ADEQUATE MODULES

Contents

1. Introduction

For any scheme X the category $QCoh(\mathcal{O}_X)$ of quasi-coherent modules is abelian and a weak Serre subcategory of the abelian category of all $\mathcal{O}_X$-modules. The same thing works for the category of quasi-coherent modules on an algebraic space X viewed as a subcategory of the category of all $\mathcal{O}_X$-modules on the small étale site of X. Moreover, for a quasi-compact and quasi-separated morphism $f : X \to Y$ the pushforward f_* and higher direct images preserve quasi-coherency.

Next, let X be a scheme and let $\mathcal{O}$ be the structure sheaf on one of the big sites of X, say, the big fppf site. The category of quasi-coherent $\mathcal{O}$-modules is abelian (in fact it is equivalent to the category of usual quasi-coherent $\mathcal{O}_X$-modules on the scheme X we mentioned above) but its imbedding into $Mod(\mathcal{O})$ is not exact. An example is the map of quasi-coherent modules

$$\mathcal{O}_{\mathbf{A}^1_k} \longrightarrow \mathcal{O}_{\mathbf{A}^1_k}$$

on $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$ given by multiplication by x. In the abelian category of quasi-coherent sheaves this map is injective, whereas in the abelian category of all $\mathcal{O}$-modules on the big site of $\mathbf{A}^1_k$ this map has a nontrivial kernel as we see by evaluating on sections over $\mathrm{Spec}(k[x]/(x)) = \mathrm{Spec}(k)$. Moreover, for a quasi-compact and quasi-separated morphism $f : X \to Y$ the functor $f_{big,*}$ does not preserve quasi-coherency.

In this chapter we introduce a larger category of modules, closely related to quasi-coherent modules, which "fixes" the two problems mentioned above.

---

2. Conventions

In this chapter we fix $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ and we fix a big τ-site Sch_τ as in Topologies, Section 2. All schemes will be objects of Sch_τ. In particular, given a scheme S we obtain sites $(Aff/S)_\tau \subset (Sch/S)_\tau$. The structure sheaf $\mathcal{O}$ on these sites is defined by the rule $\mathcal{O}(T) = \Gamma(T, \mathcal{O}_T)$.

All rings A will be such that $\mathrm{Spec}(A)$ is (isomorphic to) an object of Sch_τ. Given a ring A we denote Alg_A the category of A-algebras whose objects are the A-algebras B of the form $B = \Gamma(U, \mathcal{O}_U)$ where S is an affine object of Sch_τ. Thus given an affine scheme $S = \mathrm{Spec}(A)$ the functor

$$(Aff/S)_\tau \longrightarrow Alg_A, \quad U \longmapsto \mathcal{O}(U)$$

is an equivalence.

3. Adequate functors

In this section we discuss a topic closely related to direct images of quasi-coherent sheaves. Most of this material was taken from the paper [Jaf97].

Definition 3.1. Let A be a ring. A *module-valued functor* is a functor $F : Alg_A \rightarrow Ab$ such that

   (1) for every object B of Alg_A the group $F(B)$ is endowed with the structure of a B-module, and
   (2) for any morphism $B \rightarrow B'$ of Alg_A the map $F(B) \rightarrow F(B')$ is B-linear.

A *morphism of module-valued functors* is a transformation of functors $\varphi : F \rightarrow G$ such that $F(B) \rightarrow G(B)$ is B-linear for all $B \in \mathrm{Ob}(Alg_A)$.

Let $S = \mathrm{Spec}(A)$ be an affine scheme. The category of module-valued functors on Alg_A is equivalent to the category $PMod((Aff/S)_\tau, \mathcal{O})$ of presheaves of $\mathcal{O}$-modules. The equivalence is given by the rule which assigns to the module-valued functor F the presheaf $\mathcal{F}$ defined by the rule $\mathcal{F}(U) = F(\mathcal{O}(U))$. This is clear from the equivalence $(Aff/S)_\tau \rightarrow Alg_A$, $U \mapsto \mathcal{O}(U)$ given in Section 2. The quasi-inverse sets $F(B) = \mathcal{F}(\mathrm{Spec}(B))$.

An important special case of a module-valued functor comes about as follows. Let M be an A-module. Then we will denote $\underline{M}$ the module-valued functor $B \mapsto M \otimes_A B$ (with obvious B-module structure). Note that if $M \rightarrow N$ is a map of A-modules then there is an associated morphism $\underline{M} \rightarrow \underline{N}$ of module-valued functors. Conversely, any morphism of module-valued functors $\underline{M} \rightarrow \underline{N}$ comes from an A-module map $M \rightarrow N$ as the reader can see by evaluating on $B = A$. In other words Mod_A is a full subcategory of the category of module-valued functors on Alg_A.

Given and A-module map $\varphi : M \rightarrow N$ then $\mathrm{Coker}(\underline{M} \rightarrow \underline{N}) = \underline{Q}$ where $Q = \mathrm{Coker}(M \rightarrow N)$ because $\otimes$ is right exact. But this isn't the case for the kernel in general: for example an injective map of A-modules need not be injective after base change. Thus the following definition makes sense.

Definition 3.2. Let A be a ring. A module-valued functor F on Alg_A is called

   (1) *adequate* if there exists a map of A-modules $M \rightarrow N$ such that F is isomorphic to $\mathrm{Ker}(\underline{M} \rightarrow \underline{N})$.
   (2) *linearly adequate* if F is isomorphic to the kernel of a map $\underline{A^{\oplus n}} \rightarrow \underline{A^{\oplus m}}$.

Note that F is adequate if and only if there exists an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ and F is linearly adequate if and only if there exists an exact sequence $0 \to F \to \underline{A}^{\oplus n} \to \underline{A}^{\oplus m}$.

Let A be a ring. In this section we will show the category of adequate functors on Alg_A is abelian (Lemmas 3.10 and 3.11) and has a set of generators (Lemma 3.6). We will also see that it is a weak Serre subcategory of the category of all module-valued functors on Alg_A (Lemma 3.16) and that it has arbitrary colimits (Lemma 3.12).

Lemma 3.3. *Let A be a ring. Let F be an adequate functor on Alg_A. If $B = \text{colim } B_i$ is a filtered colimit of A-algebras, then $F(B) = \text{colim } F(B_i)$.*

Proof. This holds because for any A-module M we have $M \otimes_A B = \text{colim } M \otimes_A B_i$ (see Algebra, Lemma 11.9) and because filtered colimits commute with exact sequences, see Algebra, Lemma 8.9. $\qquad\square$

Remark 3.4. Consider the category $Alg_{fp,A}$ whose objects are A-algebras B of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and whose morphisms are A-algebra maps. Every A-algebra B is a filtered colimit of finitely presented A-algebra, i.e., a filtered colimit of objects of $Alg_{fp,A}$. By Lemma 3.3 we conclude every adequate functor F is determined by its restriction to $Alg_{fp,A}$. For some questions we can therefore restrict to functors on $Alg_{fp,A}$. For example, the category of adequate functors does not depend on the choice of the big τ-site chosen in Section 2.

Lemma 3.5. *Let A be a ring. Let F be an adequate functor on Alg_A. If $B \to B'$ is flat, then $F(B) \otimes_B B' \to F(B')$ is an isomorphism.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. This gives the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & F(B) \otimes_B B' & \longrightarrow & (M \otimes_A B) \otimes_B B' & \longrightarrow & (N \otimes_A B) \otimes_B B' \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(B') & \longrightarrow & M \otimes_A B' & \longrightarrow & N \otimes_A B'
\end{array}
$$

where the rows are exact (the top one because $B \to B'$ is flat). Since the right two vertical arrows are isomorphisms, so is the left one. $\qquad\square$

Lemma 3.6. *Let A be a ring. Let F be an adequate functor on Alg_A. Then there exists a surjection $L \to F$ with L a direct sum of linearly adequate functors.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ where $\underline{M} \to \underline{N}$ is given by $\varphi : M \to N$. By Lemma 3.3 it suffices to construct $L \to F$ such that $L(B) \to F(B)$ is surjective for every finitely presented A-algebra B. Hence it suffices to construct, given a finitely presented A-algebra B and an element $\xi \in F(B)$ a map $L \to F$ with L linearly adequate such that ξ is in the image of $L(B) \to F(B)$. (Because there is a set worth of such pairs (B, ξ) up to isomorphism.)

To do this write $\sum_{i=1,\ldots,n} m_i \otimes b_i$ the image of ξ in $\underline{M}(B) = M \otimes_A B$. We know that $\sum \varphi(m_i) \otimes b_i = 0$ in $N \otimes_A B$. As N is a filtered colimit of finitely presented A-modules, we can find a finitely presented A-module N', a commutative diagram

of A-modules

$$
\begin{array}{ccc}
A^{\oplus n} & \longrightarrow & N' \\
\Big\downarrow{\scriptstyle m_1,\ldots,m_n} & & \Big\downarrow \\
M & \longrightarrow & N
\end{array}
$$

such that $(b_1,\ldots,b_n)$ maps to zero in $N' \otimes_A B$. Choose a presentation $A^{\oplus l} \to A^{\oplus k} \to N' \to 0$. Choose a lift $A^{\oplus n} \to A^{\oplus k}$ of the map $A^{\oplus n} \to N'$ of the diagram. Then we see that there exist $(c_1,\ldots,c_l) \in B^{\oplus l}$ such that $(b_1,\ldots,b_n,c_1,\ldots,c_l)$ maps to zero in $B^{\oplus k}$ under the map $B^{\oplus n} \oplus B^{\oplus l} \to B^{\oplus k}$. Consider the commutative diagram

$$
\begin{array}{ccc}
A^{\oplus n} \oplus A^{\oplus l} & \longrightarrow & A^{\oplus k} \\
\Big\downarrow & & \Big\downarrow \\
M & \longrightarrow & N
\end{array}
$$

where the left vertical arrow is zero on the summand $A^{\oplus l}$. Then we see that L equal to the kernel of $\underline{A^{\oplus n+l}} \to \underline{A^{\oplus k}}$ works because the element $(b_1,\ldots,b_n,c_1,\ldots,c_l) \in L(B)$ maps to ξ. $\qquad\square$

Consider a graded A-algebra $B = \bigoplus_{d \geq 0} B_d$. Then there are two A-algebra maps $p, a : B \to B[t, t^{-1}]$, namely $p : b \mapsto b$ and $a : b \mapsto t^{\deg(b)} b$ where b is homogeneous. If F is a module-valued functor on Alg_A, then we define

$$
(3.6.1) \qquad F(B)^{(k)} = \{ \xi \in F(B) \mid t^k F(p)(\xi) = F(a)(\xi) \}.
$$

For functors which behave well with respect to flat ring extensions this gives a direct sum decomposition. This amounts to the fact that representations of $\mathbf{G}_m$ are completely reducible.

Lemma 3.7. *Let A be a ring. Let F be a module-valued functor on Alg_A. Assume that for $B \to B'$ flat the map $F(B) \otimes_B B' \to F(B')$ is an isomorphism. Let B be a graded A-algebra. Then*

(1) *$F(B) = \bigoplus_{k \in \mathbf{Z}} F(B)^{(k)}$, and*
(2) *the map $B \to B_0 \to B$ induces map $F(B) \to F(B)$ whose image is contained in $F(B)^{(0)}$.*

Proof. Let $x \in F(B)$. The map $p : B \to B[t, t^{-1}]$ is free hence we know that

$$
F(B[t, t^{-1}]) = \bigoplus_{k \in \mathbf{Z}} F(p)(F(B)) \cdot t^k = \bigoplus_{k \in \mathbf{Z}} F(B) \cdot t^k
$$

as indicated we drop the $F(p)$ in the rest of the proof. Write $F(a)(x) = \sum t^k x_k$ for some $x_k \in F(B)$. Denote $\epsilon : B[t, t^{-1}] \to B$ the B-algebra map $t \mapsto 1$. Note that the compositions $\epsilon \circ p, \epsilon \circ a : B \to B[t, t^{-1}] \to B$ are the identity. Hence we see that

$$
x = F(\epsilon)(F(a)(x)) = F(\epsilon)(\sum t^k x_k) = \sum x_k.
$$

On the other hand, we claim that $x_k \in F(B)^{(k)}$. Namely, consider the commutative diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\;\;a\;\;} & B[t, t^{-1}] \\
\Big\downarrow{\scriptstyle a'} & & \Big\downarrow{\scriptstyle f} \\
B[s, s^{-1}] & \xrightarrow{\;\;g\;\;} & B[t, s, t^{-1}, s^{-1}]
\end{array}
$$

where $a'(b) = s^{\deg(b)}b$, $f(b) = b$, $f(t) = st$ and $g(b) = t^{\deg(b)}b$ and $g(s) = s$. Then

$$F(g)(F(a'))(x) = F(g)(\sum s^k x_k) = \sum s^k F(a)(x_k)$$

and going the other way we see

$$F(f)(F(a))(x) = F(f)(\sum t^k x_k) = \sum (st)^k x_k.$$

Since $B \to B[s, t, s^{-1}, t^{-1}]$ is free we see that $F(B[t, s, t^{-1}, s^{-1}]) = \bigoplus_{k,l \in \mathbf{Z}} F(B) \cdot t^k s^l$ and comparing coefficients in the expressions above we find $F(a)(x_k) = t^k x_k$ as desired.

Finally, the image of $F(B_0) \to F(B)$ is contained in $F(B)^{(0)}$ because $B_0 \to B \xrightarrow{a} B[t, t^{-1}]$ is equal to $B_0 \to B \xrightarrow{p} B[t, t^{-1}]$. $\square$

As a particular case of Lemma 3.7 note that

$$\underline{M}(B)^{(k)} = M \otimes_A B_k$$

where B_k is the degree k part of the graded A-algebra B.

Lemma 3.8. *Let A be a ring. Given a solid diagram*

of module-valued functors on Alg_A with exact row there exists a dotted arrow making the diagram commute.

Proof. Suppose that the map $A^{\oplus n} \to A^{\oplus m}$ is given by the $m \times n$-matrix (a_{ij}). Consider the ring $B = A[x_1, \ldots, x_n]/(\sum a_{ij}x_j)$. The element $(x_1, \ldots, x_n) \in \underline{A}^{\oplus n}(B)$ maps to zero in $\underline{A}^{\oplus m}(B)$ hence is the image of a unique element $\xi \in L(B)$. Note that ξ has the following universal property: for any A-algebra C and any $\xi' \in L(C)$ there exists an A-algebra map $B \to C$ such that ξ maps to ξ' via the map $L(B) \to L(C)$.

Note that B is a graded A-algebra, hence we can use Lemmas 3.7 and 3.5 to decompose the values of our functors on B into graded pieces. Note that $\xi \in L(B)^{(1)}$ as $(x_1, \ldots, x_n)$ is an element of degree one in $\underline{A}^{\oplus n}(B)$. Hence we see that $\varphi(\xi) \in \underline{M}(B)^{(1)} = M \otimes_A B_1$. Since B_1 is generated by $x_1, \ldots, x_n$ as an A-module we can write $\varphi(\xi) = \sum m_i \otimes x_i$. Consider the map $A^{\oplus n} \to M$ which maps the ith basis vector to m_i. By construction the associated map $\underline{A}^{\oplus n} \to \underline{M}$ maps the element ξ to $\varphi(\xi)$. It follows from the universal property mentioned above that the diagram commutes. $\square$

Lemma 3.9. *Let A be a ring. Let $\varphi : F \to \underline{M}$ be a map of module-valued functors on Alg_A with F adequate. Then $\mathrm{Coker}(\varphi)$ is adequate.*

Proof. By Lemma 3.6 we may assume that $F = \bigoplus L_i$ is a direct sum of linearly adequate functors. Choose exact sequences $0 \to L_i \to \underline{A}^{\oplus n_i} \to \underline{A}^{\oplus m_i}$. For each i choose a map $A^{\oplus n_i} \to M$ as in Lemma 3.8. Consider the diagram

Consider the A-modules

$$Q = \mathrm{Coker}(\bigoplus A^{\oplus n_i} \to M \oplus \bigoplus A^{\oplus m_i}) \quad \text{and} \quad P = \mathrm{Coker}(\bigoplus A^{\oplus n_i} \to \bigoplus A^{\oplus m_i}).$$

Then we see that $\mathrm{Coker}(\varphi)$ is isomorphic to the kernel of $\underline{Q} \to \underline{P}$. $\square$

Lemma 3.10. *Let A be a ring. Let $\varphi : F \to G$ be a map of adequate functors on Alg_A. Then $Coker(\varphi)$ is adequate.*

Proof. Choose an injection $G \to \underline{M}$. Then we have an injection $G/F \to \underline{M}/F$. By Lemma 3.9 we see that $\underline{M}/F$ is adequate, hence we can find an injection $\underline{M}/F \to \underline{N}$. Composing we obtain an injection $G/F \to \underline{N}$. By Lemma 3.9 the cokernel of the induced map $G \to \underline{N}$ is adequate hence we can find an injection $\underline{N}/G \to \underline{K}$. Then $0 \to G/F \to \underline{N} \to \underline{K}$ is exact and we win. $\square$

Lemma 3.11. *Let A be a ring. Let $\varphi : F \to G$ be a map of adequate functors on Alg_A. Then $Ker(\varphi)$ is adequate.*

Proof. Choose an injection $F \to \underline{M}$ and an injection $G \to \underline{N}$. Denote $F \to \underline{M \oplus N}$ the diagonal map so that

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ \underline{M \oplus N} & \longrightarrow & \underline{N} \end{array}$$

commutes. By Lemma 3.10 we can find a module map $M \oplus N \to K$ such that F is the kernel of $\underline{M \oplus N} \to \underline{K}$. Then $\mathrm{Ker}(\varphi)$ is the kernel of $\underline{M \oplus N} \to \underline{K \oplus N}$. $\square$

Lemma 3.12. *Let A be a ring. An arbitrary direct sum of adequate functors on Alg_A is adequate. A colimit of adequate functors is adequate.*

Proof. The statement on direct sums is immediate. A general colimit can be written as a kernel of a map between direct sums, see Categories, Lemma 14.11. Hence this follows from Lemma 3.11. $\square$

Lemma 3.13. *Let A be a ring. Let F, G be module-valued functors on Alg_A. Let $\varphi : F \to G$ be a transformation of functors. Assume*

(1) *φ is additive,*
(2) *for every A-algebra B and $\xi \in F(B)$ and unit $u \in B^*$ we have $\varphi(u\xi) = u\varphi(\xi)$ in $G(B)$, and*
(3) *for any flat ring map $B \to B'$ we have $G(B) \otimes_B B' = G(B')$.*

Then φ is a morphism of module-valued functors.

Proof. Let B be an A-algebra, $\xi \in F(B)$, and $b \in B$. We have to show that $\varphi(b\xi) = b\varphi(\xi)$. Consider the ring map

$$B \to B' = B[x, y, x^{-1}, y^{-1}]/(x + y - b).$$

This ring map is faithfully flat, hence $G(B) \subset G(B')$. On the other hand

$$\varphi(b\xi) = \varphi((x + y)\xi) = \varphi(x\xi) + \varphi(y\xi) = x\varphi(\xi) + y\varphi(\xi) = (x + y)\varphi(\xi) = b\varphi(\xi)$$

because x, y are units in B'. Hence we win. $\square$

Lemma 3.14. *Let A be a ring. Let $0 \to \underline{M} \to G \to L \to 0$ be a short exact sequence of module-valued functors on Alg_A with L linearly adequate. Then G is adequate.*

Proof. We first point out that for any flat A-algebra map $B \to B'$ the map $G(B) \otimes_B B' \to G(B')$ is an isomorphism. Namely, this holds for $\underline{M}$ and L, see Lemma 3.5 and hence follows for G by the five lemma. In particular, by Lemma 3.7 we see that $G(B) = \bigoplus_{k \in \mathbf{Z}} G(B)^{(k)}$ for any graded A-algebra B.

Choose an exact sequence $0 \to L \to \underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$. Suppose that the map $A^{\oplus n} \to A^{\oplus m}$ is given by the $m \times n$-matrix (a_{ij}). Consider the graded A-algebra $B = A[x_1, \ldots, x_n]/(\sum a_{ij} x_j)$. The element $(x_1, \ldots, x_n) \in \underline{A^{\oplus n}}(B)$ maps to zero in $\underline{A^{\oplus m}}(B)$ hence is the image of a unique element $\xi \in L(B)$. Observe that $\xi \in L(B)^{(1)}$. The map

$$\mathrm{Hom}_A(B, C) \longrightarrow L(C), \quad f \longmapsto L(f)(\xi)$$

defines an isomorphism of functors. The reason is that f is determined by the images $c_i = f(x_i) \in C$ which have to satisfy the relations $\sum a_{ij} c_j = 0$. And $L(C)$ is the set of n-tuples $(c_1, \ldots, c_n)$ satisfying the relations $\sum a_{ij} c_j = 0$.

Since the value of each of the functors $\underline{M}, G, L$ on B is a direct sum of its weight spaces (by the lemma mentioned above) exactness of $0 \to \underline{M} \to G \to L \to 0$ implies the sequence $0 \to \underline{M}(B)^{(1)} \to G(B)^{(1)} \to L(B)^{(1)} \to 0$ is exact. Thus we may choose an element $\theta \in G(B)^{(1)}$ mapping to ξ.

Consider the graded A-algebra

$$C = A[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum a_{ij} x_j, \sum a_{ij} y_j)$$

There are three graded A-algebra homomorphisms $p_1, p_2, m : B \to C$ defined by the rules

$$p_1(x_i) = x_i, \quad p_1(x_i) = y_i, \quad m(x_i) = x_i + y_i.$$

We will show that the element

$$\tau = G(m)(\theta) - G(p_1)(\theta) - G(p_2)(\theta) \in G(C)$$

is zero. First, τ maps to zero in $L(C)$ by a direct calculation. Hence τ is an element of $\underline{M}(C)$. Moreover, since m, p_1, p_2 are graded algebra maps we see that $\tau \in G(C)^{(1)}$ and since $\underline{M} \subset G$ we conclude

$$\tau \in \underline{M}(C)^{(1)} = M \otimes_A C_1.$$

We may write uniquely $\tau = \underline{M}(p_1)(\tau_1) + \underline{M}(p_2)(\tau_2)$ with $\tau_i \in M \otimes_A B_1 = \underline{M}(B)^{(1)}$ because $C_1 = p_1(B_1) \oplus p_2(B_1)$. Consider the ring map $q_1 : C \to B$ defined by $x_i \mapsto x_i$ and $y_i \mapsto 0$. Then $\underline{M}(q_1)(\tau) = \underline{M}(q_1)(\underline{M}(p_1)(\tau_1) + \underline{M}(p_2)(\tau_2)) = \tau_1$. On the other hand, because $q_1 \circ m = q_1 \circ p_1$ we see that $G(q_1)(\tau) = -G(q_1 \circ p_2)(\tau)$. Since $q_1 \circ p_2$ factors as $B \to A \to B$ we see that $G(q_1 \circ p_2)(\tau)$ is in $G(B)^{(0)}$, see Lemma 3.7. Hence $\tau_1 = 0$ because it is in $G(B)^{(0)} \cap \underline{M}(B)^{(1)} \subset G(B)^{(0)} \cap G(B)^{(1)} = 0$. Similarly $\tau_2 = 0$, whence $\tau = 0$.

Since $\theta \in G(B)$ we obtain a transformation of functors

$$\psi : L(-) = \mathrm{Hom}_A(B, -) \longrightarrow G(-)$$

by mapping $f : B \to C$ to $G(f)(\theta)$. Since θ is a lift of ξ the map ψ is a right inverse of $G \to L$. In terms of ψ the statements proved above have the following meaning: $\tau = 0$ means that ψ is additive and $\theta \in G(B)^{(1)}$ implies that for any A-algebra D we have $\psi(ul) = u\psi(l)$ in $G(D)$ for $l \in L(D)$ and $u \in D^*$ a unit. This implies that ψ is a morphism of module-valued functors, see Lemma 3.13. Clearly this implies that $G \cong \underline{M} \oplus L$ and we win. $\square$

Remark 3.15. Let A be a ring. The proof of Lemma 3.14 shows that any extension $0 \to \underline{M} \to E \to L \to 0$ of module-valued functors on Alg_A with L linearly adequate splits. It uses only the following properties of the module-valued functor $F = \underline{M}$:

(1) $F(B) \otimes_B B' \to F(B')$ is an isomorphism for a flat ring map $B \to B'$, and
(2) $F(C)^{(1)} = F(p_1)(F(B)^{(1)}) \oplus F(p_2)(F(B)^{(1)})$ where $B = A[x_1, \ldots, x_n]/(\sum a_{ij}x_j)$ and $C = A[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum a_{ij}x_j, \sum a_{ij}y_j)$.

These two properties hold for any adequate functor F; details omitted. Hence we see that L is a projective object of the abelian category of adequate functors.

Lemma 3.16. *Let A be a ring. Let $0 \to F \to G \to H \to 0$ be a short exact sequence of module-valued functors on Alg_A. If F and H are adequate, so is G.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. If we can show that $(\underline{M} \oplus G)/F$ is adequate, then G is the kernel of the map of adequate functors $(\underline{M} \oplus G)/F \to \underline{N}$, hence adequate by Lemma 3.11. Thus we may assume $F = \underline{M}$.

We can choose a surjection $L \to H$ where L is a direct sum of linearly adequate functors, see Lemma 3.6. If we can show that the pullback $G \times_H L$ is adequate, then G is the cokernel of the map $\mathrm{Ker}(L \to H) \to G \times_H L$ hence adequate by Lemma 3.10. Thus we may assume that $H = \bigoplus L_i$ is a direct sum of linearly adequate functors. By Lemma 3.14 each of the pullbacks $G \times_H L_i$ is adequate. By Lemma 3.12 we see that $\bigoplus G \times_H L_i$ is adequate. Then G is the cokernel of

$$\bigoplus_{i \neq i'} F \longrightarrow \bigoplus G \times_H L_i$$

where ξ in the summand (i, i') maps to $(0, \ldots, 0, \xi, 0, \ldots, 0, -\xi, 0, \ldots, 0)$ with nonzero entries in the summands i and i'. Thus G is adequate by Lemma 3.10. $\square$

Lemma 3.17. *Let $A \to A'$ be a ring map. If F is an adequate functor on Alg_A, then its restriction F' to $Alg_{A'}$ is adequate too.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. Then $F'(B') = F(B') = \mathrm{Ker}(M \otimes_A B' \to N \otimes_A B')$. Since $M \otimes_A B' = M \otimes_A A' \otimes_{A'} B'$ and similarly for N we see that F' is the kernel of $\underline{M \otimes_A A'} \to \underline{N \otimes_A A'}$. $\square$

Lemma 3.18. *Let $A \to A'$ be a ring map. If F' is an adequate functor on $Alg_{A'}$, then the module-valued functor $F : B \mapsto F'(A' \otimes_A B)$ on Alg_A is adequate too.*

Proof. Choose an exact sequence $0 \to F' \to \underline{M'} \to \underline{N'}$. Then

$$\begin{aligned}
F(B) &= F'(A' \otimes_A B) \\
&= \mathrm{Ker}(M' \otimes_{A'} (A' \otimes_A B) \to N' \otimes_{A'} (A' \otimes_A B)) \\
&= \mathrm{Ker}(M' \otimes_A B \to N' \otimes_A B)
\end{aligned}$$

Thus F is the kernel of $\underline{M} \to \underline{N}$ where $M = M'$ and $N = N'$ viewed as A-modules. $\square$

Lemma 3.19. *Let $A = A_1 \times \ldots \times A_n$ be a product of rings. An adequate functor over A is the same thing as a sequence $F_1, \ldots, F_n$ of adequate functors F_i over A_i.*

Proof. This is true because an A-algebra B is canonically a product $B_1 \times \ldots \times B_n$ and the same thing holds for A-modules. Setting $F(B) = \coprod F_i(B_i)$ gives the correspondence. Details omitted. $\square$

Lemma 3.20. *Let $A \to A'$ be a ring map and let F be a module-valued functor on Alg_A such that*

(1) *the restriction F' of F to the category of A'-algebras is adequate, and*
(2) *for any A-algebra B the sequence*

$$0 \to F(B) \to F(B \otimes_A A') \to F(B \otimes_A A' \otimes_A A')$$

is exact.

Then F is adequate.

Proof. The functors $B \to F(B \otimes_A A')$ and $B \mapsto F(B \otimes_A A' \otimes_A A')$ are adequate, see Lemmas 3.18 and 3.17. Hence F as a kernel of a map of adequate functors is adequate, see Lemma 3.11. $\square$

4. Higher exts of adequate functors

Let A be a ring. In Lemma 3.16 we have seen that any extension of adequate functors in the category of module-valued functors on Alg_A is adequate. In this section we show that the same remains true for higher ext groups.

Lemma 4.1. *Let A be a ring. For every module-valued functor F on Alg_A there exists a morphism $Q(F) \to F$ of module-valued functors on Alg_A such that (1) $Q(F)$ is adequate and (2) for every adequate functor G the map $\mathrm{Hom}(G, Q(F)) \to \mathrm{Hom}(G, F)$ is a bijection.*

Proof. Choose a set $\{L_i\}_{i \in I}$ of linearly adequate functors such that every linearly adequate functor is isomorphic to one of the L_i. This is possible. Suppose that we can find $Q(F) \to F$ with (1) and (2)' or every $i \in I$ the map $\mathrm{Hom}(L_i, Q(F)) \to \mathrm{Hom}(L_i, F)$ is a bijection. Then (2) holds. Namely, combining Lemmas 3.6 and 3.11 we see that every adequate functor G sits in an exact sequence

$$K \to L \to G \to 0$$

with K and L direct sums of linearly adequate functors. Hence (2)' implies that $\mathrm{Hom}(L, Q(F)) \to \mathrm{Hom}(L, F)$ and $\mathrm{Hom}(K, Q(F)) \to \mathrm{Hom}(K, F)$ are bijections, whence the same thing for G.

Consider the category $\mathcal{I}$ whose objects are pairs (i, φ) where $i \in I$ and $\varphi : L_i \to F$ is a morphism. A morphism $(i, \varphi) \to (i', \varphi')$ is a map $\psi : L_i \to L_{i'}$ such that $\varphi' \circ \psi = \varphi$. Set

$$Q(F) = \mathrm{colim}_{(i,\varphi) \in \mathrm{Ob}(\mathcal{I})} L_i$$

There is a natural map $Q(F) \to F$, by Lemma 3.12 it is adequate, and by construction it has property (2)'. $\square$

Lemma 4.2. *Let A be a ring. Denote $\mathcal{P}$ the category of module-valued functors on Alg_A and $\mathcal{A}$ the category of adequate functors on Alg_A. Denote $i : \mathcal{A} \to \mathcal{P}$ the inclusion functor. Denote $Q : \mathcal{P} \to \mathcal{A}$ the construction of Lemma 4.1. Then*

(1) *i is fully faithful, exact, and its image is a weak Serre subcategory,*
(2) *$\mathcal{P}$ has enough injectives,*
(3) *the functor Q is a right adjoint to i hence left exact,*
(4) *Q transforms injectives into injectives,*
(5) *$\mathcal{A}$ has enough injectives.*

Proof. This lemma just collects some facts we have already seen so far. Part (1) is clear from the definitions, the characterization of weak Serre subcategories (see Homology, Lemma 9.3), and Lemmas 3.10, 3.11, and 3.16. Recall that $\mathcal{P}$ is equivalent to the category $PMod((\mathit{Aff}/\operatorname{Spec}(A))_\tau, \mathcal{O})$. Hence (2) by Injectives, Proposition 8.5. Part (3) follows from Lemma 4.1 and Categories, Lemma 24.4. Parts (4) and (5) follow from Homology, Lemmas 25.1 and 25.3. $\qquad\square$

Let A be a ring. As in Formal Deformation Theory, Section 10 given an A-algebra B and an B-module N we set $B[N]$ equal to the R-algebra with underlying B-module $B \oplus N$ with multiplication given by $(b, m)(b', m') = (bb', bm' + b'm)$. Note that this construction is functorial in the pair (B, N) where morphism $(B, N) \to (B', N')$ is given by an A-algebra map $B \to B'$ and an B-module map $N \to N'$. In some sense the functor TF of pairs defined in the following lemma is the tangent space of F. Below we will only consider pairs (B, N) such that $B[N]$ is an object of Alg_A.

Lemma 4.3. *Let A be a ring. Let F be a module valued functor. For every $B \in \operatorname{Ob}(Alg_A)$ and B-module N there is a canonical decomposition*

$$F(B[N]) = F(B) \oplus TF(B, N)$$

characterized by the following properties

(1) *$TF(B, N) = Ker(F(B[N]) \to F(B))$,*
(2) *there is a B-module structure $TF(B, N)$ compatible with $B[N]$-module structure on $F(B[N])$,*
(3) *TF is a functor from the category of pairs (B, N),*
(4) *there are canonical maps $N \otimes_B F(B) \to TF(B, N)$ inducing a transformation between functors defined on the category of pairs (B, N),*
(5) *$TF(B, 0) = 0$ and the map $TF(B, N) \to TF(B, N')$ is zero when $N \to N'$ is the zero map.*

Proof. Since $B \to B[N] \to B$ is the identity we see that $F(B) \to F(B[N])$ is a direct summand whose complement is $TF(N, B)$ as defined in (1). This construction is functorial in the pair (B, N) simply because given a morphism of pairs $(B, N) \to (B', N')$ we obtain a commutative diagram

$$\begin{array}{ccccc}
B' & \longrightarrow & B'[N'] & \longrightarrow & B' \\
\uparrow & & \uparrow & & \uparrow \\
B & \longrightarrow & B[N] & \longrightarrow & B
\end{array}$$

in Alg_A. The B-module structure comes from the $B[N]$-module structure and the ring map $B \to B[N]$. The map in (4) is the composition

$$N \otimes_B F(B) \longrightarrow B[N] \otimes_{B[N]} F(B[N]) \longrightarrow F(B[N])$$

whose image is contained in $TF(B, N)$. (The first arrow uses the inclusions $N \to B[N]$ and $F(B) \to F(B[N])$ and the second arrow is the multiplication map.) If $N = 0$, then $B = B[N]$ hence $TF(B, 0) = 0$. If $N \to N'$ is zero then it factors as $N \to 0 \to N'$ hence the induced map is zero since $TF(B, 0) = 0$. $\qquad\square$

Let A be a ring. Let M be an A-module. Then the module-valued functor $\underline{M}$ has tangent space $T\underline{M}$ given by the rule $T\underline{M}(B, N) = N \otimes_A M$. In particular, for B given, the functor $N \mapsto T\underline{M}(B, N)$ is additive and right exact. It turns out this also holds for injective module-valued functors.

Lemma 4.4. *Let A be a ring. Let I be an injective object of the category of module-valued functors. Then for any $B \in \text{Ob}(Alg_A)$ and short exact sequence $0 \to N_1 \to N \to N_2 \to 0$ of B-modules the sequence*

$$TI(B, N_1) \to TI(B, N) \to TI(B, N_2) \to 0$$

is exact.

Proof. We will use the results of Lemma 4.3 without further mention. Denote $h : Alg_A \to Sets$ the functor given by $h(C) = \text{Mor}_A(B[N], C)$. Similarly for h_1 and h_2. The map $B[N] \to B[N_2]$ corresponding to the surjection $N \to N_2$ is surjective. It corresponds to a map $h_2 \to h$ such that $h_2(C) \to h(C)$ is injective for all A-algebras C. On the other hand, there are two maps $p, q : h \to h_1$, corresponding to the zero map $N_1 \to N$ and the injection $N_1 \to N$. Note that

$$h_2 \longrightarrow h \rightrightarrows h_1$$

is an equalizer diagram. Denote $\mathcal{O}_h$ the module-valued functor $C \mapsto \bigoplus_{h(C)} C$. Similarly for $\mathcal{O}_{h_1}$ and $\mathcal{O}_{h_2}$. Note that

$$\text{Hom}_{\mathcal{P}}(\mathcal{O}_h, F) = F(B[N])$$

where $\mathcal{P}$ is the category of of module-valued functors on Alg_A. We claim there is an equalizer diagram

$$\mathcal{O}_{h_2} \longrightarrow \mathcal{O}_h \rightrightarrows \mathcal{O}_{h_1}$$

in $\mathcal{P}$. Namely, suppose that $C \in \text{Ob}(Alg_A)$ and $\xi = \sum_{i=1,\dots,n} c_i \cdot f_i$ where $c_i \in C$ and $f_i : B[N] \to C$ is an element of $\mathcal{O}_h(C)$. If $p(\xi) = q(\xi)$, then we see that

$$\sum c_i \cdot f_i \circ z = \sum c_i \cdot f_i \circ y$$

where $z, y : B[N_1] \to B[N]$ are the maps $z : (b, m_1) \mapsto (b, 0)$ and $y : (b, m_1) \mapsto (b, m_1)$. This means that for every i there exists a j such that $f_j \circ z = f_i \circ y$. Clearly, this implies that $f_i(N_1) = 0$, i.e., f_i factors through a unique map $\overline{f}_i : B[N_2] \to C$. Hence ξ is the image of $\overline{\xi} = \sum c_i \cdot \overline{f}_i$. Since I is injective, it transforms this equalizer diagram into a coequalizer diagram

$$I(B[N_1]) \rightrightarrows I(B[N]) \longrightarrow I(B[N_2])$$

This diagram is compatible with the direct sum decompositions $I(B[N]) = I(B) \oplus TI(B, N)$ and $I(B[N_i]) = I(B) \oplus TI(B, N_i)$. The zero map $N \to N_1$ induces the zero map $TI(B, N) \to TI(B, N_1)$. Thus we see that the coequalizer property above means we have an exact sequence $TI(B, N_1) \to TI(B, N) \to TI(B, N_2) \to 0$ as desired. $\square$

Lemma 4.5. *Let A be a ring. Let F be a module-valued functor such that for any $B \in \text{Ob}(Alg_A)$ the functor $TF(B, -)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Then*

(1) *$TF(B, N_1 \oplus N_2) = TF(B, N_1) \oplus TF(B, N_2)$,*
(2) *there is a second functorial B-module structure on $TF(B, N)$ defined by setting $x \cdot b = TF(B, b \cdot 1_N)(x)$ for $x \in TF(B, N)$ and $b \in B$,*
(3) *the canonical map $N \otimes_B F(B) \to TF(B, N)$ of Lemma 4.3 is B-linear also with respect to the second B-module structure,*

(4) *given a finitely presented B-module N there is a canonical isomorphism $TF(B,B) \otimes_B N \to TF(B,N)$ where the tensor product uses the second B-module structure on $TF(B,B)$.*

Proof. We will use the results of Lemma 4.3 without further mention. The maps $N_1 \to N_1 \oplus N_2$ and $N_2 \to N_1 \oplus N_2$ give a map $TF(B,N_1) \oplus TF(B,N_2) \to TF(B,N_1 \oplus N_2)$ which is injective since the maps $N_1 \oplus N_2 \to N_1$ and $N_1 \oplus N_2 \to N_2$ induce an inverse. Since TF is right exact we see that $TF(B,N_1) \to TF(B,N_1 \oplus N_2) \to TF(B,N_2) \to 0$ is exact. Hence $TF(B,N_1) \oplus TF(B,N_2) \to TF(B,N_1 \oplus N_2)$ is an isomorphism. This proves (1).

To see (2) the only thing we need to show is that $x \cdot (b_1 + b_2) = x \cdot b_1 + x \cdot b_2$. (Associativity and additivity are clear.) To see this consider

$$N \xrightarrow{(b_1,b_2)} N \oplus N \xrightarrow{+} N$$

and apply $TF(B,-)$.

Part (3) follows immediately from the fact that $N \otimes_B F(B) \to TF(B,N)$ is functorial in the pair (B,N).

Suppose N is a finitely presented B-module. Choose a presentation $B^{\oplus m} \to B^{\oplus n} \to N \to 0$. This gives an exact sequence

$$TF(B,B^{\oplus m}) \to TF(B,B^{\oplus n}) \to TF(B,N) \to 0$$

by right exactness of $TF(B,-)$. By part (1) we can write $TF(B,B^{\oplus m}) = TF(B,B)^{\oplus m}$ and $TF(B,B^{\oplus n}) = TF(B,B)^{\oplus n}$. Next, suppose that $B^{\oplus m} \to B^{\oplus n}$ is given by the matrix $T = (b_{ij})$. Then the induced map $TF(B,B)^{\oplus m} \to TF(B,B)^{\oplus n}$ is given by the matrix with entries $TF(B,b_{ij} \cdot 1_B)$. This combined with right exactness of $\otimes$ proves (4). $\qquad\square$

Example 4.6. Let F be a module-valued functor as in Lemma 4.5. It is not always the case that the two module structures on $TF(B,N)$ agree. Here is an example. Suppose $A = \mathbf{F}_p$ where p is a prime. Set $F(B) = B$ but with B-module structure given by $b \cdot x = b^p x$. Then $TF(B,N) = N$ with B-module structure given by $b \cdot x = b^p x$ for $x \in N$. However, the second B-module structure is given by $x \cdot b = bx$. Note that in this case the canonical map $N \otimes_B F(B) \to TF(B,N)$ is zero as raising an element $n \in B[N]$ to the pth power is zero.

In the following lemma we will frequently use the observation that if $0 \to F \to G \to H \to 0$ is an exact sequence of module-valued functors on Alg_A, then for any pair (B,N) the sequence $0 \to TF(B,N) \to TG(B,N) \to TH(B,N) \to 0$ is exact. This follows from the fact that $0 \to F(B[N]) \to G(B[N]) \to H(B[N]) \to 0$ is exact.

Lemma 4.7. *Let A be a ring. For F a module-valued functor on Alg_A say $(*)$ holds if for all $B \in \mathrm{Ob}(Alg_A)$ the functor $TF(B,-)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Let $0 \to F \to G \to H \to 0$ be a short exact sequence of module-valued functors on Alg_A.*

(1) *If $(*)$ holds for F,G then $(*)$ holds for H.*
(2) *If $(*)$ holds for F,H then $(*)$ holds for G.*
(3) *If $H' \to H$ is morphism of module-valued functors on Alg_A and $(*)$ holds for F, G, H, and H', then $(*)$ holds for $G \times_H H'$.*

Proof. Let B be given. Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of B-modules. Part (1) follows from a diagram chase in the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & TF(B, N_1) & \longrightarrow & TG(B, N_1) & \longrightarrow & TH(B, N_1) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_2) & \longrightarrow & TG(B, N_2) & \longrightarrow & TH(B, N_2) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_3) & \longrightarrow & TG(B, N_3) & \longrightarrow & TH(B, N_3) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

with exact horizontal rows and exact columns involving TF and TG. To prove part (2) we do a diagram chase in the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & TF(B, N_1) & \longrightarrow & TG(B, N_1) & \longrightarrow & TH(B, N_1) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_2) & \longrightarrow & TG(B, N_2) & \longrightarrow & TH(B, N_2) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_3) & \longrightarrow & TG(B, N_3) & \longrightarrow & TH(B, N_3) & \longrightarrow & 0 \\
& & \downarrow & & & & \downarrow & & \\
& & 0 & & & & 0 & &
\end{array}
$$

with exact horizontal rows and exact columns involving TF and TH. Part (3) follows from part (2) as $G \times_H H'$ sits in the exact sequence $0 \to F \to G \times_H H' \to H' \to 0$. $\square$

Most of the work in this section was done in order to prove the following key vanishing result.

Lemma 4.8. *Let A be a ring. Let M, P be A-modules with P of finite presentation. Then $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M}) = 0$ for $i > 0$ where $\mathcal{P}$ is the category of module-valued functors on Alg_A.*

Proof. Choose an injective resolution $\underline{M} \to I^\bullet$ in $\mathcal{P}$, see Lemma 4.2. By Derived Categories, Lemma 27.2 any element of $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ comes from a morphism $\varphi : \underline{P} \to I^i$ with $d^i \circ \varphi = 0$. We will prove that the Yoneda extension

$$E : 0 \to \underline{M} \to I^0 \to \ldots \to I^{i-1} \times_{\mathrm{Ker}(d^i)} \underline{P} \to \underline{P} \to 0$$

of $\underline{P}$ by $\underline{M}$ associated to φ is trivial, which will prove the lemma by Derived Categories, Lemma 27.5.

For F a module-valued functor on Alg_A say $(*)$ holds if for all $B \in \mathrm{Ob}(Alg_A)$ the functor $TF(B, -)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Recall that the module-valued functors $\underline{M}, I^n, \underline{P}$ each have property $(*)$, see Lemma 4.4 and the remarks preceding it. By splitting $0 \to \underline{M} \to I^\bullet$ into short exact sequences we find that each of the functors $\mathrm{Im}(d^{n-1}) =$

$\text{Ker}(d^n) \subset I^n$ has property $(*)$ by Lemma 4.7 and also that $I^{i-1} \times_{\text{Ker}(d^i)} \underline{P}$ has property $(*)$.

Thus we may assume the Yoneda extension is given as

$$E : 0 \to \underline{M} \to F_{i-1} \to \ldots \to F_0 \to \underline{P} \to 0$$

where each of the module-valued functors F_j has property $(*)$. Set $G_j(B) = TF_j(B,B)$ viewed as a B-module via the *second* B-module structure defined in Lemma 4.5. Since TF_j is a functor on pairs we see that G_j is a module-valued functor on Alg_A. Moreover, since E is an exact sequence the sequence $G_{j+1} \to G_j \to G_{j-1}$ is exact (see remark preceding Lemma 4.7). Observe that $T\underline{M}(B,B) = M \otimes_A B = \underline{M}(B)$ and that the two B-module structures agree on this. Thus we obtain a Yoneda extension

$$E' : 0 \to \underline{M} \to G_{i-1} \to \ldots \to G_0 \to \underline{P} \to 0$$

Moreover, the canonical maps

$$F_j(B) = B \otimes_B F_j(B) \longrightarrow TF_j(B,B) = G_j(B)$$

of Lemma 4.3 (4) are B-linear by Lemma 4.5 (3) and functorial in B. Hence a map

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \underline{M} & \longrightarrow & F_{i-1} & \longrightarrow & \ldots & \longrightarrow & F_0 & \longrightarrow & \underline{P} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle 1} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & \underline{M} & \longrightarrow & G_{i-1} & \longrightarrow & \ldots & \longrightarrow & G_0 & \longrightarrow & \underline{P} & \longrightarrow & 0
\end{array}
$$

of Yoneda extensions. In particular we see that E and E' have the same class in $\text{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ by the lemma on Yoneda Exts mentioned above. Finally, let N be a A-module of finite presentation. Then we see that

$$0 \to T\underline{M}(A,N) \to TF_{i-1}(A,N) \to \ldots \to TF_0(A,N) \to T\underline{P}(A,N) \to 0$$

is exact. By Lemma 4.5 (4) with $B = A$ this translates into the exactness of the sequence of A-modules

$$0 \to M \otimes_A N \to G_{i-1}(A) \otimes_A N \to \ldots \to G_0(A) \otimes_A N \to P \otimes_A N \to 0$$

Hence the sequence of A-modules $0 \to M \to G_{i-1}(A) \to \ldots \to G_0(A) \to P \to 0$ is universally exact, in the sense that it remains exact on tensoring with any finitely presented A-module N. Let $K = \text{Ker}(G_0(A) \to P)$ so that we have exact sequences

$$0 \to K \to G_0(A) \to P \to 0 \quad \text{and} \quad G_2(A) \to G_1(A) \to K \to 0$$

Tensoring the second sequence with N we obtain that $K \otimes_A N = \text{Coker}(G_2(A) \otimes_A N \to G_1(A) \otimes_A N)$. Exactness of $G_2(A) \otimes_A N \to G_1(A) \otimes_A N \to G_0(A) \otimes_A N$ then implies that $K \otimes_A N \to G_0(A) \otimes_A N$ is injective. By Algebra, Theorem 79.3 this means that the A-module extension $0 \to K \to G_0(A) \to P \to 0$ is exact, and because P is assumed of finite presentation this means the sequence is split, see Algebra, Lemma 79.4. Any splitting $P \to G_0(A)$ defines a map $\underline{P} \to G_0$ which splits the surjection $G_0 \to \underline{P}$. Thus the Yoneda extension E' is equivalent to the trivial Yoneda extension and we win. $\qquad\square$

Lemma 4.9. *Let A be a ring. Let M be an A-module. Let L be a linearly adequate functor on Alg_A. Then $Ext^i_{\mathcal{P}}(L, \underline{M}) = 0$ for $i > 0$ where $\mathcal{P}$ is the category of module-valued functors on Alg_A.*

Proof. Since L is linearly adequate there exists an exact sequence

$$0 \to L \to \underline{A}^{\oplus m} \to \underline{A}^{\oplus n} \to \underline{P} \to 0$$

Here $P = \mathrm{Coker}(A^{\oplus m} \to A^{\oplus n})$ is the cokernel of the map of finite free A-modules which is given by the definition of linearly adequate functors. By Lemma 4.8 we have the vanishing of $\mathrm{Ext}_{\mathcal{P}}^i(\underline{P}, \underline{M})$ and $\mathrm{Ext}_{\mathcal{P}}^i(\underline{A}, \underline{M})$ for $i > 0$. Let $K = \mathrm{Ker}(\underline{A}^{\oplus n} \to \underline{P})$. By the long exact sequence of Ext groups associated to the exact sequence $0 \to K \to \underline{A}^{\oplus n} \to \underline{P} \to 0$ we conclude that $\mathrm{Ext}_{\mathcal{P}}^i(K, \underline{M}) = 0$ for $i > 0$. Repeating with the sequence $0 \to L \to \underline{A}^{\oplus m} \to K \to 0$ we win. $\square$

Lemma 4.10. *With notation as in Lemma 4.2 we have $R^p Q(F) = 0$ for all $p > 0$ and any adequate functor F.*

Proof. Choose an exact sequence $0 \to F \to \underline{M}^0 \to \underline{M}^1$. Set $M^2 = \mathrm{Coker}(M^0 \to M^1)$ so that $0 \to F \to \underline{M}^0 \to \underline{M}^1 \to \underline{M}^2 \to 0$ is a resolution. By Derived Categories, Lemma 21.3 we obtain a spectral sequence

$$R^p Q(\underline{M}^q) \Rightarrow R^{p+q} Q(F)$$

Since $Q(\underline{M}^q) = \underline{M}^q$ it suffices to prove $R^p Q(\underline{M}) = 0$, $p > 0$ for any A-module M.

Choose an injective resolution $\underline{M} \to I^\bullet$ in the category $\mathcal{P}$. Suppose that $R^i Q(\underline{M})$ is nonzero. Then $\mathrm{Ker}(Q(I^i) \to Q(I^{i+1}))$ is strictly bigger than the image of $Q(I^{i-1}) \to Q(I^i)$. Hence by Lemma 3.6 there exists a linearly adequate functor L and a map $\varphi : L \to Q(I^i)$ mapping into the kernel of $Q(I^i) \to Q(I^{i+1})$ which does not factor through the image of $Q(I^{i-1}) \to Q(I^i)$. Because Q is a left adjoint to the inclusion functor the map φ corresponds to a map $\varphi' : L \to I^i$ with the same properties. Thus φ' gives a nonzero element of $\mathrm{Ext}_{\mathcal{P}}^i(L, \underline{M})$ contradicting Lemma 4.9. $\square$

5. Adequate modules

In Descent, Section 7 we have seen that quasi-coherent modules on a scheme S are the same as quasi-coherent modules on any of the big sites $(Sch/S)_\tau$ associated to S. We have seen that there are two issues with this identification:

(1) $QCoh(\mathcal{O}_S) \to Mod((Sch/S)_\tau, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ is not exact in general, and
(2) given a quasi-compact and quasi-separated morphism $f : X \to S$ the functor f_* does not preserve quasi-coherent sheaves on the big sites in general.

Part (1) means that we cannot define a triangulated subcategory of $D(\mathcal{O})$ consisting of complexes whose cohomology sheaves are quasi-coherent. Part (2) means that $Rf_*\mathcal{F}$ isn't a complex with quasi-coherent cohomology sheaves even when $\mathcal{F}$ is quasi-coherent and f is quasi-compact and quasi-separated. Moreover, the examples given in the proofs of Descent, Lemma 7.13 and Descent, Proposition 7.14 are not of a pathological nature.

In this section we discuss a slightly larger category of $\mathcal{O}$-modules on $(Sch/S)_\tau$ with contains the quasi-coherent modules, is abelian, and is preserved under f_* when f is quasi-compact and quasi-separated. To do this, suppose that S is a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$. For any affine object $U = \mathrm{Spec}(A)$ of $(Sch/S)_\tau$ we can restrict $\mathcal{F}$ to $(Aff/U)_\tau$ to get a presheaf of $\mathcal{O}$-modules on this site. The corresponding module-valued functor, see Section 3, will be denoted

$$F = F_{\mathcal{F},A} : Alg_A \longrightarrow Ab, \quad B \longmapsto \mathcal{F}(\mathrm{Spec}(B))$$

The assignment $\mathcal{F} \mapsto F_{\mathcal{F},A}$ is an exact functor of abelian categories.

Definition 5.1. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $(Sch/S)_\tau$ is *adequate* if there exists a τ-covering $\{\mathrm{Spec}(A_i) \to S\}_{i \in I}$ such that $F_{\mathcal{F},A_i}$ is adequate for all $i \in I$.

We will see below that the category of adequate $\mathcal{O}$-modules is independent of the chosen topology τ.

Lemma 5.2. *Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. For any affine scheme $\mathrm{Spec}(A)$ over S the functor $F_{\mathcal{F},A}$ is adequate.*

Proof. Let $\{\mathrm{Spec}(A_i) \to S\}_{i \in I}$ be a τ-covering such that $F_{\mathcal{F},A_i}$ is adequate for all $i \in I$. We can find a standard affine τ-covering $\{\mathrm{Spec}(A'_j) \to \mathrm{Spec}(A)\}_{j=1,\ldots,m}$ such that $\mathrm{Spec}(A'_j) \to \mathrm{Spec}(A) \to S$ factors through $\mathrm{Spec}(A_{i(j)})$ for some $i(j) \in I$. Then we see that $F_{\mathcal{F},A'_j}$ is the restriction of $F_{\mathcal{F},A_{i(j)}}$ to the category of A'_j-algebras. Hence $F_{\mathcal{F},A'_j}$ is adequate by Lemma 3.17. By Lemma 3.19 the sequence $F_{\mathcal{F},A'_j}$ corresponds to an adequate "product" functor F' over $A' = A'_1 \times \ldots \times A'_m$. As $\mathcal{F}$ is a sheaf (for the Zariski topology) this product functor F' is equal to $F_{\mathcal{F},A'}$, i.e., is the restriction of F to A'-algebras. Finally, $\{\mathrm{Spec}(A') \to \mathrm{Spec}(A)\}$ is a τ-covering. It follows from Lemma 3.20 that $F_{\mathcal{F},A}$ is adequate. $\square$

Lemma 5.3. *Let $S = \mathrm{Spec}(A)$ be an affine scheme. The category of adequate $\mathcal{O}$-modules on $(Sch/S)_\tau$ is equivalent to the category of adequate module-valued functors on Alg_A.*

Proof. Given an adequate module $\mathcal{F}$ the functor $F_{\mathcal{F},A}$ is adequate by Lemma 5.2. Given an adequate functor F we choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ and we consider the $\mathcal{O}$-module $\mathcal{F} = \mathrm{Ker}(M^a \to N^a)$ where M^a denotes the quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$ associated to the quasi-coherent sheaf $\widetilde{M}$ on S. Note that $F = F_{\mathcal{F},A}$, in particular the module $\mathcal{F}$ is adequate by definition. We omit the proof that the constructions define mutually inverse equivalences of categories. $\square$

Lemma 5.4. *Let $f : T \to S$ be a morphism of schemes. The pullback $f^*\mathcal{F}$ of an adequate $\mathcal{O}$-module $\mathcal{F}$ on $(Sch/S)_\tau$ is an adequate $\mathcal{O}$-module on $(Sch/T)_\tau$.*

Proof. The pullback map $f^* : Mod((Sch/S)_\tau, \mathcal{O}) \to Mod((Sch/T)_\tau, \mathcal{O})$ is given by restriction, i.e., $f^*\mathcal{F}(V) = \mathcal{F}(V)$ for any scheme V over T. Hence this lemma follows immediately from Lemma 5.2 and the definition. $\square$

Here is a characterization of the category of adequate $\mathcal{O}$-modules. To understand the significance, consider a map $\mathcal{G} \to \mathcal{H}$ of quasi-coherent $\mathcal{O}_S$-modules on a scheme S. The cokernel of the associated map $\mathcal{G}^a \to \mathcal{H}^a$ of $\mathcal{O}$-modules is quasi-coherent because it is equal to $(\mathcal{H}/\mathcal{G})^a$. But the kernel of $\mathcal{G}^a \to \mathcal{H}^a$ in general isn't quasi-coherent. However, it is adequate.

Lemma 5.5. *Let S be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $(Sch/S)_\tau$. The following are equivalent*

(1) *$\mathcal{F}$ is adequate,*
(2) *there exists an affine open covering $S = \bigcup S_i$ and maps of quasi-coherent $\mathcal{O}_{S_i}$-modules $\mathcal{G}_i \to \mathcal{H}_i$ such that $\mathcal{F}|_{(Sch/S_i)_\tau}$ is the kernel of $\mathcal{G}_i^a \to \mathcal{H}_i^a$*
(3) *there exists a τ-covering $\{S_i \to S\}_{i \in I}$ and maps of $\mathcal{O}_{S_i}$-quasi-coherent modules $\mathcal{G}_i \to \mathcal{H}_i$ such that $\mathcal{F}|_{(Sch/S_i)_\tau}$ is the kernel of $\mathcal{G}_i^a \to \mathcal{H}_i^a$,*
(4) *there exists a τ-covering $\{f_i : S_i \to S\}_{i \in I}$ such that each $f_i^*\mathcal{F}$ is adequate,*

(5) *for any affine scheme U over S the restriction $\mathcal{F}|_{(Sch/U)_\tau}$ is the kernel of a map $\mathcal{G}^a \to \mathcal{H}^a$ of quasi-coherent $\mathcal{O}_U$-modules.*

Proof. Let $U = \operatorname{Spec}(A)$ be an affine scheme over S. Set $F = F_{\mathcal{F},A}$. By definition, the functor F is adequate if and only if there exists a map of A-modules $M \to N$ such that $F = \operatorname{Ker}(\underline{M} \to \underline{N})$. Combining with Lemmas 5.2 and 5.3 we see that (1) and (5) are equivalent.

It is clear that (5) implies (2) and (2) implies (3). If (3) holds then we can refine the covering $\{S_i \to S\}$ such that each $S_i = \operatorname{Spec}(A_i)$ is affine. Then we see, by the prelimiary remarks of the proof, that $F_{\mathcal{F},A_i}$ is adequate. Thus $\mathcal{F}$ is adequate by definition. Hence (3) implies (1).

Finally, (4) is equivalent to (1) using Lemma 5.4 for one direction and that a composition of τ-coverings is a τ-covering for the other. $\square$

Just like is true for quasi-coherent sheaves the category of adequate modules is independent of the topology.

Lemma 5.6. *Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. For any surjective flat morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ of affines over S the extended Čech complex*

$$0 \to \mathcal{F}(\operatorname{Spec}(A)) \to \mathcal{F}(\operatorname{Spec}(B)) \to \mathcal{F}(\operatorname{Spec}(B \otimes_A B)) \to \dots$$

is exact. In particular $\mathcal{F}$ satisfies the sheaf condition for fpqc coverings, and is a sheaf of $\mathcal{O}$-modules on $(Sch/S)_{fppf}$.

Proof. With $A \to B$ as in the lemma let $F = F_{\mathcal{F},A}$. This functor is adequate by Lemma 5.2. By Lemma 3.5 since $A \to B$, $A \to B \otimes_A B$, etc are flat we see that $F(B) = F(A) \otimes_A B$, $F(B \otimes_A B) = F(A) \otimes_A B \otimes_A B$, etc. Exactness follows from Descent, Lemma 3.6.

Thus $\mathcal{F}$ satisfies the sheaf condition for τ-coverings (in particular Zariski coverings) and any faithfully flat covering of an affine by an affine. Arguing as in the proofs of Descent, Lemma 5.1 and Descent, Proposition 5.2 we conclude that $\mathcal{F}$ satisfies the sheaf condition for all fpqc coverings (made out of objects of $(Sch/S)_\tau$). Details omitted. $\square$

Lemma 5.6 shows in particular that for any pair of topologies τ, τ' the collection of adequate modules for the τ-topology and the τ'-topology are identical (as presheaves of modules on the underlying category Sch/S).

Definition 5.7. Let S be a scheme. The category of adequate $\mathcal{O}$-modules on $(Sch/S)_\tau$ is denoted $Adeq(\mathcal{O})$ or $Adeq((Sch/S)_\tau, \mathcal{O})$. If we want to think just about the abelian category of adequate modules without choosing a topology we simply write $Adeq(S)$.

Lemma 5.8. *Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$.*

(1) *The restriction $\mathcal{F}|_{S_{Zar}}$ is a quasi-coherent $\mathcal{O}_S$-module on the scheme S.*
(2) *The restriction $\mathcal{F}|_{S_{\acute{e}tale}}$ is the quasi-coherent module associated to $\mathcal{F}|_{S_{Zar}}$.*
(3) *For any affine scheme U over S we have $H^q(U, \mathcal{F}) = 0$ for all $q > 0$.*
(4) *There is a canonical isomorphism*

$$H^q(S, \mathcal{F}|_{S_{Zar}}) = H^q((Sch/S)_\tau, \mathcal{F}).$$

Proof. By Lemma 3.5 and Lemma 5.2 we see that for any flat morphism of affines $U \to V$ over S we have $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$. This works in particular if $U \subset V \subset S$ are affine opens of S, hence $\mathcal{F}|_{S_{Zar}}$ is quasi-coherent. Thus (1) holds.

Let $S' \to S$ be an étale morphism of schemes. Then for $U \subset S'$ affine open mapping into an affine open $V \subset S$ we see that $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$ because $U \to V$ is étale, hence flat. Therefore $\mathcal{F}|_{S'_{Zar}}$ is the pullback of $\mathcal{F}|_{S_{Zar}}$. This proves (2).

We are going to apply Cohomology on Sites, Lemma 11.9 to the site $(Sch/S)_\tau$ with $\mathcal{B}$ the set of affine schemes over S and Cov the set of standard affine τ-coverings. Assumption (3) of the lemma is satisfied by Descent, Lemma 7.8 and Lemma 5.6 for the case of a covering by a single affine. Hence we conclude that $H^p(U, \mathcal{F}) = 0$ for every affine scheme U over S. This proves (3). In exactly the same way as in the proof of Descent, Proposition 7.10 this implies the equality of cohomologies (4). $\qquad\square$

Remark 5.9. Let S be a scheme. We have functors $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ and $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$. Namely, the functor $u : \mathcal{F} \mapsto \mathcal{F}^a$ comes from taking the associated $\mathcal{O}$-module which is adequate by Lemma 5.5. Conversely, the functor v comes from restriction $v : \mathcal{G} \mapsto \mathcal{G}|_{S_{Zar}}$, see Lemma 5.8. Since $\mathcal{F}^a$ can be described as the pullback of $\mathcal{F}$ under a morphism of ringed topoi $((Sch/S)_\tau, \mathcal{O}) \to (S_{Zar}, \mathcal{O}_S)$, see Descent, Remark 7.6 and since restriction is the pushforward we see that u and v are adjoint as follows

$$\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, v\mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(u\mathcal{F}, \mathcal{G})$$

where $\mathcal{O}$ denotes the structure sheaf on the big site. It is immediate from the description that the adjunction mapping $\mathcal{F} \to vu\mathcal{F}$ is an isomorphism for all quasi-coherent sheaves.

Lemma 5.10. *Let S be a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$. If for every affine scheme $\mathrm{Spec}(A)$ over S the functor $F_{\mathcal{F},A}$ is adequate, then the sheafification of $\mathcal{F}$ is an adequate $\mathcal{O}$-module.*

Proof. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Set $F = F_{\mathcal{F},A}$. The sheafification $\mathcal{F}^\# = (\mathcal{F}^+)^+$, see Sites, Section 10. By construction

$$(\mathcal{F})^+(U) = \mathrm{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F})$$

where the colimit is over coverings in the site $(Sch/S)_\tau$. Since U is affine it suffices to take the limit over standard affine τ-coverings $\mathcal{U} = \{U_i \to U\}_{i \in I} = \{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_{i \in I}$ of U. Since each $A \to A_i$ and $A \to A_i \otimes_A A_j$ is flat we see that

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}(\prod F(A) \otimes_A A_i \to \prod F(A) \otimes_A A_i \otimes_A A_j)$$

by Lemma 3.5. Since $A \to \prod A_i$ is faithfully flat we see that this always is canonically isomorphic to $F(A)$ by Descent, Lemma 3.6. Thus the presheaf $(\mathcal{F})^+$ has the same value as $\mathcal{F}$ on all affine schemes over S. Repeating the argument once more we deduce the same thing for $\mathcal{F}^\# = ((\mathcal{F})^+)^+$. Thus $F_{\mathcal{F},A} = F_{\mathcal{F}^\#,A}$ and we conclude that $\mathcal{F}^\#$ is adequate. $\qquad\square$

Lemma 5.11. *Let S be a scheme.*
   (1) *The category $Adeq(\mathcal{O})$ is abelian.*
   (2) *The functor $Adeq(\mathcal{O}) \to Mod((Sch/S)_\tau, \mathcal{O})$ is exact.*

(3) *If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $\mathcal{O}$-modules and $\mathcal{F}_1$ and $\mathcal{F}_3$ are adequate, then $\mathcal{F}_2$ is adequate.*

(4) *The category $Adeq(\mathcal{O})$ has colimits and $Adeq(\mathcal{O}) \to Mod((Sch/S)_\tau, \mathcal{O})$ commutes with them.*

Proof. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of adequate $\mathcal{O}$-modules. To prove (1) and (2) it suffices to show that $\mathcal{K} = \mathrm{Ker}(\varphi)$ and $\mathcal{Q} = \mathrm{Coker}(\varphi)$ computed in $Mod((Sch/S)_\tau, \mathcal{O})$ are adequate. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Let $F = F_{\mathcal{F},A}$ and $G = F_{\mathcal{G},A}$. By Lemmas 3.11 and 3.10 the kernel K and cokernel Q of the induced map $F \to G$ are adequate functors. Because the kernel is computed on the level of presheaves, we see that $K = F_{\mathcal{K},A}$ and we conclude $\mathcal{K}$ is adequate. To prove the result for the cokernel, denote $\mathcal{Q}'$ the presheaf cokernel of φ. Then $Q = F_{\mathcal{Q}',A}$ and $\mathcal{Q} = (\mathcal{Q}')^\#$. Hence $\mathcal{Q}$ is adequate by Lemma 5.10.

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $\mathcal{O}$-modules and $\mathcal{F}_1$ and $\mathcal{F}_3$ are adequate. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Let $F_i = F_{\mathcal{F}_i,A}$. The sequence of functors

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is exact, because for $V = \mathrm{Spec}(B)$ affine over U we have $H^1(V, \mathcal{F}_1) = 0$ by Lemma 5.8. Since F_1 and F_3 are adequate functors by Lemma 5.2 we see that F_2 is adequate by Lemma 3.16. Thus $\mathcal{F}_2$ is adequate.

Let $\mathcal{I} \to Adeq(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Denote $\mathcal{F} = \mathrm{colim}_i \mathcal{F}_i$ the colimit computed in $Mod((Sch/S)_\tau, \mathcal{O})$. To prove (4) it suffices to show that $\mathcal{F}$ is adequate. Let $\mathcal{F}' = \mathrm{colim}_i \mathcal{F}_i$ be the colimit computed in presheaves of $\mathcal{O}$-modules. Then $\mathcal{F} = (\mathcal{F}')^\#$. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Let $F_i = F_{\mathcal{F}_i,A}$. By Lemma 3.12 the functor $\mathrm{colim}_i F_i = F_{\mathcal{F}',A}$ is adequate. Lemma 5.10 shows that $\mathcal{F}$ is adequate. $\square$

The following lemma tells us that the total direct image $Rf_*\mathcal{F}$ of an adequate module under a quasi-compact and quasi-separated morphism is a complex whose cohomology sheaves are adequate.

Lemma 5.12. *Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. For any adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ are adequate $\mathcal{O}_S$-modules on $(Sch/S)_\tau$.*

Proof. First we explain how to compute the higher direct images. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then $R^i f_*\mathcal{F}$ is the ith cohomology sheaf of the complex $f_*\mathcal{I}^\bullet$. Hence $R^i f_*\mathcal{F}$ is the sheaf associated to the presheaf which associates to an object U/S of $(Sch/S)_\tau$ the module

$$\frac{\mathrm{Ker}(f_*\mathcal{I}^i(U) \to f_*\mathcal{I}^{i+1}(U))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(U) \to f_*\mathcal{I}^i(U))} = \frac{\mathrm{Ker}(\mathcal{I}^i(U \times_S T) \to \mathcal{I}^{i+1}(U \times_S T))}{\mathrm{Im}(\mathcal{I}^{i-1}(U \times_S T) \to \mathcal{I}^i(U \times_S T))}$$

$$= H^i(U \times_S T, \mathcal{F})$$

$$= H^i((Sch/U \times_S T)_\tau, \mathcal{F}|_{(Sch/U \times_S T)_\tau})$$

$$= H^i(U \times_S T, \mathcal{F}|_{(U \times_S T)_{Zar}})$$

The first equality by Topologies, Lemma 7.12 (and its analogues for other topologies), the second equality by definition of cohomology of $\mathcal{F}$ over an object of $(Sch/T)_\tau$, the third equality by Cohomology on Sites, Lemma 8.1, and the last

equality by Lemma 5.8. Thus by Lemma 5.10 it suffices to prove the claim stated in the following paragraph.

Let A be a ring. Let T be a scheme quasi-compact and quasi-separated over A. Let $\mathcal{F}$ be an adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$. For an A-algebra B set $T_B = T \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ and denote $\mathcal{F}_B = \mathcal{F}|_{(T_B)_{Zar}}$ the restriction of $\mathcal{F}$ to the small Zariski site of T_B. (Recall that this is a "usual" quasi-coherent sheaf on the scheme T_B, see Lemma 5.8.) Claim: The functor

$$B \longmapsto H^q(T_B, \mathcal{F}_B)$$

is adequate. We will prove the lemma by the usual procedure of cutting T into pieces.

Case I: T is affine. In this case the schemes T_B are all affine and $H^q(T_B, \mathcal{F}_B) = 0$ for all $q \geq 1$. The functor $B \mapsto H^0(T_B, \mathcal{F}_B)$ is adequate by Lemma 3.18.

Case II: T is separated. Let n be the minimal number of affines needed to cover T. We argue by induction on n. The base case is Case I. Choose an affine open covering $T = V_1 \cup \ldots \cup V_n$. Set $V = V_1 \cup \ldots \cup V_{n-1}$ and $U = V_n$. Observe that

$$U \cap V = (V_1 \cap V_n) \cup \ldots \cup (V_{n-1} \cap V_n)$$

is also a union of $n-1$ affine opens as T is separated, see Schemes, Lemma 21.8. Note that for each B the base changes U_B, V_B and $(U \cap V)_B = U_B \cap V_B$ behave in the same way. Hence we see that for each B we have a long exact sequence

$$0 \to H^0(T_B, \mathcal{F}_B) \to H^0(U_B, \mathcal{F}_B) \oplus H^0(V_B, \mathcal{F}_B) \to H^0((U \cap V)_B, \mathcal{F}_B) \to H^1(T_B, \mathcal{F}_B) \to \ldots$$

functorial in B, see Cohomology, Lemma 9.2. By induction hypothesis the functors $B \mapsto H^q(U_B, \mathcal{F}_B)$, $B \mapsto H^q(V_B, \mathcal{F}_B)$, and $B \mapsto H^q((U \cap V)_B, \mathcal{F}_B)$ are adequate. Using Lemmas 3.11 and 3.10 we see that our functor $B \mapsto H^q(T_B, \mathcal{F}_B)$ sits in the middle of a short exact sequence whose outer terms are adequate. Thus the claim follows from Lemma 3.16.

Case III: General quasi-compact and quasi-separated case. The proof is again by induction on the number n of affines needed to cover T. The base case $n = 1$ is Case I. Choose an affine open covering $T = V_1 \cup \ldots \cup V_n$. Set $V = V_1 \cup \ldots \cup V_{n-1}$ and $U = V_n$. Note that since T is quasi-separated $U \cap V$ is a quasi-compact open of an affine scheme, hence Case II applies to it. The rest of the argument proceeds in exactly the same manner as in the paragraph above and is omitted. $\square$

6. Parasitic adequate modules

In this section we start comparing adequate modules and quasi-coherent modules on a scheme S. Recall that there are functors $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ and $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$ satisfying the adjunction

$$\mathcal{H}om_{QCoh(\mathcal{O}_S)}(\mathcal{F}, v\mathcal{G}) = \mathcal{H}om_{Adeq(\mathcal{O})}(u\mathcal{F}, \mathcal{G})$$

and such that $\mathcal{F} \to vu\mathcal{F}$ is an isomorphism for every quasi-coherent sheaf $\mathcal{F}$, see Remark 5.9. Hence u is a fully faithful embedding and we can identify $QCoh(\mathcal{O}_S)$ with a full subcategory of $Adeq(\mathcal{O})$. The functor v is exact but u is not left exact in general. The kernel of v is the subcategory of parasitic adequate modules.

In Descent, Definition 8.1 we give the definition of a parasitic module. For adequate modules the notion does not depend on the chosen topology.

Lemma 6.1. *Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. The following are equivalent:*

  (1) $v\mathcal{F} = 0$,
  (2) $\mathcal{F}$ *is parasitic,*
  (3) $\mathcal{F}$ *is parasitic for the τ-topology,*
  (4) $\mathcal{F}(U) = 0$ *for all $U \subset S$ open, and*
  (5) *there exists an affine open covering $S = \bigcup U_i$ such that $\mathcal{F}(U_i) = 0$ for all i.*

Proof. The implications $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ are immediate from the definitions. Assume (5). Suppose that $S = \bigcup U_i$ is an affine open covering such that $\mathcal{F}(U_i) = 0$ for all i. Let $V \to S$ be a flat morphism. There exists an affine open covering $V = \bigcup V_j$ such that each V_j maps into some U_i. As the morphism $V_j \to S$ is flat, also $V_j \to U_i$ is flat. Hence the corresponding ring map $A_i = \mathcal{O}(U_i) \to \mathcal{O}(V_j) = B_j$ is flat. Thus by Lemma 5.2 and Lemma 3.5 we see that $\mathcal{F}(U_i) \otimes_{A_i} B_j \to \mathcal{F}(V_j)$ is an isomorphism. Hence $\mathcal{F}(V_j) = 0$. Since $\mathcal{F}$ is a sheaf for the Zariski topology we conclude that $\mathcal{F}(V) = 0$. In this way we see that (5) implies (2).

This proves the equivalence of (2), (3), (4), and (5). As (1) is equivalent to (3) (see Remark 5.9) we conclude that all five conditions are equivalent. $\square$

Let S be a scheme. The subcategory of parasitic adequate modules is a Serre subcategory of $Adeq(\mathcal{O})$. The quotient is the category of quasi-coherent modules.

Lemma 6.2. *Let S be a scheme. The subcategory $\mathcal{C} \subset Adeq(\mathcal{O})$ of parasitic adequate modules is a Serre subcategory. Moreover, the functor v induces an equivalence of categories*

$$Adeq(\mathcal{O})/\mathcal{C} = QCoh(\mathcal{O}_S).$$

Proof. The category $\mathcal{C}$ is the kernel of the exact functor $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$, see Lemma 6.1. Hence it is a Serre subcategory by Homology, Lemma 9.4. By Homology, Lemma 9.6 we obtain an induced exact functor $\overline{v} : Adeq(\mathcal{O})/\mathcal{C} \to QCoh(\mathcal{O}_S)$. Because u is a right inverse to v we see right away that $\overline{v}$ is essentially surjective. We see that $\overline{v}$ is faithful by Homology, Lemma 9.7. Because u is a right inverse to v we finally conclude that $\overline{v}$ is fully faithful. $\square$

Lemma 6.3. *Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. For any parasitic adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ are parasitic adequate $\mathcal{O}_S$-modules on $(Sch/S)_\tau$.*

Proof. We have already seen in Lemma 5.12 that these higher direct images are adequate. Hence it suffices to show that $(R^i f_*\mathcal{F})(U_i) = 0$ for any τ-covering $\{U_i \to S\}$ open. And $R^i f_*\mathcal{F}$ is parasitic by Descent, Lemma 8.3. $\square$

7. Derived categories of adequate modules, I

Let S be a scheme. We continue the discussion started in Section 6. The exact functor v induces a functor

$$D(Adeq(\mathcal{O})) \longrightarrow D(QCoh(\mathcal{O}_S))$$

and similarly for bounded versions.

Lemma 7.1. *Let S be a scheme. Let $\mathcal{C} \subset Adeq(\mathcal{O})$ denote the full subcategory consisting of parasitic adequate modules. Then*

$$D(Adeq(\mathcal{O}))/D_{\mathcal{C}}(Adeq(\mathcal{O})) = D(QCoh(\mathcal{O}_S))$$

and similarly for the bounded versions.

Proof. Follows immediately from Derived Categories, Lemma 13.3. $\qquad\qquad\square$

Next, we look for a description the other way around by looking at the functors

$$K^+(QCoh(\mathcal{O}_S)) \longrightarrow K^+(Adeq(\mathcal{O})) \longrightarrow D^+(Adeq(\mathcal{O})) \longrightarrow D^+(QCoh(\mathcal{O}_S)).$$

In some cases the derived category of adequate modules is a localization of the homotopy category of complexes of quasi-coherent modules at universal quasi-isomorphisms. Let S be a scheme. A map of complexes $\varphi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of quasi-coherent $\mathcal{O}_S$-modules is said to be a *universal quasi-isomorphism* if for every morphism of schemes $f : T \to S$ the pullback $f^*\varphi$ is a quasi-isomorphism.

Lemma 7.2. *Let $U = \mathrm{Spec}(A)$ be an affine scheme. The bounded below derived category $D^+(Adeq(\mathcal{O}))$ is the localization of $K^+(QCoh(\mathcal{O}_U))$ at the multiplicative subset of universal quasi-isomorphisms.*

Proof. If $\varphi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a morphism of complexes of quasi-coherent $\mathcal{O}_U$-modules, then $u\varphi : u\mathcal{F}^\bullet \to u\mathcal{G}^\bullet$ is a quasi-isomorphism if and only if φ is a universal quasi-isomorphism. Hence the collection S of universal quasi-isomorphisms is a saturated multiplicative system compatible with the triangulated structure by Derived Categories, Lemma 5.3. Hence $S^{-1}K^+(QCoh(\mathcal{O}_U))$ exists and is a triangulated category, see Derived Categories, Proposition 5.5. We obtain a canonical functor $can : S^{-1}K^+(QCoh(\mathcal{O}_U)) \to D^+(Adeq(\mathcal{O}))$ by Derived Categories, Lemma 5.6.

Note that, almost by definition, every adequate module on U has an embedding into a quasi-coherent sheaf, see Lemma 5.5. Hence by Derived Categories, Lemma 16.4 given $\mathcal{F}^\bullet \in \mathrm{Ob}(K^+(Adeq(\mathcal{O})))$ there exists a quasi-isomorphism $\mathcal{F}^\bullet \to u\mathcal{G}^\bullet$ where $\mathcal{G}^\bullet \in \mathrm{Ob}(K^+(QCoh(\mathcal{O}_U)))$. This proves that can is essentially surjective.

Similarly, suppose that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are bounded below complexes of quasi-coherent $\mathcal{O}_U$-modules. A morphism in $D^+(Adeq(\mathcal{O}))$ between these consists of a pair $f : u\mathcal{F}^\bullet \to \mathcal{H}^\bullet$ and $s : u\mathcal{G}^\bullet \to \mathcal{H}^\bullet$ where s is a quasi-isomorphism. Pick a quasi-isomorphism $s' : \mathcal{H}^\bullet \to u\mathcal{E}^\bullet$. Then we see that $s' \circ f : \mathcal{F} \to \mathcal{E}^\bullet$ and the universal quasi-isomorphism $s' \circ s : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ give a morphism in $S^{-1}K^+(QCoh(\mathcal{O}_U))$ mapping to the given morphism. This proves the "fully" part of full faithfulness. Faithfulness is proved similarly. $\qquad\square$

Lemma 7.3. *Let $U = \mathrm{Spec}(A)$ be an affine scheme. The inclusion functor*

$$Adeq(\mathcal{O}) \to Mod((Sch/U)_\tau, \mathcal{O})$$

has a right adjoint A[1]*. Moreover, the adjunction mapping $A(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every adequate module $\mathcal{F}$.*

Proof. By Topologies, Lemma 7.11 (and similarly for the other topologies) we may work with $\mathcal{O}$-modules on $(Aff/U)_\tau$. Denote $\mathcal{P}$ the category of module-valued functors on Alg_A and $\mathcal{A}$ the category of adequate functors on Alg_A. Denote $i : \mathcal{A} \to$

---

[1]This is the "adequator".

$\mathcal{P}$ the inclusion functor. Denote $Q : \mathcal{P} \to \mathcal{A}$ the construction of Lemma 4.1. We have the commutative diagram

(7.3.1)
$$
\begin{array}{ccc}
Adeq(\mathcal{O}) \xrightarrow{\ k\ } Mod((Aff/U)_\tau, \mathcal{O}) \xrightarrow{\ j\ } PMod((Aff/U)_\tau, \mathcal{O}) \\
\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\
\mathcal{A} \xrightarrow{\qquad\qquad\qquad i \qquad\qquad\qquad} \mathcal{P}
\end{array}
$$

The left vertical equality is Lemma 5.3 and the right vertical equality was explained in Section 3. Define $A(\mathcal{F}) = Q(j(\mathcal{F}))$. Since j is fully faithful it follows immediately that A is a right adjoint of the inclusion functor k. Also, since k is fully faithful too, the final assertion follows formally. $\qquad\square$

The functor A is a right adjoint hence left exact. Since the inclusion functor is exact, see Lemma 5.11 we conclude that A transforms injectives into injectives, and that the category $Adeq(\mathcal{O})$ has enough injectives, see Homology, Lemma 25.3 and Injectives, Theorem 8.4. This also follows from the equivalence in (7.3.1) and Lemma 4.2.

Lemma 7.4. *Let $U = \mathrm{Spec}(A)$ be an affine scheme. For any object $\mathcal{F}$ of $Adeq(\mathcal{O})$ we have $R^p A(\mathcal{F}) = 0$ for all $p > 0$ where A is as in Lemma 7.3.*

Proof. With notation as in the proof of Lemma 7.3 choose an injective resolution $k(\mathcal{F}) \to \mathcal{I}^\bullet$ in the category of $\mathcal{O}$-modules on $(Aff/U)_\tau$. By Cohomology on Sites, Lemmas 12.2 and Lemma 5.8 the complex $j(\mathcal{I}^\bullet)$ is exact. On the other hand, each $j(\mathcal{I}^n)$ is an injective object of the category of presheaves of modules by Cohomology on Sites, Lemma 12.1. It follows that $R^p A(\mathcal{F}) = R^p Q(j(k(\mathcal{F})))$. Hence the result now follows from Lemma 4.10. $\qquad\square$

Let S be a scheme. By the discussion in Section 5 the embedding $Adeq(\mathcal{O}) \subset Mod((Sch/S)_\tau, \mathcal{O})$ exhibits $Adeq(\mathcal{O})$ as a weak Serre subcategory of the category of all $\mathcal{O}$-modules. Denote

$$D_{Adeq}(\mathcal{O}) \subset D(\mathcal{O}) = D(Mod((Sch/S)_\tau, \mathcal{O}))$$

the triangulated subcategory of complexes whose cohomology sheaves are adequate, see Derived Categories, Section 13. We obtain a canonical functor

$$D(Adeq(\mathcal{O})) \longrightarrow D_{Adeq}(\mathcal{O})$$

see Derived Categories, Equation (13.1.1).

Lemma 7.5. *If $U = \mathrm{Spec}(A)$ is an affine scheme, then the bounded below version*

(7.5.1)
$$D^+(Adeq(\mathcal{O})) \longrightarrow D^+_{Adeq}(\mathcal{O})$$

of the functor above is an equivalence.

Proof. Let $A : Mod(\mathcal{O}) \to Adeq(\mathcal{O})$ be the right adjoint to the inclusion functor constructed in Lemma 7.3. Since A is left exact and since $Mod(\mathcal{O})$ has enough injectives, A has a right derived functor $RA : D^+_{Adeq}(\mathcal{O}) \to D^+(Adeq(\mathcal{O}))$. We claim that RA is a quasi-inverse to (7.5.1). To see this the key fact is that if $\mathcal{F}$ is an adequate module, then the adjunction map $\mathcal{F} \to RA(\mathcal{F})$ is a quasi-isomorphism by Lemma 7.4.

Namely, to prove the lemma in full it suffices to show:

(1) Given $\mathcal{F}^\bullet \in K^+(Adeq(\mathcal{O}))$ the canonical map $\mathcal{F}^\bullet \to RA(\mathcal{F}^\bullet)$ is a quasi-isomorphism, and
(2) given $\mathcal{G}^\bullet \in K^+(Mod(\mathcal{O}))$ the canonical map $RA(\mathcal{G}^\bullet) \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Both (1) and (2) follow from the key fact via a spectral sequence argument using one of the spectral sequences of Derived Categories, Lemma 21.3. Some details omitted. $\square$

Lemma 7.6. *Let $U = \mathrm{Spec}(A)$ be an affine scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be adequate $\mathcal{O}$-modules. For any $i \geq 0$ the natural map*

$$Ext^i_{Adeq(\mathcal{O})}(\mathcal{F}, \mathcal{G}) \longrightarrow Ext^i_{Mod(\mathcal{O})}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. By definition these ext groups are computed as hom sets in the derived category. Hence this follows immediately from Lemma 7.5. $\square$

8. Pure extensions

We want to characterize extensions of quasi-coherent sheaves on the big site of an affine schemes in terms of algebra. To do this we introduce the following notion.

Definition 8.1. Let A be a ring.
(1) An A-module P is said to be *pure projective* if for every universally exact sequence $0 \to K \to M \to N \to 0$ of A-module the sequence $0 \to \mathrm{Hom}_A(P, K) \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N) \to 0$ is exact.
(2) An A-module I is said to be *pure injective* if for every universally exact sequence $0 \to K \to M \to N \to 0$ of A-module the sequence $0 \to \mathrm{Hom}_A(N, I) \to \mathrm{Hom}_A(M, I) \to \mathrm{Hom}_A(K, I) \to 0$ is exact.

Let's characterize pure projectives.

Lemma 8.2. *Let A be a ring.*
(1) *A module is pure projective if and only if it is a direct summand of a direct sum of finitely presented A-modules.*
(2) *For any module M there exists a universally exact sequence $0 \to N \to P \to M \to 0$ with P pure projective.*

Proof. First note that a finitely presented A-module is pure projective by Algebra, Theorem 79.3. Hence a direct summand of a direct sum of finitely presented A-modules is indeed pure projective. Let M be any A-module. Write $M = \mathrm{colim}_{i \in I} P_i$ as a filtered colimit of finitely presented A-modules. Consider the sequence

$$0 \to N \to \bigoplus P_i \to M \to 0.$$

For any finitely presented A-module P the map $\mathrm{Hom}_A(P, \bigoplus P_i) \to \mathrm{Hom}_A(P, M)$ is surjective, as any map $P \to M$ factors through some P_i. Hence by Algebra, Theorem 79.3 this sequence is universally exact. This proves (2). If now M is pure projective, then the sequence is split and we see that M is a direct summand of $\bigoplus P_i$. $\square$

Let's characterize pure injectives.

Lemma 8.3. *Let A be a ring. For any A-module M set $M^\wedge = \mathrm{Hom}_\mathbf{Z}(M, \mathbf{Q}/\mathbf{Z})$.*

(1) *For any A-module M the A-module $M^\wedge$ is pure injective.*
(2) *An A-module I is pure injective if and only if the map $I \to (I^\wedge)^\wedge$ splits.*
(3) *For any module M there exists a universally exact sequence $0 \to M \to I \to N \to 0$ with I pure injective.*

Proof. We will use the properties of the functor $M \mapsto M^\wedge$ found in More on Algebra, Section 42 without further mention. Part (1) holds because $\mathrm{Hom}_A(N, M^\wedge) = \mathrm{Hom}_{\mathbf{Z}}(N \otimes_A M, \mathbf{Q/Z})$ and because $\mathbf{Q/Z}$ is injective in the category of abelian groups. Hence if $I \to (I^\wedge)^\wedge$ is split, then I is pure injective. We claim that for any A-module M the evaluation map $ev : M \to (M^\wedge)^\wedge$ is universally injective. To see this note that $ev^\wedge : ((M^\wedge)^\wedge)^\wedge \to M^\wedge$ has a right inverse, namely $ev' : M^\wedge \to ((M^\wedge)^\wedge)^\wedge$. Then for any A-module N applying the exact faithful functor $^\wedge$ to the map $N \otimes_A M \to N \otimes_A (M^\wedge)^\wedge$ gives

$$\mathrm{Hom}_A(N, ((M^\wedge)^\wedge)^\wedge) = \left(N \otimes_A (M^\wedge)^\wedge \right)^\wedge \to \left(N \otimes_A M \right)^\wedge = \mathrm{Hom}_A(N, M^\wedge)$$

which is surjective by the existence of the right inverse. The claim follows. The claim implies (3) and the necessity of the condition in (2). $\qquad\square$

Before we continue we make the following observation which we will use frequently in the rest of this section.

Lemma 8.4. *Let A be a ring.*
(1) *Let $L \to M \to N$ be a universally exact sequence of A-modules. Let $K = \mathrm{Im}(M \to N)$. Then $K \to N$ is universally injective.*
(2) *Any universally exact complex can be split into universally exact short exact sequences.*

Proof. Proof of (1). For any A-module T the sequence $L \otimes_A T \to M \otimes_A T \to K \otimes_A T \to 0$ is exact by right exactness of $\otimes$. By assumption the sequence $L \otimes_A T \to M \otimes_A T \to N \otimes_A T$ is exact. Combined this shows that $K \otimes_A T \to N \otimes_A T$ is injective.

Part (2) means the following: Suppose that $M^\bullet$ is a universally exact complex of A-modules. Set $K^i = \mathrm{Ker}(d^i) \subset M^i$. Then the short exact sequences $0 \to K^i \to M^i \to K^{i+1} \to 0$ are universally exact. This follows immediately from part (1). $\quad\square$

Definition 8.5. Let A be a ring. Let M be an A-module.
(1) A *pure projective resolution* $P_\bullet \to M$ is a universally exact sequence

$$\ldots \to P_1 \to P_0 \to M \to 0$$

with each P_i pure projective.
(2) A *pure injective resolution* $M \to I^\bullet$ is a universally exact sequence

$$0 \to M \to I^0 \to I^1 \to \ldots$$

with each I^i pure injective.

These resolutions satisfy the usual uniqueness properties among the class of all universally exact left or right resolutions.

Lemma 8.6. *Let A be a ring.*
(1) *Any A-module has a pure projective resolution.*

Let $M \to N$ be a map of A-modules. Let $P_\bullet \to M$ be a pure projective resolution and let $N_\bullet \to N$ be a universally exact resolution.

 (2) *There exists a map of complexes $P_\bullet \to N_\bullet$ inducing the given map*

$$M = Coker(P_1 \to P_0) \to Coker(N_1 \to N_0) = N$$

 (3) *two maps $\alpha, \beta : P_\bullet \to N_\bullet$ inducing the same map $M \to N$ are homotopic.*

Proof. Part (1) follows immediately from Lemma 8.2. Before we prove (2) and (3) note that by Lemma 8.4 we can split the universally exact complex $N_\bullet \to N \to 0$ into universally exact short exact sequences $0 \to K_0 \to N_0 \to N \to 0$ and $0 \to K_i \to N_i \to K_{i-1} \to 0$.

Proof of (2). Because P_0 is pure projective we can find a map $P_0 \to N_0$ lifting the map $P_0 \to M \to N$. We obtain an induced map $P_1 \to F_0 \to N_0$ wich ends up in K_0. Since P_1 is pure projective we may lift this to a map $P_1 \to N_1$. This in turn induces a map $P_2 \to P_1 \to N_1$ which maps to zero into N_0, i.e., into K_1. Hence we may lift to get a map $P_2 \to N_2$. Repeat.

Proof of (3). To show that α, β are homotopic it suffices to show the difference $\gamma = \alpha - \beta$ is homotopic to zero. Note that the image of $\gamma_0 : P_0 \to N_0$ is contained in K_0. Hence we may lift γ_0 to a map $h_0 : P_0 \to N_1$. Consider the map $\gamma_1' = \gamma_1 - h_0 \circ d_{P,1} : P_1 \to N_1$. By our choice of h_0 we see that the image of γ_1' is contained in K_1. Since P_1 is pure projective may lift γ_1' to a map $h_1 : P_1 \to N_2$. At this point we have $\gamma_1 = h_0 \circ d_{F,1} + d_{N,2} \circ h_1$. Repeat. $\square$

Lemma 8.7. *Let A be a ring.*

 (1) *Any A-module has a pure injective resolution.*

Let $M \to N$ be a map of A-modules. Let $M \to M^\bullet$ be a universally exact resolution and let $N \to I^\bullet$ be a pure injective resolution.

 (2) *There exists a map of complexes $M^\bullet \to I^\bullet$ inducing the given map*

$$M = Ker(M^0 \to M^1) \to Ker(I^0 \to I^1) = N$$

 (3) *two maps $\alpha, \beta : M^\bullet \to I^\bullet$ inducing the same map $M \to N$ are homotopic.*

Proof. This lemma is dual to Lemma 8.6. The proof is identical, except one has to reverse all the arrows. $\square$

Using the material above we can define pure extension groups as follows. Let A be a ring and let M, N be A-modules. Choose a pure injective resolution $N \to I^\bullet$. By Lemma 8.7 the complex

$$\mathrm{Hom}_A(M, I^\bullet)$$

is well defined up to homotopy. Hence its ith cohomology module is a well defined invariant of M and N.

Definition 8.8. Let A be a ring and let M, N be A-modules. The ith *pure extension module* $\mathrm{Pext}_A^i(M, N)$ is the ith cohomology module of the complex $\mathrm{Hom}_A(M, I^\bullet)$ where $I^\bullet$ is a pure injective resolution of N.

Warning: It is not true that an exact sequence of A-modules gives rise to a long exact sequence of pure extensions groups. (You need a universally exact sequence for this.) We collect some facts which are obvious from the material above.

Lemma 8.9. *Let A be a ring.*

(1) $Pext_A^i(M, N) = 0$ for $i > 0$ whenever N is pure injective,
(2) $Pext_A^i(M, N) = 0$ for $i > 0$ whenever M is pure projective, in particular if M is an A-module of finite presentation,
(3) $Pext_A^i(M, N)$ is also the ith cohomology module of the complex $\mathrm{Hom}_A(P_\bullet, N)$ where $P_\bullet$ is a pure projective resolution of M.

Proof. To see (3) consider the double complex

$$A^{\bullet, \bullet} = \mathrm{Hom}_A(P_\bullet, I^\bullet)$$

Each of its rows is exact except in degree 0 where its cohomology is $\mathrm{Hom}_A(M, I^q)$. Each of its columns is exact except in degree 0 where its cohomology is $\mathrm{Hom}_A(P_p, N)$. Hence the two spectral sequences associated to this complex in Homology, Section 22 degenerate, giving the equality. $\square$

9. Higher exts of quasi-coherent sheaves on the big site

It turns out that the module-valued functor $\underline{I}$ associated to a pure injective module I gives rise to an injective object in the category of adequate functors on Alg_A. Warning: It is not true that a pure projective module gives rise to a projective object in the category of adequate functors. We do have plenty of projective objects, namely, the linearly adequate functors.

Lemma 9.1. *Let A be a ring. Let $\mathcal{A}$ be the category of adequate functors on Alg_A. The injective objects of $\mathcal{A}$ are exactly the functors $\underline{I}$ where I is a pure injective A-module.*

Proof. Let I be an injective object of $\mathcal{A}$. Choose an embedding $I \to \underline{M}$ for some A-module M. As I is injective we see that $\underline{M} = I \oplus F$ for some module-valued functor F. Then $M = I(A) \oplus F(A)$ and it follows that $I = \underline{I(A)}$. Thus we see that any injective object is of the form $\underline{I}$ for some A-module I. It is clear that the module I has to be pure injective since any universally exact sequence $0 \to M \to N \to L \to 0$ gives rise to an exact sequence $0 \to \underline{M} \to \underline{N} \to \underline{L} \to 0$ of $\mathcal{A}$.

Finally, suppose that I is a pure injective A-module. Choose an embedding $\underline{I} \to J$ into an injective object of $\mathcal{A}$ (see Lemma 4.2). We have seen above that $J = \underline{I'}$ for some A-module I' which is pure injective. As $\underline{I} \to \underline{I'}$ is injective the map $I \to I'$ is universally injective. By assumption on I it splits. Hence $\underline{I}$ is a summand of $J = \underline{I'}$ whence an injective object of the category $\mathcal{A}$. $\square$

Let $U = \mathrm{Spec}(A)$ be an affine scheme. Let M be an A-module. We will use the notation M^a to denote the quasi-coherent sheaf of $\mathcal{O}$-modules on $(Sch/U)_\tau$ associated to the quasi-coherent sheaf $\widetilde{M}$ on U. Now we have all the notation in place to formulate the following lemma.

Lemma 9.2. *Let $U = \mathrm{Spec}(A)$ be an affine scheme. Let M, N be A-modules. For all i we have a canonical isomorphism*

$$Ext^i_{Mod(\mathcal{O})}(M^a, N^a) = Pext_A^i(M, N)$$

functorial in M and N.

Proof. Let us construct a canonical arrow from right to left. Namely, if $N \to I^\bullet$ is a pure injective resolution, then $M^a \to (I^\bullet)^a$ is an exact complex of (adequate) $\mathcal{O}$-modules. Hence any element of $Pext_A^i(M, N)$ gives rise to a map $N^a \to M^a[i]$ in $D(\mathcal{O})$, i.e., an element of the group on the left.

To prove this map is an isomorphism, note that we may replace $\mathrm{Ext}^i_{Mod(\mathcal{O})}(M^a, N^a)$ by $\mathrm{Ext}^i_{Adeq(\mathcal{O})}(M^a, N^a)$, see Lemma 7.6. Let $\mathcal{A}$ be the category of adequate functors on Alg_A. We have seen that $\mathcal{A}$ is equivalent to $Adeq(\mathcal{O})$, see Lemma 5.3; see also the proof of Lemma 7.3. Hence now it suffices to prove that

$$\mathrm{Ext}^i_{\mathcal{A}}(\underline{M}, \underline{N}) = \mathrm{Pext}^i_A(M, N)$$

However, this is clear from Lemma 9.1 as a pure injective resolution $N \to I^\bullet$ exactly corresponds to an injective resolution of $\underline{N}$ in $\mathcal{A}$. $\qquad\square$

10. Derived categories of adequate modules, II

Let S be a scheme. Denote $\mathcal{O}_S$ the structure sheaf of S and $\mathcal{O}$ the structure sheaf of the big site $(Sch/S)_\tau$. In Descent, Remark 7.4 we constructed a morphism of ringed sites

$$(10.0.1) \qquad\qquad f : ((Sch/S)_\tau, \mathcal{O}) \longrightarrow (S_{Zar}, \mathcal{O}_S).$$

In the previous sections have seen that the functor $f_* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_S)$ transforms adequate sheaves into quasi-coherent sheaves, and induces an exact functor $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$, and in fact that $f_* = v$ induces an equivalence $Adeq(\mathcal{O})/\mathcal{C} \to QCoh(\mathcal{O}_S)$ where $\mathcal{C}$ is the subcategory of parasitic adequate modules. Moreover, the functor f^* transforms quasi-coherent modules into adequate modules, and induces a functor $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ which is a left adjoint to v.

There is a very similar relationship between $D_{Adeq}(\mathcal{O})$ and $D_{QCoh}(S)$. First we explain why the category $D_{Adeq}(\mathcal{O})$ is independent of the chosen topology.

Remark 10.1. Let S be a scheme. Let $\tau, \tau' \in \{Zar, \text{é}tale, smooth, syntomic, fppf\}$. Denote $\mathcal{O}_\tau$, resp. $\mathcal{O}_{\tau'}$ the structure sheaf $\mathcal{O}$ viewed as a sheaf on $(Sch/S)_\tau$, resp. $(Sch/S)_{\tau'}$. Then $D_{Adeq}(\mathcal{O}_\tau)$ and $D_{Adeq}(\mathcal{O}_{\tau'})$ are canonically isomorphic. This follows from Cohomology on Sites, Lemma 22.6. Namely, assume τ is stronger than the topology τ', let $\mathcal{C} = (Sch/S)_{fppf}$, and let $\mathcal{B}$ the collection of affine schemes over S. Assumptions (1) and (2) we've seen above. Assumption (3) is clear and assumption (4) follows from Lemma 5.8.

Remark 10.2. Let S be a scheme. The morphism f see (10.0.1) induces adjoint functors $Rf_* : D_{Adeq}(\mathcal{O}) \to D_{QCoh}(S)$ and $Lf^* : D_{QCoh}(S) \to D_{Adeq}(\mathcal{O})$. Moreover $Rf_* Lf^* \cong \mathrm{id}_{D_{QCoh}(S)}$.

We sketch the proof. By Remark 10.1 we may assume the topology τ is the Zariski topology. We will use the existence of the unbounded total derived functors Lf^* and Rf_* on $\mathcal{O}$-modules and their adjointness, see Cohomology on Sites, Lemma 19.1. In this case f_* is just the restriction to the subcategory S_{Zar} of $(Sch/S)_{Zar}$. Hence it is clear that $Rf_* = f_*$ induces $Rf_* : D_{Adeq}(\mathcal{O}) \to D_{QCoh}(S)$. Suppose that $\mathcal{G}^\bullet$ is an object of $D_{QCoh}(S)$. We may choose a system $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ of bounded above complexes of flat $\mathcal{O}_S$-modules whose transition maps are termwise split injectives and a diagram

$$\begin{array}{ccccc} \mathcal{K}_1^\bullet & \longrightarrow & \mathcal{K}_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}\mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 2}\mathcal{G}^\bullet & \longrightarrow & \ldots \end{array}$$

with the properties (1), (2), (3) listed in Derived Categories, Lemma 28.1 where $\mathcal{P}$ is the collection of flat $\mathcal{O}_S$-modules. Then $Lf^*\mathcal{G}^\bullet$ is computed by $\mathrm{colim}\, f^*\mathcal{K}_n^\bullet$, see Cohomology on Sites, Lemmas 18.1 and 18.3 (note that our sites have enough points by Étale Cohomology, Lemma 30.1). We have to see that $H^i(Lf^*\mathcal{G}^\bullet) = \mathrm{colim}\, H^i(f^*\mathcal{K}_n^\bullet)$ is adequate for each i. By Lemma 5.11 we conclude that it suffices to show that each $H^i(f^*\mathcal{K}_n^\bullet)$ is adequate.

The adequacy of $H^i(f^*\mathcal{K}_n^\bullet)$ is local on S, hence we may assume that $S = \mathrm{Spec}(A)$ is affine. Because S is affine $D_{QCoh}(S) = D(QCoh(\mathcal{O}_S))$, see the discussion in Derived Categories of Schemes, Section 3. Hence there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{K}_n^\bullet$ where $\mathcal{F}^\bullet$ is a bounded above complex of flat quasi-coherent modules. Then $f^*\mathcal{F}^\bullet \to f^*\mathcal{K}_n^\bullet$ is a quasi-isomorphism, and the cohomology sheaves of $f^*\mathcal{F}^\bullet$ are adequate.

The final assertion $Rf_*Lf^* \cong \mathrm{id}_{D_{QCoh}(S)}$ follows from the explicit description of the functors above. (In plain English: if $\mathcal{F}$ is quasi-coherent and $p > 0$, then $L_p f^*\mathcal{F}$ is a parasitic adequate module.)

Remark 10.3. Remark 10.2 above implies we have an equivalence of derived categories
$$D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O}) \longrightarrow D_{QCoh}(S)$$
where $\mathcal{C}$ is the category of parasitic adequate modules. Namely, it is clear that $D_{\mathcal{C}}(\mathcal{O})$ is the kernel of Rf_*, hence a functor as indicated. For any object X of $D_{Adeq}(\mathcal{O})$ the map $Lf^*Rf_*X \to X$ maps to a quasi-isomorphism in $D_{QCoh}(S)$, hence $Lf^*Rf_*X \to X$ is an isomorphism in $D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O})$. Finally, for X, Y objects of $D_{Adeq}(\mathcal{O})$ the map
$$Rf_* : \mathrm{Hom}_{D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O})}(X, Y) \to \mathrm{Hom}_{D_{QCoh}(S)}(Rf_*X, Rf_*Y)$$
is bijective as Lf^* gives an inverse (by the remarks above).

11. Other chapters

Preliminaries

References

[Jaf97] David Benjamin Jaffe, *Coherent functors, with application to torsion in the picard group*, Trans. Amer. Math. Soc. **349** (1997), no. 2, 481–527.