcal{F}$ such that $\mathcal{F}_n$ is quasi-coherent for all $n \geq 0$.
(2) A quasi-coherent sheaf $\mathcal{F}$ on U is *cartesian* if and only if all the maps $\mathcal{F}(\varphi) : \mathcal{F}_n \to \mathcal{F}_m$ induce isomorphisms $U(\varphi)^*\mathcal{F}_n \to \mathcal{F}_m$.

The property on pullbacks needs only be checked for the degeneracies.

Lemma 9.2. *Let S be a scheme. Let U be a simplicial scheme over S. Let $\mathcal{F}$ be a quasi-coherent module on U. Then $\mathcal{F}$ is cartesian if and only if the induced maps $(d^n_j)^*\mathcal{F}_{n-1} \to \mathcal{F}_n$ are isomorphisms.*

Proof. The category Δ is generated by the morphisms the morphisms δ^n_j and σ^n_j, see Simplicial, Lemma 2.2. Hence we only need to check the maps $(d^n_j)^*\mathcal{F}_{n-1} \to \mathcal{F}_n$ and $(s^n_j)^*\mathcal{F}_{n+1} \to \mathcal{F}_n$ are isomorphisms, see Simplicial, Lemma 3.2 for notation. But $d^{n+1}_j \circ s^n_j = \mathrm{id}_{U_n}$ so it the result for d^{n+1}_j implies the result for s^n_j. ☐

Lemma 9.3. *Let S be a scheme. Let U be a simplicial scheme over S. The category of cartesian quasi-coherent modules over U is equivalent to the category of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a quasi-coherent module over U_0 and*

$$\alpha : (d^1_1)^*\mathcal{F} \longrightarrow (d^1_0)^*\mathcal{F}$$

is an isomorphism such that $(s^0_0)^*\alpha = id_{\mathcal{F}}$ and such that

$$(d^2_1)^*\alpha = (d^2_0)^*\alpha \circ (d^2_2)^*\alpha$$

on X_2.

Proof. The statement of the displayed equality makes sense because $d^1_1 \circ d^2_2 = d^1_1 \circ d^2_1$, $d^1_1 \circ d^2_0 = d^1_0 \circ d^2_2$, and $d^1_0 \circ d^2_0 = d^1_0 \circ d^2_1$ as morphisms $X_2 \to X_0$, see Simplicial, Remark 3.3 hence we can picture these maps as follows

$$\begin{tikzcd}
& (d^2_0)^*(d^1_1)^*\mathcal{F} \arrow[r, "(d^2_0)^*\alpha"'] \arrow[dl, equal] & (d^2_0)^*(d^1_0)^*\mathcal{F} \arrow[dr, equal] & \\
(d^2_2)^*(d^1_0)^*\mathcal{F} & & & (d^2_1)^*(d^1_0)^*\mathcal{F} \\
& (d^2_2)^*(d^1_1)^*\mathcal{F} \arrow[ul, "(d^2_2)^*\alpha"] \arrow[r, equal] & (d^2_1)^*(d^1_1)^*\mathcal{F} \arrow[ur, "(d^2_1)^*\alpha"'] &
\end{tikzcd}$$

and the condition signifies the diagram is commutative. It is clear that given a cartesian quasi-coherent sheaf $\mathcal{F}$ we can set $\mathcal{F} = \mathcal{F}_0$ and α equal to the composition

$$(d^0_1)^*\mathcal{F}_0 = \mathcal{F}_1 = (d^0_0)^*\mathcal{F}_0$$

of identifications given by the cartesian structure. To prove this functor is an equivalence we construct a quasi-inverse. The construction of the quasi-inverse is analogous to the construction discussed in Descent, Section 3 from which we borrow the notation $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and $\tau^n_{ij} : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Namely, given a pair $(\mathcal{F}, \alpha)$ as in the lemma we set $\mathcal{F}_n = X(\tau^n_n)^*\mathcal{F}$. Then given $\beta : [n] \to [m]$ we define $\mathcal{F}(\beta) : \mathcal{F}_n \to \mathcal{F}_m$ as the pullback by $X(\tau^m_{\beta(n)m})$ of the map α precomposed with the canonical $X(\beta)$-map $\mathcal{F}_n \to X(\beta)^*\mathcal{F}_n$. We omit the verification that the commutativity of the displayed diagram above implies the maps compose correctly. We also omit the verification that the two functors are quasi-inverse to each other. □

Lemma 9.4. *Let $f : V \to U$ be a morphism of simplicial schemes. Given a cartesian quasi-coherent module $\mathcal{F}$ on U the pullback $f^*\mathcal{F}$ is a cartesian quasi-coherent module on V.*

Proof. This is immediate from the definitions. □

Lemma 9.5. *Let $f : V \to U$ be a cartesian morphism of simplicial schemes. Assume the morphisms $d^n_j : U_n \to U_{n-1}$ are flat and the morphisms $V_n \to U_n$ are quasi-compact and quasi-separated. For a cartesian quasi-coherent module $\mathcal{G}$ on V the pushforward $f_*\mathcal{G}$ is a cartesian quasi-coherent module on U.*

Proof. If $\mathcal{F} = f_*\mathcal{G}$, then $\mathcal{F}_n = f_{n,*}\mathcal{G}_n$ and the maps $\mathcal{F}(\varphi)$ are defined using the base change maps, see Cohomology, Section 18. The sheaves $\mathcal{F}_n$ are quasi-coherent by Schemes, Lemma 24.1. The base change maps along the degeneracies d^n_j are isomorphisms by Cohomology of Schemes, Lemma 5.2. Hence we are done by Lemma 9.2. □

Lemma 9.6. *Let $f : V \to U$ be a cartesian morphism of simplicial schemes. Assume the morphisms $d^n_j : U_n \to U_{n-1}$ are flat and the morphisms $V_n \to U_n$ are quasi-compact and quasi-separated. Then f^* and f_* form an adjoint pair of functors between the categories of cartesian quasi-coherent modules on U and V.*

Proof. We have seen in Lemmas 9.4 and 9.5 that the statement makes sense. The adjointness property follows immediately from the fact that each f_n^* is adjoint to $f_{n,*}$. □

Lemma 9.7. *Let $f : X \to S$ be a morphism of schemes which has a section*[2]*. Let $(X/S)_\bullet$ be the simplicial scheme associated to $X \to S$, see Definition 8.3. Then pullback defines an equivalence between the category of quasi-coherent $\mathcal{O}_S$-modules and the category of cartesian quasi-coherent modules on $(X/S)_\bullet$.*

Proof. Let $\sigma : S \to X$ be a section of f. Let $(\mathcal{F}, \alpha)$ be a pair as in Lemma 9.3. Set $\mathcal{G} = \sigma^*\mathcal{F}$. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "(\sigma\circ f, 1)"'] \arrow[d, "f"'] & X \times_S X \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_0"] & X \\ S \arrow[r, "\sigma"] & X & \end{tikzcd}$$

Note that $\mathrm{pr}_0 = d^1_1$ and $\mathrm{pr}_1 = d^1_0$. Hence we see that $(\sigma \circ f, 1)^*\alpha$ defines an isomorphism

$$f^*\mathcal{G} = (\sigma \circ f, 1)^*\mathrm{pr}_0^*\mathcal{F} \longrightarrow (\sigma \circ f, 1)^*\mathrm{pr}_1^*\mathcal{F} = \mathcal{F}$$

We omit the verification that this isomorphism is compatible with α and the canonical isomorphism $\mathrm{pr}_0^* f^*\mathcal{G} \to \mathrm{pr}_1^* f^*\mathcal{G}$. □

10. Groupoids and simplicial schemes

Given a groupoid in schemes we can build a simplicial scheme. It will turn out that the category of quasi-coherent sheaves on a groupoid is equivalent to the category of cartesian quasi-coherent sheaves on the associated simplicial scheme.

Lemma 10.1. *Let (U, R, s, t, c, e, i) be a groupoid scheme over S. There exists a simplicial scheme X over S with the following properties*

1. $X_0 = U$, $X_1 = R$, $X_2 = R \times_{s,U,t} R$,
2. $s^0_0 = e : X_0 \to X_1$,
3. $d^1_0 = s : X_1 \to X_0$, $d^1_1 = t : X_1 \to X_0$,
4. $s^1_0 = (e \circ t, 1) : X_1 \to X_2$, $s^1_1 = (1, e \circ t) : X_1 \to X_2$,
5. $d^2_0 = pr_1 : X_2 \to X_1$, $d^2_1 = c : X_2 \to X_1$, $d^2_2 = pr_0$, *and*
6. $X = cosk_2 sk_2 X$.

For all n we have $X_n = R \times_{s,U,t} \ldots \times_{s,U,t} R$ with n factors. The map $d^n_j : X_n \to X_{n-1}$ is given on functors of points by

$$(r_1, \ldots, r_n) \longmapsto (r_1, \ldots, c(r_j, r_{j+1}), \ldots, r_n)$$

for $1 \leq j \leq n-1$ whereas $d^n_0(r_1, \ldots, r_n) = (r_2, \ldots, r_n)$ and $d^n_n(r_1, \ldots, r_n) = (r_1, \ldots, r_{n-1})$.

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[2]In fact, it would be enough to assume that f has fpqc locally on S a section, since we have descent of quasi-coherent modules by Descent, Section 5.

Proof. We only have to verify that the rules prescribed in (1), (2), (3), (4), (5) define a 2-truncated simplicial scheme U' over S, since then (6) allows us to set $X = \text{cosk}_2 U'$, see Simplicial, Lemma 18.2. Using the functor of points approach, all we have to verify is that if $(\text{Ob}, \text{Arrows}, s, t, c, e, i)$ is a groupoid, then

$$\begin{tikzcd}
\text{Arrows} \times_{s,\text{Ob},t} \text{Arrows} \arrow[d, shift right=8, "\text{pr}_1"'] \arrow[d, "c"] \arrow[d, shift left=8, "\text{pr}_0"] \\
\text{Arrows} \arrow[u, shift left=4, "{1,e}"] \arrow[u, shift right=4, "{e,1}"'] \arrow[d, shift right=4, "s"'] \arrow[d, shift left=4, "t"] \\
\text{Ob} \arrow[u, "e"]
\end{tikzcd}$$

is a 2-truncated simplicial set. We omit the details.

Finally, the description of X_n for $n > 2$ follows by induction from the description of X_0, X_1, X_2, and Simplicial, Remark 18.9 and Lemma 18.6. Alternately, one shows that cosk_2 applied to the 2-truncated simplicial set displayed above gives a simplicial set whose nth term equals $\text{Arrows} \times_{s,\text{Ob},t} \ldots \times_{s,\text{Ob},t} \text{Arrows}$ with n factors and degeneracy maps as given in the lemma. Some details omitted. $\square$

Lemma 10.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let X be the simplicial scheme over S constructed in Lemma 10.1. Then the category of quasi-coherent modules on (U, R, s, t, c) is equivalent to the category of cartesian quasi-coherent modules on X.*

Proof. This is clear from Lemma 9.3 and Groupoids, Definition 12.1. $\square$

In the following lemma we will use the concept of a cartesian morphism $V \to U$ of simplicial schemes as defined in Definition 8.1.

Lemma 10.3. *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Let X be the simplicial scheme over S constructed in Lemma 10.1. Let $(R/U)_\bullet$ be the simplicial scheme associated to $s : R \to U$, see Definition 8.3. There exists a cartesian morphism $t_\bullet : (R/U)_\bullet \to X$ of simplicial schemes with low degree morphisms given by*

$$\begin{tikzcd}
R \times_{s,U,s} R \times_{s,U,s} R \arrow[r, shift left=4, "pr_{12}"] \arrow[r, "pr_{02}"] \arrow[r, shift right=4, "pr_{01}"'] \arrow[d, "{(r_0,r_1,r_2) \mapsto (r_0 \circ r_1^{-1}, r_1 \circ r_2^{-1})}"'] & R \times_{s,U,s} R \arrow[r, shift left=2, "pr_1"] \arrow[r, shift right=2, "pr_0"'] \arrow[d, "{(r_0,r_1) \mapsto r_0 \circ r_1^{-1}}"'] & R \arrow[d, "t"] \\
R \times_{s,U,t} R \arrow[r, shift left=4, "pr_1"] \arrow[r, "c"] \arrow[r, shift right=4, "pr_0"'] & R \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & U
\end{tikzcd}$$

Proof. For arbitrary n we define $(R/U)_\bullet \to X_n$ by the rule

$$(r_0, \ldots, r_n) \longrightarrow (r_0 \circ r_1^{-1}, \ldots, r_{n-1} \circ r_n^{-1})$$

Compatibility with degeneracy maps is clear from the description of the degeneracies in Lemma 10.1. We omit the verification that the maps respect the morphisms s_j^n. Groupoids, Lemma 11.5 (with the roles of s and t reversed) shows that the two right squares are cartesian. In exactly the same manner one shows all the other squares are cartesian too. Hence the morphism is cartesian. □

11. Descent data give equivalence relations

In Section 8 we saw how descent data relative to $X \to S$ can be formulated in terms of cartesian simplicial schemes over $(X/S)_\bullet$. Here we link this to equivalence relations as follows.

Lemma 11.1. *Let $f : X \to S$ be a morphism of schemes. Let $\pi : Y \to (X/S)_\bullet$ be a cartesian morphism of simplicial schemes, see Definitions 8.1 and 8.3. Then the morphism*

$$j = (d^1_1, d^1_0) : Y_1 \to Y_0 \times_S Y_0$$

defines an equivalence relation on Y_0 over S, see Groupoids, Definition 3.1.

Proof. Note that j is a monomorphism. Namely the composition $Y_1 \to Y_0 \times_S Y_0 \to Y_0 \times_S X$ is an isomorphism as π is cartesian.

Consider the morphism

$$(d^2_2, d^2_0) : Y_2 \to Y_1 \times_{d^1_0, Y_0, d^1_1} Y_1.$$

This works because $d_0 \circ d_2 = d_1 \circ d_0$, see Simplicial, Remark 3.3. Also, it is a morphism over $(X/S)_2$. It is an isomorphism because $Y \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $Y_0 \times_{\pi_0, X, \mathrm{pr}_1} (X \times_S X \times_S X) = X \times_S Y_0 \times_S X$ because π is cartesian. Details omitted.

As in Groupoids, Definition 3.1 we denote $t = \mathrm{pr}_0 \circ j = d^1_1$ and $s = \mathrm{pr}_1 \circ j = d^1_0$. The isomorphism above, combined with the morphism $d^2_1 : Y_2 \to Y_1$ give us a composition morphism

$$c : Y_1 \times_{s, Y_0, t} Y_1 \longrightarrow Y_1$$

over $Y_0 \times_S Y_0$. This immediately implies that for any scheme T/S the relation $Y_1(T) \subset Y_0(T) \times Y_0(T)$ is transitive.

Reflexivity follows from the fact that the restriction of the morphism j to the diagonal $\Delta : X \to X \times_S X$ is an isomorphism (again use the cartesian property of π).

To see symmetry we consider the morphism

$$(d^2_2, d^2_1) : Y_2 \to Y_1 \times_{d^1_1, Y_0, d^1_1} Y_1.$$

This works because $d_1 \circ d_2 = d_1 \circ d_1$, see Simplicial, Remark 3.3. It is an isomorphism because $Y \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $Y_0 \times_{\pi_0, X, \mathrm{pr}_0} (X \times_S X \times_S X) = Y_0 \times_S X \times_S X$ because π is cartesian. Details omitted.

Let T/S be a scheme. Let $a \sim b$ for $a, b \in Y_0(T)$ be synonymous with $(a, b) \in Y_1(T)$. The isomorphism (d^2_2, d^2_1) above implies that if $a \sim b$ and $a \sim c$, then $b \sim c$. Combined with reflexivity this shows that $\sim$ is an equivalence relation. □

12. An example case

In this section we show that disjoint unions of spectra of Artinian rings can be descended along a quasi-compact surjective flat morphism of schemes.

Lemma 12.1. *Let $X \to S$ be a morphism of schemes. Suppose $Y \to (X/S)_\bullet$ is a cartesian morphism of simplicial schemes. For $y \in Y_0$ a point define*

$$T_y = \{y' \in Y_0 \mid \exists\ y_1 \in Y_1 : d^1_1(y_1) = y, d^1_0(y_1) = y'\}$$

as a subset of Y_0. Then $y \in T_y$ and $T_y \cap T_{y'} \neq \emptyset \Rightarrow T_y = T_{y'}$.

Proof. Combine Lemma 11.1 and Groupoids, Lemma 3.4. □

Lemma 12.2. *Let $X \to S$ be a morphism of schemes. Suppose $Y \to (X/S)_\bullet$ is a cartesian morphism of simplicial schemes. Let $y \in Y_0$ be a point. If $X \to S$ is quasi-compact, then*

$$T_y = \{y' \in Y_0 \mid \exists\ y_1 \in Y_1 : d^1_1(y_1) = y, d^1_0(y_1) = y'\}$$

is a quasi-compact subset of Y_0.

Proof. Let F_y be the scheme theoretic fibre of $d^1_1 : Y_1 \to Y_0$ at y. Then we see that T_y is the image of the morphism

$$\begin{tikzcd} F_y \arrow[r] \arrow[d] & Y_1 \arrow[r, "d^1_0"] \arrow[d, "d^1_1"] & Y_0 \\ y \arrow[r] & Y_0 & \end{tikzcd}$$

Note that F_y is quasi-compact. This proves the lemma. □

Lemma 12.3. *Let $X \to S$ be a quasi-compact flat surjective morphism. Let (V, φ) be a descent datum relative to $X \to S$. If V is a disjoint union of spectra of Artinian rings, then (V, φ) is effective.*

Proof. Let $Y \to (X/S)_\bullet$ be the cartesian morphism of simplicial schemes corresponding to (V, φ) by Lemma 8.5. Observe that $Y_0 = V$. Write $V = \coprod_{i \in I} \text{Spec}(A_i)$ with each A_i local Artinian. Moreover, let $v_i \in V$ be the unique closed point of $\text{Spec}(A_i)$ for all $i \in I$. Write $i \sim j$ if and only if $v_i \in T_{v_j}$ with notation as in Lemma 12.1 above. By Lemmas 12.1 and 12.2 this is an equivalence relation with finite equivalence classes. Let $\overline{I} = I/\sim$. Then we can write $V = \coprod_{\overline{i} \in \overline{I}} V_{\overline{i}}$ with $V_{\overline{i}} = \coprod_{i \in \overline{i}} \text{Spec}(A_i)$. By construction we see that $\varphi : V \times_S X \to X \times_S V$ maps the open and closed subspaces $V_{\overline{i}} \times_S X$ into the open and closed subspaces $X \times_S V_{\overline{i}}$. In other words, we get descent data $(V_{\overline{i}}, \varphi_{\overline{i}})$, and (V, φ) is the coproduct of them in the category of descent data. Since each of the $V_{\overline{i}}$ is a finite union of spectra of Artinian local rings the morphism $V_{\overline{i}} \to X$ is affine, see Morphisms, Lemma 13.13. Since $\{X \to S\}$ is an fpqc covering we see that all the descent data $(V_{\overline{i}}, \varphi_{\overline{i}})$ are effective by Descent, Lemma 33.1. □

To be sure, the lemma above has very limited applicability!

13. Other chapters

Preliminaries

(1) Introduction

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