

PUSHOUTS OF ALGEBRAIC SPACES

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1. Introduction

The goal of this chapter is to discuss pushouts in the category of algebraic spaces. This can be done with varying assumptions. A fairly general pushout construction is given in [TT13]: one of the morphisms is affine and the other is a closed immersion. We discuss a particular case of this in Section 2 where we assume one of the morphisms is affine and the other is a thickening, a situation that often comes up in deformation theory.

In Sections 3 and 4 we discuss diagrams

$$\begin{array}{ccc} f^{-1}(X \setminus Z) & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ X \setminus Z & \longrightarrow & X \end{array}$$

where f is a quasi-compact and quasi-separated morphism of algebraic spaces, $Z \rightarrow X$ is a closed immersion of finite presentation, the map $f^{-1}(Z) \rightarrow Z$ is an isomorphism, and f is flat along $f^{-1}(Z)$. In this situation we glue quasi-coherent modules on $X \setminus Z$ and Y (in Section 3) to quasi-coherent modules on X and we glue algebraic spaces over $X \setminus Z$ and Y (in Section 4) to algebraic spaces over X .

In Section 5 we discuss how proper birational morphisms of Noetherian algebraic spaces give rise to coequalizer diagrams in algebraic spaces in some sense.

2. Pushouts in the category of algebraic spaces

This section is analogue of More on Morphisms, Section 11. We first prove a general result on colimits and algebraic spaces. To do this we discuss a bit of notation. Let S be a scheme. Let $\mathcal{I} \rightarrow (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram (see Categories, Section 14). For each i we may consider the small étale site $X_{i, \text{étale}}$. For each morphism $i \rightarrow j$ of \mathcal{I} we have the morphism $X_i \rightarrow X_j$ and hence a pullback functor

This is a chapter of the Stacks Project, version 714e994, compiled on Oct 28, 2014.

$X_{j,\acute{e}tale} \rightarrow X_{i,\acute{e}tale}$. Hence we obtain a pseudo functor from \mathcal{I}^{opp} into the 2-category of categories. Denote

$$\lim_i X_{i,\acute{e}tale}$$

the 2-limit (see insert future reference here). What does this mean concretely? An object of this limit is a system of étale morphisms $U_i \rightarrow X_i$ over \mathcal{I} such that for each $i \rightarrow j$ in \mathcal{I} the diagram

$$\begin{array}{ccc} U_i & \longrightarrow & U_j \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_j \end{array}$$

is cartesian. Morphisms between objects are defined in the obvious manner. Suppose that $f_i : X_i \rightarrow T$ is a family of morphisms such that for each $i \rightarrow j$ the composition $X_i \rightarrow X_j \rightarrow T$ is equal to f_i . Then we get a functor $T_{\acute{e}tale} \rightarrow \lim X_{i,\acute{e}tale}$. With this notation in hand we can formulate our lemma.

Lemma 2.1. *Let S be a scheme. Let $\mathcal{I} \rightarrow (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram as above. Assume that*

- (1) $X = \text{colim } X_i$ exists in the category of schemes,
- (2) $\coprod X_i \rightarrow X$ is surjective,
- (3) if $U \rightarrow X$ is étale and $U_i = X_i \times_X U$, then $U = \text{colim } U_i$ in the category of schemes, and
- (4) the functor $X_{\acute{e}tale} \rightarrow \lim X_{i,\acute{e}tale}$ is an equivalence.

Then $X = \text{colim } X_i$ in the category of algebraic spaces over S also.

Proof. Let Z be an algebraic space over S . Suppose that $f_i : X_i \rightarrow Z$ is a family of morphisms such that for each $i \rightarrow j$ the composition $X_i \rightarrow X_j \rightarrow Z$ is equal to f_i . We have to construct a morphism of algebraic spaces $f : X \rightarrow Z$ such that we can recover f_i as the composition $X_i \rightarrow X \rightarrow Z$. Let $W \rightarrow Z$ be a surjective étale morphism of a scheme to Z . For each i set $U_i = W \times_{Z,f_i} X_i$ and denote $h_i : U_i \rightarrow W$ the projection. Then $U_i \rightarrow X_i$ forms an object of $\lim X_{i,\acute{e}tale}$. By assumption (4) we can find an étale morphism $U \rightarrow X$ and (functorial) isomorphisms $U_i = X_i \times_X U$. By assumption (3) there exists a morphism $h : U \rightarrow W$ such that the compositions $U_i \rightarrow U \rightarrow W$ are h_i . Let $g : U \rightarrow Z$ be the composition of h with the map $W \rightarrow Z$. To finish the proof we have to show that $g : U \rightarrow Z$ descends to a morphism $X \rightarrow Z$. To do this, consider the morphism $(h, h) : U \times_X U \rightarrow W \times_Z W$. Composing with $U_i \times_{X_i} U_i \rightarrow U \times_X U$ we obtain (h_i, h_i) which factors through $W \times_Z W$. Since $U \times_X U$ is the colimit of the schemes $U_i \times_{X_i} U_i$ by (3) we see that (h, h) factors through $W \times_Z W$. Hence the two compositions $U \times_X U \rightarrow U \rightarrow W \rightarrow Z$ are equal. Because each $U_i \rightarrow X_i$ is surjective and assumption (2) we see that $U \rightarrow X$ is surjective. As Z is a sheaf for the étale topology, we conclude that $g : U \rightarrow Z$ descends to $f : X \rightarrow Z$ as desired. \square

Lemma 2.2. *Let S be a scheme. Let $X \rightarrow X'$ be a thickening of schemes over S and let $X \rightarrow Y$ be an affine morphism of schemes over S . Let $Y' = Y \amalg_X X'$ be the pushout in the category of schemes (see More on Morphisms, Lemma 11.1). Then Y' is also a pushout in the category of algebraic spaces over S .*

Proof. This is an immediate consequence of Lemma 2.1 and More on Morphisms, Lemmas 11.1, 11.2, and 11.4. \square

Lemma 2.3. *Let S be a scheme. Let $X \rightarrow X'$ be a thickening of algebraic spaces over S and let $X \rightarrow Y$ be an affine morphism of algebraic spaces over S . Then there exists a pushout*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ f \downarrow & & \downarrow f' \\ Y & \longrightarrow & Y \amalg_X X' \end{array}$$

in the category of algebraic spaces over S . Moreover $Y' = Y \amalg_X X'$ is a thickening of Y and

$$\mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_* \mathcal{O}_X} f'_* \mathcal{O}_{X'}$$

as sheaves on $Y_{\acute{e}tale} = (Y')_{\acute{e}tale}$.

Proof. Choose a scheme V and a surjective étale morphism $V \rightarrow Y$. Set $U = V \times_Y X$. This is a scheme affine over V with a surjective étale morphism $U \rightarrow X$. By More on Morphisms of Spaces, Lemma 8.6 there exists a $U' \rightarrow X'$ surjective étale with $U = U' \times_{X'} X$. In particular the morphism of schemes $U \rightarrow U'$ is a thickening too. Apply More on Morphisms, Lemma 11.1 to obtain a pushout $V' = V \amalg_U U'$ in the category of schemes.

We repeat this procedure to construct a pushout

$$\begin{array}{ccc} U \times_X U & \longrightarrow & U' \times_{X'} U' \\ \downarrow & & \downarrow \\ V \times_Y V & \longrightarrow & R' \end{array}$$

in the category of schemes. Consider the morphisms

$$U \times_X U \rightarrow U \rightarrow V', \quad U' \times_{X'} U' \rightarrow U' \rightarrow V', \quad V \times_Y V \rightarrow V \rightarrow V'$$

where we use the first projection in each case. Clearly these glue to give a morphism $t' : R' \rightarrow V'$ which is étale by More on Morphisms, Lemma 11.4. Similarly, we obtain $s' : R' \rightarrow V'$ étale. The morphism $j' = (t', s') : R' \rightarrow V' \times_S V'$ is unramified (as t' is étale) and a monomorphism when restricted to the closed subscheme $V \times_Y V \subset R'$. As $V \times_Y V \subset R'$ is a thickening it follows that j' is a monomorphism too. Finally, j' is an equivalence relation as we can use the functoriality of pushouts of schemes to construct a morphism $c' : R' \times_{s', V', t'} R' \rightarrow R'$ (details omitted). At this point we set $Y' = U'/R'$, see Spaces, Theorem 10.5.

We have morphisms $X' = U'/U' \times_{X'} U' \rightarrow V'/R' = Y'$ and $Y = V/V \times_Y V \rightarrow V'/R' = Y'$. By construction these fit into the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ f \downarrow & & \downarrow f' \\ Y & \longrightarrow & Y' \end{array}$$

Since $Y \rightarrow Y'$ is a thickening we have $Y_{\acute{e}tale} = (Y')_{\acute{e}tale}$, see More on Morphisms of Spaces, Lemma 8.6. The commutativity of the diagram gives a map of sheaves

$$\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_Y \times_{f_* \mathcal{O}_X} f'_* \mathcal{O}_{X'}$$

on this set. By More on Morphisms, Lemma 11.1 this map is an isomorphism when we restrict to the scheme V' , hence it is an isomorphism.

To finish the proof we show that the diagram above is a pushout in the category of algebraic spaces. To see this, let Z be an algebraic space and let $a' : X' \rightarrow Z$ and $b : Y \rightarrow Z$ be morphisms of algebraic spaces. By Lemma 2.2 we obtain a unique morphism $h : V' \rightarrow Z$ fitting into the commutative diagrams

$$\begin{array}{ccc} U' & \longrightarrow & V' \\ \downarrow & & \downarrow h \\ X' & \xrightarrow{a'} & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} V & \longrightarrow & V' \\ \downarrow & & \downarrow h \\ Y & \xrightarrow{b} & Z \end{array}$$

The uniqueness shows that $h \circ t' = h \circ s'$. Hence h factors uniquely as $V' \rightarrow Y' \rightarrow Z$ and we win. \square

In the following lemma we use the fibre product of categories as defined in Categories, Example 29.3.

Lemma 2.4. *Let S be a base scheme. Let $X \rightarrow X'$ be a thickening of algebraic spaces over S and let $X \rightarrow Y$ be an affine morphism of algebraic spaces over S . Let $Y' = Y \amalg_{X'} X'$ be the pushout (see Lemma 2.3). Base change gives a functor*

$$F : (\text{Spaces}/Y') \longrightarrow (\text{Spaces}/Y) \times_{(\text{Spaces}/Y')} (\text{Spaces}/X')$$

given by $V' \mapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ which sends (Sch/Y') into $(\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$. The functor F has a left adjoint

$$G : (\text{Spaces}/Y) \times_{(\text{Spaces}/Y')} (\text{Spaces}/X') \longrightarrow (\text{Spaces}/Y')$$

which sends the triple (V, U', φ) to the pushout $V \amalg_{(V \times_Y X)} U'$ in the category of algebraic spaces over S . The functor G sends $(\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$ into (Sch/Y') .

Proof. The proof is completely formal. Since the morphisms $X \rightarrow X'$ and $X \rightarrow Y$ are representable it is clear that F sends (Sch/Y') into $(\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$.

Let us construct G . Let (V, U', φ) be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \rightarrow U'$ is a thickening. Since $\varphi : V \times_Y X \rightarrow U' \times_{X'} X = U$ is an isomorphism we have a morphism $U \rightarrow V$ over $X \rightarrow Y$ which identifies U with the fibre product $X \times_Y V$. In particular $U \rightarrow V$ is affine, see Morphisms of Spaces, Lemma 20.5. Hence we can apply Lemma 2.3 to get a pushout $V' = V \amalg_U U'$. Denote $V' \rightarrow Y'$ the morphism we obtain in virtue of the fact that V' is a pushout and because we are given morphisms $V \rightarrow Y$ and $U' \rightarrow X'$ agreeing on U as morphisms into Y' . Setting $G(V, U', \varphi) = V'$ gives the functor G .

If (V, U', φ) is an object of $(\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$ then $U = U' \times_{X'} X$ is a scheme too and we can form the pushout $V' = V \amalg_U U'$ in the category of schemes by More on Morphisms, Lemma 11.1. By Lemma 2.2 this is also a pushout in the category of schemes, hence G sends $(\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$ into (Sch/Y') .

Let us prove that G is a left adjoint to F . Let Z be an algebraic space over Y' . We have to show that

$$\text{Mor}(V', Z) = \text{Mor}((V, U', \varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \rightarrow Z$ be a morphism. Denote \tilde{g} , resp. \tilde{f}' the composition of g' with the morphism $V \rightarrow V'$, resp. $U' \rightarrow V'$. Base change \tilde{g} , resp. \tilde{f}' by $Y \rightarrow Y'$, resp. $X' \rightarrow Y'$ to get a morphism $g : V \rightarrow Z \times_{Y'} Y$, resp. $f' : U' \rightarrow Z \times_{Y'} X'$. Then (g, f') is an element of the

right hand side of the equation above (details omitted). Conversely, suppose that $(g, f') : (V, U', \varphi) \rightarrow F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \rightarrow Z$, resp. $\tilde{f}' : U' \rightarrow Z$ of g , resp. f' by $Z \times_{Y'} X' \rightarrow Z$, resp. $Z \times_{Y'} Y \rightarrow Z$. Then \tilde{g} and \tilde{f}' agree as morphism from U to Z . By the universal property of pushout, we obtain a morphism $g' : V' \rightarrow Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse. \square

Lemma 2.5. *Let S be a scheme. Let*

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & D & \longrightarrow & F \end{array}$$

be a commutative diagram of algebraic spaces over S . Assume that A, B, C, D and A, B, E, F form cartesian squares and that $B \rightarrow D$ is surjective étale. Then C, D, E, F is a cartesian square.

Proof. This is formal. \square

Lemma 2.6. *In the situation of Lemma 2.4 the functor $F \circ G$ is isomorphic to the identity functor.*

Proof. We will prove that $F \circ G$ is isomorphic to the identity by reducing this to the corresponding statement of More on Morphisms, Lemma 11.2.

Choose a scheme Y_1 and a surjective étale morphism $Y_1 \rightarrow Y$. Set $X_1 = Y_1 \times_Y X$. This is a scheme affine over Y_1 with a surjective étale morphism $X_1 \rightarrow X$. By More on Morphisms of Spaces, Lemma 8.6 there exists a $X'_1 \rightarrow X'$ surjective étale with $X_1 = X'_1 \times_{X'} X$. In particular the morphism of schemes $X_1 \rightarrow X'_1$ is a thickening too. Apply More on Morphisms, Lemma 11.1 to obtain a pushout $Y'_1 = Y_1 \amalg_{X_1} X'_1$ in the category of schemes. In the proof of Lemma 2.3 we constructed Y' as a quotient of an étale equivalence relation on Y'_1 such that we get a commutative diagram

(2.6.1)
$$\begin{array}{ccccc} & & X & \longrightarrow & X' \\ & \nearrow & \downarrow & & \downarrow \\ X_1 & \longrightarrow & X'_1 & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ & \nearrow & Y & \longrightarrow & Y' \\ Y_1 & \longrightarrow & Y'_1 & \longrightarrow & Y' \end{array}$$

where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. Denote F_1, G_1 the functors constructed in More on Morphisms, Lemma 11.2 for the front

square. Then the diagram of categories

$$\begin{array}{ccc}
 (Sch/Y'_1) & \xleftarrow{G_1} & (Sch/Y_1) \times_{(Sch/Y'_1)} (Sch/X'_1) \\
 \downarrow & \xrightarrow{F_1} & \downarrow \\
 (Spaces/Y') & \xleftarrow{G} & (Spaces/Y) \times_{(Spaces/Y')} (Spaces/X') \\
 & \xrightarrow{F} &
 \end{array}$$

is commutative by simple considerations regarding base change functors and the agreement of pushouts in schemes with pushouts in spaces of Lemma 2.2.

Let (V, U', φ) be an object of $(Spaces/Y) \times_{(Spaces/Y')} (Spaces/X')$. Denote $U = U' \times_{X'} X$ so that $G(V, U', \varphi) = V \amalg_U U'$. Choose a scheme V_1 and a surjective étale morphism $V_1 \rightarrow Y_1 \times_Y V$. Set $U_1 = V_1 \times_Y X$. Then

$$U_1 = V_1 \times_Y X \longrightarrow (Y_1 \times_Y V) \times_Y X = X_1 \times_Y V = X_1 \times_X X \times_Y V = X_1 \times_X U$$

is surjective étale too. By More on Morphisms of Spaces, Lemma 8.6 there exists a thickening $U_1 \rightarrow U'_1$ and a surjective étale morphism $U'_1 \rightarrow X'_1 \times_{X'} U'$ whose base change to $X_1 \times_X U$ is the displayed morphism. At this point (V_1, U'_1, φ_1) is an object of $(Sch/Y_1) \times_{(Sch/Y'_1)} (Sch/X'_1)$. In the proof of Lemma 2.3 we constructed $G(V, U', \varphi) = V \amalg_U U'$ as a quotient of an étale equivalence relation on $G_1(V_1, U'_1, \varphi_1) = V_1 \amalg_{U_1} U'_1$ such that we get a commutative diagram

$$(2.6.2) \quad \begin{array}{ccccc}
 & & U & \xrightarrow{\quad} & U' \\
 & & \downarrow & & \downarrow \\
 U_1 & \xrightarrow{\quad} & U'_1 & \xrightarrow{\quad} & U' \\
 \downarrow & & \downarrow & & \downarrow \\
 & & V & \xrightarrow{\quad} & G(V, U', \varphi) \\
 \downarrow & & \downarrow & & \downarrow \\
 V_1 & \xrightarrow{\quad} & G_1(V_1, U'_1, \varphi_1) & \xrightarrow{\quad} & G(V, U', \varphi)
 \end{array}$$

where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. In particular

$$G_1(V_1, U'_1, \varphi_1) \rightarrow G(V, U', \varphi)$$

is surjective étale.

Finally, we come to the proof of the lemma. We have to show that the adjunction mapping $(V, U', \varphi) \rightarrow F(G(V, U', \varphi))$ is an isomorphism. We know $(V_1, U'_1, \varphi_1) \rightarrow F_1(G_1(V_1, U'_1, \varphi_1))$ is an isomorphism by More on Morphisms, Lemma 11.2. Recall that F and F_1 are given by base change. Using the properties of (2.6.2) and Lemma 2.5 we see that $V \rightarrow G(V, U', \varphi) \times_{Y'} Y$ and $U' \rightarrow G(V, U', \varphi) \times_{Y'} X'$ are isomorphisms, i.e., $(V, U', \varphi) \rightarrow F(G(V, U', \varphi))$ is an isomorphism. \square

Lemma 2.7. *Let S be a base scheme. Let $X \rightarrow X'$ be a thickening of algebraic spaces over S and let $X \rightarrow Y$ be an affine morphism of algebraic spaces over S . Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 2.3). Let $V' \rightarrow Y'$ be a morphism of*

algebraic spaces over S . Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$, and $U = X \times_{Y'} V'$. There is an equivalence of categories between

- (1) quasi-coherent $\mathcal{O}_{V'}$ -modules flat over Y' , and
- (2) the category of triples $(\mathcal{G}, \mathcal{F}', \varphi)$ where
 - (a) \mathcal{G} is a quasi-coherent \mathcal{O}_V -module flat over Y ,
 - (b) \mathcal{F}' is a quasi-coherent $\mathcal{O}_{U'}$ -module flat over X , and
 - (c) $\varphi : (U \rightarrow V)^* \mathcal{G} \rightarrow (U \rightarrow U')^* \mathcal{F}'$ is an isomorphism of \mathcal{O}_U -modules.

The equivalence maps \mathcal{G}' to $((V \rightarrow V')^* \mathcal{G}', (U' \rightarrow V')^* \mathcal{G}', \text{can})$. Suppose \mathcal{G}' corresponds to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$. Then

- (a) \mathcal{G}' is a finite type $\mathcal{O}_{V'}$ -module if and only if \mathcal{G} and \mathcal{F}' are finite type \mathcal{O}_Y and $\mathcal{O}_{U'}$ -modules.
- (b) if $V' \rightarrow Y'$ is locally of finite presentation, then \mathcal{G}' is an $\mathcal{O}_{V'}$ -module of finite presentation if and only if \mathcal{G} and \mathcal{F}' are \mathcal{O}_Y and $\mathcal{O}_{U'}$ -modules of finite presentation.

Proof. A quasi-inverse functor assigns to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$ the fibre product

$$(V \rightarrow V')^* \mathcal{G} \times_{(U \rightarrow V')^* \mathcal{F}'} (U' \rightarrow V')^* \mathcal{F}'$$

where $\mathcal{F} = (U \rightarrow U')^* \mathcal{F}'$. This works, because on affines étale over V' and Y' we recover the equivalence of More on Algebra, Lemma 4.12. Details omitted.

Parts (a) and (b) reduce by étale localization (Properties of Spaces, Section 28) to the case where V' and Y' are affine in which case the result follows from More on Algebra, Lemmas 4.11 and 4.13. \square

Lemma 2.8. *In the situation of Lemma 2.6. If $V' = G(V, U', \varphi)$ for some triple (V, U', φ) , then*

- (1) $V' \rightarrow Y'$ is locally of finite type if and only if $V \rightarrow Y$ and $U' \rightarrow X'$ are locally of finite type,
- (2) $V' \rightarrow Y'$ is flat if and only if $V \rightarrow Y$ and $U' \rightarrow X'$ are flat,
- (3) $V' \rightarrow Y'$ is flat and locally of finite presentation if and only if $V \rightarrow Y$ and $U' \rightarrow X'$ are flat and locally of finite presentation,
- (4) $V' \rightarrow Y'$ is smooth if and only if $V \rightarrow Y$ and $U' \rightarrow X'$ are smooth,
- (5) $V' \rightarrow Y'$ is étale if and only if $V \rightarrow Y$ and $U' \rightarrow X'$ are étale, and
- (6) add more here as needed.

If W' is flat over Y' , then the adjunction mapping $G(F(W')) \rightarrow W'$ is an isomorphism. Hence F and G define mutually quasi-inverse functors between the category of spaces flat over Y' and the category of triples (V, U', φ) with $V \rightarrow Y$ and $U' \rightarrow X'$ flat.

Proof. Choose a diagram (2.6.1) as in the proof of Lemma 2.6.

Proof of (1) – (5). Let (V, U', φ) be an object of $(\text{Spaces}/Y) \times_{(\text{Spaces}/Y')} (\text{Spaces}/X')$. Construct a diagram (2.6.2) as in the proof of Lemma 2.6. Then the base change of $G(V, U', \varphi) \rightarrow Y'$ to Y'_1 is $G_1(V_1, U'_1, \varphi_1) \rightarrow Y'_1$. Hence (1) – (5) follow immediately from the corresponding statements of More on Morphisms, Lemma 11.4 for schemes.

Suppose that $W' \rightarrow Y'$ is flat. Choose a scheme W'_1 and a surjective étale morphism $W'_1 \rightarrow Y'_1 \times_{Y'} W'$. Observe that $W'_1 \rightarrow W'$ is surjective étale as a composition of surjective étale morphisms. We know that $G_1(F_1(W'_1)) \rightarrow W'_1$ is an isomorphism by More on Morphisms, Lemma 11.4 applied to W'_1 over Y'_1 and the front of the

diagram (with functors G_1 and F_1 as in the proof of Lemma 2.6). Then the construction of $G(F(W'))$ (as a pushout, i.e., as constructed in Lemma 2.3) shows that $G_1(F_1(W'_1)) \rightarrow G(F(W))$ is surjective étale. Whereupon we conclude that $G(F(W)) \rightarrow W$ is étale, see for example Properties of Spaces, Lemma 13.3. But $G(F(W)) \rightarrow W$ is an isomorphism on underlying reduced algebraic spaces (by construction), hence it is an isomorphism. \square

3. Formal glueing of quasi-coherent modules

This section is the analogue of More on Algebra, Section 63. In the case of morphisms of schemes, the result can be found in the paper by Joyet [Joy96]; this is a good place to start reading. For a discussion of applications to descent problems for stacks, see the paper by Moret-Bailly [MB96]. In the case of an affine morphism of schemes there is a statement in the appendix of the paper [FR70] but one needs to add the hypothesis that the closed subscheme is cut out by a finitely generated ideal (as in the paper by Joyet) since otherwise the result does not hold. A generalization of this material to (higher) derived categories with potential applications to nonflat situations can be found in [Bha14, Section 5].

We start with a lemma on abelian sheaves supported on closed subsets.

Lemma 3.1. *Let S be a scheme. Let $f : Y \rightarrow X$ be a morphism of algebraic spaces over S . Let $Z \subset X$ closed subspace such that $f^{-1}Z \rightarrow Z$ is integral and universally injective. Let \bar{y} be a geometric point of Y and $\bar{x} = f(\bar{y})$. We have*

$$(Rf_*Q)_{\bar{x}} = Q_{\bar{y}}$$

in $D(\text{Ab})$ for any object Q of $D(Y_{\text{étale}})$ supported on $|f^{-1}Z|$.

Proof. Consider the commutative diagram of algebraic spaces

$$\begin{array}{ccc} f^{-1}Z & \xrightarrow{\quad} & Y \\ f' \downarrow & & \downarrow f \\ Z & \xrightarrow{\quad i \quad} & X \end{array}$$

By Cohomology of Spaces, Lemma 8.4 we can write $Q = Ri'_*K'$ for some object K' of $D(f^{-1}Z_{\text{étale}})$. By Morphisms of Spaces, Lemma 47.5 we have $K' = (f')^{-1}K$ with $K = Rf'_*K'$. Then we have $Rf_*Q = Rf_*Ri'_*K' = Ri_*Rf'_*K' = Ri_*K$. Let \bar{z} be the geometric point of Z corresponding to \bar{x} and let \bar{z}' be the geometric point of $f^{-1}Z$ corresponding to \bar{y} . We obtain the result of the lemma as follows

$$Q_{\bar{y}} = (Ri'_*K')_{\bar{y}} = K'_{\bar{z}'} = (f')^{-1}K_{\bar{z}} = K_{\bar{z}} = Ri_*K_{\bar{x}} = Rf_*Q_{\bar{x}}$$

The middle equality holds because of the description of the stalk of a pullback given in Properties of Spaces, Lemma 16.9. \square

Lemma 3.2. *Let S be a scheme. Let $f : Y \rightarrow X$ be a morphism of algebraic spaces over S . Let $Z \subset X$ closed subspace such that $f^{-1}Z \rightarrow Z$ is integral and universally injective. Let \bar{y} be a geometric point of Y and $\bar{x} = f(\bar{y})$. Let \mathcal{G} be an abelian sheaf on Y . Then the map of two term complexes*

$$(f_*\mathcal{G}_{\bar{x}} \rightarrow (f \circ j')_*(\mathcal{G}|_V)_{\bar{x}}) \longrightarrow (\mathcal{G}_{\bar{y}} \rightarrow j'_*(\mathcal{G}|_V)_{\bar{y}})$$

induces an isomorphism on kernels and an injection on cokernels. Here $V = Y \setminus f^{-1}Z$ and $j' : V \rightarrow Y$ is the inclusion.

Proof. Choose a distinguished triangle

$$\mathcal{G} \rightarrow Rj'_*\mathcal{G}|_V \rightarrow Q \rightarrow \mathcal{G}[1]$$

in $D(Y_{\acute{e}tale})$. The cohomology sheaves of Q are supported on $|f^{-1}Z|$. We apply Rf_* and we obtain

$$Rf_*\mathcal{G} \rightarrow Rf_*Rj'_*\mathcal{G}|_V \rightarrow Rf_*Q \rightarrow Rf_*\mathcal{G}[1]$$

Taking stalks at \bar{x} we obtain an exact sequence

$$0 \rightarrow (R^{-1}f_*Q)_{\bar{x}} \rightarrow f_*\mathcal{G}_{\bar{x}} \rightarrow (f \circ j')_*(\mathcal{G}|_V)_{\bar{x}} \rightarrow (R^0f_*Q)_{\bar{x}}$$

We can compare this with the exact sequence

$$0 \rightarrow H^{-1}(Q)_{\bar{y}} \rightarrow \mathcal{G}_{\bar{y}} \rightarrow j'_*(\mathcal{G}|_V)_{\bar{y}} \rightarrow H^0(Q)_{\bar{y}}$$

Thus we see that the lemma follows because $Q_{\bar{y}} = Rf_*Q_{\bar{x}}$ by Lemma 3.1. \square

Lemma 3.3. *Let S be a scheme. Let X be an algebraic space over S . Let $f : Y \rightarrow X$ be a quasi-compact and quasi-separated morphism. Let \bar{x} be a geometric point of X and let $\text{Spec}(\mathcal{O}_{X,\bar{x}}) \rightarrow X$ be the canonical morphism. For a quasi-coherent module \mathcal{G} on Y we have*

$$f_*\mathcal{G}_{\bar{x}} = \Gamma(Y \times_X \text{Spec}(\mathcal{O}_{X,\bar{x}}), p^*\mathcal{F})$$

where $p : Y \times_X \text{Spec}(\mathcal{O}_{X,\bar{x}}) \rightarrow Y$ is the projection.

Proof. Observe that $f_*\mathcal{G}_{\bar{x}} = \Gamma(\text{Spec}(\mathcal{O}_{X,\bar{x}}), h^*f_*\mathcal{G})$ where $h : \text{Spec}(\mathcal{O}_{X,\bar{x}}) \rightarrow X$. Hence the result is true because h is flat so that Cohomology of Spaces, Lemma 10.1 applies. \square

Lemma 3.4. *Let S be a scheme. Let X be an algebraic space over S . Let $i : Z \rightarrow X$ be a closed immersion of finite presentation. Let $Q \in D_{QCoh}(\mathcal{O}_X)$ be supported on $|Z|$. Let \bar{x} be a geometric point of X and let $I_{\bar{x}} \subset \mathcal{O}_{X,\bar{x}}$ be the stalk of the ideal sheaf of Z . Then the cohomology modules $H^n(Q_{\bar{x}})$ are $I_{\bar{x}}$ -power torsion (see More on Algebra, Definition 62.1).*

Proof. Choose an affine scheme U and an étale morphism $U \rightarrow X$ such that \bar{x} lifts to a geometric point \bar{u} of U . Then we can replace X by U , Z by $U \times_X Z$, Q by the restriction $Q|_U$, and \bar{x} by \bar{u} . Thus we may assume that $X = \text{Spec}(A)$ is affine. Let $I \subset A$ be the ideal defining Z . Since $i : Z \rightarrow X$ is of finite presentation, the ideal $I = (f_1, \dots, f_r)$ is finitely generated. The object Q comes from a complex of A -modules M^\bullet , see Derived Categories of Spaces, Lemma 4.2 and Derived Categories of Schemes, Lemma 3.4. Since the cohomology sheaves of Q are supported on Z we see that the localization M_f^\bullet is acyclic for each $f \in I$. Take $x \in H^p(M^\bullet)$. By the above we can find n_i such that $f_i^{n_i}x = 0$ in $H^p(M^\bullet)$ for each i . Then with $n = \sum n_i$ we see that I^n annihilates x . Thus $H^p(M^\bullet)$ is I -power torsion. Since the ring map $A \rightarrow \mathcal{O}_{X,\bar{x}}$ is flat and since $I_{\bar{x}} = I\mathcal{O}_{X,\bar{x}}$ we conclude. \square

Lemma 3.5. *Let S be a scheme. Let $f : Y \rightarrow X$ be a morphism of algebraic spaces over S . Let $Z \subset X$ be a closed subspace. Assume $f^{-1}Z \rightarrow Z$ is an isomorphism and that f is flat in every point of $f^{-1}Z$. For any Q in $D_{QCoh}(\mathcal{O}_Y)$ supported on $|f^{-1}Z|$ we have $Lf^*Rf_*Q = Q$.*

Proof. We show the canonical map $Lf^*Rf_*Q \rightarrow Q$ is an isomorphism by checking on stalks at \bar{y} . If \bar{y} is not in $f^{-1}Z$, then both sides are zero and the result is true. Assume the image \bar{x} of \bar{y} is in Z . By Lemma 3.1 we have $Rf_*Q_{\bar{x}} = Q_{\bar{y}}$ and since f is flat at \bar{y} we see that

$$(Lf^*Rf_*Q)_{\bar{y}} = (Rf_*Q)_{\bar{x}} \otimes_{\mathcal{O}_{X,\bar{x}}} \mathcal{O}_{Y,\bar{y}} = Q_{\bar{y}} \otimes_{\mathcal{O}_{X,\bar{x}}} \mathcal{O}_{Y,\bar{y}}$$

Thus we have to check that the canonical map

$$Q_{\bar{y}} \otimes_{\mathcal{O}_{X,\bar{x}}} \mathcal{O}_{Y,\bar{y}} \longrightarrow Q_{\bar{y}}$$

is an isomorphism in the derived category. Let $I_{\bar{x}} \subset \mathcal{O}_{X,\bar{x}}$ be the stalk of the ideal sheaf defining Z . Since $Z \rightarrow X$ is locally of finite presentation this ideal is finitely generated and the cohomology groups of $Q_{\bar{y}}$ are $I_{\bar{y}} = I_{\bar{x}}\mathcal{O}_{Y,\bar{y}}$ -power torsion by Lemma 3.4 applied to Q on Y . It follows that they are also $I_{\bar{x}}$ -power torsion. The ring map $\mathcal{O}_{X,\bar{x}} \rightarrow \mathcal{O}_{Y,\bar{y}}$ is flat and induces an isomorphism after dividing by $I_{\bar{x}}$ and $I_{\bar{y}}$ because we assumed that $f^{-1}Z \rightarrow Z$ is an isomorphism. Hence we see that the cohomology modules of $Q_{\bar{y}} \otimes_{\mathcal{O}_{X,\bar{x}}} \mathcal{O}_{Y,\bar{y}}$ are equal to the cohomology modules of $Q_{\bar{y}}$ by More on Algebra, Lemma 63.2 which finishes the proof. \square

Situation 3.6. Here S is a base scheme, $f : Y \rightarrow X$ is a quasi-compact and quasi-separated morphism of algebraic spaces over S , and $Z \rightarrow X$ is a closed immersion of finite presentation. We assume that $f^{-1}(Z) \rightarrow Z$ is an isomorphism and that f is flat in every point $x \in |f^{-1}Z|$. We set $U = X \setminus Z$ and $V = Y \setminus f^{-1}(Z)$. Picture

$$\begin{array}{ccc} V & \xrightarrow{\quad} & Y \\ f|_V \downarrow & j' & \downarrow f \\ U & \xrightarrow{\quad} & X \end{array}$$

In Situation 3.6 we define $QCoh(Y \rightarrow X, Z)$ as the category of triples $(\mathcal{H}, \mathcal{G}, \varphi)$ where \mathcal{H} is a quasi-coherent sheaf of \mathcal{O}_U -modules, \mathcal{G} is a quasi-coherent sheaf of \mathcal{O}_Y -modules, and $\varphi : f^*\mathcal{H} \rightarrow \mathcal{G}|_V$ is an isomorphism of \mathcal{O}_V -modules. There is a canonical functor

$$(3.6.1) \quad QCoh(\mathcal{O}_X) \longrightarrow QCoh(Y \rightarrow X, Z)$$

which maps \mathcal{F} to the system $(\mathcal{F}|_U, f^*\mathcal{F}, \text{can})$. By analogy with the proof given in the affine case, we construct a functor in the opposite direction. To an object $(\mathcal{H}, \mathcal{G}, \varphi)$ we assign the \mathcal{O}_X -module

$$(3.6.2) \quad \text{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \rightarrow (f \circ j')_*\mathcal{G}|_V)$$

Observe that j and j' are quasi-compact morphisms as $Z \rightarrow X$ is of finite presentation. Hence f_* , j_* , and $(f \circ j')_*$ transform quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 11.2). Thus the module (3.6.2) is quasi-coherent.

Lemma 3.7. *In Situation 3.6. The functor (3.6.2) is right adjoint to the functor (3.6.1).*

Proof. This follows easily from the adjointness of f^* to f_* and j^* to j_* . Details omitted. \square

Lemma 3.8. *In Situation 3.6. Let $X' \rightarrow X$ be a flat morphism of algebraic spaces. Set $Z' = X' \times_X Z$ and $Y' = X' \times_X Y$. The pullbacks $QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_{X'})$*

and $QCoh(Y \rightarrow X, Z) \rightarrow QCoh(Y' \rightarrow X', Z')$ are compatible with the functors (3.6.2) and 3.6.1).

Proof. This is true because pullback commutes with pullback and because flat pullback commutes with pushforward along quasi-compact and quasi-separated morphisms, see Cohomology of Spaces, Lemma 10.1. \square

Proposition 3.9. *In Situation 3.6 the functor (3.6.1) is an equivalence with quasi-inverse given by (3.6.2).*

Proof. We first treat the special case where X and Y are affine schemes and where the morphism f is flat. Say $X = \text{Spec}(R)$ and $Y = \text{Spec}(S)$. Then f corresponds to a flat ring map $R \rightarrow S$. Moreover, $Z \subset X$ is cut out by a finitely generated ideal $I \subset R$. Choose generators $f_1, \dots, f_t \in I$. By the description of quasi-coherent modules in terms of modules (Schemes, Section 7), we see that the category $QCoh(Y \rightarrow X, Z)$ is canonically equivalent to the category $\text{Glue}(R \rightarrow S, f_1, \dots, f_t)$ of More on Algebra, Remark 63.10 such that the functors (3.6.1) and (3.6.2) correspond to the functors Can and H^0 . Hence the result follows from More on Algebra, Proposition 63.15 in this case.

We return to the general case. Let \mathcal{F} be a quasi-coherent module on X . We will show that

$$\alpha : \mathcal{F} \longrightarrow \text{Ker}(j_*\mathcal{F}|_U \oplus f_*f^*\mathcal{F} \rightarrow (f \circ j')_*f^*\mathcal{F}|_V)$$

is an isomorphism. Let $(\mathcal{H}, \mathcal{G}, \varphi)$ be an object of $QCoh(Y \rightarrow X, Z)$. We will show that

$$\beta : f^*\text{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \rightarrow (f \circ j')_*\mathcal{G}|_V) \longrightarrow \mathcal{G}$$

and

$$\gamma : j^*\text{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \rightarrow (f \circ j')_*\mathcal{G}|_V) \longrightarrow \mathcal{H}$$

are isomorphisms. To see these statements are true it suffices to look at stalks. Let \bar{y} be a geometric point of Y mapping to the geometric point \bar{x} of X .

Fix an object $(\mathcal{H}, \mathcal{G}, \varphi)$ of $QCoh(Y \rightarrow X, Z)$. By Lemma 3.2 and a diagram chase (omitted) the canonical map

$$\text{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \rightarrow (f \circ j')_*\mathcal{G}|_V)_{\bar{x}} \longrightarrow \text{Ker}(j_*\mathcal{H}_{\bar{x}} \oplus \mathcal{G}_{\bar{y}} \rightarrow j'_*\mathcal{G}_{\bar{y}})$$

is an isomorphism.

In particular, if \bar{y} is a geometric point of V , then we see that $j'_*\mathcal{G}_{\bar{y}} = \mathcal{G}_{\bar{y}}$ and hence that this kernel is equal to $\mathcal{H}_{\bar{x}}$. This easily implies that $\alpha_{\bar{x}}$, $\beta_{\bar{x}}$, and $\beta_{\bar{y}}$ are isomorphisms in this case.

Next, assume that \bar{y} is a point of $f^{-1}Z$. Let $I_{\bar{x}} \subset \mathcal{O}_{X, \bar{x}}$, resp. $I_{\bar{y}} \subset \mathcal{O}_{Y, \bar{y}}$ be the stalk of the ideal cutting out Z , resp. $f^{-1}Z$. Then $I_{\bar{x}}$ is a finitely generated ideal, $I_{\bar{y}} = I_{\bar{x}}\mathcal{O}_{Y, \bar{y}}$, and $\mathcal{O}_{X, \bar{x}} \rightarrow \mathcal{O}_{Y, \bar{y}}$ is a flat local homomorphism inducing an isomorphism $\mathcal{O}_{X, \bar{x}}/I_{\bar{x}} = \mathcal{O}_{Y, \bar{y}}/I_{\bar{y}}$. At this point we can bootstrap using the diagram

of categories

$$\begin{array}{ccc}
& \xleftarrow{(3.6.2)} & \\
Q\text{Coh}(\mathcal{O}_X) & \xrightarrow{(3.6.1)} & Q\text{Coh}(Y \rightarrow X, Z) \\
\downarrow & & \downarrow \\
\text{Mod}_{\mathcal{O}_{X,\bar{x}}} & \xrightarrow{\text{Can}} & \text{Glue}(\mathcal{O}_{X,\bar{x}} \rightarrow \mathcal{O}_{Y,\bar{y}}, f_1, \dots, f_t) \\
& \xleftarrow{H^0} &
\end{array}$$

Namely, as in the first paragraph of the proof we identify

$$\text{Glue}(\mathcal{O}_{X,\bar{x}} \rightarrow \mathcal{O}_{Y,\bar{y}}, f_1, \dots, f_t) = Q\text{Coh}(\text{Spec}(\mathcal{O}_{Y,\bar{y}}) \rightarrow \text{Spec}(\mathcal{O}_{X,\bar{x}}), V(I_{\bar{x}}))$$

The right vertical functor is given by pullback, and it is clear that the inner square is commutative. Our computation of the stalk of the kernel in the third paragraph of the proof combined with Lemma 3.3 implies that the outer square (using the curved arrows) commutes. Thus we conclude using the case of a flat morphism of affine schemes which we handled in the first paragraph of the proof. \square

Lemma 3.10. *In Situation 3.6 the functor Rf_* induces an equivalence between $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$ and $D_{QCoh,|Z|}(\mathcal{O}_X)$ with quasi-inverse given by Lf^* .*

Proof. Since f is quasi-compact and quasi-separated we see that Rf_* defines a functor from $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$ to $D_{QCoh,|Z|}(\mathcal{O}_X)$, see Derived Categories of Spaces, Lemma 6.1. By Derived Categories of Spaces, Lemma 5.4 we see that Lf^* maps $D_{QCoh,|Z|}(\mathcal{O}_X)$ into $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$. In Lemma 3.5 we have seen that $Lf^*Rf_*Q = Q$ for Q in $D_{QCoh,|f^{-1}Z|}(\mathcal{O}_Y)$. By the dual of Derived Categories, Lemma 7.2 to finish the proof it suffices to show that $Lf^*K = 0$ implies $K = 0$ for K in $D_{QCoh,|Z|}(\mathcal{O}_X)$. This follows from the fact that f is flat at all points of $f^{-1}Z$ and the fact that $f^{-1}Z \rightarrow Z$ is surjective. \square

Lemma 3.11. *In Situation 3.6 there exists an fpqc covering $\{X_i \rightarrow X\}_{i \in I}$ refining the family $\{U \rightarrow X, Y \rightarrow X\}$.*

Proof. For the definition and general properties of fpqc coverings we refer to Topologies, Section 8. In particular, we can first choose an étale covering $\{X_i \rightarrow X\}$ with X_i affine and by base changing Y , Z , and U to each X_i we reduce to the case where X is affine. In this case U is quasi-compact and hence a finite union $U = U_1 \cup \dots \cup U_n$ of affine opens. Then Z is quasi-compact hence also $f^{-1}Z$ is quasi-compact. Thus we can choose an affine scheme W and an étale morphism $h : W \rightarrow Y$ such that $h^{-1}f^{-1}Z \rightarrow f^{-1}Z$ is surjective. Say $W = \text{Spec}(B)$ and $h^{-1}f^{-1}Z = V(J)$ where $J \subset B$ is an ideal of finite type. By Pro-étale Cohomology, Lemma 5.1 there exists a localization $B \rightarrow B'$ such that points of $\text{Spec}(B')$ correspond exactly to points of $W = \text{Spec}(B)$ specializing to $h^{-1}f^{-1}Z = V(J)$. It follows that the composition $\text{Spec}(B') \rightarrow \text{Spec}(B) = W \rightarrow Y \rightarrow X$ is flat as by assumption $f : Y \rightarrow X$ is flat at all the points of $f^{-1}Z$. Then $\{\text{Spec}(B') \rightarrow X, U_1 \rightarrow X, \dots, U_n \rightarrow X\}$ is an fpqc covering by Topologies, Lemma 8.2. \square

4. Formal glueing of algebraic spaces

In Situation 3.6 we consider the category $\mathit{Spaces}(X \rightarrow Y, Z)$ of commutative diagrams of algebraic spaces over S of the form

$$\begin{array}{ccccc} U' & \longleftarrow & V' & \longrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & V & \longrightarrow & Y \end{array}$$

where both squares are cartesian. There is a canonical functor

$$(4.0.1) \quad \mathit{Spaces}/X \longrightarrow \mathit{Spaces}(Y \rightarrow X, Z)$$

which maps $X' \rightarrow X$ to the morphisms $U \times_X X' \leftarrow V \times_X X' \rightarrow Y \times_X X'$.

Lemma 4.1. *In Situation 3.6 the functor (4.0.1) restricts to an equivalence*

- (1) *from the category of algebraic spaces affine over X to the full subcategory of $\mathit{Spaces}(Y \rightarrow X, Z)$ consisting of $(U' \leftarrow V' \rightarrow Y')$ with $U' \rightarrow U$, $V' \rightarrow V$, and $Y' \rightarrow Y$ affine,*
- (2) *from the category of closed immersions $X' \rightarrow X$ to the full subcategory of $\mathit{Spaces}(Y \rightarrow X, Z)$ consisting of $(U' \leftarrow V' \rightarrow Y')$ with $U' \rightarrow U$, $V' \rightarrow V$, and $Y' \rightarrow Y$ closed immersions, and*
- (3) *same statement as in (2) for finite morphisms.*

Proof. The category of algebraic spaces affine over X is equivalent to the category of quasi-coherent sheaves \mathcal{A} of \mathcal{O}_X -algebras. The full subcategory of $\mathit{Spaces}(Y \rightarrow X, Z)$ consisting of $(U' \leftarrow V' \rightarrow Y')$ with $U' \rightarrow U$, $V' \rightarrow V$, and $Y' \rightarrow Y$ affine is equivalent to the category of algebra objects of $QCoh(Y \rightarrow X, Z)$. In both cases this follows from Morphisms of Spaces, Lemma 20.7 with quasi-inverse given by the relative spectrum construction (Morphisms of Spaces, Definition 20.8) which commutes with arbitrary base change. Thus part (1) of the lemma follows from Proposition 3.9.

Fully faithfulness in part (2) follows from part (1). For essential surjectivity, we reduce by part (1) to proving that $X' \rightarrow X$ is a closed immersion if and only if both $U \times_X X' \rightarrow U$ and $Y \times_X X' \rightarrow Y$ are closed immersions. By Lemma 3.11 $\{U \rightarrow X, Y \rightarrow X\}$ can be refined by an fpqc covering. Hence the result follows from Descent on Spaces, Lemma 10.15.

For (3) use the argument proving (2) and Descent on Spaces, Lemma 10.21. \square

Lemma 4.2. *In Situation 3.6 the functor (4.0.1) reflects isomorphisms.*

Proof. By a formal argument with base change, this reduces to the following question: A morphism $a : X' \rightarrow X$ of algebraic spaces such that $U \times_X X' \rightarrow U$ and $Y \times_X X' \rightarrow Y$ are isomorphisms, is an isomorphism. The family $\{U \rightarrow X, Y \rightarrow X\}$ can be refined by an fpqc covering by Lemma 3.11. Hence the result follows from Descent on Spaces, Lemma 10.13. \square

Lemma 4.3. *In Situation 3.6 the functor (4.0.1) is fully faithful on algebraic spaces separated over X . More precisely, it induces a bijection*

$$\mathrm{Mor}_X(X'_1, X'_2) \longrightarrow \mathrm{Mor}_{\mathit{Spaces}(Y \rightarrow X, Z)}(F(X'_1), F(X'_2))$$

whenever $X'_2 \rightarrow X$ is separated.

Proof. Since $X'_2 \rightarrow X$ is separated, the graph $i : X'_1 \rightarrow X'_1 \times_X X'_2$ of a morphism $X'_1 \rightarrow X'_2$ over X is a closed immersion, see Morphisms of Spaces, Lemma 4.6. Moreover a closed immersion $i : T \rightarrow X'_1 \times_X X'_2$ is the graph of a morphism if and only if $\text{pr}_1 \circ i$ is an isomorphism. The same is true for

- (1) the graph of a morphism $U \times_X X'_1 \rightarrow U \times_X X'_2$ over U ,
- (2) the graph of a morphism $V \times_X X'_1 \rightarrow V \times_X X'_2$ over V , and
- (3) the graph of a morphism $Y \times_X X'_1 \rightarrow Y \times_X X'_2$ over Y .

Moreover, if morphisms as in (1), (2), (3) fit together to form a morphism in the category $\text{Spaces}(Y \rightarrow X, Z)$, then these graphs fit together to give an object of $\text{Spaces}(Y \times_X (X'_1 \times_X X'_2) \rightarrow X'_1 \times_X X'_2, Z \times_X (X'_1 \times_X X'_2))$ whose triple of morphisms are closed immersions. The proof is finished by applying Lemmas 4.1 and 4.2. \square

5. Coequalizers and glueing

Let X be a Noetherian algebraic space and $Z \rightarrow X$ a closed subscheme. Let $X' \rightarrow X$ be the blowing up in Z . In this section we show that X can be recovered from X' , Z_n and glueing data where Z_n is the n th infinitesimal neighbourhood of Z in X .

Lemma 5.1. *Let S be a scheme. Let*

$$\begin{array}{ccc} Y & \xrightarrow{\quad g \quad} & X \\ & \searrow & \swarrow \\ & & B \end{array}$$

be a commutative diagram of algebraic spaces over S . Assume B Noetherian, g proper and surjective, and $X \rightarrow B$ separated of finite type. Let $R = Y \times_X Y$ with projection morphisms $t, s : R \rightarrow Y$. There exists a coequalizer X' of $s, t : R \rightarrow Y$ in the category of algebraic spaces separated over B . The morphism $X' \rightarrow X$ is a finite universal homeomorphism.

Proof. Denote $h : R \rightarrow X$ the given morphism. The sheaves

$$g_*\mathcal{O}_Y \quad \text{and} \quad h_*\mathcal{O}_R$$

are coherent \mathcal{O}_X -algebras (Cohomology of Spaces, Lemma 19.2). The X -morphisms s, t induce \mathcal{O}_X -algebra maps $s^\#, t^\#$ from the first to the second. Set

$$\mathcal{A} = \text{Equalizer}(s^\#, t^\# : g_*\mathcal{O}_Y \rightarrow h_*\mathcal{O}_R)$$

Then \mathcal{A} is a coherent \mathcal{O}_X -algebra and we can define

$$X' = \underline{\text{Spec}}_X(\mathcal{A})$$

as in Morphisms of Spaces, Definition 20.8. By Morphisms of Spaces, Remark 20.9 and functoriality of the $\underline{\text{Spec}}$ construction there is a factorization

$$Y \longrightarrow X' \longrightarrow X$$

and the morphism $g' : Y \rightarrow X'$ equalizes s and t . Since \mathcal{A} is a coherent \mathcal{O}_X -module it is clear that $X' \rightarrow X$ is a finite morphism of algebraic spaces. Since the surjective morphism $g : Y \rightarrow X$ factors through X' we see that $X' \rightarrow X$ is surjective.

To check that $X' \rightarrow X$ is a universal homeomorphism, it suffices to check that it is universally injective (as we've already seen that it is universally surjective and universally closed). To check this it suffices to check that $|X' \times_X U| \rightarrow |U|$ is injective, for all $U \rightarrow X$ étale, see More on Morphisms of Spaces, Lemma 3.6. It

suffices to check this in all cases where U is an affine scheme (minor detail omitted). Since the construction of X' commutes with étale localization, we may replace U by X . Hence it suffices to check that $|X'| \rightarrow |X|$ is injective when X is moreover an affine scheme. First observe that $|Y| \rightarrow |X'|$ is surjective, because $g' : Y \rightarrow X'$ is proper by Morphisms of Spaces, Lemma 37.6 (hence the image is closed) and $\mathcal{O}_{X'} \subset g'_* \mathcal{O}_Y$ by construction. Thus if $x_1, x_2 \in |X'|$ map to the same point in $|X|$, then we can lift x_1, x_2 to points $y_1, y_2 \in |Y|$ mapping to the same point of $|X|$. Then we can find an $r \in |R|$ with $s(r) = y_1$ and $t(r) = y_2$, see Properties of Spaces, Lemma 4.3). Since g' coequalizes s and t we conclude that $x_1 = x_2$ as desired.

To prove that X' is the coequalizer, let $W \rightarrow B$ be a separated morphism of algebraic spaces over S and let $a : Y \rightarrow W$ be a morphism over B which equalizes s and t . We will show that a factors in a unique manner through the morphism $g' : Y \rightarrow X'$. We will first reduce this to the case where $W \rightarrow B$ is separated of finite type by a limit argument (we recommend the reader skip this argument). Since Y is quasi-compact we can find a quasi-compact open subspace $W' \subset W$ such that a factors through W' . After replacing W by W' we may assume W is quasi-compact. By Limits of Spaces, Lemma 10.1 we can write $W = \lim_{i \in I} W_i$ as a cofiltered limit with affine transition morphisms with W_i of finite type over B . After shrinking I we may assume $W_i \rightarrow B$ is separated as well, see Limits of Spaces, Lemma 6.8. Since $W = \lim W_i$ we have $a = \lim a_i$ for some morphisms $a_i : Y \rightarrow W_i$. If we can prove a_i factors through g' for all i , then the same thing is true for a . This proves the reduction to the case of a finite type W .

Assume we have $a : Y \rightarrow W$ equalizing s and t with $W \rightarrow B$ separated and of finite type. Consider

$$\Gamma \subset X \times_B W$$

the scheme theoretic image of $(g, a) : Y \rightarrow X \times_B W$. Since g is proper we conclude $Y \rightarrow \Gamma$ is surjective and the projection $p : \Gamma \rightarrow X$ is proper, see Morphisms of Spaces, Lemma 37.8. Since both g and a equalize s and t , the morphism $Y \rightarrow \Gamma$ also equalizes s and t .

We claim that $p : \Gamma \rightarrow X$ is a universal homeomorphism. As in the proof of the corresponding fact for $X' \rightarrow X$, it suffices to show that p is universally injective. By More on Morphisms of Spaces, Lemma 3.6 it suffices to check $|\Gamma \times_X U| \rightarrow |U|$ is injective for every $U \rightarrow X$ étale. It suffices to check this for U affine (minor details omitted). Taking scheme theoretic image commutes with étale localization (Morphisms of Spaces, Lemma 16.3). Hence we may replace X by V and we conclude it suffices to show that $|\Gamma| \rightarrow |X|$ is injective. If $\gamma_1, \gamma_2 \in |\Gamma|$ map to the same point in $|X|$, then we can lift γ_1, γ_2 to points $y_1, y_2 \in |Y|$ mapping to the same point of $|X|$ (by surjectivity of $Y \rightarrow \Gamma$ we've seen above). Then we can find an $r \in |R|$ with $s(r) = y_1$ and $t(r) = y_2$, see Properties of Spaces, Lemma 4.3). Since $Y \rightarrow \Gamma$ coequalizes s and t we conclude that $\gamma_1 = \gamma_2$ as desired.

As a proper universal homeomorphism the morphism p is finite (see for example More on Morphisms of Spaces, Lemma 24.5). We conclude that

$$\Gamma = \underline{\text{Spec}}(p_* \mathcal{O}_\Gamma).$$

Since $Y \rightarrow \Gamma$ equalizes s and t the map $p_* \mathcal{O}_\Gamma \rightarrow g_* \mathcal{O}_Y$ factors through \mathcal{A} and we obtain a morphism $X' \rightarrow \Gamma$ by functoriality of the Spec construction. We can

compose this morphism with the projection $q : \Gamma \rightarrow W$ to get the desired morphism $X' \rightarrow W$. We omit the proof of uniqueness of the factorization. \square

We will work in the following situation.

Situation 5.2. Let S be a scheme. Let $X \rightarrow B$ be a separated finite type morphism of algebraic spaces over S with B Noetherian. Let $Z \rightarrow X$ be a closed immersion and let $U \subset X$ be the complementary open subspace. Finally, let $f : X' \rightarrow X$ be a proper morphism of algebraic spaces such that $f^{-1}(U) \rightarrow U$ is an isomorphism.

Lemma 5.3. *In Situation 5.2 let $Y = X' \amalg Z$ and $R = Y \times_X Y$ with projections $t, s : R \rightarrow Y$. There exists a coequalizer X_1 of $s, t : R \rightarrow Y$ in the category of algebraic spaces separated over B . The morphism $X_1 \rightarrow X$ is a finite universal homeomorphism, an isomorphism over U and $Z \rightarrow X$ lifts to X_1 .*

Proof. Existence of X_1 and the fact that $X_1 \rightarrow X$ is a finite universal homeomorphism is a special case of Lemma 5.1. The formation of X_1 commutes with étale localization on X (see proof of Lemma 5.1). Thus the morphisms $X_n \rightarrow X$ are isomorphisms over U . It is immediate from the construction that $Z \rightarrow X$ lifts to X_1 . \square

In Situation 5.2 for $n \geq 1$ let $Z_n \subset X$ be the n th order infinitesimal neighbourhood of Z in X , i.e., the closed subscheme defined by the n th power of the sheaf of ideals cutting out Z . Consider $Y_n = X' \amalg Z_n$ and $R_n = Y_n \times_X Y_n$ and the coequalizer

$$R_n \rightrightarrows Y_n \longrightarrow X_n \longrightarrow X$$

as in Lemma 5.3. The maps $Y_n \rightarrow Y_{n+1}$ and $R_n \rightarrow R_{n+1}$ induce morphisms

$$(5.3.1) \quad X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \dots \rightarrow X$$

Each of these morphisms is a universal homeomorphism as the morphisms $X_n \rightarrow X$ are universal homeomorphisms.

Lemma 5.4. *In (5.3.1) for all n large enough, there exists an m such that $X_n \rightarrow X_{n+m}$ factors through a closed immersion $X \rightarrow X_{n+m}$.*

Proof. Let's look a bit more closely at the construction of X_n and how it changes as we increase n . We have $X_n = \underline{\text{Spec}}(\mathcal{A}_n)$ where \mathcal{A}_n is the equalizer of $s_n^\#$ and $t_n^\#$ going from $g_{n,*}\mathcal{O}_{Y_n}$ to $h_{n,*}\mathcal{O}_{R_n}$. Here $g_n : Y_n = X' \amalg Z_n \rightarrow X$ and $h_n : R_n = Y_n \times_X Y_n \rightarrow X$ are the given morphisms. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals corresponding to Z . Then

$$g_{n,*}\mathcal{O}_{Y_n} = f_*\mathcal{O}_{X'} \times \mathcal{O}_X/\mathcal{I}^n$$

Similarly, we have a decomposition

$$R_n = X' \times_X X' \amalg X'' \times_X Z_n \amalg Z_n \times_X Z_n$$

Denote $f_n : X' \times_X Z_n \rightarrow X$ the restriction of f and denote

$$\mathcal{A} = \text{Equalizer}(f_*\mathcal{O}_{X'} \rightrightarrows (f \times f)_*\mathcal{O}_{X' \times_X X'})$$

Then we see that

$$\mathcal{A}_n = \text{Equalizer}(\mathcal{A} \times \mathcal{O}_X/\mathcal{I}^n \rightrightarrows f_{n,*}\mathcal{O}_{X' \times_X Z_n})$$

We have canonical maps

$$\mathcal{O}_X \rightarrow \dots \rightarrow \mathcal{A}_3 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_1$$

of coherent \mathcal{O}_X -algebras. The statement of the lemma means that for n large enough there exists an $m \geq 0$ such that the image of $\mathcal{A}_{n+m} \rightarrow \mathcal{A}_n$ is isomorphic to \mathcal{O}_X .

Since $X_n \rightarrow X$ is an isomorphism over U we see that the kernel of $\mathcal{O}_X \rightarrow \mathcal{A}_n$ is supported on $|Z|$. Since X is Noetherian, the sequence of kernels $\mathcal{J}_n = \text{Ker}(\mathcal{O}_X \rightarrow \mathcal{A}_n)$ stabilizes (Cohomology of Spaces, Lemma 12.1). Say $\mathcal{J}_{n_0} = \mathcal{J}_{n_0+1} = \dots = \mathcal{J}$. By Cohomology of Spaces, Lemma 12.2 we find that $\mathcal{I}^t \mathcal{J} = 0$ for some $t \geq 0$. On the other hand, there is an \mathcal{O}_X -algebra map $\mathcal{A}_n \rightarrow \mathcal{O}_X/\mathcal{I}^n$ and hence $\mathcal{J} \subset \mathcal{I}^n$ for all n . By Artin-Rees (Cohomology of Spaces, Lemma 12.3) we find that $\mathcal{J} \cap \mathcal{I}^n \subset \mathcal{I}^{n-c} \mathcal{J}$ for some $c \geq 0$ and all $n \gg 0$. We conclude that $\mathcal{J} = 0$.

Pick $n \geq n_0$ as in the previous paragraph. Then $\mathcal{O}_X \rightarrow \mathcal{A}_n$ is injective. Hence it now suffices to find $m \geq 0$ such that the image of $\mathcal{A}_{n+m} \rightarrow \mathcal{A}_n$ is equal to the image of \mathcal{O}_X . Observe that \mathcal{A}_n sits in a short exact sequence

$$0 \rightarrow \text{Ker}(\mathcal{A} \rightarrow f_{n,*} \mathcal{O}_{X' \times_X Z_n}) \rightarrow \mathcal{A}_n \rightarrow \mathcal{O}_X/\mathcal{I}^n \rightarrow 0$$

and similarly for \mathcal{A}_{n+m} . Hence it suffices to show

$$\text{Ker}(\mathcal{A} \rightarrow f_{n+m,*} \mathcal{O}_{X' \times_X Z_{n+m}}) \subset \text{Im}(\mathcal{I}^n \rightarrow \mathcal{A})$$

for some $m \geq 0$. To do this we may work étale locally on X and since X is Noetherian we may assume that X is a Noetherian affine scheme. Say $X = \text{Spec}(R)$ and \mathcal{I} corresponds to the ideal $I \subset R$. Let $\mathcal{A} = \tilde{A}$ for a finite R -algebra A . Let $f_* \mathcal{O}_{X'} = \tilde{B}$ for a finite R -algebra B . Then $R \rightarrow A \subset B$ and these maps become isomorphisms on inverting any element of I .

Note that $f_{n,*} \mathcal{O}_{X' \times_X Z_n}$ is equal to $f_*(\mathcal{O}_{X'}/\mathcal{I}^n \mathcal{O}_{X'})$ in the notation used in Cohomology of Spaces, Section 20. By Cohomology of Spaces, Lemma 20.4 we see that there exists a $c \geq 0$ such that

$$\text{Ker}(B \rightarrow \Gamma(X, f_*(\mathcal{O}_{X'}/\mathcal{I}^{n+m+c} \mathcal{O}_{X'})))$$

is contained in $\mathcal{I}^{n+m} B$. On the other hand, as $R \rightarrow B$ is finite and an isomorphism after inverting any element of I we see that $\mathcal{I}^{n+m} B \subset \text{Im}(\mathcal{I}^n \rightarrow B)$ for m large enough (can be chosen independent of n). This finishes the proof as $A \subset B$. \square

Remark 5.5. The meaning of Lemma 5.4 is the the system $X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \dots$ is essentially constant with value X . See Categories, Definition 22.1.

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