COHOMOLOGY ON SITES

Contents

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1. Introduction

In this document we work out some topics on cohomology of sheaves. We work out what happens for sheaves on sites, although often we will simply duplicate the discussion, the constructions, and the proofs from the topological case in the case. Basic references are [AGV71], [God73] and [Ive86].

2. Topics

Here are some topics that should be discussed in this chapter, and have not yet been written.

(1) Cohomology of a sheaf of modules on a site is the same as the cohomology of the underlying abelian sheaf.
(2) Hypercohomology on a site.
(3) Ext-groups.
(4) Ext sheaves.
(5) Tor functors.
(6) Higher direct images for a morphism of sites.
(7) Derived pullback for morphisms between ringed sites.
(8) Cup-product.
(9) Group cohomology.
(10) Comparison of group cohomology and cohomology on $\mathcal{T}_G$.
(11) Cech cohomology on sites.
(12) Cech to cohomology spectral sequence on sites.
(13) Leray Spectral sequence for a morphism between ringed sites.
(14) Etc, etc, etc.

3. Cohomology of sheaves

Let $\mathcal{C}$ be a site, see Sites, Definition 6.2. Let $\mathcal{F}$ be a abelian sheaf on $\mathcal{C}$. We know that the category of abelian sheaves on $\mathcal{C}$ has enough injectives, see Injectives, Theorem 7.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

$$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet)) \tag{3.0.1}$$

to be the *ith cohomology group of the abelian sheaf $\mathcal{F}$ over the object U*. In other words, these are the right derived functors of the functor $\mathcal{F} \mapsto \mathcal{F}(U)$. The family of functors $H^i(U, -)$ forms a universal δ-functor $Ab(\mathcal{C}) \to Ab$.

It sometimes happens that the site $\mathcal{C}$ does not have a final object. In this case we define the *global sections* of a presheaf of sets $\mathcal{F}$ over $\mathcal{C}$ to be the set

$$\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(e, \mathcal{F}) \tag{3.0.2}$$

where e is a final object in the category of presheaves on $\mathcal{C}$. In this case, given an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$, we define the *ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$* as follows

$$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet)) \tag{3.0.3}$$

in other words, it is the ith right derived functor of the global sections functor. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $Ab(\mathcal{C}) \to Ab$.

Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi, see Sites, Definition 16.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{3.0.4}$$

to be the *ith higher direct image of $\mathcal{F}$*. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $Ab(\mathcal{C}) \to Ab(\mathcal{D})$.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, see Modules on Sites, Definition 6.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. We know that the category of $\mathcal{O}$-modules has enough injectives, see Injectives, Theorem 8.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

$$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet)) \tag{3.0.5}$$

to be the *the ith cohomology group of $\mathcal{F}$ over U*. The family of functors $H^i(U, -)$ forms a universal δ-functor $Mod(\mathcal{O}) \to \text{Mod}_{\mathcal{O}(U)}$. Similarly

$$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet)) \tag{3.0.6}$$

it the *ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$*. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $Mod(\mathcal{C}) \to \text{Mod}_{\Gamma(\mathcal{C}, \mathcal{O})}$.

Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi, see Modules on Sites, Definition 7.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{3.0.7}$$

to be the *ith higher direct image of $\mathcal{F}$*. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $Mod(\mathcal{O}) \to Mod(\mathcal{O}')$.

4. Derived functors

We briefly explain an approach to right derived functors using resolution functors. Namely, suppose that $(\mathcal{C}, \mathcal{O})$ is a ringed site. In this chapter we will write

$$K(\mathcal{O}) = K(Mod(\mathcal{O})) \quad \text{and} \quad D(\mathcal{O}) = D(Mod(\mathcal{O}))$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 8.1 and Definition 11.3. By Derived Categories, Remark 24.3 there exists a resolution functor

$$j = j_{(\mathcal{C}, \mathcal{O})} : K^+(Mod(\mathcal{O})) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $Mod(\mathcal{O})$ which consists of injective $\mathcal{O}$-modules. For any left exact functor $F : Mod(\mathcal{O}) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote RF the right derived functor of Derived Categories, Section 20 constructed using the resolution functor j just described:

$$RF = F \circ j' : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{B}) \tag{4.0.8}$$

see Derived Categories, Lemma 25.1 for notation. Note that we may think of RF as defined on $Mod(\mathcal{O})$, $\text{Comp}^+(Mod(\mathcal{O}))$, or $K^+(\mathcal{O})$ depending on the situation.

According to Derived Categories, Definition 17.2 we obtain the $i$the right derived functor

$$R^iF = H^i \circ RF : \mathit{Mod}(\mathcal{O}) \longrightarrow \mathcal{B} \tag{4.0.9}$$

so that $R^0F = F$ and $\{R^iF, \delta\}_{i \geq 0}$ is universal δ-functor, see Derived Categories, Lemma 20.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\text{Mod}_R)$ and $D(R) = D(\text{Mod}_R)$ and similarly for the bounded versions. For any object U of $\mathcal{C}$ have a left exact functor $\Gamma(U, -) : \mathit{Mod}(\mathcal{O}) \longrightarrow \text{Mod}_{\mathcal{O}(U)}$ which gives rise to

$$R\Gamma(U, -) : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{O}(U))$$

by the discussion above. Note that $H^i(U, -) = R^i\Gamma(U, -)$ is compatible with (3.0.5) above. We similarly have

$$R\Gamma(\mathcal{C}, -) : D^+(\mathcal{O}) \longrightarrow D^+(\Gamma(\mathcal{C}, \mathcal{O}))$$

compatible with (3.0.6). If $f : (\mathit{Sh}(\mathcal{C}), \mathcal{O}) \to (\mathit{Sh}(\mathcal{D}), \mathcal{O}')$ is a morphism of ringed topoi then we get a left exact functor $f_* : \mathit{Mod}(\mathcal{O}) \to \mathit{Mod}(\mathcal{O}')$ which gives rise to *derived pushforward*

$$Rf_* : D^+(\mathcal{O}) \to D^+(\mathcal{O}')$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^if_*\mathcal{F}^\bullet$ and called the ith *higher direct image* in accordance with (3.0.7). The displayed functors above are exact functor of derived categories.

5. First cohomology and torsors

Definition 5.1. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A *pseudo torsor*, or more precisely a *pseudo $\mathcal{G}$-torsor*, is a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive.

A *morphism of pseudo $\mathcal{G}$-torsors* $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. A *torsor*, or more precisely a *$\mathcal{G}$-torsor*, is a pseudo G-torsor such that in addition

(2) for every $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i \in I}$ of U such that $\mathcal{F}(U_i)$ is nonempty for all $i \in I$.

A *morphism of G-torsors* is simply a morphism of pseudo G-torsors. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

Lemma 5.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\Gamma(\mathcal{C}, \mathcal{F}) \neq \emptyset$.*

Proof. Omitted. □

Lemma 5.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{H}$ be an abelian sheaf on $\mathcal{C}$. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(\mathcal{C}, \mathcal{H})$.*

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \in \mathrm{Ob}(\mathcal{C})$ the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by sections of the form $[s]-[s']$. There is a canonical map $a : \mathrm{Ker}(\sigma) \to \mathcal{H}$ which maps $[s] - [s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s = s'$. Consider the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\sigma) \arrow[r] \arrow[d, "a"] & \mathbf{Z}[\mathcal{F}] \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{E} \arrow[r] & \underline{\mathbf{Z}} \arrow[r] & 0
\end{tikzcd}$$

Here $\mathcal{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_{\mathcal{F}} \in H^1(\mathcal{C}, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(\mathcal{C}, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(\mathcal{C}, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(\mathcal{C}, \mathcal{Q})$ because $H^1(\mathcal{C}, \mathcal{I}) = 0$ (see Derived Categories, Lemma 20.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a $\mathcal{H}$-torsor.

We omit the verification that the two constructions given above are mutually inverse. $\square$

6. First cohomology and extensions

Lemma 6.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. There is a canonical bijection*

$$Ext^1_{Mod(\mathcal{O})}(\mathcal{O}, \mathcal{F}) \longrightarrow H^1(\mathcal{C}, \mathcal{F})$$

which associates to the extension

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$$

the image of $1 \in \Gamma(\mathcal{C}, \mathcal{O})$ in $H^1(\mathcal{C}, \mathcal{F})$.

Proof. Let us construct the inverse of the map given in the lemma. Let $\xi \in H^1(\mathcal{C}, \mathcal{F})$. Choose an injection $\mathcal{F} \subset \mathcal{I}$ with $\mathcal{I}$ injective in $Mod(\mathcal{O})$. Set $\mathcal{Q} = \mathcal{I}/\mathcal{F}$. By the long exact sequence of cohomology, we see that ξ is the image of of a section $\tilde{\xi} \in \Gamma(\mathcal{C}, \mathcal{Q}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{Q})$. Now, we just form the pullback

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, "\tilde{\xi}"] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{Q} \arrow[r] & 0
\end{tikzcd}$$

see Homology, Section 6. $\square$

The following lemma will be superseded by the more general Lemma 12.4.

Lemma 6.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $\mathcal{F}_{ab}$ denote the underlying sheaf of abelian groups. Then there is a functorial isomorphism*

$$H^1(\mathcal{C}, \mathcal{F}_{ab}) = H^1(\mathcal{C}, \mathcal{F})$$

where the left hand side is cohomology computed in $Ab(\mathcal{C})$ and the right hand side is cohomology computed in $Mod(\mathcal{O})$.

Proof. Let $\underline{\mathbf{Z}}$ denote the constant sheaf $\mathbf{Z}$. As $Ab(\mathcal{C}) = Mod(\underline{\mathbf{Z}})$ we may apply Lemma 6.1 twice, and it follows that we have to show

$$\mathrm{Ext}^1_{Mod(\mathcal{O})}(\mathcal{O}, \mathcal{F}) = \mathrm{Ext}^1_{Mod(\underline{\mathbf{Z}})}(\underline{\mathbf{Z}}, \mathcal{F}_{ab}).$$

Suppose that $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$ is an extension in $Mod(\mathcal{O})$. Then we can use the obvious map of abelian sheaves $1 : \underline{\mathbf{Z}} \to \mathcal{O}$ and pullback to obtain an extension $\mathcal{E}_{ab}$, like so:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_{ab} \arrow[r] \arrow[d, equal] & \mathcal{E}_{ab} \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O} \arrow[r] & 0 \end{tikzcd}$$

The converse is a little more fun. Suppose that $0 \to \mathcal{F}_{ab} \to \mathcal{E}_{ab} \to \underline{\mathbf{Z}} \to 0$ is an extension in $Mod(\underline{\mathbf{Z}})$. Since $\underline{\mathbf{Z}}$ is a flat $\underline{\mathbf{Z}}$-module we see that the sequence

$$0 \to \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to 0$$

is exact, see Modules on Sites, Lemma 28.7. Of course $\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} = \mathcal{O}$. Hence we can form the pushout via the ($\mathcal{O}$-linear) multiplication map $\mu : \mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{F}$ to get an extension of $\mathcal{O}$ by $\mathcal{F}$, like this

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \arrow[r] \arrow[d, "\mu"] & \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O} \arrow[r] & 0 \end{tikzcd}$$

which is the desired extension. We omit the verification that these constructions are mutually inverse. $\square$

7. First cohomology and invertible sheaves

The Picard group of a ringed site is defined in Modules on Sites, Section 31.

Lemma 7.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. There is a canonical isomorphism*

$$H^1(\mathcal{C}, \mathcal{O}^*) = Pic(\mathcal{O}).$$

of abelian groups.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s\cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally by the very definition of an invertible sheaf. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}^*$-torsor. Thus we get a map

$$\begin{matrix} \text{set of invertible sheaves on } (\mathcal{C}, \mathcal{O}) \\ \text{up to isomorphism} \end{matrix} \longrightarrow \begin{matrix} \text{set of } \mathcal{O}^*\text{-torsors} \\ \text{up to isomorphism} \end{matrix}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 5.3 the right hand side is canonically bijective to $H^1(\mathcal{C}, \mathcal{O}^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 5.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}$ is the neutral element in $\text{Pic}(\mathcal{O})$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}(U))/\mathcal{O}^*(U)$$

where the action of $f \in \mathcal{O}^*(U)$ on (s, g) is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}$-modules by setting $(s, g)+(s', g') = (s, g+(s'/s)g')$ where s'/s is the local section f of $\mathcal{O}^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for h a local section of $\mathcal{O}$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}$-module whose associated $\mathcal{O}^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. □

8. Locality of cohomology

The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an object of the site.

Lemma 8.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.*

1. *If $\mathcal{I}$ is an injective $\mathcal{O}$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*
2. *For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(\mathcal{C}/U, \mathcal{F}|_U)$.*

Proof. Recall that the functor j_U^{-1} of restriction to U is a right adjoint to the functor $j_{U!}$ of extension by 0, see Modules on Sites, Section 19. Moreover, $j_{U!}$ is exact. Hence (1) follows from Homology, Lemma 25.1.

By definition $H^p(U, \mathcal{F}) = H^p(\mathcal{I}^\bullet(U))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $\textit{Mod}(\mathcal{O})$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $\textit{Mod}(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\mathcal{I}^\bullet|_U(U))$. Of course $\mathcal{F}(U) = \mathcal{F}|_U(U)$ for any sheaf $\mathcal{F}$ on $\mathcal{C}$. Hence the equality in (2). □

The following lemma will be use to see what happens if we change a partial universe, or to compare cohomology of the small and big étale sites.

Lemma 8.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume u satisfies the hypotheses of Sites, Lemma 20.8. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. For any abelian sheaf $\mathcal{F}$ on $\mathcal{D}$ we have isomorphisms*

$$R\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = R\Gamma(\mathcal{D}, \mathcal{F}),$$

in particular $H^p(\mathcal{C}, g^{-1}\mathcal{F}) = H^p(\mathcal{D}, \mathcal{F})$ and for any $U \in \text{Ob}(\mathcal{C})$ we have isomorphisms

$$R\Gamma(U, g^{-1}\mathcal{F}) = R\Gamma(u(U), \mathcal{F}),$$

in particular $H^p(U, g^{-1}\mathcal{F}) = H^p(u(U), \mathcal{F})$. All of these isomorphisms are functorial in $\mathcal{F}$.

Proof. Since it is clear that $\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = \Gamma(\mathcal{D}, \mathcal{F})$ by hypothesis (e), it suffices to show that g^{-1} transforms injective abelian sheaves into injective abelian sheaves. As usual we use Homology, Lemma 25.1 to see this. The left adjoint to g^{-1} is $g_! = f^{-1}$ with the notation of Sites, Lemma 20.8 which is an exact functor. Hence the lemma does indeed apply. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\varphi : U \to V$ be a morphism of $\mathcal{O}$. Then there is a canonical *restriction mapping*

$$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U \tag{8.2.1}$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 11.5.

The following lemma says that it is possible to kill higher cohomology classes by going to a covering.

Lemma 8.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let U be an object of $\mathcal{C}$. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists a covering $\{U_i \to U\}$ of $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\mathrm{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \mathrm{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists a covering $\{U_i \to U\}$ of the site such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. $\square$

Lemma 8.4. *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. For any $\mathcal{F} \in \mathrm{Ob}(Mod(\mathcal{O}_\mathcal{C}))$ the sheaf $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto H^i(u(V), \mathcal{F})$$

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^if_*\mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_\mathcal{D}$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(u(V)) \to \mathcal{I}^{i+1}(u(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(u(V)) \to \mathcal{I}^i(u(V)))}$$

which is equal to $H^i(u(V), \mathcal{F})$ and we win. $\square$

9. The Cech complex and Cech cohomology

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target, see Sites, Definition 6.1. Assume that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in the factor $\mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p})$. We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_p}|_{U_{i_0} \times_U \ldots \times_U U_{i_{p+1}}} \tag{9.0.1}$$

where the restriction is via the projection map

$$U_{i_0} \times_U \ldots \times_U U_{i_{p+1}} \longrightarrow U_{i_0} \times_U \ldots \times_U \widehat{U_{i_j}} \times_U \ldots \times_U U_{i_{p+1}}.$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 9.1. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Cech complex* associated to $\mathcal{F}$ and the family $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Cech cohomology groups* of $\mathcal{F}$ with respect to $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

We observe that any covering $\{U_i \to U\}$ of a site $\mathcal{C}$ is a family of morphisms with fixed target to which the definition applies.

Lemma 9.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The following are equivalent*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{C}$ and*
2. *for every covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$ the natural map*
$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$
(see Sites, Section 10) is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every covering of $\mathcal{C}$. $\square$

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be another. Let $f : U \to V$, $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$ be a morphism of families of morphisms with fixed target, see Sites, Section 8. In this case we get a map of Cech complexes

$$\varphi : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \tag{9.2.1}$$

which in degree p is given by

$$\varphi(s)_{i_0\ldots i_p} = (f_{i_0} \times \ldots \times f_{i_p})^* s_{\alpha(i_0)\ldots\alpha(i_p)}$$

10. Cech cohomology as a functor on presheaves

Warning: In this section we work exclusively with abelian presheaves on a category. The results are completely wrong in the setting of sheaves and categories of sheaves!

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow \text{Comp}^+(Ab) \tag{10.0.2}$$

see Derived Categories, Definition 8.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 12.9.

Lemma 10.1. *The functor given by Equation (10.0.2) is an exact functor (see Homology, Lemma 7.1).*

Proof. For any object W of $\mathcal{C}$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $PAb(\mathcal{C})$ to Ab. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. □

Lemma 10.2. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category $PAb(\mathcal{C})$ to the category of $\mathbf{Z}$-modules (see Homology, Definition 11.1).*

Proof. By Lemma 10.1 a short exact sequence of abelian presheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathbf{Z}$-modules. Hence we can use Homology, Lemma 12.12 to get the boundary maps $\delta_{\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. □

Lemma 10.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Consider the chain complex $\mathbf{Z}_{\mathcal{U},\bullet}$ of abelian presheaves*

$$\ldots \to \bigoplus_{i_0i_1i_2} \mathbf{Z}_{U_{i_0}\times_U U_{i_1}\times_U U_{i_2}} \to \bigoplus_{i_0i_1} \mathbf{Z}_{U_{i_0}\times_U U_{i_1}} \to \bigoplus_{i_0} \mathbf{Z}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$\mathbf{Z}_{U_{i_0}\times_U\ldots\times_u U_{i_{p+1}}} \longrightarrow \mathbf{Z}_{U_{i_0}\times_U\ldots\widehat{U_{i_j}}\ldots\times_U U_{i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$\text{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U,\bullet}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in \text{Ob}(PAb(\mathcal{C}))$.

Proof. This is a tautology based on the fact that

$$\begin{aligned}\mathrm{Hom}_{PAb(\mathcal{C})}(\bigoplus_{i_0\ldots i_p} \mathbf{Z}_{U_{i_0}\times_U \ldots \times_U U_{i_p}}, \mathcal{F}) &= \prod_{i_0\ldots i_p} \mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0}\times_U \ldots \times_U U_{i_p}}, \mathcal{F}) \\ &= \prod_{i_0\ldots i_p} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p})\end{aligned}$$

see Modules on Sites, Lemma 4.2. □

Lemma 10.4. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The chain complex $\mathbf{Z}_{\mathcal{U},\bullet}$ of presheaves of Lemma 10.3 above is exact in positive degrees, i.e., the homology presheaves $H_i(\mathbf{Z}_{\mathcal{U},\bullet})$ are zero for $i > 0$.*

Proof. Let V be an object of $\mathcal{C}$. We have to show that the chain complex of abelian groups $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is exact in degrees > 0. This is the complex

$$\begin{array}{c} \ldots \\ \downarrow \\ \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[\mathrm{Mor}_\mathcal{C}(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \\ \downarrow \\ \bigoplus_{i_0 i_1} \mathbf{Z}[\mathrm{Mor}_\mathcal{C}(V, U_{i_0} \times_U U_{i_1})] \\ \downarrow \\ \bigoplus_{i_0} \mathbf{Z}[\mathrm{Mor}_\mathcal{C}(V, U_{i_0})] \\ \downarrow \\ 0 \end{array}$$

For any morphism $\varphi : V \to U$ denote $\mathrm{Mor}_\varphi(V, U_i) = \{\varphi_i : V \to U_i \mid f_i \circ \varphi_i = \varphi\}$. We will use a similar notation for $\mathrm{Mor}_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p})$. Note that composing with the various projection maps between the fibred products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ preserves these morphism sets. Hence we see that the complex above is the same as the complex

$$\begin{array}{c} \ldots \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0 i_1} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0} \times_U U_{i_1})] \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0})] \\ \downarrow \\ 0 \end{array}$$

Next, we make the remark that we have

$$\mathrm{Mor}_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p}) = \mathrm{Mor}_\varphi(V, U_{i_0}) \times \ldots \times \mathrm{Mor}_\varphi(V, U_{i_p})$$

Using this and the fact that $\mathbf{Z}[A] \oplus \mathbf{Z}[B] = \mathbf{Z}[A \coprod B]$ we see that the complex becomes

$$\begin{array}{c}
\ldots \\
\downarrow \\
\bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1 i_2} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_2})\right] \\
\downarrow \\
\bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_1})\right] \\
\downarrow \\
\bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0} \mathrm{Mor}_\varphi(V, U_{i_0})\right] \\
\downarrow \\
0
\end{array}$$

Finally, on setting $S_\varphi = \coprod_{i \in I} \mathrm{Mor}_\varphi(V, U_i)$ we see that we get

$$\bigoplus\nolimits_\varphi (\ldots \to \mathbf{Z}[S_\varphi \times S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi] \to 0 \to \ldots)$$

Thus we have simplified our task. Namely, it suffices to show that for any nonempty set S the (extended) complex of free abelian groups

$$\ldots \to \mathbf{Z}[S \times S \times S] \to \mathbf{Z}[S \times S] \to \mathbf{Z}[S] \xrightarrow{\Sigma} \mathbf{Z} \to 0 \to \ldots$$

is exact in all degrees. To see this fix an element $s \in S$, and use the homotopy

$$n_{(s_0, \ldots, s_p)} \longmapsto n_{(s, s_0, \ldots, s_p)}$$

with obvious notations. $\square$

Lemma 10.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The chain complex*

$$\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}$$

is exact in positive degrees. Here $\mathbf{Z}_{\mathcal{U},\bullet}$ is the cochain complex of Lemma 10.3, and the tensor product is over the constant presheaf of rings with value $\mathbf{Z}$.

Proof. Let V be an object of $\mathcal{C}$. In the proof of Lemma 10.4 we saw that $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathbf{Z}$ placed in degree zero. Hence also $\mathbf{Z}_{\mathcal{U},\bullet}(V) \otimes_{\mathbf{Z}} \mathcal{O}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathcal{O}(V)$ placed in degree zero. Or you can use Modules on Sites, Lemma 28.9, which applies since the presheaves $\mathbf{Z}_{\mathcal{U},i}$ are flat, and the proof of Lemma 10.4 shows that $H_0(\mathbf{Z}_{\mathcal{U},\bullet})$ is a flat presheaf also. $\square$

Lemma 10.6. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The Cech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow Ab.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(PAb(\mathcal{C})) \longrightarrow D^+(\mathbf{Z})$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of abelian presheaves has enough injectives, see Injectives, Proposition 6.1. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of abelian presheaves to the category of $\mathbf{Z}$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 20.

Let $\mathcal{I}$ be a injective abelian presheaf. In this case the functor $\mathrm{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$ is exact on $PAb(\mathcal{C})$. By Lemma 10.3 we have

$$\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 10.4 we have that $\mathbf{Z}_{\mathcal{U},\bullet}$ is exact in positive degrees. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 10.2) satisfies the assumptions of Homology, Lemma 11.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 20.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 11.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any abelian presheaf on $\mathcal{C}$. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $PAb(\mathcal{C})$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q).$$

Consider the simple complex $sA^\bullet$ associated to this double complex. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to A^{p,0} = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to A^{0,q} = \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 22.7. Namely, the columns of the double complex are exact in positive degrees because the Cech complex as a functor is exact (Lemma 10.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Cech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathbf{Z})$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. $\square$

11. Cech cohomology and cohomology

The relationship between cohomology and Cech cohomology comes from the fact that the Cech cohomology of an injective abelian sheaf is zero. To see this we note that an injective abelian sheaf is an injective abelian presheaf and then we apply results in Cech cohomology in the preceding section.

Lemma 11.1. *Let $\mathcal{C}$ be a site. An injective abelian sheaf is also injective as an object in the category $PAb(\mathcal{C})$.*

Proof. Apply Homology, Lemma 25.1 to the categories $\mathcal{A} = Ab(\mathcal{C})$, $\mathcal{B} = PAb(\mathcal{C})$, the inclusion functor and sheafification. (See Modules on Sites, Section 3 to see that all assumptions of the lemma are satisfied.) □

Lemma 11.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective abelian sheaf, i.e., an injective object of $Ab(\mathcal{C})$. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

Proof. By Lemma 11.1 we see that $\mathcal{I}$ is an injective object in $PAb(\mathcal{C})$. Hence we can apply Lemma 10.6 (or its proof) to see the vanishing of higher Cech cohomology group. For the zeroth see Lemma 9.2. □

Lemma 11.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $Ab(\mathcal{C}) \to D^+(\mathbf{Z})$. In particular this gives a transformation of functors $\check{H}^p(U, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $Ab(\mathcal{C})$.

Proof. Let $\mathcal{F}$ be an abelian sheaf. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $A^{\bullet,\bullet}$ with terms $A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Moreover, consider the associated simple complex $sA^\bullet$, see Homology, Definition 22.3. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 22.7 to see that α is a quasi-isomorphism. Namely, Lemma 11.2 implies that the qth row of the double complex $A^{\bullet,\bullet}$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathbf{Z})$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

Lemma 11.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be an abelian sheaf on $\mathcal{C}$. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. The map*

$$\check{H}^1(\mathcal{U}, \mathcal{G}) \longrightarrow H^1(U, \mathcal{G})$$

is injective and identifies $\check{H}^1(\mathcal{U}, \mathcal{G})$ via the bijection of Lemma 5.3 with the set of isomorphism classes of $\mathcal{G}|_U$-torsors which restrict to trivial torsors over each U_i.

Proof. To see this we construct an inverse map. Namely, let $\mathcal{F}$ be a $\mathcal{G}|_U$-torsor on $\mathcal{C}/U$ whose restriction to $\mathcal{C}/U_i$ is trivial. By Lemma 5.2 this means there exists a section $s_i \in \mathcal{F}(U_i)$. On $U_{i_0} \times_U U_{i_1}$ there is a unique section $s_{i_0i_1}$ of $\mathcal{G}$ such that $s_{i_0i_1} \cdot s_{i_0}|_{U_{i_0} \times_U U_{i_1}} = s_{i_1}|_{U_{i_0} \times_U U_{i_1}}$. An easy computation shows that $s_{i_0i_1}$ is a Čech cocycle and that its class is well defined (i.e., does not depend on the choice of the sections s_i). The inverse maps the isomorphism class of $\mathcal{F}$ to the cohomology class of the cocycle $(s_{i_0i_1})$.

We omit the verification that this map is indeed an inverse. □

Lemma 11.5. *Let $\mathcal{C}$ be a site. Consider the functor $i : Ab(\mathcal{C}) \to PAb(\mathcal{C})$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 8.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an object U of $\mathcal{C}$ are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

Lemma 11.6. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. For any abelian sheaf $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 22.2) for the functors

$$i : Ab(\mathcal{C}) \to PAb(\mathcal{C}) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \to Ab.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 9.2. We have that $i(\mathcal{I})$ is Cech acyclic by Lemma 11.2. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $PAb(\mathcal{C})$ by Lemma 10.6. Putting everything together gives the lemma. □

Lemma 11.7. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Assume that $H^i(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$.*

Proof. We will use the spectral sequence of Lemma 11.6. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

Lemma 11.8. *Let $\mathcal{C}$ be a site. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let U be an object of $\mathcal{C}$. If there exists a cofinal system of coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0} \times_U U_{i_1}} - s_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0} \times_U U_{i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0} \times_U U_{i_1}} - t_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. ☐

Lemma 11.9. *(Variant of Cohomology, Lemma 12.7.) Let $\mathcal{C}$ be a site. Let $Cov_\mathcal{C}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 6.2). Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$, and $Cov \subset Cov_\mathcal{C}$ be subsets. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Assume that*

(1) *For every $\mathcal{U} \in Cov$, $\mathcal{U} = \{U_i \to U\}_{i \in I}$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
(2) *For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.*
(3) *For every $\mathcal{U} \in Cov$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Cech cohomology for any $\mathcal{U} \in$ Cov". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective abelian sheaf. By Lemma 11.2 $\mathcal{I}$ has vanishing higher Cech cohomology for any $\mathcal{U} \in$ Cov. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 11.8 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in$ Cov we get a short exact sequence of Cech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Cech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Cech cohomology groups for any covering $\mathcal{U} \in$ Cov. This implies that $\mathcal{Q}$ is also an abelian sheaf with vanishing higher Cech cohomology for all $\mathcal{U} \in$ Cov.

Next, we look at the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \to & H^0(U, \mathcal{F}) & \to & H^0(U, \mathcal{I}) & \to & H^0(U, \mathcal{Q}) \\ & \swarrow & & & & & \\ & & H^1(U, \mathcal{F}) & \to & H^1(U, \mathcal{I}) & \to & H^1(U, \mathcal{Q}) \\ & \swarrow & & & & & \\ & & \ldots & & \ldots & & \ldots \end{array}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary abelian sheaf with vanishing higher Cech cohomology for all $\mathcal{U} \in \text{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

12. Cohomology of modules

Everything that was said for cohomology of abelian sheaves goes for cohomology of modules, since the two agree.

Lemma 12.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. An injective sheaf of modules is also injective as an object in the category $PMod(\mathcal{O})$.*

Proof. Apply Homology, Lemma 25.1 to the categories $\mathcal{A} = Mod(\mathcal{O})$, $\mathcal{B} = PMod(\mathcal{O})$, the inclusion functor and sheafification. (See Modules on Sites, Section 11 to see that all assumptions of the lemma are satisfied.) □

Lemma 12.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the functor $i : Mod(\mathcal{C}) \to PMod(\mathcal{C})$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 8.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Mod(\mathcal{O})$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an object U of $\mathcal{C}$ are given by

$$\frac{\text{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\text{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

Lemma 12.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective $\mathcal{O}$-module, i.e., an injective object of $Mod(\mathcal{O})$. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

Proof. Lemma 10.3 gives the first equality in the following sequence of equalities

$$\begin{aligned} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) &= \text{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= \text{Mor}_{PMod(\mathbf{Z})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= \text{Mor}_{PMod(\mathcal{O})}(\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}, \mathcal{I}) \end{aligned}$$

The third equality by Modules on Sites, Lemma 9.2. By Lemma 12.1 we see that $\mathcal{I}$ is an injective object in $PMod(\mathcal{O})$. Hence $\text{Hom}_{PMod(\mathcal{O})}(-, \mathcal{I})$ is an exact functor. By Lemma 10.5 we see the vanishing of higher Cech cohomology groups. For the zeroth see Lemma 9.2. □

Lemma 12.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module, and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then we have*

$$H^i(\mathcal{C}, \mathcal{F}_{ab}) = H^i(\mathcal{C}, \mathcal{F})$$

and for any object U of $\mathcal{C}$ we also have

$$H^i(U, \mathcal{F}_{ab}) = H^i(U, \mathcal{F}).$$

Here the left hand side is cohomology computed in $Ab(\mathcal{C})$ and the right hand side is cohomology computed in $Mod(\mathcal{O})$.

Proof. By Derived Categories, Lemma 20.4 the δ-functor $(\mathcal{F} \mapsto H^p(U, \mathcal{F}))_{p \geq 0}$ is universal. The functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}_{ab}$ is exact. Hence $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is a δ-functor also. Suppose we show that $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is also universal. This will imply the second statement of the lemma by uniqueness of universal δ-functors, see Homology, Lemma 11.5. Since $Mod(\mathcal{O})$ has enough injectives, it suffices to show that $H^i(U, \mathcal{I}_{ab}) = 0$ for any injective object $\mathcal{I}$ in $Mod(\mathcal{O})$, see Homology, Lemma 11.4.

Let $\mathcal{I}$ be an injective object of $Mod(\mathcal{O})$. Apply Lemma 11.9 with $\mathcal{F} = \mathcal{I}$, $\mathcal{B} = \mathcal{C}$ and $\text{Cov} = \text{Cov}_{\mathcal{C}}$. Assumption (3) of that lemma holds by Lemma 12.3. Hence we see that $H^i(U, \mathcal{I}_{ab}) = 0$ for every object U of $\mathcal{C}$.

If $\mathcal{C}$ has a final object then this also implies the first equality. If not, then according to Sites, Lemma 28.5 we see that the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ is equivalent to a ringed topos where the underlying site does have a final object. Hence the lemma follows. $\square$

Lemma 12.5. *Let $\mathcal{C}$ be a site. Let I be a set. For $i \in I$ let $\mathcal{F}_i$ be an abelian sheaf on $\mathcal{C}$. Let $U \in \text{Ob}(\mathcal{C})$. The canonical map*

$$H^p(U, \prod\nolimits_{i \in I} \mathcal{F}_i) \longrightarrow \prod\nolimits_{i \in I} H^p(U, \mathcal{F}_i)$$

is an isomorphism for $p = 0$ and injective for $p = 1$.

Proof. The statement for $p = 0$ is true because the product of sheaves is equal to the product of the underlying presheaves, see Sites, Lemma 10.1. Proof for $p = 1$. Set $\mathcal{F} = \prod \mathcal{F}_i$. Let $\xi \in H^1(U, \mathcal{F})$ map to zero in $\prod H^1(U, \mathcal{F}_i)$. By locality of cohomology, see Lemma 8.3, there exists a covering $\mathcal{U} = \{U_j \to U\}$ such that $\xi|_{U_j} = 0$ for all j. By Lemma 11.4 this means ξ comes from an element $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$. Since the maps $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \to H^1(U, \mathcal{F}_i)$ are injective for all i (by Lemma 11.4), and since the image of ξ is zero in $\prod H^1(U, \mathcal{F}_i)$ we see that the image $\check{\xi}_i = 0$ in $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$. However, since $\mathcal{F} = \prod \mathcal{F}_i$ we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the product of the complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_i)$, hence by Homology, Lemma 28.1 we conclude that $\check{\xi} = 0$ as desired. $\square$

Lemma 12.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a : U' \to U$ be a monomorphism in $\mathcal{C}$. Then for any injective $\mathcal{O}$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.*

Proof. Let $j : \mathcal{C}/U \to \mathcal{C}$ and $j' : \mathcal{C}/U' \to \mathcal{C}$ be the localization morphisms (Modules on Sites, Section 19). Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}$-modules

$$\text{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, \mathcal{F}) = \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

Similarly, the sheaf $j'_!\mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover below we describe a canonical map of $\mathcal{O}$-modules

$$j'_!\mathcal{O}_{U'} \longrightarrow j_!\mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 3.5). It suffices to prove the displayed map of modules is injective, see Homology, Lemma 23.2.

To construct our map it suffices to construct a map between the presheaves which assign to an object V of $\mathcal{C}$ the $\mathcal{O}(V)$-module

$$\bigoplus\nolimits_{\varphi' \in \mathrm{Mor}_\mathcal{C}(V, U')} \mathcal{O}(V) \quad \text{and} \quad \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V, U)} \mathcal{O}(V)$$

see Modules on Sites, Lemma 19.2. We take the map which maps the summand corresponding to φ' to the summand corresponding to $\varphi = a \circ \varphi'$ by the identity map on $\mathcal{O}(V)$. As a is a monomorphism, this map is injective. As sheafification is exact, the result follows. $\square$

13. Limp sheaves

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a sheaf of sets on $\mathcal{C}$ (we intentionally use a roman capital here to distinguish from abelian sheaves). Given an abelian sheaf $\mathcal{F}$ we denote $\mathcal{F}(K) = \mathrm{Mor}_{Sh(\mathcal{C})}(K, \mathcal{F})$. The functor $\mathcal{F} \mapsto \mathcal{F}(K)$ is a left exact functor $\mathit{Mod}(\mathcal{O}) \to \mathit{Ab}$ hence we have its right derived functors. We will denote these $H^p(K, \mathcal{F})$ so that $H^0(K, \mathcal{F}) = \mathcal{F}(K)$.

We mention two special cases. The first is the case where $K = h_U^\#$ for some object U of $\mathcal{C}$. In this case $H^p(K, \mathcal{F}) = H^p(U, \mathcal{F})$, because $\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$, see Sites, Section 13. The second is the case $\mathcal{O} = \mathbf{Z}$ (the constant sheaf). In this case the cohomology groups are functors $H^p(K, -) : \mathit{Ab}(\mathcal{C}) \to \mathit{Ab}$. Here is the analogue of Lemma 12.4.

Lemma 13.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a sheaf of sets on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then $H^p(K, \mathcal{F}) = H^p(K, \mathcal{F}_{ab})$.*

Proof. Note that both $H^p(K, \mathcal{F})$ and $H^p(K, \mathcal{F}_{ab})$ depend only on the topos, not on the underlying site. Hence by Sites, Lemma 28.5 we may replace $\mathcal{C}$ by a "larger" site such that $K = h_U$ for some object U of $\mathcal{C}$. In this case the result follows from Lemma 12.4. $\square$

Lemma 13.2. *Let $\mathcal{C}$ be a site. Let $K' \to K$ be a surjective map of sheaves of sets on $\mathcal{C}$. Set $K'_p = K' \times_K \ldots \times_K K'$ ($p+1$-factors). For every abelian sheaf $\mathcal{F}$ there is a spectral sequence with $E_1^{p,q} = H^q(K'_p, \mathcal{F})$ converging to $H^{p+q}(K, \mathcal{F})$.*

Proof. After replacing $\mathcal{C}$ by a "larger" site as in Sites, Lemma 28.5 we may assume that K, K' are objects of $\mathcal{C}$ and that $\mathcal{U} = \{K' \to K\}$ is a covering. Then we have the Čech to cohomology spectral sequence of Lemma 11.6 whose E_1 page is as indicated in the statement of the lemma. $\square$

Lemma 13.3. *Let $\mathcal{C}$ be a site. Let K be a sheaf of sets on $\mathcal{C}$. Consider the morphism of topoi $j : Sh(\mathcal{C}/K) \to Sh(\mathcal{C})$, see Sites, Lemma 29.3. Then j^{-1} preserves injectives and $H^p(K, \mathcal{F}) = H^p(\mathcal{C}/K, j^{-1}\mathcal{F})$ for any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$.*

Proof. By Sites, Lemmas 29.1 and 29.3 the morphism of topoi j is equivalent to a localization. Hence this follows from Lemma 8.1. $\square$

Keeping in mind Lemma 13.1 we see that the following definition is the "correct one" also for sheaves of modules on ringed sites.

Definition 13.4. Let $\mathcal{C}$ be a site. We say an abelian sheaf $\mathcal{F}$ is *limp*[1] if for every sheaf of sets K we have $H^p(K, \mathcal{F}) = 0$ for all $p \geq 1$.

It is clear that being limp is an intrinsic property, i.e., preserved under equivalences of topoi. A limp sheaf has vanishing higher cohomology on all objects of the site, but in general the condition of being limp is strictly stronger. Here is a characterization of limp sheaves which is sometimes useful.

Lemma 13.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian sheaf. If*

1. *$H^p(U, \mathcal{F}) = 0$ for $p > 0$ and $U \in \mathrm{Ob}(\mathcal{C})$, and*
2. *for every surjection $K' \to K$ of sheaves of sets the extended Čech complex*
$$0 \to H^0(K, \mathcal{F}) \to H^0(K', \mathcal{F}) \to H^0(K' \times_K K', \mathcal{F}) \to \ldots$$
is exact,

then $\mathcal{F}$ is limp (and the converse holds too).

Proof. By assumption (1) we have $H^p(h_U^\#, g^{-1}\mathcal{I}) = 0$ for all $p > 0$ and all objects U of $\mathcal{C}$. Note that if $K = \coprod K_i$ is a coproduct of sheaves of sets on $\mathcal{C}$ then $H^p(K, g^{-1}\mathcal{I}) = \prod H^p(K_i, g^{-1}\mathcal{I})$. For any sheaf of sets K there exists a surjection
$$K' = \coprod h_{U_i}^\# \longrightarrow K$$
see Sites, Lemma 13.5. Thus we conclude that: (*) for every sheaf of sets K there exists a surjection $K' \to K$ of sheaves of sets such that $H^p(K', \mathcal{F}) = 0$ for $p > 0$. We claim that (*) and condition (2) imply that $\mathcal{F}$ is limp. Note that conditions (*) and (2) only depend on $\mathcal{F}$ as an object of the topos $\mathit{Sh}(\mathcal{C})$ and not on the underlying site. (We will not use property (1) in the rest of the proof.)

We are going to prove by induction on $n \geq 0$ that (*) and (2) imply the following induction hypothesis IH_n: $H^p(K, \mathcal{F}) = 0$ for all $0 < p \leq n$ and all sheaves of sets K. Note that IH_0 holds. Assume IH_n. Pick a sheaf of sets K. Pick a surjection $K' \to K$ such that $H^p(K', \mathcal{F}) = 0$ for all $p > 0$. We have a spectral sequence with
$$E_1^{p,q} = H^q(K'_p, \mathcal{F})$$
covering to $H^{p+q}(K, \mathcal{F})$, see Lemma 13.2. By IH_n we see that $E_1^{p,q} = 0$ for $0 < q \leq n$ and by assumption (2) we see that $E_2^{p,0} = 0$ for $p > 0$. Finally, we have $E_1^{0,q} = 0$ for $q > 0$ because $H^q(K', \mathcal{F}) = 0$ by choice of K'. Hence we conclude that $H^{n+1}(K, \mathcal{F}) = 0$ because all the terms $E_2^{p,q}$ with $p + q = n + 1$ are zero. $\square$

14. The Leray spectral sequence

The key to proving the existence of the Leray spectral sequence is the following lemma.

Lemma 14.1. *Let $f : (\mathit{Sh}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (\mathit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Then for any injective object $\mathcal{I}$ in $\mathit{Mod}(\mathcal{O}_\mathcal{C})$ the pushforward $f_*\mathcal{I}$ is limp.*

Proof. Let K be a sheaf of sets on $\mathcal{D}$. By Modules on Sites, Lemma 7.2 we may replace $\mathcal{C}$, $\mathcal{D}$ by "larger" sites such that f comes from a morphism of ringed sites induced by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that $K = h_V$ for some object V of $\mathcal{D}$.

[1] This is probably nonstandard notation. Please email stacks.project@gmail.com if you know the correct terminology.

Thus we have to show that $H^q(V, f_*\mathcal{I})$ is zero for $q > 0$ and all objects V of $\mathcal{D}$ when f is given by a morphism of ringed sites. Let $\mathcal{V} = \{V_j \to V\}$ be any covering of $\mathcal{D}$. Since u is continuous we see that $\mathcal{U} = \{u(V_j) \to u(v)\}$ is a covering of $\mathcal{C}$. Then we have an equality of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I})$$

by the definition of f_*. By Lemma 12.3 we see that the cohomology of this complex is zero in positive degrees. We win by Lemma 11.9. $\square$

For flat morphisms the functor f_* preserves injective modules. In particular the functor $f_* : \textit{Ab}(\mathcal{C}) \to \textit{Ab}(\mathcal{D})$ always transforms injective abelian sheaves into injective abelian sheaves.

Lemma 14.2. *Let $f : (\textit{Sh}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (\textit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. If f is flat, then $f_*\mathcal{I}$ is an injective $\mathcal{O}_\mathcal{D}$-module for any injective $\mathcal{O}_\mathcal{C}$-module $\mathcal{I}$.*

Proof. In this case the functor f^* is exact, see Modules on Sites, Lemma 30.2. Hence the result follows from Homology, Lemma 25.1. $\square$

Lemma 14.3. *Let $(\textit{Sh}(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be a ringed topos. A limp sheaf is right acyclic for the following functors:*

1. *the functor $H^0(U, -)$ for any object U of $\mathcal{C}$,*
2. *the functor $\mathcal{F} \mapsto \mathcal{F}(K)$ for any presheaf of sets K,*
3. *the functor $\Gamma(\mathcal{C}, -)$ of global sections,*
4. *the functor f_* for any morphism $f : (\textit{Sh}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (\textit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi.*

Proof. Part (2) is the definition of a limp sheaf. Part (1) is a consequence of (2) as pointed out in the discussion following the definition of limp sheaves. Part (3) is a special case of (2) where $K = e$ is the final object of $\textit{Sh}(\mathcal{C})$.

To prove (4) we may assume, by Modules on Sites, Lemma 7.2 that f is given by a morphism of sites. In this case we see that R^if_*, $i > 0$ of a limp sheaf are zero by the description of higher direct images in Lemma 8.4. $\square$

Remark 14.4. As a consequence of the results above we find that Derived Categories, Lemma 22.1 applies to a number of situations. For example, given a morphism $f : (\textit{Sh}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (\textit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi we have

$$R\Gamma(\mathcal{D}, Rf_*\mathcal{F}) = R\Gamma(\mathcal{C}, \mathcal{F})$$

for any sheaf of $\mathcal{O}_\mathcal{C}$-modules $\mathcal{F}$. Namely, for an injective $\mathcal{O}_\mathcal{X}$-module $\mathcal{I}$ the $\mathcal{O}_\mathcal{D}$-module $f_*\mathcal{I}$ is limp by Lemma 14.1 and a limp sheaf is acyclic for $\Gamma(\mathcal{D}, -)$ by Lemma 14.3.

Lemma 14.5 (Leray spectral sequence)**.** *Let $f : (\textit{Sh}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (\textit{Sh}(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{D}, R^qf_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{C}, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 22.2 coming from the composition of functors $\Gamma(\mathcal{C}, -) = \Gamma(\mathcal{D}, -) \circ f_*$. To see that the assumptions of Derived Categories, Lemma 22.2 are satisfied, see Lemmas 14.1 and 14.3. $\square$

Lemma 14.6. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{C}$-module.*

1. *If $R^qf_*\mathcal{F} = 0$ for $q > 0$, then $H^p(\mathcal{C}, \mathcal{F}) = H^p(\mathcal{D}, f_*\mathcal{F})$ for all p.*
2. *If $H^p(\mathcal{D}, R^qf_*\mathcal{F}) = 0$ for all q and $p > 0$, then $H^q(\mathcal{C}, \mathcal{F}) = H^0(\mathcal{D}, R^qf_*\mathcal{F})$ for all q.*

Proof. These are two simple conditions that force the Leray spectral sequence to converge. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. $\square$

Lemma 14.7 (Relative Leray spectral sequence)**.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$ be morphisms of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{C}$-module. There is a spectral sequence with*

$$E_2^{p,q} = R^pg_*(R^qf_*\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_*\mathcal{F}$. This spectral sequence is functorial in $\mathcal{F}$, and there is a version for bounded below complexes of $\mathcal{O}_\mathcal{C}$-modules.

Proof. This is a Grothendieck spectral sequence for composition of functors, see Derived Categories, Lemma 22.2 and Lemmas 14.1 and 14.3. $\square$

15. The base change map

In this section we construct the base change map in some cases; the general case is treated in Remark 19.2. The discussion in this section avoids using derived pullback by restricting to the case of a base change by a flat morphism of ringed sites. Before we state the result, let us discuss flat pullback on the derived category. Suppose $g : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is a flat morphism of ringed topoi. By Modules on Sites, Lemma 30.2 the functor $g^* : Mod(\mathcal{O}_\mathcal{D}) \to Mod(\mathcal{O}_\mathcal{C})$ is exact. Hence it has a derived functor

$$g^* : D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_\mathcal{D})$$

which is computed by simply pulling back an representative of a given object in $D(\mathcal{O}_\mathcal{D})$, see Derived Categories, Lemma 17.8. It preserved the bounded (above, below) subcategories. Hence as indicated we indicate this functor by g^* rather than Lg^*.

Lemma 15.1. *Let*

$$\begin{tikzcd}
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g'"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "f"] \\
(Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})
\end{tikzcd}$$

be a commutative diagram of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. Assume both g and g' are flat. Then there exists a canonical base change map

$$g^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*(g')^*\mathcal{F}^\bullet$$

in $D^+(\mathcal{O}_{\mathcal{D}'})$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^*\mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 14.2 we see that $(g')_*\mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^*\mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 18.6 and 18.7 the arrow β in the diagram

$$\begin{tikzcd} (g')_*(g')^*\mathcal{F}^\bullet \arrow[r] & (g')_*\mathcal{J}^\bullet \\ \mathcal{F}^\bullet \arrow[u, "adjunction"] \arrow[r] & \mathcal{I}^\bullet \arrow[u, "\beta"'] \end{tikzcd}$$

exists and is unique up to homotopy. Pushing down to $\mathcal{D}$ we get

$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$

By adjunction of g^* and g_* we get a map of complexes $g^*f_*\mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. $\square$

16. Cohomology and colimits

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \to \textit{Mod}(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram over the index category $\mathcal{I}$, see Categories, Section 14. For each i there is a canonical map $\mathcal{F}_i \to \operatorname{colim}_i \mathcal{F}_i$ which induces a map on cohomology. Hence we get a canonical map

$$\operatorname{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \operatorname{colim}_i \mathcal{F}_i)$$

for every $p \geq 0$ and every object U of $\mathcal{C}$. These maps are in general not isomorphisms, even for $p = 0$.

The following lemma is the analogue of Sites, Lemma 11.2 for cohomology.

Lemma 16.1. *Let $\mathcal{C}$ be a site. Let $Cov_\mathcal{C}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 6.2). Let $\mathcal{B} \subset \operatorname{Ob}(\mathcal{C})$, and $Cov \subset Cov_\mathcal{C}$ be subsets. Assume that*

1. *For every $\mathcal{U} \in Cov$ we have $\mathcal{U} = \{U_i \to U\}_{i \in I}$ with I finite, $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.*

Then the map

$$\operatorname{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \operatorname{colim}_i \mathcal{F}_i)$$

is an isomorphism for every $p \geq 0$, every $U \in \mathcal{B}$, and every filtered diagram $\mathcal{I} \to Ab(\mathcal{C})$.

Proof. To prove the lemma we will argue by induction on p. Note that we require in (1) the coverings $\mathcal{U} \in \text{Cov}$ to be finite, so that all the elements of $\mathcal{B}$ are quasi-compact. Hence (2) and (1) imply that any $U \in \mathcal{B}$ satisfies the hypothesis of Sites, Lemma 11.2 (4). Thus we see that the result holds for $p = 0$. Now we assume the lemma holds for p and prove it for $p + 1$.

Choose a filtered diagram $\mathcal{F} : \mathcal{I} \to Ab(\mathcal{C})$, $i \mapsto \mathcal{F}_i$. Since $Ab(\mathcal{C})$ has functorial injective embeddings, see Injectives, Theorem 7.4, we can find a morphism of filtered diagrams $\mathcal{F} \to \mathcal{I}$ such that each $\mathcal{F}_i \to \mathcal{I}_i$ is an injective map of abelian sheaves into an injective abelian sheaf. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

Since colimits of sheaves are the sheafification of colimits on the level of presheaves, since sheafification is exact, and since filtered colimits of abelian groups are exact (see Algebra, Lemma 8.9), we see the sequence

$$0 \to \mathrm{colim}_i\, \mathcal{F}_i \to \mathrm{colim}_i\, \mathcal{I}_i \to \mathrm{colim}_i\, \mathcal{Q}_i \to 0.$$

is also a short exact sequence. We claim that $H^q(U, \mathrm{colim}_i\, \mathcal{I}_i) = 0$ for all $U \in \mathcal{B}$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{tikzcd} \mathrm{colim}_i\, H^p(U, \mathcal{I}_i) \arrow[r] \arrow[d] & \mathrm{colim}_i\, H^p(U, \mathcal{Q}_i) \arrow[r] \arrow[d] & \mathrm{colim}_i\, H^{p+1}(U, \mathcal{F}_i) \arrow[r] \arrow[d] & 0 \arrow[d] \\ H^p(U, \mathrm{colim}_i\, \mathcal{I}_i) \arrow[r] & H^p(U, \mathrm{colim}_i\, \mathcal{Q}_i) \arrow[r] & H^{p+1}(U, \mathrm{colim}_i\, \mathcal{F}_i) \arrow[r] & 0 \end{tikzcd}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 8.9. Hence by the snake lemma we deduce the result for $p + 1$.

It remains to show that the claim is true. We will use Lemma 11.9. By the result for $p = 0$ we see that for $\mathcal{U} \in \mathrm{Cov}$ we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{colim}_i\, \mathcal{I}_i) = \mathrm{colim}_i\, \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the $U_{j_0} \times_U \ldots \times_U U_{j_p}$ are in $\mathcal{B}$. By Lemma 11.2 each of the complexes in the colimit of Cech complexes is acyclic in degree ≥ 1. Hence by Algebra, Lemma 8.9 we see that also the Cech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{colim}_i\, \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, \mathrm{colim}_i\, \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 11.9. are satisfied and the claim follows. □

Let $\mathcal{C}$ be a limit of sites $\mathcal{C}_i$ as in Sites, Situation 11.3 and Lemmas 11.4, 11.5, and 11.6. In particular, all coverings in $\mathcal{C}$ and $\mathcal{C}_i$ have finite index sets. Moreover, assume given

(1) an abelian sheaf $\mathcal{F}_i$ on $\mathcal{C}_i$ for all $i \in \mathrm{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ a map $\varphi_a : f_a^{-1}\mathcal{F}_i \to \mathcal{F}_j$ of abelian sheaves on $\mathcal{C}_j$

such that $\varphi_c = \varphi_b \circ f_b^{-1}\varphi_a$ whenever $c = a \circ b$.

Lemma 16.2. *In the situation discussed above set $\mathcal{F} = \mathrm{colim}\, f_i^{-1}\mathcal{F}_i$. Let $i \in \mathrm{Ob}(\mathcal{I})$, $X_i \in Ob(\mathcal{C}_i)$. Then*

$$\mathrm{colim}_{a:j\to i}\, H^p(u_a(X_i), \mathcal{F}_j) = H^p(u_i(X_i), \mathcal{F})$$

for all $p \geq 0$.

Proof. The case $p = 0$ is Sites, Lemma 11.6.

In this paragraph we show that we can find a map of systems $(\gamma_i) : (\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ with $\mathcal{G}_i$ an injective abelian sheaf and γ_i injective. For each i we pick an injection $\mathcal{F}_i \to \mathcal{I}_i$ where $\mathcal{I}_i$ is an injective abelian sheaf on $\mathcal{C}_i$. Then we can consider the family of maps

$$\gamma_i : \mathcal{F}_i \longrightarrow \prod\nolimits_{b:k\to i} f_{b,*}\mathcal{I}_k = \mathcal{G}_i$$

where the component maps are the maps adjoint to the maps $f_b^{-1}\mathcal{F}_i \to \mathcal{F}_k \to \mathcal{I}_k$. For $a : j \to i$ in $\mathcal{I}$ there is a canonical map

$$\psi_a : f_a^{-1}\mathcal{G}_i \to \mathcal{G}_j$$

whose components are the canonical maps $f_b^{-1} f_{a \circ b, *} \mathcal{I}_k \to f_{b, *} \mathcal{I}_k$ for $b : k \to j$. Thus we find an injection $\{\gamma_i\} : \{\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ of systems of abelian sheaves. Note that $\mathcal{G}_i$ is an injective sheaf of abelian groups on $\mathcal{C}_i$, see Lemma 14.2 and Homology, Lemma 23.3. This finishes the construction.

Arguing exactly as in the proof of Lemma 16.1 we see that it suffices to prove that $H^p(X, \operatorname{colim} f_i^{-1}\mathcal{G}_i) = 0$ for $p > 0$.

Set $\mathcal{G} = \operatorname{colim} f_i^{-1}\mathcal{G}_i$. To show vanishing of cohomology of $\mathcal{G}$ on every object of $\mathcal{C}$ we show that the Čech cohomology of $\mathcal{G}$ for any covering $\mathcal{U}$ of $\mathcal{C}$ is zero (Lemma 11.9). The covering $\mathcal{U}$ comes from a covering $\mathcal{U}_i$ of $\mathcal{C}_i$ for some i. We have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) = \operatorname{colim}_{a : j \to i} \check{\mathcal{C}}^\bullet(u_a(\mathcal{U}_i), \mathcal{G}_j)$$

by the case $p = 0$. The right hand side is acyclic in positive degrees as a filtered colimit of acyclic complexes by Lemma 11.2. See Algebra, Lemma 8.9. □

17. Flat resolutions

In this section we redo the arguments of Cohomology, Section 27 in the setting of ringed sites and ringed topoi.

Lemma 17.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}$-modules. The functor*

$$K(\textit{Mod}(\mathcal{O})) \longrightarrow K(\textit{Mod}(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto \textit{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{G}^\bullet)$$

is an exact functor of triangulated categories.

Proof. Omitted. Hint: See More on Algebra, Lemmas 45.1 and 45.2. □

Definition 17.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the complex

$$\operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

is acyclic.

Lemma 17.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(\textit{Mod}(\mathcal{O})) \longrightarrow K(\textit{Mod}(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto \textit{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 17.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. □

Lemma 17.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}$-modules, then $\textit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}$-modules.*

Proof. Follows from the isomorphism

$$\operatorname{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)) = \operatorname{Tot}(\operatorname{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$$

and the definition. □

Lemma 17.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(\textit{Mod}(\mathcal{O}))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 17.1 and the fact that in a distinguished triangle in $K(\textit{Mod}(\mathcal{O}))$ if two out of three are acyclic, so is the third. □

Lemma 17.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A bounded above complex of flat $\mathcal{O}$-modules is K-flat.*

Proof. Let $\mathcal{K}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be an acyclic complex of $\mathcal{O}$-modules. Note that $\mathcal{L}^\bullet = \text{colim}_m \tau_{\leq m}\mathcal{L}^\bullet$ where we take termwise colimits. Hence also

$$\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) = \text{colim}_m \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \tau_{\leq m}\mathcal{L}^\bullet)$$

termwise. Hence to prove the complex on the left is acyclic it suffices to show each of the complexes on the right is acyclic. Since $\tau_{\leq m}\mathcal{L}^\bullet$ is acyclic this reduces us to the case where $\mathcal{L}^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 22.6 has

$$'E_1^{p,q} = H^p(\mathcal{L}^\bullet \otimes_R \mathcal{K}^q)$$

which is zero as $\mathcal{K}^q$ is flat and $\mathcal{L}^\bullet$ acyclic. Hence we win. □

Lemma 17.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\text{colim}_i \mathcal{K}_i^\bullet$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$\text{colim}_i \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}_i^\bullet) = \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \text{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. □

Lemma 17.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}$-modules there exists a commutative diagram of complexes of $\mathcal{O}$-modules*

$$\begin{tikzcd}
\mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\
\tau_{\leq 1}\mathcal{G}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{G}^\bullet \arrow[r] & \ldots
\end{tikzcd}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $\text{colim}\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules on Sites, Lemma 28.6 and Derived Categories, Lemma 28.1. The induced map $\text{colim}\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. □

Lemma 17.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}$-modules there exists a K-flat complex $\mathcal{K}^\bullet$ and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$.*

Proof. Choose a diagram as in Lemma 17.8. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 28.5. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 17.6. The induced map $\text{colim}\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism by construction. Since $\text{colim}\, \mathcal{K}_n^\bullet$ is K-flat by Lemma 17.7 we win. □

Lemma 17.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the induced map*

$$Tot(id_{\mathcal{F}^\bullet} \otimes \alpha) : Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) \longrightarrow Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 17.9. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) \arrow[r] \arrow[d] & \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet) \arrow[d] \\ \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) \arrow[r] & \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet) \end{tikzcd}$$

The result follows as by Lemma 17.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 17.9. By Lemma 17.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}) \longrightarrow K(\mathcal{O}), \quad \mathcal{G}^\bullet \longmapsto \mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

By Lemma 17.3 this functor induces a functor $D(\mathcal{O}) \to D(\mathcal{O})$ simply because $D(\mathcal{O})$ is the localization of $K(\mathcal{O})$ at quasi-isomorphisms. By Lemma 17.10 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

Definition 17.11. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. The *derived tensor product*

$$- \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}^\bullet : D(\mathcal{O}) \longrightarrow D(\mathcal{O})$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O})$. Hence when we write $\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

Definition 17.12. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules. The *Tor*'s of $\mathcal{F}$ and $\mathcal{G}$ are define by the formula

$$\mathrm{Tor}_p^{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we have a long exact cohomology sequence

$$\begin{tikzcd} & \mathcal{F}_1 \otimes_{\mathcal{O}} \mathcal{G} \arrow[r] & \mathcal{F}_2 \otimes_{\mathcal{O}} \mathcal{G} \arrow[r] & \mathcal{F}_3 \otimes_{\mathcal{O}} \mathcal{G} \arrow[r] & 0 \\ \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{F}_1, \mathcal{G}) \arrow[r] & \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{F}_2, \mathcal{G}) \arrow[r] & \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{F}_3, \mathcal{G}) \arrow[ul] & & \end{tikzcd}$$

for every $\mathcal{O}$-module $\mathcal{G}$. This will be called the long exact sequence of Tor associated to the situation.

Lemma 17.13. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module. The following are equivalent*

1. *$\mathcal{F}$ is a flat $\mathcal{O}$-module, and*
2. *$Tor_1^{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{O}$-module $\mathcal{G}$.*

Proof. If $\mathcal{F}$ is flat, then $\mathcal{F} \otimes_{\mathcal{O}} -$ is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if $\mathcal{G} \to \mathcal{H}$ is injective with cokernel $\mathcal{Q}$, the long exact sequence of Tor shows that the kernel of $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}$ is a quotient of $\text{Tor}_1^{\mathcal{O}}(\mathcal{F}, \mathcal{Q})$ which is zero by assumption. Hence $\mathcal{F}$ is flat. $\square$

18. Derived pullback

Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}') \to D(\mathcal{O})$$

However, we have to be a little careful since we haven't yet proved the pullback of a flat module is flat in complete generality, see Modules on Sites, Section 38. In this section, we will use the hypothesis that our sites have enough points, but once we improve the result of the aforementioned section, all of the results in this section will hold without the assumption on the existence of points.

Lemma 18.1. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be a morphism of topoi. Let $\mathcal{O}'$ be a sheaf of rings on $\mathcal{C}'$. Assume $\mathcal{C}$ has enough points. For any complex of $\mathcal{O}'$-modules $\mathcal{G}^\bullet$, there exists a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ such that $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}'$-modules and $f^{-1}\mathcal{K}^\bullet$ is a K-flat complex of $f^{-1}\mathcal{O}'$-modules.*

Proof. In the proof of Lemma 17.9 we find a quasi-isomorphism $\mathcal{K}^\bullet = \text{colim}_i \mathcal{K}_i^\bullet \to \mathcal{G}^\bullet$ where each $\mathcal{K}_i^\bullet$ is a bounded above complex of flat $\mathcal{O}'$-modules. By Modules on Sites, Lemma 38.3 applied to the morphism of ringed topoi $(Sh(\mathcal{C}), f^{-1}\mathcal{O}') \to (Sh(\mathcal{C}'), \mathcal{O}')$ we see that $f^{-1}\mathcal{F}_i^\bullet$ is a bounded above complex of flat $f^{-1}\mathcal{O}'$-modules. Hence $f^{-1}\mathcal{K}^\bullet = \text{colim}_i f^{-1}\mathcal{K}_i^\bullet$ is K-flat by Lemmas 17.6 and 17.7. $\square$

Remark 18.2. It is straightforward to show that the pullback of a K-flat complex is K-flat for a morphism of ringed topoi with enough points; this slightly improves the result of Lemma 18.1. However, in applications it seems rather that the explicit form of the K-flat complexes constructed in Lemma 17.9 is what is useful (as in the proof above) and not the plain fact that they are K-flat. Note for example that the terms of the complex constructed are each direct sums of modules of the form $j_{U!}\mathcal{O}_U$, see Lemma 17.8.

Lemma 18.3. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Assume $\mathcal{C}$ has enough points. There exists an exact functor*

$$Lf^* : D(\mathcal{O}') \longrightarrow D(\mathcal{O})$$

of triangulated categories so that $Lf^*\mathcal{K}^\bullet = f^*\mathcal{K}^\bullet$ for any complex as in Lemma 18.1 in particular for any bounded above complex of flat $\mathcal{O}'$-modules.

Proof. To see this we use the general theory developed in Derived Categories, Section 15. Set $\mathcal{D} = K(\mathcal{O}')$ and $\mathcal{D}' = D(\mathcal{O})$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}')$. This gives a situation as in Derived Categories, Situation 15.1 so that Derived Categories, Definition 15.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 15.15 with $\mathcal{P} \subset \text{Ob}(\mathcal{D})$ the collection of complexes $\mathcal{K}^\bullet$ such that $f^{-1}\mathcal{K}^\bullet$ is a K-flat complex of $f^{-1}\mathcal{O}'$-modules: (1) follows from Lemma 18.1 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between elements of $\mathcal{P}$ the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O} \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O}$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. The complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}'$-modules by our choice of $\mathcal{P}$. Hence Lemma 17.10 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}') = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O})$$

see Derived Categories, Equation (15.9.1). Finally, Derived Categories, Lemma 15.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is in $\mathcal{P}$. Since the proof of Lemma 18.1 shows that bounded above complexes of flat modules are in $\mathcal{P}$ we win. □

Lemma 18.4. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Assume $\mathcal{C}$ has enough points. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes_{\mathcal{O}'}^{\mathbf{L}} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \text{Ob}(D(\mathcal{O}'))$.

Proof. By Lemma 18.1 we may assume that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat complexes of $\mathcal{O}'$-modules such that $f^*\mathcal{F}^\bullet$ and $f^*\mathcal{G}^\bullet$ are K-flat complexes of $\mathcal{O}$-modules. In this case $\mathcal{F}^\bullet \otimes_{\mathcal{O}'}^{\mathbf{L}} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet$. By Lemma 17.4 $\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$ is K-flat also. Hence the isomorphism of the lemma comes from the isomorphism

$$\text{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}} f^*\mathcal{G}^\bullet) \longrightarrow f^*\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}'} \mathcal{G}^q)$ of Modules on Sites, Lemma 26.1. □

Lemma 18.5. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. There is a canonical bifunctorial isomorphism*

$$\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} Lf^*\mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_Y}^{\mathbf{L}} f^{-1}\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet$ in $D(\mathcal{O})$ and $\mathcal{G}^\bullet$ in $D(\mathcal{O}')$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}$-module and let $\mathcal{G}$ be an $\mathcal{O}'$-module. Then $\mathcal{F} \otimes_{\mathcal{O}} f^*\mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}$ because $f^*\mathcal{G} = \mathcal{O} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}$. The lemma follows from this and the definitions. □

19. Cohomology of unbounded complexes

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The category $Mod(\mathcal{O})$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \in \text{Ob}(\mathcal{C})} j_{U!}\mathcal{O}_U,$$

see Modules on Sites, Section 14 and Lemmas 28.5 and 28.6. By Injectives, Theorem 12.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}$-modules. Hence we can define

$$R\Gamma(\mathcal{C}, \mathcal{F}^\bullet) = \Gamma(\mathcal{C}, \mathcal{I}^\bullet)$$

and similarly for any left exact functor, see Derived Categories, Lemma 29.6. For any morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ we obtain

$$Rf_* : D(\mathcal{O}) \longrightarrow D(\mathcal{O}')$$

on the unbounded derived categories.

Lemma 19.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Assume $\mathcal{C}$ has enough points. The functor Rf_* defined above and the functor Lf^* defined in Lemma 18.3 are adjoint:*

$$\mathrm{Hom}_{D(\mathcal{O})}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}')}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in \mathrm{Ob}(D(\mathcal{O}))$ and $\mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}'))$.

Proof. This follows formally from the fact that Rf_* and Lf^* exist, see Derived Categories, Lemma 28.4. $\square$

Remark 19.2. The construction of unbounded derived functor Lf^* and Rf_* allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{tikzcd}
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g'"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "f"] \\
(Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})
\end{tikzcd}$$

is a commutative diagram of ringed topoi. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_{\mathcal{C}}$-modules. Then there exists a canonical base change map

$$Lg^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*L(g')^*\mathcal{F}^\bullet$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, this map is adjoint to a map $L(f')^*Lg^*Rf_*\mathcal{F}^\bullet \to L(g')^*\mathcal{F}^\bullet$ Since $L(f')^*Lg^* = L(g')^*Lf^*$ we see this is the same as a map $L(g')^*Lf^*Rf_*\mathcal{F}^\bullet \to L(g')^*\mathcal{F}^\bullet$ which we can take to be $L(g')^*$ of the adjunction map $Lf^*Rf_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet$.

20. Some properties of K-injective complexes

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Denote $j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ the corresponding localization morphism. The pullback functor j^* is exact as it is just the restriction functor. Thus derived pullback Lj^* is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}) \to D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero $j_! : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ (see Modules on Sites, Definition 19.1) is an exact functor (Modules on Sites, Lemma 19.3). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \to D(\mathcal{O}), \quad F \mapsto j_!F$$

by simply applying $j_!$ to any complex representing the object F.

Lemma 20.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. The restriction of a K-injective complex of $\mathcal{O}$-modules to $\mathcal{C}/U$ is a K-injective complex of $\mathcal{O}_U$-modules.*

Proof. Follows immediately from Derived Categories, Lemma 29.10 and the fact that the restriction functor has the exact left adjoint $j_!$. See discussion above. $\square$

Lemma 20.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Denote $j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ the corresponding localization morphism. The restriction functor $D(\mathcal{O}) \to D(\mathcal{O}_U)$ is a right adjoint to extension by zero $j_! : D(\mathcal{O}_U) \to D(\mathcal{O})$.*

Proof. We have to show that

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!E, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E, F|_U)$$

Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_U$-modules representing E and choose a K-injective complex $\mathcal{I}^\bullet$ representing F. By Lemma 20.1 the complex $\mathcal{I}^\bullet|_U$ is K-injective as well. Hence we see that the formula above becomes

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!\mathcal{E}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{I}^\bullet|_U)$$

which holds as $|_U$ and $j_!$ are adjoint functors (Modules on Sites, Lemma 19.2) and Derived Categories, Lemma 29.2. $\square$

Lemma 20.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a flat map of sheaves of rings. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}'$-modules, then $\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}$-modules.*

Proof. This is true because $\mathrm{Hom}_{K(\mathcal{O})}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{K(\mathcal{O}')}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}', \mathcal{I}^\bullet)$ by Modules on Sites, Lemma 11.3 and the fact that tensoring with $\mathcal{O}'$ is exact. $\square$

Lemma 20.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules, then $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}', I^\bullet)$ is a K-injective complex of $\mathcal{O}'$-modules.*

Proof. This is true because $\mathrm{Hom}_{K(\mathcal{O}')}(\mathcal{G}^\bullet, \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}', \mathcal{I}^\bullet)) = \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{I}^\bullet)$ by Modules on Sites, Lemma 27.5. $\square$

21. Derived and homotopy limits

Let $\mathcal{C}$ be a site. Consider the category $\mathcal{C} \times \mathbf{N}$ with $\mathrm{Mor}((U, n), (V, m)) = \emptyset$ if $n > m$ and $\mathrm{Mor}((U, n), (V, m)) = \mathrm{Mor}(U, V)$ else. We endow this with the structure of a site by letting coverings be families $\{(U_i, n) \to (U, n)\}$ such that $\{U_i \to U\}$ is a covering of $\mathcal{C}$. Then the reader verifies immediately that sheaves on $\mathcal{C} \times \mathbf{N}$ are the same thing as inverse systems of sheaves on $\mathcal{C}$. In particular $Ab(\mathcal{C} \times \mathbf{N})$ is inverse systems of abelian sheaves on $\mathcal{C}$. Consider now the functor

$$\lim : Ab(\mathcal{C} \times \mathbf{N}) \to Ab(\mathcal{C})$$

which takes an inverse system to its limit. This is nothing but g_* where $g : Sh(\mathcal{C} \times \mathbf{N}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to the continuous and cocontinuous functor $\mathcal{C} \times \mathbf{N} \to \mathcal{C}$. (Observe that g^{-1} assigns to a sheaf on $\mathcal{C}$ the corresponding constant inverse system.)

By the general machinery explained above we obtain a derived functor

$$R\lim = Rg_* : D(\mathcal{C} \times \mathbf{N}) \to D(\mathcal{C}).$$

As indicated this functor is often denoted $R\lim$.

On the other hand, the continuous and cocontinuous functors $\mathcal{C} \to \mathcal{C} \times \mathbf{N}$, $U \mapsto (U, n)$ define morphisms of topoi $i_n : \mathit{Sh}(\mathcal{C}) \to \mathit{Sh}(\mathcal{C} \times \mathbf{N})$. Of course i_n^{-1} is the functor which picks the nth term of the inverse system. Thus there are transformations of functors $i_{n+1}^{-1} \to i_n^{-1}$. Hence given $K \in D(\mathcal{C} \times \mathbf{N})$ we get $K_n = i_n^{-1}K \in D(\mathcal{C})$ and maps $K_{n+1} \to K_n$. In Derived Categories, Definition 32.1 we have defined the notion of a homotopy limit

$$R\lim K_n \in D(\mathcal{C})$$

We claim the two notions agree (as far as it makes sense).

Lemma 21.1. *Let $\mathcal{C}$ be a site. Let K be an object of $D(\mathcal{C} \times \mathbf{N})$. Set $K_n = i_n^{-1}K$ as above. Then*

$$R\lim K \cong R\lim K_n$$

in $D(\mathcal{C})$.

Proof. To calculate $R\lim$ on an object K of $D(\mathcal{C} \times \mathbf{N})$ we choose a K-injective representative $\mathcal{I}^\bullet$ whose terms are injective objects of $\mathit{Ab}(\mathcal{C} \times \mathbf{N})$, see Injectives, Theorem 12.6. We may and do think of $\mathcal{I}^\bullet$ as an inverse system of complexes $(\mathcal{I}_n^\bullet)$ and then we see that

$$R\lim K = \lim \mathcal{I}_n^\bullet$$

where the right hand side is the termwise inverse limit.

Let $\mathcal{J} = (\mathcal{J}_n)$ be an injective object of $\mathit{Ab}(\mathcal{C} \times \mathbf{N})$. The morphisms $(U, n) \to (U, n+1)$ are monomorphisms of $\mathcal{C} \times \mathbf{N}$, hence $\mathcal{J}(U, n+1) \to \mathcal{J}(U, n)$ is surjective (Lemma 12.6). It follows that $\mathcal{J}_{n+1} \to \mathcal{J}_n$ is surjective as a map of presheaves.

Note that the functor i_n^{-1} has an exact left adjoint $i_{n,!}$. Namely, $i_{n,!}\mathcal{F}$ is the inverse system $\ldots 0 \to 0 \to \mathcal{F} \to \ldots \to \mathcal{F}$. Thus the complexes $i_n^{-1}\mathcal{I}^\bullet = \mathcal{I}_n^\bullet$ are K-injective by Derived Categories, Lemma 29.10.

Because we chose our K-injective complex to have injective terms we conclude that

$$0 \to \lim \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to 0$$

is a short exact sequence of complexes of abelian sheaves as it is a short exact sequence of complexes of abelian presheaves. Moreover, the products in the middle and the right represent the products in $D(\mathcal{C})$, see Injectives, Lemma 13.4 and its proof (this is where we use that $\mathcal{I}_n^\bullet$ is K-injective). Thus $R\lim K$ is a homotopy limit of the inverse system (K_n) by definition of homotopy limits in triangulated categories. $\square$

Lemma 21.2. *Let $f : (\mathit{Sh}(\mathcal{C}), \mathcal{O}) \to (\mathit{Sh}(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Then Rf_* commutes with $R\lim$, i.e., Rf_* commutes with derived limits.*

Proof. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. By induction on n we may choose actual complexes $\mathcal{K}_n^\bullet$ of $\mathcal{O}$-modules and maps of complexes $\mathcal{K}_{n+1}^\bullet \to \mathcal{K}_n^\bullet$ representing the maps $K_{n+1} \to K_n$ in $D(\mathcal{O})$. In other words, there exists an object K in $D(\mathcal{C} \times \mathbf{N})$ whose associated inverse system is the given one. Next, consider the commutative diagram

$$\begin{tikzcd} Sh(\mathcal{C} \times \mathbf{N}) \arrow[r, "g"'] \arrow[d, "f \times 1"'] & Sh(\mathcal{C}) \arrow[d, "f"'] \\ Sh(\mathcal{C}' \times \mathbf{N}) \arrow[r, "g'"] & Sh(\mathcal{C}') \end{tikzcd}$$

of morphisms of topoi. It follows that $R\lim R(f \times 1)_*K = Rf_*R\lim K$. Working through the definitions and using Lemma 21.1 we obtain that $R\lim(Rf_*K_n) = Rf_*(R\lim K_n)$.

Alternate proof in case $\mathcal{C}$ has enough points. Consider the defining distinguished triangle

$$R\lim K_n \to \prod K_n \to \prod K_n$$

in $D(\mathcal{O})$. Applying the exact functor Rf_* we obtain the distinguished triangle

$$Rf_*(R\lim K_n) \to Rf_*\left(\prod K_n\right) \to Rf_*\left(\prod K_n\right)$$

in $D(\mathcal{O}')$. Thus we see that it suffices to prove that Rf_* commutes with products in the derived category (which are not just given by products of complexes, see Injectives, Lemma 13.4). However, since Rf_* is a right adjoint by Lemma 19.1 this follows formally (see Categories, Lemma 24.4). Caution: Note that we cannot apply Categories, Lemma 24.4 directly as $R\lim K_n$ is not a limit in $D(\mathcal{O})$. □

22. Producing K-injective resolutions

First a technical lemma about cohomology sheaves of termwise limits of inverse systems of complexes of modules.

Lemma 22.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{F}_n^\bullet)$ be an inverse system of complexes of $\mathcal{O}$-modules. Let $m \in \mathbf{Z}$. Suppose given $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ and an integer n_0 such that*

1. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
2. *for every $U \in \mathcal{B}$*
   (a) *the systems of abelian groups $\mathcal{F}_n^{m-2}(U)$ and $\mathcal{F}_n^{m-1}(U)$ have vanishing $R^1\lim$ (for example these have the Mittag-Leffler property),*
   (b) *the system of abelian groups $H^{m-1}(\mathcal{F}_n^\bullet(U))$ has vanishing $R^1\lim$ (for example it has the Mittag-Leffler property), and*
   (c) *we have $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$ for all $n \geq n_0$.*

Then the maps $H^m(\mathcal{F}^\bullet) \to \lim H^m(\mathcal{F}_n^\bullet) \to H^m(\mathcal{F}_{n_0}^\bullet)$ are isomorphisms of sheaves where $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$ be the termwise inverse limit.

Proof. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{F}^\bullet(U))$ is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \to \lim_n \mathcal{F}_n^{m-1}(U) \to \lim_n \mathcal{F}_n^m(U) \to \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply More on Algebra, Lemma 61.2 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all $n \geq n_0$. By assumption (1) we conclude that the sheafification of $U \mapsto H^m(\mathcal{F}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$ for all $n \geq n_0$. Thus the inverse system of sheaves $H^m(\mathcal{F}_n^\bullet)$ is constant for $n \geq n_0$ with value $H^m(\mathcal{F}^\bullet)$ which proves the lemma. $\square$

The following lemma computes the cohomology sheaves of the derived limit in a special case.

Lemma 22.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $d \in \mathbf{N}$. Assume*

1. *K_n is an object of $D^-(\mathcal{O})$ for all n,*
2. *for $q \in \mathbf{Z}$ there exists $n(q)$ such that $H^q(K_{n+1}) \to H^q(K_n)$ is an isomorphism for $n \geq n(p)$,*
3. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
4. *for every $U \in \mathcal{B}$ we have $H^p(U, H^q(K_n)) = 0$ for $p > d$ and all q.*

Then we have $H^m(R\lim K_n) = \lim H^m(K_n)$ for all $m \in \mathbf{Z}$.

Proof. Set $K = R\lim K_n$. Let $U \in \mathcal{B}$. For each n there is a spectral sequence

$$H^p(U, H^q(K_n)) \Rightarrow H^{p+q}(U, K_n)$$

which converges as K_n is bounded below, see Derived Categories, Lemma 21.3. If we fix $m \in \mathbf{Z}$, then we see from our assumption (4) that only $H^p(U, H^q(K_n))$ contribute to $H^m(U, K_n)$ for $0 \leq p \leq d$ and $m - d \leq q \leq m$. By assumption (2) this implies that $H^m(U, K_{n+1}) \to H^m(U, K_n)$ is an isomorphism as soon as $n \geq \max n(m), \ldots, n(m-d)$. The functor $R\Gamma(U, -)$ commutes with derived limits by Injectives, Lemma 13.6. Thus we have

$$H^m(U, K) = H^m(R\lim R\Gamma(U, K_n))$$

On the other hand we have just seen that the complexes $R\Gamma(U, K_n)$ have eventually constant cohomology groups. Thus by More on Algebra, Remark 61.16 we find that $H^m(U, K)$ is equal to $H^m(U, K_n)$ for all $n \gg 0$ for some bound independent of $U \in \mathcal{B}$. Pick such an n. Finally, recall that $H^m(K)$ is the sheafification of the presheaf $U \mapsto H^m(U, K)$ and $H^m(K_n)$ is the sheafification of the presheaf $U \mapsto H^m(U, K_n)$. On the elements of $\mathcal{B}$ these presheaves have the same values. Therefore assumption (3) guarantees that the sheafifications are the same too. The lemma follows. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules. The category $\mathit{Mod}(\mathcal{O})$ has enough injectives, hence we can use Derived Categories, Lemma 28.3 produce a diagram

$$\begin{tikzcd} \ldots \arrow[r] & \tau_{\geq -2}\mathcal{F}^\bullet \arrow[r] \arrow[d] & \tau_{\geq -1}\mathcal{F}^\bullet \arrow[d] \\ \ldots \arrow[r] & \mathcal{I}_2^\bullet \arrow[r] & \mathcal{I}_1^\bullet \end{tikzcd}$$

in the category of complexes of $\mathcal{O}$-modules such that

1. the vertical arrows are quasi-isomorphisms,
2. $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,
3. the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 29.4 and 29.7 this is a K-injective complex. In general the canonical map

$$\mathcal{F}^\bullet \to \mathcal{I}^\bullet \tag{22.2.1}$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

Lemma 22.3. *In the situation described above. Denote $\mathcal{H}^i = H^i(\mathcal{F}^\bullet)$ the ith cohomology sheaf. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $d \in \mathbf{N}$. Assume*

1. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
2. *for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$*[2].

Then (22.2.1) is a quasi-isomorphism.

Proof. Let $m \in \mathbf{Z}$. We have to show that the map $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism $\mathcal{H}^m \to H^m(\mathcal{I}^\bullet)$. Since $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}\mathcal{F}^\bullet$ it suffices to show that $H^m(\mathcal{I}^\bullet) \to H^m(\mathcal{I}_n^\bullet)$ is an isomorphism for n large enough. To do this we will verify the hypotheses (1), (2)(a), (2)(b), (2)(c) of Lemma 22.1.

Hypothesis (1) is assumption (1) above. Hypothesis (2)(a) follows from the fact that the maps $\mathcal{I}_{n+1}^k \to \mathcal{I}_n^k$ are split surjections. We will prove hypothesis (2)(b) and (2)(c) simultaneously by proving that for $U \in \mathcal{B}$ the system $H^m(\mathcal{I}_n^\bullet(U))$ becomes constant for $n \geq -m + d$. Namely, recalling that $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}\mathcal{F}^\bullet$ we obtain for all n a distinguished triangle

$$\mathcal{H}^{-n}[n] \to \mathcal{I}_n^\bullet \to \mathcal{I}_{n-1}^\bullet \to \mathcal{H}^{-n}[n+1]$$

(Derived Categories, Remark 12.4) in $D(\mathcal{O})$. By assumption (2) we see that if $m > d - n$ then

$$H^m(U, \mathcal{H}^{-n}[n]) = 0 \quad \text{and} \quad H^m(U, \mathcal{H}^{-n}[n+1]) = 0.$$

Observe that $H^m(\mathcal{I}_n^\bullet(U)) = H^m(U, \mathcal{I}_n^\bullet)$ as $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives. Unwinding the long exact sequence of cohomology associated to the distinguished triangle above this implies that

$$H^m(\mathcal{I}_n^\bullet(U)) \to H^m(\mathcal{I}_{n-1}^\bullet(U))$$

is an isomorphism for $m > d - n$, i.e., $n > d - m$ and we win. $\square$

Lemma 22.4. *With assumptions and notation as in Lemma 22.3. Let K denote the object of $D(\mathcal{O})$ represented by the complex $\mathcal{F}^\bullet$. Then $K = R\lim \tau_{\geq -n} K$, i.e., K is the derived limit of its canonical truncations.*

Proof. First proof. Injectives, Lemma 13.4 shows that $\prod \tau_{\geq -n} K$ is represented by the complex $\prod \mathcal{I}_n^\bullet$. Because the transition maps $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections, we have a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to \prod \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to 0$$

Since $\mathcal{I}^\bullet$ represents K by Lemma 22.3 the distinguished triangle of the lemma is the distinguished triangle associated to the short exact sequence above (Derived Categories, Lemma 12.1).

[2] In fact, analyzing the proof we see that it suffices if there exists a function $d : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ such that $H^p(U, \mathcal{H}^q) = 0$ for $p > d(q)$ where $q + d(q) \to -\infty$ as $q \to -\infty$

Second proof. Apply Lemma 22.2 to see that the cohomology sheaves of $R\lim \tau_{\geq -n}K$ are isomorphic to the cohomology sheaves of K. $\square$

Here is another case where we can describe the derived limit.

Lemma 22.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

1. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
2. *for all $U \in \mathcal{B}$ and all $q \in \mathbf{Z}$ we have*
   (a) *$H^p(U, H^q(K_n)) = 0$ for $p > 0$,*
   (b) *the inverse system $H^0(U, H^q(K_n))$ has vanishing $R^1\lim$.*

Then $H^q(R\lim K_n) = \lim H^q(K_n)$ for $q \in \mathbf{Z}$ and $R^t \lim H^q(K_n) = 0$ for $t > 0$.

Proof. Observe that $K_n = R\lim_m \tau_{\geq -m}K_n$ by Lemma 22.4. Let $U \in \mathcal{B}$. Then we get $H^q(U, K_n) = H^q(R\lim_m R\Gamma(U, \tau_{\geq -m}K_n))$ because $R\Gamma(U, -)$ commutes with derived limits by Injectives, Lemma 13.6. For each m condition (2)(a) imply $H^q(U, \tau_{\geq -m}K_n) = H^0(U, H^q(\tau_{\geq -m}K_n))$ for all q, n by using the spectral sequence of Derived Categories, Lemma 21.3. The spectral sequence converges because $\tau_{\geq -m}K_n$ is bounded below (and so this argument simplifies considerably when K_n is bounded below). This value is constant and equal to $H^0(U, H^q(K_n))$ for $m > |q|$. We conclude that $H^q(U, K_n) = H^0(U, H^q(K_n))$.

Using again that the functor $R\Gamma(U, -)$ commutes with derived limits we have

$$H^q(U, K) = H^q(R\lim R\Gamma(U, K_n)) = \lim H^0(U, H^q(K_n))$$

where the final equality follows from More on Algebra, Remark 61.16 and assumption (2)(b). Thus $H^q(U, K)$ is the inverse limit the sections of the sheaves $H^q(K_n)$ over U. Since $\lim H^q(K_n)$ is a sheaf we find using assumption (1) that $H^q(K)$, which is the sheafification of the presheaf $U \mapsto H^q(U, K)$, is equal to $\lim H^q(K_n)$. This proves the first statement. Applying this to the inverse system $(H^q(K_n)[0])$ the second assertion follows also. $\square$

The construction above can be used in the following setting. Let $\mathcal{C}$ be a category. Let $\mathrm{Cov}(\mathcal{C}) \supset \mathrm{Cov}'(\mathcal{C})$ be two ways to endow $\mathcal{C}$ with the structure of a site. Denote τ the topology corresponding to $\mathrm{Cov}(\mathcal{C})$ and τ' the topology corresponding to $\mathrm{Cov}'(\mathcal{C})$. Then the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

where ϵ_* is the identity functor on underlying presheaves and where ϵ^{-1} is the τ-sheafification of a τ'-sheaf (hence clearly exact). Let $\mathcal{O}$ be a sheaf of rings for the τ-topology. Then $\mathcal{O}$ is also a sheaf for the τ'-topology and ϵ becomes a morphism of ringed sites

$$\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$$

In this situation we can sometimes point out subcategories of $D(\mathcal{O}_\tau)$ and $D(\mathcal{O}_{\tau'})$ which are identified by the functors ϵ^* and $R\epsilon_*$.

Lemma 22.6. *With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $\mathcal{A} \subset PMod(\mathcal{O})$ be a full subcategory. Assume*

1. *every object of $\mathcal{A}$ is a sheaf for the τ-topology,*
2. *$\mathcal{A}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\tau)$,*
3. *every object of $\mathcal{C}$ has a τ'-covering whose members are elements of $\mathcal{B}$, and*
4. *for every $U \in \mathcal{B}$ we have $H^p_\tau(U, \mathcal{F}) = 0$, $p > 0$ for all $\mathcal{F} \in \mathcal{A}$.*

Then $\mathcal{A}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\tau)$ and there is an equivalence of triangulated categories $D_\mathcal{A}(\mathcal{O}_\tau) = D_\mathcal{A}(\mathcal{O}_{\tau'})$ given by ϵ^* and $R\epsilon_*$.

Proof. Note that for $A \in \mathcal{A}$ we can think of A as a sheaf in either topology and (abusing notation) that $\epsilon_*A = A$ and $\epsilon^*A = A$. Consider an exact sequence

$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$

in $Mod(\mathcal{O}_{\tau'})$ with A_0, A_1, A_3, A_4 in $\mathcal{A}$. We have to show that A_2 is an element of $\mathcal{A}$, see Homology, Definition 9.1. Apply the exact functor $\epsilon^* = \epsilon^{-1}$ to conclude that ϵ^*A_2 is an object of $\mathcal{A}$. Consider the map of sequences

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & A_4 \arrow[d] \\ A_0 \arrow[r] & A_1 \arrow[r] & \epsilon_*\epsilon^*A_2 \arrow[r] & A_3 \arrow[r] & A_4 \end{tikzcd}$$

to conclude that $A_2 = \epsilon_*\epsilon^*A_2$ is an object of $\mathcal{A}$. At this point it makes sense to talk about the derived categories $D_\mathcal{A}(\mathcal{O}_\tau)$ and $D_\mathcal{A}(\mathcal{O}_{\tau'})$, see Derived Categories, Section 13.

Since ϵ^* is exact and preserves $\mathcal{A}$, it is clear that we obtain a functor $\epsilon^* : D_\mathcal{A}(\mathcal{O}_{\tau'}) \to D_\mathcal{A}(\mathcal{O}_\tau)$. We claim that $R\epsilon_*$ is a quasi-inverse. Namely, let $\mathcal{F}^\bullet$ be an object of $D_\mathcal{A}(\mathcal{O}_\tau)$. Construct a map $\mathcal{F}^\bullet \to \mathcal{I}^\bullet = \lim \mathcal{I}_n^\bullet$ as in (22.2.1). By Lemma 22.3 and assumption (4) we see that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism. Then

$$R\epsilon_*\mathcal{F}^\bullet = \epsilon_*\mathcal{I}^\bullet = \lim_n \epsilon_*\mathcal{I}_n^\bullet$$

For every $U \in \mathcal{B}$ we have

$$H^m(\epsilon_*\mathcal{I}_n^\bullet(U)) = H^m(\mathcal{I}_n^\bullet(U)) = \begin{cases} H^m(\mathcal{F}^\bullet)(U) & \text{if} \quad m \geq -n \\ 0 & \text{if} \quad m < n \end{cases}$$

by the assumed vanishing of (4), the spectral sequence Derived Categories, Lemma 21.3, and the fact that $\tau_{\geq -n}\mathcal{F}^\bullet \to \mathcal{I}_n^\bullet$ is a quasi-isomorphism. The maps $\epsilon_*\mathcal{I}_{n+1}^\bullet \to \epsilon_*\mathcal{I}_n^\bullet$ are termwise split surjections as ϵ_* is a functor. Hence we can apply Homology, Lemma 27.7 to the sequence of complexes

$$\lim_n \epsilon_*\mathcal{I}_n^{m-2}(U) \to \lim_n \epsilon_*\mathcal{I}_n^{m-1}(U) \to \lim_n \epsilon_*\mathcal{I}_n^{m}(U) \to \lim_n \epsilon_*\mathcal{I}_n^{m+1}(U)$$

to conclude that $H^m(\epsilon_*\mathcal{I}^\bullet(U)) = H^m(\mathcal{F}^\bullet)(U)$ for $U \in \mathcal{B}$. Sheafifying and using property (3) this proves that $H^m(\epsilon_*\mathcal{I}^\bullet)$ is isomorphic to $\epsilon_*H^m(\mathcal{F}^\bullet)$, i.e., is an object of $\mathcal{A}$. Thus $R\epsilon_*$ indeed gives rise to a functor

$$R\epsilon_* : D_\mathcal{A}(\mathcal{O}_\tau) \longrightarrow D_\mathcal{A}(\mathcal{O}_{\tau'})$$

For $\mathcal{F}^\bullet \in D_\mathcal{A}(\mathcal{O}_\tau)$ the adjunction map $\epsilon^*R\epsilon_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet$ is a quasi-isomorphism as we've seen above that the cohomology sheaves of $R\epsilon_*\mathcal{F}^\bullet$ are $\epsilon_*H^m(\mathcal{F}^\bullet)$. For $\mathcal{G}^\bullet \in D_\mathcal{A}(\mathcal{O}_{\tau'})$ the adjunction map $\mathcal{G}^\bullet \to R\epsilon_*\epsilon^*\mathcal{G}^\bullet$ is a quasi-isomorphism for the same reason, i.e., because the cohomology sheaves of $R\epsilon_*\epsilon^*\mathcal{G}^\bullet$ are isomorphic to $\epsilon_*H^m(\epsilon^*\mathcal{G}) = H^m(\mathcal{G}^\bullet)$. $\square$

23. Cohomology on Hausdorff and locally quasi-compact spaces

We continue our convention to say "Hausdorff and locally quasi-compact" instead of saying "locally compact" as is often done in the literature. Let LC denote the category whose objects are Hausdorff and locally quasi-compact topological spaces and whose morphisms are continuous maps.

Lemma 23.1. *The category LC has fibre products and a final object and hence has arbitrary finite limits. Given morphisms $X \to Z$ and $Y \to Z$ in LC with X and Y quasi-compact, then $X \times_Z Y$ is quasi-compact.*

Proof. The final object is the singleton space. Given morphisms $X \to Z$ and $Y \to Z$ of LC the fibre product $X \times_Z Y$ is a subspace of $X \times Y$. Hence $X \times_Z Y$ is Hausdorff as $X \times Y$ is Hausdorff by Topology, Section 3.

If X and Y are quasi-compact, then $X \times Y$ is quasi-compact by Topology, Theorem 13.4. Since $X \times_Z Y$ is a closed subset of $X \times Y$ (Topology, Lemma 3.4) we find that $X \times_Z Y$ is quasi-compact by Topology, Lemma 11.3.

Finally, returning to the general case, if $x \in X$ and $y \in Y$ we can pick quasi-compact neighbourhoods $x \in E \subset X$ and $y \in F \subset Y$ and we find that $E \times_Z F$ is a quasi-compact neighbourhood of (x, y) by the result above. Thus $X \times_Z Y$ is an object of LC by Topology, Lemma 12.2. $\square$

We can endow LC with a stronger topology than the usual one.

Definition 23.2. Let $\{f_i : X_i \to X\}$ be a family of morphisms with fixed target in the category LC. We say this family is a *qc covering*[3] if for every $x \in X$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x.

Observe that an open covering $X = \bigcup U_i$ of an object of LC gives a qc covering $\{U_i \to X\}$ because X is locally quasi-compact. We start with the obligatory lemma.

Lemma 23.3. *Let X be a Hausdorff and locally quasi-compact space, in other words, an object of LC.*

1. *If $X' \to X$ is an isomorphism in LC then $\{X' \to X\}$ is a qc covering.*
2. *If $\{f_i : X_i \to X\}_{i \in I}$ is a qc covering and for each i we have a qc covering $\{g_{ij} : X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a qc covering.*
3. *If $\{X_i \to X\}_{i \in I}$ is a qc covering and $X' \to X$ is a morphism of LC then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a qc covering.*

Proof. Part (1) holds by the remark above that open coverings are qc coverings.

Proof of (2). Let $x \in X$. Choose $i_1, \ldots, i_n \in I$ and $E_a \subset X_{i_a}$ quasi-compact such that $\bigcup f_{i_a}(E_a)$ is a neighbourhood of x. For every $e \in E_a$ we can find a finite subset $J_e \subset J_{i_a}$ and quasi-compact $F_{e,j} \subset X_{ij}$, $j \in J_e$ such that $\bigcup g_{ij}(F_{e,j})$ is a neighbourhood of e. Since E_a is quasi-compact we find a finite collection $e_1, \ldots, e_{m_a}$ such that

$$E_a \subset \bigcup\nolimits_{k=1,\ldots,m_a} \bigcup\nolimits_{j \in J_{e_k}} g_{ij}(F_{e_k,j})$$

Then we find that

$$\bigcup\nolimits_{a=1,\ldots,n} \bigcup\nolimits_{k=1,\ldots,m_a} \bigcup\nolimits_{j \in J_{e_k}} f_i(g_{ij}(F_{e_k,j}))$$

[3] This is nonstandard notation. We chose it to remind the reader of fpqc coverings of schemes.

is a neighbourhood of x.

Proof of (3). Let $x' \in X'$ be a point. Let $x \in X$ be its image. Choose $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x. Choose a quasi-compact neighbourhood $F \subset X'$ of x' which maps into the quasi-compact neighbourhood $\bigcup f_{i_j}(E_j)$ of x. Then $F \times_X E_j \subset X' \times_X X_{i_j}$ is a quasi-compact subset and F is the image of the map $\coprod F \times_X E_j \to F$. Hence the base change is a qc covering and the proof is finished. $\square$

Besides some set theoretic issues the lemma above shows that LC with the collection of qc coverings forms a site. We will denote this site (suitably modified to overcome the set theoretical issues) LC_{qc}.

Remark 23.4 (Set theoretic issues)**.** The category LC is a "big" category as its objects form a proper class. Similarly, the coverings form a proper class. Let us define the *size* of a topological space X to be the cardinality of the set of points of X. Choose a function *Bound* on cardinals, for example as in Sets, Equation (9.1.1). Finally, let S_0 be an initial set of objects objects of LC, for example $S_0 = \{(\mathbf{R}, \text{euclidean topology})\}$. Exactly as in Sets, Lemma 9.2 we can choose a limit ordinal α such that $LC_\alpha = LC \cap V_\alpha$ contains S_0 and is preserved under all countable limits and colimits which exist in LC. Moreover, if $X \in LC_\alpha$ and if $Y \in LC$ and $\text{size}(Y) \leq Bound(\text{size}(X))$, then Y is isomorphic to an object of LC_α. Next, we apply Sets, Lemma 11.1 to choose set Cov of qc covering on LC_α such that every qc covering in LC_α is combinatorially equivalent to a covering this set. In this way we obtain a site (LC_α, Cov) which we will denote LC_{qc}.

There is a second topology on the site LC_{qc} of Remark 23.4. Namely, given an object X we can consider all coverings $\{X_i \to X\}$ of LC_{qc} such that $X_i \to X$ is an open immersion. We denote this site LC_{Zar}. The identity functor $LC_{Zar} \to LC_{qc}$ is continuous and defines a morphism of sites

$$\epsilon : LC_{qc} \to LC_{Zar}$$

by an application of Sites, Proposition 15.6.

Consider an object X of the site LC_{qc} constructed in Remark 23.4. (Translation for those not worried about set theoretic issues: Let X be a Hausdorff and locally quasi-compact space.) Let X_{Zar} be the site whose objects are opens of X, see Sites, Example 6.4. There is a morphism of sites

$$\pi : LC_{Zar}/X \to X_{Zar}$$

given by the continuous functor

$$X_{Zar} \longrightarrow LC_{Zar}/X, \quad U \longmapsto U$$

Namely, X_{Zar} has fibre products and a final object and the functor above commutes with these and Sites, Proposition 15.6 applies.

Lemma 23.5. *Let X be an object of LC_{qc}. Let $\mathcal{F}$ be a sheaf on X_{Zar}. Then the sheaf $\pi^{-1}\mathcal{F}$ on LC_{Zar}/X is given by the rule*

$$\pi^{-1}\mathcal{F}(Y) = \Gamma(Y_{Zar}, f^{-1}\mathcal{F})$$

for $f : Y \to X$ in LC_{qc}. Moreover $\pi^{-1}\mathcal{F}$ is a sheaf for the qc topology, i.e., the sheaf $\epsilon^{-1}\pi^{-1}\mathcal{F}$ on LC_{qc} is given by the same formula.

Proof. Of course the pullback f^{-1} on the right hand side indicates usual pullback of sheaves on topological spaces (Sites, Example 15.2). The equality of the lemma follows directly from the defintions.

Let $\mathcal{V} = \{g_i : Y_i \to Y\}_{i \in I}$ be a covering of LC_{qc}/X. It suffices to show that $\pi^{-1}\mathcal{F}(Y) \to H^0(\mathcal{V}, \pi^{-1}\mathcal{F})$ is an isomorphism, see Sites, Section 10. We first point out that the map is injective as a qc covering is surjective and we can detect equality of sections at stalks (use Sheaves, Lemmas 11.1 and 21.4). Thus we see that $\pi^{-1}\mathcal{F}$ is a separated presheaf on LC_{qc} hence it suffices to show that any element $(s_i) \in H^0(\mathcal{V}, \pi^{-1}\mathcal{F})$ maps to an element in the image of $\pi^{-1}\mathcal{F}(Y)$ after replacing $\mathcal{V}$ by a refinement (Sites, Theorem 10.10).

Observe that $\pi^{-1}\mathcal{F}|_{Y_{i,Zar}}$ is the pullback of $f^{-1}\mathcal{F} = \pi^{-1}\mathcal{F}|_{Y_{Zar}}$ under the continuous map $g_i : Y_i \to Y$. Thus we can choose an open covering $Y_i = \bigcup V_{ij}$ such that for each j there is an open $W_{ij} \subset Y$ and a section $t_{ij} \in \pi^{-1}\mathcal{F}(W_{ij})$ such that $s|_{U_{ij}}$ is the pullback of t_{ij}. In other words, after refining the covering $\{Y_i \to Y\}$ we may assume there are opens $W_i \subset Y$ such that $Y_i \to Y$ factors through W_i and sections t_i of $\pi^{-1}\mathcal{F}$ over W_i which restrict to the given sections s_i. Moreover, if $y \in Y$ is in the image of both $Y_i \to Y$ and $Y_j \to Y$, then the images $t_{i,y}$ and $t_{j,y}$ in the stalk $f^{-1}\mathcal{F}_y$ agree (because s_i and s_j agree over $Y_i \times_Y Y_j$). Thus for $y \in Y$ there is a well defined element t_y of $f^{-1}\mathcal{F}_y$ agreeing with $t_{i,y}$ whenever $y \in Y_i$. We will show that the element (t_y) comes from a global section of $f^{-1}\mathcal{F}$ over Y which will finish the proof of the lemma.

It suffices to show that this is true locally on Y, see Sheaves, Section 17. Let $y_0 \in Y$. Pick $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset Y_{i_j}$ such that $\bigcup g_{i_j}(E_j)$ is a neighbourhood of y_0. Then we can find an open neighbourhood $V \subset Y$ of y_0 contained in $W_{i_1} \cap \ldots \cap W_{i_n}$ such that the sections $t_{i_j}|_V$, $j = 1, \ldots, n$ agree. Hence we see that $(t_y)_{y \in V}$ comes from this section and the proof is finished. $\square$

Lemma 23.6. *Let X be an object of LC_{qc}. Let $\mathcal{F}$ be an abelian sheaf on X_{Zar}. Then we have*
$$H^q(X_{Zar}, \mathcal{F}) = H^q(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F})$$
In particular, if A is an abelian group, then we have $H^q(X, \underline{A}) = H^q(LC_{qc}/X, \underline{A})$.

Proof. The statement is more precisely that the canonical map
$$H^q(X_{Zar}, \mathcal{F}) \longrightarrow H^q(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F})$$
is an isomorphism for all q. The result holds for $q = 0$ by Lemma 23.5. We argue by induction on q. Pick $q_0 > 0$. We will assume the result holds for $q < q_0$ and prove it for q_0.

Injective. Let $\xi \in H^{q_0}(X, \mathcal{F})$. We may choose an open covering $\mathcal{U} : X = \bigcup U_i$ such that $\xi|_{U_i}$ is zero for all i (Cohomology, Lemma 7.2). Then $\mathcal{U}$ is also a covering for the qc topology. Hence we obtain a map
$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \longrightarrow E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\epsilon^{-1}\pi^{-1}\mathcal{F}))$$
between the spectral sequences of Cohomology, Lemma 12.4 and Lemma 11.6. Since the maps $\underline{H}^q(\mathcal{F})(U_{i_0 \ldots i_p}) \to \underline{H}^q(\epsilon^{-1}\pi^{-1}\mathcal{F}))(U_{i_0 \ldots i_p})$ are isomorphisms for $q < q_0$ we see that
$$\mathrm{Ker}(H^{q_0}(X, \mathcal{F}) \to \prod H^{q_0}(U_i, \mathcal{F}))$$

maps isomorphically to the corresponding subgroup of $H^{q_0}(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F})$. In this way we conclude that our map is injective for q_0.

Surjective. Let $\xi \in H^{q_0}(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F})$. If for every $x \in X$ we can find a neighbourhood $x \in U \subset X$ such that $\xi|_U = 0$, then we can use the Čech complex argument of the previous paragraph to conclude that ξ is in the image of our map. Fix $x \in X$. We can find a qc covering $\{f_i : X_i \to X\}_{i \in I}$ such that $\xi|_{X_i}$ is zero (Lemma 8.3). Pick $i_1, \ldots, i_n \in I$ and $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x. We may replace X by $\bigcup f_{i_j}(E_j)$ and set $Y = \coprod E_{i_j}$. Then $Y \to X$ is a surjective continuous map of Hausdorff and quasi-compact topological spaces, $\xi \in H^{q_0}(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F})$, and $\xi|_Y = 0$. Set $Y_p = Y \times_X \ldots \times_X Y$ ($p+1$-factors) and denote $\mathcal{F}_p$ the pullback of $\mathcal{F}$ to Y_p. Then the spectral sequence

$$E_1^{p,q} = \check{C}^p(\{Y \to X\}, \underline{H}^q(\epsilon^{-1}\pi^{-1}\mathcal{F}))$$

of Lemma 11.6 has rows for $q < q_0$ which are (by induction) the complexes

$$H^q(Y_0, \mathcal{F}_0) \to H^q(Y_1, \mathcal{F}_1) \to H^q(Y_2, \mathcal{F}_2) \to \ldots$$

If these complexes were exact in degree $p = q_0 - q$, then the spectral sequence would collapse and ξ would be zero. This is not true in general, but we don't need to show ξ is zero, we just need to show ξ becomes zero after restricting X to a neighbourhood of x. Thus it suffices to show that the complexes

$$\text{colim}_{x \in U \subset X} \left(H^q(Y_0 \times_X U, \mathcal{F}_0) \to H^q(Y_1 \times_X U, \mathcal{F}_1) \to H^q(Y_2 \times_X U, \mathcal{F}_2) \to \ldots\right)$$

are exact (some details omitted). By the proper base change theorem in topology (for example Cohomology, Lemma 19.1) the colimit is equal to

$$H^q(Y_x, \underline{\mathcal{F}_x}) \to H^q(Y_x^2, \underline{\mathcal{F}_x}) \to H^q(Y_x^3, \underline{\mathcal{F}_x}) \to \ldots$$

where $Y_x \subset Y$ is the fibre of $Y \to X$ over x and where $\underline{\mathcal{F}_x}$ denotes the constant sheaf with value $\mathcal{F}_x$. But the simplicial topological space (Y_x^n) is homotopy equivalent to the constant simplicial space on the singleton $\{x\}$, see Simplicial, Lemma 25.9. Since $H^q(-, \underline{\mathcal{F}_x})$ is a functor on the category of topological spaces, we conclude that the cosimplicial abelian group with values $H^q(Y_x^n, \underline{\mathcal{F}_x})$ is homotopy equivalent to the constant cosimplicial abelian group with value

$$H^q(\{x\}, \mathcal{F}_x) = \begin{cases} \mathcal{F}_x & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$$

As the complex associated to a constant cosimplicial group has the required exactness properties this finishes the proof of the lemma. □

Lemma 23.7. *Let $f : X \to Y$ be a morphism of LC. If f is proper and surjective, then $\{f : X \to Y\}$ is a qc covering.*

Proof. Let $y \in Y$ be a point. For each $x \in X_y$ choose a quasi-compact neighbourhood $E_x \subset X$. Choose $x \in U_x \subset E_x$ open. Since f is proper the fibre X_y is quasi-compact and we find $x_1, \ldots, x_n \in X_y$ such that $X_y \subset U_{x_1} \cup \ldots \cup U_{x_n}$. We claim that $f(E_{x_1}) \cup \ldots \cup f(E_{x_n})$ is a neighbourhood of y. Namely, as f is closed (Topology, Theorem 16.5) we see that $Z = f(X \setminus U_{x_1} \cup \ldots \cup U_{x_n})$ is a closed subset of Y not containing y. As f is surjective we see that $Y \setminus Z$ is contained in $f(E_{x_1}) \cup \ldots \cup f(E_{x_n})$ as desired. □

24. Spectral sequences for Ext

In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of complexes $\mathcal{G}^\bullet, \mathcal{F}^\bullet$ of complexes of modules on a ringed site $(\mathcal{C}, \mathcal{O})$ we denote

$$\mathrm{Ext}^n_\mathcal{O}(\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{F}^\bullet[n])$$

according to our general conventions in Derived Categories, Section 27.

Example 24.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a bounded above complex of $\mathcal{O}$-modules. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Then there is a spectral sequence with E_2-page

$$E_2^{i,j} = \mathrm{Ext}^i_\mathcal{O}(H^{-j}(\mathcal{K}^\bullet), \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_\mathcal{O}(\mathcal{K}^\bullet, \mathcal{F})$$

and another spectral sequence with E_1-page

$$E_1^{i,j} = \mathrm{Ext}^j_\mathcal{O}(\mathcal{K}^{-i}, \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_\mathcal{O}(\mathcal{K}^\bullet, \mathcal{F}).$$

To construct these spectral sequences choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and consider the two spectral sequences coming from the double complex $\mathrm{Hom}_\mathcal{O}(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$, see Homology, Section 22.

25. Hom complexes

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ be two complexes of $\mathcal{O}$-modules. We construct a complex of $\mathcal{O}$-modules $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. Namely, for each n we set

$$\mathcal{H}om^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = \prod\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

It is a good idea to think of $\mathcal{H}om^n$ as the sheaf of $\mathcal{O}$-modules of all $\mathcal{O}$-linear maps from $\mathcal{L}^\bullet$ to $\mathcal{M}^\bullet$ (viewed as graded $\mathcal{O}$-modules) which are homogenous of degree n. In this terminology, we define the differential by the rule

$$\mathrm{d}(f) = \mathrm{d}_\mathcal{M} \circ f - (-1)^n f \circ \mathrm{d}_\mathcal{L}$$

for $f \in \mathcal{H}om^n_\mathcal{O}(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. We omit the verification that $\mathrm{d}^2 = 0$. This construction is a special case of Differential Graded Algebra, Example 19.6. It follows immediately from the construction that we have

$$H^n(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n]) \tag{25.0.1}$$

for all $n \in \mathbf{Z}$ and every $U \in \mathrm{Ob}(\mathcal{C})$. Similarly, we have

$$H^n(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n]) \tag{25.0.2}$$

for the complex of global sections.

Lemma 25.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is an isomorphism*

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)) = \mathcal{H}om^\bullet(\mathit{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules functorial in $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 54.1. $\square$

Lemma 25.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$\mathit{Tot}(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 54.2. □

Lemma 25.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$\mathit{Tot}(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 54.3. □

Lemma 25.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism*

$$\mathcal{K}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}$-modules functorial in both complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 54.5. □

Lemma 25.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be a complex of $\mathcal{O}$-modules. Then*

$$H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

for all $U \in \mathrm{Ob}(\mathcal{C})$. Similarly, $H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L, M)$.

Proof. We have

$$\begin{aligned} H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) &= \mathrm{Hom}_{K(\mathcal{O}_U)}(L|_U, M|_U) \\ &= \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U) \end{aligned}$$

The first equality is (25.0.1). The second equality is true because $\mathcal{I}^\bullet|_U$ is K-injective by Lemma 20.1. The proof of the last equation is similar except that it uses (25.0.2). □

Lemma 25.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{I}')^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism of K-injective complexes of $\mathcal{O}$-modules. Let $(\mathcal{L}')^\bullet \to \mathcal{L}^\bullet$ be a quasi-isomorphism of complexes of $\mathcal{O}$-modules. Then*

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet) \longrightarrow \mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet)$$

is a quasi-isomorphism.

Proof. Let M be the object of $D(\mathcal{O})$ represented by $\mathcal{I}^\bullet$ and $(\mathcal{I}')^\bullet$. Let L be the object of $D(\mathcal{O})$ represented by $\mathcal{L}^\bullet$ and $(\mathcal{L}')^\bullet$. By Lemma 25.5 we see that the sheaves

$$H^0(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet)) \quad \text{and} \quad H^0(\mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet))$$

are both equal to the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

Thus the map is a quasi-isomorphism. □

Lemma 25.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be a K-flat complex of $\mathcal{O}$-modules. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}$-modules.*

Proof. Namely, if $\mathcal{K}^\bullet$ is an acyclic complex of $\mathcal{O}$-modules, then

$$\begin{aligned}
\mathrm{Hom}_{K(\mathcal{O})}(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) &= H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)))) \\
&= H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet))) \\
&= \mathrm{Hom}_{K(\mathcal{O})}(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet) \\
&= 0
\end{aligned}$$

The first equality by (25.0.2). The second equality by Lemma 25.1. The third equality by (25.0.2). The final equality because $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$ is acyclic because $\mathcal{L}^\bullet$ is K-flat (Definition 17.2) and because $\mathcal{I}^\bullet$ is K-injective. □

26. Internal hom in the derived category

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L, M be objects of $D(\mathcal{O})$. We would like to construct an object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O})$ such that for every third object K of $D(\mathcal{O})$ there exists a canonical bijection

(26.0.1) $$\mathrm{Hom}_{D(\mathcal{O})}(K, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O})}(K \otimes^{\mathbf{L}}_{\mathcal{O}} L, M)$$

Observe that this formula defines $R\mathcal{H}om(L, M)$ up to unique isomorphism by the Yoneda lemma (Categories, Lemma 3.5).

To construct such an object, choose a K-injective complex of $\mathcal{O}$-modules $\mathcal{I}^\bullet$ representing M and any complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. Then we set Then we set

$$R\mathcal{H}om(L, M) = \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where the right hand side is the complex of $\mathcal{O}$-modules constructed in Section 25. This is well defined by Lemma 25.6. We get a functor

$$D(\mathcal{O})^{opp} \times D(\mathcal{O}) \longrightarrow D(\mathcal{O}), \quad (K, L) \longmapsto R\mathcal{H}om(K, L)$$

As a prelude to proving (26.0.1) we compute the cohomology groups of $R\mathcal{H}om(K, L)$.

Lemma 26.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. For every object U of $\mathcal{C}$ we have*

$$H^0(U, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

and we have $H^0(\mathcal{C}, R\mathcal{H}om(L, M) = \mathrm{Hom}_{D(\mathcal{O})}(L, M)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}$-modules representing M and a K-flat complex $\mathcal{L}^\bullet$ representing L. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is K-injective by Lemma 25.7. Hence we can compute cohomology over U by simply taking sections over U and the result follows from Lemma 25.5. □

Lemma 26.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L, M be objects of $D(\mathcal{O})$. With the construction as described above there is a canonical isomorphism*

$$R\mathcal{H}om(K, R\mathcal{H}om(L, M)) = R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}} L, M)$$

in $D(\mathcal{O})$ functorial in K, L, M which recovers (26.0.1) on taking $H^0(\mathcal{C}, -)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M and a K-flat complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. Let $\mathcal{H}^\bullet$ be the complex described above. For any complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ we have

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) = \mathcal{H}om^\bullet(\text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet), \mathcal{I}^\bullet)$$

by Lemma 25.1. Note that the left hand side represents $R\mathcal{H}om(K, R\mathcal{H}om(L, M))$ (use Lemma 25.7) and that the right hand side represents $R\mathcal{H}om(K \otimes_\mathcal{O}^\mathbf{L} L, M)$. This proves the displayed formula of the lemma. Taking global sections and using Lemma 26.1 we obtain (26.0.1). $\square$

Lemma 26.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. The construction of $R\mathcal{H}om(K, L)$ commutes with restrictions, i.e., for every object U of $\mathcal{C}$ we have $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$.*

Proof. This is clear from the construction and Lemma 20.1. $\square$

Lemma 26.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The bifunctor $R\mathcal{H}om(-,-)$ transforms distinguished triangles into distinguished triangles in both variables.*

Proof. This follows from the observation that the assignment

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \longmapsto \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$$

transforms a termwise split short exact sequences of complexes in either variable into a termwise split short exact sequence. Details omitted. $\square$

Lemma 26.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L, M be objects of $D(\mathcal{O})$. There is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes_\mathcal{O}^\mathbf{L} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

in $D(\mathcal{O})$ functorial in K, L, M.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and a K-flat complex $\mathcal{K}^\bullet$ representing K. The map is defined using the map

$$\text{Tot}(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_\mathcal{O} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet), \mathcal{I}^\bullet)$$

of Lemma 25.3. By our particular choice of complexes the left hand side represents $R\mathcal{H}om(L, M) \otimes_\mathcal{O}^\mathbf{L} K$ and the right hand side represents $R\mathcal{H}om(R\mathcal{H}om(K, L), M)$. We omit the proof that this is functorial in all three objects of $D(\mathcal{O})$. $\square$

Lemma 26.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L, M in $D(\mathcal{O})$ there is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes_\mathcal{O}^\mathbf{L} R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, M)$$

in $D(\mathcal{O})$.

Proof. In general (without suitable finiteness conditions) we do not see how to get this map from Lemma 25.2. Instead, we use the maps

$$\begin{tikzcd} R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(K, L) \otimes_{\mathcal{O}}^{\mathbf{L}} K \arrow[d] \\ R\mathcal{H}om(R\mathcal{H}om(K, L), M) \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(K, L) \arrow[d] \\ M \end{tikzcd}$$

gotten by applying Lemma 26.5 twice. Finally, we use Lemma 26.2 to translate the composition

$$R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(K, L) \otimes_{\mathcal{O}}^{\mathbf{L}} K \longrightarrow M$$

into a map as in the statement of the lemma. □

Lemma 26.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L in $D(\mathcal{O})$ there is a canonical morphism*

$$K \longrightarrow R\mathcal{H}om(L, K \otimes_{\mathcal{O}}^{\mathbf{L}} L)$$

in $D(\mathcal{O})$ functorial in both K and L.

Proof. Choose K-flat complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ represeting K and L. Choose a K-injective complex $\mathcal{I}^\bullet$ and a quasi-isomorphism $\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) \to \mathcal{I}^\bullet$. Then we use

$$\mathcal{K}^\bullet \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where the first map comes from Lemma 25.4. □

Lemma 26.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L be an object of $D(\mathcal{O})$. Set $L^\wedge = R\mathcal{H}om(L, \mathcal{O})$. For M in $D(\mathcal{O})$ there is a canonical map*

$$L^\wedge \otimes_{\mathcal{O}}^{\mathbf{L}} M \longrightarrow R\mathcal{H}om(L, M) \tag{26.8.1}$$

which induces a canonical map

$$H^0(\mathcal{C}, L^\wedge \otimes_{\mathcal{O}}^{\mathbf{L}} M) \longrightarrow \text{Hom}_{D(\mathcal{O})}(L, M)$$

functorial in M in $D(\mathcal{O})$.

Proof. The map (26.8.1) is a special case of Lemma 26.6 using the identification $M = R\mathcal{H}om(\mathcal{O}, M)$. □

Remark 26.9. Let $h : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let K, L be objects of $D(\mathcal{O}')$. We claim there is a canonical map

$$Lh^* R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O})$. Namely, by (26.0.1) proved in Lemma 26.2 such a map is the same thing as a map

$$Lh^* R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} Lh^*K \longrightarrow Lh^*L$$

The source of this arrow is $Lh^*(\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K)$ by Lemma 18.4 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\text{id} : R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, L)$$

via (26.0.1).

Remark 26.10. Suppose that

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "h"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "f"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{tikzcd}$$

is a commutative diagram of ringed topoi. Let K, L be objects of $D(\mathcal{O}_\mathcal{C})$. We claim there exists a canonical base change map

$$Lg^*Rf_*R\mathcal{H}om(K, L) \longrightarrow R(f')_*R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, we take the map adjoint to the composition

$$\begin{aligned} L(f')^*Lg^*Rf_*R\mathcal{H}om(K, L) &= Lh^*Lf^*Rf_*R\mathcal{H}om(K, L) \\ &\to Lh^*R\mathcal{H}om(K, L) \\ &\to R\mathcal{H}om(Lh^*K, Lh^*L) \end{aligned}$$

where the first arrow uses the adjunction mapping $Lf^*Rf_* \to \text{id}$ and the second arrow is the canonical map constructed in Remark 26.9.

27. Derived lower shriek

In this section we study some situations where besides Lf^* and Rf_* there also a derived functor $Lf_!$.

Lemma 27.1. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites which induces a morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. The functor $g_! : Mod(\mathcal{O}_\mathcal{C}) \to Mod(\mathcal{O}_\mathcal{D})$ (see Modules on Sites, Lemma 40.1) has a left derived functor*

$$Lg_! : D(\mathcal{O}_\mathcal{C}) \longrightarrow D(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*. Moreover, for $U \in \text{Ob}(\mathcal{C})$ we have

$$Lg_!(j_{U!}\mathcal{O}_U) = g_!j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}.$$

where $j_{U!}$ and $j_{u(U)!}$ are extension by zero associated to the localization morphism $j_U : \mathcal{C}/U \to \mathcal{C}$ and $j_{u(U)} : \mathcal{D}/u(U) \to \mathcal{D}$.

Proof. We are going to use Derived Categories, Proposition 28.2 to construct $Lg_!$. To do this we have to verify assumptions (1), (2), (3), (4), and (5) of that proposition. First, since $g_!$ is a left adjoint we see that it is right exact and commutes with all colimits, so (5) holds. Conditions (3) and (4) hold because the category of modules on a ringed site is a Grothendieck abelian category. Let $\mathcal{P} \subset \text{Ob}(Mod(\mathcal{O}_\mathcal{C}))$ be the collection of $\mathcal{O}_\mathcal{C}$-modules which are direct sums of modules of the form $j_{U!}\mathcal{O}_U$. Note that $g_!j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$, see proof of Modules on Sites, Lemma 40.1. Every $\mathcal{O}_\mathcal{C}$-module is a quotient of an object of $\mathcal{P}$, see Modules on Sites, Lemma 28.6. Thus (1) holds. Finally, we have to prove (2). Let $\mathcal{K}^\bullet$ be a bounded above acyclic complex of $\mathcal{O}_\mathcal{C}$-modules with $\mathcal{K}^n \in \mathcal{P}$ for all n. We have to show that $g_!\mathcal{K}^\bullet$ is exact. To do this it suffices to show, for every injective $\mathcal{O}_\mathcal{D}$-module $\mathcal{I}$ that

$$\text{Hom}_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) = 0$$

for all $n \in \mathbf{Z}$. Since $\mathcal{I}$ is injective we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) &= \mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(g_!\mathcal{K}^\bullet, \mathcal{I})) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}))\end{aligned}$$

the last equality by the adjointness of $g_!$ and g^{-1}.

The vanishing of this group would be clear if $g^{-1}\mathcal{I}$ were an injective $\mathcal{O}_\mathcal{C}$-module. But $g^{-1}\mathcal{I}$ isn't necessarily an injective $\mathcal{O}_\mathcal{C}$-module as $g_!$ isn't exact in general. We do know that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{I}) = H^p(U, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1$$

Namely, the first equality follows from $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, \mathcal{H}) = \mathcal{H}(U)$ and taking derived functors. The vanishing of $H^p(U, g^{-1}\mathcal{I})$ for all $U \in \mathrm{Ob}(\mathcal{C})$ comes from the vanishing of all higher Čech cohomology groups $\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I})$ via Lemma 11.9. Namely, for a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ in $\mathcal{C}$ we have $\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I}) = \check{H}^p(u(\mathcal{U}), \mathcal{I})$. Since $\mathcal{I}$ is an injective $\mathcal{O}$-module these Čech cohomology groups vanish, see Lemma 12.3. Since each $\mathcal{K}^{-q}$ is a direct sum of modules of the form $j_{U!}\mathcal{O}_U$ we see that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1 \text{ and all } q$$

Let us use the spectral sequence (see Example 24.1)

$$E_1^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}) = 0.$$

Note that the spectral sequence abuts to zero as $\mathcal{K}^\bullet$ is acyclic (hence vanishes in the derived category, hence produces vanishing ext groups). By the vanishing of higher exts proved above the only nonzero terms on the E_1 page are the terms $E_1^{0,q} = \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I})$. We conclude that the complex $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})$ is acyclic as desired.

Thus the left derived functor $Lg_!$ exists. We still have to show that it is left adjoint to $g^{-1} = g^* = Rg^* = Lg^*$, i.e., that we have

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(\mathcal{H}^\bullet, g^{-1}\mathcal{E}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(Lg_!\mathcal{H}^\bullet, \mathcal{E}^\bullet) \tag{27.1.1}$$

This is actually a formal consequence of the discussion above. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{H}^\bullet$ such that $\mathcal{K}^\bullet$ computes $Lg_!$. Moreover, choose a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex of $\mathcal{O}_\mathcal{D}$-modules $\mathcal{I}^\bullet$. Then the RHS of (27.1.1) is

$$\mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

On the other hand, by the definition of morphisms in the derived category the LHS of (27.1.1) is

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}^\bullet) &= \mathrm{colim}_{s:\mathcal{L}^\bullet \to \mathcal{K}^\bullet} \mathrm{Hom}_{K(\mathcal{O}_\mathcal{C})}(\mathcal{L}^\bullet, g^{-1}\mathcal{I}^\bullet) \\ &= \mathrm{colim}_{s:\mathcal{L}^\bullet \to \mathcal{K}^\bullet} \mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{L}^\bullet, \mathcal{I}^\bullet)\end{aligned}$$

by the adjointness of $g_!$ and g^* on the level of sheaves of modules. The colimit is over all quasi-isomorphisms with target $\mathcal{K}^\bullet$. Since for every complex $\mathcal{L}^\bullet$ there exists a quasi-isomorphism $(\mathcal{K}')^\bullet \to \mathcal{L}^\bullet$ such that $(\mathcal{K}')^\bullet$ computes $Lg_!$ we see that we may as well take the colimit over quasi-isomorphisms of the form $s : (\mathcal{K}')^\bullet \to \mathcal{K}^\bullet$ where $(\mathcal{K}')^\bullet$ computes $Lg_!$. In this case

$$\mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!(\mathcal{K}')^\bullet, \mathcal{I}^\bullet)$$

is an isomorphism as $g_!(\mathcal{K}')^\bullet \to g_!\mathcal{K}^\bullet$ is a quasi-isomorphism and $\mathcal{I}^\bullet$ is K-injective. This finishes the proof. $\square$

Remark 27.2. Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 27.1. In general it is **not** the case that the diagram

$$\begin{tikzcd} D(\mathcal{O}_\mathcal{C}) \arrow[r, "Lg_!"'] \arrow[d, "forget"'] & D(\mathcal{O}_\mathcal{D}) \arrow[d, "forget"] \\ D(\mathcal{C}) \arrow[r, "Lg_!^{Ab}"] & D(\mathcal{D}) \end{tikzcd}$$

commutes where the functor $Lg_!^{Ab}$ is the one constructed in Lemma 27.1 but using the constant sheaf $\mathbf{Z}$ as the structure sheaf on both $\mathcal{C}$ and $\mathcal{D}$. In general it isn't even the case that $g_! = g_!^{Ab}$ (see Modules on Sites, Remark 40.2), but this phenomenon **can occur even if** $g_! = g_!^{Ab}$! Namely, the construction of $Lg_!$ in the proof of Lemma 27.1 shows that $Lg_!$ agrees with $Lg_!^{Ab}$ if and only if the canonical maps

$$Lg_!^{Ab} j_{U!}\mathcal{O}_U \longrightarrow j_{u(U)!}\mathcal{O}_{u(U)}$$

are isomorphisms in $D(\mathcal{D})$ for all objects U in $\mathcal{C}$. In general all we can say is that there exists a natural transformation

$$Lg_!^{Ab} \circ forget \longrightarrow forget \circ Lg_!$$

28. Derived lower shriek for fibred categories

In this section we work out some special cases of the situation discussed in Section 27. We make sure that we have equality between lower shriek on modules and sheaves of abelian groups. We encourage the reader to skip this section on a first reading.

Situation 28.1. Here $(\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a ringed site and $p : \mathcal{C} \to \mathcal{D}$ is a fibred category. We endow $\mathcal{C}$ with the topology inherited from $\mathcal{D}$ (Stacks, Section 10). We denote $\pi : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ the morphism of topoi associated to p (Stacks, Lemma 10.3). We set $\mathcal{O}_\mathcal{C} = \pi^{-1}\mathcal{O}_\mathcal{D}$ so that we obtain a morphism of ringed topoi

$$\pi : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

Lemma 28.2. *Assumptions and notation as in Situation 28.1. For $U \in \mathrm{Ob}(\mathcal{C})$ consider the induced morphism of topoi*

$$\pi_U : Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{D}/p(U))$$

Then there exists a morphism of topoi

$$\sigma : Sh(\mathcal{D}/p(U)) \to Sh(\mathcal{C}/U)$$

such that $\pi_U \circ \sigma = id$ and $\sigma^{-1} = \pi_{U,*}$.

Proof. Observe that π_U is the restriction of π to the localizations, see Sites, Lemma 27.4. For an object $V \to p(U)$ of $\mathcal{D}/p(U)$ denote $V \times_{p(U)} U \to U$ the strongly cartesian morphism of $\mathcal{C}$ over $\mathcal{D}$ which exists as p is a fibred category. The functor

$$v : \mathcal{D}/p(U) \to \mathcal{C}/U, \quad V/p(U) \mapsto V \times_{p(U)} U/U$$

is continuous by the definition of the topology on $\mathcal{C}$. Moreover, it is a right adjoint to p by the definition of strongly cartesian morphisms. Hence we are in the situation discussed in Sites, Section 21 and we see that the sheaf $\pi_{U,*}\mathcal{F}$ is equal to $V \mapsto \mathcal{F}(V \times_{p(U)} U)$ (see especially Sites, Lemma 21.2).

But here we have more. Namely, the functor v is also cocontinuous (as all morphisms in coverings of $\mathcal{C}$ are strongly cartesian). Hence v defines a morphism σ as indicated in the lemma. The equality $\sigma^{-1} = \pi_{U,*}$ is immediate from the definition. Since $\pi_U^{-1}\mathcal{G}$ is given by the rule $U'/U \mapsto \mathcal{G}(p(U')/p(U))$ it follows that $\sigma^{-1} \circ \pi_U^{-1} = \text{id}$ which proves the equality $\pi_U \circ \sigma = \text{id}$. □

Situation 28.3. Let $(\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a ringed site. Let $u : \mathcal{C}' \to \mathcal{C}$ be a 1-morphism of fibred categories over $\mathcal{D}$ (Categories, Definition 31.9). Endow $\mathcal{C}$ and $\mathcal{C}'$ with their inherited topologies (Stacks, Definition 10.2) and let $\pi : Sh(\mathcal{C}) \to Sh(\mathcal{D})$, $\pi' : Sh(\mathcal{C}') \to Sh(\mathcal{D})$, and $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ be the corresponding morphisms of topoi (Stacks, Lemma 10.3). Set $\mathcal{O}_\mathcal{C} = \pi^{-1}\mathcal{O}_\mathcal{D}$ and $\mathcal{O}_{\mathcal{C}'} = (\pi')^{-1}\mathcal{O}_\mathcal{D}$. Observe that $g^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$ so that

$$\begin{tikzcd}
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[rr, "g"] \arrow[dr, "\pi'"'] & & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[dl, "\pi"] \\
& (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) &
\end{tikzcd}$$

is a commutative diagram of morphisms of ringed topoi.

Lemma 28.4. *Assumptions and notation as in Situation 28.3. For $U' \in \text{Ob}(\mathcal{C}')$ set $U = u(U')$ and $V = p'(U')$ and consider the induced morphisms of ringed topoi*

$$\begin{tikzcd}
(Sh(\mathcal{C}'/U'), \mathcal{O}_{U'}) \arrow[rr, "g'"] \arrow[dr, "\pi'_{U'}"'] & & (Sh(\mathcal{C}), \mathcal{O}_U) \arrow[dl, "\pi_U"] \\
& (Sh(\mathcal{D}/V), \mathcal{O}_V) &
\end{tikzcd}$$

Then there exists a morphism of topoi

$$\sigma' : Sh(\mathcal{D}/V) \to Sh(\mathcal{C}'/U'),$$

such that setting $\sigma = g' \circ \sigma'$ we have $\pi'_{U'} \circ \sigma' = id$, $\pi_U \circ \sigma = id$, $(\sigma')^{-1} = \pi'_{U',*}$, and $\sigma^{-1} = \pi_{U,*}$.

Proof. Let $v' : \mathcal{D}/V \to \mathcal{C}'/U'$ be the functor constructed in the proof of Lemma 28.2 starting with $p' : \mathcal{C}' \to \mathcal{D}'$ and the object U'. Since u is a 1-morphism of fibred categories over $\mathcal{D}$ it transforms strongly cartesian morphisms into strongly cartesian morphisms, hence the functor $v = u \circ v'$ is the functor of the proof of Lemma 28.2 relative to $p : \mathcal{C} \to \mathcal{D}$ and U. Thus our lemma follows from that lemma. □

Lemma 28.5. *Assumption and notation as in Situation 28.3.*

(1) *There are left adjoints $g_! : Mod(\mathcal{O}_{\mathcal{C}'}) \to Mod(\mathcal{O}_\mathcal{C})$ and $g_!^{Ab} : Ab(\mathcal{C}') \to Ab(\mathcal{C})$ to $g^* = g^{-1}$ on modules and on abelian sheaves.*

(2) *The diagram*

$$\begin{tikzcd}
Mod(\mathcal{O}_{\mathcal{C}'}) \arrow[r, "g_!"'] \arrow[d] & Mod(\mathcal{O}_\mathcal{C}) \arrow[d] \\
Ab(\mathcal{C}') \arrow[r, "g_!^{Ab}"] & Ab(\mathcal{C})
\end{tikzcd}$$

commutes.

(3) *There are left adjoints* $Lg_! : D(\mathcal{O}_{\mathcal{C}'}) \to D(\mathcal{O}_\mathcal{C})$ *and* $Lg_!^{Ab} : D(\mathcal{C}') \to D(\mathcal{C})$ *to* $g^* = g^{-1}$ *on derived categories of modules and abelian sheaves.*
(4) *The diagram*

$$\begin{tikzcd} D(\mathcal{O}_{\mathcal{C}'}) \arrow[r, "Lg_!"'] \arrow[d] & D(\mathcal{O}_\mathcal{C}) \arrow[d] \\ D(\mathcal{C}') \arrow[r, "Lg_!^{Ab}"] & D(\mathcal{C}) \end{tikzcd}$$

commutes.

Proof. The functor u is continuous and cocontinuous Stacks, Lemma 10.3. Hence the existence of the functors $g_!$, $g_!^{Ab}$, $Lg_!$, and $Lg_!^{Ab}$ can be found in Modules on Sites, Sections 16 and 40 and Section 27.

To prove (2) it suffices to show that the canonical map

$$g_!^{Ab} j_{U'!}\mathcal{O}_{U'} \to j_{u(U')!}\mathcal{O}_{u(U')}$$

is an isomorphism for all objects U' of $\mathcal{C}'$, see Modules on Sites, Remark 40.2. Similarly, to prove (4) it suffices to show that the canonical map

$$Lg_!^{Ab} j_{U'!}\mathcal{O}_{U'} \to j_{u(U')!}\mathcal{O}_{u(U')}$$

is an isomorphism in $D(\mathcal{C})$ for all objects U' of $\mathcal{C}'$, see Remark 27.2. This will also imply the previous formula hence this is what we will show.

We will use that for a localization morphism j the functors $j_!$ and $j_!^{Ab}$ agree (see Modules on Sites, Remark 19.5) and that $j_!$ is exact (Modules on Sites, Lemma 19.3). Let us adopt the notation of Lemma 28.4. Since $Lg_!^{Ab} \circ j_{U'!} = j_{U!} \circ L(g')_!^{Ab}$ (by commutativity of Sites, Lemma 27.4 and uniqueness of adjoint functors) it suffices to prove that $L(g')_!^{Ab}\mathcal{O}_{U'} = \mathcal{O}_U$. Using the results of Lemma 28.4 we have for any object E of $D(\mathcal{C}/u(U'))$ the following sequence of equalities

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{C}/U)}(L(g')_!^{Ab}\mathcal{O}_{U'}, E) &= \mathrm{Hom}_{D(\mathcal{C}'/U')}(\mathcal{O}_{U'}, (g')^{-1}E) \\ &= \mathrm{Hom}_{D(\mathcal{C}'/U')}((\pi'_{U'})^{-1}\mathcal{O}_V, (g')^{-1}E) \\ &= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, R\pi'_{U',*}(g')^{-1}E) \\ &= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, (\sigma')^{-1}(g')^{-1}E) \\ &= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \sigma^{-1}E) \\ &= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \pi_{U,*}E) \\ &= \mathrm{Hom}_{D(\mathcal{C}/U)}(\pi_U^{-1}\mathcal{O}_V, E) \\ &= \mathrm{Hom}_{D(\mathcal{C}/U)}(\mathcal{O}_U, E) \end{aligned}$$

By Yoneda's lemma we conclude. □

Remark 28.6. Assumptions and notation as in Situation 28.1. Note that setting $\mathcal{C}' = \mathcal{D}$ and u equal to the structure functor of $\mathcal{C}$ gives a situation as in Situation 28.3. Hence Lemma 28.5 tells us we have functors $\pi_!$, $\pi_!^{Ab}$, $L\pi_!$, and $L\pi_!^{Ab}$ such that $forget \circ \pi_! = \pi_!^{Ab} \circ forget$ and $forget \circ L\pi_! = L\pi_!^{Ab} \circ forget$.

Remark 28.7. Assumptions and notation as in Situation 28.3. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$, let $\mathcal{F}'$ be an abelian sheaf on $\mathcal{C}'$, and let $t : \mathcal{F}' \to g^{-1}\mathcal{F}$ be a map. Then we obtain a canonical map

$$L\pi'_!(\mathcal{F}') \longrightarrow L\pi_!(\mathcal{F})$$

by using the adjoint $g_!\mathcal{F}' \to \mathcal{F}$ of t, the map $Lg_!(\mathcal{F}') \to g_!\mathcal{F}'$, and the equality $L\pi'_! = L\pi_! \circ Lg_!$.

Lemma 28.8. *Assumptions and notation as in Situation 28.1. For $\mathcal{F}$ in $Ab(\mathcal{C})$ the sheaf $\pi_!\mathcal{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto \mathrm{colim}_{\mathcal{C}_V^{opp}} \mathcal{F}|_{\mathcal{C}_V}$$

with restriction maps as indicated in the proof.

Proof. Denote $\mathcal{H}$ be the rule of the lemma. For a morphism $h : V' \to V$ of $\mathcal{D}$ there is a pullback functor $h^* : \mathcal{C}_V \to \mathcal{C}_{V'}$ of fibre categories (Categories, Definition 31.6). Moreover for $U \in \mathrm{Ob}(\mathcal{C}_V)$ there is a strongly cartesian morphism $h^*U \to U$ covering h. Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

Hence a map $\mathcal{H}(V) \to \mathcal{H}(V')$ between colimits, see Categories, Lemma 14.7.

To prove the lemma we show that

$$\mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{H}, \mathcal{G}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \pi^{-1}\mathcal{G})$$

for every sheaf $\mathcal{G}$ on $\mathcal{C}$. An element of the left hand side is a compatible system of maps $\mathcal{F}(U) \to \mathcal{G}(p(U))$ for all U in $\mathcal{C}$. Since $\pi^{-1}\mathcal{G}(U) = \mathcal{G}(p(U))$ by our choice of topology on $\mathcal{C}$ we see the same thing is true for the right hand side and we win. $\square$

29. Homology on a category

In the case of a category over a point we will baptize the left derived lower shriek functors the homology functors.

Example 29.1 (Category over point)**.** Let $\mathcal{C}$ be a category. Endow $\mathcal{C}$ with the chaotic topology (Sites, Example 6.6). Thus presheaves and sheaves agree on $\mathcal{C}$. The functor $p : \mathcal{C} \to *$ where $*$ is the category with a single object and a single morphism is cocontinuous and continuous. Let $\pi : Sh(\mathcal{C}) \to Sh(*)$ be the corresponding morphism of topoi. Let B be a ring. We endow $*$ with the sheaf of rings B and $\mathcal{C}$ with $\mathcal{O}_\mathcal{C} = \pi^{-1}B$ which we will denote $\underline{B}$. In this way

$$\pi : (Sh(\mathcal{C}), \underline{B}) \to (*, B)$$

is an example of Situation 28.1. By Remark 28.6 we do not need to distinguish between $\pi_!$ on modules or abelian sheaves. By Lemma 28.8 we see that $\pi_!\mathcal{F} = \mathrm{colim}_{\mathcal{C}^{opp}} \mathcal{F}$. Thus $L_n\pi_!$ is the nth left derived functor of taking colimits. In the following, we write

$$H_n(\mathcal{C}, \mathcal{F}) = L_n\pi_!(\mathcal{F})$$

and we will name this the *nth homology group of $\mathcal{F}$* on $\mathcal{C}$.

Example 29.2 (Computing homology)**.** In Example 29.1 we can compute the functors $H_n(\mathcal{C}, -)$ as follows. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Consider the chain complex

$$K_\bullet(\mathcal{F}) : \ \ldots \to \bigoplus\nolimits_{U_2 \to U_1 \to U_0} \mathcal{F}(U_0) \to \bigoplus\nolimits_{U_1 \to U_0} \mathcal{F}(U_0) \to \bigoplus\nolimits_{U_0} \mathcal{F}(U_0)$$

where the transition maps are given by

$$(U_2 \to U_1 \to U_0, s) \longmapsto (U_1 \to U_0, s) - (U_2 \to U_0, s) + (U_2 \to U_1, s|_{U_1})$$

and similarly in other degrees. By construction

$$H_0(\mathcal{C}, \mathcal{F}) = \text{colim}_{\mathcal{C}^{opp}} \mathcal{F} = H_0(K_\bullet(\mathcal{F})),$$

see Categories, Lemma 14.11. The construction of $K_\bullet(\mathcal{F})$ is functorial in $\mathcal{F}$ and transforms short exact sequences of $Ab(\mathcal{C})$ into short exact sequences of complexes. Thus the sequence of functors $\mathcal{F} \mapsto H_n(K_\bullet(\mathcal{F}))$ forms a δ-functor, see Homology, Definition 11.1 and Lemma 12.12. For $\mathcal{F} = j_{U!}\mathbf{Z}_U$ the complex $K_\bullet(\mathcal{F})$ is the complex associated to the free $\mathbf{Z}$-module on the simplicial set $X_\bullet$ with terms

$$X_n = \coprod\nolimits_{U_n \to \ldots \to U_1 \to U_0} \text{Mor}_\mathcal{C}(U_0, U)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$. Namely, the map $X_\bullet \to \{*\}$ is obvious, the map $\{*\} \to X_n$ is given by mapping $*$ to $(U \to \ldots \to U, \text{id}_U)$, and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 25.2) defining the homotopy between the two maps $X_\bullet \to X_\bullet$ are given by the rule

$$h_{n,i} : (U_n \to \ldots \to U_0, f) \longmapsto (U_n \to \ldots \to U_i \to U \to \ldots \to U, \text{id})$$

for $i > 0$ and $h_{n,0} = \text{id}$. Verifications omitted. This implies that $K_\bullet(j_{U!}\mathbf{Z}_U)$ has trivial cohomology in negative degrees (by the functoriality of Simplicial, Remark 25.4 and the result of Simplicial, Lemma 26.1). Thus $K_\bullet(\mathcal{F})$ computes the left derived functors $H_n(\mathcal{C}, -)$ of $H_0(\mathcal{C}, -)$ for example by (the duals of) Homology, Lemma 11.4 and Derived Categories, Lemma 17.6.

Example 29.3. Let $u : \mathcal{C}' \to \mathcal{C}$ be a functor. Endow $\mathcal{C}'$ and $\mathcal{C}$ with the chaotic topology as in Example 29.1. The functors u, $\mathcal{C}' \to *$, and $\mathcal{C} \to *$ where $*$ is the category with a single object and a single morphism are cocontinuous and continuous. Let $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$, $\pi' : Sh(\mathcal{C}') \to Sh(*)$, and $\pi : Sh(\mathcal{C}) \to Sh(*)$, be the corresponding morphisms of topoi. Let B be a ring. We endow $*$ with the sheaf of rings B and $\mathcal{C}'$, $\mathcal{C}$ with the constant sheaf $\underline{B}$. In this way

$$\begin{tikzcd} (Sh(\mathcal{C}'), \underline{B}) \arrow[rr, "g"] \arrow[dr, "\pi'"'] & & (Sh(\mathcal{C}), \underline{B}) \arrow[dl, "\pi"] \\ & (Sh(*), B) & \end{tikzcd}$$

is an example of Situation 28.3. Thus Lemma 28.5 applies to g so we do not need to distinguish between $g_!$ on modules or abelian sheaves. In particular Remark 28.7 produces canonical maps

$$H_n(\mathcal{C}', \mathcal{F}') \longrightarrow H_n(\mathcal{C}, \mathcal{F})$$

whenever we have $\mathcal{F}$ in $Ab(\mathcal{C})$, $\mathcal{F}'$ in $Ab(\mathcal{C}')$, and a map $t : \mathcal{F}' \to g^{-1}\mathcal{F}$. In terms of the computation of homology given in Example 29.2 we see that these maps come from a map of complexes

$$K_\bullet(\mathcal{F}') \longrightarrow K_\bullet(\mathcal{F})$$

given by the rule

$$(U'_n \to \ldots \to U'_0, s') \longmapsto (u(U'_n) \to \ldots \to u(U'_0), t(s'))$$

with obvious notation.

Remark 29.4. Notation and assumptions as in Example 29.1. Let $\mathcal{F}^\bullet$ be a bounded complex of abelian sheaves on $\mathcal{C}$. For any object U of $\mathcal{C}$ there is a canonical map

$$\mathcal{F}^\bullet(U) \longrightarrow L\pi_!(\mathcal{F}^\bullet)$$

in $D(Ab)$. If $\mathcal{F}^\bullet$ is a complex of $\underline{B}$-modules then this map is in $D(B)$. To prove this, note that we compute $L\pi_!(\mathcal{F}^\bullet)$ by taking a quasi-isomorphism $\mathcal{P}^\bullet \to \mathcal{F}^\bullet$ where $\mathcal{P}^\bullet$ is a complex of projectives. However, since the topology is chaotic this means that $\mathcal{P}^\bullet(U) \to \mathcal{F}^\bullet(U)$ is a quasi-isomorphism hence can be inverted in $D(Ab)$, resp. $D(B)$. Composing with the canonical map $\mathcal{P}^\bullet(U) \to \pi_!(\mathcal{P}^\bullet)$ coming from the computation of $\pi_!$ as a colimit we obtain the desired arrow.

Lemma 29.5. *Notation and assumptions as in Example 29.1. If $\mathcal{C}$ has either an initial or a final object, then $L\pi_! \circ \pi^{-1} = id$ on $D(Ab)$, resp. $D(B)$.*

Proof. If $\mathcal{C}$ has an initial object, then $\pi_!$ is computed by evaluating on this object and the statement is clear. If $\mathcal{C}$ has a final object, then $R\pi_*$ is computed by evaluating on this object, hence $R\pi_* \circ \pi^{-1} \cong \text{id}$ on $D(Ab)$, resp. $D(B)$. This implies that $\pi^{-1} : D(Ab) \to D(\mathcal{C})$, resp. $\pi^{-1} : D(B) \to D(\underline{B})$ is fully faithful, see Categories, Lemma 24.3. Then the same lemma implies that $L\pi_! \circ \pi^{-1} = \text{id}$ as desired. $\square$

Lemma 29.6. *Notation and assumptions as in Example 29.1. Let $B \to B'$ be a ring map. Consider the commutative diagram of ringed topoi*

$$\begin{tikzcd}
(Sh(\mathcal{C}), \underline{B}) \arrow[d, "\pi"'] & (Sh(\mathcal{C}), \underline{B'}) \arrow[l, "h"'] \arrow[d, "\pi'"] \\
(*, B) & (*, B') \arrow[l, "f"]
\end{tikzcd}$$

Then $L\pi_! \circ Lh^* = Lf^* \circ L\pi'_!$.

Proof. Both functors are right adjoint to the obvious functor $D(B') \to D(\underline{B})$. $\square$

Lemma 29.7. *Notation and assumptions as in Example 29.1. Let $U_\bullet$ be a cosimplicial object in $\mathcal{C}$ such that for every $U \in \text{Ob}(\mathcal{C})$ the simplicial set $\text{Mor}_\mathcal{C}(U_\bullet, U)$ is homotopy equivalent to the constant simplicial set on a singleton. Then*

$$L\pi_!(\mathcal{F}) = \mathcal{F}(U_\bullet)$$

in $D(Ab)$, resp. $D(B)$ functorially in $\mathcal{F}$ in $Ab(\mathcal{C})$, resp. $Mod(\underline{B})$.

Proof. As $L\pi_!$ agrees for modules and abelian sheaves by Lemma 28.5 it suffices to prove this when $\mathcal{F}$ is an abelian sheaf. For $U \in \text{Ob}(\mathcal{C})$ the abelian sheaf $j_{U!}\mathbf{Z}_U$ is a projective object of $Ab(\mathcal{C})$ since $\text{Hom}(j_{U!}\mathbf{Z}_U, \mathcal{F}) = \mathcal{F}(U)$ and taking sections is an exact functor as the topology is chaotic. Every abelian sheaf is a quotient of a direct sum of $j_{U!}\mathbf{Z}_U$ by Modules on Sites, Lemma 28.6. Thus we can compute $L\pi_!(\mathcal{F})$ by choosing a resolution

$$\ldots \to \mathcal{G}^{-1} \to \mathcal{G}^0 \to \mathcal{F} \to 0$$

whose terms are direct sums of sheaves of the form above and taking $L\pi_!(\mathcal{F}) = \pi_!(\mathcal{G}^\bullet)$. Consider the double complex $A^{\bullet,\bullet} = \mathcal{G}^\bullet(U_\bullet)$. The map $\mathcal{G}^0 \to \mathcal{F}$ gives a map of complexes $A^{0,\bullet} \to \mathcal{F}(U_\bullet)$. Since $\pi_!$ is computed by taking the colimit over

$\mathcal{C}^{opp}$ (Lemma 28.8) we see that the two compositions $\mathcal{G}^m(U_1) \to \mathcal{G}^m(U_0) \to \pi_!\mathcal{G}^m$ are equal. Thus we obtain a canonical map of complexes

$$\mathrm{Tot}(A^{\bullet,\bullet}) \longrightarrow \pi_!(\mathcal{G}^\bullet) = L\pi_!(\mathcal{F})$$

To prove the lemma it suffices to show that the complexes

$$\ldots \to \mathcal{G}^m(U_1) \to \mathcal{G}^m(U_0) \to \pi_!\mathcal{G}^m \to 0$$

are exact, see Homology, Lemma 22.7. Since the sheaves $\mathcal{G}^m$ are direct sums of the sheaves $j_{U!}\mathbf{Z}_U$ we reduce to $\mathcal{G} = j_{U!}\mathbf{Z}_U$. The complex $j_{U!}\mathbf{Z}_U(U_\bullet)$ is the complex of abelian groups associated to the free $\mathbf{Z}$-module on the simplicial set $\mathrm{Mor}_\mathcal{C}(U_\bullet, U)$ which we assumed to be homotopy equivalent to a singleton. We conclude that

$$j_{U!}\mathbf{Z}_U(U_\bullet) \to \mathbf{Z}$$

is a homotopy equivalence of abelian groups hence a quasi-isomorphism (Simplicial, Remark 25.4 and Lemma 26.1). This finishes the proof since $\pi_!j_{U!}\mathbf{Z}_U = \mathbf{Z}$ as was shown in the proof of Lemma 28.5. □

Lemma 29.8. *Notation and assumptions as in Example 29.3. If there exists a cosimplicial object $U'_\bullet$ of $\mathcal{C}'$ such that Lemma 29.7 applies to both $U'_\bullet$ in $\mathcal{C}'$ and $u(U'_\bullet)$ in $\mathcal{C}$, then we have $L\pi'_! \circ g^{-1} = L\pi_!$ as functors $D(\mathcal{C}) \to D(Ab)$, resp. $D(\mathcal{C}, \underline{B}) \to D(B)$.*

Proof. Follows immediately from Lemma 29.7 and the fact that g^{-1} is given by precomposing with u. □

Lemma 29.9. *Let $\mathcal{C}_i$, $i = 1, 2$ be categories. Let $u_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$ be the projection functors. Let B be a ring. Let $g_i : (Sh(\mathcal{C}_1 \times \mathcal{C}_2), \underline{B}) \to (Sh(\mathcal{C}_i), \underline{B})$ be the corresponding morphisms of ringed topoi, see Example 29.3. For $K_i \in D(\mathcal{C}_i, B)$ we have*

$$L(\pi_1 \times \pi_2)_!(g_1^{-1}K_1 \otimes_{\underline{B}}^{\mathbf{L}} g_2^{-1}K_2) = L\pi_{1,!}(K_1) \otimes_B^{\mathbf{L}} L\pi_{2,!}(K_2)$$

in $D(B)$ with obvious notation.

Proof. As both sides commute with colimits, it suffices to prove this for $K_1 = j_{U!}\underline{B}_U$ and $K_2 = j_{V!}\underline{B}_V$ for $U \in \mathrm{Ob}(\mathcal{C}_1)$ and $V \in \mathrm{Ob}(\mathcal{C}_2)$. See construction of $L\pi_!$ in Lemma 27.1. In this case

$$g_1^{-1}K_1 \otimes_{\underline{B}}^{\mathbf{L}} g_2^{-1}K_2 = g_1^{-1}K_1 \otimes_{\underline{B}} g_2^{-1}K_2 = j_{(U,V)!}\underline{B}_{(U,V)}$$

Verification omitted. Hence the result follows as both the left and the right hand side of the formula of the lemma evaluate to B, see construction of $L\pi_!$ in Lemma 27.1. □

Lemma 29.10. *Notation and assumptions as in Example 29.1. If there exists a cosimplicial object $U_\bullet$ of $\mathcal{C}$ such that Lemma 29.7 applies, then*

$$L\pi_!(K_1 \otimes_{\underline{B}}^{\mathbf{L}} K_2) = L\pi_!(K_1) \otimes_B^{\mathbf{L}} L\pi_!(K_2)$$

for all $K_i \in D(\underline{B})$.

Proof. Consider the diagram of categories and functors

$$\begin{tikzcd} & & \mathcal{C} \\ \mathcal{C} \arrow[r, "u"] & \mathcal{C} \times \mathcal{C} \arrow[ru, "u_1"] \arrow[rd, "u_2"] & \\ & & \mathcal{C} \end{tikzcd}$$

where u is the diagonal functor and u_i are the projection functors. This gives morphisms of ringed topoi g, g_1, g_2. For any object (U_1, U_2) of $\mathcal{C}$ we have

$$\mathrm{Mor}_{\mathcal{C}\times\mathcal{C}}(u(U_\bullet), (U_1, U_2)) = \mathrm{Mor}_{\mathcal{C}}(U_\bullet, U_1) \times \mathrm{Mor}_{\mathcal{C}}(U_\bullet, U_2)$$

which is homotopy equivalent to a point by Simplicial, Lemma 25.10. Thus Lemma 29.8 gives $L\pi_!(g^{-1}K) = L(\pi \times \pi)_!(K)$ for any K in $D(\mathcal{C} \times \mathcal{C}, B)$. Take $K = g_1^{-1}K_1 \otimes_B^{\mathbf{L}} g_2^{-1}K_2$. Then $g^{-1}K = K_1 \otimes_{\underline{B}}^{\mathbf{L}} K_2$ because $g^{-1} = g^* = Lg^*$ commutes with derived tensor product (Lemma 18.4 – a site with chaotic topology has enough points). To finish we apply Lemma 29.9. $\square$

Remark 29.11 (Simplicial modules)**.** Let $\mathcal{C} = \Delta$ and let B be any ring. This is a special case of Example 29.1 where the assumptions of Lemma 29.7 hold. Namely, let $U_\bullet$ be the cosimplicial object of Δ given by the identity functor. To verify the condition we have to show that for $[m] \in \mathrm{Ob}(\Delta)$ the simplicial set $\Delta[m] : n \mapsto \mathrm{Mor}_\Delta([n], [m])$ is homotopy equivalent to a point. This is explained in Simplicial, Example 25.7.

In this situation the category $\mathit{Mod}(\underline{B})$ is just the category of simplicial B-modules and the functor $L\pi_!$ sends a simplicial B-module $M_\bullet$ to its associated complex $s(M_\bullet)$ of B-modules. Thus the results above can be reinterpreted in terms of results on simplicial modules. For example a special case of Lemma 29.10 is: if $M_\bullet$, $M'_\bullet$ are flat simplicial B-modules, then the complex $s(M_\bullet \otimes_B M'_\bullet)$ is quasi-isomorphic to the total complex associated to the double complex $s(M_\bullet) \otimes_B s(M'_\bullet)$. (Hint: use flatness to convert from derived tensor products to usual tensor products.) This is a special case of the Eilenberg-Zilber theorem which can be found in [EZ53].

Lemma 29.12. *Let $\mathcal{C}$ be a category (endowed with chaotic topology). Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings on $\mathcal{C}$. Assume*

1. *there exists a cosimplicial object $U_\bullet$ in $\mathcal{C}$ as in Lemma 29.7, and*
2. *$L\pi_!\mathcal{O} \to L\pi_!\mathcal{O}'$ is an isomorphism.*

For K in $D(\mathcal{O})$ we have

$$L\pi_!(K) = L\pi_!(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}')$$

in $D(Ab)$.

Proof. Note: in this proof $L\pi_!$ denotes the left derived functor of $\pi_!$ on abelian sheaves. Since $L\pi_!$ commutes with colimits, it suffices to prove this for bounded above complexes of $\mathcal{O}$-modules (compare with argument of Derived Categories, Proposition 28.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are direct sums of

$j_{U!}\mathcal{O}_U$ with $U \in \mathrm{Ob}(\mathcal{C})$, see Modules on Sites, Lemma 28.6. Thus it suffices to prove the lemma for $j_{U!}\mathcal{O}_U$. By assumption

$$S_\bullet = \mathrm{Mor}_\mathcal{C}(U_\bullet, U)$$

is a simplicial set homotopy equivalent to the constant simplicial set on a singleton. Set $P_n = \mathcal{O}(U_n)$ and $P'_n = \mathcal{O}'(U_n)$. Observe that the complex associated to the simplicial abelian group

$$X_\bullet : n \longmapsto \bigoplus\nolimits_{s \in S_n} P_n$$

computes $L\pi_!(j_{U!}\mathcal{O}_U)$ by Lemma 29.7. Since $j_{U!}\mathcal{O}_U$ is a flat $\mathcal{O}$-module we have $j_{U!}\mathcal{O}_U \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}' = j_{U!}\mathcal{O}'_U$ and $L\pi_!$ of this is computed by the complex associated to the simplicial abelian group

$$X'_\bullet : n \longmapsto \bigoplus\nolimits_{s \in S_n} P'_n$$

As the rule which to a simplicial set $T_\bullet$ associated the simplicial abelian group with terms $\bigoplus_{t \in T_n} P_n$ is a functor, we see that $X_\bullet \to P_\bullet$ is a homotopy equivalence of simplicial abelian groups. Similarly, the rule which to a simplicial set $T_\bullet$ associates the simplicial abelian group with terms $\bigoplus_{t \in T_n} P'_n$ is a functor. Hence $X'_\bullet \to P'_\bullet$ is a homotopy equivalence of simplicial abelian groups. By assumption $P_\bullet \to P'_\bullet$ is a quasi-isomorphism (since $P_\bullet$, resp. $P'_\bullet$ computes $L\pi_!\mathcal{O}$, resp. $L\pi_!\mathcal{O}'$ by Lemma 29.7). We conclude that $X_\bullet$ and $X'_\bullet$ are quasi-isomorphic as desired. $\square$

Remark 29.13. Let $\mathcal{C}$ and B be as in Example 29.1. Assume there exists a cosimplicial object as in Lemma 29.7. Let $\mathcal{O} \to \underline{B}$ be a map sheaf of rings on $\mathcal{C}$ which induces an isomorphism $L\pi_!\mathcal{O} \to L\pi_!\underline{B}$. In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{O}) \longrightarrow D(B)$$

Namely, for any object K of $D(\mathcal{O})$ we have $L\pi_!^{Ab}(K) = L\pi_!^{Ab}(K \otimes_\mathcal{O}^\mathbf{L} \underline{B})$ by Lemma 29.12. Thus we can define the displayed functor as the composition of $- \otimes_\mathcal{O}^\mathbf{L} \underline{B}$ with the functor $L\pi_! : D(\underline{B}) \to D(B)$. In other words, we obtain a B-module structure on $L\pi_!(K)$ coming from the (canonical, functorial) identification of $L\pi_!(K)$ with $L\pi_!(K \otimes_\mathcal{O}^\mathbf{L} \underline{B})$ of the lemma.

30. Calculating derived lower shriek

In this section we apply the results from Section 29 to compute $L\pi_!$ in Situation 28.1 and $Lg_!$ in Situation 28.3.

Lemma 30.1. *Assumptions and notation as in Situation 28.1. For $\mathcal{F}$ in $PAb(\mathcal{C})$ and $n \geq 0$ consider the abelian sheaf $L_n(\mathcal{F})$ on $\mathcal{D}$ which is the sheaf associated to the presheaf*

$$V \longmapsto H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V})$$

with restriction maps as indicated in the proof. Then $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$.

Proof. For a morphism $h : V' \to V$ of $\mathcal{D}$ there is a pullback functor $h^* : \mathcal{C}_V \to \mathcal{C}_{V'}$ of fibre categories (Categories, Definition 31.6). Moreover for $U \in \mathrm{Ob}(\mathcal{C}_V)$ there is a strongly cartesian morphism $h^*U \to U$ covering h. Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

By Example 29.3 we obtain the desired restriction map

$$H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V}) \longrightarrow H_n(\mathcal{C}_{V'}, \mathcal{F}|_{\mathcal{C}_{V'}})$$

Let us denote $L_{n,p}(\mathcal{F})$ this presheaf, so that $L_n(\mathcal{F}) = L_{n,p}(\mathcal{F})^{\#}$. The canonical map $\gamma : \mathcal{F} \to \mathcal{F}^+$ (Sites, Theorem 10.10) defines a canonical map $L_{n,p}(\mathcal{F}) \to L_{n,p}(\mathcal{F}^+)$. We have to prove this map becomes an isomorphism after sheafification.

Let us use the computation of homology given in Example 29.2. Denote $K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$ the complex associated to the restriction of $\mathcal{F}$ to the fibre category $\mathcal{C}_V$. By the remarks above we obtain a presheaf $K_\bullet(\mathcal{F})$ of complexes

$$V \longmapsto K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$$

whose cohomology presheaves are the presheaves $L_{n,p}(\mathcal{F})$. Thus it suffices to show that

$$K_\bullet(\mathcal{F}) \longrightarrow K_\bullet(\mathcal{F}^+)$$

becomes an isomorphism on sheafification.

Injectivity. Let V be an object of $\mathcal{D}$ and let $\xi \in K_n(\mathcal{F})(V)$ be an element which maps to zero in $K_n(\mathcal{F}^+)(V)$. We have to show there exists a covering $\{V_j \to V\}$ such that $\xi|_{V_j}$ is zero in $K_n(\mathcal{F})(V_j)$. We write

$$\xi = \sum (U_{i,n+1} \to \ldots \to U_{i,0}, \sigma_i)$$

with $\sigma_i \in \mathcal{F}(U_{i,0})$. We arrange it so that each sequence of morphisms $U_n \to \ldots \to U_0$ of $\mathcal{C}_V$ occurs are most once. Since the sums in the definition of the complex $K_\bullet$ are direct sums, the only way this can map to zero in $K_\bullet(\mathcal{F}^+)(V)$ is if all σ_i map to zero in $\mathcal{F}^+(U_{i,0})$. By construction of $\mathcal{F}^+$ there exist coverings $\{U_{i,0,j} \to U_{i,0}\}$ such that $\sigma_i|_{U_{i,0,j}}$ is zero. By our construction of the topology on $\mathcal{C}$ we can write $U_{i,0,j} \to U_{i,0}$ as the pullback (Categories, Definition 31.6) of some morphisms $V_{i,j} \to V$ and moreover each $\{V_{i,j} \to V\}$ is a covering. Choose a covering $\{V_j \to V\}$ dominating each of the coverings $\{V_{i,j} \to V\}$. Then it is clear that $\xi|_{V_j} = 0$.

Surjectivity. Proof omitted. Hint: Argue as in the proof of injectivity. $\square$

Lemma 30.2. *Assumptions and notation as in Situation 28.1. For $\mathcal{F}$ in $Ab(\mathcal{C})$ and $n \geq 0$ the sheaf $L_n\pi_!(\mathcal{F})$ is equal to the sheaf $L_n(\mathcal{F})$ constructed in Lemma 30.1.*

Proof. Consider the sequence of functors $\mathcal{F} \mapsto L_n(\mathcal{F})$ from $PAb(\mathcal{C}) \to Ab(\mathcal{C})$. Since for each $V \in \mathrm{Ob}(\mathcal{D})$ the sequence of functors $H_n(\mathcal{C}_V, -)$ forms a δ-functor so do the functors $\mathcal{F} \mapsto L_n(\mathcal{F})$. Our goal is to show these form a universal δ-functor. In order to do this we construct some abelian presheaves on which these functors vanish.

For $U' \in \mathrm{Ob}(\mathcal{C})$ consider the abelian presheaf $\mathcal{F}_{U'} = j_{U'!}^{PAb}\mathbf{Z}_{U'}$ (Modules on Sites, Remark 19.6). Recall that

$$\mathcal{F}_{U'}(U) = \bigoplus\nolimits_{\mathrm{Mor}_\mathcal{C}(U,U')} \mathbf{Z}$$

If U lies over $V = p(U)$ in $\mathcal{D}$) and U' lies over $V' = p(U')$ then any morphism $a : U \to U'$ factors uniquely as $U \to h^*U' \to U'$ where $h = p(a) : V \to V'$ (see

Categories, Definition 31.6). Hence we see that

$$\mathcal{F}_{U'}|_{\mathcal{C}_V} = \bigoplus\nolimits_{h \in \mathrm{Mor}_{\mathcal{D}}(V, V')} j_{h^*U'!}\mathbf{Z}_{h^*U'}$$

where $j_{h^*U'} : Sh(\mathcal{C}_V/h^*U') \to Sh(\mathcal{C}_V)$ is the localization morphism. The sheaves $j_{h^*U'!}\mathbf{Z}_{h^*U'}$ have vanishing higher homology groups (see Example 29.2). We conclude that $L_n(\mathcal{F}_{U'}) = 0$ for all $n > 0$ and all U'. It follows that any abelian presheaf $\mathcal{F}$ is a quotient of an abelian presheaf $\mathcal{G}$ with $L_n(\mathcal{G}) = 0$ for all $n > 0$ (Modules on Sites, Lemma 28.6). Since $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$ we see that the same thing is true for abelian sheaves. Thus the sequence of functors $L_n(-)$ is a universal delta functor on $Ab(\mathcal{C})$ (Homology, Lemma 11.4). Since we have agreement with $H^{-n}(L\pi_!(-))$ for $n = 0$ by Lemma 28.8 we conclude by uniqueness of universal δ-functors (Homology, Lemma 11.5) and Derived Categories, Lemma 17.6. □

Lemma 30.3. *Assumptions and notation as in Situation 28.3. For an abelian sheaf $\mathcal{F}'$ on $\mathcal{C}'$ the sheaf $L_n g_!(\mathcal{F}')$ is the sheaf associated to the presheaf*

$$U \longmapsto H_n(\mathcal{I}_U, \mathcal{F}'_U)$$

For notation and restriction maps see proof.

Proof. Say $p(U) = V$. The category $\mathcal{I}_U$ is the category of pairs (U', φ) where $\varphi : U \to u(U')$ is a morphism of $\mathcal{C}$ with $p(\varphi) = \mathrm{id}_V$, i.e., φ is a morphism of the fibre category $\mathcal{C}_V$. Morphisms $(U'_1, \varphi_1) \to (U'_2, \varphi_2)$ are given by morphisms $a : U'_1 \to U'_2$ of the fibre category $\mathcal{C}'_V$ such that $\varphi_2 = u(a) \circ \varphi_1$. The presheaf $\mathcal{F}'_U$ sends (U', φ) to $\mathcal{F}'(U')$. We will construct the restriction mappings below.

Choose a factorization

$$\begin{tikzcd} \mathcal{C}' \arrow[r, shift left, "u'"] & \mathcal{C}'' \arrow[l, shift left, "w"] \arrow[r, "u''"] & \mathcal{C} \end{tikzcd}$$

of u as in Categories, Lemma 31.14. Then $g_! = g''_! \circ g'_!$ and similarly for derived functors. On the other hand, the functor $g'_!$ is exact, see Modules on Sites, Lemma 16.6. Thus we get $Lg_!(\mathcal{F}') = Lg''_!(\mathcal{F}'')$ where $\mathcal{F}'' = g'_!\mathcal{F}'$. Note that $\mathcal{F}'' = h^{-1}\mathcal{F}'$ where $h : Sh(\mathcal{C}'') \to Sh(\mathcal{C}')$ is the morphism of topoi associated to w, see Sites, Lemma 22.1. The functor u'' turns $\mathcal{C}''$ into a fibred category over $\mathcal{C}$, hence Lemma 30.2 applies to the computation of $L_n g''_!$. The result follows as the construction of $\mathcal{C}''$ in the proof of Categories, Lemma 31.14 shows that the fibre category $\mathcal{C}''_U$ is equal to $\mathcal{I}_U$. Moreover, $h^{-1}\mathcal{F}'|_{\mathcal{C}''_U}$ is given by the rule described above (as w is continuous and cocontinuous by Stacks, Lemma 10.3 so we may apply Sites, Lemma 20.5). □

31. Simplicial modules

Let $A_\bullet$ be a simplicial ring. Recall that we may think of $A_\bullet$ as a sheaf on Δ (endowed with the chaotic topology), see Simplicial, Section 4. Then a simplicial module $M_\bullet$ over $A_\bullet$ is just a sheaf of $A_\bullet$-modules on Δ. In other words, for every $n \geq 0$ we have an A_n-module M_n and for every map $\varphi : [n] \to [m]$ we have a corresponding map

$$M_\bullet(\varphi) : M_m \longrightarrow M_n$$

which is $A_\bullet(\varphi)$-linear such that these maps compose in the usual manner.

Let $\mathcal{C}$ be a site. A *simplicial sheaf of rings* $\mathcal{A}_\bullet$ on $\mathcal{C}$ is a simplicial object in the category of sheaves of rings on $\mathcal{C}$. In this case the assignment $U \mapsto \mathcal{A}_\bullet(U)$ is a sheaf of simplicial rings and in fact the two notions are equivalent. A similar discussion holds for simplicial abelian sheaves, simplicial sheaves of Lie algebras, and so on.

However, as in the case of simplicial rings above, there is another way to think about simplicial sheaves. Namely, consider the projection

$$p : \Delta \times \mathcal{C} \longrightarrow \mathcal{C}$$

This defines a fibred category with strongly cartesian morphisms exactly the morphisms of the form $([n], U) \to ([n], V)$. We endow the category $\Delta \times \mathcal{C}$ with the topology inherited from $\mathcal{C}$ (see Stacks, Section 10). The simple description of the coverings in $\Delta \times \mathcal{C}$ (Stacks, Lemma 10.1) immediately implies that a simplicial sheaf of rings on $\mathcal{C}$ is the same thing as a sheaf of rings on $\Delta \times \mathcal{C}$.

By analogy with the case of simplicial modules over a simplicial ring, we define simplicial modules over simplicial sheaves of rings as follows.

Definition 31.1. Let $\mathcal{C}$ be a site. Let $\mathcal{A}_\bullet$ be a simplicial sheaf of rings on $\mathcal{C}$. A *simplicial $\mathcal{A}_\bullet$-module* $\mathcal{F}_\bullet$ (sometimes called a *simplicial sheaf of $\mathcal{A}_\bullet$-modules*) is a sheaf of modules over the sheaf of rings on $\Delta \times \mathcal{C}$ associated to $\mathcal{A}_\bullet$.

We obtain a category $\textit{Mod}(\mathcal{A}_\bullet)$ of simplicial modules and a corresponding derived category $D(\mathcal{A}_\bullet)$. Given a map $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ of simplicial sheaves of rings we obtain a functor

$$- \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{B}_\bullet : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{B}_\bullet)$$

Moreover, the material of the preceding sections determines a functor

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{C})$$

Given a simplicial module $\mathcal{F}_\bullet$ the object $L\pi_!(\mathcal{F}_\bullet)$ is represented by the associated chain complex $s(\mathcal{F}_\bullet)$ (Simplicial, Section 22). This follows from Lemmas 30.2 and 29.7.

Lemma 31.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ be a homomorphism of simplicial sheaves of rings on $\mathcal{C}$. If $L\pi_!\mathcal{A}_\bullet \to L\pi_!\mathcal{B}_\bullet$ is an isomorphism in $D(\mathcal{C})$, then we have*

$$L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{B}_\bullet)$$

for all K in $D(\mathcal{A}_\bullet)$.

Proof. Let $([n], U)$ be an object of $\Delta \times \mathcal{C}$. Since $L\pi_!$ commutes with colimits, it suffices to prove this for bounded above complexes of $\mathcal{O}$-modules (compare with argument of Derived Categories, Proposition 28.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are flat modules, see Modules on Sites, Lemma 28.6. Thus it suffices to prove the lemma for a flat $\mathcal{A}_\bullet$-module $\mathcal{F}$. In this case the derived tensor product is the usual tensor product and is a sheaf also. Hence by Lemma 30.2 we can compute the cohomology sheaves of both sides of the equation by the procedure of Lemma 30.1. Thus it suffices to prove the result for the restriction of $\mathcal{F}$ to the fibre categories (i.e., to $\Delta \times U$). In this case the result follows from Lemma 29.12. $\square$

Remark 31.3. Let $\mathcal{C}$ be a site. Let $\epsilon : \mathcal{A}_\bullet \to \mathcal{O}$ be an augmentation (Simplicial, Definition 19.1) in the category of sheaves of rings. Assume ϵ induces a quasi-isomorphism $s(\mathcal{A}_\bullet) \to \mathcal{O}$. In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{O})$$

Namely, for any object K of $D(\mathcal{A}_\bullet)$ we have $L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O})$ by Lemma 31.2. Thus we can define the displayed functor as the composition of $- \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O}$ with the functor $L\pi_! : D(\Delta \times \mathcal{C}, \pi^{-1}\mathcal{O}) \to D(\mathcal{O})$ of Remark 28.6. In other words, we obtain a $\mathcal{O}$-module structure on $L\pi_!(K)$ coming from the (canonical, functorial) identification of $L\pi_!(K)$ with $L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O})$ of the lemma.

32. Cohomology on a category

In the situation of Example 29.1 in addition to the derived functor $L\pi_!$, we also have the functor $R\pi_*$. For an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $H_n(\mathcal{C}, \mathcal{F}) = H^{-n}(L\pi_!\mathcal{F})$ and $H^n(\mathcal{C}, \mathcal{F}) = H^n(R\pi_*\mathcal{F})$.

Example 32.1 (Computing cohomology)**.** In Example 29.1 we can compute the functors $H^n(\mathcal{C}, -)$ as follows. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Consider the cochain complex

$$K^\bullet(\mathcal{F}) : \prod\nolimits_{U_0} \mathcal{F}(U_0) \to \prod\nolimits_{U_0 \to U_1} \mathcal{F}(U_0) \to \prod\nolimits_{U_0 \to U_1 \to U_2} \mathcal{F}(U_0) \to \ldots$$

where the transition maps are given by

$$(s_{U_0 \to U_1}) \longmapsto ((U_0 \to U_1 \to U_2) \mapsto s_{U_0 \to U_1} - s_{U_0 \to U_2} + s_{U_1 \to U_2}|_{U_0})$$

and similarly in other degrees. By construction

$$H^0(\mathcal{C}, \mathcal{F}) = \lim\nolimits_{\mathcal{C}^{opp}} \mathcal{F} = H^0(K^\bullet(\mathcal{F})),$$

see Categories, Lemma 14.10. The construction of $K^\bullet(\mathcal{F})$ is functorial in $\mathcal{F}$ and transforms short exact sequences of $Ab(\mathcal{C})$ into short exact sequences of complexes. Thus the sequence of functors $\mathcal{F} \mapsto H^n(K^\bullet(\mathcal{F}))$ forms a δ-functor, see Homology, Definition 11.1 and Lemma 12.12. For an object U of $\mathcal{C}$ denote $p_U : Sh(*) \to Sh(\mathcal{C})$ the corresponding point with p_U^{-1} equal to evaluation at U, see Sites, Example 32.7. Let A be an abelian group and set $\mathcal{F} = p_{U,*}A$. In this case the complex $K^\bullet(\mathcal{F})$ is the complex with terms $\mathrm{Map}(X_n, A)$ where

$$X_n = \coprod\nolimits_{U_0 \to \ldots \to U_{n-1} \to U_n} \mathrm{Mor}_\mathcal{C}(U, U_0)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$. Namely, the map $X_\bullet \to \{*\}$ is obvious, the map $\{*\} \to X_n$ is given by mapping $*$ to $(U \to \ldots \to U, \mathrm{id}_U)$, and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 25.2) defining the homotopy between the two maps $X_\bullet \to X_\bullet$ are given by the rule

$$h_{n,i} : (U_0 \to \ldots \to U_n, f) \longmapsto (U \to \ldots \to U \to U_i \to \ldots \to U_n, \mathrm{id})$$

for $i > 0$ and $h_{n,0} = \mathrm{id}$. Verifications omitted. Since $\mathrm{Map}(-, A)$ is a contravariant functor, implies that $K^\bullet(p_{U,*}A)$ has trivial cohomology in positive degrees (by the functoriality of Simplicial, Remark 25.4 and the result of Simplicial, Lemma 27.5). This implies that $K^\bullet(\mathcal{F})$ is acyclic in positive degrees also if $\mathcal{F}$ is a product of sheaves of the form $p_{U,*}A$. As every abelian sheaf on $\mathcal{C}$ embeds into such a product we conclude that $K^\bullet(\mathcal{F})$ computes the left derived functors $H^n(\mathcal{C}, -)$ of $H^0(\mathcal{C}, -)$ for example by Homology, Lemma 11.4 and Derived Categories, Lemma 17.6.

Example 32.2 (Computing Exts)**.** In Example 29.1 assume we are moreover given a sheaf of rings $\mathcal{O}$ on $\mathcal{C}$. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules. Consider the complex $K^\bullet(\mathcal{G}, \mathcal{F})$ with degree n term

$$\prod\nolimits_{U_0 \to U_1 \to \ldots \to U_n} \mathrm{Hom}_{\mathcal{O}(U_n)}(\mathcal{G}(U_n), \mathcal{F}(U_0))$$

and transition map given by

$$(\varphi_{U_0 \to U_1}) \longmapsto ((U_0 \to U_1 \to U_2) \mapsto \varphi_{U_0 \to U_1} \circ \rho^{U_2}_{U_1} - \varphi_{U_0 \to U_2} + \rho^{U_1}_{U_0} \circ \varphi_{U_1 \to U_2}$$

and similarly in other degrees. Here the ρ's indicate restriction maps. By construction

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = H^0(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for all pairs of $\mathcal{O}$-modules $\mathcal{F}, \mathcal{G}$. The assignment $(\mathcal{G}, \mathcal{F}) \mapsto K^\bullet(\mathcal{G}, \mathcal{F})$ is a bifunctor which transforms direct sums in the first variable into products and commutes with products in the second variable. We claim that

$$\mathrm{Ext}^i_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = H^i(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for $i \geq 0$ provided either

(1) $\mathcal{G}(U)$ is a projective $\mathcal{O}(U)$-module for all $U \in \mathrm{Ob}(\mathcal{C})$, or
(2) $\mathcal{F}(U)$ is an injective $\mathcal{O}(U)$-module for all $U \in \mathrm{Ob}(\mathcal{C})$.

Namely, case (1) the functor $K^\bullet(\mathcal{G}, -)$ is an exact functor from the category of $\mathcal{O}$-modules to the category of cochain complexes of abelian groups. Thus, arguing as in Example 32.1, it suffices to show that $K^\bullet(\mathcal{G}, \mathcal{F})$ is acyclic in positive degrees when $\mathcal{F}$ is $p_{U,*}A$ for an $\mathcal{O}(U)$-module A. Choose a short exact sequence

$$0 \to \mathcal{G}' \to \bigoplus j_{U_i!}\mathcal{O}_{U_i} \to \mathcal{G} \to 0 \tag{32.2.1}$$

see Modules on Sites, Lemma 28.6. Since (1) holds for the middle and right sheaves, it also holds for $\mathcal{G}'$ and evaluating (32.2.1) on an object of $\mathcal{C}$ gives a split exact sequence of modules. We obtain a short exact sequence of complexes

$$0 \to K^\bullet(\mathcal{G}, \mathcal{F}) \to \prod K^\bullet(j_{U_i!}\mathcal{O}_{U_i}, \mathcal{F}) \to K^\bullet(\mathcal{G}', \mathcal{F}) \to 0$$

for any $\mathcal{F}$, in particular $\mathcal{F} = p_{U,*}A$. On H^0 we obtain

$$0 \to \mathrm{Hom}(\mathcal{G}, p_{U,*}A) \to \mathrm{Hom}(\prod j_{U_i!}\mathcal{O}_{U_i}, p_{U,*}A) \to \mathrm{Hom}(\mathcal{G}', p_{U,*}A) \to 0$$

which is exact as $\mathrm{Hom}(\mathcal{H}, p_{U,*}A) = \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{H}(U), A)$ and the sequence of sections of (32.2.1) over U is split exact. Thus we can use dimension shifting to see that it suffices to prove $K^\bullet(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is acyclic in positive degrees for all $U, U' \in \mathrm{Ob}(\mathcal{C})$. In this case $K^n(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is equal to

$$\prod\nolimits_{U \to U_0 \to U_1 \to \ldots \to U_n \to U'} A$$

In other words, $K^\bullet(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is the complex with terms $\mathrm{Map}(X_\bullet, A)$ where

$$X_n = \coprod\nolimits_{U_0 \to \ldots \to U_{n-1} \to U_n} \mathrm{Mor}_{\mathcal{C}}(U, U_0) \times \mathrm{Mor}_{\mathcal{C}}(U_n, U')$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$ as can be proved in exactly the same way as the corresponding statement in Example 32.1. This finishes the proof of the claim.

The argument in case (2) is similar (but dual).

33. Strictly perfect complexes

This section is the analogue of Cohomology, Section 35.

Definition 33.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. We say $\mathcal{E}^\bullet$ is *strictly perfect* if $\mathcal{E}^i$ is zero for all but finitely many i and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}$-module for all i.

Let U be an object of $\mathcal{C}$. We will often say "Let $\mathcal{E}^\bullet$ be a strictly perfect complex of $\mathcal{O}_U$-modules" to mean $\mathcal{E}^\bullet$ is a strictly perfect complex of modules on the ringed site $(\mathcal{C}/U, \mathcal{O}_U)$, see Modules on Sites, Definition 19.1.

Lemma 33.2. *The cone on a morphism of strictly perfect complexes is strictly perfect.*

Proof. This is immediate from the definitions. $\square$

Lemma 33.3. *The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.*

Proof. Omitted. $\square$

Lemma 33.4. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_\mathcal{D}$-modules, then $f^*\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_\mathcal{C}$-modules.*

Proof. We have seen in Modules on Sites, Lemma 17.2 that the pullback of a finite free module is finite free. The functor f^* is additive functor hence preserves direct summands. The lemma follows. $\square$

Lemma 33.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Given a solid diagram of $\mathcal{O}_U$-modules*

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[dr, dotted] & \mathcal{F} \\ & \mathcal{G} \arrow[u, "p"'] \end{tikzcd}$$

with $\mathcal{E}$ a direct summand of a finite free $\mathcal{O}_U$-module and p surjective, then there exists a covering $\{U_i \to U\}$ such that a dotted arrow making the diagram commute exists over each U_i.

Proof. We may assume $\mathcal{E} = \mathcal{O}_U^{\oplus n}$ for some n. In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in $\Gamma(U, \mathcal{F})$. This is locally possible by the characterization of surjective maps of sheaves (Sites, Section 12). $\square$

Lemma 33.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.*

(1) *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{F}^\bullet$ acyclic. Then there exists a covering $\{U_i \to U\}$ such that each $\alpha|_{U_i}$ is homotopic to zero.*
(2) *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^i = 0$ for $i < a$, and $H^i(\mathcal{F}^\bullet) = 0$ for $i \geq a$. Then there exists a covering $\{U_i \to U\}$ such that each $\alpha|_{U_i}$ is homotopic to zero.*

Proof. The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex $\mathcal{E}^\bullet$. If $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$ for some direct summand $\mathcal{E}$ of a finite free $\mathcal{O}$-module and integer $n \geq a$, then the result follows from Lemma 33.5 and the fact that $\mathcal{F}^{n-1} \to \mathrm{Ker}(\mathcal{F}^n \to \mathcal{F}^{n+1})$ is surjective by the assumed vanishing of $H^n(\mathcal{F}^\bullet)$. If $\mathcal{E}^i$ is zero except for $i \in [a, b]$, then we have a split exact sequence of complexes

$$0 \to \mathcal{E}^b[-b] \to \mathcal{E}^\bullet \to \sigma_{\leq b-1}\mathcal{E}^\bullet \to 0$$

which determines a distinguished triangle in $K(\mathcal{O}_U)$. Hence an exact sequence

$$\mathrm{Hom}_{K(\mathcal{O}_U)}(\sigma_{\leq b-1}\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition $\mathcal{E}^b[-b] \to \mathcal{F}^\bullet$ is homotopic to zero on the members of a covering of U by the above, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is zero on the members of a covering of U by induction hypothesis. □

Lemma 33.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Given a solid diagram of complexes of $\mathcal{O}_U$-modules*

$$\begin{tikzcd} \mathcal{E}^\bullet \arrow[r, "\alpha"'] \arrow[dr, dotted] & \mathcal{F}^\bullet \\ & \mathcal{G}^\bullet \arrow[u, "f"'] \end{tikzcd}$$

with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^j = 0$ for $j < a$ and $H^j(f)$ an isomorphism for $j > a$ and surjective for $j = a$, then there exists a covering $\{U_i \to U\}$ and for each i a dotted arrow over U_i making the diagram commute up to homotopy.

Proof. Our assumptions on f imply the cone $C(f)^\bullet$ has vanishing cohomology sheaves in degrees $\geq a$. Hence Lemma 33.6 guarantees there is a covering $\{U_i \to U\}$ such that the composition $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet$ is homotopic to zero over U_i. Since

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet \to \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in $K(\mathcal{O}_{U_i})$ we see that we can lift $\alpha|_{U_i}$ up to homotopy to a map $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{G}^\bullet|_{U_i}$ as desired. □

Lemma 33.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect.*

1. *For any element $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ there exists a covering $\{U_i \to U\}$ such that $\alpha|_{U_i}$ is given by a morphism of complexes $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.*
2. *Given a morphism of complexes $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ whose image in the group $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is zero, there exists a covering $\{U_i \to U\}$ such that $\alpha|_{U_i}$ is homotopic to zero.*

Proof. Proof of (1). By the construction of the derived category we can find a quasi-isomorphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and a map of complexes $\beta : \mathcal{E}^\bullet \to \mathcal{G}^\bullet$ such that $\alpha = f^{-1}\beta$. Thus the result follows from Lemma 33.7. We omit the proof of (2). □

Lemma 33.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}$-modules with $\mathcal{E}^\bullet$ strictly perfect. Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms*

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 26.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex. Let $(\mathcal{H}')^\bullet$ be the complex with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{L}^{-q}, \mathcal{I}^p)$$

which represents $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ by the construction in Section 26. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an object U of $\mathcal{C}$ we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U) \to H^0((\mathcal{H}')^\bullet(U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U)$$

By Lemma 33.8 the sheafification of $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^0((\mathcal{H}')^\bullet(U))$. A similar argument can be given for the other cohomology sheaves. Thus $\mathcal{H}^\bullet$ is quasi-isomorphic to $(\mathcal{H}')^\bullet$ which proves the lemma. $\square$

Lemma 33.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}$-modules with*

1. *$\mathcal{F}^n = 0$ for $n \ll 0$,*
2. *$\mathcal{E}^n = 0$ for $n \gg 0$, and*
3. *$\mathcal{E}^n$ isomorphic to a direct summand of a finite free $\mathcal{O}$-module.*

Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 26.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives. Note that $\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 29.4). Hence the construction in Section 26 shows that $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $(\mathcal{H}')^\bullet$ with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{I}^p) = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

(equality because there are only finitely many nonzero terms). Note that $\mathcal{H}^\bullet$ is the total complex associated to the double complex with terms $\mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^p)$ and similarly for $(\mathcal{H}')^\bullet$. The natural map $(\mathcal{H}')^\bullet \to \mathcal{H}^\bullet$ comes from a map of double complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 22.6)

$${}'E_1^{p,q} = H^p(\mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{H}^\bullet)$ and similarly for $(\mathcal{H}')^\bullet$. To finish the proof of the lemma it suffices to show that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism

$$H^p(\mathcal{H}om_\mathcal{O}(\mathcal{E}, \mathcal{F}^\bullet)) \longrightarrow H^p(\mathcal{H}om_\mathcal{O}(\mathcal{E}, \mathcal{I}^\bullet))$$

on cohomology sheaves whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}$-module. Since this is clear when $\mathcal{E}$ is finite free the result follows. $\square$

34. Pseudo-coherent modules

In this section we discuss pseudo-coherent complexes.

Definition 34.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. Let $m \in \mathbf{Z}$.

(1) We say $\mathcal{E}^\bullet$ is *m-pseudo-coherent* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ and for each i a morphism of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ where $\mathcal{E}_i$ is a strictly perfect complex of $\mathcal{O}_{U_i}$-modules and $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective.
(2) We say $\mathcal{E}^\bullet$ is *pseudo-coherent* if it is m-pseudo-coherent for all m.
(3) We say an object E of $D(\mathcal{O})$ is *m-pseudo-coherent* (resp. *pseudo-coherent*) if and only if it can be represented by a m-pseudo-coherent (resp. pseudo-coherent) complex of $\mathcal{O}$-modules.

If $\mathcal{C}$ has a final object X which is quasi-compact (i.e., every covering of X can be refined by a finite covering), then an m-pseudo-coherent object of $D(\mathcal{O})$ is in $D^-(\mathcal{O})$. But this need not be the case in general.

Lemma 34.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$.*

(1) *If $\mathcal{C}$ has a final object X and if there exist a covering $\{U_i \to X\}$, strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$-modules, and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $H^j(\alpha_i)$ an isomorphism for $j > m$ and $H^m(\alpha_i)$ surjective, then E is m-pseudo-coherent.*
(2) *If E is m-pseudo-coherent, then any complex of $\mathcal{O}$-modules representing E is m-pseudo-coherent.*
(3) *If for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ is m-pseudo-coherent, then E is m-pseudo-coherent.*

Proof. Let $\mathcal{F}^\bullet$ be any complex representing E and let X, $\{U_i \to X\}$, and $\alpha_i : \mathcal{E}_i \to E|_{U_i}$ be as in (1). We will show that $\mathcal{F}^\bullet$ is m-pseudo-coherent as a complex, which will prove (1) and (2) in case $\mathcal{C}$ has a final object. By Lemma 33.8 we can after refining the covering $\{U_i \to X\}$ represent the maps α_i by maps of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{F}^\bullet|_{U_i}$. By assumption $H^j(\alpha_i)$ are isomorphisms for $j > m$, and $H^m(\alpha_i)$ is surjective whence $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof of (2). By the above we see that $\mathcal{F}^\bullet|_U$ is m-pseudo-coherent as a complex of $\mathcal{O}_U$-modules for all objects U of $\mathcal{C}$. It is a formal consequence of the definitions that $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof of (3). Follows from the definitions and Sites, Definition 6.2 part (2). □

Lemma 34.3. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{C})$. If E is m-pseudo-coherent, then Lf^*E is m-pseudo-coherent.*

Proof. Say f is given by the functor $u : \mathcal{D} \to \mathcal{C}$. Let U be an object of $\mathcal{C}$. By Sites, Lemma 15.9 we can find a covering $\{U_i \to U\}$ and for each i a morphism $U_i \to u(V_i)$ for some object V_i of $\mathcal{D}$. By Lemma 34.2 it suffices to show that $Lf^*E|_{U_i}$ is m-pseudo-coherent. To do this it is enough to show that $Lf^*E|_{u(V_i)}$ is m-pseudo-coherent, since $Lf^*E|_{U_i}$ is the restriction of $Lf^*E|_{u(V_i)}$ to $\mathcal{C}/U_i$ (via Modules on Sites, Lemma 19.4). By the commutative diagram of Modules on Sites, Lemma 20.1 it suffices to prove the lemma for the morphism of ringed sites

$(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \to (\mathcal{D}/V_i, \mathcal{O}_{V_i})$. Thus we may assume $\mathcal{D}$ has a final object Y such that $X = u(Y)$ is a final object of $\mathcal{C}$.

Let $\{V_i \to Y\}$ be a covering such that for each i there exists a strictly perfect complex $\mathcal{F}_i^\bullet$ of $\mathcal{O}_{V_i}$-modules and a morphism $\alpha_i : \mathcal{F}_i^\bullet \to E|_{V_i}$ of $D(\mathcal{O}_{V_i})$ such that $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective. Arguing as above it suffices to prove the result for $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \to (\mathcal{D}/V_i, \mathcal{O}_{V_i})$. Hence we may assume that there exists a strictly perfect complex $\mathcal{F}^\bullet$ of $\mathcal{O}_\mathcal{D}$-modules and a morphism $\alpha : \mathcal{F}^\bullet \to E$ of $D(\mathcal{O}_\mathcal{D})$ such that $H^j(\alpha)$ is an isomorphism for $j > m$ and $H^m(\alpha)$ is surjective. In this case, choose a distinguished triangle

$$\mathcal{F}^\bullet \to E \to C \to \mathcal{F}^\bullet[1]$$

The assumption on α means exactly that the cohomology sheaves $H^j(C)$ are zero for all $j \geq m$. Applying Lf^* we obtain the distinguished triangle

$$Lf^*\mathcal{F}^\bullet \to Lf^*E \to Lf^*C \to Lf^*\mathcal{F}^\bullet[1]$$

By the construction of Lf^* as a left derived functor we see that $H^j(Lf^*C) = 0$ for $j \geq m$ (by the dual of Derived Categories, Lemma 17.1). Hence $H^j(Lf^*\alpha)$ is an isomorphism for $j > m$ and $H^m(Lf^*\alpha)$ is surjective. On the other hand, since $\mathcal{F}^\bullet$ is a bounded above complex of flat $\mathcal{O}_\mathcal{D}$-modules we see that $Lf^*\mathcal{F}^\bullet = f^*\mathcal{F}^\bullet$. Applying Lemma 33.4 we conclude. □

Lemma 34.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site and $m \in \mathbf{Z}$. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$.*

1. *If K is $(m+1)$-pseudo-coherent and L is m-pseudo-coherent then M is m-pseudo-coherent.*
2. *If K anf M are m-pseudo-coherent, then L is m-pseudo-coherent.*
3. *If L is $(m+1)$-pseudo-coherent and M is m-pseudo-coherent, then K is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. Choose a covering $\{U_i \to U\}$ and maps $\alpha_i : \mathcal{K}_i^\bullet \to K|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{K}_i^\bullet$ strictly perfect and $H^j(\alpha_i)$ isomorphisms for $j > m+1$ and surjective for $j = m+1$. We may replace $\mathcal{K}_i^\bullet$ by $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$ and hence we may assume that $\mathcal{K}_i^j = 0$ for $j < m+1$. After refining the covering we may choose maps $\beta_i : \mathcal{L}_i^\bullet \to L|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{L}_i^\bullet$ strictly perfect such that $H^j(\beta)$ is an isomorphism for $j > m$ and surjective for $j = m$. By Lemma 33.7 we can, after refining the covering, find maps of complexes $\gamma_i : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ such that the diagrams

$$\begin{tikzcd} K|_{U_i} \arrow[r] & L|_{U_i} \\ \mathcal{K}_i^\bullet \arrow[u, "\alpha_i"] \arrow[r, "\gamma_i"] & \mathcal{L}_i^\bullet \arrow[u, "\beta_i"'] \end{tikzcd}$$

are commutative in $D(\mathcal{O}_{U_i})$ (this requires representing the maps α_i, β_i and $K|_{U_i} \to L|_{U_i}$ by actual maps of complexes; some details omitted). The cone $C(\gamma_i)^\bullet$ is strictly perfect (Lemma 33.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 5.19 and 5.20 that $C(\gamma_i)^\bullet \to M|_{U_i}$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence M is m-pseudo-coherent by Lemma 34.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. $\square$

Lemma 34.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$.*

1. *If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes_{\mathcal{O}}^{\mathbf{L}} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.*
2. *If K and L are pseudo-coherent, then $K \otimes_{\mathcal{O}}^{\mathbf{L}} L$ is pseudo-coherent.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. By replacing U by the members of a covering and replacing $\mathcal{C}$ by the localization $\mathcal{C}/U$ we may assume there exist strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ and maps $\alpha : \mathcal{K}^\bullet \to K$ and $\beta : \mathcal{L}^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) \to K \otimes_{\mathcal{O}}^{\mathbf{L}} L$$

induces isomorphisms on cohomology sheaves in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted).

Proof of (2). Let U be an object of $\mathcal{C}$. We may first replace U by the members of a covering and $\mathcal{C}$ by the localization $\mathcal{C}/U$ to reduce to the case that K and L are bounded above. Then the statement follows immediately from case (1). $\square$

Lemma 34.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $m \in \mathbf{Z}$. If $K \oplus L$ is m-pseudo-coherent (resp. pseudo-coherent) in $D(\mathcal{O})$ so are K and L.*

Proof. Assume that $K \oplus L$ is m-pseudo-coherent. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume $K \oplus L \in D^-(\mathcal{O}_U)$, hence $L \in D^-(\mathcal{O}_U)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 4.9. By Lemma 34.4 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. Since L is bounded above we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n-1], L[n-2], \ldots, L$ are m-pseudo-coherent as desired. $\square$

Lemma 34.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. Let $m \in \mathbf{Z}$.*

1. *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m$, then $H^m(K)$ is a finite type $\mathcal{O}$-module.*
2. *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m + 1$, then $H^{m+1}(K)$ is a finitely presented $\mathcal{O}$-module.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. We have to show that $H^m(K)$ is can be generated by finitely many sections over the members of a covering of U (see Modules on Sites, Definition 23.1). Thus during the proof we may (finitely often) choose a covering $\{U_i \to U\}$ and replace $\mathcal{C}$ by $\mathcal{C}/U_i$ and U by U_i. In particular, by our definitions we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$

and a surjection in degree m. It suffices to prove the result for $\mathcal{E}^\bullet$. Let n be the largest integer such that $\mathcal{E}^n \neq 0$. If $n = m$, then $H^m(\mathcal{E}^\bullet)$ is a quotient of $\mathcal{E}^n$ and the result is clear. If $n > m$, then $\mathcal{E}^{n-1} \to \mathcal{E}^n$ is surjective as $H^n(E^\bullet) = 0$. By Lemma 33.5 we can (after replacing U by the members of a covering) find a section of this surjection and write $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$. Hence it suffices to prove the result for the complex $(\mathcal{E}')^\bullet$ which is the same as $\mathcal{E}^\bullet$ except has $\mathcal{E}'$ in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). Pick an object U of $\mathcal{C}$. As in the proof of (1) we may work locally on U. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $\mathcal{E}^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \mathrm{Coker}(\mathcal{E}^m \to \mathcal{E}^{m+1})$ which is of finite presentation. □

35. Tor dimension

In this section we take a closer look at resolutions by flat modules.

Definition 35.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$ with $a \leq b$.

(1) We say E has *tor-amplitude in* $[a, b]$ if $H^i(E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}) = 0$ for all $\mathcal{O}$-modules $\mathcal{F}$ and all $i \notin [a, b]$.
(2) We say E has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.
(3) We say E *locally has finite tor dimension* if for any object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ has finite tor dimension for all i.

Note that if E has finite tor dimension, then E is an object of $D^b(\mathcal{O})$ as can be seen by taking $\mathcal{F} = \mathcal{O}$ in the definition above.

Lemma 35.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules with tor-amplitude in $[a, b]$. Then Coker$(d_{\mathcal{E}^\bullet}^{a-1})$ is a flat $\mathcal{O}$-module.*

Proof. As $\mathcal{E}^\bullet$ is a bounded above complex of flat modules we see that $\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$. Hence for every $\mathcal{O}$-module $\mathcal{F}$ the sequence

$$\mathcal{E}^{a-2} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}} \mathcal{F}$$

is exact in the middle. Since $\mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to \mathcal{E}^a \to \mathrm{Coker}(d^{a-1}) \to 0$ is a flat resolution this implies that $\mathrm{Tor}_1^{\mathcal{O}}(\mathrm{Coker}(d^{a-1}), \mathcal{F}) = 0$ for all $\mathcal{O}$-modules $\mathcal{F}$. This means that $\mathrm{Coker}(d^{a-1})$ is flat, see Lemma 17.13. □

Lemma 35.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent*

(1) *E has tor-amplitude in $[a, b]$.*
(2) *E is represented by a complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$.*

Proof. If (2) holds, then we may compute $E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}$ and it is clear that (1) holds.

Assume that (1) holds. We may represent E by a bounded above complex of flat $\mathcal{O}$-modules $\mathcal{K}^\bullet$, see Section 17. Let n be the largest integer such that $\mathcal{K}^n \neq 0$. If $n > b$, then $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is surjective as $H^n(\mathcal{K}^\bullet) = 0$. As $\mathcal{K}^n$ is flat we see that

$\mathrm{Ker}(\mathcal{K}^{n-1} \to \mathcal{K}^n)$ is flat (Modules on Sites, Lemma 28.8). Hence we may replace $\mathcal{K}^\bullet$ by $\tau_{\leq n-1}\mathcal{K}^\bullet$. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of flat $\mathcal{O}$-modules with $\mathcal{K}^i = 0$ for $i > b$.

Set $\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet$. Everything is clear except that $\mathcal{E}^a$ is flat which follows immediately from Lemma 35.2 and the definitions. □

Lemma 35.4. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Assume $\mathcal{C}$ has enough points. Let E be an object of $D(\mathcal{O}_\mathcal{D})$. If E has tor amplitude in $[a, b]$, then Lf^*E has tor amplitude in $[a, b]$.*

Proof. Assume E has tor amplitude in $[a, b]$. By Lemma 35.3 we can represent E by a complex of $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$. Then Lf^*E is represented by $f^*\mathcal{E}^\bullet$. By Modules on Sites, Lemma 38.3 the module $f^*\mathcal{E}^i$ are flat (this is where we need the assumption on the existence of points). Thus by Lemma 35.3 we conclude that Lf^*E has tor amplitude in $[a, b]$. □

Lemma 35.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$.*

1. *If K has tor-amplitude in $[a + 1, b + 1]$ and L has tor-amplitude in $[a, b]$ then M has tor-amplitude in $[a, b]$.*
2. *If K and M have tor-amplitude in $[a, b]$, then L has tor-amplitude in $[a, b]$.*
3. *If L has tor-amplitude in $[a + 1, b + 1]$ and M has tor-amplitude in $[a, b]$, then K has tor-amplitude in $[a + 1, b + 1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes_\mathcal{O}^\mathbf{L} \mathcal{F}$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

Lemma 35.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. If K has tor-amplitude in $[a, b]$ and L has tor-amplitude in $[c, d]$ then $K \otimes_\mathcal{O}^\mathbf{L} L$ has tor amplitude in $[a + c, b + d]$.*

Proof. Omitted. Hint: use the spectral sequence for tors. □

Lemma 35.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a, b \in \mathbf{Z}$. For K, L objects of $D(\mathcal{O})$ if $K \oplus L$ has tor amplitude in $[a, b]$ so do K and L.*

Proof. Clear from the fact that the Tor functors are additive. □

Lemma 35.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let K be an object of $D(\mathcal{O})$.*

1. *If $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}$ is bounded above, then $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n$ is uniformly bounded above for all n.*
2. *If $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}$ as an object of $D(\mathcal{O}/\mathcal{I})$ has tor amplitude in $[a, b]$, then $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n$ as an object of $D(\mathcal{O}/\mathcal{I}^n)$ has tor amplitude in $[a, b]$ for all n.*

Proof. Proof of (1). Assume that $K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}$ is bounded above, say $H^i(K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}) = 0$ for $i > b$. Note that we have distinguished triangles

$$K \otimes_\mathcal{O}^\mathbf{L} \mathcal{I}^n/\mathcal{I}^{n+1} \to K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^{n+1} \to K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}^n \to K \otimes_\mathcal{O}^\mathbf{L} \mathcal{I}^n/\mathcal{I}^{n+1}[1]$$

and that

$$K \otimes_\mathcal{O}^\mathbf{L} \mathcal{I}^n/\mathcal{I}^{n+1} = \left(K \otimes_\mathcal{O}^\mathbf{L} \mathcal{O}/\mathcal{I}\right) \otimes_{\mathcal{O}/\mathcal{I}}^\mathbf{L} \mathcal{I}^n/\mathcal{I}^{n+1}$$

By induction we conclude that $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n) = 0$ for $i > b$ for all n.

Proof of (2). Assume $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$ as an object of $D(\mathcal{O}/\mathcal{I})$ has tor amplitude in $[a, b]$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}/\mathcal{I}^n$-modules. Then we have a finite filtration

$$0 \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

whose successive quotients are sheaves of $\mathcal{O}/\mathcal{I}$-modules. Thus to prove that $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$ has tor amplitude in $[a, b]$ it suffices to show $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n \otimes_{\mathcal{O}/\mathcal{I}^n}^{\mathbf{L}} \mathcal{G})$ is zero for $i \notin [a, b]$ for all $\mathcal{O}/\mathcal{I}$-modules $\mathcal{G}$. Since

$$\left(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n\right) \otimes_{\mathcal{O}/\mathcal{I}^n}^{\mathbf{L}} \mathcal{G} = \left(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}\right) \otimes_{\mathcal{O}/\mathcal{I}}^{\mathbf{L}} \mathcal{G}$$

for every sheaf of $\mathcal{O}/\mathcal{I}$-modules $\mathcal{G}$ the result follows. □

36. Perfect complexes

In this section we discuss properties of perfect complexes on ringed sites.

Definition 36.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. We say $\mathcal{E}^\bullet$ is *perfect* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that for each i there exists a morphism of complexes $\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ which is a quasi-isomorphism with $\mathcal{E}_i^\bullet$ strictly perfect. An object E of of $D(\mathcal{O})$ is *perfect* if it can be represented by a perfect complex of $\mathcal{O}$-modules.

Lemma 36.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$.*

1. *If $\mathcal{C}$ has a final object X and there exist a covering $\{U_i \to X\}$, strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$-modules, and isomorphisms $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$, then E is perfect.*
2. *If E is perfect, then any complex representing E is perfect.*

Proof. Identical to the proof of Lemma 34.2. □

Lemma 36.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a \leq b$ be integers. If E has tor amplitude in $[a, b]$ and is $(a-1)$-pseudo-coherent, then E is perfect.*

Proof. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering and $\mathcal{C}$ by the localization $\mathcal{C}/U$ we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to E$ such that $H^i(\alpha)$ is an isomorphism for $i \geq a$. We may and do replace $\mathcal{E}^\bullet$ by $\sigma_{\geq a-1}\mathcal{E}^\bullet$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to E \to C \to \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of E and $\mathcal{E}^\bullet$ and the assumption on α we obtain $C \cong \mathcal{K}[a-2]$ with $\mathcal{K} = \text{Ker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Applying $- \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}$ the assumption that E has tor amplitude in $[a, b]$ implies $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F}$ has image $\text{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}} \mathcal{F})$. It follows that $\text{Tor}_1^{\mathcal{O}}(\mathcal{E}', \mathcal{F}) = 0$ where $\mathcal{E}' = \text{Coker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Hence $\mathcal{E}'$ is flat (Lemma 17.13). Thus there exists a covering $\{U_i \to U\}$ such that $\mathcal{E}'|_{U_i}$ is a direct summand of a finite free module by Modules on Sites, Lemma 28.12. Thus the complex

$$\mathcal{E}'|_{U_i} \to \mathcal{E}^{a-1}|_{U_i} \to \ldots \to \mathcal{E}^b|_{U_i}$$

is quasi-isomorphic to $E|_{U_i}$ and E is perfect. □

Lemma 36.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. The following are equivalent*

(1) *E is perfect, and*
(2) *E is pseudo-coherent and locally has finite tor dimension.*

Proof. Assume (1). Let U be an object of $\mathcal{C}$. By definition there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ is represented by a strictly perfect complex. Thus E is pseudo-coherent (i.e., m-pseudo-coherent for all m) by Lemma 34.2. Moreover, a direct summand of a finite free module is flat, hence $E|_{U_i}$ has finite Tor dimension by Lemma 35.3. Thus (2) holds.

Assume (2). Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume there exist integers $a \leq b$ such that $E|_U$ has tor amplitude in $[a, b]$. Since $E|_U$ is m-pseudo-coherent for all m we conclude using Lemma 36.3. □

Lemma 36.5. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Assume $\mathcal{C}$ has enough points. Let E be an object of $D(\mathcal{O}_\mathcal{D})$. If E is perfect in $D(\mathcal{O}_\mathcal{D})$, then Lf^*E is perfect in $D(\mathcal{O}_\mathcal{C})$.*

Proof. This follows from Lemma 36.4, 35.4, and 34.3. (An alternative proof is to copy the proof of Lemma 34.3. This gives a proof of the result without assuming the site $\mathcal{C}$ has enough points.) □

Lemma 36.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$. If two out of three of K, L, M are perfect then the third is also perfect.*

Proof. First proof: Combine Lemmas 36.4, 34.4, and 35.5. Second proof (sketch): Say K and L are perfect. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume that $K|_U$ and $L|_U$ are represented by strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$. After replacing U by the members of a covering we may assume the map $K|_U \to L|_U$ is given by a map of complexes $\alpha : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, see Lemma 33.8. Then $M|_U$ is isomorphic to the cone of α which is strictly perfect by Lemma 33.2. □

Lemma 36.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If K, L are perfect objects of $D(\mathcal{O})$, then so is $K \otimes_\mathcal{O}^\mathbf{L} L$.*

Proof. Follows from Lemmas 36.4, 34.5, and 35.6. □

Lemma 36.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $K \oplus L$ is a perfect object of $D(\mathcal{O})$, then so are K and L.*

Proof. Follows from Lemmas 36.4, 34.6, and 35.7. □

Lemma 36.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a perfect object of $D(\mathcal{O})$. Then $K^\wedge = R\mathcal{H}om(K, \mathcal{O})$ is a perfect object too and $(K^\wedge)^\wedge = K$. There are functorial isomorphisms*

$$K^\wedge \otimes_\mathcal{O}^\mathbf{L} M = R\mathcal{H}om_\mathcal{O}(K, M)$$

and

$$H^0(\mathcal{C}, K^\wedge \otimes_\mathcal{O}^\mathbf{L} M) = \mathrm{Hom}_{D(\mathcal{O})}(K, M)$$

for M in $D(\mathcal{O})$.

Proof. We will us without further mention that formation of internal hom commutes with restriction (Lemma 26.3). In particular we may check the first two statements locally, i.e., given any object U of $\mathcal{C}$ it suffices to prove there is a covering $\{U_i \to U\}$ such that the statement is true after restricting to $\mathcal{C}/U_i$ for each i. By Lemma 26.8 to see the final statement it suffices to check that the map (26.8.1)

$$K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}} M \longrightarrow R\mathcal{H}om(K, M)$$

is an isomorphism. This is a local question as well. Hence it suffices to prove the lemma when K is represented by a strictly perfect complex.

Assume K is represented by the strictly perfect complex $\mathcal{E}^\bullet$. Then it follows from Lemma 33.9 that $K^\wedge$ is represented by the complex whose terms are $(\mathcal{E}^n)^\wedge = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^n, \mathcal{O})$ in degree $-n$. Since $\mathcal{E}^n$ is a direct summand of a finite free $\mathcal{O}$-module, so is $(\mathcal{E}^n)^\wedge$. Hence $K^\wedge$ is represented by a strictly perfect complex too. It is also clear that $(K^\wedge)^\wedge = K$ as we have $((\mathcal{E}^n)^\wedge)^\wedge = \mathcal{E}^n$. To see that (26.8.1) is an isomorphism, represent M by a K-flat complex $\mathcal{F}^\bullet$. By Lemma 33.9 the complex $R\mathcal{H}om(K, M)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

On the other hand, the object $K^\wedge \otimes^{\mathbf{L}} M$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{F}^p \otimes_{\mathcal{O}} (\mathcal{E}^{-q})^\wedge$$

Thus the assertion that (26.8.1) is an isomorphism reduces to the assertion that the canonical map

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F})$$

is an isomorphism when $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}$-module and $\mathcal{F}$ is any $\mathcal{O}$-module. This follows immediately from the corresponding statement when $\mathcal{E}$ is finite free. □

Lemma 36.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(K_n)_{n \in \mathbf{N}}$ be a system of perfect objects of $D(\mathcal{O})$. Let $K = hocolim K_n$ be the derived colimit (Derived Categories, Definition 31.1). Then for any object E of $D(\mathcal{O})$ we have*

$$R\mathcal{H}om(K, E) = R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}} K_n^\wedge$$

where $(K_n^\wedge)$ is the inverse system of dual perfect complexes.

Proof. By Lemma 36.9 we have $R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}} K_n^\wedge = R\lim R\mathcal{H}om(K_n, E)$ which fits into the distinguished triangle

$$R\lim R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E)$$

Because K similarly fits into the distinguished triangle $\bigoplus K_n \to \bigoplus K_n \to K$ it suffices to show that $\prod R\mathcal{H}om(K_n, E) = R\mathcal{H}om(\bigoplus K_n, E)$. This is a formal consequence of (26.0.1) and the fact that derived tensor product commutes with direct sums. □

37. Projection formula

A general version of the projection formula is the following.

Lemma 37.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $E \in D(\mathcal{O}_\mathcal{C})$ and $K \in D(\mathcal{O}_\mathcal{D})$. If K is perfect, then*

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K)$$

in $D(\mathcal{O}_\mathcal{D})$.

Proof. Without any assumptions there is a map $Rf_*(E) \otimes^{\mathbf{L}} K \to Rf_*(E \otimes^{\mathbf{L}} Lf^*K)$. Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*(E) \otimes^{\mathbf{L}} K) = Lf^*(Rf_*(E)) \otimes^{\mathbf{L}} Lf^*K \longrightarrow E \otimes^{\mathbf{L}} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$. See Lemmas 18.4 and 19.1. To check it is an isomorphism we may work locally on $\mathcal{D}$, i.e., for any object V of $\mathcal{D}$ we have to find a covering $\{V_j \to V\}$ such that the map restricts to an isomorphism on V_j. By definition of perfect objects, this means we may assume K is represented by a strictly perfect complex of $\mathcal{O}_\mathcal{D}$-modules. Note that, completely generally, the statement is true for $K = K_1 \oplus K_2$, if and only if the statement is true for K_1 and K_2. Hence we may assume K is a finite complex of finite free $\mathcal{O}_\mathcal{D}$-modules. In this case a simple argument involving stupid truncations reduces the statement to the case where K is represented by a finite free $\mathcal{O}_\mathcal{D}$-module. Since the statement is invariant under finite direct summands in the K variable, we conclude it suffices to prove it for $K = \mathcal{O}_\mathcal{D}[n]$ in which case it is trivial. $\square$

38. Weakly contractible objects

An object U of a site is *weakly contractible* if every surjection $\mathcal{F} \to \mathcal{G}$ of sheaves of sets gives rise to a surjection $\mathcal{F}(U) \to \mathcal{G}(U)$, see Sites, Definition 39.2.

Lemma 38.1. *Let $\mathcal{C}$ be a site. Let U be a weakly contractible object of $\mathcal{C}$. Then*

1. *the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor $Ab(\mathcal{C}) \to Ab$,*
2. *$H^p(U, \mathcal{F}) = 0$ for every abelian sheaf $\mathcal{F}$ and all $p \geq 1$, and*
3. *for any sheaf of groups $\mathcal{G}$ any $\mathcal{G}$-torsor has a section over U.*

Proof. The first statement follows immediately from the definition (see also Homology, Section 7). The higher derived functors vanish by Derived Categories, Lemma 17.8. Let $\mathcal{F}$ be a $\mathcal{G}$-torsor. Then $\mathcal{F} \to *$ is a surjective map of sheaves. Hence (3) follows from the definition as well. $\square$

It is convenient to list some consequences of having enough weakly contractible objects here.

Proposition 38.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ such that every $U \in \mathcal{B}$ is weakly contractible and every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Then*

1. *A complex $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ of $\mathcal{O}$-modules is exact, if and only if $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact for all $U \in \mathcal{B}$.*
2. *Every object K of $D(\mathcal{O})$ is a derived limit of its canonical truncations: $K = R\lim \tau_{\geq -n}K$.*
3. *Given an inverse system $\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$ with surjective transition maps, the projection $\lim \mathcal{F}_n \to \mathcal{F}_1$ is surjective.*

(4) *Products are exact on $Mod(\mathcal{O})$.*
(5) *Products on $D(\mathcal{O})$ can be computed by taking products of any representative complexes.*
(6) *If $(\mathcal{F}_n)$ is an inverse system of $\mathcal{O}$-modules, then $R^p \lim \mathcal{F}_n = 0$ for all $p > 1$ and*
$$R^1 \lim \mathcal{F}_n = Coker(\prod \mathcal{F}_n \to \prod \mathcal{F}_n)$$
where the map is $(x_n) \mapsto (x_n - f(x_{n+1}))$.
(7) *If (K_n) is an inverse system of objects of $D(\mathcal{O})$, then there are short exact sequences*
$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(R\lim K_n) \to \lim H^p(K_n) \to 0$$

Proof. Proof of (1). If the sequence is exact, then evaluating at any weakly contractible element of $\mathcal{C}$ gives an exact sequence by Lemma 38.1. Conversely, assume that $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact for all $U \in \mathcal{B}$. Let V be an object of $\mathcal{C}$ and let $s \in \mathcal{F}_2(V)$ be an element of the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$. By assumption there exists a covering $\{U_i \to V\}$ with $U_i \in \mathcal{B}$. Then $s|_{U_i}$ lifts to a section $s_i \in \mathcal{F}_1(U_i)$. Thus s is a section of the image sheaf $\mathrm{Im}(\mathcal{F}_1 \to \mathcal{F}_2)$. In other words, the sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact.

Proof of (2). Lemma 22.3 applies to every complex of sheaves on $\mathcal{C}$. Thus (1) holds by Lemma 22.4.

Proof of (3). Let $(\mathcal{F}_n)$ be a system as in (2) and set $\mathcal{F} = \lim \mathcal{F}_n$. If $U \in \mathcal{B}$, then $\mathcal{F}(U) = \lim \mathcal{F}_n(U)$ surjects onto $\mathcal{F}_1(U)$ as all the transition maps $\mathcal{F}_{n+1}(U) \to \mathcal{F}_n(U)$ are surjective. Thus $\mathcal{F} \to \mathcal{F}_1$ is surjective by Sites, Definition 12.1 and the assumption that every object has a covering by elements of $\mathcal{B}$.

Proof of (4). Let $\mathcal{F}_{i,1} \to \mathcal{F}_{i,2} \to \mathcal{F}_{i,3}$ be a family of exact sequences of $\mathcal{O}$-modules. We want to show that $\prod \mathcal{F}_{i,1} \to \prod \mathcal{F}_{i,2} \to \prod \mathcal{F}_{i,3}$ is exact. We use the criterion of (1). Let $U \in \mathcal{B}$. Then
$$(\prod \mathcal{F}_{i,1})(U) \to (\prod \mathcal{F}_{i,2})(U) \to (\prod \mathcal{F}_{i,3})(U)$$
is the same as
$$\prod \mathcal{F}_{i,1}(U) \to \prod \mathcal{F}_{i,2}(U) \to \prod \mathcal{F}_{i,3}(U)$$
Each of the sequences $\mathcal{F}_{i,1}(U) \to \mathcal{F}_{i,2}(U) \to \mathcal{F}_{i,3}(U)$ are exact by (1). Thus the displayed sequences are exact by Homology, Lemma 28.1. We conclude by (1) again.

Proof of (5). Follows from (4) and (slightly generalized) Derived Categories, Lemma 32.2.

Proof of (6) and (7). We refer to Section 21 for a discussion of derived and homotopy limits and their relationship. By Derived Categories, Definition 32.1 we have a distinguished triangle
$$R\lim K_n \to \prod K_n \to \prod K_n \to R\lim K_n[1]$$
Taking the long exact sequence of cohomology sheaves we obtain
$$H^{p-1}(\prod K_n) \to H^{p-1}(\prod K_n) \to H^p(R\lim K_n) \to H^p(\prod K_n) \to H^p(\prod K_n)$$
Since products are exact by (4) this becomes
$$\prod H^{p-1}(K_n) \to \prod H^{p-1}(K_n) \to H^p(R\lim K_n) \to \prod H^p(K_n) \to \prod H^p(K_n)$$

Now we first apply this to the case $K_n = \mathcal{F}_n[0]$ where $(\mathcal{F}_n)$ is as in (6). We conclude that (6) holds. Next we apply it to (K_n) as in (7) and we conclude (7) holds. $\square$

39. Compact objects

In this section we study compact objects in the derived category of modules on a ringed site. We recall that compact objects are defined in Derived Categories, Definition 34.1.

Lemma 39.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Assume $\mathcal{C}$ has the following properties*

1. *$\mathcal{C}$ has a quasi-compact final object X,*
2. *every object of $\mathcal{C}$ can be covered by quasi-compact objects,*
3. *for a finite covering $\{U_i \to U\}_{i \in I}$ with U, U_i quasi-compact the fibre products $U_i \times_U U_j$ are quasi-compact.*

Then any perfect object of $D(\mathcal{O})$ is compact.

Proof. Let K be a perfect object and let $K^\wedge$ be its dual, see Lemma 36.9. Then we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M in $D(\mathcal{O}_X)$. Since $K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums (by construction) and H^0 does by Lemma 16.1 and the construction of direct sums in Injectives, Lemma 13.4 we obtain the result of the lemma. $\square$

Lemma 39.2. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $S \subset \mathrm{Ob}(\mathcal{A})$ be a set of objects such that*

1. *any object of $\mathcal{A}$ is a quotient of a direct sum of elements of S, and*
2. *for any $E \in S$ the functor $\mathrm{Hom}_\mathcal{A}(E, -)$ commutes with direct sums.*

Then every compact object of $D(\mathcal{A})$ is a direct summand in $D(\mathcal{A})$ of a finite complex of finite direct sums of elements of S.

Proof. Assume $K \in D(\mathcal{A})$ is a compact object. Represent K by a complex $K^\bullet$ and consider the map

$$K^\bullet \longrightarrow \bigoplus\nolimits_{n \geq 0} \tau_{\geq n} K^\bullet$$

where we have used the canonical truncations, see Homology, Section 13. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n} K$ is zero for at least one n, i.e., K is in $D^-(R)$.

We may represent K by a bounded above complex $K^\bullet$ each of whose terms is a direct sum of objects from S, see Derived Categories, Lemma 16.5. Note that we have

$$K^\bullet = \bigcup\nolimits_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 13. Hence by Derived Categories, Lemmas 31.4 and 31.5 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n} K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(\mathcal{A})$ for some complex $L^\bullet$ which is bounded and whose terms are direct sums of elements of S. Say L^i is zero for $i \notin [a, b]$. Let c be the largest integer $\leq b + 1$ such that L^i a finite direct sum of elements of S for $i < c$. Claim: if $c < b + 1$, then we can modify $L^\bullet$ to increase c. By induction this claim will show we have a factorization of 1_K as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(\mathcal{A})$ where L can be represented by a finite complex of finite direct sums of elements of S. Note that $e = \varphi \circ \psi \in \mathrm{End}_{D(\mathcal{A})}(L)$ is an idempotent. By Derived Categories, Lemma 4.12 we see that $L = \mathrm{Ker}(e) \oplus \mathrm{Ker}(1-e)$. The map $\varphi : K \to L$ induces an isomorphism with $\mathrm{Ker}(1-e)$ in $D(R)$ and we conclude.

Proof of the claim. Write $L^c = \bigoplus_{\lambda \in \Lambda} E_\lambda$. Since L^{c-1} is a finite direct sum of elements of S we can by assumption (2) find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda \in \Lambda'} E_\lambda \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow (\bigoplus\nolimits_{\lambda \in \Lambda \setminus \Lambda'} E_\lambda)[-i]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree i. By our assumption on K we see that, after possibly replacing Λ' by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(\mathcal{A})$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of π. Since π is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(\mathcal{A})$ (see Derived Categories, Lemma 12.1). Since $\mathrm{Hom}_{D(\mathcal{A})}(K, -)$ is homological (see Derived Categories, Lemma 4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(\mathcal{A})$ whose composition with $(L')^\bullet \to L^\bullet$ gives φ. Setting ψ' equal to the composition of ψ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree c and since $(L')^c = \bigoplus_{\lambda \in \Lambda'} E_\lambda$ the claim is proved. □

Lemma 39.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Assume every object of $\mathcal{C}$ has a covering by quasi-compact objects. Then every compact object of $D(\mathcal{O})$ is a direct summand in $D(\mathcal{O})$ of a finite complex whose terms are finite direct sums of $\mathcal{O}$-modules of the form $j_!\mathcal{O}_U$ where U is a quasi-compact object of $\mathcal{C}$.*

Proof. Apply Lemma 39.2 where $S \subset \mathrm{Ob}(\textit{Mod}(\mathcal{O}))$ is the set of modules of the form $j_!\mathcal{O}_U$ with $U \in \mathrm{Ob}(\mathcal{C})$ quasi-compact. Assumption (1) holds by Modules on Sites, Lemma 28.6 and the assumption that every U can be covered by quasi-compact objects. Assumption (2) follows as

$$\mathrm{Hom}_\mathcal{O}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$$

which commutes with direct sums by Sites, Lemma 11.2. □

In the situation of the lemma above it is not always true that the modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O})$ (even if U is a quasi-compact object of $\mathcal{C}$). Here is a criterion.

Lemma 39.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. The $\mathcal{O}$-module $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O})$ if there exists an integer d such that*

1. *$H^p(U, \mathcal{F}) = 0$ for all $p > d$, and*
2. *the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums.*

Proof. Assume (1) and (2). The first means that the functor $F = H^0(U, -)$ has finite cohomological dimension. Moreover, any direct sum of injective modules is acyclic for F by (2). Since we may compute RF by applying F to any complex of acyclics (Derived Categories, Lemma 30.2). Thus, if K_i be a family of objects of

$D(\mathcal{O})$, then we can choose K-injective representatives $I_i^\bullet$ and we see that $\bigoplus K_i$ is represented by $\bigoplus I_i^\bullet$. Thus $H^0(U, -)$ commutes with direct sums. $\square$

Lemma 39.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$ which is quasi-compact and weakly contractible. Then $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O})$.*

Proof. Combine Lemmas 39.4 and 38.1 with Modules on Sites, Lemma 29.2. $\square$

40. Complexes with locally constant cohomology sheaves

Locally constant sheaves are introduced in Modules on Sites, Section 42. Let $\mathcal{C}$ be a site. Let Λ be a ring. We denote $D(\mathcal{C}, \Lambda)$ the derived category of the abelian category of $\underline{\Lambda}$-modules on $\mathcal{C}$.

Lemma 40.1. *Let $\mathcal{C}$ be a site with final object X. Let Λ be a Noetherian ring. Let $K \in D^b(\mathcal{C}, \Lambda)$ with $H^i(K)$ locally constant sheaves of Λ-modules of finite type. Then there exists a covering $\{U_i \to X\}$ such that each $K|_{U_i}$ is represented by a complex of locally constant sheaves of Λ-modules of finite type.*

Proof. Let $a \leq b$ be such that $H^i(K) = 0$ for $i \not\in [a, b]$. By induction on $b - a$ we will prove there exists a covering $\{U_i \to X\}$ such that $K|_{U_i}$ can be represented by a complex $\underline{M^\bullet}_{U_i}$ with M^p a finite type Λ-module and $M^p = 0$ for $p \not\in [a, b]$. If $b = a$, then this is clear. In general, we may replace X by the members of a covering and assume that $H^b(K)$ is constant, say $H^b(K) = \underline{M}$. By Modules on Sites, Lemma 41.5 the module M is a finite Λ-module. Choose a surjection $\Lambda^{\oplus r} \to M$ given by generators $x_1, \ldots, x_r$ of M.

By a slight generalization of Lemma 8.3 (details omitted) there exists a covering $\{U_i \to X\}$ such that $x_i \in H^0(X, H^b(K))$ lifts to an element of $H^b(U_i, K)$. Thus, after replacing X by the U_i we reach the situation where there is a map $\underline{\Lambda^{\oplus r}}[-b] \to K$ inducing a surjection on cohomology sheaves in degree b. Choose a distinguished triangle

$$\underline{\Lambda^{\oplus r}}[-b] \to K \to L \to \underline{\Lambda^{\oplus r}}[-b+1]$$

Now the cohomology sheaves of L are nonzero only in the interval $[a, b-1]$, agree with the cohomology sheaves of K in the interval $[a, b-2]$ and there is a short exact sequence

$$0 \to H^{b-1}(K) \to H^{b-1}(L) \to \underline{\mathrm{Ker}(\Lambda^{\oplus r} \to M)} \to 0$$

in degree $b-1$. By Modules on Sites, Lemma 42.5 we see that $H^{b-1}(L)$ is locally constant of finite type. By induction hypothesis we obtain an isomorphism $\underline{M^\bullet} \to L$ in $D(\mathcal{C}, \underline{\Lambda})$ with M^p a finite Λ-module and $M^p = 0$ for $p \not\in [a, b-1]$. The map $L \to \Lambda^{\oplus r}[-b+1]$ gives a map $\underline{M^{b-1}} \to \underline{\Lambda^{\oplus r}}$ which locally is constant (Modules on Sites, Lemma 42.3). Thus we may assume it is given by a map $M^{b-1} \to \Lambda^{\oplus r}$. The distinguished triangle shows that the composition $M^{b-2} \to M^{b-1} \to \Lambda^{\oplus r}$ is zero and the axioms of triangulated categories produce an isomorphism

$$\underline{M^a \to \ldots \to M^{b-1} \to \Lambda^{\oplus r}} \longrightarrow K$$

in $D(\mathcal{C}, \Lambda)$. $\square$

Let $\mathcal{C}$ be a site. Let Λ be a ring. Using the morphism $Sh(\mathcal{C}) \to Sh(pt)$ we see that there is a functor $D(\Lambda) \to D(\mathcal{C}, \Lambda)$, $K \mapsto \underline{K}$.

Lemma 40.2. *Let $\mathcal{C}$ be a site with final object X. Let Λ be a ring. Let*

(1) *K a perfect object of $D(\Lambda)$,*

(2) *a finite complex* $K^\bullet$ *of finite projective* Λ*-modules representing* K*,*
(3) $\mathcal{L}^\bullet$ *a complex of sheaves of* Λ*-modules, and*
(4) $\varphi : \underline{K} \to \mathcal{L}^\bullet$ *a map in* $D(\mathcal{C}, \Lambda)$*.*

Then there exists a covering $\{U_i \to X\}$ *and maps of complexes* $\alpha_i : \underline{K}^\bullet|_{U_i} \to \mathcal{L}^\bullet|_{U_i}$ *representing* $\varphi|_{U_i}$*.*

Proof. Follows immediately from Lemma 33.8. $\square$

Lemma 40.3. *Let* $\mathcal{C}$ *be a site with final object* X*. Let* Λ *be a ring. Let* K, L *be objects of* $D(\Lambda)$ *with* K *perfect. Let* $\varphi : \underline{K} \to \underline{L}$ *be map in* $D(\mathcal{C}, \Lambda)$*. There exists a covering* $\{U_i \to X\}$ *such that* $\varphi|_{U_i}$ *is equal to* $\underline{\alpha_i}$ *for some map* $\alpha_i : K \to L$ *in* $D(\Lambda)$*.*

Proof. Follows from Lemma 40.2 and Modules on Sites, Lemma 42.3. $\square$

Lemma 40.4. *Let* $\mathcal{C}$ *be a site. Let* Λ *be a Noetherian ring. Let* $K, L \in D^-(\mathcal{C}, \Lambda)$*. If the cohomology sheaves of* K *and* L *are locally constant sheaves of* Λ*-modules of finite type, then the cohomology sheaves of* $K \otimes_\Lambda^{\mathbf{L}} L$ *are locally constant sheaves of* Λ*-modules of finite type.*

Proof. We'll prove this as an application of Lemma 40.1. Note that $H^i(K \otimes_\Lambda^{\mathbf{L}} L)$ is the same as $H^i(\tau_{\geq i-1}K \otimes_\Lambda^{\mathbf{L}} \tau_{\geq i-1}L)$. Thus we may assume K and L are bounded. By Lemma 40.1 we may assume that K and L are represented by complexes of locally constant sheaves of Λ-modules of finite type. Then we can replace these complexes by bounded above complexes of finite free Λ-modules. In this case the result is clear. $\square$

Lemma 40.5. *Let* $\mathcal{C}$ *be a site. Let* Λ *be a Noetherian ring. Let* $I \subset \Lambda$ *be an ideal. Let* $K \in D^-(\mathcal{C}, \Lambda)$*. If the cohomology sheaves of* $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ *are locally constant sheaves of* Λ/I*-modules of finite type, then the cohomology sheaves of* $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$ *are locally constant sheaves of* Λ/I^n*-modules of finite type for all* $n \geq 1$*.*

Proof. Recall that the locally constant sheaves of Λ-modules of finite type form a weak Serre subcategory of all $\underline{\Lambda}$-modules, see Modules on Sites, Lemma 42.5. Thus the subcategory of $D(\mathcal{C}, \Lambda)$ consisting of complexes whose cohomology sheaves are locally constant sheaves of Λ-modules of finite type forms a strictly full, saturated triangulated subcategory of $D(\mathcal{C}, \Lambda)$, see Derived Categories, Lemma 13.1. Next, consider the distinguished triangles

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} \to K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^{n+1}} \to K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n} \to K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}}[1]$$

and the isomorphisms

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} = \left(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}\right) \otimes_{\Lambda/I}^{\mathbf{L}} \underline{I^n/I^{n+1}}$$

Combined with Lemma 40.4 we obtain the result. $\square$

41. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields

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