

# **GROUPOID SCHEMES**

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## **1. Introduction**

This chapter is devoted to generalities concerning groupoid schemes. See for example the beautiful paper [KM97] by Keel and Mori.

## **2. Notation**

Let  $S$  be a scheme. If  $U, T$  are schemes over  $S$  we denote  $U(T)$  for the set of  $T$ -valued points of  $U$  over  $S$ . In a formula:  $U(T) = \text{Mor}_S(T, U)$ . We try to reserve the letter  $T$  to denote a “test scheme” over  $S$ , as in the discussion that follows. Suppose we are given schemes  $X, Y$  over  $S$  and a morphism of schemes  $f : X \rightarrow Y$  over  $S$ . For any scheme  $T$  over  $S$  we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

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which as indicated we denote by  $f$  also. In fact this construction is functorial in the scheme  $T/S$ . Yoneda's Lemma, see Categories, Lemma 3.5, says that  $f$  determines and is determined by this transformation of functors  $f : h_X \rightarrow h_Y$ . More generally, we use the same notation for maps between fibre products. For example, if  $X, Y, Z$  are schemes over  $S$ , and if  $m : X \times_S Y \rightarrow Z \times_S Z$  is a morphism of schemes over  $S$ , then we think of  $m$  as corresponding to a collection of maps between  $T$ -valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

We continue our convention to label projection maps starting with index 0, so we have  $\text{pr}_0 : X \times_S Y \rightarrow X$  and  $\text{pr}_1 : X \times_S Y \rightarrow Y$ .

### 3. Equivalence relations

Recall that a *relation*  $R$  on a set  $A$  is just a subset of  $R \subset A \times A$ . We usually write  $aRb$  to indicate  $(a, b) \in R$ . We say the relation is *transitive* if  $aRb, bRc \Rightarrow aRc$ . We say the relation is *reflexive* if  $aRa$  for all  $a \in A$ . We say the relation is *symmetric* if  $aRb \Rightarrow bRa$ . A relation is called an *equivalence relation* if it is transitive, reflexive and symmetric.

In the setting of schemes we are going to relax the notion of a relation a little bit and just require  $R \rightarrow A \times A$  to be a map. Here is the definition.

**Definition 3.1.** Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ .

- (1) A *pre-relation* on  $U$  over  $S$  is any morphism  $j : R \rightarrow U \times_S U$ . In this case we set  $t = \text{pr}_0 \circ j$  and  $s = \text{pr}_1 \circ j$ , so that  $j = (t, s)$ .
- (2) A *relation* on  $U$  over  $S$  is a monomorphism  $j : R \rightarrow U \times_S U$ .
- (3) A *pre-equivalence relation* is a pre-relation  $j : R \rightarrow U \times_S U$  such that the image of  $j : R(T) \rightarrow U(T) \times U(T)$  is an equivalence relation for all  $T/S$ .
- (4) We say a morphism  $R \rightarrow U \times_S U$  is an *equivalence relation on  $U$  over  $S$*  if and only if for every  $T/S$  the  $T$ -valued points of  $R$  define an equivalence relation on the set of  $T$ -valued points of  $U$ .

In other words, an equivalence relation is a pre-equivalence relation such that  $j$  is a relation.

**Lemma 3.2.** Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ . Let  $j : R \rightarrow U \times_S U$  be a pre-relation. Let  $g : U' \rightarrow U$  be a morphism of schemes. Finally, set

$$R' = (U' \times_S U') \times_{U \times_S U} R \xrightarrow{j'} U' \times_S U'$$

Then  $j'$  is a pre-relation on  $U'$  over  $S$ . If  $j$  is a relation, then  $j'$  is a relation. If  $j$  is a pre-equivalence relation, then  $j'$  is a pre-equivalence relation. If  $j$  is an equivalence relation, then  $j'$  is an equivalence relation.

**Proof.** Omitted. □

**Definition 3.3.** Let  $S$  be a scheme. Let  $U$  be a scheme over  $S$ . Let  $j : R \rightarrow U \times_S U$  be a pre-relation. Let  $g : U' \rightarrow U$  be a morphism of schemes. The pre-relation  $j' : R' \rightarrow U' \times_S U'$  is called the *restriction*, or *pullback* of the pre-relation  $j$  to  $U'$ . In this situation we sometimes write  $R' = R|_{U'}$ .

**Lemma 3.4.** *Let  $j : R \rightarrow U \times_S U$  be a pre-relation. Consider the relation on points of the scheme  $U$  defined by the rule*

$$x \sim y \Leftrightarrow \exists r \in R : t(r) = x, s(r) = y.$$

*If  $j$  is a pre-equivalence relation then this is an equivalence relation.*

**Proof.** Suppose that  $x \sim y$  and  $y \sim z$ . Pick  $r \in R$  with  $t(r) = x$ ,  $s(r) = y$  and pick  $r' \in R$  with  $t(r') = y$ ,  $s(r') = z$ . Pick a field  $K$  fitting into the following commutative diagram

$$\begin{array}{ccc} \kappa(r) & \longrightarrow & K \\ \uparrow & & \uparrow \\ \kappa(y) & \longrightarrow & \kappa(r') \end{array}$$

Denote  $x_K, y_K, z_K : \text{Spec}(K) \rightarrow U$  the morphisms

$$\begin{aligned} \text{Spec}(K) &\rightarrow \text{Spec}(\kappa(r)) \rightarrow \text{Spec}(\kappa(x)) \rightarrow U \\ \text{Spec}(K) &\rightarrow \text{Spec}(\kappa(r)) \rightarrow \text{Spec}(\kappa(y)) \rightarrow U \\ \text{Spec}(K) &\rightarrow \text{Spec}(\kappa(r')) \rightarrow \text{Spec}(\kappa(z)) \rightarrow U \end{aligned}$$

By construction  $(x_K, y_K) \in j(R(K))$  and  $(y_K, z_K) \in j(R(K))$ . Since  $j$  is a pre-equivalence relation we see that also  $(x_K, z_K) \in j(R(K))$ . This clearly implies that  $x \sim z$ .

The proof that  $\sim$  is reflexive and symmetric is omitted.  $\square$

#### 4. Group schemes

Let us recall that a *group* is a pair  $(G, m)$  where  $G$  is a set, and  $m : G \times G \rightarrow G$  is a map of sets with the following properties:

- (1) (associativity)  $m(g, m(g', g'')) = m(m(g, g'), g'')$  for all  $g, g', g'' \in G$ ,
- (2) (identity) there exists a unique element  $e \in G$  (called the *identity*, *unit*, or *1* of  $G$ ) such that  $m(g, e) = m(e, g) = g$  for all  $g \in G$ , and
- (3) (inverse) for all  $g \in G$  there exists a  $i(g) \in G$  such that  $m(g, i(g)) = m(i(g), g) = e$ , where  $e$  is the identity.

Thus we obtain a map  $e : \{*\} \rightarrow G$  and a map  $i : G \rightarrow G$  so that the quadruple  $(G, m, e, i)$  satisfies the axioms listed above.

A *homomorphism of groups*  $\psi : (G, m) \rightarrow (G', m')$  is a map of sets  $\psi : G \rightarrow G'$  such that  $m'(\psi(g), \psi(g')) = \psi(m(g, g'))$ . This automatically insures that  $\psi(e) = e'$  and  $i'(\psi(g)) = \psi(i(g))$ . (Obvious notation.) We will use this below.

**Definition 4.1.** Let  $S$  be a scheme.

- (1) A *group scheme over  $S$*  is a pair  $(G, m)$ , where  $G$  is a scheme over  $S$  and  $m : G \times_S G \rightarrow G$  is a morphism of schemes over  $S$  with the following property: For every scheme  $T$  over  $S$  the pair  $(G(T), m)$  is a group.
- (2) A *morphism  $\psi : (G, m) \rightarrow (G', m')$  of group schemes over  $S$*  is a morphism  $\psi : G \rightarrow G'$  of schemes over  $S$  such that for every  $T/S$  the induced map  $\psi : G(T) \rightarrow G'(T)$  is a homomorphism of groups.

Let  $(G, m)$  be a group scheme over the scheme  $S$ . By the discussion above (and the discussion in Section 2) we obtain morphisms of schemes over  $S$ : (identity)  $e :$

$S \rightarrow G$  and (inverse)  $i : G \rightarrow G$  such that for every  $T$  the quadruple  $(G(T), m, e, i)$  satisfies the axioms of a group listed above.

Let  $(G, m)$ ,  $(G', m')$  be group schemes over  $S$ . Let  $f : G \rightarrow G'$  be a morphism of schemes over  $S$ . It follows from the definition that  $f$  is a morphism of group schemes over  $S$  if and only if the following diagram is commutative:

$$\begin{array}{ccc} G \times_S G & \xrightarrow{f \times f} & G' \times_S G' \\ m \downarrow & & \downarrow m \\ G & \xrightarrow{f} & G' \end{array}$$

**Lemma 4.2.** *Let  $(G, m)$  be a group scheme over  $S$ . Let  $S' \rightarrow S$  be a morphism of schemes. The pullback  $(G_{S'}, m_{S'})$  is a group scheme over  $S'$ .*

**Proof.** Omitted. □

**Definition 4.3.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ .

- (1) A *closed subgroup scheme* of  $G$  is a closed subscheme  $H \subset G$  such that  $m|_{H \times_S H}$  factors through  $H$  and induces a group scheme structure on  $H$  over  $S$ .
- (2) An *open subgroup scheme* of  $G$  is an open subscheme  $G' \subset G$  such that  $m|_{G' \times_S G'}$  factors through  $G'$  and induces a group scheme structure on  $G'$  over  $S$ .

Alternatively, we could say that  $H$  is a closed subgroup scheme of  $G$  if it is a group scheme over  $S$  endowed with a morphism of group schemes  $i : H \rightarrow G$  over  $S$  which identifies  $H$  with a closed subscheme of  $G$ .

**Definition 4.4.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ .

- (1) We say  $G$  is a *smooth group scheme* if the structure morphism  $G \rightarrow S$  is smooth.
- (2) We say  $G$  is a *flat group scheme* if the structure morphism  $G \rightarrow S$  is flat.
- (3) We say  $G$  is a *separated group scheme* if the structure morphism  $G \rightarrow S$  is separated.

Add more as needed.

## 5. Examples of group schemes

**Example 5.1** (Multiplicative group scheme). Consider the functor which associates to any scheme  $T$  the group  $\Gamma(T, \mathcal{O}_T^*)$  of units in the global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_m = \text{Spec}(\mathbf{Z}[x, x^{-1}])$$

The morphism giving the group structure is the morphism

$$\begin{array}{rcl} \mathbf{G}_m \times \mathbf{G}_m & \rightarrow & \mathbf{G}_m \\ \text{Spec}(\mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}]) & \rightarrow & \text{Spec}(\mathbf{Z}[x, x^{-1}]) \\ \mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}] & \leftarrow & \mathbf{Z}[x, x^{-1}] \\ x \otimes x & \leftarrow & x \end{array}$$

Hence we see that  $\mathbf{G}_m$  is a group scheme over  $\mathbf{Z}$ . For any scheme  $S$  the base change  $\mathbf{G}_{m,S}$  is a group scheme over  $S$  whose functor of points is

$$T/S \mapsto \mathbf{G}_{m,S}(T) = \mathbf{G}_m(T) = \Gamma(T, \mathcal{O}_T^*)$$

as before.

**Example 5.2** (Roots of unity). Let  $n \in \mathbf{N}$ . Consider the functor which associates to any scheme  $T$  the subgroup of  $\Gamma(T, \mathcal{O}_T^*)$  consisting of  $n$ th roots of unity. This is representable by the scheme

$$\mu_n = \operatorname{Spec}(\mathbf{Z}[x]/(x^n - 1)).$$

The morphism giving the group structure is the morphism

$$\begin{array}{ccc} \mu_n \times \mu_n & \rightarrow & \mu_n \\ \operatorname{Spec}(\mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1)) & \rightarrow & \operatorname{Spec}(\mathbf{Z}[x]/(x^n - 1)) \\ \mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1) & \leftarrow & \mathbf{Z}[x]/(x^n - 1) \\ x \otimes x & \leftarrow & x \end{array}$$

Hence we see that  $\mu_n$  is a group scheme over  $\mathbf{Z}$ . For any scheme  $S$  the base change  $\mu_{n,S}$  is a group scheme over  $S$  whose functor of points is

$$T/S \mapsto \mu_{n,S}(T) = \mu_n(T) = \{f \in \Gamma(T, \mathcal{O}_T^*) \mid f^n = 1\}$$

as before.

**Example 5.3** (Additive group scheme). Consider the functor which associates to any scheme  $T$  the group  $\Gamma(T, \mathcal{O}_T)$  of global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_a = \operatorname{Spec}(\mathbf{Z}[x])$$

The morphism giving the group structure is the morphism

$$\begin{array}{ccc} \mathbf{G}_a \times \mathbf{G}_a & \rightarrow & \mathbf{G}_a \\ \operatorname{Spec}(\mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x]) & \rightarrow & \operatorname{Spec}(\mathbf{Z}[x]) \\ \mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x] & \leftarrow & \mathbf{Z}[x] \\ x \otimes 1 + 1 \otimes x & \leftarrow & x \end{array}$$

Hence we see that  $\mathbf{G}_a$  is a group scheme over  $\mathbf{Z}$ . For any scheme  $S$  the base change  $\mathbf{G}_{a,S}$  is a group scheme over  $S$  whose functor of points is

$$T/S \mapsto \mathbf{G}_{a,S}(T) = \mathbf{G}_a(T) = \Gamma(T, \mathcal{O}_T)$$

as before.

**Example 5.4** (General linear group scheme). Let  $n \geq 1$ . Consider the functor which associates to any scheme  $T$  the group

$$\operatorname{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible  $n \times n$  matrices over the global sections of the structure sheaf. This is representable by the scheme

$$\operatorname{GL}_n = \operatorname{Spec}(\mathbf{Z}[\{x_{ij}\}_{1 \leq i,j \leq n}][1/d])$$

where  $d = \det((x_{ij}))$  with  $(x_{ij})$  the  $n \times n$  matrix with entry  $x_{ij}$  in the  $(i, j)$ -spot. The morphism giving the group structure is the morphism

$$\begin{aligned} \mathrm{GL}_n \times \mathrm{GL}_n &\rightarrow \mathrm{GL}_n \\ \mathrm{Spec}(\mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d]) &\rightarrow \mathrm{Spec}(\mathbf{Z}[x_{ij}, 1/d]) \\ \mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d] &\leftarrow \mathbf{Z}[x_{ij}, 1/d] \\ \sum x_{ik} \otimes x_{kj} &\leftarrow x_{ij} \end{aligned}$$

Hence we see that  $\mathrm{GL}_n$  is a group scheme over  $\mathbf{Z}$ . For any scheme  $S$  the base change  $\mathrm{GL}_{n,S}$  is a group scheme over  $S$  whose functor of points is

$$T/S \mapsto \mathrm{GL}_{n,S}(T) = \mathrm{GL}_n(T) = \mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

as before.

**Example 5.5.** The determinant defines a morphisms of group schemes

$$\det : \mathrm{GL}_n \longrightarrow \mathbf{G}_m$$

over  $\mathbf{Z}$ . By base change it gives a morphism of group schemes  $\mathrm{GL}_{n,S} \rightarrow \mathbf{G}_{m,S}$  over any base scheme  $S$ .

**Example 5.6** (Constant group). Let  $G$  be an abstract group. Consider the functor which associates to any scheme  $T$  the group of locally constant maps  $T \rightarrow G$  (where  $T$  has the Zariski topology and  $G$  the discrete topology). This is representable by the scheme

$$G_{\mathrm{Spec}(\mathbf{Z})} = \coprod_{g \in G} \mathrm{Spec}(\mathbf{Z}).$$

The morphism giving the group structure is the morphism

$$G_{\mathrm{Spec}(\mathbf{Z})} \times_{\mathrm{Spec}(\mathbf{Z})} G_{\mathrm{Spec}(\mathbf{Z})} \longrightarrow G_{\mathrm{Spec}(\mathbf{Z})}$$

which maps the component corresponding to the pair  $(g, g')$  to the component corresponding to  $gg'$ . For any scheme  $S$  the base change  $G_S$  is a group scheme over  $S$  whose functor of points is

$$T/S \mapsto G_S(T) = \{f : T \rightarrow G \text{ locally constant}\}$$

as before.

## 6. Properties of group schemes

In this section we collect some simple properties of group schemes which hold over any base.

**Lemma 6.1.** *Let  $S$  be a scheme. Let  $G$  be a group scheme over  $S$ . Then  $G \rightarrow S$  is separated (resp. quasi-separated) if and only if the identity morphism  $e : S \rightarrow G$  is a closed immersion (resp. quasi-compact).*

**Proof.** We recall that by Schemes, Lemma 21.12 we have that  $e$  is an immersion which is a closed immersion (resp. quasi-compact) if  $G \rightarrow S$  is separated (resp. quasi-separated). For the converse, consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\Delta_{G/S}} & G \times_S G \\ \downarrow & & \downarrow (g, g') \mapsto m(i(g), g') \\ S & \xrightarrow{e} & G \end{array}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that  $\Delta_{G/S}$  is a base change of  $e$ . Hence if  $e$  is a closed immersion (resp. quasi-compact) so is  $\Delta_{G/S}$ , see Schemes, Lemma 18.2 (resp. Schemes, Lemma 19.3).  $\square$

**Lemma 6.2.** *Let  $S$  be a scheme. Let  $G$  be a group scheme over  $S$ . Let  $T$  be a scheme over  $S$  and let  $\psi : T \rightarrow G$  be a morphism over  $S$ . If  $T$  is flat over  $S$ , then the morphism*

$$T \times_S G \longrightarrow G, \quad (t, g) \longmapsto m(\psi(t), g)$$

*is flat. In particular, if  $G$  is flat over  $S$ , then  $m : G \times_S G \rightarrow G$  is flat.*

**Proof.** Consider the diagram

$$\begin{array}{ccccc} T \times_S G & \xrightarrow{(t,g) \mapsto (t, m(\psi(t), g))} & T \times_S G & \xrightarrow{\text{pr}} & G \\ & & \downarrow & & \downarrow \\ & & T & \longrightarrow & S \end{array}$$

The left top horizontal arrow is an isomorphism and the square is cartesian. Hence the lemma follows from Morphisms, Lemma 26.7.  $\square$

**Lemma 6.3.** *Let  $(G, m, e, i)$  be a group scheme over the scheme  $S$ . Denote  $f : G \rightarrow S$  the structure morphism. Assume  $f$  is flat. Then there exist canonical isomorphisms*

$$\Omega_{G/S} \cong f^* \mathcal{C}_{S/G} \cong f^* e^* \Omega_{G/S}$$

*where  $\mathcal{C}_{S/G}$  denotes the conormal sheaf of the immersion  $e$ . In particular, if  $S$  is the spectrum of a field, then  $\Omega_{G/S}$  is a free  $\mathcal{O}_G$ -module.*

**Proof.** In Morphisms, Lemma 34.5 we identified  $\Omega_{G/S}$  with the conormal sheaf of the diagonal morphism  $\Delta_{G/S}$ . In the proof of Lemma 6.1 we showed that  $\Delta_{G/S}$  is a base change of the immersion  $e$  by the morphism  $(g, g') \mapsto m(i(g), g')$ . This morphism is isomorphic to the morphism  $(g, g') \mapsto m(g, g')$  hence is flat by Lemma 6.2. Hence we get the first isomorphism by Morphisms, Lemma 33.4. By Morphisms, Lemma 34.16 we have  $\mathcal{C}_{S/G} \cong e^* \Omega_{G/S}$ .

If  $S$  is the spectrum of a field, then  $G \rightarrow S$  is flat, and any  $\mathcal{O}_S$ -module on  $S$  is free.  $\square$

## 7. Properties of group schemes over a field

In this section we collect some simple properties of group schemes over a field.

**Lemma 7.1.** *If  $(G, m)$  is a group scheme over a field  $k$ , then the multiplication map  $m : G \times_k G \rightarrow G$  is open.*

**Proof.** The multiplication map is isomorphic to the projection map  $\text{pr}_0 : G \times_k G \rightarrow G$  because the diagram

$$\begin{array}{ccc} G \times_k G & \xrightarrow{(g,g') \mapsto (m(g,g'), g')} & G \times_k G \\ \downarrow m & & \downarrow (g,g') \mapsto g \\ G & \xrightarrow{\text{id}} & G \end{array}$$

is commutative with isomorphisms as horizontal arrows. The projection is open by Morphisms, Lemma 24.4.  $\square$

**Lemma 7.2.** *Let  $G$  be a group scheme over a field. Then  $G$  is a separated scheme.*

**Proof.** Say  $S = \operatorname{Spec}(k)$  with  $k$  a field, and let  $G$  be a group scheme over  $S$ . By Lemma 6.1 we have to show that  $e : S \rightarrow G$  is a closed immersion. By Morphisms, Lemma 21.2 the image of  $e : S \rightarrow G$  is a closed point of  $G$ . It is clear that  $\mathcal{O}_G \rightarrow e_*\mathcal{O}_S$  is surjective, since  $e_*\mathcal{O}_S$  is a skyscraper sheaf supported at the neutral element of  $G$  with value  $k$ . We conclude that  $e$  is a closed immersion by Schemes, Lemma 24.2.  $\square$

**Lemma 7.3.** *Let  $G$  be a group scheme over a field  $k$ . Then*

- (1) *every local ring  $\mathcal{O}_{G,g}$  of  $G$  has a unique minimal prime ideal,*
- (2) *there is exactly one irreducible component  $Z$  of  $G$  passing through  $e$ , and*
- (3)  *$Z$  is geometrically irreducible over  $k$ .*

**Proof.** For any point  $g \in G$  there exists a field extension  $k \subset K$  and a  $K$ -valued point  $g' \in G(K)$  mapping to  $g$ . If we think of  $g'$  as a  $K$ -rational point of the group scheme  $G_K$ , then we see that  $\mathcal{O}_{G,g} \rightarrow \mathcal{O}_{G_K,g'}$  is a faithfully flat local ring map (as  $G_K \rightarrow G$  is flat, and a local flat ring map is faithfully flat, see Algebra, Lemma 38.16). The result for  $\mathcal{O}_{G_K,g'}$  implies the result for  $\mathcal{O}_{G,g}$ , see Algebra, Lemma 29.5. Hence in order to prove (1) it suffices to prove it for  $k$ -rational points  $g$  of  $G$ . In this case translation by  $g$  defines an automorphism  $G \rightarrow G$  which maps  $e$  to  $g$ . Hence  $\mathcal{O}_{G,g} \cong \mathcal{O}_{G,e}$ . In this way we see that (2) implies (1), since irreducible components passing through  $e$  correspond one to one with minimal prime ideals of  $\mathcal{O}_{G,e}$ .

In order to prove (2) and (3) it suffices to prove (2) when  $k$  is algebraically closed. In this case, let  $Z_1, Z_2$  be two irreducible components of  $G$  passing through  $e$ . Since  $k$  is algebraically closed the closed subscheme  $Z_1 \times_k Z_2 \subset G \times_k G$  is irreducible too, see Varieties, Lemma 6.4. Hence  $m(Z_1 \times_k Z_2)$  is contained in an irreducible component of  $G$ . On the other hand it contains  $Z_1$  and  $Z_2$  since  $m|_{e \times G} = \operatorname{id}_G$  and  $m|_{G \times e} = \operatorname{id}_G$ . We conclude  $Z_1 = Z_2$  as desired.  $\square$

**Remark 7.4.** Warning: The result of Lemma 7.3 does not mean that every irreducible component of  $G/k$  is geometrically irreducible. For example the group scheme  $\mu_{3,\mathbf{Q}} = \operatorname{Spec}(\mathbf{Q}[x]/(x^3 - 1))$  over  $\mathbf{Q}$  has two irreducible components corresponding to the factorization  $x^3 - 1 = (x - 1)(x^2 + x + 1)$ . The first factor corresponds to the irreducible component passing through the identity, and the second irreducible component is not geometrically irreducible over  $\operatorname{Spec}(\mathbf{Q})$ .

**Lemma 7.5.** *Let  $G$  be a group scheme which is locally of finite type over a field  $k$ . Then  $G$  is equidimensional and  $\dim(G) = \dim_g(G)$  for all  $g \in G$ . For any closed point  $g \in G$  we have  $\dim(G) = \dim(\mathcal{O}_{G,g})$ .*

**Proof.** Let us first prove that  $\dim_g(G) = \dim_{g'}(G)$  for any pair of points  $g, g' \in G$ . By Morphisms, Lemma 29.3 we may extend the ground field at will. Hence we may assume that both  $g$  and  $g'$  are defined over  $k$ . Hence there exists an automorphism of  $G$  mapping  $g$  to  $g'$ , whence the equality. By Morphisms, Lemma 29.1 we have  $\dim_g(G) = \dim(\mathcal{O}_{G,g}) + \operatorname{trdeg}_k(\kappa(g))$ . On the other hand, the dimension of  $G$  (or any open subset of  $G$ ) is the supremum of the dimensions of the local rings of  $G$ , see Properties, Lemma 11.4. Clearly this is maximal for closed points  $g$  in which case  $\operatorname{trdeg}_k(\kappa(g)) = 0$  (by the Hilbert Nullstellensatz, see Morphisms, Section 17). Hence the lemma follows.  $\square$

The following result is sometimes referred to as Cartier's theorem.



**Lemma 7.6.** *Let  $G$  be a group scheme which is locally of finite type over a field  $k$  of characteristic zero. Then the structure morphism  $G \rightarrow \operatorname{Spec}(k)$  is smooth, i.e.,  $G$  is a smooth group scheme.*

**Proof.** By Lemma 6.3 the module of differentials of  $G$  over  $k$  is free. Hence smoothness follows from Varieties, Lemma 15.1.  $\square$

**Remark 7.7.** Any group scheme over a field of characteristic 0 is reduced, see [Per75, I, Theorem 1.1 and I, Corollary 3.9, and II, Theorem 2.4] and also [Per76, Proposition 4.2.8]. This was a question raised in [Oor66, page 80]. We have seen in Lemma 7.6 that this holds when the group scheme is locally of finite type.

**Lemma 7.8.** *Let  $G$  be a group scheme which is locally of finite type over a perfect field  $k$  of characteristic  $p > 0$  (see Lemma 7.6 for the characteristic zero case). If  $G$  is reduced then the structure morphism  $G \rightarrow \operatorname{Spec}(k)$  is smooth, i.e.,  $G$  is a smooth group scheme.*

**Proof.** By Lemma 6.3 the sheaf  $\Omega_{G/k}$  is free. Hence the lemma follows from Varieties, Lemma 15.2.  $\square$

**Remark 7.9.** Let  $k$  be a field of characteristic  $p > 0$ . Let  $\alpha \in k$  be an element which is not a  $p$ th power. The closed subgroup scheme

$$G = V(x^p + \alpha y^p) \subset \mathbf{G}_{a,k}^2$$

is reduced and irreducible but not smooth (not even normal).

**Lemma 7.10.** *Let  $G$  be a group scheme over a perfect field  $k$ . Then the reduction  $G_{\text{red}}$  of  $G$  is a closed subgroup scheme of  $G$ .*

**Proof.** Omitted. Hint: Use that  $G_{\text{red}} \times_k G_{\text{red}}$  is reduced by Varieties, Lemmas 4.3 and 4.7.  $\square$

The next lemma will be generalized slightly in More on Groupoids, Lemma 10.2. Namely, if  $G' \rightarrow G$  is a morphism of group schemes over a field whose image is open, then its image is closed.

**Lemma 7.11.** *Let  $G$  be group scheme over a field  $k$ . Let  $G' \subset G$  be an open subgroup scheme. Then  $G'$  is open and closed in  $G$ .*

**Proof.** Suppose that  $k \subset K$  is a field extension such that  $G'_K \subset G_K$  is closed. Then it follows from Morphisms, Lemma 26.10 that  $G'$  is closed (as  $G_K \rightarrow G$  is flat, quasi-compact and surjective). Hence it suffices to prove the lemma after replacing  $k$  by some extension. Choose  $K$  to be an algebraically closed field extension of very large cardinality. Then by Varieties, Lemma 12.2, we see that  $G_K$  is a Jacobson scheme all of whose closed points have residue field equal to  $K$ . In other words we may assume  $G$  is a Jacobson scheme all of whose closed points have residue field  $k$ .

Let  $Z = G \setminus G'$ . We have to show that  $Z$  is open. Because  $G$  is Jacobson and  $Z$  is closed the closed points of  $Z$  are dense in  $Z$ . Moreover any closed point  $z \in Z$  is a  $k$ -rational point and hence we translation by  $z$  defines an automorphism  $L_z : G \rightarrow G$ ,  $g \mapsto m(z, g)$  with  $e \mapsto z$ . As  $G'$  is a subgroup scheme we conclude that  $L_z(G') \subset Z$ . Altogether we see that

$$Z = \bigcup_{z \in Z(k)} L_z(G')$$

is a union of open subsets, and hence open as desired.  $\square$

**Lemma 7.12.** *Let  $i : G' \rightarrow G$  be an immersion of group schemes over a field  $k$ . Then  $i$  is a closed immersion, i.e.,  $i(G')$  is a closed subgroup scheme of  $G$ .*

**Proof.** To show that  $i$  is a closed immersion it suffices to show that  $i(G')$  is a closed subset of  $G$ . Let  $k \subset k'$  be a perfect extension of  $k$ . If  $i(G'_{k'}) \subset G_{k'}$  is closed, then  $i(G') \subset G$  is closed by Morphisms, Lemma 26.10 (as  $G_{k'} \rightarrow G$  is flat, quasi-compact and surjective). Hence we may and do assume  $k$  is perfect. We will use without further mention that products of reduced schemes over  $k$  are reduced. We may replace  $G'$  and  $G$  by their reductions, see Lemma 7.10. Let  $\overline{G'} \subset G$  be the closure of  $i(G')$  viewed as a reduced closed subscheme. By Varieties, Lemma 14.1 we conclude that  $\overline{G'} \times_k \overline{G'}$  is the closure of the image of  $G' \times_k G' \rightarrow G \times_k G$ . Hence

$$m(\overline{G'} \times_k \overline{G'}) \subset \overline{G'}$$

as  $m$  is continuous. It follows that  $\overline{G'} \subset G$  is a (reduced) closed subgroup scheme. By Lemma 7.11 we see that  $i(G') \subset \overline{G'}$  is also closed which implies that  $i(G') = \overline{G'}$  as desired.  $\square$

**Lemma 7.13.** *Let  $G$  be a group scheme over a field. There exists an open and closed subscheme  $G' \subset G$  which is a countable union of affines.*

**Proof.** Let  $e \in U(k)$  be a quasi-compact open neighbourhood of the identity element. By replacing  $U$  by  $U \cap i(U)$  we may assume that  $U$  is invariant under the inverse map. As  $G$  is separated this is still a quasi-compact set. Set

$$G' = \bigcup_{n \geq 1} m_n(U \times_k \dots \times_k U)$$

where  $m_n : G \times_k \dots \times_k G \rightarrow G$  is the  $n$ -slot multiplication map  $(g_1, \dots, g_n) \mapsto m(\dots(m(g_1, g_2), g_3), \dots), g_n)$ . Each of these maps are open (see Lemma 7.1) hence  $G'$  is an open subgroup scheme. By Lemma 7.11 it is also a closed subgroup scheme.  $\square$

**Remark 7.14.** If  $G$  is a group scheme over a field, is there always a quasi-compact open and closed subgroup scheme? Or is there a counter example?

## 8. Actions of group schemes

Let  $(G, m)$  be a group and let  $V$  be a set. Recall that a (left) action of  $G$  on  $V$  is given by a map  $a : G \times V \rightarrow V$  such that

- (1) (associativity)  $a(m(g, g'), v) = a(g, a(g', v))$  for all  $g, g' \in G$  and  $v \in V$ , and
- (2) (identity)  $a(e, v) = v$  for all  $v \in V$ .

We also say that  $V$  is a  $G$ -set (this usually means we drop the  $a$  from the notation – which is abuse of notation). A map of  $G$ -sets  $\psi : V \rightarrow V'$  is any set map such that  $\psi(a(g, v)) = a(g, \psi(v))$  for all  $v \in V$ .

**Definition 8.1.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ .

- (1) An action of  $G$  on the scheme  $X/S$  is a morphism  $a : G \times_S X \rightarrow X$  over  $S$  such that for every  $T/S$  the map  $a : G(T) \times X(T) \rightarrow X(T)$  defines the structure of a  $G(T)$ -set on  $X(T)$ .
- (2) Suppose that  $X, Y$  are schemes over  $S$  each endowed with an action of  $G$ . An equivariant or more precisely a  $G$ -equivariant morphism  $\psi : X \rightarrow Y$  is a morphism of schemes over  $S$  such that for every  $T/S$  the map  $\psi : X(T) \rightarrow Y(T)$  is a morphism of  $G(T)$ -sets.

In situation (1) this means that the diagrams

$$(8.1.1) \quad \begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{1_G \times a} & G \times_S X \\ m \times 1_X \downarrow & & \downarrow a \\ G \times_S X & \xrightarrow{a} & X \end{array} \quad \begin{array}{ccc} G \times_S X & \xrightarrow{a} & X \\ e \times 1_X \uparrow & \nearrow 1_X & \\ X & & \end{array}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\text{id} \times f} & G \times_S Y \\ a \downarrow & & \downarrow a \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

**Definition 8.2.** Let  $S, G \rightarrow S$ , and  $X \rightarrow S$  as in Definition 8.1. Let  $a : G \times_S X \rightarrow X$  be an action of  $G$  on  $X/S$ . We say the action is *free* if for every scheme  $T$  over  $S$  the action  $a : G(T) \times X(T) \rightarrow X(T)$  is a free action of the group  $G(T)$  on the set  $X(T)$ .

**Lemma 8.3.** *Situation as in Definition 8.2, The action  $a$  is free if and only if*

$$G \times_S X \rightarrow X \times_S X, \quad (g, x) \mapsto (a(g, x), x)$$

*is a monomorphism.*

**Proof.** Immediate from the definitions.  $\square$

## 9. Principal homogeneous spaces

In Cohomology on Sites, Definition 5.1 we have defined a torsor for a sheaf of groups on a site. Suppose  $\tau \in \{\text{Zariski}, \text{étale}, \text{smooth}, \text{syntomic}, \text{fppf}\}$  is a topology and  $(G, m)$  is a group scheme over  $S$ . Since  $\tau$  is stronger than the canonical topology (see Descent, Lemma 9.3) we see that  $\underline{G}$  (see Sites, Definition 13.3) is a sheaf of groups on  $(\text{Sch}/S)_\tau$ . Hence we already know what it means to have a torsor for  $\underline{G}$  on  $(\text{Sch}/S)_\tau$ . A special situation arises if this sheaf is representable. In the following definitions we define directly what it means for the representing scheme to be a  $G$ -torsor.

**Definition 9.1.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ . Let  $X$  be a scheme over  $S$ , and let  $a : G \times_S X \rightarrow X$  be an action of  $G$  on  $X$ .

- (1) We say  $X$  is a *pseudo  $G$ -torsor* or that  $X$  is *formally principally homogeneous under  $G$*  if the induced morphism of schemes  $G \times_S X \rightarrow X \times_S X$ ,  $(g, x) \mapsto (a(g, x), x)$  is an isomorphism of schemes over  $S$ .
- (2) A pseudo  $G$ -torsor  $X$  is called *trivial* if there exists an  $G$ -equivariant isomorphism  $G \rightarrow X$  over  $S$  where  $G$  acts on  $G$  by left multiplication.

It is clear that if  $S' \rightarrow S$  is a morphism of schemes then the pullback  $X_{S'}$  of a pseudo  $G$ -torsor over  $S$  is a pseudo  $G_{S'}$ -torsor over  $S'$ .

**Lemma 9.2.** *In the situation of Definition 9.1.*

- (1) *The scheme  $X$  is a pseudo  $G$ -torsor if and only if for every scheme  $T$  over  $S$  the set  $X(T)$  is either empty or the action of the group  $G(T)$  on  $X(T)$  is simply transitive.*

- (2) A pseudo  $G$ -torsor  $X$  is trivial if and only if the morphism  $X \rightarrow S$  has a section.

**Proof.** Omitted.  $\square$

**Definition 9.3.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ . Let  $X$  be a pseudo  $G$ -torsor over  $S$ .

- (1) We say  $X$  is a *principal homogeneous space* or a  $G$ -torsor if there exists a fpqc covering<sup>1</sup>  $\{S_i \rightarrow S\}_{i \in I}$  such that each  $X_{S_i} \rightarrow S_i$  has a section (i.e., is a trivial pseudo  $G_{S_i}$ -torsor).
- (2) Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{smooth}, \text{syntomic}, \text{fppf}\}$ . We say  $X$  is a  $G$ -torsor in the  $\tau$  topology, or a  $\tau$   $G$ -torsor, or simply a  $\tau$  torsor if there exists a  $\tau$  covering  $\{S_i \rightarrow S\}_{i \in I}$  such that each  $X_{S_i} \rightarrow S_i$  has a section.
- (3) If  $X$  is a  $G$ -torsor, then we say that it is *quasi-isotrivial* if it is a torsor for the étale topology.
- (4) If  $X$  is a  $G$ -torsor, then we say that it is *locally trivial* if it is a torsor for the Zariski topology.

We sometimes say “let  $X$  be a  $G$ -torsor over  $S$ ” to indicate that  $X$  is a scheme over  $S$  equipped with an action of  $G$  which turns it into a principal homogeneous space over  $S$ . Next we show that this agrees with the notation introduced earlier when both apply.

**Lemma 9.4.** Let  $S$  be a scheme. Let  $(G, m)$  be a group scheme over  $S$ . Let  $X$  be a scheme over  $S$ , and let  $a : G \times_S X \rightarrow X$  be an action of  $G$  on  $X$ . Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{smooth}, \text{syntomic}, \text{fppf}\}$ . Then  $X$  is a  $G$ -torsor in the  $\tau$ -topology if and only if  $\underline{X}$  is a  $\underline{G}$ -torsor on  $(\text{Sch}/S)_\tau$ .

**Proof.** Omitted.  $\square$

**Remark 9.5.** Let  $(G, m)$  be a group scheme over the scheme  $S$ . In this situation we have the following natural types of questions:

- (1) If  $X \rightarrow S$  is a pseudo  $G$ -torsor and  $X \rightarrow S$  is surjective, then is  $X$  necessarily a  $G$ -torsor?
- (2) Is every  $\underline{G}$ -torsor on  $(\text{Sch}/S)_{\text{fppf}}$  representable? In other words, does every  $\underline{G}$ -torsor come from a fppf  $G$ -torsor?
- (3) Is every  $G$ -torsor an fppf (resp. smooth, resp. étale, resp. Zariski) torsor?

In general the answers to these questions is no. To get a positive answer we need to impose additional conditions on  $G \rightarrow S$ . For example: If  $S$  is the spectrum of a field, then the answer to (1) is yes because then  $\{X \rightarrow S\}$  is a fpqc covering trivializing  $X$ . If  $G \rightarrow S$  is affine, then the answer to (2) is yes (insert future reference here). If  $G = \text{GL}_{n,S}$  then the answer to (3) is yes and in fact any  $\text{GL}_{n,S}$ -torsor is locally trivial (insert future reference here).

## 10. Equivariant quasi-coherent sheaves

We think of “functions” as dual to “space”. Thus for a morphism of spaces the map on functions goes the other way. Moreover, we think of the sections of a sheaf

<sup>1</sup>This means that the default type of torsor is a pseudo torsor which is trivial on an fpqc covering. This is the definition in [ABD<sup>+</sup>66, Exposé IV, 6.5]. It is a little bit inconvenient for us as we most often work in the fppf topology.

of modules as “functions”. This leads us naturally to the direction of the arrows chosen in the following definition.

**Definition 10.1.** Let  $S$  be a scheme, let  $(G, m)$  be a group scheme over  $S$ , and let  $a : G \times_S X \rightarrow X$  be an action of the group scheme  $G$  on  $X/S$ . An  $G$ -equivariant quasi-coherent  $\mathcal{O}_X$ -module, or simply a *equivariant quasi-coherent  $\mathcal{O}_X$ -module*, is a pair  $(\mathcal{F}, \alpha)$ , where  $\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_X$ -module, and  $\alpha$  is a  $\mathcal{O}_{G \times_S X}$ -module map

$$\alpha : a^* \mathcal{F} \longrightarrow \mathrm{pr}_1^* \mathcal{F}$$

where  $\mathrm{pr}_1 : G \times_S X \rightarrow X$  is the projection such that

(1) the diagram

$$\begin{array}{ccc} (1_G \times a)^* \mathrm{pr}_2^* \mathcal{F} & \xrightarrow{\mathrm{pr}_{12}^* \alpha} & \mathrm{pr}_2^* \mathcal{F} \\ (1_G \times a)^* \alpha \uparrow & & \uparrow (m \times 1_X)^* \alpha \\ (1_G \times a)^* a^* \mathcal{F} & \xlongequal{\quad} & (m \times 1_X)^* a^* \mathcal{F} \end{array}$$

is commutative in the category of  $\mathcal{O}_{G \times_S G \times_S X}$ -modules, and

(2) the pullback

$$(e \times 1_X)^* \alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

For explanation compare with the relevant diagrams of Equation (8.1.1).

Note that the commutativity of the first diagram guarantees that  $(e \times 1_X)^* \alpha$  is an idempotent operator on  $\mathcal{F}$ , and hence condition (2) is just the condition that it is an isomorphism.

**Lemma 10.2.** *Let  $S$  be a scheme. Let  $G$  be a group scheme over  $S$ . Let  $f : X \rightarrow Y$  be a  $G$ -equivariant morphism between  $S$ -schemes endowed with  $G$ -actions. Then pullback  $f^*$  given by  $(\mathcal{F}, \alpha) \mapsto (f^* \mathcal{F}, (1_G \times f)^* \alpha)$  defines a functor from the category of  $G$ -equivariant sheaves on  $X$  to the category of quasi-coherent  $G$ -equivariant sheaves on  $Y$ .*

**Proof.** Omitted. □

## 11. Groupoids

Recall that a groupoid is a category in which every morphism is an isomorphism, see Categories, Definition 2.5. Hence a groupoid has a set of objects  $\mathrm{Ob}$ , a set of arrows  $\mathrm{Arrows}$ , a *source* and *target* map  $s, t : \mathrm{Arrows} \rightarrow \mathrm{Ob}$ , and a *composition law*  $c : \mathrm{Arrows} \times_{s, \mathrm{Ob}, t} \mathrm{Arrows} \rightarrow \mathrm{Arrows}$ . These maps satisfy exactly the following axioms

- (1) (associativity)  $c \circ (1, c) = c \circ (c, 1)$  as maps  $\mathrm{Arrows} \times_{s, \mathrm{Ob}, t} \mathrm{Arrows} \times_{s, \mathrm{Ob}, t} \mathrm{Arrows} \rightarrow \mathrm{Arrows}$ ,
- (2) (identity) there exists a map  $e : \mathrm{Ob} \rightarrow \mathrm{Arrows}$  such that
  - (a)  $s \circ e = t \circ e = \mathrm{id}$  as maps  $\mathrm{Ob} \rightarrow \mathrm{Ob}$ ,
  - (b)  $c \circ (1, e \circ s) = c \circ (e \circ t, 1) = 1$  as maps  $\mathrm{Arrows} \rightarrow \mathrm{Arrows}$ ,
- (3) (inverse) there exists a map  $i : \mathrm{Arrows} \rightarrow \mathrm{Arrows}$  such that
  - (a)  $s \circ i = t, t \circ i = s$  as maps  $\mathrm{Arrows} \rightarrow \mathrm{Ob}$ , and
  - (b)  $c \circ (1, i) = e \circ t$  and  $c \circ (i, 1) = e \circ s$  as maps  $\mathrm{Arrows} \rightarrow \mathrm{Arrows}$ .

If this is the case the maps  $e$  and  $i$  are uniquely determined and  $i$  is a bijection. Note that if  $(\text{Ob}', \text{Arrows}', s', t', c')$  is a second groupoid category, then a functor  $f : (\text{Ob}, \text{Arrows}, s, t, c) \rightarrow (\text{Ob}', \text{Arrows}', s', t', c')$  is given by a pair of set maps  $f : \text{Ob} \rightarrow \text{Ob}'$  and  $f : \text{Arrows} \rightarrow \text{Arrows}'$  such that  $s' \circ f = f \circ s$ ,  $t' \circ f = f \circ t$ , and  $c' \circ (f, f) = f \circ c$ . The compatibility with identity and inverse is automatic. We will use this below. (Warning: The compatibility with identity has to be imposed in the case of general categories.)

**Definition 11.1.** Let  $S$  be a scheme.

- (1) A *groupoid scheme over  $S$* , or simply a *groupoid over  $S$*  is a quintuple  $(U, R, s, t, c)$  where  $U$  and  $R$  are schemes over  $S$ , and  $s, t : R \rightarrow U$  and  $c : R \times_{s, U, t} R \rightarrow R$  are morphisms of schemes over  $S$  with the following property: For any scheme  $T$  over  $S$  the quintuple

$$(U(T), R(T), s, t, c)$$

is a groupoid category in the sense described above.

- (2) A *morphism  $f : (U, R, s, t, c) \rightarrow (U', R', s', t', c')$  of groupoid schemes over  $S$*  is given by morphisms of schemes  $f : U \rightarrow U'$  and  $f : R \rightarrow R'$  with the following property: For any scheme  $T$  over  $S$  the maps  $f$  define a functor from the groupoid category  $(U(T), R(T), s, t, c)$  to the groupoid category  $(U'(T), R'(T), s', t', c')$ .

Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . Note that, by the remarks preceding the definition and the Yoneda lemma, there are unique morphisms of schemes  $e : U \rightarrow R$  and  $i : R \rightarrow R$  over  $S$  such that for every scheme  $T$  over  $S$  the induced map  $e : U(T) \rightarrow R(T)$  is the identity, and  $i : R(T) \rightarrow R(T)$  is the inverse of the groupoid category. The septuple  $(U, R, s, t, c, e, i)$  satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) above, and conversely given a septuple with this property the quintuple  $(U, R, s, t, c)$  is a groupoid scheme. Note that  $i$  is an isomorphism, and  $e$  is a section of both  $s$  and  $t$ . Moreover, given a groupoid scheme over  $S$  we denote

$$j = (t, s) : R \longrightarrow U \times_S U$$

which is compatible with our conventions in Section 3 above. We sometimes say “let  $(U, R, s, t, c, e, i)$  be a groupoid over  $S$ ” to stress the existence of identity and inverse.

**Lemma 11.2.** *Given a groupoid scheme  $(U, R, s, t, c)$  over  $S$  the morphism  $j : R \rightarrow U \times_S U$  is a pre-equivalence relation.*

**Proof.** Omitted. This is a nice exercise in the definitions. □

**Lemma 11.3.** *Given an equivalence relation  $j : R \rightarrow U$  over  $S$  there is a unique way to extend it to a groupoid  $(U, R, s, t, c)$  over  $S$ .*

**Proof.** Omitted. This is a nice exercise in the definitions. □

**Lemma 11.4.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . In the commutative diagram*

$$\begin{array}{ccccc}
 & & U & & \\
 & \swarrow t & & \nwarrow t & \\
 R & \xleftarrow{pr_0} & R \times_{s,U,t} R & \xrightarrow{c} & R \\
 \downarrow s & & \downarrow pr_1 & & \downarrow s \\
 U & \xleftarrow{t} & R & \xrightarrow{s} & U
 \end{array}$$

*the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.*

**Proof.** Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry.  $\square$

**Lemma 11.5.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c, e, i)$  be a groupoid over  $S$ . The diagram*

$$\begin{array}{ccccc}
 & & R \times_{t,U,t} R & \xrightarrow{pr_1} & R & \xrightarrow{t} & U \\
 & & \downarrow pr_0 \times co(i,1) & \searrow pr_0 & \downarrow id_R & & \downarrow id_U \\
 & & R \times_{s,U,t} R & \xrightarrow{c} & R & \xrightarrow{t} & U \\
 & & \downarrow pr_1 & \searrow pr_0 & \downarrow s & & \\
 & & R & \xrightarrow{s} & U & & \\
 & & & \downarrow t & & & 
 \end{array}$$

*is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.*

**Proof.** The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms  $(\alpha, \beta)$  with the same target, the pair of morphisms  $(\alpha, \alpha^{-1} \circ \beta)$ . In any groupoid this defines a bijection between  $\text{Arrows} \times_{t, \text{Ob}, t} \text{Arrows}$  and  $\text{Arrows} \times_{s, \text{Ob}, t} \text{Arrows}$ . Hence the second assertion of the lemma. The last assertion follows from Lemma 11.4.  $\square$

## 12. Quasi-coherent sheaves on groupoids

See the introduction of Section 10 for our choices in direction of arrows.

**Definition 12.1.** Let  $S$  be a scheme, let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . A *quasi-coherent module* on  $(U, R, s, t, c)$  is a pair  $(\mathcal{F}, \alpha)$ , where  $\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_U$ -module, and  $\alpha$  is a  $\mathcal{O}_R$ -module map

$$\alpha : t^* \mathcal{F} \longrightarrow s^* \mathcal{F}$$

such that

(1) the diagram

$$\begin{array}{ccccc}
 & \text{pr}_1^* t^* \mathcal{F} & \xrightarrow{\text{pr}_1^* \alpha} & \text{pr}_1^* s^* \mathcal{F} & \\
 & \parallel & & \parallel & \\
 \text{pr}_0^* s^* \mathcal{F} & & & & c^* s^* \mathcal{F} \\
 & \nwarrow \text{pr}_0^* \alpha & & \nearrow c^* \alpha & \\
 & \text{pr}_0^* t^* \mathcal{F} & \xlongequal{\quad} & c^* t^* \mathcal{F} &
 \end{array}$$

is a commutative in the category of  $\mathcal{O}_{R \times_{s,U,t} R}$ -modules, and

(2) the pullback

$$e^* \alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

Compare with the commutative diagrams of Lemma 11.4.

The commutativity of the first diagram forces the operator  $e^* \alpha$  to be idempotent. Hence the second condition can be reformulated as saying that  $e^* \alpha$  is an isomorphism. In fact, the condition implies that  $\alpha$  is an isomorphism.

**Lemma 12.2.** *Let  $S$  be a scheme, let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . If  $(\mathcal{F}, \alpha)$  is a quasi-coherent module on  $(U, R, s, t, c)$  then  $\alpha$  is an isomorphism.*

**Proof.** Pull back the commutative diagram of Definition 12.1 by the morphism  $(i, 1) : R \rightarrow R \times_{s,U,t} R$ . Then we see that  $i^* \alpha \circ \alpha = s^* e^* \alpha$ . Pulling back by the morphism  $(1, i)$  we obtain the relation  $\alpha \circ i^* \alpha = t^* e^* \alpha$ . By the second assumption these morphisms are the identity. Hence  $i^* \alpha$  is an inverse of  $\alpha$ .  $\square$

**Lemma 12.3.** *Let  $S$  be a scheme. Consider a morphism  $f : (U, R, s, t, c) \rightarrow (U', R', s', t', c')$  of groupoid schemes over  $S$ . Then pullback  $f^*$  given by*

$$(\mathcal{F}, \alpha) \mapsto (f^* \mathcal{F}, f^* \alpha)$$

*defines a functor from the category of quasi-coherent sheaves on  $(U', R', s', t', c')$  to the category of quasi-coherent sheaves on  $(U, R, s, t, c)$ .*

**Proof.** Omitted.  $\square$

**Lemma 12.4.** *Let  $S$  be a scheme. Consider a morphism  $f : (U, R, s, t, c) \rightarrow (U', R', s', t', c')$  of groupoid schemes over  $S$ . Assume that*

- (1)  $f : U \rightarrow U'$  is quasi-compact and quasi-separated,
- (2) the square

$$\begin{array}{ccc}
 R & \xrightarrow{f} & R' \\
 t \downarrow & & \downarrow t' \\
 U & \xrightarrow{f} & U'
 \end{array}$$

*is cartesian, and*

- (3)  $s'$  and  $t'$  are flat.

*Then pushforward  $f_*$  given by*

$$(\mathcal{F}, \alpha) \mapsto (f_* \mathcal{F}, f_* \alpha)$$



defines a functor from the category of quasi-coherent sheaves on  $(U, R, s, t, c)$  to the category of quasi-coherent sheaves on  $(U, R, s, t, c)$  which is right adjoint to pullback as defined in Lemma 12.3.

**Proof.** Since  $U \rightarrow U'$  is quasi-compact and quasi-separated we see that  $f_*$  transforms quasi-coherent sheaves into quasi-coherent sheaves (Schemes, Lemma 24.1). Moreover, since the squares

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ t \downarrow & & \downarrow t' \\ U & \xrightarrow{f} & U' \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \xrightarrow{f} & R' \\ s \downarrow & & \downarrow s' \\ U & \xrightarrow{f} & U' \end{array}$$

are cartesian we find that  $(t')^* f_* \mathcal{F} = f_* t^* \mathcal{F}$  and  $(s')^* f_* \mathcal{F} = f_* s^* \mathcal{F}$ , see Cohomology of Schemes, Lemma 5.2. Thus it makes sense to think of  $f_*$  as a map  $(t')^* f_* \mathcal{F} \rightarrow (s')^* f_* \mathcal{F}$ . A similar argument shows that  $f_* \alpha$  satisfies the cocycle condition. The functor is adjoint to the pullback functor since pullback and push-forward on modules on ringed spaces are adjoint. Some details omitted.  $\square$

**Lemma 12.5.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . The category of quasi-coherent modules on  $(U, R, s, t, c)$  has colimits.*

**Proof.** Let  $i \mapsto (\mathcal{F}_i, \alpha_i)$  be a diagram over the index category  $\mathcal{I}$ . We can form the colimit  $\mathcal{F} = \text{colim } \mathcal{F}_i$  which is a quasi-coherent sheaf on  $U$ , see Schemes, Section 24. Since colimits commute with pullback we see that  $s^* \mathcal{F} = \text{colim } s^* \mathcal{F}_i$  and similarly  $t^* \mathcal{F} = \text{colim } t^* \mathcal{F}_i$ . Hence we can set  $\alpha = \text{colim } \alpha_i$ . We omit the proof that  $(\mathcal{F}, \alpha)$  is the colimit of the diagram in the category of quasi-coherent modules on  $(U, R, s, t, c)$ .  $\square$

**Lemma 12.6.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . If  $s, t$  are flat, then the category of quasi-coherent modules on  $(U, R, s, t, c)$  is abelian.*

**Proof.** Let  $\varphi : (\mathcal{F}, \alpha) \rightarrow (\mathcal{G}, \beta)$  be a homomorphism of quasi-coherent modules on  $(U, R, s, t, c)$ . Since  $s$  is flat we see that

$$0 \rightarrow s^* \text{Ker}(\varphi) \rightarrow s^* \mathcal{F} \rightarrow s^* \mathcal{G} \rightarrow s^* \text{Coker}(\varphi) \rightarrow 0$$

is exact and similarly for pullback by  $t$ . Hence  $\alpha$  and  $\beta$  induce isomorphisms  $\kappa : t^* \text{Ker}(\varphi) \rightarrow s^* \text{Ker}(\varphi)$  and  $\lambda : t^* \text{Coker}(\varphi) \rightarrow s^* \text{Coker}(\varphi)$  which satisfy the cocycle condition. Then it is straightforward to verify that  $(\text{Ker}(\varphi), \kappa)$  and  $(\text{Coker}(\varphi), \lambda)$  are a kernel and cokernel in the category of quasi-coherent modules on  $(U, R, s, t, c)$ . Moreover, the condition  $\text{Coim}(\varphi) = \text{Im}(\varphi)$  follows because it holds over  $U$ .  $\square$

### 13. Colimits of quasi-coherent modules

In this section we prove some technical results saying that under suitable assumptions every quasi-coherent module on a groupoid is a filtered colimit of “small” quasi-coherent modules.

**Lemma 13.1.** *Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $s, t$  are flat, quasi-compact, and quasi-separated. For any quasi-coherent module  $\mathcal{G}$  on  $U$ , there exists a canonical isomorphism  $\alpha : t^* t_* s^* \mathcal{G} \rightarrow s^* t_* s^* \mathcal{G}$  which turns  $(t_* s^* \mathcal{G}, \alpha)$  into a quasi-coherent module on  $(U, R, s, t, c)$ . This construction defines a functor*

$$QCoh(\mathcal{O}_U) \longrightarrow QCoh(U, R, s, t, c)$$

which is a right adjoint to the forgetful functor  $(\mathcal{F}, \beta) \mapsto \mathcal{F}$ .

**Proof.** The pushforward of a quasi-coherent module along a quasi-compact and quasi-separated morphism is quasi-coherent, see Schemes, Lemma 24.1. Hence  $t_* s^* \mathcal{G}$  is quasi-coherent. With notation as in Lemma 11.4 we have

$$t^* t_* s^* \mathcal{G} = \text{pr}_{0,*} c^* s^* \mathcal{G} = \text{pr}_{0,*} \text{pr}_1^* s^* \mathcal{G} = s^* t_* s^* \mathcal{G}$$

The middle equality because  $s \circ c = s \circ \text{pr}_1$  as morphisms  $R \times_{s,U,t} R \rightarrow U$ , and the first and the last equality because we know that base change and pushforward commute in these steps by Cohomology of Schemes, Lemma 5.2.

To verify the cocycle condition of Definition 12.1 for  $\alpha$  and the adjointness property we describe the construction  $\mathcal{G} \mapsto (\mathcal{G}, \alpha)$  in another way. Consider the groupoid scheme  $(R, R \times_{s,U,s} R, \text{pr}_0, \text{pr}_1, \text{pr}_{02})$  associated to the equivalence relation  $R \times_{s,U,s} R$  on  $R$ , see Lemma 11.3. There is a morphism

$$f : (R, R \times_{s,U,s} R, \text{pr}_1, \text{pr}_0, \text{pr}_{02}) \longrightarrow (U, R, s, t, c)$$

of groupoid schemes given by  $t : R \rightarrow U$  and  $R \times_{t,U,t} R \rightarrow R$  given by  $(r_0, r_1) \mapsto r_0 \circ r_1^{-1}$  (we omit the verification of the commutativity of the required diagrams). Since  $t, s : R \rightarrow U$  are quasi-compact, quasi-separated, and flat, and since we have a cartesian square

$$\begin{array}{ccc} R \times_{s,U,s} R & \xrightarrow{(r_0, r_1) \mapsto r_0 \circ r_1^{-1}} & R \\ \text{pr}_0 \downarrow & & \downarrow t \\ R & \xrightarrow{t} & U \end{array}$$

by Lemma 11.5 it follows that Lemma 12.4 applies to  $f$ . Note that

$$QCoh(R, R \times_{s,U,s} R, \text{pr}_0, \text{pr}_1, \text{pr}_{02}) = QCoh(\mathcal{O}_U)$$

by the theory of descent of quasi-coherent sheaves as  $\{t : R \rightarrow U\}$  is an fpqc covering, see Descent, Proposition 5.2. Observe that pullback along  $f$  agrees with the forgetful functor and that pushforward agrees with the construction that assigns to  $\mathcal{G}$  the pair  $(\mathcal{G}, \alpha)$ . We omit the precise verifications. Thus the lemma follows from Lemma 12.4.  $\square$

**Lemma 13.2.** *Let  $f : Y \rightarrow X$  be a morphism of schemes. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module, let  $\mathcal{G}$  be a quasi-coherent  $\mathcal{O}_Y$ -module, and let  $\varphi : \mathcal{G} \rightarrow f^* \mathcal{F}$  be a module map. Assume*

- (1)  $\varphi$  is injective,
- (2)  $f$  is quasi-compact, quasi-separated, flat, and surjective,
- (3)  $X, Y$  are locally Noetherian, and
- (4)  $\mathcal{G}$  is a coherent  $\mathcal{O}_Y$ -module.

Then  $\mathcal{F} \cap f_* \mathcal{G}$  defined as the pullback

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & f_* f^* \mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F} \cap f_* \mathcal{G} & \longrightarrow & f_* \mathcal{G} \end{array}$$

is a coherent  $\mathcal{O}_X$ -module.

**Proof.** We will freely use the characterization of coherent modules of Cohomology of Schemes, Lemma 9.1 as well as the fact that coherent modules form a Serre subcategory of  $QCoh(\mathcal{O}_X)$ , see Cohomology of Schemes, Lemma 9.3. If  $f$  has a section  $\sigma$ , then we see that  $\mathcal{F} \cap f_*\mathcal{G}$  is contained in the image of  $\sigma^*\mathcal{G} \rightarrow \sigma^*f^*\mathcal{F} = \mathcal{F}$ , hence coherent. In general, to show that  $\mathcal{F} \cap f_*\mathcal{G}$  is coherent, it suffices to show that  $f^*(\mathcal{F} \cap f_*\mathcal{G})$  is coherent (see Descent, Lemma 6.1). Since  $f$  is flat this is equal to  $f^*\mathcal{F} \cap f^*f_*\mathcal{G}$ . Since  $f$  is flat, quasi-compact, and quasi-separated we see  $f^*f_*\mathcal{G} = p_*q^*\mathcal{G}$  where  $p, q : Y \times_X Y \rightarrow Y$  are the projections, see Cohomology of Schemes, Lemma 5.2. Since  $p$  has a section we win.  $\square$

Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid in schemes over  $S$ . Assume that  $U$  is locally Noetherian. In the lemma below we say that a quasi-coherent sheaf  $(\mathcal{F}, \alpha)$  on  $(U, R, s, t, c)$  is *coherent* if  $\mathcal{F}$  is a coherent  $\mathcal{O}_U$ -module.

**Lemma 13.3.** *Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume that*

- (1)  $U, R$  are Noetherian,
- (2)  $s, t$  are flat, quasi-compact, and quasi-separated.

*Then every quasi-coherent module  $(\mathcal{F}, \alpha)$  on  $(U, R, s, t, c)$  is a filtered colimit of coherent modules.*

**Proof.** We will use the characterization of Cohomology of Schemes, Lemma 9.1 of coherent modules on locally Noetherian scheme without further mention. Write  $\mathcal{F} = \text{colim } \mathcal{H}_i$  with  $\mathcal{H}_i$  coherent, see Properties, Lemma 20.6. Given a quasi-coherent sheaf  $\mathcal{H}$  on  $U$  we denote  $t_*s^*\mathcal{H}$  the quasi-coherent sheaf on  $(U, R, s, t, c)$  of Lemma 13.1. There is an adjunction map  $\mathcal{F} \rightarrow t_*s^*\mathcal{F}$  in  $QCoh(U, R, s, t, c)$ . Consider the pullback diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & t_*s^*\mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F}_i & \longrightarrow & t_*s^*\mathcal{H}_i \end{array}$$

in other words  $\mathcal{F}_i = \mathcal{F} \cap t_*s^*\mathcal{H}_i$ . Then  $\mathcal{F}_i$  is coherent by Lemma 13.2. On the other hand, the diagram above is a pullback diagram in  $QCoh(U, R, s, t, c)$  also as restriction to  $U$  is an exact functor by (the proof of) Lemma 12.6. Finally, because  $t$  is quasi-compact and quasi-separated we see that  $t_*$  commutes with colimits (see Cohomology of Schemes, Lemma 6.1). Hence  $t_*s^*\mathcal{F} = \text{colim } t_*s^*\mathcal{H}_i$  and hence  $\mathcal{F} = \text{colim } \mathcal{F}_i$  as desired.  $\square$

Here is a curious lemma that is useful when working with groupoids on fields. In fact, this is the standard argument to prove that any representation of an algebraic group is a colimit of finite dimensional representations.

**Lemma 13.4.** *Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume that*

- (1)  $U, R$  are affine,
- (2) *there exist  $e_i \in \mathcal{O}_R(R)$  such that every element  $g \in \mathcal{O}_R(R)$  can be uniquely written as  $\sum s^*(f_i)e_i$  for some  $f_i \in \mathcal{O}_U(U)$ .*

*Then every quasi-coherent module  $(\mathcal{F}, \alpha)$  on  $(U, R, s, t, c)$  is a filtered colimit of finite type quasi-coherent modules.*

**Proof.** The assumption means that  $\mathcal{O}_R(R)$  is a free  $\mathcal{O}_U(U)$ -module via  $s$  with basis  $e_i$ . Hence for any quasi-coherent  $\mathcal{O}_U$ -module  $\mathcal{G}$  we see that  $s^*\mathcal{G}(R) = \bigoplus_i \mathcal{G}(U)e_i$ .

We will write  $s(-)$  to indicate pullback of sections by  $s$  and similarly for other morphisms. Let  $(\mathcal{F}, \alpha)$  be a quasi-coherent module on  $(U, R, s, t, c)$ . Let  $\sigma \in \mathcal{F}(U)$ . By the above we can write

$$\alpha(t(\sigma)) = \sum s(\sigma_i)e_i$$

for some unique  $\sigma_i \in \mathcal{F}(U)$  (all but finitely many are zero of course). We can also write

$$c(e_i) = \sum \text{pr}_1(f_{ij})\text{pr}_0(e_j)$$

as functions on  $R \times_{s,U,t} R$ . Then the commutativity of the diagram in Definition 12.1 means that

$$\sum \text{pr}_1(\alpha(t(\sigma_i)))\text{pr}_0(e_i) = \sum \text{pr}_1(s(\sigma_i)f_{ij})\text{pr}_0(e_j)$$

(calculation omitted). Picking off the coefficients of  $\text{pr}_0(e_l)$  we see that  $\alpha(t(\sigma_l)) = \sum s(\sigma_i)f_{il}$ . Hence the submodule  $\mathcal{G} \subset \mathcal{F}$  generated by the elements  $\sigma_i$  defines a finite type quasi-coherent module preserved by  $\alpha$ . Hence it is a subobject of  $\mathcal{F}$  in  $QCoh(U, R, s, t, c)$ . This submodule contains  $\sigma$  (as one sees by pulling back the first relation by  $e$ ). Hence we win.  $\square$

We suggest the reader skip the rest of this section. Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid in schemes over  $S$ . Let  $\kappa$  be a cardinal. In the following we will say that a quasi-coherent sheaf  $(\mathcal{F}, \alpha)$  on  $(U, R, s, t, c)$  is  $\kappa$ -generated if  $\mathcal{F}$  is a  $\kappa$ -generated  $\mathcal{O}_U$ -module, see Properties, Definition 21.1.

**Lemma 13.5.** *Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $\kappa$  be a cardinal. There exists a set  $T$  and a family  $(\mathcal{F}_t, \alpha_t)_{t \in T}$  of  $\kappa$ -generated quasi-coherent modules on  $(U, R, s, t, c)$  such that every  $\kappa$ -generated quasi-coherent module on  $(U, R, s, t, c)$  is isomorphic to one of the  $(\mathcal{F}_t, \alpha_t)$ .*

**Proof.** For each quasi-coherent module  $\mathcal{F}$  on  $U$  there is a (possibly empty) set of maps  $\alpha : t^*\mathcal{F} \rightarrow s^*\mathcal{F}$  such that  $(\mathcal{F}, \alpha)$  is a quasi-coherent modules on  $(U, R, s, t, c)$ . By Properties, Lemma 21.2 there exists a set of isomorphism classes of  $\kappa$ -generated quasi-coherent  $\mathcal{O}_U$ -modules.  $\square$

**Lemma 13.6.** *Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume that  $s, t$  are flat. There exists a cardinal  $\kappa$  such that every quasi-coherent module  $(\mathcal{F}, \alpha)$  on  $(U, R, s, t, c)$  is the directed colimit of its  $\kappa$ -generated quasi-coherent submodules.*

**Proof.** In the statement of the lemma and in this proof a *submodule* of a quasi-coherent module  $(\mathcal{F}, \alpha)$  is a quasi-coherent submodule  $\mathcal{G} \subset \mathcal{F}$  such that  $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$  as subsheaves of  $s^*\mathcal{F}$ . This makes sense because since  $s, t$  are flat the pullbacks  $s^*$  and  $t^*$  are exact, i.e., preserve subsheaves. The proof will be a repeat of the proof of Properties, Lemma 21.3. We urge the reader to read that proof first.

Choose an affine open covering  $U = \bigcup_{i \in I} U_i$ . For each pair  $i, j$  choose affine open coverings

$$U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk} \quad \text{and} \quad s^{-1}(U_i) \cap t^{-1}(U_j) = \bigcup_{k \in J_{ij}} W_{ijk}.$$

Write  $U_i = \text{Spec}(A_i)$ ,  $U_{ijk} = \text{Spec}(A_{ijk})$ ,  $W_{ijk} = \text{Spec}(B_{ijk})$ . Let  $\kappa$  be any infinite cardinal  $\geq$  than the cardinality of any of the sets  $I, I_{ij}, J_{ij}$ .

Let  $(\mathcal{F}, \alpha)$  be a quasi-coherent module on  $(U, R, s, t, c)$ . Set  $M_i = \mathcal{F}(U_i)$ ,  $M_{ijk} = \mathcal{F}(U_{ijk})$ . Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}$$

and that  $\alpha$  gives isomorphisms

$$\alpha|_{W_{ijk}} : M_i \otimes_{A_i, t} B_{ijk} \longrightarrow M_j \otimes_{A_j, s} B_{ijk}$$

see Schemes, Lemma 7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every  $i, j \in I$ ,  $k \in I_{ij}$  or  $k \in J_{ij}$  and  $m \in M_i$  a finite subset  $S(i, j, k, m) \subset M_j$  such that we have

$$m \otimes 1 = \sum_{m' \in S(i, j, k, m)} m' \otimes a_{m'} \quad \text{or} \quad \alpha(m \otimes 1) = \sum_{m' \in S(i, j, k, m)} m' \otimes b_{m'}$$

in  $M_{ijk}$  for some  $a_{m'} \in A_{ijk}$  or  $b_{m'} \in B_{ijk}$ . Moreover, let's agree that  $S(i, i, k, m) = \{m\}$  for all  $i, j = i, k, m$  when  $k \in I_{ij}$ . Fix such a collection  $S(i, j, k, m)$

Given a family  $\mathcal{S} = (S_i)_{i \in I}$  of subsets  $S_i \subset M_i$  of cardinality at most  $\kappa$  we set  $\mathcal{S}' = (S'_i)$  where

$$S'_j = \bigcup_{(i, j, k, m) \text{ such that } m \in S_i} S(i, j, k, m)$$

Note that  $S_i \subset S'_i$ . Note that  $S'_i$  has cardinality at most  $\kappa$  because it is a union over a set of cardinality at most  $\kappa$  of finite sets. Set  $\mathcal{S}^{(0)} = \mathcal{S}$ ,  $\mathcal{S}^{(1)} = \mathcal{S}'$  and by induction  $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$ . Then set  $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$ . Writing  $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$  we see that for any element  $m \in S_i^{(\infty)}$  the image of  $m$  in  $M_{ijk}$  can be written as a finite sum  $\sum m' \otimes a_{m'}$  with  $m' \in S_j^{(\infty)}$ . In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk} \quad \text{and} \quad \alpha(N_i \otimes_{A_i, t} B_{ijk}) = N_j \otimes_{A_j, s} B_{ijk}$$

as submodules of  $M_{ijk}$  or  $M_j \otimes_{A_j, s} B_{ijk}$ . Thus there exists a quasi-coherent submodule  $\mathcal{G} \subset \mathcal{F}$  with  $\mathcal{G}(U_i) = N_i$  such that  $\alpha(t^* \mathcal{G}) = s^* \mathcal{G}$  as submodules of  $s^* \mathcal{F}$ . In other words,  $(\mathcal{G}, \alpha|_{t^* \mathcal{G}})$  is a submodule of  $(\mathcal{F}, \alpha)$ . Moreover, by construction  $\mathcal{G}$  is  $\kappa$ -generated.

Let  $\{(\mathcal{G}_t, \alpha_t)\}_{t \in T}$  be the set of  $\kappa$ -generated quasi-coherent submodules of  $(\mathcal{F}, \alpha)$ . If  $t, t' \in T$  then  $\mathcal{G}_t + \mathcal{G}_{t'}$  is also a  $\kappa$ -generated quasi-coherent submodule as it is the image of the map  $\mathcal{G}_t \oplus \mathcal{G}_{t'} \rightarrow \mathcal{F}$ . Hence the system (ordered by inclusion) is directed. The arguments above show that every section of  $\mathcal{F}$  over  $U_i$  is in one of the  $\mathcal{G}_t$  (because we can start with  $\mathcal{S}$  such that the given section is an element of  $S_i$ ). Hence  $\text{colim}_t \mathcal{G}_t \rightarrow \mathcal{F}$  is both injective and surjective as desired.  $\square$

#### 14. Groupoids and group schemes

There are many ways to construct a groupoid out of an action  $a$  of a group  $G$  on a set  $V$ . We choose the one where we think of an element  $g \in G$  as an arrow with source  $v$  and target  $a(g, v)$ . This leads to the following construction for group actions of schemes.

**Lemma 14.1.** *Let  $S$  be a scheme. Let  $Y$  be a scheme over  $S$ . Let  $(G, m)$  be a group scheme over  $Y$  with identity  $e_G$  and inverse  $i_G$ . Let  $X/Y$  be a scheme over  $Y$  and let  $a : G \times_Y X \rightarrow X$  be an action of  $G$  on  $X/Y$ . Then we get a groupoid scheme  $(U, R, s, t, c, e, i)$  over  $S$  in the following manner:*

- (1) *We set  $U = X$ , and  $R = G \times_Y X$ .*
- (2) *We set  $s : R \rightarrow U$  equal to  $(g, x) \mapsto x$ .*
- (3) *We set  $t : R \rightarrow U$  equal to  $(g, x) \mapsto a(g, x)$ .*
- (4) *We set  $c : R \times_{s, U, t} R \rightarrow R$  equal to  $((g, x), (g', x')) \mapsto (m(g, g'), x')$ .*
- (5) *We set  $e : U \rightarrow R$  equal to  $x \mapsto (e_G(x), x)$ .*
- (6) *We set  $i : R \rightarrow R$  equal to  $(g, x) \mapsto (i_G(g), a(g, x))$ .*

**Proof.** Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing  $g$  as an arrow from  $v$  to  $a(g, v)$ .  $\square$

**Lemma 14.2.** *Let  $S$  be a scheme. Let  $Y$  be a scheme over  $S$ . Let  $(G, m)$  be a group scheme over  $Y$ . Let  $X$  be a scheme over  $Y$  and let  $a : G \times_Y X \rightarrow X$  be an action of  $G$  on  $X$  over  $Y$ . Let  $(U, R, s, t, c)$  be the groupoid scheme constructed in Lemma 14.1. The rule  $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$  defines an equivalence of categories between  $G$ -equivariant  $\mathcal{O}_X$ -modules and the category of quasi-coherent modules on  $(U, R, s, t, c)$ .*

**Proof.** The assertion makes sense because  $t = a$  and  $s = \text{pr}_1$  as morphisms  $R = G \times_Y X \rightarrow X$ , see Definitions 10.1 and 12.1. Using the translation in Lemma 14.1 the commutativity requirements of the two definitions match up exactly.  $\square$

## 15. The stabilizer group scheme

Given a groupoid scheme we get a group scheme as follows.

**Lemma 15.1.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . The scheme  $G$  defined by the cartesian square*

$$\begin{array}{ccc} G & \longrightarrow & R \\ \downarrow & & \downarrow j=(t,s) \\ U & \xrightarrow{\Delta} & U \times_S U \end{array}$$

*is a group scheme over  $U$  with composition law  $m$  induced by the composition law  $c$ .*

**Proof.** This is true because in a groupoid category the set of self maps of any object forms a group.  $\square$

Since  $\Delta$  is an immersion we see that  $G = j^{-1}(\Delta_{U/S})$  is a locally closed subscheme of  $R$ . Thinking of it in this way, the structure morphism  $j^{-1}(\Delta_{U/S}) \rightarrow U$  is induced by either  $s$  or  $t$  (it is the same), and  $m$  is induced by  $c$ .

**Definition 15.2.** Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . The group scheme  $j^{-1}(\Delta_{U/S}) \rightarrow U$  is called the *stabilizer of the groupoid scheme*  $(U, R, s, t, c)$ .

In the literature the stabilizer group scheme is often denoted  $S$  (because the word stabilizer starts with an “s” presumably); we cannot do this since we have already used  $S$  for the base scheme.

**Lemma 15.3.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ , and let  $G/U$  be its stabilizer. Denote  $R_t/U$  the scheme  $R$  seen as a scheme over  $U$  via the morphism  $t : R \rightarrow U$ . There is a canonical left action*

$$a : G \times_U R_t \longrightarrow R_t$$

*induced by the composition law  $c$ .*

**Proof.** In terms of points over  $T/S$  we define  $a(g, r) = c(g, r)$ .  $\square$

**Lemma 15.4.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $G$  be the stabilizer group scheme of  $R$ . Let*

$$G_0 = G \times_{U, pr_0} (U \times_S U) = G \times_S U$$

*as a group scheme over  $U \times_S U$ . The action of  $G$  on  $R$  of Lemma 15.3 induces an action of  $G_0$  on  $R$  over  $U \times_S U$  which turns  $R$  into a pseudo  $G_0$ -torsor over  $U \times_S U$ .*

**Proof.** This is true because in a groupoid category  $\mathcal{C}$  the set  $\text{Mor}_{\mathcal{C}}(x, y)$  is a principal homogeneous set under the group  $\text{Mor}_{\mathcal{C}}(y, y)$ .  $\square$

**Lemma 15.5.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $p \in U \times_S U$  be a point. Denote  $R_p$  the scheme theoretic fibre of  $j = (t, s) : R \rightarrow U \times_S U$ . If  $R_p \neq \emptyset$ , then the action*

$$G_{0, \kappa(p)} \times_{\kappa(p)} R_p \longrightarrow R_p$$

*(see Lemma 15.4) which turns  $R_p$  into a  $G_{\kappa(p)}$ -torsor over  $\kappa(p)$ .*

**Proof.** The action is a pseudo-torsor by the lemma cited in the statement. And if  $R_p$  is not the empty scheme, then  $\{R_p \rightarrow p\}$  is an fpqc covering which trivializes the pseudo-torsor.  $\square$

## 16. Restricting groupoids

Consider a (usual) groupoid  $\mathcal{C} = (\text{Ob}, \text{Arrows}, s, t, c)$ . Suppose we have a map of sets  $g : \text{Ob}' \rightarrow \text{Ob}$ . Then we can construct a groupoid  $\mathcal{C}' = (\text{Ob}', \text{Arrows}', s', t', c')$  by thinking of a morphism between elements  $x', y'$  of  $\text{Ob}'$  as a morphisms in  $\mathcal{C}$  between  $g(x'), g(y')$ . In other words we set

$$\text{Arrows}' = \text{Ob}' \times_{g, \text{Ob}, t} \text{Arrows} \times_{s, \text{Ob}, g} \text{Ob}'.$$

with obvious choices for  $s', t'$ , and  $c'$ . There is a canonical functor  $\mathcal{C}' \rightarrow \mathcal{C}$  which is fully faithful, but not necessarily essentially surjective. This groupoid  $\mathcal{C}'$  endowed with the functor  $\mathcal{C}' \rightarrow \mathcal{C}$  is called the *restriction* of the groupoid  $\mathcal{C}$  to  $\text{Ob}'$ .

**Lemma 16.1.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $g : U' \rightarrow U$  be a morphism of schemes. Consider the following diagram*

$$\begin{array}{ccccc}
 & & s' & & \\
 & & \curvearrowright & & \\
 R' & \xrightarrow{\quad} & R \times_{s, U} U' & \xrightarrow{\quad} & U' \\
 \downarrow & & \downarrow & & \downarrow g \\
 U' \times_{t, R} R & \xrightarrow{\quad} & R & \xrightarrow{s} & U \\
 \downarrow & & \downarrow t & & \\
 U' & \xrightarrow{g} & U & & 
 \end{array}$$

$t'$  (curved arrow from  $R'$  to  $U'$ )

where all the squares are fibre product squares. Then there is a canonical composition law  $c' : R' \times_{s', U', t'} R' \rightarrow R'$  such that  $(U', R', s', t', c')$  is a groupoid scheme over  $S$  and such that  $U' \rightarrow U$ ,  $R' \rightarrow R$  defines a morphism  $(U', R', s', t', c') \rightarrow (U, R, s, t, c)$  of groupoid schemes over  $S$ . Moreover, for any scheme  $T$  over  $S$  the functor of groupoids

$$(U'(T), R'(T), s', t', c') \rightarrow (U(T), R(T), s, t, c)$$

is the restriction (see above) of  $(U(T), R(T), s, t, c)$  via the map  $U'(T) \rightarrow U(T)$ .

**Proof.** Omitted.  $\square$

**Definition 16.2.** Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $g : U' \rightarrow U$  be a morphism of schemes. The morphism of groupoids  $(U', R', s', t', c') \rightarrow (U, R, s, t, c)$  constructed in Lemma 16.1 is called the *restriction of  $(U, R, s, t, c)$  to  $U'$* . We sometime use the notation  $R' = R|_{U'}$  in this case.

**Lemma 16.3.** The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 16.2 and 3.3 agree via the constructions of Lemmas 11.2 and 11.3.

**Proof.** What we are saying here is that  $R'$  of Lemma 16.1 is also equal to

$$R' = (U' \times_S U') \times_{U \times_S U} R \longrightarrow U' \times_S U'$$

In fact this might have been a clearer way to state that lemma.  $\square$

**Lemma 16.4.** Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $g : U' \rightarrow U$  be a morphism of schemes. Let  $(U', R', s', t', c')$  be the restriction of  $(U, R, s, t, c)$  via  $g$ . Let  $G$  be the stabilizer of  $(U, R, s, t, c)$  and let  $G'$  be the stabilizer of  $(U', R', s', t', c')$ . Then  $G'$  is the base change of  $G$  by  $g$ , i.e., there is a canonical identification  $G' = U' \times_{g, U} G$ .

**Proof.** Omitted.  $\square$

## 17. Invariant subschemes

In this section we discuss briefly the notion of an invariant subscheme.

**Definition 17.1.** Let  $(U, R, s, t, c)$  be a groupoid scheme over the base scheme  $S$ .

- (1) A subset  $W \subset U$  is *set-theoretically  $R$ -invariant* if  $t(s^{-1}(W)) \subset W$ .
- (2) An open  $W \subset U$  is  *$R$ -invariant* if  $t(s^{-1}(W)) \subset W$ .
- (3) A closed subscheme  $Z \subset U$  is called  *$R$ -invariant* if  $t^{-1}(Z) = s^{-1}(Z)$ . Here we use the scheme theoretic inverse image, see Schemes, Definition 17.7.
- (4) A monomorphism of schemes  $T \rightarrow U$  is  *$R$ -invariant* if  $T \times_{U, t} R = R \times_{s, U} T$  as schemes over  $R$ .

For subsets and open subschemes  $W \subset U$  the  $R$ -invariance is also equivalent to requiring that  $s^{-1}(W) = t^{-1}(W)$  as subsets of  $R$ . If  $W \subset U$  is an  $R$ -equivariant open subscheme then the restriction of  $R$  to  $W$  is just  $R_W = s^{-1}(W) = t^{-1}(W)$ . Similarly, if  $Z \subset U$  is an  $R$ -invariant closed subscheme, then the restriction of  $R$  to  $Z$  is just  $R_Z = s^{-1}(Z) = t^{-1}(Z)$ .

**Lemma 17.2.** Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ .

- (1) For any subset  $W \subset U$  the subset  $t(s^{-1}(W))$  is set-theoretically  $R$ -invariant.



- (2) If  $s$  and  $t$  are open, then for every open  $W \subset U$  the open  $t(s^{-1}(W))$  is an  $R$ -invariant open subscheme.
- (3) If  $s$  and  $t$  are open and quasi-compact, then  $U$  has an open covering consisting of  $R$ -invariant quasi-compact open subschemes.

**Proof.** Part (1) follows from Lemmas 3.4 and 11.2, namely,  $t(s^{-1}(W))$  is the set of points of  $U$  equivalent to a point of  $W$ . Next, assume  $s$  and  $t$  open and  $W \subset U$  open. Since  $s$  is open the set  $W' = t(s^{-1}(W))$  is an open subset of  $U$ . Finally, assume that  $s, t$  are both open and quasi-compact. Then, if  $W \subset U$  is a quasi-compact open, then also  $W' = t(s^{-1}(W))$  is a quasi-compact open, and invariant by the discussion above. Letting  $W$  range over all affine opens of  $U$  we see (3).  $\square$

**Lemma 17.3.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $s$  and  $t$  quasi-compact and flat and  $U$  quasi-separated. Let  $W \subset U$  be quasi-compact open. Then  $t(s^{-1}(W))$  is an intersection of a nonempty family of quasi-compact open subsets of  $U$ .*

**Proof.** Note that  $s^{-1}(W)$  is quasi-compact open in  $R$ . As a continuous map  $t$  maps the quasi-compact subset  $s^{-1}(W)$  to a quasi-compact subset  $t(s^{-1}(W))$ . As  $t$  is flat and  $s^{-1}(W)$  is closed under generalization, so is  $t(s^{-1}(W))$ , see (Morphisms, Lemma 26.8 and Topology, Lemma 18.5). Pick a quasi-compact open  $W' \subset U$  containing  $t(s^{-1}(W))$ . By Properties, Lemma 2.4 we see that  $W'$  is a spectral space (here we use that  $U$  is quasi-separated). Then the lemma follows from Topology, Lemma 23.7 applied to  $t(s^{-1}(W)) \subset W'$ .  $\square$

**Lemma 17.4.** *Assumptions and notation as in Lemma 17.3. There exists an  $R$ -invariant open  $V \subset U$  and a quasi-compact open  $W'$  such that  $W \subset V \subset W' \subset U$ .*

**Proof.** Set  $E = t(s^{-1}(W))$ . Recall that  $E$  is set-theoretically  $R$ -invariant (Lemma 17.2). By Lemma 17.3 there exists a quasi-compact open  $W'$  containing  $E$ . Let  $Z = U \setminus W'$  and consider  $T = t(s^{-1}(Z))$ . Observe that  $Z \subset T$  and that  $E \cap T = \emptyset$  because  $s^{-1}(E) = t^{-1}(E)$  is disjoint from  $s^{-1}(Z)$ . Since  $T$  is the image of the closed subset  $s^{-1}(Z) \subset R$  under the quasi-compact morphism  $t : R \rightarrow U$  we see that any point  $\xi$  in the closure  $\overline{T}$  is the specialization of a point of  $T$ , see Morphisms, Lemma 6.5 (and Morphisms, Lemma 6.3 to see that the scheme theoretic image is the closure of the image). Say  $\xi' \rightsquigarrow \xi$  with  $\xi' \in T$ . Suppose that  $r \in R$  and  $s(r) = \xi$ . Since  $s$  is flat we can find a specialization  $r' \rightsquigarrow r$  in  $R$  such that  $s(r') = \xi'$  (Morphisms, Lemma 26.8). Then  $t(r') \rightsquigarrow t(r)$ . We conclude that  $t(r') \in T$  as  $T$  is set-theoretically invariant by Lemma 17.2. Thus  $\overline{T}$  is a set-theoretically  $R$ -invariant closed subset and  $V = U \setminus \overline{T}$  is the open we are looking for. It is contained in  $W'$  which finishes the proof.  $\square$

## 18. Quotient sheaves

Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{smooth}, \text{syntomic}\}$ . Let  $S$  be a scheme. Let  $j : R \rightarrow U \times_S U$  be a pre-relation over  $S$ . Say  $U, R, S$  are objects of a  $\tau$ -site  $\text{Sch}_\tau$  (see Topologies, Section 2). Then we can consider the functors

$$h_U, h_R : (\text{Sch}/S)_\tau^{\text{opp}} \longrightarrow \text{Sets}.$$

These are sheaves, see Descent, Lemma 9.3. The morphism  $j$  induces a map  $j : h_R \rightarrow h_U \times h_U$ . For each object  $T \in \text{Ob}((\text{Sch}/S)_\tau)$  we can take the equivalence

relation  $\sim_T$  generated by  $j(T) : R(T) \rightarrow U(T) \times U(T)$  and consider the quotient. Hence we get a presheaf

$$(18.0.1) \quad (Sch/S)_\tau^{opp} \longrightarrow Sets, \quad T \longmapsto U(T)/\sim_T$$

**Definition 18.1.** Let  $\tau$ ,  $S$ , and the pre-relation  $j : R \rightarrow U \times_S U$  be as above. In this setting the *quotient sheaf*  $U/R$  associated to  $j$  is the sheafification of the presheaf (18.0.1) in the  $\tau$ -topology. If  $j : R \rightarrow U \times_S U$  comes from the action of a group scheme  $G/S$  on  $U$  as in Lemma 14.1 then we sometimes denote the quotient sheaf  $U/G$ .

This means exactly that the diagram

$$\begin{array}{ccc} h_R & \xrightarrow{\quad} & h_U \\ & \searrow & \downarrow \\ & & U/R \end{array}$$

is a coequalizer diagram in the category of sheaves of sets on  $(Sch/S)_\tau$ . Using the Yoneda embedding we may view  $(Sch/S)_\tau$  as a full subcategory of sheaves on  $(Sch/S)_\tau$  and hence identify schemes with representable functors. Using this abuse of notation we will often depict the diagram above simply

$$\begin{array}{ccc} R & \xrightarrow{s} & U \\ & \searrow t & \downarrow \\ & & U/R \end{array}$$

We will mostly work with the fppf topology when considering quotient sheaves of groupoids/equivalence relations.

**Definition 18.2.** In the situation of Definition 18.1. We say that the pre-relation  $j$  has a *representable quotient* if the sheaf  $U/R$  is representable. We will say a groupoid  $(U, R, s, t, c)$  has a *representable quotient* if the quotient  $U/R$  with  $j = (t, s)$  is representable.

The following lemma characterizes schemes  $M$  representing the quotient. It applies for example if  $\tau = fppf$ ,  $U \rightarrow M$  is flat, of finite presentation and surjective, and  $R \cong U \times_M U$ .

**Lemma 18.3.** *In the situation of Definition 18.1. Assume there is a scheme  $M$ , and a morphism  $U \rightarrow M$  such that*

- (1) *the morphism  $U \rightarrow M$  equalizes  $s, t$ ,*
- (2) *the morphism  $U \rightarrow M$  induces a surjection of sheaves  $h_U \rightarrow h_M$  in the  $\tau$ -topology, and*
- (3) *the induced map  $(t, s) : R \rightarrow U \times_M U$  induces a surjection of sheaves  $h_R \rightarrow h_{U \times_M U}$  in the  $\tau$ -topology.*

*In this case  $M$  represents the quotient sheaf  $U/R$ .*

**Proof.** Condition (1) says that  $h_U \rightarrow h_M$  factors through  $U/R$ . Condition (2) says that  $U/R \rightarrow h_M$  is surjective as a map of sheaves. Condition (3) says that  $U/R \rightarrow h_M$  is injective as a map of sheaves. Hence the lemma follows.  $\square$

The following lemma is wrong if we do not require  $j$  to be a pre-equivalence relation (but just a pre-relation say).

**Lemma 18.4.** *Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{smooth}, \text{syntomic}\}$ . Let  $S$  be a scheme. Let  $j : R \rightarrow U \times_S U$  be a pre-equivalence relation over  $S$ . Assume  $U, R, S$  are objects of a  $\tau$ -site  $Sch_\tau$ . For  $T \in \text{Ob}((Sch/S)_\tau)$  and  $a, b \in U(T)$  the following are equivalent:*

- (1)  $a$  and  $b$  map to the same element of  $(U/R)(T)$ , and
- (2) there exists a  $\tau$ -covering  $\{f_i : T_i \rightarrow T\}$  of  $T$  and morphisms  $r_i : T_i \rightarrow R$  such that  $a \circ f_i = s \circ r_i$  and  $b \circ f_i = t \circ r_i$ .

In other words, in this case the map of  $\tau$ -sheaves

$$h_R \longrightarrow h_U \times_{U/R} h_U$$

is surjective.

**Proof.** Omitted. Hint: The reason this works is that the presheaf (18.0.1) in this case is really given by  $T \mapsto U(T)/j(R(T))$  as  $j(R(T)) \subset U(T) \times U(T)$  is an equivalence relation, see Definition 3.1.  $\square$

**Lemma 18.5.** *Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{smooth}, \text{syntomic}\}$ . Let  $S$  be a scheme. Let  $j : R \rightarrow U \times_S U$  be a pre-equivalence relation over  $S$  and  $g : U' \rightarrow U$  a morphism of schemes over  $S$ . Let  $j' : R' \rightarrow U' \times_S U'$  be the restriction of  $j$  to  $U'$ . Assume  $U, U', R, S$  are objects of a  $\tau$ -site  $\text{Sch}_\tau$ . The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

*is injective. If  $g$  defines a surjection  $h_{U'} \rightarrow h_U$  of sheaves in the  $\tau$ -topology (for example if  $\{g : U' \rightarrow U\}$  is a  $\tau$ -covering), then  $U'/R' \rightarrow U/R$  is an isomorphism.*

**Proof.** Suppose  $\xi, \xi' \in (U'/R')(T)$  are sections which map to the same section of  $U/R$ . Then we can find a  $\tau$ -covering  $\mathcal{T} = \{T_i \rightarrow T\}$  of  $T$  such that  $\xi|_{T_i}, \xi'|_{T_i}$  are given by  $a_i, a'_i \in U'(T_i)$ . By Lemma 18.4 and the axioms of a site we may after refining  $\mathcal{T}$  assume there exist morphisms  $r_i : T_i \rightarrow R$  such that  $g \circ a_i = s \circ r_i$ ,  $g \circ a'_i = t \circ r_i$ . Since by construction  $R' = R \times_{U \times_S U} (U' \times_S U')$  we see that  $(r_i, (a_i, a'_i)) \in R'(T_i)$  and this shows that  $a_i$  and  $a'_i$  define the same section of  $U'/R'$  over  $T_i$ . By the sheaf condition this implies  $\xi = \xi'$ .

If  $h_{U'} \rightarrow h_U$  is a surjection of sheaves, then of course  $U'/R' \rightarrow U/R$  is surjective also. If  $\{g : U' \rightarrow U\}$  is a  $\tau$ -covering, then the map of sheaves  $h_{U'} \rightarrow h_U$  is surjective, see Sites, Lemma 13.4. Hence  $U'/R' \rightarrow U/R$  is surjective also in this case.  $\square$

**Lemma 18.6.** *Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{smooth}, \text{syntomic}\}$ . Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Let  $g : U' \rightarrow U$  a morphism of schemes over  $S$ . Let  $(U', R', s', t', c')$  be the restriction of  $(U, R, s, t, c)$  to  $U'$ . Assume  $U, U', R, S$  are objects of a  $\tau$ -site  $\text{Sch}_\tau$ . The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

*is injective. If the composition*

$$U' \times_{g, U, t} R \xrightarrow{\quad h \quad} R \xrightarrow[\quad s \quad]{\quad pr_1 \quad} U$$

*defines a surjection of sheaves in the  $\tau$ -topology then the map is bijective. This holds for example if  $\{h : U' \times_{g, U, t} R \rightarrow U\}$  is a  $\tau$ -covering, or if  $U' \rightarrow U$  defines a surjection of sheaves in the  $\tau$ -topology, or if  $\{g : U' \rightarrow U\}$  is a covering in the  $\tau$ -topology.*

**Proof.** Injectivity follows on combining Lemmas 11.2 and 18.5. To see surjectivity (see Sites, Section 12 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that  $T$  is a scheme and  $\sigma \in U/R(T)$ . There exists a covering  $\{T_i \rightarrow T\}$  such that  $\sigma|_{T_i}$  is the image of some element  $f_i \in U(T_i)$ . Hence we may assume that  $\sigma$  is the image of  $f \in U(T)$ . By the assumption that  $h$  is a surjection of sheaves, we can find a  $\tau$ -covering  $\{\varphi_i : T_i \rightarrow T\}$  and morphisms  $f_i : T_i \rightarrow U' \times_{g,U,t} R$  such that  $f \circ \varphi_i = h \circ f_i$ . Denote  $f'_i = \text{pr}_0 \circ f_i : T_i \rightarrow U'$ . Then we see that  $f'_i \in U'(T_i)$  maps to  $g \circ f'_i \in U(T_i)$  and that  $g \circ f'_i \sim_{T_i} h \circ f_i = f \circ \varphi_i$  notation as in (18.0.1). Namely, the element of  $R(T_i)$  giving the relation is  $\text{pr}_1 \circ f_i$ . This means that the restriction of  $\sigma$  to  $T_i$  is in the image of  $U'/R'(T_i) \rightarrow U/R(T_i)$  as desired.

If  $\{h\}$  is a  $\tau$ -covering, then it induces a surjection of sheaves, see Sites, Lemma 13.4. If  $U' \rightarrow U$  is surjective, then also  $h$  is surjective as  $s$  has a section (namely the neutral element  $e$  of the groupoid scheme).  $\square$

**Lemma 18.7.** *Let  $S$  be a scheme. Let  $f : (U, R, j) \rightarrow (U', R', j')$  be a morphism between equivalence relations over  $S$ . Assume that*

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ s \downarrow & & \downarrow s' \\ U & \xrightarrow{f} & U' \end{array}$$

*is cartesian. For any  $\tau \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{smooth}, \text{syntomic}\}$  the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\quad} & U/R \\ \downarrow & & \downarrow f \\ U' & \xrightarrow{\quad} & U'/R' \end{array}$$

*is a fibre product square of  $\tau$ -sheaves.*

**Proof.** By Lemma 18.4 the quotient sheaves have a simple description which we will use below without further mention. We first show that

$$U \longrightarrow U' \times_{U'/R'} U/R$$

is injective. Namely, assume  $a, b \in U(T)$  map to the same element on the right hand side. Then  $f(a) = f(b)$ . After replacing  $T$  by the members of a  $\tau$ -covering we may assume that there exists an  $r \in R(T)$  such that  $a = s(r)$  and  $b = t(r)$ . Then  $r' = f(r)$  is a  $T$ -valued point of  $R'$  with  $s'(r') = t'(r')$ . Hence  $r' = e'(f(a))$  (where  $e'$  is the identity of the groupoid scheme associated to  $j'$ , see Lemma 11.3). Because the first diagram of the lemma is cartesian this implies that  $r$  has to equal  $e(a)$ . Thus  $a = b$ .

Finally, we show that the displayed arrow is surjective. Let  $T$  be a scheme over  $S$  and let  $(a', \bar{b})$  be a section of the sheaf  $U' \times_{U'/R'} U/R$  over  $T$ . After replacing  $T$  by the members of a  $\tau$ -covering we may assume that  $\bar{b}$  is the class of an element  $b \in U(T)$ . After replacing  $T$  by the members of a  $\tau$ -covering we may assume that there exists an  $r' \in R'(T)$  such that  $a' = t(r')$  and  $s'(r') = f(b)$ . Because the first diagram of the lemma is cartesian we can find  $r \in R(T)$  such that  $s(r) = b$  and  $f(r) = r'$ . Then it is clear that  $a = t(r) \in U(T)$  is a section which maps to  $(a', \bar{b})$ .  $\square$

### 19. Descent in terms of groupoids

Cartesian morphisms are defined as follows.

**Definition 19.1.** Let  $S$  be a scheme. Let  $f : (U', R', s', t', c') \rightarrow (U, R, s, t, c)$  be a morphism of groupoid schemes over  $S$ . We say  $f$  is *cartesian*, or that  $(U', R', s', t', c')$  is *cartesian over*  $(U, R, s, t, c)$ , if the diagram

$$\begin{array}{ccc} R' & \xrightarrow{\quad} & R \\ s' \downarrow & f & \downarrow s \\ U' & \xrightarrow{\quad} & U \end{array}$$

is a fibre square in the category of schemes. A *morphism of groupoid schemes cartesian over*  $(U, R, s, t, c)$  is a morphism of groupoid schemes compatible with the structure morphisms towards  $(U, R, s, t, c)$ .

Cartesian morphisms are related to descent data. First we prove a general lemma describing the category of cartesian groupoid schemes over a fixed groupoid scheme.

**Lemma 19.2.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . The category of groupoid schemes cartesian over  $(U, R, s, t, c)$  is equivalent to the category of pairs  $(V, \varphi)$  where  $V$  is a scheme over  $U$  and*

$$\varphi : V \times_{U, t} R \longrightarrow R \times_{s, U} V$$

*is an isomorphism over  $R$  such that  $e^* \varphi = \text{id}_V$  and such that*

$$c^* \varphi = \text{pr}_1^* \varphi \circ \text{pr}_0^* \varphi$$

*as morphisms of schemes over  $R \times_{s, U, t} R$ .*

**Proof.** The pullback notation in the lemma signifies base change. The displayed formula makes sense because

$$(R \times_{s, U, t} R) \times_{\text{pr}_1, R, \text{pr}_1} (V \times_{U, t} R) = (R \times_{s, U, t} R) \times_{\text{pr}_0, R, \text{pr}_0} (R \times_{s, U} V)$$

as schemes over  $R \times_{s, U, t} R$ .

Given  $(V, \varphi)$  we set  $U' = V$  and  $R' = V \times_{U, t} R$ . We set  $t' : R' \rightarrow U'$  equal to the projection  $V \times_{U, t} R \rightarrow V$ . We set  $s'$  equal to  $\varphi$  followed by the projection  $R \times_{s, U} V \rightarrow V$ . We set  $c'$  equal to the composition

$$\begin{aligned} R' \times_{s', U', t'} R' &\xrightarrow{\varphi, 1} (R \times_{s, U} V) \times_V (V \times_{U, t} R) \\ &\rightarrow R \times_{s, U} V \times_{U, t} R \\ &\xrightarrow{\varphi^{-1}, 1} V \times_{U, t} (R \times_{s, U, t} R) \\ &\xrightarrow{1, c} V \times_{U, t} R = R' \end{aligned}$$

A computation, which we omit shows that we obtain a groupoid scheme over  $(U, R, s, t, c)$ . It is clear that this groupoid scheme is cartesian over  $(U, R, s, t, c)$ .

Conversely, given  $f : (U', R', s', t', c') \rightarrow (U, R, s, t, c)$  cartesian then the morphisms

$$U' \times_{U, t} R \xleftarrow{t', f} R' \xrightarrow{f, s'} R \times_{s, U} U'$$

are isomorphisms and we can set  $V = U'$  and  $\varphi$  equal to the composition  $(f, s') \circ (t', f)^{-1}$ . We omit the proof that  $\varphi$  satisfies the conditions in the lemma. We omit the proof that these constructions are mutually inverse.  $\square$

Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of schemes over  $S$ . Then we obtain a groupoid scheme  $(X, X \times_Y X, \text{pr}_1, \text{pr}_0, c)$  over  $S$ . Namely,  $j : X \times_Y X \rightarrow X \times_S X$  is an equivalence relation and we can take the associated groupoid, see Lemma 11.3.

**Lemma 19.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of schemes over  $S$ . The construction of Lemma 19.2 determines an equivalence*

$$\begin{array}{ccc} \text{category of groupoids schemes} & \longrightarrow & \text{category of descent data} \\ \text{cartesian over } (X, X \times_Y X, \dots) & & \text{relative to } X/Y \end{array}$$

**Proof.** This is clear from Lemma 19.2 and the definition of descent data on schemes in Descent, Definition 30.1.  $\square$

## 20. Separation conditions

This really means conditions on the morphism  $j : R \rightarrow U \times_S U$  when given a groupoid  $(U, R, s, t, c)$  over  $S$ . As in the previous section we first formulate the corresponding diagram.

**Lemma 20.1.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . Let  $G \rightarrow U$  be the stabilizer group scheme. The commutative diagram*

$$\begin{array}{ccccc} R & \xrightarrow{f \mapsto (f, s(f))} & R \times_{s, U} U & \longrightarrow & U \\ \downarrow \Delta_{R/U \times_S U} & & \downarrow & & \downarrow \\ R \times_{(U \times_S U)} R & \xrightarrow{(f, g) \mapsto (f, f^{-1} \circ g)} & R \times_{s, U} G & \longrightarrow & G \end{array}$$

*the two left horizontal arrows are isomorphisms and the right square is a fibre product square.*

**Proof.** Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry.  $\square$

**Lemma 20.2.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid over  $S$ . Let  $G \rightarrow U$  be the stabilizer group scheme.*

- (1) *The following are equivalent*
  - (a)  $j : R \rightarrow U \times_S U$  is separated,
  - (b)  $G \rightarrow U$  is separated, and
  - (c)  $e : U \rightarrow G$  is a closed immersion.
- (2) *The following are equivalent*
  - (a)  $j : R \rightarrow U \times_S U$  is quasi-separated,
  - (b)  $G \rightarrow U$  is quasi-separated, and
  - (c)  $e : U \rightarrow G$  is quasi-compact.

**Proof.** The group scheme  $G \rightarrow U$  is the base change of  $R \rightarrow U \times_S U$  by the diagonal morphism  $U \rightarrow U \times_S U$ , see Lemma 15.1. Hence if  $j$  is separated (resp. quasi-separated), then  $G \rightarrow U$  is separated (resp. quasi-separated). (See Schemes, Lemma 21.13). Thus (a)  $\Rightarrow$  (b) in both (1) and (2).

If  $G \rightarrow U$  is separated (resp. quasi-separated), then the morphism  $U \rightarrow G$ , as a section of the structure morphism  $G \rightarrow U$  is a closed immersion (resp. quasi-compact), see Schemes, Lemma 21.12. Thus (b)  $\Rightarrow$  (a) in both (1) and (2).

By the result of Lemma 20.1 (and Schemes, Lemmas 18.2 and 19.3) we see that if  $e$  is a closed immersion (resp. quasi-compact)  $\Delta_{R/U \times_S U}$  is a closed immersion (resp. quasi-compact). Thus (c)  $\Rightarrow$  (a) in both (1) and (2).  $\square$

## 21. Finite flat groupoids, affine case

Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $U = \text{Spec}(A)$ , and  $R = \text{Spec}(B)$  are affine. In this case we get two ring maps  $s^\#, t^\# : A \rightarrow B$ . Let  $C$  be the equalizer of  $s^\#$  and  $t^\#$ . In a formula

$$(21.0.1) \quad C = \{a \in A \mid t^\#(a) = s^\#(a)\}.$$

We will sometimes call this the *ring of  $R$ -invariant functions on  $U$* . What properties does  $M = \text{Spec}(C)$  have? The first observation is that the diagram

$$\begin{array}{ccc} R & \xrightarrow{s} & U \\ t \downarrow & & \downarrow \\ U & \longrightarrow & M \end{array}$$

is commutative, i.e., the morphism  $U \rightarrow M$  equalizes  $s, t$ . Moreover, if  $T$  is any affine scheme, and if  $U \rightarrow T$  is a morphism which equalizes  $s, t$ , then  $U \rightarrow T$  factors through  $U \rightarrow M$ . In other words,  $U \rightarrow M$  is a coequalizer in the category of affine schemes.

We would like to find conditions that guarantee the morphism  $U \rightarrow M$  is really a “quotient” in the category of schemes. We will discuss this at length elsewhere (insert future reference here); here we just discuss some special cases. Namely, we will focus on the case where  $s, t$  are finite locally free.

**Example 21.1.** Let  $k$  be a field. Let  $U = \text{GL}_{2,k}$ . Let  $B \subset \text{GL}_2$  be the closed subgroup scheme of upper triangular matrices. Then the quotient sheaf  $\text{GL}_{2,k}/B$  (in the Zariski, étale or fppf topology, see Definition 18.1) is representable by the projective line:  $\mathbf{P}^1 = \text{GL}_{2,k}/B$ . (Details omitted.) On the other hand, the ring of invariant functions in this case is just  $k$ . Note that in this case the morphisms  $s, t : R = \text{GL}_{2,k} \times_k B \rightarrow \text{GL}_{2,k} = U$  are smooth of relative dimension 3.

Recall that in Exercises, Exercises 15.6 and 15.7 we have defined the determinant and the norm for finitely locally free modules and finite locally free ring extensions. If  $\varphi : A \rightarrow B$  is a finite locally free ring map, then we will denote  $\text{Norm}_\varphi(b) \in A$  the norm of  $b \in B$ .

**Lemma 21.2.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $U = \text{Spec}(A)$ , and  $R = \text{Spec}(B)$  are affine, and  $s, t : R \rightarrow U$  finite locally free. Let  $C$  be as in (21.0.1). Let  $f \in A$ . Then  $\text{Norm}_{s^\#}(t^\#(f)) \in C$ .*

**Proof.** Consider the commutative diagram

$$\begin{array}{ccccc} & & U & & \\ & t \nearrow & & \nwarrow t & \\ R & \xleftarrow{\text{pr}_0} & R \times_{s,U,t} R & \xrightarrow{c} & R \\ s \downarrow & & \downarrow \text{pr}_1 & & \downarrow s \\ U & \xleftarrow{t} & R & \xrightarrow{s} & U \end{array}$$

of Lemma 11.4. Think of  $f \in \Gamma(U, \mathcal{O}_U)$ . The commutativity of the top part of the diagram shows that  $pr_0^\#(t^\#(f)) = c^\#(t^\#(f))$  as elements of  $\Gamma(R \times_{S, U, t} R, \mathcal{O})$ . Looking at the right lower cartesian square the compatibility of the norm construction with base change shows that  $s^\#(\text{Norm}_{s^\#}(t^\#(f))) = \text{Norm}_{pr_1}(c^\#(t^\#(f)))$ . Similarly we get  $t^\#(\text{Norm}_{s^\#}(t^\#(f))) = \text{Norm}_{pr_1}(pr_0^\#(t^\#(f)))$ . Hence by the first equality of this proof we see that  $s^\#(\text{Norm}_{s^\#}(t^\#(f))) = t^\#(\text{Norm}_{s^\#}(t^\#(f)))$  as desired.  $\square$

**Lemma 21.3.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $s, t : R \rightarrow U$  finite locally free. Then*

$$U = \coprod_{r \geq 1} U_r$$

*is a disjoint union of  $R$ -invariant opens such that the restriction  $R_r$  of  $R$  to  $U_r$  has the property that  $s, t : R_r \rightarrow U_r$  are finite locally free of rank 1.*

**Proof.** By Morphisms, Lemma 46.5 there exists a decomposition  $U = \coprod_{r \geq 0} U_r$  such that  $s : s^{-1}(U_r) \rightarrow U_r$  is finite locally free of rank  $r$ . As  $s$  is surjective we see that  $U_0 = \emptyset$ . Note that  $u \in U_r \Leftrightarrow$  the scheme theoretic fibre  $s^{-1}(u)$  has degree  $r$  over  $\kappa(u)$ . Now, if  $z \in R$  with  $s(z) = u$  and  $t(z) = u'$  then  $pr_1^{-1}(z)$  see diagram of Lemma 11.4 is a scheme over  $\kappa(z)$  which is the base change of both  $s^{-1}(u)$  and  $s^{-1}(u')$  via  $\kappa(u) \rightarrow \kappa(z)$  and  $\kappa(u') \rightarrow \kappa(z)$  by the properties of that diagram. Hence we see that the open subsets  $U_r$  are  $R$ -invariant. In particular the restriction of  $R$  to  $U_r$  is just  $s^{-1}(U_r)$  and  $s : R_r \rightarrow U_r$  is finite locally free of rank  $r$ . As  $t : R_r \rightarrow U_r$  is isomorphic to  $s$  by the inverse of  $R_r$ , we see that it has also rank  $r$ .  $\square$

**Lemma 21.4.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $U = \text{Spec}(A)$ , and  $R = \text{Spec}(B)$  are affine, and  $s, t : R \rightarrow U$  finite locally free. Let  $C \subset A$  be as in (21.0.1). Then  $A$  is integral over  $C$ .*

**Proof.** First, by Lemma 21.3 we know that  $(U, R, s, t, c)$  is a disjoint union of groupoid schemes  $(U_r, R_r, s, t, c)$  such that each  $s, t : R_r \rightarrow U_r$  has constant rank  $r$ . As  $U$  is quasi-compact, we have  $U_r = \emptyset$  for almost all  $r$ . It suffices to prove the lemma for each  $(U_r, R_r, s, t, c)$  and hence we may assume that  $s, t$  are finite locally free of rank  $r$ .

Assume that  $s, t$  are finite locally free of rank  $r$ . Let  $f \in A$ . Consider the element  $x - f \in A[x]$ , where we think of  $x$  as the coordinate on  $\mathbf{A}^1$ . Since

$$(U \times \mathbf{A}^1, R \times \mathbf{A}^1, s \times \text{id}_{\mathbf{A}^1}, t \times \text{id}_{\mathbf{A}^1}, c \times \text{id}_{\mathbf{A}^1})$$

is also a groupoid scheme with finite source and target, we may apply Lemma 21.2 to it and we see that  $P(x) = \text{Norm}_{s^\#}(t^\#(x - f))$  is an element of  $C[x]$ . Because  $s^\# : A \rightarrow B$  is finite locally free of rank  $r$  we see that  $P$  is monic of degree  $r$ . Moreover  $P(f) = 0$  by Cayley-Hamilton (Algebra, Lemma 15.1).  $\square$

**Lemma 21.5.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $U = \text{Spec}(A)$ , and  $R = \text{Spec}(B)$  are affine, and  $s, t : R \rightarrow U$  finite locally free. Let  $C \subset A$  be as in (21.0.1). Let  $C \rightarrow C'$  be a ring map, and set  $U' = \text{Spec}(A \otimes_C C')$ ,  $R' = \text{Spec}(B \otimes_C C')$ . Then*

- (1) *the maps  $s, t, c$  induce maps  $s', t', c'$  such that  $(U', R', s', t', c')$  is a groupoid scheme, and*
- (2) *there is a canonical map  $\varphi : C' \rightarrow C^1$  of  $C'$  into the  $R'$ -invariant functions  $C^1$  on  $U'$  with the properties*



- (a) for every  $f \in C^1$  there exists an  $n > 0$  such that  $f^n$  is in the image of  $\varphi$ , and
- (b) for every  $f \in \text{Ker}(\varphi)$  there exists an  $n > 0$  such that  $f^n = 0$ .
- (3) if  $C \rightarrow C'$  is flat then  $\varphi$  is an isomorphism.

**Proof.** The proof of part (1) is omitted. Let us denote  $A' = A \otimes_C C'$  and  $B' = B \otimes_C C'$ . Then we have

$$C^1 = \{x \in A' \mid (t')^\sharp(x) = (s')^\sharp(x)\} = \{a \in A \otimes_C C' \mid t^\sharp \otimes 1(x) = s^\sharp \otimes 1(x)\}.$$

In other words,  $C^1$  is the kernel of the difference map  $(t^\sharp - s^\sharp) \otimes 1$  which is just the base change of the  $C$ -linear map  $t^\sharp - s^\sharp : A \rightarrow B$  by  $C \rightarrow C'$ . Hence (3) follows.

Proof of part (2)(b). Since  $C \rightarrow A$  is integral (Lemma 21.4) and injective we see that  $\text{Spec}(A) \rightarrow \text{Spec}(C)$  is surjective, see Algebra, Lemma 35.15. Thus also  $\text{Spec}(A') \rightarrow \text{Spec}(C')$  is surjective as a base change of a surjective morphism (Morphisms, Lemma 11.4). Hence  $\text{Spec}(C^1) \rightarrow \text{Spec}(C')$  is surjective also. This implies that the kernel of  $\varphi$  is contained in the radical of the ring  $C'$ , i.e., (2)(b) holds.

Proof of part (2)(a). By Lemma 21.3 we know that  $A$  is a finite product of rings  $A_r$  and  $B$  is a finite product of rings  $B_r$  such that the groupoid scheme decomposes accordingly (see the proof of Lemma 21.4). Then also  $C$  is a product of rings  $C_r$  and correspondingly  $C'$  decomposes as a product. Hence we may and do assume that the ring maps  $s^\sharp, t^\sharp : A \rightarrow B$  are finite locally free of a fixed rank  $r$ . Let  $f \in C^1 \subset A' = A \otimes_C C'$ . We may replace  $C'$  by a finitely generated  $C$ -subalgebra of  $C'$  and hence we may assume that  $C' = C[X_1, \dots, X_n]/I$  for some ideal  $I$ . Choose a lift  $\tilde{f} \in A \otimes_C C[X_i] = A[X_i]$  of the element  $f$ . Note that  $f^r = \text{Norm}_{(s')^\sharp}((t')^\sharp(f))$  in  $A$  as  $t^\sharp(f) = s^\sharp(f)$ . Hence we see that

$$h = \text{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(f)) \in A[X_i]$$

is invariant according to Lemma 21.2 and maps to  $f^r$  in  $A'$ . Since  $C \rightarrow C[X_i]$  is flat we see from (3) that  $h \in C[X_i]$ . Hence it follows that  $f^r$  is in the image of  $\varphi$ .  $\square$

**Lemma 21.6.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $U = \text{Spec}(A)$ , and  $R = \text{Spec}(B)$  are affine, and  $s, t : R \rightarrow U$  finite locally free. Let  $C \subset A$  be as in (21.0.1). Then  $U \rightarrow M = \text{Spec}(C)$  has the following properties:*

- (1) *the map on points  $|U| \rightarrow |M|$  is surjective and  $u_0, u_1 \in |U|$  map to the same point if and only if there exists a  $r \in |R|$  with  $t(r) = u_0$  and  $s(r) = u_1$ , in a formula*

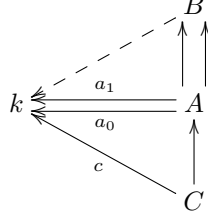
$$|M| = |U|/|R|$$

- (2) *for any algebraically closed field  $k$  we have*

$$M(k) = U(k)/R(k)$$

**Proof.** Let  $k$  be an algebraically closed field. Since  $C \rightarrow A$  is integral (Lemma 21.4) and injective we see that  $\text{Spec}(A) \rightarrow \text{Spec}(C)$  is surjective, see Algebra, Lemma 35.15. Thus  $|M| \rightarrow |U|$  is surjective. Let  $C \rightarrow k$  be a ring map. Since surjective morphisms are preserved under base change (Morphisms, Lemma 11.4) we see that  $A \otimes_C k$  is not zero. Now  $k \subset A \otimes_C k$  is a nonzero integral extension. Hence any residue field of  $A \otimes_C k$  is an algebraic extension of  $k$ , hence equal to  $k$ . Thus we see that  $U(k) \rightarrow M(k)$  is surjective.

Let  $a_0, a_1 : A \rightarrow k$  be ring maps. If there exists a ring map  $b : B \rightarrow k$  such that  $a_0 = b \circ t^\sharp$  and  $a_1 = b \circ s^\sharp$  then we see that  $a_0|_C = a_1|_C$  by definition. Conversely, suppose that  $a_0|_C = a_1|_C$ . Let us name this algebra map  $c : C \rightarrow k$ . Consider the diagram



We are trying to construct the dotted arrow, and if we do then part (2) follows, which in turn implies part (1). Since  $A \rightarrow B$  is finite and faithfully flat there exist finitely many ring maps  $b_1, \dots, b_n : B \rightarrow k$  such that  $b_i \circ s^\sharp = a_1$ . If the dotted arrow does not exist, then we see that none of the  $a'_i = b_i \circ t^\sharp$ ,  $i = 1, \dots, n$  is equal to  $a_0$ . Hence the maximal ideals

$$\mathfrak{m}'_i = \text{Ker}(a'_i \otimes 1 : A \otimes_C k \rightarrow k)$$

of  $A \otimes_C k$  are distinct from  $\mathfrak{m} = \text{Ker}(a_0 \otimes 1 : A \otimes_C k \rightarrow k)$ . By Algebra, Lemma 14.2 we would get an element  $f \in A \otimes_C k$  with  $f \in \mathfrak{m}$ , but  $f \notin \mathfrak{m}'_i$  for  $i = 1, \dots, n$ . Consider the norm

$$g = \text{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(f)) \in A \otimes_C k$$

By Lemma 21.2 this lies in the invariants  $C^1 \subset A \otimes_C k$  of the base change groupoid (base change via the map  $c : C \rightarrow k$ ). On the one hand,  $a_1(g) \in k^*$  since the value of  $t^\sharp(f)$  at all the points (which correspond to  $b_1, \dots, b_n$ ) lying over  $a_1$  is invertible (insert future reference on property determinant here). On the other hand, since  $f \in \mathfrak{m}$ , we see that  $f$  is not a unit, hence  $t^\sharp(f)$  is not a unit (as  $t^\sharp \otimes 1$  is faithfully flat), hence its norm is not a unit (insert future reference on property determinant here). We conclude that  $C^1$  contains an element which is not nilpotent and not a unit. We will now show that this leads to a contradiction. Namely, apply Lemma 21.5 to the map  $c : C \rightarrow C' = k$ , then we see that the map of  $k$  into the invariants  $C^1$  is injective and moreover, that for any element  $x \in C^1$  there exists an integer  $n > 0$  such that  $x^n \in k$ . Hence every element of  $C^1$  is either a unit or nilpotent.  $\square$

**Lemma 21.7.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume*

- (1)  $U = \text{Spec}(A)$ , and  $R = \text{Spec}(B)$  are affine, and
- (2) there exist elements  $x_i \in A$ ,  $i \in I$  such that  $B = \bigoplus_{i \in I} s^\sharp(A)t^\sharp(x_i)$ .

*Then  $A = \bigoplus_{i \in I} Cx_i$ , and  $B \cong A \otimes_C A$  where  $C \subset A$  is the  $R$ -invariant functions on  $U$  as in (21.0.1).*

**Proof.** During this proof we will write  $s, t : A \rightarrow B$  instead of  $s^\sharp, t^\sharp$ , and similarly  $c : B \rightarrow B \otimes_{s, A, t} B$ . We write  $p_0 : B \rightarrow B \otimes_{s, A, t} B$ ,  $b \mapsto b \otimes 1$  and  $p_1 : B \rightarrow B \otimes_{s, A, t} B$ ,  $b \mapsto 1 \otimes b$ . By Lemma 11.5 and the definition of  $C$  we have the

following commutative diagram

$$\begin{array}{ccccc}
 B \otimes_{s,A,t} B & \xleftarrow{c} & B & \xleftarrow{t} & A \\
 \uparrow p_1 & & \uparrow s & & \uparrow \\
 B & \xleftarrow{s} & A & \xleftarrow{t} & C
 \end{array}$$

Moreover the two left squares are cocartesian in the category of rings, and the top row is isomorphic to the diagram

$$B \otimes_{t,A,t} B \xleftarrow[p_0]{p_1} B \xleftarrow{t} A$$

which is an equalizer diagram according to Descent, Lemma 3.6 because condition (2) implies in particular that  $s$  (and hence also then isomorphic arrow  $t$ ) is faithfully flat. The lower row is an equalizer diagram by definition of  $C$ . We can use the  $x_i$  and get a commutative diagram

$$\begin{array}{ccccc}
 B \otimes_{s,A,t} B & \xleftarrow{c} & B & \xleftarrow{t} & A \\
 \uparrow p_1 & & \uparrow s & & \uparrow \\
 \bigoplus_{i \in I} Bx_i & \xleftarrow[s]{t} & \bigoplus_{i \in I} Ax_i & \xleftarrow{} & \bigoplus_{i \in I} Cx_i
 \end{array}$$

where in the right vertical arrow we map  $x_i$  to  $x_i$ , in the middle vertical arrow we map  $x_i$  to  $t(x_i)$  and in the left vertical arrow we map  $x_i$  to  $c(t(x_i)) = t(x_i) \otimes 1 = p_0(t(x_i))$  (equality by the commutativity of the top part of the diagram in Lemma 11.4). Then the diagram commutes. Moreover the middle vertical arrow is an isomorphism by assumption. Since the left two squares are cocartesian we conclude that also the left vertical arrow is an isomorphism. On the other hand, the horizontal rows are exact (i.e., they are equalizers). Hence we conclude that also the right vertical arrow is an isomorphism.  $\square$

**Proposition 21.8.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume*

- (1)  $U = \text{Spec}(A)$ , and  $R = \text{Spec}(B)$  are affine,
- (2)  $s, t : R \rightarrow U$  finite locally free, and
- (3)  $j = (t, s)$  is an equivalence.

*In this case, let  $C \subset A$  be as in (21.0.1). Then  $U \rightarrow M = \text{Spec}(C)$  is finite locally free and  $R = U \times_M U$ . Moreover,  $M$  represents the quotient sheaf  $U/R$  in the fppf topology (see Definition 18.1).*

**Proof.** During this proof we use the notation  $s, t : A \rightarrow B$  instead of the notation  $s^\sharp, t^\sharp$ . By Lemma 18.3 it suffices to show that  $C \rightarrow A$  is finite locally free and that the map

$$t \otimes s : A \otimes_C A \longrightarrow B$$

is an isomorphism. First, note that  $j$  is a monomorphism, and also finite (since already  $s$  and  $t$  are finite). Hence we see that  $j$  is a closed immersion by Morphisms, Lemma 44.13. Hence  $A \otimes_C A \rightarrow B$  is surjective.

We will perform base change by flat ring maps  $C \rightarrow C'$  as in Lemma 21.5, and we will use that formation of invariants commutes with flat base change, see part (3) of the lemma cited. We will show below that for every prime  $\mathfrak{p} \subset C$ , there exists a local flat ring map  $C_{\mathfrak{p}} \rightarrow C'_{\mathfrak{p}}$  such that the result holds after a base change to  $C'_{\mathfrak{p}}$ . This implies immediately that  $A \otimes_C C \rightarrow B$  is injective (use Algebra, Lemma 23.1). It also implies that  $C \rightarrow A$  is flat, by combining Algebra, Lemmas 38.16, 38.19, and 38.7. Then since  $U \rightarrow \text{Spec}(C)$  is surjective also (Lemma 21.6) we conclude that  $C \rightarrow A$  is faithfully flat. Then the isomorphism  $B \cong A \otimes_C C$  implies that  $A$  is a finitely presented  $C$ -module, see Algebra, Lemma 80.2. Hence  $A$  is finite locally free over  $C$ , see Algebra, Lemma 75.2.

By Lemma 21.3 we know that  $A$  is a finite product of rings  $A_r$  and  $B$  is a finite product of rings  $B_r$  such that the groupoid scheme decomposes accordingly (see the proof of Lemma 21.4). Then also  $C$  is a product of rings  $C_r$  and correspondingly  $C'$  decomposes as a product. Hence we may and do assume that the ring maps  $s, t : A \rightarrow B$  are finite locally free of a fixed rank  $r$ .

The local ring maps  $C_{\mathfrak{p}} \rightarrow C'_{\mathfrak{p}}$  we are going to use are any local flat ring maps such that the residue field of  $C'_{\mathfrak{p}}$  is infinite. By Algebra, Lemma 148.1 such local ring maps exist.

Assume  $C$  is a local ring with maximal ideal  $\mathfrak{m}$  and infinite residue field, and assume that  $s, t : A \rightarrow B$  is finite locally free of constant rank  $r > 0$ . Since  $C \subset A$  is integral (Lemma 21.4) all primes lying over  $\mathfrak{m}$  are maximal, and all maximal ideals of  $A$  lie over  $\mathfrak{m}$ . Similarly for  $C \subset B$ . Pick a maximal ideal  $\mathfrak{m}'$  of  $A$  lying over  $\mathfrak{m}$  (exists by Lemma 21.6). Since  $t : A \rightarrow B$  is finite locally free there exist at most finitely many maximal ideals of  $B$  lying over  $\mathfrak{m}'$ . Hence we conclude (by Lemma 21.6 again) that  $A$  has finitely many maximal ideals, i.e.,  $A$  is semi-local. This in turn implies that  $B$  is semi-local as well. OK, and now, because  $t \otimes s : A \otimes_C A \rightarrow B$  is surjective, we can apply Algebra, Lemma 75.7 to the ring map  $C \rightarrow A$ , the  $A$ -module  $M = B$  (seen as an  $A$ -module via  $t$ ) and the  $C$ -submodule  $s(A) \subset B$ . This lemma implies that there exist  $x_1, \dots, x_r \in A$  such that  $M$  is free over  $A$  on the basis  $s(x_1), \dots, s(x_r)$ . Hence we conclude that  $C \rightarrow A$  is finite free and  $B \cong A \otimes_C A$  by applying Lemma 21.7.  $\square$

## 22. Finite flat groupoids

In this section we prove a lemma that will help to show that the quotient of a scheme by a finite flat equivalence relation is a scheme, provided that each equivalence class is contained in an affine. See Properties of Spaces, Proposition 11.1.

**Lemma 22.1.** *Let  $S$  be a scheme. Let  $(U, R, s, t, c)$  be a groupoid scheme over  $S$ . Assume  $s, t$  are finite locally free. Let  $u \in U$  be a point such that  $t(s^{-1}(\{u\}))$  is contained in an affine open of  $U$ . Then there exists an  $R$ -invariant affine open neighbourhood of  $u$  in  $U$ .*

**Proof.** Since  $s$  is finite locally free it has finite fibres. Hence  $t(s^{-1}(\{u\})) = \{u_1, \dots, u_n\}$  is a finite set. Note that  $u \in \{u_1, \dots, u_n\}$ . Let  $W \subset U$  be an affine open containing  $\{u_1, \dots, u_n\}$ , in particular  $u \in W$ . Consider  $Z = R \setminus s^{-1}(W) \cap t^{-1}(W)$ . This is a closed subset of  $R$ . The image  $t(Z)$  is a closed subset of  $U$  which can be loosely described as the set of points of  $U$  which are  $R$ -equivalent to a point

of  $U \setminus W$ . Hence  $W' = U \setminus t(Z)$  is an  $R$ -invariant, open subscheme of  $U$  contained in  $W$ , and  $\{u_1, \dots, u_n\} \subset W'$ . Picture

$$\{u_1, \dots, u_n\} \subset W' \subset W \subset U.$$

Let  $f \in \Gamma(W, \mathcal{O}_W)$  be an element such that  $\{u_1, \dots, u_n\} \subset D(f) \subset W'$ . Such an  $f$  exists by Algebra, Lemma 14.2. By our choice of  $W'$  we have  $s^{-1}(W') \subset t^{-1}(W)$ , and hence we get a diagram

$$\begin{array}{ccc} s^{-1}(W') & \xrightarrow{t} & W \\ s \downarrow & & \\ W' & & \end{array}$$

The vertical arrow is finite locally free by assumption. Set

$$g = \text{Norm}_s(t^\# f) \in \Gamma(W', \mathcal{O}_{W'})$$

By construction  $g$  is a function on  $W'$  which is nonzero in  $u$ , as  $t^\#(f)$  is nonzero in each of the points of  $R$  lying over  $u$ , since  $f$  is nonzero in  $u_1, \dots, u_n$ . Similarly,  $D(g) \subset W'$  is equal to the set of points  $w$  such that  $f$  is not zero in any of the points equivalent to  $w$ . This means that  $D(g)$  is an  $R$ -invariant affine open of  $W'$ . The final picture is

$$\{u_1, \dots, u_n\} \subset D(g) \subset D(f) \subset W' \subset W \subset U$$

and hence we win. □

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## References

- [ABD<sup>+</sup>66] Michael Artin, Jean-Etienne Bertin, Michel Demazure, Alexander Grothendieck, Pierre Gabriel, Michel Raynaud, and Jean-Pierre Serre, *Schémas en groupes*, Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques, Institut des Hautes Études Scientifiques, Paris, 1963/1966.
- [KM97] Sean Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), 193–213.
- [Oor66] Frans Oort, *Algebraic group schemes in characteristic zero are reduced*, Invent. Math. **2** (1966), 79–80.
- [Per75] Daniel Perrin, *Schémas en groupes quasi-compacts sur un corps et groupes henséliens*, Publications Mathématiques d'Orsay **165**, **75-46** (1975), 148.
- [Per76] ———, *Approximation des schémas en groupes, quasi compacts sur un corps*, Bull. Soc. Math. France **104** (1976), no. 3, 323–335.