# DERIVED CATEGORIES OF SCHEMES

## Contents



## 1. Introduction

In this chapter we discuss derived categories of modules on schemes. Most of the material discussed here can be found in [TT90], [BN93], [BV03], and [LN07]. Of course there are many other references.

## 2. Conventions

If $\mathcal{A}$ is an abelian category and $M$ is an object of $\mathcal{A}$ then we also denote $M$ the object of $K(\mathcal{A})$ and/or $D(\mathcal{A})$ corresponding to the complex which has $M$ in degree 0 and is zero in all other degrees.

If we have a ring $A$, then $K(A)$ denotes the homotopy category of complexes of $A$-modules and $D(A)$ the associated derived category. Similarly, if we have a ringed space $(X, \mathcal{O}_X)$ the symbol $K(\mathcal{O}_X)$ denotes the homotopy category of complexes of $\mathcal{O}_X$-modules and $D(\mathcal{O}_X)$ the associated derived category.


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## 3. Derived category of quasi-coherent modules

In this section we discuss the relationship between quasi-coherent modules and all modules on a scheme $X$. A reference is [TT90, Appendix B]. By the discussion in Schemes, Section 24 the embedding $QCoh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ exhibits $QCoh(\mathcal{O}_X)$ as a weak Serre subcategory of the category of $\mathcal{O}_X$-modules. Denote

$$D_{QCoh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are quasi-coherent, see Derived Categories, Section 13. Thus we obtain a canonical functor

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X) \tag{3.0.1}$$

see Derived Categories, Equation (13.1.1).

**Lemma 3.1.** *Let $X$ be a scheme. Then $D_{QCoh}(\mathcal{O}_X)$ has direct sums.*

**Proof.** By Injectives, Lemma 13.4 the derived category $D(\mathcal{O}_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 24. $\square$

The following lemma will help us to "compute" a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

**Lemma 3.2.** *Let $X$ be a scheme. Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. Then there exists an inverse system $\mathcal{I}_n^\bullet$ of complexes of $\mathcal{O}_X$-modules such that*

1. *$\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$ represents $E$,*
2. *$\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,*
3. *$\mathcal{I}^\bullet \to \mathcal{I}_n^\bullet$ induces an identification $\tau_{\geq -n}E \to \mathcal{I}_n^\bullet$ in $D(\mathcal{O}_X)$,*
4. *the transition maps $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections, and*
5. *$\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules.*

*Moreover, $E$ is the derived limit of the inverse system of its canonical truncations $\tau_{\geq -n}E$.*

**Proof.** Denote $\mathcal{H}^i = H^i(E)$ the $i$th cohomology sheaf of $E$. Let $\mathcal{B}$ be the set of affine open subsets of $X$. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Cohomology of Schemes, Lemma 2.2. Thus the lemma follows from Cohomology, Lemmas 31.2 and 31.3. $\square$

**Lemma 3.3.** *Let $X$ be a scheme. Let $F : Mod(\mathcal{O}_X) \to Ab$ be an additive functor and $N \geq 0$ an integer. Assume that*

1. *$F$ commutes with countable direct products,*
2. *$R^pF(\mathcal{F}) = 0$ for all $p \geq N$ and $\mathcal{F}$ quasi-coherent.*

*Then for $E \in D_{QCoh}(\mathcal{O}_X)$ the maps $R^pF(E) \to R^pF(\tau_{\geq p-N+1}E)$ are isomorphisms.*

**Proof.** By shifting the complex we see it suffices to prove the assertion for $p = 0$. Write $E_n = \tau_{\geq -n}E$. We have $E = R\lim E_n$, see Lemma 3.2. Thus $RF(E) = R\lim RF(E_n)$ in $D(Ab)$ by Injectives, Lemma 13.6. Thus we have a short exact sequence

$$0 \to R^1 \lim R^{-1}F(E_n) \to R^0F(E) \to \lim R^0F(E_n) \to 0$$

see More on Algebra, Remark 61.16. To finish the proof we will show that the term on the left is zero and that the term on the right equals $R^0F(E_{N-1})$.

We have a distinguished triangle

$$H^{-n}(E)[n] \to E_n \to E_{n-1} \to H^{-n}(E)[n+1]$$

(Derived Categories, Remark 12.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$R^pF(H^{-n}(E)[n]) = R^{p+n}F(H^{-n}(E)) = 0$$

for $p + n \geq N$ and

$$R^pF(H^{-n}(E)[n+1]) = R^{p+n+1}F(H^{-n}(E)) = 0$$

for $p + n + 1 \geq N$. We conclude that

$$R^pF(E_n) \to R^pF(E_{n-1})$$

is an isomorphism for all $n \gg p$ and an isomorphism for $n \geq N$ for $p = 0$. Thus the systems $R^pF(E_n)$ all satisfy the ML condition and $R^1\lim$ gives zero (see discussion in More on Algebra, Section 61). Moreover, the system $R^0F(\tau_{\geq -n}E)$ is constant starting with $n = N - 1$ as desired. $\square$

The following lemma is the key ingredient to many of the results in this chapter.

**Lemma 3.4.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. All the functors in the diagram*

$$\begin{tikzcd}
D(QCoh(\mathcal{O}_X)) \arrow[rr, "(3.0.1)"'] & & D_{QCoh}(\mathcal{O}_X) \arrow[dl, "R\Gamma(X{,}-)"] \\
& D(A) \arrow[ul, "\sim"] &
\end{tikzcd}$$

*are equivalences of triangulated categories. Moreover, for $E$ in $D_{QCoh}(\mathcal{O}_X)$ we have $H^0(X, E) = H^0(X, H^0(E))$.*

**Proof.** The functor $R\Gamma(X, -)$ gives a functor $D(\mathcal{O}_X) \to D(A)$ and hence by restriction a functor

$$R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \longrightarrow D(A). \tag{3.4.1}$$

We will show this functor is quasi-inverse to (3.0.1) via the equivalence between quasi-coherent modules on $X$ and the category of $A$-modules.

Elucidation. Denote $(Y, \mathcal{O}_Y)$ the one point space with sheaf of rings given by $A$. Denote $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ the obvious morphism of ringed spaces. Then $R\Gamma(X, -)$ can be identified with $R\pi_*$ and the functor (3.0.1) via the equivalence $Mod(\mathcal{O}_Y) = \mathrm{Mod}_A = QCoh(\mathcal{O}_X)$ can be identified with $L\pi^* = \pi^* = \widetilde{\ }$(see Modules, Lemma 10.5 and Schemes, Lemmas 7.1 and 7.5). Thus the functors

$$\begin{tikzcd} D(A) \arrow[r, shift left] & D_{QCoh}(\mathcal{O}_X) \arrow[l, shift left] \end{tikzcd}$$

are adjoint (by Cohomology, Lemma 29.1). In particular we obtain canonical adjunction mappings

$$a : \widetilde{R\Gamma(X, E)} \longrightarrow E$$

for $E$ in $D(\mathcal{O}_X)$ and

$$b : M^\bullet \longrightarrow R\Gamma(X, \widetilde{M^\bullet})$$

for $M^\bullet$ a complex of $A$-modules.

Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. We may apply Lemma 3.3 to the functor $F(-) = \Gamma(X, -)$ with $N = 1$ by Cohomology of Schemes, Lemma 2.2. Hence

$$R^0\Gamma(X, E) = R^0\Gamma(X, \tau_{\geq 0}E) = \Gamma(X, H^0(E))$$

(the last equality by definition of the canonical truncation). Using this we will show that the adjunction mappings $a$ and $b$ induce isomorphisms $H^0(a)$ and $H^0(b)$. Thus $a$ and $b$ are quasi-isomorphisms (as the statement is invariant under shifts) and the lemma is proved.

In both cases we use that $\widetilde{\ }$ is an exact functor (Schemes, Lemma 5.4). Namely, this implies that

$$H^0\left(R\widetilde{\Gamma(X, E)}\right) = R^0\widetilde{\Gamma(X, E)} = \Gamma(\widetilde{X, H^0}(E))$$

which is equal to $H^0(E)$ because $H^0(E)$ is quasi-coherent. Thus $H^0(a)$ is an isomorphism. For the other direction we have

$$H^0(R\Gamma(X, \widetilde{M^\bullet})) = R^0\Gamma(X, \widetilde{M^\bullet}) = \Gamma(X, H^0(\widetilde{M^\bullet})) = \Gamma(X, \widetilde{H^0(M^\bullet)}) = H^0(M^\bullet)$$

which proves that $H^0(b)$ is an isomorphism. □

**Lemma 3.5.** *Let $X = \text{Spec}(A)$ be an affine scheme. If $K^\bullet$ is a K-flat complex of $A$-modules, then $\widetilde{K^\bullet}$ is a K-flat complex of $\mathcal{O}_X$-modules.*

**Proof.** By More on Algebra, Lemma 45.5 we see that $K^\bullet \otimes_A A_\mathfrak{p}$ is a K-flat complex of $A_\mathfrak{p}$-modules for every $\mathfrak{p} \in \text{Spec}(A)$. Hence we conclude from Cohomology, Lemma 27.4 (and Schemes, Lemma 5.4) that $\widetilde{K^\bullet}$ is K-flat. □

**Lemma 3.6.** *Let $f : Y \to X$ be a morphism of schemes.*

1. *The functor $Lf^*$ sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*
2. *If $X$ and $Y$ are affine and $f$ is given by the ring map $A \to B$, then the diagram*

$$\begin{tikzcd} D(B) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \\ D(A) \arrow[r] \arrow[u, "- \otimes_A^{\mathbf{L}} B"] & D_{QCoh}(\mathcal{O}_X) \arrow[u, "Lf^*"'] \end{tikzcd}$$

*commutes.*

**Proof.** We first prove the diagram

$$\begin{tikzcd} D(B) \arrow[r] & D(\mathcal{O}_Y) \\ D(A) \arrow[r] \arrow[u, "- \otimes_A^{\mathbf{L}} B"] & D(\mathcal{O}_X) \arrow[u, "Lf^*"'] \end{tikzcd}$$

commutes. This is clear from Lemma 3.5 and the constructions of the functors in question. To see (1) let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. To see that $Lf^*E$ has quasi-coherent cohomology sheaves we may work locally on $X$. Note that $Lf^*$ is compatible with restricting to open subschemes. Hence we can assume that $f$ is a morphism of affine schemes as in (2). Then we can apply Lemma 3.4 to see that $E$ comes from a complex of $A$-modules. By the commutativity of the first diagram of the proof the same holds for $Lf^*E$ and we conclude (1) is true. □

**Lemma 3.7.** *Let $X$ be a scheme.*

1. *For objects $K, L$ of $D_{QCoh}(\mathcal{O}_X)$ the derived tensor product $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(\mathcal{O}_X)$.*
2. *If $X = \mathrm{Spec}(A)$ is affine then*

$$\widetilde{M^\bullet} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{K^\bullet} = \widetilde{M^\bullet \otimes^{\mathbf{L}}_A K^\bullet}$$

*for any pair of complexes of $A$-modules $K^\bullet$, $M^\bullet$.*

**Proof.** The equality of (2) follows immediately from Lemma 3.5 and the construction of the derived tensor product. To see (1) let $K, L$ be objects of $D_{QCoh}(\mathcal{O}_X)$. To check that $K \otimes^{\mathbf{L}} L$ is in $D_{QCoh}(\mathcal{O}_X)$ we may work locally on $X$, hence we may assume $X = \mathrm{Spec}(A)$ is affine. By Lemma 3.4 we may represent $K$ and $L$ by complexes of $A$-modules. Then part (2) implies the result. □

## 4. Total direct image

The following lemma is the analogue of Cohomology of Schemes, Lemma 4.4.

**Lemma 4.1.** *Let $f : X \to S$ be a morphism of schemes. Assume that $f$ is quasi-separated and quasi-compact.*

1. *The functor $Rf_*$ sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_S)$.*
2. *If $S$ is quasi-compact, there exists an integer $N = N(X, S, f)$ such that for an object $E$ of $D_{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m > N$.*
3. *In fact, if $S$ is quasi-compact we can find $N = N(X, S, f)$ such that for every morphism of schemes $S' \to S$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \to S'$ is the base change of $f$.*

**Proof.** Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that $Rf_*E$ has quasi-coherent cohomology sheaves. This question is local on $S$, hence we may assume $S$ is quasi-compact. Pick $N = N(X, S, f)$ as in Cohomology of Schemes, Lemma 4.4. Thus $R^pf_*\mathcal{F} = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $p \geq N$. In particular, for any affine open $U \subset S$ we have $H^p(f^{-1}(U), \mathcal{F}) = 0$ for $p \geq N$, see Cohomology of Schemes, Lemma 4.5.

Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. Choose $\mathcal{I}^\bullet = \lim \mathcal{I}_n^\bullet$ as in Lemma 3.2. As $\mathcal{I}^\bullet$ is K-injective $Rf_*E$ is represented by $f_*\mathcal{I}^\bullet = \lim f_*\mathcal{I}_n^\bullet$. Let $U \subset S$ be any affine open. The cohomology $H^m(f_*\mathcal{I}_n^\bullet(U))$ of

$$f_*\mathcal{I}_n^{m-1}(U) \to f_*\mathcal{I}_n^m(U) \to f_*\mathcal{I}_n^{m+1}(U)$$

is equal to $H^m(f^{-1}(U), \tau_{\geq -n}E)$ because $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives representing $\tau_{\geq -n}E$. We have a distinguished triangle

$$H^{-n}(E)[n] \to \tau_{\geq -n}E \to \tau_{\geq -n+1}E \to H^{-n}(E)[n+1]$$

(Derived Categories, Remark 12.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have $H^m(f^{-1}(U), H^{-n}(E)[n]) = 0$ for $n + m \geq N$ by our choice of $N$. Similarly, $H^m(f^{-1}(U), H^{-n}(E)[n+1]) = 0$ for $n + m + 1 \geq N$. We conclude that

$$H^m(f_*\mathcal{I}_n^\bullet(U)) \to H^m(f_*\mathcal{I}_{n-1}^\bullet(U))$$

is an isomorphism for all $n \geq N - m$. Thus Cohomology, Lemma 31.1 applies to show that the $m$th cohomology sheaf of $\lim f_*\mathcal{I}_n^\bullet$ agrees with the $m$th cohomology sheaf of $f_*\mathcal{I}_n^\bullet$ for $n \geq N - m$. Since these cohomology sheaves are quasi-coherent by Cohomology of Schemes, Lemma 4.4 we get (1).

Finally, we show that (2) and (3) hold with our choice of $N$. Namely, the stabilization proven above gives that $H^m(Rf_*E)$ is equal to $H^m(Rf_*(\tau_{\geq -n}E))$ for all $n$ large enough which means we can work with objects in $D^+(\mathcal{O}_X)$ in order to prove (2) and (3). In this case we can for example use the spectral sequence

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E$$

(Derived Categories, Lemma 21.3) and the vanishing of $R^pf_*H^q(E)$ for $p \geq N$ to conclude. Some details omitted. $\square$

**Lemma 4.2.** *Let $f : X \to S$ be a quasi-separated and quasi-compact morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ commutes with direct sums.*

**Proof.** Let $E_i$ be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_*E_i \longrightarrow Rf_*E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer $N$ as in Lemma 4.1. Then $R^0f_*E = R^0f_*\tau_{\geq -N}E$ and $R^0f_*E_i = R^0f_*\tau_{\geq -N}E_i$ by the lemma cited. Observe that $\tau_{\geq -N}E = \bigoplus \tau_{\geq -N}E_i$. Thus we may assume all of the $E_i$ have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E \quad \text{and} \quad R^pf_*H^q(E_i) \Rightarrow R^{p+q}f_*E_i$$

(Derived Categories, Lemma 21.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Schemes, Lemma 6.1. $\square$

**Lemma 4.3.** *Let $f : X \to S$ be an affine morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ reflects isomorphisms.*

**Proof.** The statement means that a morphism $\alpha : E \to F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_*\alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is local on $S$. Hence we may assume $S$ and therefore $X$ is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ given in Lemma 3.4. Some details omitted. $\square$

**Lemma 4.4.** *Let $f : X \to S$ be an affine morphism of schemes. For $E$ in $D_{QCoh}(\mathcal{O}_S)$ we have $Rf_*Lf^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_S} f_*\mathcal{O}_X$.*

**Proof.** Since $f$ is affine the map $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$ is an isomorphism (Cohomology of Schemes, Lemma 2.3). There is a canonical map $E \otimes^{\mathbf{L}} f_*\mathcal{O}_X = E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X \to Rf_*Lf^*E$ adjoint to the map

$$Lf^*(E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X) = Lf^*E \otimes^{\mathbf{L}} Lf^*Rf_*\mathcal{O}_X \longrightarrow Lf^*E \otimes^{\mathbf{L}} \mathcal{O}_X = Lf^*E$$

coming from $1 : Lf^*E \to Lf^*E$ and the canonical map $Lf^*Rf_*\mathcal{O}_X \to \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on $S$. Hence we may assume $S$ and therefore $X$ is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ and the functor $Lf^*$ given in Lemmas 3.4 and 3.6. Some details omitted. $\square$

## 5. Derived category of coherent modules

Let $X$ be a locally Noetherian scheme. In this case the category $\textit{Coh}(\mathcal{O}_X) \subset \textit{Mod}(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules is a weak Serre subcategory, see Homology, Section 9 and Cohomology of Schemes, Lemma 9.2. Denote

$$D_{\textit{Coh}}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 13. Thus we obtain a canonical functor

$$D(\textit{Coh}(\mathcal{O}_X)) \longrightarrow D_{\textit{Coh}}(\mathcal{O}_X) \tag{5.0.1}$$

see Derived Categories, Equation (13.1.1).

**Lemma 5.1.** *Let $S$ be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E$ be an object of $D^b_{\textit{Coh}}(\mathcal{O}_X)$ such that the scheme theoretic support of $H^i(E)$ is proper over $S$ for all $i$. Then $Rf_*E$ is an object of $D^b_{\textit{Coh}}(\mathcal{O}_S)$.*

**Proof.** Consider the spectral sequence

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E$$

see Derived Categories, Lemma 21.3. By assumption and Cohomology of Schemes, Remark 17.3 the sheaves $R^pf_*H^q(E)$ are coherent. Hence $R^{p+q}f_*E$ is coherent, i.e., $E \in D_{\textit{Coh}}(\mathcal{O}_S)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Schemes, Lemma 4.4 or from Lemma 4.1. $\square$

## 6. The coherator

Let $X$ be a scheme. The *coherator* is a functor

$$Q_X : \textit{Mod}(\mathcal{O}_X) \longrightarrow \textit{QCoh}(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor $\textit{QCoh}(\mathcal{O}_X) \to \textit{Mod}(\mathcal{O}_X)$. It exists for any scheme $X$ and moreover the adjunction mapping $Q_X(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every quasi-coherent module $\mathcal{F}$, see Properties, Proposition 21.4. Since $Q_X$ is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(\textit{QCoh}(\mathcal{O}_X)).$$

As this functor is constructed by applying $Q_X$ to a K-injective replacement we see that $RQ_X$ is a right adjoint to the canonical functor $D(\textit{QCoh}(\mathcal{O}_X)) \to D(\mathcal{O}_X)$.

**Lemma 6.1.** *Let $f : X \to Y$ be an affine morphism of schemes. Then $f_*$ defines a derived functor $f_* : D(\textit{QCoh}(\mathcal{O}_X)) \to D(\textit{QCoh}(\mathcal{O}_Y))$. This functor has the property that*

$$\begin{tikzcd} D(\textit{QCoh}(\mathcal{O}_X)) \arrow[r] \arrow[d, "f_*"'] & D_{\textit{QCoh}}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(\textit{QCoh}(\mathcal{O}_Y)) \arrow[r] & D_{\textit{QCoh}}(\mathcal{O}_Y) \end{tikzcd}$$

*commutes.*

**Proof.** The functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ is exact, see Cohomology of Schemes, Lemma 2.3. Hence $f_*$ defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by simply applying $f_*$ to any representative complex, see Derived Categories, Lemma 17.8. For any complex of $\mathcal{O}_X$-modules $\mathcal{F}^\bullet$ there is a canonical map $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$. To finish the proof we show this is a quasi-isomorphism when $\mathcal{F}^\bullet$ is a complex with each $\mathcal{F}^n$ quasi-coherent. As the statement is invariant under shifts it suffices to show that $H^0(f_*(\mathcal{F}^\bullet)) \to R^0f_*\mathcal{F}^\bullet$ is an isomorphism. The statement is local on $Y$ hence we may assume $Y$ affine. By Lemma 4.1 we have $R^0f_*\mathcal{F}^\bullet = R^0f_*\tau_{\geq -n}\mathcal{F}^\bullet$ for all sufficiently large $n$. Thus we may assume $\mathcal{F}^\bullet$ bounded below. As each $\mathcal{F}^n$ is $f_*$-acyclic by Cohomology of Schemes, Lemma 2.3 we see that $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$ is a quasi-isomorphism by Leray's acyclicity lemma (Derived Categories, Lemma 17.7). □

**Lemma 6.2.** *Let $f : X \to Y$ be a morphism of schemes. Assume that*

(1) *$f$ is quasi-compact, quasi-separated, and flat, and*
(2) *denoting*
$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$
*the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ the diagram*
$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$
*commutes.*

*Then $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.*

**Proof.** Since $f$ is quasi-compact and quasi-separated, we see that $f_*$ preserve quasi-coherence, see Schemes, Lemma 24.1. Recall that $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 21.4). Hence any $K$ in $D(QCoh(\mathcal{O}_X))$ can be represented by a K-injective complex $\mathcal{I}^\bullet$ of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 12.6. Then we can define $\Phi(K) = f_*\mathcal{I}^\bullet$.

Since $f$ is flat, the functor $f^*$ is exact. Hence $f^*$ defines $f^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ and also $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$. The functor $f^* = Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ is left adjoint to $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$, see Cohomology, Lemma 29.1. Similarly, the functor $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$ is left adjoint to $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by Derived Categories, Lemma 28.4.

Let $A$ be an object of $D(QCoh(\mathcal{O}_Y))$ and $E$ an object of $D(\mathcal{O}_X)$. Then
$$\begin{aligned} \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_*E)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(A, Rf_*E) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*A, E) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(f^*A, RQ_X(E)) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E))) \end{aligned}$$
This implies what we want. □

**Lemma 6.3.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then*

(1) *$Q_X : Mod(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$ is the functor which sends $\mathcal{F}$ to the quasi-coherent $\mathcal{O}_X$-module associated to the $A$-module $\Gamma(X, \mathcal{F})$,*

(2) $RQ_X : D(\mathcal{O}_X) \to D(QCoh(\mathcal{O}_X))$ *is the functor which sends $E$ to the complex of quasi-coherent $\mathcal{O}_X$-modules associated to the object $R\Gamma(X, E)$ of $D(A)$,*
(3) *restricted to $D_{QCoh}(\mathcal{O}_X)$ the functor $RQ_X$ defines a quasi-inverse to (3.0.1).*

**Proof.** The functor $Q_X$ is the functor

$$\mathcal{F} \mapsto \widetilde{\Gamma(X, \mathcal{F})}$$

by Schemes, Lemma 7.1. This immediately implies (1) and (2). The third assertion follows from (the proof of) Lemma 3.4. $\square$

**Definition 6.4.** Let $X$ be a scheme. Let $E$ be an object of $D(\mathcal{O}_X)$. Let $T \subset X$ be a closed subset. We say $E$ is *supported on $T$* if the cohomology sheaves $H^i(E)$ are supported on $T$.

**Proposition 6.5.** *Let $X$ be a quasi-compact scheme with affine diagonal. Then the functor (3.0.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

*is an equivalence with quasi-inverse given by $RQ_X$.*

**Proof.** In this proof we will denote $i_X : D(QCoh(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ the functor of the lemma. Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$ and let $A$ be an object of $D(QCoh(\mathcal{O}_X))$. We have to show that the adjunction maps

$$RQ_X(i_X(A)) \to A \quad \text{and} \quad E \to i_X(RQ_X(E))$$

are isomorphisms. We will prove this by induction on $n$: the smallest integer $n \geq 0$ such that $E$ and $i_X(A)$ are supported on a closed subset of $X$ which is contained in the union of $n$ affine opens of $X$.

Base case: $n = 0$. In this case $E = 0$, hence the map $E \to i_X(RQ_X(E))$ is an isomorphism. Similarly $i_X(A) = 0$. Thus the cohomology sheaves of $i_X(A)$ are zero. Since the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ is fully faithful and exact, we conclude that the cohomology objects of $A$ are zero, i.e., $A = 0$ and $RQ_X(i_X(A)) \to A$ is an isomorphism as well.

Induction step. Suppose that $E$ and $i_X(A)$ are supported on a closed subset $T$ of $X$ contained in $U_1 \cup \ldots \cup U_n$ with $U_i \subset X$ affine open. Set $U = U_n$. The inclusion morphism $j : U \to X$ is flat and affine (Morphisms, Lemma 13.11). Consider the distinguished triangles

$$A \to j_*(A|_U) \to A' \to A[1] \quad \text{and} \quad E \to Rj_*(E|_U) \to E' \to E[1]$$

where $j_*$ is as in Lemma 6.1. Note that $E \to Rj_*(E|_U)$ is a quasi-isomorphism over $U = U_n$. Since $i_X \circ j_* = Rj_* \circ i_U$ by Lemma 6.1 and since $i_X(A)|_U = i_U(A|_U)$ we see that $i_X(A) \to i_X(j_*(A|_U))$ is a quasi-isomorphism over $U$. Hence $i_X(A')$ and $E'$ are supported on the closed subset $T \setminus U$ of $X$ which is contained in $U_1 \cup \ldots \cup U_{n-1}$. By induction hypothesis the statement is true for $A'$ and $E'$. By Derived Categories, Lemma 4.3 it suffices to prove the maps

$$RQ_X(i_X(j_*(A|_U))) \to j_*(A|_U) \quad \text{and} \quad Rj_*(E|_U) \to i_X(RQ_X(Rj_*E|_U))$$

are isomorphisms. By Lemmas 6.1 and 6.2 we have

$$RQ_X(i_X(j_*(A|_U))) = RQ_X(Rj_*(i_U(A|_U))) = j_*RQ_U(i_U(A|_U))$$

and

$$i_X(RQ_X(Rj_*(E|_U))) = i_X(j_*RQ_U(E|_U)) = Rj_*(i_U(RQ_U(E|_U)))$$

Finally, the maps

$$RQ_U(i_U(A|_U)) \to A|_U \quad \text{and} \quad E|_U \to i_U(RQ_U(E|_U))$$

are isomorphisms by Lemma 6.3. The result follows. $\square$

**Remark 6.6.** Analyzing the proof of Proposition 6.5 we see that we have shown the following. Let $X$ be a quasi-compact and quasi-separated scheme. Suppose that for every affine open $U \subset X$ the right derived functor

$$\Phi : D(QCoh(\mathcal{O}_U)) \to D(QCoh(\mathcal{O}_X))$$

of the left exact functor $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$ fits into a commutative diagram

$$\begin{tikzcd} D(QCoh(\mathcal{O}_U)) \arrow[r, "i_U"'] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_U) \arrow[d, "Rj_*"] \\ D(QCoh(\mathcal{O}_X)) \arrow[r, "i_X"] & D_{QCoh}(\mathcal{O}_X) \end{tikzcd}$$

Then the functor (3.0.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by $RQ_X$.

## 7. The coherator for Noetherian schemes

In the case of Noetherian schemes we can use the following lemma.

**Lemma 7.1.** *Let $X$ be a Noetherian scheme. Let $\mathcal{J}$ be an injective object of $QCoh(\mathcal{O}_X)$. Then $\mathcal{J}$ is a flasque sheaf of $\mathcal{O}_X$-modules.*

**Proof.** Let $U \subset X$ be an open subset and let $s \in \mathcal{J}(U)$ be a section. Let $\mathcal{I} \subset X$ be the quasi-coherent sheaf of ideals defining the reduced induced scheme structure on $X \setminus U$ (see Schemes, Definition 12.5). By Cohomology of Schemes, Lemma 10.4 the section $s$ corresponds to a map $\sigma : \mathcal{I}^n \to \mathcal{J}$ for some $n$. As $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$ we can extend $\sigma$ to a map $\tilde{s} : \mathcal{O}_X \to \mathcal{J}$. Then $\tilde{s}$ corresponds to a global section of $\mathcal{J}$ restricting to $s$. $\square$

**Lemma 7.2.** *Let $f : X \to Y$ be a morphism of Noetherian schemes. Then $f_*$ on quasi-coherent sheaves has a right derived extension $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ such that the diagram*

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

*commutes.*

**Proof.** Since $X$ and $Y$ are Noetherian schemes the morphism is quasi-compact and quasi-separated (see Properties, Lemma 5.4 and Schemes, Remark 21.18). Thus $f_*$ preserve quasi-coherence, see Schemes, Lemma 24.1. Next, Let $K$ be an object of $D(QCoh(\mathcal{O}_X))$. Since $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 21.4), we can represent $K$ by a K-injective complex $\mathcal{I}^\bullet$ such that each $\mathcal{I}^n$ is an injective object of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 12.6. Thus we see that the functor $\Phi$ is defined by setting

$$\Phi(K) = f_*\mathcal{I}^\bullet$$

where the right hand side is viewed as an object of $D(QCoh(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$f_*\mathcal{I}^\bullet \longrightarrow Rf_*\mathcal{I}^\bullet$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this it suffices to prove the map induces an isomorphism on cohomology sheaves. Pick any $m \in \mathbf{Z}$. Let $N = N(X, Y, f)$ be as in Lemma 4.1. Consider the short exact sequence

$$0 \to \sigma_{\geq m-N-1}\mathcal{I}^\bullet \to \mathcal{I}^\bullet \to \sigma_{\leq m-N-2}\mathcal{I}^\bullet \to 0$$

of complexes of quasi-coherent sheaves on $X$. By Lemma 4.1 we see that the cohomology sheaves of $Rf_*\sigma_{\leq m-N-2}\mathcal{I}^\bullet$ are zero in degrees $\geq m-1$. Thus we see that $R^mf_*\mathcal{I}^\bullet$ is isomorphic to $R^mf_*\sigma_{\geq m-N-1}\mathcal{I}^\bullet$. In other words, we may assume that $\mathcal{I}^\bullet$ is a bounded below complex of injective objects of $QCoh(\mathcal{O}_X)$. This follows from Leray's acyclicity lemma (Derived Categories, Lemma 17.7) via Cohomology, Lemma 13.5 and Lemma 7.1. □

**Proposition 7.3.** *Let $X$ be a Noetherian scheme. Then the functor (3.0.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

*is an equivalence with quasi-inverse given by $RQ_X$.*

**Proof.** This follows using the exact same argument as in the proof of Proposition 6.5 using Lemma 7.2. See discussion in Remark 6.6. □

## 8. Koszul complexes

Let $A$ be a ring and let $f_1, \ldots, f_r$ be a sequence of elements of $A$. We have defined the Koszul complex $K_\bullet(f_1, \ldots, f_r)$ in More on Algebra, Definition 20.2. It is a chain complex sitting in degrees $r, \ldots, 0$. We turn this into a cochain complex $K^\bullet(f_1, \ldots, f_r)$ by setting $K^{-n}(f_1, \ldots, f_r) = K_n(f_1, \ldots, f_r)$ and using the same differentials. In the rest of this section all the complexes will be cochain complexes.

We define a complex $I^\bullet(f_1, \ldots, f_r)$ such that we have a distinguished triangle

$$I^\bullet(f_1, \ldots, f_r) \to A \to K^\bullet(f_1, \ldots, f_r) \to I^\bullet(f_1, \ldots, f_r)[1]$$

in $K(A)$. In other words, we set

$$I^i(f_1, \ldots, f_r) = \begin{cases} K^{i-1}(f_1, \ldots, f_r) & \text{if } i \leq 0 \\ 0 & \text{else} \end{cases}$$

and we use the negative of the differential on $K^\bullet(f_1, \ldots, f_r)$. The maps in the distinguished triangle are the obvious ones. Note that $I^0(f_1, \ldots, f_r) = A^{\oplus r} \to A$ is given by multiplication by $f_i$ on the $i$th factor. Hence $I^\bullet(f_1, \ldots, f_r) \to A$ factors as

$$I^\bullet(f_1, \ldots, f_r) \to I \to A$$

where $I = (f_1, \ldots, f_r)$. In fact, there is a short exact sequence

$$0 \to H^{-1}(K^\bullet(f_1, \ldots, f_s)) \to H^0(I^\bullet(f_1, \ldots, f_s)) \to I \to 0$$

and for every $i < 0$ we have $H^i(I^\bullet(f_1, \ldots, f_r)) = H^{i-1}(K^\bullet(f_1, \ldots, f_r)$. Observe that given a second sequence $g_1, \ldots, g_r$ of elements of $A$ there are canonical maps

$$I^\bullet(f_1g_1, \ldots, f_rg_r) \to I^\bullet(f_1, \ldots, f_r) \quad \text{and} \quad K^\bullet(f_1g_1, \ldots, f_rg_r) \to K^\bullet(f_1, \ldots, f_r)$$

compatible with the maps described above. The first of these maps is given by multiplication by $g_i$ on the $i$th summand of $I^0(f_1g_1, \ldots, f_rg_r) = A^{\oplus r}$. In particular, given $f_1, \ldots, f_r$ we obtain an inverse system of complexes

$$I^\bullet(f_1, \ldots, f_r) \leftarrow I^\bullet(f_1^2, \ldots, f_r^2) \leftarrow I^\bullet(f_1^3, \ldots, f_r^3) \leftarrow \ldots \tag{8.0.1}$$

which will play an important role in that which is to follow. To easily formulate the following lemmas we fix some notation.

**Situation 8.1.** Here $A$ is a ring and $f_1, \ldots, f_r$ is a sequence of elements of $A$. We set $X = \mathrm{Spec}(A)$ and $U = D(f_1) \cup \ldots \cup D(f_r) \subset X$. We denote $\mathcal{U} : U = \bigcup_{i=1,\ldots,r} D(f_i)$ the given open covering of $U$.

Our first lemma is that the complexes above can be used to compute the cohomology of quasi-coherent sheaves on $U$. Suppose given a complex $I^\bullet$ of $A$-modules and an $A$-module $M$. Then we define $\mathrm{Hom}_A(I^\bullet, M)$ to be the complex with $n$th term $\mathrm{Hom}_A(I^{-n}, M)$ and differentials given as the contragredients of the differentials on $I^\bullet$.

**Lemma 8.2.** *In Situation 8.1. Let $M$ be an $A$-module and denote $\mathcal{F}$ the associated $\mathcal{O}_X$-module. Then there is a canonical isomorphism of complexes*

$$\mathrm{colim}_e \mathrm{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M) \longrightarrow \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

*functorial in $M$.*

**Proof.** Recall that the alternating Čech complex is the subcomplex of the usual Čech complex given by alternating cochains, see Cohomology, Section 24. As usual we view a $p$-cochain in $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ as an alternating function $s$ on $\{1, \ldots, r\}^{p+1}$ whose value $s_{i_0 \ldots i_p}$ at $(i_0, \ldots, i_p)$ lies in $M_{f_{i_0} \ldots f_{i_p}} = \mathcal{F}(U_{i_0 \ldots i_p})$. On the other hand, a $p$-cochain $t$ in $\mathrm{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M)$ is given by a map $t : \wedge^{p+1}(A^{\oplus r}) \to M$. Write $[i] \in A^{\oplus r}$ for the $i$th basis element and write

$$[i_0, \ldots, i_p] = [i_0] \wedge \ldots \wedge [i_p] \in \wedge^{p+1}(A^{\oplus r})$$

Then we send $t$ as above to $s$ with

$$s_{i_0 \ldots i_p} = \frac{t([i_0, \ldots, i_p])}{f_{i_0}^e \ldots f_{i_p}^e}$$

It is clear that $s$ so defined is an alternating cochain. The construction of this map is compatible with the transition maps of the system as the transition map

$$I^\bullet(f_1^e, \ldots, f_r^e) \leftarrow I^\bullet(f_1^{e+1}, \ldots, f_r^{e+1}),$$

of the (8.0.1) sends $[i_0, \ldots, i_p]$ to $f_{i_0} \ldots f_{i_p}[i_0, \ldots, i_p]$. It is clear from the description of the localizations $M_{f_{i_0} \ldots f_{i_p}}$ in Algebra, Lemma 9.9 that these maps define an isomorphism of cochain modules in degree $p$ in the limit. To finish the proof we have to show that the map is compatible with differentials. To see this recall that

$$\begin{aligned} d(s)_{i_0\ldots i_{p+1}} &= \sum\nolimits_{j=0}^{p+1}(-1)^j s_{i_0\ldots\hat{i}_j\ldots i_p} \\ &= \sum\nolimits_{j=0}^{p+1}(-1)^j \frac{t([i_0,\ldots,\hat{i}_j,\ldots i_{p+1}])}{f_{i_0}^e\ldots\hat{f}_{i_j}^e\ldots f_{i_{p+1}}^e} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{d(t)([i_0,\ldots,i_{p+1}])}{f_{i_0}^e\ldots f_{i_{p+1}}^e} &= \frac{t(d[i_0,\ldots,i_{p+1}])}{f_{i_0}^e\ldots f_{i_{p+1}}^e} \\ &= \frac{\sum_j(-1)^j f_{i_j}^e t([i_0,\ldots,\hat{i}_j,\ldots i_{p+1}])}{f_{i_0}^e\ldots f_{i_{p+1}}^e} \end{aligned}$$

The two formulas agree by inspection. □

Suppose given a finite complex $I^\bullet$ of $A$-modules and a complex of $A$-modules $M^\bullet$. We obtain a double complex $H^{\bullet,\bullet} = \mathrm{Hom}_A(I^\bullet, M^\bullet)$ where $H^{p,q} = \mathrm{Hom}_A(I^p, M^q)$. The first differential comes from the differential on $\mathrm{Hom}_A(I^\bullet, M^q)$ and the second from the differential on $M^\bullet$. Associated to this double complex is the total complex with degree $n$ term given by

$$\bigoplus\nolimits_{p+q=n}\mathrm{Hom}_A(I^p, M^q)$$

and differential as in Homology, Definition 22.3. As our complex $I^\bullet$ has only finitely many nonzero terms, the direct sum displayed above is finite. The conventions for taking the total complex associated to a Čech complex of a complex are as in Cohomology, Section 26.

**Lemma 8.3.** *In Situation 8.1. Let $M^\bullet$ be a complex of $A$-modules and denote $\mathcal{F}^\bullet$ the associated complex of $\mathcal{O}_X$-modules. Then there is a canonical isomorphism of complexes*

$$\mathrm{colim}_e\ \mathit{Tot}(\mathrm{Hom}_A(I^\bullet(f_1^e,\ldots,f_r^e), M^\bullet)) \longrightarrow \mathit{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U},\mathcal{F}^\bullet))$$

*functorial in $M^\bullet$.*

**Proof.** Immediate from Lemma 8.2 and our conventions for taking associated total complexes. □

**Lemma 8.4.** *In Situation 8.1. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then there is a canonical isomorphism*

$$\mathit{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U},\mathcal{F}^\bullet)) \longrightarrow R\Gamma(U,\mathcal{F}^\bullet)$$

*in $D(A)$ functorial in $\mathcal{F}^\bullet$.*

**Proof.** Let $\mathcal{B}$ be the set of affine opens of $U$. Since the higher cohomology groups of a quasi-coherent module on an affine scheme are zero (Cohomology of Schemes, Lemma 2.2) this is a special case of Cohomology, Lemma 32.2. □

In Situation 8.1 denote $I_e$ the object of $D(\mathcal{O}_X)$ corresponding to the complex of $A$-modules $I^\bullet(f_1^e,\ldots,f_r^e)$ via the equivalence of Lemma 3.4. The maps (8.0.1) give a system

$$I_1 \leftarrow I_2 \leftarrow I_3 \leftarrow \ldots$$

Moreover, there is a compatible system of maps $I_e \to \mathcal{O}_X$ which become isomorphisms when restricted to $U$. Thus we see that for every object $E$ of $D(\mathcal{O}_X)$ there is a canonical map

$$\operatorname{colim}_e \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E) \longrightarrow H^0(U, E) \tag{8.4.1}$$

constructed by sending a map $I_e \to E$ to its restriction to $U$ and using that $\operatorname{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_U, E|_U) = H^0(U, E)$.

**Proposition 8.5.** *In Situation 8.1. For every object $E$ of $D_{QCoh}(\mathcal{O}_X)$ the map (8.4.1) is an isomorphism.*

**Proof.** By Lemma 3.4 we may assume that $E$ is given by a complex of quasi-coherent sheaves $\mathcal{F}^\bullet$. Let $M^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ be the corresponding complex of $A$-modules. By Lemmas 8.3 and 8.4 we have quasi-isomorphisms

$$\operatorname{colim}_e \operatorname{Tot}(\operatorname{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet)) \longrightarrow \operatorname{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

Taking $H^0$ on both sides we obtain

$$\operatorname{colim}_e \operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) = H^0(U, E)$$

Since $\operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) = \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E)$ by Lemma 3.4 the lemma follows. $\square$

In Situation 8.1 denote $K_e$ the object of $D(\mathcal{O}_X)$ corresponding to the complex of $A$-modules $K^\bullet(f_1^e, \ldots, f_r^e)$ via the equivalence of Lemma 3.4. Thus we have distinguished triangles

$$I_e \to \mathcal{O}_X \to K_e \to I_e[1]$$

and a system

$$K_1 \leftarrow K_2 \leftarrow K_3 \leftarrow \ldots$$

compatible with the system $(I_e)$. Moreover, there is a compatible system of maps

$$K_e \to H^0(K_e) = \mathcal{O}_X/(f_1^e, \ldots, f_r^e)$$

**Lemma 8.6.** *In Situation 8.1. Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. Assume that $H^i(E)|_U = 0$ for $i = -r+1, \ldots, 0$. Then given $s \in H^0(X, E)$ there exists an $e \geq 0$ and a morphism $K_e \to E$ such that $s$ is in the image of $H^0(X, K_e) \to H^0(X, E)$.*

**Proof.** Since $U$ is covered by $r$ affine opens we have $H^j(U, \mathcal{F}) = 0$ for $j \geq r$ and any quasi-coherent module (Cohomology of Schemes, Lemma 4.2). By Lemma 3.3 we see that $H^0(U, E)$ is equal to $H^0(U, \tau_{\geq -r+1}E)$. There is a spectral sequence

$$H^j(U, H^i(\tau_{\geq -r+1}E)) \Rightarrow H^{i+j}(U, \tau_{\geq -N}E)$$

see Derived Categories, Lemma 21.3. Hence $H^0(U, E) = 0$ by our assumed vanishing of cohomology sheaves of $E$. We conclude that $s|_U = 0$. Think of $s$ as a morphism $\mathcal{O}_X \to E$ in $D(\mathcal{O}_X)$. By Proposition 8.5 the composition $I_e \to \mathcal{O}_X \to E$ is zero for some $e$. By the distinguished triangle $I_e \to \mathcal{O}_X \to K_e \to I_e[1]$ we obtain a morphism $K_e \to E$ such that $s$ is the composition $\mathcal{O}_X \to K_e \to E$. $\square$

## 9. Pseudo-coherent and perfect complexes

In this section we make the connection between the general notions defined in Cohomology, Sections 35, 36, 37, and 38 and the corresponding notions for complexes of modules in More on Algebra, Sections 50, 51, and 56.

**Lemma 9.1.** *Let $X$ be a scheme. If $E$ is an $m$-pseudo-coherent object of $D(\mathcal{O}_X)$, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$. If $E$ is pseudo-coherent, then $E$ is an object of $D_{QCoh}(\mathcal{O}_X)$.*

**Proof.** Locally $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ with $\mathcal{E}^\bullet$ strictly perfect. The sheaves $\mathcal{E}^i$ are direct summands of finite free modules, hence quasi-coherent. The lemma follows. $\square$

**Lemma 9.2.** *Let $X$ be a locally ringed space. A direct summand of a free $\mathcal{O}_X$-module is finite locally free.*

**Proof.** Omitted. $\square$

**Lemma 9.3.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of $A$-modules and let $E$ be the corresponding object of $D(\mathcal{O}_X)$. Then $E$ is an $m$-pseudo-coherent (resp. pseudo-coherent) as an object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) as a complex of $A$-modules.*

**Proof.** It is immediate from the definitions that if $M^\bullet$ is $m$-pseudo-coherent, so is $E$. To prove the converse, assume $E$ is $m$-pseudo-coherent. As $X = \mathrm{Spec}(A)$ is quasi-compact with a basis for the topology given by standard opens, we can find a standard open covering $X = D(f_1) \cup \ldots \cup D(f_n)$ and strictly perfect complexes $\mathcal{E}_i^\bullet$ on $D(f_i)$ and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ inducing isomorphisms on $H^j$ for $j > m$ and surjections on $H^m$. By Cohomology, Lemma 35.8 after refining the open covering we may assume $\alpha_i$ is given by a map of complexes $\mathcal{E}_i^\bullet \to \widetilde{M^\bullet}|_{U_i}$ for each $i$. By Lemma 9.2 the terms $\mathcal{E}_i^n$ are finite locally free modules. Hence after refining the open covering we may assume each $\mathcal{E}_i^n$ is a finite free $\mathcal{O}_{U_i}$-module. From the definition it follows that $M_{f_i}^\bullet$ is an $m$-pseudo-coherent complex of $A_{f_i}$-modules. We conclude by applying More on Algebra, Lemma 50.14.

The case "pseudo-coherent" follows from the fact that $E$ is pseudo-coherent if and only if $E$ is $m$-pseudo-coherent for all $m$ (by definition) and the same is true for $M^\bullet$ by More on Algebra, Lemma 50.5. $\square$

**Lemma 9.4.** *Let $X$ be a Noetherian scheme. Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. For $m \in \mathbf{Z}$ the following are equivalent*

1. *$H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and*
2. *$E$ is $m$-pseudo-coherent.*

*In particular, $E$ is pseudo-coherent if and only if $E$ is an object of $D^-_{Coh}(\mathcal{O}_X)$.*

**Proof.** As $X$ is quasi-compact we see that in both (1) and (2) the object $E$ is bounded above. Thus the question is local on $X$ and we may assume $X$ is affine. Say $X = \mathrm{Spec}(A)$ for some Noetherian ring $A$. In this case $E$ corresponds to a complex of $A$-modules $M^\bullet$ by Lemma 3.4. By Lemma 9.3 we see that $E$ is $m$-pseudo-coherent if and only if $M^\bullet$ is $m$-pseudo-coherent. On the other hand, $H^i(E)$ is coherent if and only if $H^i(M^\bullet)$ is a finite $A$-module (Properties, Lemma 16.1). Thus the result follows from More on Algebra, Lemma 50.16. $\square$

**Lemma 9.5.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of $A$-modules and let $E$ be the corresponding object of $D(\mathcal{O}_X)$. Then*

1. *$E$ has tor amplitude in $[a, b]$ if and only if $M^\bullet$ has tor amplitude in $[a, b]$.*
2. *$E$ has finite tor dimension if and only if $M^\bullet$ has finite tor dimension.*

**Proof.** Part (2) follows trivially from part (1). In the proof of (1) we will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 3.4 without further mention. Assume $M^\bullet$ has tor amplitude in $[a, b]$. Then $K^\bullet$ is isomorphic in $D(A)$ to a complex $K^\bullet$ of flat $A$-modules with $K^i = 0$ for $i \not\in [a, b]$, see More on Algebra, Lemma 51.3. Then $E$ is isomorphic to $\widetilde{K^\bullet}$. Since each $\widetilde{K^i}$ is a flat $\mathcal{O}_X$-module, we see that $E$ has tor amplitude in $[a, b]$ by Cohomology, Lemma 37.3.

Assume that $E$ has tor amplitude in $[a, b]$. Then $E$ is bounded whence $M^\bullet$ is in $K^-(A)$. Thus we may replace $M^\bullet$ by a bounded above complex of $A$-modules. We may even choose a projective resolution and assume that $M^\bullet$ is a bounded above complex of free $A$-modules. Then for any $A$-module $N$ we have

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M^\bullet} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M^\bullet \otimes_A N}$$

in $D(\mathcal{O}_X)$. Thus the vanishing of cohomology sheaves of the left hand side implies $M^\bullet$ has tor amplitude in $[a, b]$. □

**Lemma 9.6.** *Let $X$ be a quasi-separated scheme. Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent*

1. *$E$ has tor amplitude in $[a, b]$, and*
2. *for all $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for $i \not\in [a, b]$.*

**Proof.** It is clear that (1) implies (2). Assume (2). Let $U \subset X$ be an affine open. As $X$ is quasi-separated the morphism $j : U \to X$ is quasi-compact and separated, hence $j_*$ transforms quasi-coherent modules into quasi-coherent modules (Schemes, Lemma 24.1). Thus the functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{G})$ for $i \not\in [a, b]$ for all quasi-coherent $\mathcal{O}_U$-modules $\mathcal{G}$. Write $U = \mathrm{Spec}(A)$ and let $M^\bullet$ be the complex of $A$-modules corresponding to $E|_U$ by Lemma 3.4. We have just shown that $M^\bullet \otimes^{\mathbf{L}}_A N$ has vanishing cohomology groups outside the range $[a, b]$, in other words $M^\bullet$ has tor amplitude in $[a, b]$. By Lemma 9.5 we conclude that $E|_U$ has tor amplitude in $[a, b]$. This proves the lemma. □

**Lemma 9.7.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of $A$-modules and let $E$ be the corresponding object of $D(\mathcal{O}_X)$. Then $E$ is a perfect object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is perfect as an object of $D(A)$.*

**Proof.** This is a logical consequence of Lemmas 9.3 and 9.5, Cohomology, Lemma 38.4, and More on Algebra, Lemma 56.2. □

As a consequence of our description of pseudo-coherent complexes on schemes we can prove certain internal homs are quasi-coherent.

**Lemma 9.8.** *Let $X$ be a scheme.*

1. *If $L$ is in $D^+_{QCoh}(\mathcal{O}_X)$ and $K$ in $D(\mathcal{O}_X)$ is pseudo-coherent, then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$.*
2. *If $L$ is in $D_{QCoh}(\mathcal{O}_X)$ and $K$ in $D(\mathcal{O}_X)$ is perfect, then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$.*

(3) *If $X = \mathrm{Spec}(A)$ is affine and $K, L \in D(A)$ then*

$$R\mathcal{H}om(\widetilde{K}, \widetilde{L}) = R\widetilde{\mathrm{Hom}(K, L)}$$

*in the following two cases*
 (a) *$K$ is pseudo-coherent and $L$ is bounded below,*
 (b) *$K$ is perfect and $L$ arbitrary.*

(4) *If $X = \mathrm{Spec}(A)$ and $K, L$ are in $D(A)$, then the $n$th cohomology sheaf of $R\mathcal{H}om(\widetilde{K}, \widetilde{L})$ is the sheaf associated to the presheaf*

$$X \supset D(f) \longmapsto Ext^n_{A_f}(K \otimes_A A_f, L \otimes_A A_f)$$

*for $f \in A$.*

**Proof.** The construction of the internal hom in the derived category of $\mathcal{O}_X$ commutes with localization (see Cohomology, Section 34). Hence to prove (1) and (2) we may replace $X$ by an affine open. By Lemmas 3.4, 9.3, and 9.7 in order to prove (1) and (2) it suffices to prove (3).

Part (3) follows from the computation of the internal hom of Cohomology, Lemma 35.10 by representing $K$ by a bounded above (resp. finite) complex of finite projective $A$-modules and $L$ by a bounded above (resp. arbitrary) complex of $A$-modules.

To prove (4) recall that on any ringed space the $n$th cohomology sheaf of $R\mathcal{H}om(A, B)$ is the sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_{D(U)}(A|_U, B|_U[n]) = \mathrm{Ext}^n_{D(\mathcal{O}_U)}(A|_U, B|_U)$$

See Cohomology, Section 34. On the other hand, the restriction of $\widetilde{K}$ to a principal open $D(f)$ is the image of $K \otimes_A A_f$ and similarly for $L$. Hence (4) follows from the equivalence of categories of Lemma 3.4. □

## 10. Descent finiteness properties of complexes

This section is the analogue of Descent, Section 6 for objects of the derived category of a scheme. The easiest such result is probably the following.

**Lemma 10.1.** *Let $f : X \to Y$ be a surjective flat morphism of schemes (or more generally locally ringed spaces). Let $E \in D(\mathcal{O}_Y)$. Let $a, b \in \mathbf{Z}$. Then $E$ has tor-amplitude in $[a, b]$ if and only if $Lf^*E$ has tor-amplitude in $[a, b]$.*

**Proof.** Pullback always preserves tor-amplitude, see Cohomology, Lemma 37.4. We may check tor-amplitude in $[a, b]$ on stalks, see Cohomology, Lemma 37.5. A flat local ring homomorphism is faithfully flat by Algebra, Lemma 38.16. Thus the result follows from More on Algebra, Lemma 51.14. □

**Lemma 10.2.** *Let $\{f_i : X_i \to X\}$ be an fpqc covering of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then $E$ is $m$-pseudo-coherent if and only if each $Lf_i^*E$ is $m$-pseudo-coherent.*

**Proof.** Pullback always preserves $m$-pseudo-coherence, see Cohomology, Lemma 36.3. Conversely, assume that $Lf_i^*E$ is $m$-pseudo-coherent for all $i$. Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is $m$-pseudo-coherent. Since $\{f_i : X_i \to X\}$ is an fpqc covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace $X$ by $U$ and $\{f_i : X_i \to X\}$ by $\{V \to U\}$ and assume that $X$ is affine and our covering is given by a single surjective flat morphism $\{f : Y \to X\}$ of affine schemes. In this case the result follows from More on Algebra, Lemma 50.15 via Lemmas 3.4 and 9.3. $\square$

**Lemma 10.3.** *Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. Let $E \in D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then $E$ is $m$-pseudo-coherent if and only if each $Lf_i^*E$ is $m$-pseudo-coherent.*

**Proof.** Pullback always preserves $m$-pseudo-coherence, see Cohomology, Lemma 36.3. Conversely, assume that $Lf_i^*E$ is $m$-pseudo-coherent for all $i$. Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is $m$-pseudo-coherent. Since $\{f_i : X_i \to X\}$ is an fppf covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace $X$ by $U$ and $\{f_i : X_i \to X\}$ by $\{V \to U\}$ and assume that $X$ is affine and our covering is given by a single surjective flat morphism $\{f : Y \to X\}$ of finite presentation.

Since $f$ is flat the derived functor $Lf^*$ is just given by $f^*$ and $f^*$ is exact. Hence $H^i(Lf^*E) = f^*H^i(E)$. Since $Lf^*E$ is $m$-pseudo-coherent, we see that $Lf^*E \in D^-(\mathcal{O}_Y)$. Since $f$ is surjective and flat, we see that $E \in D^-(\mathcal{O}_X)$. Let $i \in \mathbf{Z}$ be the largest integer such that $H^i(E)$ is nonzero. If $i < m$, then we are done. Otherwise, $f^*H^i(E)$ is a finite type $\mathcal{O}_Y$-module by Cohomology, Lemma 36.9. Then by Descent, Lemma 6.2 the $\mathcal{O}_X$-module $H^i(E)$ is of finite type. Thus, after replacing $X$ by the members of a finite affine open covering, we may assume there exists a map

$$\alpha : \mathcal{O}_X^{\oplus n}[-i] \longrightarrow E$$

such that $H^i(\alpha)$ is a surjection. Let $C$ be the cone of $\alpha$ in $D(\mathcal{O}_X)$. Pulling back to $Y$ and using Cohomology, Lemma 36.4 we find that $Lf^*C$ is $m$-pseudo-coherent. Moreover $H^j(C) = 0$ for $j \geq i$. Thus by induction on $i$ we see that $C$ is $m$-pseudo-coherent. Using Cohomology, Lemma 36.4 again we conclude. $\square$

**Lemma 10.4.** *Let $\{f_i : X_i \to X\}$ be an fpqc covering of schemes. Let $E \in D(\mathcal{O}_X)$. Then $E$ is perfect if and only if each $Lf_i^*E$ is perfect.*

**Proof.** Pullback always preserves perfect complexes, see Cohomology, Lemma 38.5. Conversely, assume that $Lf_i^*E$ is perfect for all $i$. Then the cohomology sheaves of each $Lf_i^*E$ are quasi-coherent, see Lemma 9.1 and Cohomology, Lemma 38.4. Since the morphisms $f_i$ is flat we see that $H^p(Lf_i^*E) = f_i^*H^p(E)$. Thus the cohomology sheaves of $E$ are quasi-coherent by Descent, Proposition 5.2. Having said this the lemma follows formally from Cohomology, Lemma 38.4 and Lemmas 10.1 and 10.2. $\square$

**Lemma 10.5.** *Let $i : Z \to X$ be a morphism of ringed spaces such that $i$ is a closed immersion of underlying topological spaces and such that $i_*\mathcal{O}_Z$ is pseudo-coherent as an $\mathcal{O}_X$-module. Let $E \in D(\mathcal{O}_X)$. Then $E$ is $m$-pseudo-coherent if and only if $Ri_*E$ is $m$-pseudo-coherent.*

**Proof.** Throughout this proof we will use that $i_*$ is an exact functor, and hence that $Ri_* = i_*$, see Modules, Lemma 6.1.

Assume $E$ is $m$-pseudo-coherent. Let $x \in X$. We will find a neighbourhood of $x$ such that $i_*E$ is $m$-peudo-coherent on it. If $x \notin Z$ then this is clear. Thus we may assume $x \in Z$. We will use that $U \cap Z$ for $x \in U \subset X$ open form a fundamental system of neighbourhoods of $x$ in $Z$. After shrinking $X$ we may assume $E$ is bounded above. We will argue by induction on the largest integer $p$ such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(E)$ is an $\mathcal{O}_Z$-module of finite type, see Cohomology, Lemma 36.9. Thus we may choose, after shrinking $X$, a map $\mathcal{O}_Z^{\oplus n}[-p] \to E$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \to H^p(E)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \to E \to C \to \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$ and that $C$ is $m$-pseudo-coherent by Cohomology, Lemma 36.4. By induction we see that $i_*C$ is $m$-pseudo-coherent on $X$. Since $i_*\mathcal{O}_Z$ is $m$-pseudo-coherent on $X$ as well, we conclude from the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

and Cohomology, Lemma 36.4 that $i_*E$ is $m$-pseudo-coherent.

Assume that $i_*E$ is $m$-pseudo-coherent. Let $z \in Z$. We will find a neighbourhood of $z$ such that $E$ is $m$-peudo-coherent on it. We will use that $U \cap Z$ for $z \in U \subset X$ open form a fundamental system of neighbourhoods of $z$ in $Z$. After shrinking $X$ we may assume $i_*E$ and hence $E$ is bounded above. We will argue by induction on the largest integer $p$ such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(i_*E) = i_*H^p(E)$ is an $\mathcal{O}_X$-module of finite type, see Cohomology, Lemma 36.9. Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Z$-modules representing $E$. We may choose, after shrinking $X$, a map $\alpha : \mathcal{O}_X^{\oplus n}[-p] \to i_*\mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_X^{\oplus n} \to i_*H^p(\mathcal{E}^\bullet)$. By adjunction we find a map $\alpha : \mathcal{O}_Z^{\oplus n}[-p] \to \mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \to H^p(\mathcal{E}^\bullet)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \to E \to C \to \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$. From the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

the fact that $i_*\mathcal{O}_Z$ is pseudo-coherent and Cohomology, Lemma 36.4 we conclude that $i_*C$ is $m$-pseudo-coherent. By induction we conclude that $C$ is $m$-pseudo-coherent. By Cohomology, Lemma 36.4 again we conclude that $E$ is $m$-pseudo-coherent. $\square$

**Lemma 10.6.** *Let $f : X \to Y$ be a finite morphism of schemes such that $f_*\mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_Y$-module*[1]*. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Then $E$ is $m$-pseudo-coherent if and only if $Rf_*E$ is $m$-pseudo-coherent.*

**Proof.** This is a translation of More on Algebra, Lemma 50.11 into the language of schemes. To do the translation, use Lemmas 3.4 and 9.3. $\square$

## 11. Lifting complexes

Let $U \subset X$ be an open subspace of a ringed space and denote $j : U \to X$ the inclusion morphism. The functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is essentially surjective as $Rj_*$ is a right inverse to restriction. In this section we extend this to complexes with quasi-coherent cohomology sheaves, etc.

[1]This means that $f$ is pseudo-coherent, see More on Morphisms, Definition 40.2.

**Lemma 11.1.** *Let $X$ be a scheme and let $j : U \to X$ be a quasi-compact open immersion. The functors*

$$D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_U) \quad \text{and} \quad D^+_{QCoh}(\mathcal{O}_X) \to D^+_{QCoh}(\mathcal{O}_U)$$

*are essentially surjective. If $X$ is quasi-compact, then the functors*

$$D^-_{QCoh}(\mathcal{O}_X) \to D^-_{QCoh}(\mathcal{O}_U) \quad \text{and} \quad D^b_{QCoh}(\mathcal{O}_X) \to D^b_{QCoh}(\mathcal{O}_U)$$

*are essentially surjective.*

**Proof.** The argument preceding the lemma applies for the first case because $Rj_*$ maps $D_{QCoh}(\mathcal{O}_U)$ into $D_{QCoh}(\mathcal{O}_X)$ by Lemma 4.1. It is clear that $Rj_*$ maps $D^+_{QCoh}(\mathcal{O}_U)$ into $D^+_{QCoh}(\mathcal{O}_X)$ which implies the statement on bounded below complexes. Finally, Lemma 4.1 guarantees that $Rj_*$ maps $D^-_{QCoh}(\mathcal{O}_U)$ into $D^-_{QCoh}(\mathcal{O}_X)$ if $X$ is quasi-compact. Combining these two we obtain the last statement. $\square$

**Lemma 11.2.** *Let $X$ be an affine scheme and let $U \subset X$ be a quasi-compact open subscheme. For any pseudo-coherent object $E$ of $D(\mathcal{O}_U)$ there exists a bounded above complex of finite free $\mathcal{O}_X$-modules whose restriction to $U$ is isomorphic to $E$.*

**Proof.** By Lemma 9.1 we see that $E$ is an object of $D_{QCoh}(\mathcal{O}_U)$. By Lemma 11.1 we may assume $E = E'|U$ for some object $E'$ of $D_{QCoh}(\mathcal{O}_X)$. Write $X = \mathrm{Spec}(A)$. By Lemma 3.4 we can find a complex $M^\bullet$ of $A$-modules whose associated complex of $\mathcal{O}_X$-modules is a representative of $E'$.

Choose $f_1, \ldots, f_r \in A$ such that $U = D(f_1) \cup \ldots \cup D(f_r)$. By Lemma 9.3 the complexes $M^\bullet_{f_j}$ are pseudo-coherent complexes of $A_{f_j}$-modules. Let $n$ be an integer. Assume we have a map of complexes $\alpha : F^\bullet \to M^\bullet$ where $F^\bullet$ is bounded above, $F^i = 0$ for $i < n$, each $F^i$ is a finite free $R$-module, such that

$$H^i(\alpha_{f_j}) : H^i(F^\bullet_{f_j}) \to H^i(M^\bullet_{f_j})$$

is an isomorphism for $i > n$ and surjective for $i = n$. Picture

$$\begin{tikzcd} & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Since each $M^\bullet_{f_j}$ has vanishing cohomology in large degrees we can find such a map for $n \gg 0$. By induction on $n$ we are going to extend this to a map of complexes $F^\bullet \to M^\bullet$ such that $H^i(\alpha_{f_j})$ is an isomorphism for all $i$. The lemma will follow by taking $\widetilde{F^\bullet}$.

The induction step will be to extend the diagram above by adding $F^{n-1}$. Let $C^\bullet$ be the cone on $\alpha$ (Derived Categories, Definition 9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet_{f_j}) = 0$ for $i \geq n$. By More on Algebra, Lemma 50.2 we see that $C^\bullet_{f_j}$ is $(n-1)$-pseudo-coherent. By More on Algebra, Lemma 50.3 we see that $H^{-1}(C^\bullet_{f_j})$ is a finite $A_{f_j}$-module. Choose a finite free $A$-module $F^{n-1}$ and an $A$-module $\beta : F^{n-1} \to C^{-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that $F^{n-1}_{f_j}$ surjects onto $H^{n-1}(C^\bullet_{f_j})$. (Some details omitted; hint: clear denominators.) Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$.

The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

$$\begin{tikzcd} & F^{n-1} \arrow[r, "d^{n-1}"'] \arrow[d, "\alpha^{n-1}"] & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ \ldots \arrow[r] & M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{tikzcd} (F^n \to \ldots) \arrow[r] \arrow[d] & (F^{n-1} \to \ldots) \arrow[r] \arrow[d] & F^{n-1} \arrow[r] \arrow[d, "\beta"'] & (F^n \to \ldots)[1] \arrow[d] \\ (F^n \to \ldots) \arrow[r] & M^\bullet \arrow[r] & C^\bullet \arrow[r] & (F^n \to \ldots)[1] \end{tikzcd}$$

Hence our choice of $\beta$ implies that the map of complexes $(F^{-1} \to \ldots) \to M^\bullet$ induces an isomorphism on cohomology localized at $f_j$ in degrees $\geq n$ and a surjection in degree $-1$. This finishes the proof of the lemma. $\square$

**Lemma 11.3.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $E \in D^b_{QCoh}(\mathcal{O}_X)$. There exists an integer $n_0 > 0$ such that $Ext^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = 0$ for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and every $n \geq n_0$.*

**Proof.** Recall that $\text{Ext}^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n])$. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 30.6. Thus if $X = U \cup V$ and the result of the lemma holds for $E|_U$, $E|_V$, and $E|_{U\cap V}$ for some bound $n_0$, then the result holds for $E$ with bound $n_0 + 1$. Thus it suffices to prove the lemma when $X$ is affine, see Cohomology of Schemes, Lemma 4.1.

Assume $X = \text{Spec}(A)$ is affine. Choose a complex of $A$-modules $M^\bullet$ whose associated complex of quasi-coherent modules represents $E$, see Lemma 3.4. Write $\mathcal{E} = \widetilde{P}$ for some $A$-module $P$. Since $\mathcal{E}$ is finite locally free, we see that $P$ is a finite projective $A$-module. We have

$$\begin{aligned} \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n]) &= \text{Hom}_{D(A)}(P, M^\bullet[n]) \\ &= \text{Hom}_{K(A)}(P, M^\bullet[n]) \\ &= \text{Hom}_A(P, H^n(M^\bullet)) \end{aligned}$$

The first equality by Lemma 3.4, the second equality by Derived Categories, Lemma 19.8, and the final equality because $\text{Hom}_A(P, -)$ is an exact functor. As $E$ and hence $M^\bullet$ is bounded we get zero for all sufficiently large $n$. $\square$

**Lemma 11.4.** *Let $X$ be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object $E$ of $D(\mathcal{O}_U)$ there exists an integer $r$ and a finite locally free sheaf $\mathcal{F}$ on $U$ such that $\mathcal{F}[-r] \oplus E$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.*

**Proof.** Say $X = \text{Spec}(A)$. Recall that a perfect complex is pseudo-coherent, see Cohomology, Lemma 38.4. By Lemma 11.2 we can find a bounded above complex $\mathcal{F}^\bullet$ of finite free $A$-modules such that $E$ is isomorphic to $\mathcal{F}^\bullet|_U$ in $D(\mathcal{O}_U)$. By Cohomology, Lemma 38.4 and since $U$ is quasi-compact, we see that $E$ has finite tor dimension, say $E$ has tor amplitude in $[a, b]$. Pick $r < a$ and set

$$\mathcal{F} = \text{Ker}(\mathcal{F}^r \to \mathcal{F}^{r+1}) = \text{Im}(\mathcal{F}^{r-1} \to \mathcal{F}^r).$$

Since $E$ has tor amplitude in $[a, b]$ we see that $\mathcal{F}|_U$ is flat (Cohomology, Lemma 37.2). Hence $\mathcal{F}|_U$ is flat and of finite presentation, thus finite locally free (Properties, Lemma 18.2). It follows that

$$(\mathcal{F} \to \mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)|_U$$

is a strictly perfect complex on $U$ representing $E$. We obtain a distinguished triangle

$$\mathcal{F}|_U[-r-1] \to E \to (\mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)|_U \to \mathcal{F}|_U[-r]$$

Note that $(\mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)$ is a perfect complex on $X$. To finish the proof it suffices to pick $r$ such that the map $\mathcal{F}|_U[-r-1] \to E$ is zero in $D(\mathcal{O}_U)$, see Derived Categories, Lemma 4.10. By Lemma 11.3 this holds if $r \ll 0$. $\square$

**Lemma 11.5.** *Let $X$ be an affine scheme. Let $U \subset X$ be a quasi-compact open. Let $E, E'$ be objects of $D_{QCoh}(\mathcal{O}_X)$ with $E$ perfect. For every map $\alpha : E|_U \to E'|_U$ there exist maps*

$$E \xleftarrow{\beta} E_1 \xrightarrow{\gamma} E'$$

*of perfect complexes on $X$ such that $\beta : E_1 \to E$ restricts to an isomorphism on $U$ and such that $\alpha = \gamma|_U \circ \beta|_U^{-1}$. Moreover we can assume $E_1 = E \otimes_{\mathcal{O}_X}^{\mathbf{L}} I$ for some perfect complex $I$ on $X$.*

**Proof.** Write $X = \mathrm{Spec}(A)$. Write $U = D(f_1) \cup \ldots \cup D(f_r)$. Choose finite complex of finite projective $A$-modules $M^\bullet$ representing $E$ (Lemma 9.7). Choose a complex of $A$-modules $(M')^\bullet$ representing $E'$ (Lemma 3.4). In this case the complex $H^\bullet = \mathrm{Hom}_A(M^\bullet, (M')^\bullet)$ is a complex of $A$-modules whose associated complex of quasi-coherent $\mathcal{O}_X$-modules represents $R\mathcal{H}om(E, E')$, see Cohomology, Lemma 35.9. Then $\alpha$ determines an element $s$ of $H^0(U, R\mathcal{H}om(E, E'))$, see Cohomology, Lemma 34.1. There exists an $e$ and a map

$$\xi : I^\bullet(f_1^e, \ldots, f_r^e) \to \mathrm{Hom}_A(M^\bullet, (M')^\bullet)$$

corresponding to $s$, see Proposition 8.5. Letting $E_1$ be the object corresponding to complex of quasi-coherent $\mathcal{O}_X$-modules associated to

$$\mathrm{Tot}(I^\bullet(f_1^e, \ldots, f_r^e) \otimes_A M^\bullet)$$

we obtain $E_1 \to E$ using the canonical map $I^\bullet(f_1^e, \ldots, f_r^e) \to A$ and $E_1 \to E'$ using $\xi$ and Cohomology, Lemma 34.1. $\square$

**Lemma 11.6.** *Let $X$ be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object $F$ of $D(\mathcal{O}_U)$ the object $F \oplus F[1]$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.*

**Proof.** By Lemma 11.4 we can find a perfect object $E$ of $D(\mathcal{O}_X)$ such that $E|_U = \mathcal{F}[r] \oplus F$ for some finite locally free $\mathcal{O}_U$-module $\mathcal{F}$. By Lemma 11.5 we can find a morphism of perfect complexes $\alpha : E_1 \to E$ such that $(E_1)|_U \cong E|_U$ and such that $\alpha|_U$ is the map

$$\begin{pmatrix} \mathrm{id}_{\mathcal{F}[r]} & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{F}[r] \oplus F \to \mathcal{F}[r] \oplus F$$

Then the cone on $\alpha$ is a solution. $\square$

**Lemma 11.7.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. For any morphism $\alpha : E \to E'$ in $D_{QCoh}(\mathcal{O}_X)$ such that*

(1) *$E$ is perfect, and*

(2) *$E'$ is supported on $T = V(f)$*

*there exists an $n \geq 0$ such that $f^n\alpha = 0$.*

**Proof.** We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 30.6. Thus if $X = U \cup V$ and the result of the lemma holds for $f|_U$, $f|_V$, and $f|_{U\cap V}$, then the result holds for $f$. Thus it suffices to prove the lemma when $X$ is affine, see Cohomology of Schemes, Lemma 4.1.

Let $X = \text{Spec}(A)$. Then $f \in A$. We will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 3.4 without further mention. Represent $E$ by a finite complex of finite projective $A$-modules $P^\bullet$. This is possible by Lemma 9.7. Let $t$ be the largest integer such that $P^t$ is nonzero. The distinguished triangle

$$P^t[-t] \to P^\bullet \to \sigma_{\leq t-1}P^\bullet \to P^t[-t+1]$$

shows that by induction on the length of the complex $P^\bullet$ we can reduce to the case where $P^\bullet$ has a single nonzero term. This and the shift functor reduces us to the case where $P^\bullet$ consists of a single finite projective $A$-module $P$ in degree 0. Represent $E'$ by a complex $M^\bullet$ of $A$-modules. Then $\alpha$ corresponds to a map $P \to H^0(M^\bullet)$. Since the module $H^0(M^\bullet)$ is supported on $V(f)$ by assumption (2) we see that every element of $H^0(M^\bullet)$ is annihilated by a power of $f$. Since $P$ is a finite $A$-module the map $f^n\alpha : P \to H^0(M^\bullet)$ is zero for some $n$ as desired. □

**Lemma 11.8.** *Let $X$ be an affine scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $U \subset X$ be a quasi-compact open. For every perfect object $F$ of $D(\mathcal{O}_U)$ supported on $T \cap U$ the object $F \oplus F[1]$ is the restriction of a perfect object $E$ of $D(\mathcal{O}_X)$ supported in $T$.*

**Proof.** Say $T = V(g_1, \ldots, g_s)$. After replacing $g_j$ by a power we may assume multiplication by $g_j$ is zero on $F$, see Lemma 11.7. Choose $E$ as in Lemma 11.6. Note that $g_j : E \to E$ restricts to zero on $U$. Choose a distinguished triangle

$$E \xrightarrow{g_1} E \to C_1 \to E[1]$$

By Derived Categories, Lemma 4.10 the object $C_1$ restricts to $F \oplus F[1] \oplus F[1] \oplus F[2]$ on $U$. Moreover, $g_1 : C_1 \to C_1$ has square zero by Derived Categories, Lemma 4.5. Namely, the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "0"'] & C_1 \arrow[r] \arrow[d, "g_1"'] & E[1] \arrow[d, "0"'] \\ E \arrow[r] & C_1 \arrow[r] & E[1] \end{tikzcd}$$

is commutative since the compositions $E \xrightarrow{g_1} E \to C_1$ and $C_1 \to E[1] \xrightarrow{g_1} E[1]$ are zero. Continuing, setting $C_{i+1}$ equal to the cone of the map $g_i : C_i \to C_i$ we obtain a perfect complex $C_s$ on $X$ supported on $T$ whose restriction to $U$ gives

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s]$$

Choose a morphisms of perfect complexes $\beta : C' \to C_s$ and $\gamma : C' \to C_s$ as in Lemma 11.5 such that $\beta|_U$ is an isomorphism and such that $\gamma|_U \circ \beta|_U^{-1}$ is the morphism

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s] \to F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s]$$

which is the identity on all summands except for $F$ where it is zero. By Lemma 11.5 we also have $C' = C_s \otimes^{\mathbf{L}} I$ for some perfect complex $I$ on $X$. Hence the nullity of $g_j^2 \mathrm{id}_{C_s}$ implies the same thing for $C'$. Thus $C'$ is supported on $T$ as well. Then $\mathrm{Cone}(\gamma)$ is a solution. $\square$

A special case of the following lemma can be found in [Nee96].

**Lemma 11.9.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in $X$. Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\alpha : P \to E|_U$ be a map where $P$ is a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$. Then there exists a map $\beta : R \to E$ where $R$ is a perfect object of $D(\mathcal{O}_X)$ supported on $T$ such that $P$ is a direct summand of $R|_U$ in $D(\mathcal{O}_U)$ compatible $\alpha$ and $\beta|_U$.*

**Proof.** Since $X$ is quasi-compact there exists an integer $m$ such that $X = U \cup V_1 \cup \ldots \cup V_m$ for some affine opens $V_j$ of $X$. Arguing by induction on $m$ we see that we may assume $m = 1$. In other words, we may assume that $X = U \cup V$ with $V$ affine. By Lemma 11.8 we can choose a perfect object $Q$ in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V}$. By Lemma 11.5 we can replace $Q$ by $Q \otimes^{\mathbf{L}} I$ (still supported on $T \cap V$) and assume that the map

$$Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V} \longrightarrow P|_{U \cap V} \longrightarrow E|_{U \cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 30.10 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus $R$ is perfect and supported on $T$ as desired. $\square$

**Remark 11.10.** The proof of Lemma 11.9 shows that

$$R|_U = P \oplus P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_U$ equals that of $P$. By repeating the construction for the map $P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m] \to R|_U$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_U$ and $P$ above any given degree.

## 12. Approximation by perfect complexes

In this section we discuss the observation, due to Neeman and Lipman, that a pseudo-coherent complex can be "approximated" by perfect complexes.

**Definition 12.1.** Let $X$ be a scheme. Consider triples $(T, E, m)$ where

(1) $T \subset X$ is a closed subset,
(2) $E$ is an object of $D_{QCoh}(\mathcal{O}_X)$, and
(3) $m \in \mathbf{Z}$.

We say *approximation holds for the triple* $(T, E, m)$ if there exists a perfect object $P$ of $D(\mathcal{O}_X)$ supported on $T$ and a map $\alpha : P \to E$ which induces isomorphisms $H^i(P) \to H^i(E)$ for $i > m$ and a surjection $H^m(P) \to H^m(E)$.

Approximation cannot hold for every triple. Namely, it is clear that if approximation holds for the triple $(T, E, m)$, then

(1) $E$ is $m$-pseudo-coherent, see Cohomology, Definition 36.1, and
(2) the cohomology sheaves $H^i(E)$ are supported on $T$ for $i \geq m$.

Moreover, the "support" of a perfect complex is a closed subscheme whose complement is retrocompact in $X$ (details omitted). Hence we cannot expect approximation to hold without this assumption on $T$. This partly explains the conditions in the following definition.

**Definition 12.2.** Let $X$ be a scheme. We say *approximation by perfect complexes holds* on $X$ if for any closed subset $T \subset X$ with $X \setminus T$ retro-compact in $X$ there exists an integer $r$ such that for every triple $(T, E, m)$ as in Definition 12.1 with

(1) $E$ is $(m-r)$-pseudo-coherent, and
(2) $H^i(E)$ is supported on $T$ for $i \geq m - r$

approximation holds.

We will prove that approximation by perfect complexes holds for quasi-compact and quasi-separated schemes. It seems that the second condition is necessary for our method of proof. It is possible that the first condition may be weakened to "$E$ is $m$-pseudo-coherent" by carefuly analyzing the arguments below.

**Lemma 12.3.** *Let $X$ be a scheme. Let $U \subset X$ be an open subscheme. Let $(T, E, m)$ be a triple as in Definition 12.1. If*

(1) *$T \subset U$,*
(2) *approximation holds for $(T, E|_U, m)$, and*
(3) *the sheaves $H^i(E)$ for $i \geq m$ are supported on $T$,*

*then approximation holds for $(T, E, m)$.*

**Proof.** Let $j : U \to X$ be the inclusion morphism. If $P \to E|_U$ is an approximation of the triple $(T, E|_U, m)$ over $U$, then $j_!P = Rj_*P \to j_!(E|_U) \to E$ is an approximation of $(T, E, m)$ over $X$. See Cohomology, Lemmas 30.9 and 38.9. $\square$

**Lemma 12.4.** *Let $X$ be an affine scheme. Then approximation holds for every triple $(T, E, m)$ as in Definition 12.1 such that there exists an integer $r \geq 0$ with*

(1) *$E$ is $m$-pseudo-coherent,*
(2) *$H^i(E)$ is supported on $T$ for $i \geq m - r + 1$,*
(3) *$X \setminus T$ is the union of $r$ affine opens.*

*In particular, approximation by perfect complexes holds for affine schemes.*

**Proof.** Say $X = \mathrm{Spec}(A)$. Write $T = V(f_1, \ldots, f_r)$. (The case $r = 0$, i.e., $T = X$ follows immediately from Lemma 9.3 and the definitions.) Let $(T, E, m)$ be a triple as in the lemma. Let $t$ be the largest integer such that $H^t(E)$ is nonzero. We will proceed by induction on $t$. The base case is $t < m$; in this case the result is trivial. Now suppose that $t \geq m$. By Cohomology, Lemma 36.9 the sheaf $H^t(E)$ is of finite type. Since it is quasi-coherent it is generated by finitely many sections (Properties, Lemma 16.1). For every $s \in \Gamma(X, H^t(E)) = H^t(X, E)$ (see proof of Lemma 3.4) we can find an $e > 0$ and a morphism $K_e[-t] \to E$ such that $s$ is in the image of $H^0(K_e) = H^t(K_e[-t]) \to H^t(E)$, see Lemma 8.6. Taking a finite direct sum of these maps we obtain a map $P \to E$ where $P$ is a perfect complex supported on $T$, where $H^i(P) = 0$ for $i > t$, and where $H^t(P) \to E$ is surjective. Choose a distinguished triangle

$$P \to E \to E' \to P[1]$$

Then $E'$ is $m$-pseudo-coherent (Cohomology, Lemma 36.4), $H^i(E') = 0$ for $i \geq t$, and $H^i(E')$ is supported on $T$ for $i \geq m - r + 1$. By induction we find an approximation $P' \to E'$ of $(T, E', m)$. Fit the composition $P' \to E' \to P[1]$ into a distringuished triangle $P \to P'' \to P' \to P[1]$ and extend the morphisms $P' \to E'$ and $P[1] \to P[1]$ into a morphism of distinguished triangles

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P'' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & P[1] \arrow[d] \\ P \arrow[r] & E \arrow[r] & E' \arrow[r] & P[1] \end{tikzcd}$$

using TR3. Then $P''$ is a perfect complex (Cohomology, Lemma 38.6) supported on $T$. An easy diagram chase shows that $P'' \to E$ is the desired approximation. $\square$

**Lemma 12.5.** *Let $X$ be a scheme. Let $X = U \cup V$ be an open covering with $U$ quasi-compact, $V$ affine, and $U \cap V$ quasi-compact. If approximation by perfect complexes holds on $U$, then approximation holds on $X$.*

**Proof.** Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in $X$. Let $r_U$ be the integer of Definition 12.2 adapted to the pair $(U, T \cap U)$. Set $T' = T \setminus U$. Note that $T' \subset V$ and that $V \setminus T' = (X \setminus T) \cap U \cap V$ is quasi-compact by our assumption on $T$. Let $r'$ be the number of affines needed to cover $V \setminus T'$. We claim that $r = \max(r_U, r')$ works for the pair $(X, T)$.

To see this choose a triple $(T, E, m)$ such that $E$ is $(m-r)$-pseudo-coherent and $H^i(E)$ is supported on $T$ for $i \geq m - r$. Let $t$ be the largest integer such that $H^t(E)|_U$ is nonzero. (Such an integer exists as $U$ is quasi-compact and $E|_U$ is $(m-r)$-pseudo-coherent.) We will prove that $E$ can be approximated by induction on $t$.

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on $T'$ for $i \geq m - r'$. Hence Lemma 12.4 guarantees the existence of an approximation $P \to E|_V$ of $(T', E|_V, m)$ on $V$. Applying Lemma 12.3 we see that $(T', E, m)$ can be approximated. Such an approximation is also an approximation of $(T, E, m)$.

Induction step. Choose an approximation $P \to E|_U$ of $(T \cap U, E|_U, m)$. This in particular gives a surjection $H^t(P) \to H^t(E|_U)$. By Lemma 11.8 we can choose a perfect object $Q$ in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U\cap V} \to (P \oplus P[1])|_{U \cap V}$. By Lemma 11.5 we can replace $Q$ by $Q \otimes^{\mathbf{L}} I$ and assume that the map

$$Q|_{U\cap V} \to (P \oplus P[1])|_{U\cap V} \longrightarrow P|_{U \cap V} \longrightarrow E|_{U\cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 30.10 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus $R$ is perfect and supported on $T$ and the map $H^t(R) \to H^t(E)$ is surjective on restriction to $U$. Choose a distinguised triangle

$$R \to E \to E' \to R[1]$$

Then $E'$ is $(m-r)$-pseudo-coherent (Cohomology, Lemma 36.4), $H^i(E')|_U = 0$ for $i \geq t$, and $H^i(E')$ is supported on $T$ for $i \geq m - r$. By induction we find an approximation $R' \to E'$ of $(T, E', m)$. Fit the composition $R' \to E' \to R[1]$ into a distringuished triangle $R \to R'' \to R' \to R[1]$ and extend the morphisms $R' \to E'$

and $R[1] \to R[1]$ into a morphism of distinguished triangles

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R'' \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d] & R[1] \arrow[d] \\ R \arrow[r] & E \arrow[r] & E' \arrow[r] & R[1] \end{tikzcd}$$

using TR3. Then $R''$ is a perfect complex (Cohomology, Lemma 38.6) supported on $T$. An easy diagram chase shows that $R'' \to E$ is the desired approximation. $\square$

**Theorem 12.6.** *Let $X$ be a quasi-compact and quasi-separated scheme. Then approximation by perfect complexes holds on $X$.*

**Proof.** This follows from the induction principle of Cohomology of Schemes, Lemma 4.1 and Lemmas 12.5 and 12.4. $\square$

## 13. Generating derived categories

In this section we prove that the derived category $D_{QCoh}(\mathcal{O}_X)$ of a quasi-compact and quasi-separated scheme can be generated by a single perfect object. We urge the reader to read the proof of this result in the wonderful paper by Bondal and van den Bergh, see [BV03].

**Lemma 13.1.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $U$ be a quasi-compact open subscheme. Let $P$ be a perfect object of $D(\mathcal{O}_U)$. Then $P$ is a direct summand of the restriction of a perfect object of $D(\mathcal{O}_X)$.*

**Proof.** Special case of Lemma 11.9. $\square$

**Lemma 13.2.** *In Situation 8.1 denote $j : U \to X$ the open immersion and let $K$ be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over $A$. For $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

1. *$E = Rj_*(E|_U)$, and*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$.*

**Proof.** Choose a distinguished triangle $E \to Rj_*(E|_U) \to N \to E[1]$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[n], E) = 0$$

for all $n$ as $K|_U = 0$. Thus it suffices to prove the result for $N$. In other words, we may assume that $E$ restricts to zero on $U$. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i + e''_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e''_i}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 20.11. Hence if $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$ then the same thing is true for the $K$ replaced by $K_e$ as in Lemma 8.6. Thus our lemma follows immediately from that one and the fact that $E$ is determined by the complex of $A$-modules $R\Gamma(X, E)$, see Lemma 3.4. $\square$

**Theorem 13.3.** *Let $X$ be a quasi-compact and quasi-separated scheme. The category $D_{QCoh}(\mathcal{O}_X)$ can be generated by a single perfect object. More precisely, there exists a perfect object $P$ of $D(\mathcal{O}_X)$ such that for $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

1. *$E = 0$, and*

(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$ *for all* $n \in \mathbf{Z}$.

**Proof.** We will prove this using the induction principle of Cohomology of Schemes, Lemma 4.1

If $X$ is affine, then $\mathcal{O}_X$ is a perfect generator. This follows from Lemma 3.4.

Assume that $X = U \cup V$ is an open covering with $U$ quasi-compact such that the theorem holds for $U$ and $V$ is an affine open. Let $P$ be a perfect object of $D(\mathcal{O}_U)$ which is a generator for $D_{QCoh}(\mathcal{O}_U)$. Using Lemma 13.1 we may choose a perfect object $Q$ of $D(\mathcal{O}_X)$ whose restriction to $U$ is a direct sum one of whose summands is $P$. Say $V = \mathrm{Spec}(A)$. Let $Z = X \setminus U$. This is a closed subset of $V$ with $V \setminus Z$ quasi-compact. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over $A$. Note that since $K$ is supported on $Z \subset V$ closed, the pushforward $K' = R(V \to X)_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to $V$ is $K$ (see Cohomology, Lemma 38.9). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into $E$. By Cohomology, Lemma 30.9 we have $K' = R(V \to X)_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 13.2 the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \cap V \to V)_*E|_{U\cap V}$. This implies that $E = R(U \to X)_*E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of $P$ the vanishing of these groups implies that $E|_U$ is zero. Whence $E$ is zero. □

Here is an example.

**Lemma 13.4.** *Let $A$ be a ring. Let $X = \mathbf{P}^1_A$. Then*

$$E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$$

*is a generator (Derived Categories, Definition 33.1) of $D_{QCoh}(X)$.*

**Proof.** Write $X = \mathrm{Proj}(A[X_0, X_1])$. Let $U = D_+(X_0) = \mathrm{Spec}(A[x])$ where $x = X_0/X_1$. Let $j : V = D_+(X_1) \to \mathbf{P}^1$ be the inclusion morphism. Consider the complex

$$K = (\mathcal{O}_X(-1) \xrightarrow{X_1} \mathcal{O}_X)$$

The restriction of $K$ to $U = \mathrm{Spec}(A[x]$ is isomorphic to the Koszul complex $A[x] \xrightarrow{x} A[x]$ and the restriction to $V$ is zero.

Let $L$ be an object of $D_{QCoh}(X)$ with $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L[n]) = 0$ for all $n \in \mathbf{Z}$. By Derived Categories of Schemes, Lemma 13.2 this implies that $L|_U$ is the pushforward of a complex living on $U \cap V$. This implies $L = Rj_*(L|_V)$ (small argument omitted). Then

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X, L) &= \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X, Rj_*(L|_V)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_V)}(\mathcal{O}_V, L|_V) \\ &= H^n(V, L|_V) \end{aligned}$$

Thus if in addition $\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X, L[n]) = 0$ for all $n$, then we find $H^n(V, L|_V) = 0$ for all $n$ and since $V$ is affine this means $L|_V = 0$ which in turn implies $L = 0$. The lemma follows as $K$ and $\mathcal{O}_X$ are in $\langle E \rangle$, see Derived Categories, Lemma 33.2. $\square$

The following result is an strengthening of Theorem 13.3 proved using exactly the same methods. Let $T \subset X$ be a closed subset of a scheme $X$. Let's denote $D_T(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on $T$.

**Lemma 13.5.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. With notation as above, the category $D_{QCoh,T}(\mathcal{O}_X)$ is generated by a single perfect object.*

**Proof.** We will prove this using the induction principle of Cohomology of Schemes, Lemma 4.1.

Assume $X = \mathrm{Spec}(A)$ is affine. In this case there exist $f_1, \ldots, f_r \in A$ such that $T = V(f_1, \ldots, f_r)$. Let $K$ be the Koszul complex on $f_1, \ldots, f_r$ as in Lemma 13.2. Then $K$ is a perfect object with cohomology supported on $T$ and hence a perfect object of $D_{QCoh,T}(\mathcal{O}_X)$. On the other hand, if $E \in D_{QCoh,T}(\mathcal{O}_X)$ and $\mathrm{Hom}(K, E[n]) = 0$ for all $n$, then Lemma 13.2 tells us that $E = Rj_*(E|_{X \setminus T}) = 0$. Hence $K$ generates $D_{QCoh,T}(\mathcal{O}_X)$, (by our definition of generators of triangulated categories in Derived Categories, Definition 33.1).

Assume that $X = U \cup V$ is an open covering with $U$ quasi-compact such that the lemma holds for $U$ and $V$ is an affine open. Let $P$ be a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$ which is a generator for $D_{QCoh,T\cap U}(\mathcal{O}_U)$. Using Lemma 11.9 we may choose a perfect object $Q$ of $D(\mathcal{O}_X)$ supported on $T$ whose restriction to $U$ is a direct sum one of whose summands is $P$. Write $V = \mathrm{Spec}(B)$. Let $Z = X \setminus U$. Then $Z$ is a closed subset of $V$ such that $V \setminus Z$ is quasi-compact. As $X$ is quasi-separated, it follows that $Z \cap T$ is a closed subset of $V$ such that $W = V \setminus (Z \cap T)$ is quasi-compact. Thus we can choose $g_1, \ldots, g_s \in B$ such that $Z \cap T = V(g_1, \ldots, g_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $g_1, \ldots, g_s$ over $B$. Note that since $K$ is supported on $(Z \cap T) \subset V$ closed, the pushforward $K' = R(V \to X)_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to $V$ is $K$ (see Cohomology, Lemma 38.9). We claim that $Q \oplus K'$ is a generator for $D_{QCoh,T}(\mathcal{O}_X)$.

Let $E$ be an object of $D_{QCoh,T}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into $E$. By Cohomology, Lemma 30.9 we have $K' = R(V \to X)_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 13.2 we have $E|_V = Rj_*E|_W$ where $j : W \to V$ is the inclusion. Picture

$$\begin{tikzcd}
W \arrow[r, "j"] & V & Z \cap T \arrow[l] \arrow[d] \\
U \cap V \arrow[u, "j'"] \arrow[ur, "j''"'] & & Z \arrow[ul]
\end{tikzcd}$$

Since $E$ is supported on $T$ we see that $E|_W$ is supported on $T \cap W = T \cap U \cap V$ which is closed in $W$. We conclude that

$$E|_V = Rj_*(E|_W) = Rj_*(Rj'_*(E|_{U\cap V})) = Rj''_*(E|_{U\cap V})$$

where the second equality is part (1) of Cohomology, Lemma 30.9. This implies that $E = R(U \to X)_* E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of $P$ the vanishing of these groups implies that $E|_U$ is zero. Whence $E$ is zero. $\square$

## 14. Compact and perfect objects

Let $X$ be a Noetherian scheme of finite dimension. By Cohomology, Proposition 21.6 and Cohomology on Sites, Lemma 39.4 the sheaves of modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O}_X)$ for all opens $U \subset X$. These sheaves are typically not quasi-coherent, hence these do not give perfect object of the derived category $D(\mathcal{O}_X)$. However, if we restrict ourselves to complexes with quasi-coherent cohomology sheaves, then this does not happen. Here is the precise statement.

**Proposition 14.1.** *Let $X$ be a quasi-compact and quasi-separated scheme. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.*

**Proof.** By Cohomology, Lemma 39.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$. Conversely, let $K$ be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that $K$ is perfect, it suffices to show that $K|_U$ is perfect for every affine open $U \subset X$, see Cohomology, Lemma 38.2. Observe that $j : U \to X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 4.2. Thus the adjointness of restriction to $U$ and $Rj_*$ implies that $K|_U$ is a compact object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that $X$ is affine.

Assume $X = \mathrm{Spec}(A)$ is affine. By Lemma 3.4 the problem is translated into the same problem for $D(A)$. For $D(A)$ the result is More on Algebra, Proposition 57.2. $\square$

The following result is a strengthening of Proposition 14.1. Let $T \subset X$ be a closed subset of a scheme $X$. As before $D_T(\mathcal{O}_X)$ denotes the the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on $T$. Since taking direct sums commutes with taking cohomology sheaves, it follows that $D_T(\mathcal{O}_X)$ has direct sums and that they are equal to direct sums in $D(\mathcal{O}_X)$.

**Lemma 14.2.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. An object of $D_{QCoh,T}(\mathcal{O}_X)$ is compact if and only if it is perfect as an object of $D(\mathcal{O}_X)$.*

**Proof.** We observe that $D_{QCoh,T}(\mathcal{O}_X)$ is a triangulated category with direct sums by the remark preceding the lemma. By Cohomology, Lemma 39.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$ hence a fortiori of any subcategory preserved under taking direct sums. For the converse we will use there exists a generator $E \in D_{QCoh,T}(\mathcal{O}_X)$ which is a perfect complex of $\mathcal{O}_X$-modules, see Lemma 13.5. Hence by the above, $E$ is compact. Then it follows from Derived Categories, Proposition 34.6 that $E$ is a classical generator of the full subcategory of compact objects of $D_{QCoh,T}(\mathcal{O}_X)$. Thus any compact object can be constructed out of $E$ by a finite sequence of operations consisting of (a) taking shifts, (b) taking finite direct sums, (c) taking cones, and (d) taking direct summands. Each of these operations preserves the property of being perfect and the result follows. ☐

The following lemma is an application of the ideas that go into the proof of the preceding lemma.

**Lemma 14.3.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $U = X \setminus T$ is quasi-compact. Let $\alpha : P \to E$ be a morphism of $D_{QCoh}(\mathcal{O}_X)$ with either*

1. *$P$ is perfect and $E$ supported on $T$, or*
2. *$P$ pseudo-coherent, $E$ supported on $T$, and $E$ bounded below.*

*Then there exists a perfect complex of $\mathcal{O}_X$-modules $I$ and a map $I \to \mathcal{O}_X[0]$ such that $I \otimes^{\mathbf{L}} P \to E$ is zero and such that $I|_U \to \mathcal{O}_U[0]$ is an isomorphism.*

**Proof.** Set $\mathcal{D} = D_{QCoh,T}(\mathcal{O}_X)$. In both cases the complex $K = R\mathcal{H}om(P, E)$ is an object of $\mathcal{D}$. See Lemma 9.8 for quasi-coherence. It is clear that $K$ is supported on $T$ as formation of $R\mathcal{H}om$ commutes with restriction to opens. The map $\alpha$ defines an element of $H^0(K) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[0], K)$. Then it suffices to prove the result for the map $\alpha : \mathcal{O}_X[0] \to K$.

Let $E \in \mathcal{D}$ be a perfect generator, see Lemma 13.5. Write

$$K = \mathrm{hocolim} K_n$$

as in Derived Categories, Lemma 34.3 using the generator $E$. Since the functor $\mathcal{D} \to D(\mathcal{O}_X)$ commutes with direct sums, we see that $K = \mathrm{hocolim} K_n$ also in $D(\mathcal{O}_X)$. Since $\mathcal{O}_X$ is a compact object of $D(\mathcal{O}_X)$ we find an $n$ and a morphism $\alpha_n : \mathcal{O}_X \to K_n$ which gives rise to $\alpha$. By Derived Categories, Lemma 34.4 applied to the morphism $\mathcal{O}_X[0] \to K_n$ in the ambient category $D(\mathcal{O}_X)$ we see that $\alpha_n$ factors as $\mathcal{O}_X[0] \to Q \to K_n$ where $Q$ is an object of $\langle E \rangle$. We conclude that $Q$ is a perfect complex supported on $T$.

Choose a distinguished triangle

$$I \to \mathcal{O}_X[0] \to Q \to I[1]$$

By construction $I$ is perfect, the map $I \to \mathcal{O}_X[0]$ restricts to an isomorphism over $U$, and the composition $I \to K$ is zero as $\alpha$ factors through $Q$. This proves the lemma. ☐

## 15. Derived categories as module categories

In this section we draw some conclusions of what has gone before. Before we do so we need a couple more lemmas.

**Lemma 15.1.** *Let $X$ be a scheme. Let $K^\bullet$ be a complex of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Let $(E, d) = \mathrm{Hom}_{Comp^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor*

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

*of Differential Graded Algebra, Lemma 25.3 has image contained in $D_{QCoh}(\mathcal{O}_X)$.*

**Proof.** Let $P$ be a differential graded $E$-module with property (P) and let $F_\bullet$ be a filtration on $P$ as in Differential Graded Algebra, Section 13. Then we have

$$P \otimes_E K^\bullet = \mathrm{hocolim}\ F_i P \otimes_E K^\bullet$$

Each of the $F_iP$ has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. $\square$

The following lemma can be strengthened (there is a uniformity in the vanishing over all $L$ with nonzero cohomology sheaves only in a fixed range).

**Lemma 15.2.** *Let $X$ be a quasi-compact and quasi-separated scheme. Let $K$, $L$ be objects of $D(\mathcal{O}_X)$ with $K$ perfect and $L$ in $D^b_{QCoh}(\mathcal{O}_X)$. Then $Ext^n_{D(\mathcal{O}_X)}(K, L)$ is nonzero for only a finite number of $n$.*

**Proof.** Since $K$ is perfect we have

$$\mathrm{Ext}^i_{D(\mathcal{O}_X)}(K, L) = H^i(X, K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

where $K^\wedge$ is the "dual" perfect complex to $K$, see Cohomology, Lemma 38.10. Note that $P = K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(X)$ by Lemmas 3.7 and 9.1 (to see that a perfect complex has quasi-coherent cohomology sheaves). On the other hand, the spectral sequence

$$E_1^{p,q} = H^p(K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(L)) \Rightarrow H^{p+q}(K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = H^{p+q}(P),$$

the boundedness of $L$, and the finite tor amplitude of $K^\wedge$ show that $P$ has only finitely many nonzero cohomology sheaves. It follows that $H^n(X, P) = 0$ for $n \ll 0$. But also $H^n(X, P) = 0$ for $n \gg 0$ by Cohomology of Schemes, Lemma 4.3 and the spectral sequence expressing $H^n(X, P^\bullet)$ in terms of $H^p(X, H^q(P^\bullet))$ using that the cohomology sheaves of $P$ are quasi-coherent. $\square$

The following result is taken from [BV03].

**Theorem 15.3.** *Let $X$ be a quasi-compact and quasi-separated scheme. Then there exist a differential graded algebra $(E, d)$ with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{QCoh}(\mathcal{O}_X)$ is equivalent to $D(E, d)$.*

**Proof.** Let $K^\bullet$ be a K-injective complex of $\mathcal{O}$-modules which is perfect and generates $D_{QCoh}(\mathcal{O}_X)$. Such a thing exists by Theorem 13.3 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where $\mathrm{Comp}^{dg}(\mathcal{O}_X)$ is the differential graded category of complexes of $\mathcal{O}$-modules. Please see Differential Graded Algebra, Section 25. Since $K^\bullet$ is K-injective we have

$$H^n(E) = \mathrm{Ext}^n_{D(\mathcal{O}_X)}(K^\bullet, K^\bullet) \tag{15.3.1}$$

for all $n \in \mathbf{Z}$. Only a finite number of these Exts are nonzero by Lemma 15.2. Consider the functor

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 25.3. Since $K^\bullet$ is perfect, it defines a compact object of $D(\mathcal{O}_X)$, see Proposition 14.1. Combined with (15.3.1) the functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 25.5. It has a right adjoint

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, \mathrm{d})$$

by Differential Graded Algebra, Lemmas 25.4 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 15.1 that we obtain

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of $K^\bullet$ as a generator of $D_{QCoh}(\mathcal{O}_X)$ the kernel of the adjoint restricted to $D_{QCoh}(\mathcal{O}_X)$ is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 7.2. $\square$

**Remark 15.4.** Let $X$ be a quasi-compact and quasi-separated scheme over a ring $R$. By the construction of the proof of Theorem 15.3 there exists a differential graded algebra $(A, \text{d})$ over $R$ such that $D_{QCoh}(X)$ is $R$-linearly equivalent to $D(A, \text{d})$ as a triangulated category. One may ask: how unique is $(A, \text{d})$? The answer is (only) slightly better than just saying that $(A, \text{d})$ is well defined up to derived equivalence. Namely, suppose that $(B, \text{d})$ is a second such pair. Then we have

$$(A, \text{d}) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

and

$$(B, \text{d}) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, L^\bullet)$$

for some K-injective complexes $K^\bullet$ and $L^\bullet$ of $\mathcal{O}_X$-modules corresponding to perfect generators of $D_{QCoh}(\mathcal{O}_X)$. Set

$$\Omega = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, L^\bullet) \quad \Omega' = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, K^\bullet)$$

Then $\Omega$ is a differential graded $B^{opp} \otimes_R A$-module and $\Omega'$ is a differential graded $A^{opp} \otimes_R B$-module. Moreover, the equivalence

$$D(A, \text{d}) \to D_{QCoh}(\mathcal{O}_X) \to D(B, \text{d})$$

is given by the functor $- \otimes_A^{\mathbf{L}} \Omega'$ and similarly for the quasi-inverse. Thus we are in the situation of Differential Graded Algebra, Remark 27.10. If we ever need this remark we will provide a precise statement with a detailed proof here.

## 16. Cohomology and base change, IV

This section continues the discussion of Cohomology of Schemes, Section 20.

**Lemma 16.1.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. For $E$ in $D_{QCoh}(\mathcal{O}_X)$ and $K$ in $D_{QCoh}(\mathcal{O}_Y)$ we have*

$$Rf_*(E) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K = Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*K)$$

**Proof.** Without any assumptions there is a map $Rf_*(E) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K \to Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*K)$. Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*(E) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K) = Lf^*(Rf_*(E)) \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*K \longrightarrow E \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$. See Cohomology, Lemmas 28.2 and 29.1. To check it is an isomorphism we may work locally on $Y$. Hence we reduce to the case that $Y$ is affine.

Suppose that $K = \bigoplus K_i$ is a direct sum of some complexes $K_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each $K_i$, then it holds for $K$. Namely, the functors $Lf^*$ and $\otimes^{\mathbf{L}}$ preserve direct sums by construction and $Rf_*$ commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 4.2. Moreover, suppose that $K \to L \to M \to K[1]$ is a distinguished triangle in $D_{QCoh}(Y)$. Then if the statement of the lemma holds for two of $K, L, M$, then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume $Y$ affine, say $Y = \text{Spec}(A)$. The functor $\widetilde{\ } : D(A) \to D_{QCoh}(\mathcal{O}_Y)$ is an equivalence (Lemma 3.4). Let $T$ be the property for $K \in D(A)$ that the statement of the lemma holds for $\widetilde{K}$. The discussion above and More on Algebra, Remark 45.11 shows that it suffices to prove $T$ holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. $\square$

**Definition 16.2.** Let $S$ be a scheme. Let $X, Y$ be schemes over $S$. We say $X$ and $Y$ are *Tor independent over $S$* if for every $x \in X$ and $y \in Y$ mapping to the same point $s \in S$ the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are Tor independent over $\mathcal{O}_{S,s}$ (see More on Algebra, Definition 47.1).

**Lemma 16.3.** *Let $g : S' \to S$ be a morphism of schemes. Let $f : X \to S$ be quasi-compact and quasi-separated. Consider the base change diagram*

$$\begin{tikzcd} X' \arrow[r, "h"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

*If $X$ and $S'$ are Tor independent over $S$, then for all $E \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf'_*Lh^*E = Lg^*Rf_*E$.*

**Proof.** For any object $E$ of $D(\mathcal{O}_X)$ we can use Cohomology, Remark 29.2 to get a canonical base change map $Lg^*Rf_*E \to Rf'_*Lh^*E$. To check this is an isomorphism we may work locally on $S'$. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In particular, $g$ is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*Lh^*E = Rf_*(Rh_*Lh^*E)$$

is an isomorphism, see Lemma 4.3 (and use Lemmas 3.6, 3.7, and 4.1 to see that the objects $Rf'_*Lh^*E$ and $Lg^*Rf_*E$ have quasi-coherent cohomology sheaves). Note that $h$ is affine as well (Morphisms, Lemma 13.8). By Lemma 4.4 the map becomes a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_S} g_*\mathcal{O}_{S'} \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} h_*\mathcal{O}_{X'})$$

Observe that $h_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{S'}$. Thus by Lemma 16.1 it suffices to prove that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$. This follows from our assumption that $X$ and $S'$ are Tor independent over $S$. Namely, to check it we may work locally on $X$, hence we may also assume $X$ is affine. Say $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. Our assumption implies that $A$ and $R'$ are Tor independent over $R$ (More on Algebra, Lemma 47.4), i.e., $\mathrm{Tor}_i^R(A, R') = 0$ for $i > 0$. In other words $A \otimes^{\mathbf{L}}_R R' = A \otimes_R R'$ which exactly means that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$ (use Lemma 3.6). $\square$

The following two lemmas remain true if we replace $\mathcal{G}$ with a bounded complex of quasi-coherent $\mathcal{O}_X$-modules each flat over $S$.

**Lemma 16.4.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module flat over $S$. Then formation of*

$$Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$$

*commutes with arbitrary base change (see proof for precise statement).*

**Proof.** The statement means the following. Let $g : S' \to S$ be a morphism of schemes and consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

in other words $X' = S' \times_S X$. Set $E' = Lh^*E$ and $\mathcal{G}' = h^*\mathcal{G}$ (here we do **not** use the derived pullback). The lemma asserts that we have

$$Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) = Rf'_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{G}')$$

To prove this, note that in Cohomology, Remark 29.2 we have constructed an arrow

$$Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) \longrightarrow R(f')_*(Lh^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})) = R(f')_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lh^*\mathcal{G})$$

which we can compose with the map $Lh^*\mathcal{G} \to h^*\mathcal{G}$ to get a canonical map

$$Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) \longrightarrow Rf'_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{G}')$$

To check this map is an isomorphism we may work locally on $S'$. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In this case, we will use the induction principle to prove this map is always an isomorphism for any quasi-compact and quasi-separated $X$ over $S$ (Cohomology of Schemes, Lemma 4.1).

Suppose $X = \text{Spec}(A)$ is affine. The functor $\widetilde{\ } : D(A) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence (Lemma 3.4). Let $T$ be the property for $K \in D(A)$ that the canonical arrow above is an isomorphism for $E = \widetilde{K}$. If we have $T(K_i)$ for a family of objects $K_i$, then we have $T(\bigoplus K_i)$. Namely, derived tensor product and derived pullback commute with direct sums and the same holds for total direct image in this case by Lemma 4.2. Moreover, if $T$ holds for two out of three objects of a distinguished triangle, then it holds for the third (Derived Categories, Lemma 4.3). By More on Algebra, Remark 45.11 this shows that it suffices to prove $T$ holds for $A[k]$. This reduces us to the case $E = \mathcal{O}_X$. In this case we are saying that $Lg^*f_*\mathcal{G} = g^*f_*\mathcal{G}$ (by flatness of $\mathcal{G}$ over $S$) equals $f'_*h^*\mathcal{G}$ which holds by Cohomology of Schemes, Lemma 5.1.

The induction step. Suppose that $X = U \cup V$ is an open covering with $U$, $V$, $U \cap V$ quasi-compact such that the result holds for the restriction of $E$ and $\mathcal{G}$ to $U$, $V$, and $U \cap V$. Denote $a = f|_U$, $b = f|_V$ and $c = f|_{U \cap V}$. Let $a' : U' \to S'$, $b' : V' \to S'$ and $c' : U' \cap V' \to S'$ be the base changes of $a$, $b$, and $c$. Note that formation of $- \otimes^{\mathbf{L}} -$ commutes with restriction to opens. Set $H = E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}$ and $H' = E' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{G}'$. Using the distinguished triangles from relative Mayer-Vietoris

(Cohomology, Lemma 30.8) we obtain a commutative diagram

$$\begin{tikzcd}
Lg^*Rf_*H \arrow[r] \arrow[d] & Rf'_*H' \arrow[d] \\
Lg^*Ra_*H|_U \oplus Lg^*Rb_*H|_V \arrow[r] \arrow[d] & Ra'_*H'|_{U'} \oplus Rb'_*H'|_{V'} \arrow[d] \\
Lg^*Rc_*H|_{U\cap V} \arrow[r] \arrow[d] & Rc'_*H'|_{U'\cap V'} \arrow[d] \\
Lg^*Rf_*H[1] \arrow[r] & Rf'_*H'[1]
\end{tikzcd}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 4.3) and the proof of the lemma is finished. □

**Lemma 16.5.** *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module flat over $S$. Then formation of*

$$Rf_*R\mathcal{H}om(E, \mathcal{G})$$

*commutes with arbitrary base change (see proof for precise statement).*

**Proof.** The statement means the following. Let $g : S' \to S$ be a morphism of schemes and consider the base change diagram

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

in other words $X' = S' \times_S X$. Set $E' = Lh^*E$ and $\mathcal{G}' = h^*\mathcal{G}$ (here we do **not** use the derived pullback). The lemma asserts that we have

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) = Rf'_*R\mathcal{H}om(E', \mathcal{G}')$$

To prove this, note that in Cohomology, Remark 34.10 we have constructed an arrow

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) \longrightarrow R(f')_*R\mathcal{H}om(Lh^*E, Lh^*\mathcal{G})$$

which we can compose with the map $Lh^*\mathcal{G} \to h^*\mathcal{G}$ to get a canonical map

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) \to Rf'_*R\mathcal{H}om(E', \mathcal{G}')$$

With these preliminaries out of the way, we deduce the result from Lemma 16.4. Namely, since $E$ is a perfect complex there exists a dual perfect complex $E_{dual}$, see Cohomology, Lemma 38.10, such that $R\mathcal{H}om(E, \mathcal{G}) = E_{dual} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}$. We omit the verification that the base change map of Lemma 16.4 for $E_{dual}$ agrees with the base change map for $E$ constructed above. □

## 17. Producing perfect complexes

The following lemma is our main technical tool for producing perfect complexes. Later versions of this result will reduce to this by Noetherian approximation, see Section 20.

**Lemma 17.1.** *Let $S$ be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ such that*

1. *$E \in D^b_{Coh}(\mathcal{O}_X)$,*
2. *the scheme theoretic support of $H^i(E)$ is proper over $S$ for all $i$,*
3. *$E$ has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$.*

*Then $Rf_*E$ is a perfect object of $D(\mathcal{O}_S)$.*

**Proof.** By Lemma 5.1 we see that $Rf_*E$ is an object of $D^b_{Coh}(\mathcal{O}_S)$. Hence $Rf_*E$ is pseudo-coherent (Lemma 9.4). Hence it suffices to show that $Rf_*E$ has finite tor dimension, see Cohomology, Lemma 38.4. By Lemma 9.6 it suffices to check that $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}$ has universally bounded cohomology for all quasi-coherent sheaves $\mathcal{F}$ on $S$. Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset X$ be the union of the supports of $H^i(E)$ for all $i$. Then $T$ is proper over $S$ by assumptions (1) and (2). In particular there exists a quasi-compact open $X' \subset X$ containing $T$. Setting $f' = f|_{X'}$ we have $Rf_*(E) = Rf'_*(E|_{X'})$ because $E$ restricts to zero on $X \setminus T$. Thus we may replace $X$ by $X'$ and assume $f$ is quasi-compact. Moreover, $f$ is quasi-separated by Morphisms, Lemma 16.7. Now

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F} = Rf_*\left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}\right) = Rf_*\left(E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F}\right)$$

by Lemma 16.1 and Cohomology, Lemma 28.3. By assumption (3) the complex $E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F}$ has cohomology sheaves in a given finite range, say $[a, b]$. Then $Rf_*$ of it has cohomology in the range $[a, \infty)$ and we win. □

## 18. Cohomology, Ext groups, and base change

The results in this section will be used to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems.

**Lemma 18.1.** *Let $S$ be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module flat over $S$ with scheme theoretic support proper over $S$. Then $K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$ is a perfect object of $D(\mathcal{O}_S)$ and there are functorial isomorphisms*

$$H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \longrightarrow H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

*for $\mathcal{F}$ quasi-coherent on $S$ compatible with boundary maps (see proof).*

**Proof.** We have

$$\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F} = \mathcal{G} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G} \otimes_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}$$

the first equality by Cohomology, Lemma 28.3, the second as $\mathcal{G}$ is a flat $f^{-1}\mathcal{O}_S$-module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})) &= H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}) \\ &= H^i(S, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})) \\ &= H^i(S, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \\ &= H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology, Lemma 14.1), and the third equality by Lemma 16.1. The object $K$ is perfect by Lemma 17.1. We check the lemma applies: Locally $E$ is isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules. Hence locally $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}$ is isomorphic to a finite complex whose terms are finite direct sums of copies $\mathcal{G}$. This immediately implies the hypotheses on the cohomology sheaves $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$. The hypothesis on the tor dimension also follows as $\mathcal{G}$ is flat over $f^{-1}\mathcal{O}_S$.

The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of quasi-coherent $\mathcal{O}_S$-modules, the isomorphisms fit into commutative diagrams

$$\begin{tikzcd} H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3) \arrow[r] \arrow[d, "\delta"'] & H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3)) \arrow[d, "\delta"] \\ H^{i+1}(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1) \arrow[r] & H^{i+1}(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1)) \end{tikzcd}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

This sequence is exact because $\mathcal{G}$ is flat over $S$. We omit the verification of the commutativity of the displayed diagram. □

**Lemma 18.2.** *Let $S$ be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module flat over $S$ with scheme theoretic support proper over $S$. Then $K = Rf_*R\mathcal{H}om(E, \mathcal{G})$ is a perfect object of $D(\mathcal{O}_S)$ and there are functorial isomorphisms*

$$H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \longrightarrow Ext^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

*for $\mathcal{F}$ quasi-coherent on $S$ compatible with boundary maps (see proof).*

**Proof.** Since $E$ is a perfect complex there exists a dual perfect complex $E_{dual}$, see Cohomology, Lemma 38.10. Observe that $R\mathcal{H}om(E, \mathcal{G}) = E_{dual} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}$ and that

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) = H^i(X, E_{dual} \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

by construction of $E_{dual}$. Thus the perfectness of $K$ and the isomorphisms follow from the corresponding results of Lemma 18.1 applied to $E_{dual}$ and $\mathcal{G}$.

The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{tikzcd} H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3) \arrow[r] \arrow[d, "\delta"'] & \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3) \arrow[d, "\delta"] \\ H^{i+1}(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1) \end{tikzcd}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

This sequence is exact because $\mathcal{G}$ is flat over $S$. We omit the verification of the commutativity of the displayed diagram. $\square$

**Lemma 18.3.** *Let $S$ be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ and $\mathcal{G}$ an $\mathcal{O}_X$-module. Assume*

1. *$E \in D^-_{Coh}(\mathcal{O}_X)$, and*
2. *$\mathcal{G}$ is a coherent $\mathcal{O}_X$-module flat over $S$ with scheme theoretic support is proper over $S$.*

*Then for every $m \in \mathbf{Z}$ there exists a perfect object $K$ of $D(\mathcal{O}_S)$ and functorial maps*

$$\alpha^i_{\mathcal{F}} : Ext^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F})$$

*for $\mathcal{F}$ quasi-coherent on $S$ compatible with boundary maps (see proof) such that $\alpha^i_{\mathcal{F}}$ is an isomorphism for $i \leq m$.*

**Proof.** We may replace $X$ by a quasi-compact open neighbourhood of the support of $\mathcal{G}$, hence we may assume $X$ is Noetherian. In this case $X$ and $f$ are quasi-compact and quasi-separated. Choose an approximation $P \to E$ by a perfect complex $P$ of $(X, E, -m-1)$ (possible by Theorem 12.6). Then the induced map

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(P, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\mathrm{Ext}^i_{\mathcal{O}_X}(C, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \quad \text{resp.} \quad \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(C, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

where $C$ is the cone of $P \to E$. Since $C$ has vanishing cohomology sheaves in degrees $\geq -m-1$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 27.3. This reduces us to the case that $E$ is a perfect complex which is Lemma 18.2.

The statement on boundaries is explained in the proof of Lemma 18.2. $\square$

## 19. Limits and derived categories

In this section we collect some results about the derived category of a scheme which is the limit of an inverse system of schemes. More precisely, we will work in the following setting.

**Situation 19.1.** Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes over $S$ with affine transition morphisms $f_{i'i} : S_{i'} \to S_i$. We assume that $S_i$ is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also fix an element $0 \in I$.

**Lemma 19.2.** *In Situation 19.1. Let $E_0$ and $K_0$ be objects of $D(\mathcal{O}_{S_0})$. Set $E_i = Lf_{i0}^*E_0$ and $K_i = Lf_{i0}^*K_0$ for $i \geq 0$ and set $E = Lf_0^*E_0$ and $K = Lf_0^*K_0$. Then the map*

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{S_i})}(E_i, K_i) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_S)}(E, K)$$

*is an isomorphism if either*

1. *$E_0$ is perfect and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$, or*
2. *$E_0$ is pseudo-coherent and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$ has finite tor dimension.*

**Proof.** For every open $U_0 \subset S_0$ consider the condition $P$ that the canonical map

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove $P$ holds for all quasi-compact opens $U_0$ by the induction principle of Cohomology of Schemes, Lemma 4.1. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 30.6. Thus it suffices to prove the lemma when $S_0$ is affine.

Assume $S_0$ is affine. Say $S_0 = \mathrm{Spec}(A_0)$, $S_i = \mathrm{Spec}(A_i)$, and $S = \mathrm{Spec}(A)$. We will use Lemma 3.4 without further mention.

In case (1) the object $E_0^\bullet$ corresponds to a finite complex of finite projective $A_0$-modules, see Lemma 9.7. We may represent the object $K_0$ by a K-flat complex $K_0^\bullet$ of $A_0$-modules. In this situation we are trying to prove

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(A_i)}(E_0^\bullet \otimes_{A_0} A_i, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \mathrm{Hom}_{D(A)}(E_0^\bullet \otimes_{A_0} A, K_0^\bullet \otimes_{A_0} A)$$

Because $E_0^\bullet$ is a bounded above complex of projective modules we can rewrite this as

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

Since there are only a finite number of nonzero modules $E_0^n$ and since these are all finitely presented modules, this map is an isomorphism.

In case (2) the object $E_0$ corresponds to a bounded above complex $E_0^\bullet$ of finite free $A_0$-modules, see Lemma 9.3. We may represent $K_0$ by a finite complex $K_0^\bullet$ of flat $A_0$-modules, see Lemma 9.5 and More on Algebra, Lemma 51.3. In particular $K_0^\bullet$ is K-flat and we can argue as before to arrive at the map

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

It is clear that this map is an isomorphism (only a finite number of terms are involved since $K_0^\bullet$ is bounded). ☐

**Lemma 19.3.** *In Situation 19.1 the category of perfect objects of $D(\mathcal{O}_S)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{S_i})$.*

**Proof.** For every open $U_0 \subset S_0$ consider the condition $P$ that the functor

$$\operatorname{colim}_{i \geq 0} D_{perf}(\mathcal{O}_{U_i}) \longrightarrow D_{perf}(\mathcal{O}_U)$$

is an equivalence where $_{perf}$ indicates the full subcategory of perfect objects and where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove $P$ holds for all quasi-compact opens $U_0$ by the induction principle of Cohomology of Schemes, Lemma 4.1. First, we observe that we already know the functor is fully faithful by Lemma 19.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have $S_0 = U_0 \cup V_0$ and that $P$ holds for $U_0$, $V_0$, and $U_0 \cap V_0$. Let $E$ be a perfect object of $D(\mathcal{O}_S)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on $U_i$ and $E_{V,i}$ perfect on $V_i$ whose pullback to $U$ and $V$ are isomorphic to $E|_U$ and $E|_V$. Denote

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing $i$, we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \to E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \cap V} \to E|_{U \cap V} \to Lf_i^*E_{V,i}|_{U \cap V}.$$

Apply Cohomology, Lemma 30.10 to get an object $E_i$ on $S_i$ and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps $a$ and $b$ over $U_i$ and $V_i$. Then it is clear that $E_i$ is perfect and that $d$ is adjoint to an isomorphism $Lf_i^*E_i \to E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when $S_0$ is affine. Say $S_0 = \operatorname{Spec}(A_0)$, $S_i = \operatorname{Spec}(A_i)$, and $S = \operatorname{Spec}(A)$. Using Lemmas 3.4 and 9.7 we see that we have to show that

$$D_{perf}(A) = \operatorname{colim} D_{perf}(A_i)$$

This is clear from the fact that perfect complexes over rings are given by finite complexes of finite projective (hence finitely presented) modules. $\square$

## 20. Cohomology and base change, V

A final section on cohomology and base change continueing the discussion of Sections 16 and 17. An easy to grok special case is given in Remark 20.2.

**Lemma 20.1.** *Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module, flat over $S$, with support proper over $S$. Then*

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$$

*is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

**Proof.** The statement on base change is Lemma 16.4. Thus it suffices to show that $K$ is a perfect object. If $S$ is Noetherian, then this follows from Lemma 18.1. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on $S$, hence we may assume $S$ is affine. Say $S = \operatorname{Spec}(R)$. We write $R = \operatorname{colim} R_i$ as a filtered colimit of Noetherian rings $R_i$. By Limits, Lemma 9.1 there exists an $i$ and a scheme $X_i$ of finite presentation over $R_i$ whose base change to $R$ is $X$. By Limits, Lemma 9.2 we may assume after increasing $i$, that there exists a finitely presented $\mathcal{O}_{X_i}$-module $\mathcal{G}_i$ whose pullback to $X$ is $\mathcal{G}$.

After increasing $i$ we may assume $\mathcal{G}_i$ is flat over $R_i$, see Limits, Lemma 9.3. After increasing $i$ we may assume the support of $\mathcal{G}_i$ is proper over $R_i$, see Limits, Lemma 12.7. Finally, by Lemma 19.3 we may, after increasing $i$, assume there exists a perfect object $E_i$ of $D(\mathcal{O}_{X_i})$ whose pullback to $X$ is $E$. Applying Lemma 18.1 to $X_i \to \text{Spec}(R_i)$, $E_i$, $\mathcal{G}_i$ and using the base change property already shown we obtain the result. □

**Remark 20.2.** Let $R$ be a ring. Let $X$ be a scheme of finite presentation over $R$. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module flat over $R$ with scheme theoretic support proper over $R$. By Lemma 20.1 there exists a finite complex of finite projective $R$-modules $M^\bullet$ such that we have

$$R\Gamma(X_{R'}, \mathcal{G}_{R'}) = M^\bullet \otimes_R R'$$

functorially in the $R$-algebra $R'$.

**Lemma 20.3.** *Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module, flat over $S$, with support proper over $S$. Then*

$$K = Rf_*R\mathcal{H}om(E, \mathcal{G})$$

*is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.*

**Proof.** The statement on base change is Lemma 16.5. Thus it suffices to show that $K$ is a perfect object. If $S$ is Noetherian, then this follows from Lemma 18.2. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on $S$, hence we may assume $S$ is affine. Say $S = \text{Spec}(R)$. We write $R = \text{colim}\, R_i$ as a filtered colimit of Noetherian rings $R_i$. By Limits, Lemma 9.1 there exists an $i$ and a scheme $X_i$ of finite presentation over $R_i$ whose base change to $R$ is $X$. By Limits, Lemma 9.2 we may assume after increasing $i$, that there exists a finitely presented $\mathcal{O}_{X_i}$-module $\mathcal{G}_i$ whose pullback to $X$ is $\mathcal{G}$. After increasing $i$ we may assume $\mathcal{G}_i$ is flat over $R_i$, see Limits, Lemma 9.3. After increasing $i$ we may assume the support of $\mathcal{G}_i$ is proper over $R_i$, see Limits, Lemma 12.7. Finally, by Lemma 19.3 we may, after increasing $i$, assume there exists a perfect object $E_i$ of $D(\mathcal{O}_{X_i})$ whose pullback to $X$ is $E$. Applying Lemma 18.2 to $X_i \to \text{Spec}(R_i)$, $E_i$, $\mathcal{G}_i$ and using the base change property already shown we obtain the result. □

## 21. Other chapters

Preliminaries

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