

STACKS

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1. Introduction

In this very short chapter we introduce stacks, and stacks in groupoids. See [DM69], and [Vis04].

2. Presheaves of morphisms associated to fibred categories

Let \mathcal{C} be a category. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category, see Categories, Section 31. Suppose that $x, y \in \text{Ob}(\mathcal{S}_U)$ are objects in the fibre category over U . We are going to define a functor

$$\text{Mor}(x, y) : (\mathcal{C}/U)^{opp} \longrightarrow \text{Sets}.$$

In other words this will be a presheaf on \mathcal{C}/U , see Sites, Definition 2.2. Make a choice of pullbacks as in Categories, Definition 31.6. Then, for $f : V \rightarrow U$ we set

$$\text{Mor}(x, y)(f : V \rightarrow U) = \text{Mor}_{\mathcal{S}_V}(f^*x, f^*y).$$

Let $f' : V' \rightarrow U$ be a second object of \mathcal{C}/U . We also have to define the restriction map corresponding to a morphism $g : V'/U \rightarrow V/U$ in \mathcal{C}/U , in other words $g : V' \rightarrow V$ and $f' = f \circ g$. This will be a map

$$\text{Mor}_{\mathcal{S}_V}(f^*x, f^*y) \longrightarrow \text{Mor}_{\mathcal{S}_{V'}}(f'^*x, f'^*y), \quad \phi \longmapsto \phi|_{V'}$$

This map will basically be g^* , except that this transforms an element ϕ of the left hand side into an element $g^*\phi$ of $\text{Mor}_{\mathcal{S}_{V'}}(g^*f^*x, g^*f^*y)$. At this point we use the

This is a chapter of the Stacks Project, version 714e994, compiled on Oct 28, 2014.

transformation $\alpha_{g,f}$ of Categories, Lemma 31.7. In a formula, the restriction map is described by

$$\phi|_{V'} = (\alpha_{g,f})_y^{-1} \circ g^* \phi \circ (\alpha_{g,f})_x.$$

Of course, nobody thinks of this restriction map in this way. We will only do this once in order to verify the following lemma.

Lemma 2.1. *This actually does give a presheaf.*

Proof. Let $g : V'/U \rightarrow V/U$ be as above and similarly $g' : V''/U \rightarrow V'/U$ be morphisms in \mathcal{C}/U . So $f' = f \circ g$ and $f'' = f' \circ g' = f \circ g \circ g'$. Let $\phi \in \text{Mor}_{S_V}(f^*x, f^*y)$. Then we have

$$\begin{aligned} & (\alpha_{g \circ g', f})_y^{-1} \circ (g \circ g')^* \phi \circ (\alpha_{g \circ g', f})_x \\ &= (\alpha_{g \circ g', f})_y^{-1} \circ (\alpha_{g', g})_{f^*y}^{-1} \circ (g')^* g^* \phi \circ (\alpha_{g', g})_{f^*x} \circ (\alpha_{g \circ g', f})_x \\ &= (\alpha_{g', f'})_y^{-1} \circ (g')^* (\alpha_{g, f})_y^{-1} \circ (g')^* g^* \phi \circ (g')^* (\alpha_{g, f})_x \circ (\alpha_{g', f'})_x \\ &= (\alpha_{g', f'})_y^{-1} \circ (g')^* \left((\alpha_{g, f})_y^{-1} \circ g^* \phi \circ (\alpha_{g, f})_x \right) \circ (\alpha_{g', f'})_x \end{aligned}$$

which is what we want, namely $\phi|_{V''} = (\phi|_{V'})|_{V''}$. The first equality holds because $\alpha_{g', g}$ is a transformation of functors, and hence

$$\begin{array}{ccc} (g \circ g')^* f^* x & \xrightarrow{(g \circ g')^* \phi} & (g \circ g')^* f^* y \\ (\alpha_{g', g})_{f^* x} \downarrow & & \downarrow (\alpha_{g', g})_{f^* y} \\ (g')^* g^* f^* x & \xrightarrow{(g')^* g^* \phi} & (g')^* g^* f^* y \end{array}$$

commutes. The second equality holds because of property (d) of a pseudo functor since $f' = f \circ g$ (see Categories, Definition 27.5). The last equality follows from the fact that $(g')^*$ is a functor. \square

From now on we often omit mentioning the transformations $\alpha_{g,f}$ and we simply identify the functors $g^* \circ f^*$ and $(f \circ g)^*$. In particular, given $g : V'/U \rightarrow V/U$ the restriction mappings for the presheaf $\text{Mor}(x, y)$ will sometimes be denoted $\phi \mapsto g^* \phi$. We formalize the construction in a definition.

Definition 2.2. Let \mathcal{C} be a category. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category, see Categories, Section 31. Given an object U of \mathcal{C} and objects x, y of the fibre category, the *presheaf of morphisms from x to y* is the presheaf

$$(f : V \rightarrow U) \mapsto \text{Mor}_{S_V}(f^*x, f^*y)$$

described above. It is denoted $\text{Mor}(x, y)$. The subpresheaf $\text{Isom}(x, y)$ whose values over V is the set of isomorphisms $f^*x \rightarrow f^*y$ in the fibre category S_V is called the *presheaf of isomorphisms from x to y* .

If \mathcal{S} is fibred in groupoids then of course $\text{Isom}(x, y) = \text{Mor}(x, y)$, and it is customary to use the *Isom* notation.

Lemma 2.3. *Let $F : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ be a 1-morphism of fibred categories over the category \mathcal{C} . Let $U \in \text{Ob}(\mathcal{C})$ and $x, y \in \text{Ob}(S_U)$. Then F defines a canonical morphism of presheaves*

$$\text{Mor}_{S_1}(x, y) \longrightarrow \text{Mor}_{S_2}(F(x), F(y))$$

on \mathcal{C}/U .

Proof. By Categories, Definition 31.9 the functor F maps strongly cartesian morphisms to strongly cartesian morphisms. Hence if $f : V \rightarrow U$ is a morphism in \mathcal{C} , then there are canonical isomorphisms $\alpha_V : f^*F(x) \rightarrow F(f^*x)$, $\beta_V : f^*F(y) \rightarrow F(f^*y)$ such that $f^*F(x) \rightarrow F(f^*x) \rightarrow F(x)$ is the canonical morphism $f^*F(x) \rightarrow F(x)$, and similarly for β_V . Thus we may define

$$\begin{array}{ccc} \text{Mor}_{\mathcal{S}_1}(x, y)(f : V \rightarrow U) & \xlongequal{\quad} & \text{Mor}_{\mathcal{S}_{1,V}}(f^*x, f^*y) \\ & & \downarrow \\ \text{Mor}_{\mathcal{S}_2}(F(x), F(y))(f : V \rightarrow U) & \xlongequal{\quad} & \text{Mor}_{\mathcal{S}_{2,V}}(f^*F(x), f^*F(y)) \end{array}$$

by $\phi \mapsto \beta_V^{-1} \circ F(\phi) \circ \alpha_V$. We omit the verification that this is compatible with the restriction mappings. \square

Remark 2.4. Suppose that $p : \mathcal{S} \rightarrow \mathcal{C}$ is fibred in groupoids. In this case we can prove Lemma 2.1 using Categories, Lemma 34.4 which says that $\mathcal{S} \rightarrow \mathcal{C}$ is equivalent to the category associated to a contravariant functor $F : \mathcal{C} \rightarrow \text{Groupoids}$. In the case of the fibred category associated to F we have $g^* \circ f^* = (f \circ g)^*$ on the nose and there is no need to use the maps $\alpha_{g,f}$. In this case the lemma is (even more) trivial. Of course then one uses that the $\text{Mor}(x, y)$ presheaf is unchanged when passing to an equivalent fibred category which follows from Lemma 2.3.

Lemma 2.5. *Let \mathcal{C} be a category. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category, see Categories, Section 31. Let $U \in \text{Ob}(\mathcal{C})$ and let $x, y \in \text{Ob}(\mathcal{S}_U)$. Denote $x, y : \mathcal{C}/U \rightarrow \mathcal{S}$ also the corresponding 1-morphisms, see Categories, Lemma 39.1. Then*

- (1) *the 2-fibre product $\mathcal{S} \times_{\mathcal{S} \times \mathcal{S}, (x, y)} \mathcal{C}/U$ is fibred in setoids over \mathcal{C}/U , and*
- (2) *$\text{Isom}(x, y)$ is the presheaf of sets corresponding to this category fibred in setoids, see Categories, Lemma 37.6.*

Proof. Omitted. Hint: Objects of the 2-fibre product are $(a : V \rightarrow U, z, a : V \rightarrow U, (\alpha, \beta))$ where $\alpha : z \rightarrow a^*x$ and $\beta : z \rightarrow a^*y$ are isomorphisms in \mathcal{S}_V . Thus the relationship with $\text{Isom}(x, y)$ comes by assigning to such an object the isomorphism $\beta \circ \alpha^{-1}$. \square

3. Descent data in fibred categories

In this section we define the notion of a descent datum in the abstract setting of a fibred category. Before we do so we point out that this is completely analogous to descent data for quasi-coherent sheaves (Descent, Section 2) and descent data for schemes over schemes (Descent, Section 30).

We will use the convention where the projection maps $\text{pr}_i : X \times \dots \times X \rightarrow X$ are labeled starting with $i = 0$. Hence we have $\text{pr}_0, \text{pr}_1 : X \times X \rightarrow X$, $\text{pr}_0, \text{pr}_1, \text{pr}_2 : X \times X \times X \rightarrow X$, etc.

Definition 3.1. Let \mathcal{C} be a category. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category. Make a choice of pullbacks as in Categories, Definition 31.6. Let $\mathcal{U} = \{f_i : U_i \rightarrow U\}_{i \in I}$ be a family of morphisms of \mathcal{C} . Assume all the fibre products $U_i \times_U U_j$, and $U_i \times_U U_j \times_U U_k$ exist.

- (1) A descent datum (X_i, φ_{ij}) in \mathcal{S} relative to the family $\{f_i : U_i \rightarrow U\}$ is given by an object X_i of \mathcal{S}_{U_i} for each $i \in I$, an isomorphism $\varphi_{ij} : \text{pr}_0^* X_i \rightarrow \text{pr}_1^* X_j$

in $\mathcal{S}_{U_i \times_U U_j}$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{array}{ccc} \mathrm{pr}_0^* X_i & \xrightarrow{\quad \mathrm{pr}_{02}^* \varphi_{ik} \quad} & \mathrm{pr}_2^* X_k \\ & \searrow \mathrm{pr}_{01}^* \varphi_{ij} \quad \nearrow \mathrm{pr}_{12}^* \varphi_{jk} & \\ & \mathrm{pr}_1^* X_j & \end{array}$$

in the category $\mathcal{S}_{U_i \times_U U_j \times_U U_k}$ commutes. This is called the *cocycle condition*.

- (2) A *morphism* $\psi : (X_i, \varphi_{ij}) \rightarrow (X'_i, \varphi'_{ij})$ of descent data is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms $\psi_i : X_i \rightarrow X'_i$ in \mathcal{S}_{U_i} such that all the diagrams

$$\begin{array}{ccc} \mathrm{pr}_0^* X_i & \xrightarrow{\quad \varphi_{ij} \quad} & \mathrm{pr}_1^* X_j \\ \mathrm{pr}_0^* \psi_i \downarrow & & \downarrow \mathrm{pr}_1^* \psi_j \\ \mathrm{pr}_0^* X'_i & \xrightarrow{\quad \varphi'_{ij} \quad} & \mathrm{pr}_1^* X'_j \end{array}$$

in the categories $\mathcal{S}_{U_i \times_U U_j}$ commute.

- (3) The category of descent data relative to \mathcal{U} is denoted $DD(\mathcal{U})$.

The fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ will exist if each of the morphisms $f_i : U_i \rightarrow U$ is *representable*, see Categories, Definition 6.4. Recall that in a site one of the conditions for a covering $\{U_i \rightarrow U\}$ is that each of the morphisms is representable, see Sites, Definition 6.2 part (3). In fact the main interest in the definition above is where \mathcal{C} is a site and $\{U_i \rightarrow U\}$ is a covering of \mathcal{C} . However, a descent datum is just an abstract gadget that can be defined as above. This is useful: for example, given a fibred category over \mathcal{C} one can look at the collection of families with respect to which descent data are effective, and try to use these as the family of coverings for a site.

Remarks 3.2. Two remarks on Definition 3.1 are in order. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category. Let $\{f_i : U_i \rightarrow U\}_{i \in I}$, and (X_i, φ_{ij}) be as in Definition 3.1.

- (1) There is a diagonal morphism $\Delta : U_i \rightarrow U_i \times_U U_i$. We can pull back φ_{ii} via this morphism to get an automorphism $\Delta^* \varphi_{ii} \in \mathrm{Aut}_{U_i}(x_i)$. On pulling back the cocycle condition for the triple (i, i, i) by $\Delta_{123} : U_i \rightarrow U_i \times_U U_i \times_U U_i$ we deduce that $\Delta^* \varphi_{ii} \circ \Delta^* \varphi_{ii} = \Delta^* \varphi_{ii}$; thus $\Delta^* \varphi_{ii} = \mathrm{id}_{x_i}$.
- (2) There is a morphism $\Delta_{13} : U_i \times_U U_j \rightarrow U_i \times_U U_j \times_U U_i$ and we can pull back the cocycle condition for the triple (i, j, i) to get the identity $(\sigma^* \varphi_{ji}) \circ \varphi_{ij} = \mathrm{id}_{\mathrm{pr}_0^* x_i}$, where $\sigma : U_i \times_U U_j \rightarrow U_j \times_U U_i$ is the switching morphism.

Lemma 3.3. (*Pullback of descent data.*) Let \mathcal{C} be a category. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category. Make a choice pullbacks as in Categories, Definition 31.6. Let $\mathcal{U} = \{f_i : U_i \rightarrow U\}_{i \in I}$, and $\mathcal{V} = \{V_j \rightarrow V\}_{j \in J}$ be a families of morphisms of \mathcal{C} with fixed target. Assume all the fibre products $U_i \times_U U_{i'}$, $U_i \times_U U_{i'} \times_U U_{i''}$, $V_j \times_V V_{j'}$, and $V_j \times_V V_{j'} \times_V V_{j''}$ exist. Let $\alpha : I \rightarrow J$, $h : U \rightarrow V$ and $g_i : U_i \rightarrow V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 8.1.

- (1) Let $(Y_j, \varphi_{jj'})$ be a descent datum relative to the family $\{V_j \rightarrow V\}$. The system

$$(g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')})$$

is a descent datum relative to \mathcal{U} .

- (2) This construction defines a functor between descent data relative to \mathcal{V} and descent data relative to \mathcal{U} .
- (3) Given a second $\alpha' : I \rightarrow J$, $h' : U \rightarrow V$ and $g'_i : U_i \rightarrow V_{\alpha'(i)}$ morphism of families of maps with fixed target, then if $h = h'$ the two resulting functors between descent data are canonically isomorphic.

Proof. Omitted. □

Definition 3.4. With $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$, $\mathcal{V} = \{V_j \rightarrow V\}_{j \in J}$, $\alpha : I \rightarrow J$, $h : U \rightarrow V$, and $g_i : U_i \rightarrow V_{\alpha(i)}$ as in Lemma 3.3 the functor

$$(Y_j, \varphi_{jj'}) \longmapsto (g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')})$$

constructed in that lemma is called the *pullback functor* on descent data.

Given $h : U \rightarrow V$, if there exists a morphism $\tilde{h} : \mathcal{U} \rightarrow \mathcal{V}$ covering h then \tilde{h}^* is independent of the choice of \tilde{h} as we saw in Lemma 3.3. Hence we will sometimes simply write h^* to indicate the pullback functor.

Definition 3.5. Let \mathcal{C} be a category. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category. Make a choice of pullbacks as in Categories, Definition 31.6. Let $\mathcal{U} = \{f_i : U_i \rightarrow U\}_{i \in I}$ be a family of morphisms with target U . Assume all the fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ exist.

- (1) Given an object X of \mathcal{S}_U the *trivial descent datum* is the descent datum (X, id_X) with respect to the family $\{\text{id}_U : U \rightarrow U\}$.
- (2) Given an object X of \mathcal{S}_U we have a *canonical descent datum* on the family of objects $f_i^* X$ by pulling back the trivial descent datum (X, id_X) via the obvious map $\{f_i : U_i \rightarrow U\} \rightarrow \{\text{id}_U : U \rightarrow U\}$. We denote this descent datum $(f_i^* X, \text{can})$.
- (3) A descent datum (X_i, φ_{ij}) relative to $\{f_i : U_i \rightarrow U\}$ is called *effective* if there exists an object X of \mathcal{S}_U such that (X_i, φ_{ij}) is isomorphic to $(f_i^* X, \text{can})$.

Note that the rule that associates to $X \in \mathcal{S}_U$ its canonical descent datum relative to \mathcal{U} defines a functor

$$\mathcal{S}_U \longrightarrow DD(\mathcal{U}).$$

A descent datum is effective if and only if it is in the essential image of this functor. Let us make explicit the canonical descent datum as follows.

Lemma 3.6. *In the situation of Definition 3.5 part (2) the maps $\text{can}_{ij} : pr_0^* f_i^* X \rightarrow pr_1^* f_j^* X$ are equal to $(\alpha_{pr_1, f_j})_X \circ (\alpha_{pr_0, f_i})_X^{-1}$ where $\alpha_{\cdot, \cdot}$ is as in Categories, Lemma 31.7 and where we use the equality $f_i \circ pr_0 = f_j \circ pr_1$ as maps $U_i \times_U U_j \rightarrow U$.*

Proof. Omitted. □

4. Stacks

Here is the definition of a stack. It mixes the notion of a fibred category with the notion of descent.

Definition 4.1. Let \mathcal{C} be a site. A *stack* over \mathcal{C} is a category $p : \mathcal{S} \rightarrow \mathcal{C}$ over \mathcal{C} which satisfies the following conditions:

- (1) $p : \mathcal{S} \rightarrow \mathcal{C}$ is a fibred category, see Categories, Definition 31.5,

- (2) for any $U \in \text{Ob}(\mathcal{C})$ and any $x, y \in \mathcal{S}_U$ the presheaf $\text{Mor}(x, y)$ (see Definition 2.2) is a sheaf on the site \mathcal{C}/U , and
- (3) for any covering $\mathcal{U} = \{f_i : U_i \rightarrow U\}_{i \in I}$ of the site \mathcal{C} , any descent datum in \mathcal{S} relative to \mathcal{U} is effective.

We find the formulation above the most convenient way to think about a stack. Namely, given a category over \mathcal{C} in order to verify that it is a stack you proceed to check properties (1), (2) and (3) in that order. Certainly properties (2) and (3) do not make sense if the category isn't fibred. Without (2) we cannot prove that the descent in (3) is unique up to unique isomorphism and functorial.

The following lemma provides an alternative definition.

Lemma 4.2. *Let \mathcal{C} be a site. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category over \mathcal{C} . The following are equivalent*

- (1) \mathcal{S} is a stack over \mathcal{C} , and
- (2) for any covering $\mathcal{U} = \{f_i : U_i \rightarrow U\}_{i \in I}$ of the site \mathcal{C} the functor

$$\mathcal{S}_U \longrightarrow DD(\mathcal{U})$$

which associates to an object its canonical descent datum is an equivalence.

Proof. Omitted. □

Lemma 4.3. *Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a stack over the site \mathcal{C} . Let \mathcal{S}' be a subcategory of \mathcal{S} . Assume*

- (1) if $\varphi : y \rightarrow x$ is a strongly cartesian morphism of \mathcal{S} and x is an object of \mathcal{S}' , then y is isomorphic to an object of \mathcal{S}' ,
- (2) \mathcal{S}' is a full subcategory of \mathcal{S} , and
- (3) if $\{f_i : U_i \rightarrow U\}$ is a covering of \mathcal{C} , and x an object of \mathcal{S} over U such that f_i^*x is isomorphic to an object of \mathcal{S}' for each i , then x is isomorphic to an object of \mathcal{S}' .

Then $\mathcal{S}' \rightarrow \mathcal{C}$ is a stack.

Proof. Omitted. Hints: The first condition guarantees that \mathcal{S}' is a fibred category. The second condition guarantees that the *Isom*-presheaves of \mathcal{S}' are sheaves (as they are identical to their counter parts in \mathcal{S}). The third condition guarantees that the descent condition holds in \mathcal{S}' as we can first descend in \mathcal{S} and then (3) implies the resulting object is isomorphic to an object of \mathcal{S}' . □

Lemma 4.4. *Let \mathcal{C} be a site. Let $\mathcal{S}_1, \mathcal{S}_2$ be categories over \mathcal{C} . Suppose that \mathcal{S}_1 and \mathcal{S}_2 are equivalent as categories over \mathcal{C} . Then \mathcal{S}_1 is a stack over \mathcal{C} if and only if \mathcal{S}_2 is a stack over \mathcal{C} .*

Proof. Let $F : \mathcal{S}_1 \rightarrow \mathcal{S}_2, G : \mathcal{S}_2 \rightarrow \mathcal{S}_1$ be functors over \mathcal{C} , and let $i : F \circ G \rightarrow \text{id}_{\mathcal{S}_2}, j : G \circ F \rightarrow \text{id}_{\mathcal{S}_1}$ be isomorphisms of functors over \mathcal{C} . By Categories, Lemma 31.8 we see that \mathcal{S}_1 is fibred if and only if \mathcal{S}_2 is fibred over \mathcal{C} . Hence we may assume that both \mathcal{S}_1 and \mathcal{S}_2 are fibred. Moreover, the proof of Categories, Lemma 31.8 shows that F and G map strongly cartesian morphisms to strongly cartesian morphisms, i.e., F and G are 1-morphisms of fibred categories over \mathcal{C} . This means that given $U \in \text{Ob}(\mathcal{C})$, and $x, y \in \mathcal{S}_{1,U}$ then the presheaves

$$\text{Mor}_{\mathcal{S}_1}(x, y), \text{Mor}_{\mathcal{S}_1}(F(x), F(y)) : (\mathcal{C}/U)^{opp} \longrightarrow \text{Sets}.$$

are identified, see Lemma 2.3. Hence the first is a sheaf if and only if the second is a sheaf. Finally, we have to show that if every descent datum in \mathcal{S}_1 is effective, then so is every descent datum in \mathcal{S}_2 . To do this, let $(X_i, \varphi_{ii'})$ be a descent datum in \mathcal{S}_2 relative the covering $\{U_i \rightarrow U\}$ of the site \mathcal{C} . Then $(G(X_i), G(\varphi_{ii'}))$ is a descent datum in \mathcal{S}_1 relative the covering $\{U_i \rightarrow U\}$. Let X be an object of $\mathcal{S}_{1,U}$ such that the descent datum $(f_i^* X, \text{can})$ is isomorphic to $(G(X_i), G(\varphi_{ii'}))$. Then $F(X)$ is an object of $\mathcal{S}_{2,U}$ such that the descent datum $(f_i^* F(X), \text{can})$ is isomorphic to $(F(G(X_i)), F(G(\varphi_{ii'})))$ which in turn is isomorphic to the original descent datum $(X_i, \varphi_{ii'})$ using i . \square

The 2-category of stacks over \mathcal{C} is defined as follows.

Definition 4.5. Let \mathcal{C} be a site. The 2-category of stacks over \mathcal{C} is the sub 2-category of the 2-category of fibred categories over \mathcal{C} (see Categories, Definition 31.9) defined as follows:

- (1) Its objects will be stacks $p : \mathcal{S} \rightarrow \mathcal{C}$.
- (2) Its 1-morphisms $(\mathcal{S}, p) \rightarrow (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \rightarrow \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.
- (3) Its 2-morphisms $t : G \rightarrow H$ for $G, H : (\mathcal{S}, p) \rightarrow (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \text{id}_{p(x)}$ for all $x \in \text{Ob}(\mathcal{S})$.

Lemma 4.6. Let \mathcal{C} be a site. The (2,1)-category of stacks over \mathcal{C} has 2-fibre products, and they are described as in Categories, Lemma 30.3.

Proof. Let $f : \mathcal{X} \rightarrow \mathcal{S}$ and $g : \mathcal{Y} \rightarrow \mathcal{S}$ be 1-morphisms of stacks over \mathcal{C} as defined above. The category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ described in Categories, Lemma 30.3 is a fibred category according to Categories, Lemma 31.10. (This is where we use that f and g preserve strongly cartesian morphisms.) It remains to show that the morphism presheaves are sheaves and that descent relative to coverings of \mathcal{C} is effective.

Recall that an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by a quadruple (U, x, y, ϕ) . It lies over the object U of \mathcal{C} . Next, let (U, x', y', ϕ') be second object lying over U . Recall that $\phi : f(x) \rightarrow g(y)$, and $\phi' : f(x') \rightarrow g(y')$ are isomorphisms in the category \mathcal{S}_U . Let us use these isomorphisms to identify $z = f(x) = g(y)$ and $z' = f(x') = g(y')$. With this identifications it is clear that

$$\text{Mor}((U, x, y, \phi), (U, x', y', \phi')) = \text{Mor}(x, x') \times_{\text{Mor}(z, z')} \text{Mor}(y, y')$$

as presheaves. However, as the fibred product in the category of presheaves preserves sheaves (Sites, Lemma 10.1) we see that this is a sheaf.

Let $\mathcal{U} = \{f_i : U_i \rightarrow U\}_{i \in I}$ be a covering of the site \mathcal{C} . Let (X_i, χ_{ij}) be a descent datum in $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ relative to \mathcal{U} . Write $X_i = (U_i, x_i, y_i, \phi_i)$ as above. Write $\chi_{ij} = (\varphi_{ij}, \psi_{ij})$ as in the definition of the category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ (see Categories, Lemma 30.3). It is clear that (x_i, φ_{ij}) is a descent datum in \mathcal{X} and that (y_i, ψ_{ij}) is a descent datum in \mathcal{Y} . Since \mathcal{X} and \mathcal{Y} are stacks these descent data are effective. Thus we get $x \in \text{Ob}(\mathcal{X}_U)$, and $y \in \text{Ob}(\mathcal{Y}_U)$ with $x_i = x|_{U_i}$, and $y_i = y|_{U_i}$ compatibly with descent data. Set $z = f(x)$ and $z' = g(y)$ which are both objects of \mathcal{S}_U . The morphisms ϕ_i are elements of $\text{Isom}(z, z')(U_i)$ with the property that $\phi_i|_{U_i \times_U U_j} = \phi_j|_{U_i \times_U U_j}$. Hence by the sheaf property of $\text{Isom}(z, z')$ we obtain an isomorphism $\phi : z = f(x) \rightarrow z' = g(y)$. We omit the verification that the canonical descent

datum associated to the object (U, x, y, ϕ) of $(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})_U$ is isomorphic to the descent datum we started with. \square

Lemma 4.7. *Let \mathcal{C} be a site. Let $\mathcal{S}_1, \mathcal{S}_2$ be stacks over \mathcal{C} . Let $F : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ be a 1-morphism. Then the following are equivalent*

- (1) *F is fully faithful,*
- (2) *for every $U \in \text{Ob}(\mathcal{C})$ and for every $x, y \in \text{Ob}(\mathcal{S}_{1,U})$ the map*

$$F : \text{Mor}_{\mathcal{S}_1}(x, y) \longrightarrow \text{Mor}_{\mathcal{S}_2}(F(x), F(y))$$

is an isomorphism of sheaves on \mathcal{C}/U .

Proof. Assume (1). For U, x, y as in (2) the displayed map F evaluates to the map $F : \text{Mor}_{\mathcal{S}_{1,V}}(x|_V, y|_V) \rightarrow \text{Mor}_{\mathcal{S}_{2,V}}(F(x|_V), F(y|_V))$ on an object V of \mathcal{C} lying over U . Now, since F is fully faithful, the corresponding map $\text{Mor}_{\mathcal{S}_1}(x|_V, y|_V) \rightarrow \text{Mor}_{\mathcal{S}_2}(F(x|_V), F(y|_V))$ is a bijection. Morphisms in the fibre category $\mathcal{S}_{1,V}$ are exactly those morphisms between $x|_V$ and $y|_V$ in \mathcal{S}_1 lying over id_V . Similarly, morphisms in the fibre category $\mathcal{S}_{2,V}$ are exactly those morphisms between $F(x|_V)$ and $F(y|_V)$ in \mathcal{S}_2 lying over id_V . Thus we find that F induces a bijection between these also. Hence (2) holds.

Assume (2). Suppose given objects U, V of \mathcal{C} and $x \in \text{Ob}(\mathcal{S}_{1,U})$ and $y \in \text{Ob}(\mathcal{S}_{1,V})$. To show that F is fully faithful, it suffices to prove it induces a bijection on morphisms lying over a fixed $f : U \rightarrow V$. Choose a strongly Cartesian $f^*y \rightarrow y$ in \mathcal{S}_1 lying above f . This results in a bijection between the set of morphisms $x \rightarrow y$ in \mathcal{S}_1 lying over f and $\text{Mor}_{\mathcal{S}_{1,U}}(x, f^*y)$. Since F preserves strongly Cartesian morphisms as a 1-morphism in the 2-category of stacks over \mathcal{C} , we also get a bijection between the set of morphisms $F(x) \rightarrow F(y)$ in \mathcal{S}_2 lying over f and $\text{Mor}_{\mathcal{S}_{2,U}}(F(x), F(f^*y))$. Since F induces a bijection $\text{Mor}_{\mathcal{S}_{1,U}}(x, f^*y) \rightarrow \text{Mor}_{\mathcal{S}_{2,U}}(F(x), F(f^*y))$ we conclude (1) holds. \square

Lemma 4.8. *Let \mathcal{C} be a site. Let $\mathcal{S}_1, \mathcal{S}_2$ be stacks over \mathcal{C} . Let $F : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ be a 1-morphism which is fully faithful. Then the following are equivalent*

- (1) *F is an equivalence,*
- (2) *for every $U \in \text{Ob}(\mathcal{C})$ and for every $x \in \text{Ob}(\mathcal{S}_{2,U})$ there exists a covering $\{f_i : U_i \rightarrow U\}$ such that f_i^*x is in the essential image of the functor $F : \mathcal{S}_{1,U_i} \rightarrow \mathcal{S}_{2,U_i}$.*

Proof. The implication (1) \Rightarrow (2) is immediate. To see that (2) implies (1) we have to show that every x as in (2) is in the essential image of the functor F . To do this choose a covering as in (2), $x_i \in \text{Ob}(\mathcal{S}_{1,U_i})$, and isomorphisms $\varphi_i : F(x_i) \rightarrow f_i^*x$. Then we get a descent datum for \mathcal{S}_1 relative to $\{f_i : U_i \rightarrow U\}$ by taking

$$\varphi_{ij} : x_i|_{U_i \times_U U_j} \longrightarrow x_j|_{U_i \times_U U_j}$$

the arrow such that $F(\varphi_{ij}) = \varphi_j^{-1} \circ \varphi_i$. This descent datum is effective by the axioms of a stack, and hence we obtain an object x_1 of \mathcal{S}_1 over U . We omit the verification that $F(x_1)$ is isomorphic to x over U . \square

Remark 4.9. (Cutting down a “big” stack to get a stack.) Let \mathcal{C} be a site. Suppose that $p : \mathcal{S} \rightarrow \mathcal{C}$ is functor from a “big” category to \mathcal{C} , i.e., suppose that the collection of objects of \mathcal{S} forms a proper class. Finally, suppose that $p : \mathcal{S} \rightarrow \mathcal{C}$ satisfies conditions (1), (2), (3) of Definition 4.1. In general there is no way to replace $p : \mathcal{S} \rightarrow \mathcal{C}$ by a equivalent category such that we obtain a stack. The

reason is that it can happen that a fibre categories \mathcal{S}_U may have a proper class of isomorphism classes of objects. On the other hand, suppose that

- (4) for every $U \in \text{Ob}(\mathcal{C})$ there exists a set $S_U \subset \text{Ob}(\mathcal{S}_U)$ such that every object of \mathcal{S}_U is isomorphic in \mathcal{S}_U to an element of S_U .

In this case we can find a full subcategory $\mathcal{S}_{\text{small}}$ of \mathcal{S} such that, setting $p_{\text{small}} = p|_{\mathcal{S}_{\text{small}}}$, we have

- (a) the functor $p_{\text{small}} : \mathcal{S}_{\text{small}} \rightarrow \mathcal{C}$ defines a stack, and
- (b) the inclusion $\mathcal{S}_{\text{small}} \rightarrow \mathcal{S}$ is fully faithful and essentially surjective.

(Hint: For every $U \in \text{Ob}(\mathcal{C})$ let $\alpha(U)$ denote the smallest ordinal such that $\text{Ob}(\mathcal{S}_U) \cap V_{\alpha(U)}$ surjects onto the set of isomorphism classes of \mathcal{S}_U , and set $\alpha = \sup_{U \in \text{Ob}(\mathcal{C})} \alpha(U)$. Then take $\text{Ob}(\mathcal{S}_{\text{small}}) = \text{Ob}(\mathcal{S}) \cap V_\alpha$. For notation used see Sets, Section 5.)

5. Stacks in groupoids

Among stacks those which are fibred in groupoids are somewhat easier to comprehend. We redefine them as follows.

Definition 5.1. A *stack in groupoids* over a site \mathcal{C} is a category $p : \mathcal{S} \rightarrow \mathcal{C}$ over \mathcal{C} such that

- (1) $p : \mathcal{S} \rightarrow \mathcal{C}$ is fibred in groupoids over \mathcal{C} (see Categories, Definition 33.1),
- (2) for all $U \in \text{Ob}(\mathcal{C})$, for all $x, y \in \text{Ob}(\mathcal{S}_U)$ the presheaf $\text{Isom}(x, y)$ is a sheaf on the site \mathcal{C}/U , and
- (3) for all coverings $\mathcal{U} = \{U_i \rightarrow U\}$ in \mathcal{C} , all descent data (x_i, ϕ_{ij}) for \mathcal{U} are effective.

Usually the hardest part to check is the third condition. Here is the lemma comparing this with the notion of a stack.

Lemma 5.2. Let \mathcal{C} be a site. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a category over \mathcal{C} . The following are equivalent

- (1) \mathcal{S} is a stack in groupoids over \mathcal{C} ,
- (2) \mathcal{S} is a stack over \mathcal{C} and all fibre categories are groupoids, and
- (3) \mathcal{S} is fibred in groupoids over \mathcal{C} and is a stack over \mathcal{C} .

Proof. Omitted, but see Categories, Lemma 33.2. □

Lemma 5.3. Let \mathcal{C} be a site. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a stack. Let $p' : \mathcal{S}' \rightarrow \mathcal{C}$ be the category fibred in groupoids associated to \mathcal{S} constructed in Categories, Lemma 33.3. Then $p' : \mathcal{S}' \rightarrow \mathcal{C}$ is a stack in groupoids.

Proof. Recall that the morphisms in \mathcal{S}' are exactly the strongly cartesian morphisms of \mathcal{S} , and that any isomorphism of \mathcal{S} is such a morphism. Hence descent data in \mathcal{S}' are exactly the same thing as descent data in \mathcal{S} . Now apply Lemma 4.2. Some details omitted. □

Lemma 5.4. Let \mathcal{C} be a site. Let $\mathcal{S}_1, \mathcal{S}_2$ be categories over \mathcal{C} . Suppose that \mathcal{S}_1 and \mathcal{S}_2 are equivalent as categories over \mathcal{C} . Then \mathcal{S}_1 is a stack in groupoids over \mathcal{C} if and only if \mathcal{S}_2 is a stack in groupoids over \mathcal{C} .

Proof. Follows by combining Lemmas 5.2 and 4.4. □

The 2-category of stacks in groupoids over \mathcal{C} is defined as follows.

Definition 5.5. Let \mathcal{C} be a site. The 2-category of stacks in groupoids over \mathcal{C} is the sub 2-category of the 2-category of stacks over \mathcal{C} (see Definition 4.5) defined as follows:

- (1) Its objects will be stacks in groupoids $p : \mathcal{S} \rightarrow \mathcal{C}$.
- (2) Its 1-morphisms $(\mathcal{S}, p) \rightarrow (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \rightarrow \mathcal{S}'$ such that $p' \circ G = p$. (Since every morphism is strongly cartesian every functor preserves them.)
- (3) Its 2-morphisms $t : G \rightarrow H$ for $G, H : (\mathcal{S}, p) \rightarrow (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \text{id}_{p(x)}$ for all $x \in \text{Ob}(\mathcal{S})$.

Note that any 2-morphism is automatically an isomorphism, so that in fact the 2-category of stacks in groupoids over \mathcal{C} is a (strict) $(2, 1)$ -category.

Lemma 5.6. *Let \mathcal{C} be a category. The 2-category of stacks in groupoids over \mathcal{C} has 2-fibre products, and they are described as in Categories, Lemma 30.3.*

Proof. This is clear from Categories, Lemma 33.7 and Lemmas 5.2 and 4.6. \square

6. Stacks in setoids

This is just a brief section saying that a stack in sets is the same thing as a sheaf of sets. Please consult Categories, Section 37 for notation.

Definition 6.1. Let \mathcal{C} be a site.

- (1) A *stack in setoids* over \mathcal{C} is a stack over \mathcal{C} all of whose fibre categories are setoids.
- (2) A *stack in sets*, or a *stack in discrete categories* is a stack over \mathcal{C} all of whose fibre categories are discrete.

From the discussion in Section 5 this is the same thing as a stack in groupoids whose fibre categories are setoids (resp. discrete). Moreover, it is also the same thing as a category fibred in setoids (resp. sets) which is a stack.

Lemma 6.2. *Let \mathcal{C} be a site. Under the equivalence*

$$\left\{ \begin{array}{c} \text{the category of presheaves} \\ \text{of sets over } \mathcal{C} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{the category of categories} \\ \text{fibred in sets over } \mathcal{C} \end{array} \right\}$$

of Categories, Lemma 36.6 the stacks in sets correspond precisely to the sheaves.

Proof. Omitted. Hint: Show that effectivity of descent corresponds exactly to the sheaf condition. \square

Lemma 6.3. *Let \mathcal{C} be a site. Let \mathcal{S} be a category fibred in setoids over \mathcal{C} . Then \mathcal{S} is a stack in setoids if and only if the unique equivalent category \mathcal{S}' fibred in sets (see Categories, Lemma 37.5) is a stack in sets. In other words, if and only if the presheaf*

$$U \mapsto \text{Ob}(\mathcal{S}_U) / \cong$$

is a sheaf.

Proof. Omitted. \square

Lemma 6.4. *Let \mathcal{C} be a site. Let $\mathcal{S}_1, \mathcal{S}_2$ be categories over \mathcal{C} . Suppose that \mathcal{S}_1 and \mathcal{S}_2 are equivalent as categories over \mathcal{C} . Then \mathcal{S}_1 is a stack in setoids over \mathcal{C} if and only if \mathcal{S}_2 is a stack in setoids over \mathcal{C} .*

Proof. By Categories, Lemma 37.5 we see that a category \mathcal{S} over \mathcal{C} is fibred in setoids over \mathcal{C} if and only if it is equivalent over \mathcal{C} to a category fibred in sets. Hence we see that \mathcal{S}_1 is fibred in setoids over \mathcal{C} if and only if \mathcal{S}_2 is fibred in setoids over \mathcal{C} . Hence now the lemma follows from Lemma 6.3. \square

The 2-category of stacks in setoids over \mathcal{C} is defined as follows.

Definition 6.5. Let \mathcal{C} be a site. The 2-category of stacks in setoids over \mathcal{C} is the sub 2-category of the 2-category of stacks over \mathcal{C} (see Definition 4.5) defined as follows:

- (1) Its objects will be stacks in setoids $p : \mathcal{S} \rightarrow \mathcal{C}$.
- (2) Its 1-morphisms $(\mathcal{S}, p) \rightarrow (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \rightarrow \mathcal{S}'$ such that $p' \circ G = p$. (Since every morphism is strongly cartesian every functor preserves them.)
- (3) Its 2-morphisms $t : G \rightarrow H$ for $G, H : (\mathcal{S}, p) \rightarrow (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \text{id}_{p(x)}$ for all $x \in \text{Ob}(\mathcal{S})$.

Note that any 2-morphism is automatically an isomorphism, so that in fact the 2-category of stacks in setoids over \mathcal{C} is a (strict) (2, 1)-category.

Lemma 6.6. Let \mathcal{C} be a site. The 2-category of stacks in setoids over \mathcal{C} has 2-fibre products, and they are described as in Categories, Lemma 30.3.

Proof. This is clear from Categories, Lemmas 33.7 and 37.4 and Lemmas 5.2 and 4.6. \square

Lemma 6.7. Let \mathcal{C} be a site. Let \mathcal{S}, \mathcal{T} be stacks in groupoids over \mathcal{C} and let \mathcal{R} be a stack in setoids over \mathcal{C} . Let $f : \mathcal{T} \rightarrow \mathcal{S}$ and $g : \mathcal{R} \rightarrow \mathcal{S}$ be 1-morphisms. If f is faithful, then the 2-fibre product

$$\mathcal{T} \times_{f, \mathcal{S}, g} \mathcal{R}$$

is a stack in setoids over \mathcal{C} .

Proof. Immediate from the explicit description of the 2-fibre product in Categories, Lemma 30.3. \square

Lemma 6.8. Let \mathcal{C} be a site. Let \mathcal{S} be a stack in groupoids over \mathcal{C} and let \mathcal{S}_i , $i = 1, 2$ be stacks in setoids over \mathcal{C} . Let $f_i : \mathcal{S}_i \rightarrow \mathcal{S}$ be 1-morphisms. Then the 2-fibre product

$$\mathcal{S}_1 \times_{f_1, \mathcal{S}, f_2} \mathcal{S}_2$$

is a stack in setoids over \mathcal{C} .

Proof. This is a special case of Lemma 6.7 as f_2 is faithful. \square

Lemma 6.9. Let \mathcal{C} be a site. Let

$$\begin{array}{ccc} \mathcal{T}_2 & \longrightarrow & \mathcal{T}_1 \\ G' \downarrow & & \downarrow G \\ \mathcal{S}_2 & \xrightarrow{F} & \mathcal{S}_1 \end{array}$$

be a 2-cartesian diagram of stacks in groupoids over \mathcal{C} . Assume

- (1) for every $U \in \text{Ob}(\mathcal{C})$ and $x \in \text{Ob}((\mathcal{S}_1)_U)$ there exists a covering $\{U_i \rightarrow U\}$ such that $x|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \rightarrow (\mathcal{S}_1)_{U_i}$, and

(2) G' is faithful,

then G is faithful.

Proof. We may assume that \mathcal{T}_2 is the category $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ described in Categories, Lemma 30.3. By Categories, Lemma 33.8 the faithfulness of G, G' can be checked on fibre categories. Suppose that y, y' are objects of \mathcal{T}_1 over the object U of \mathcal{C} . Let $\alpha, \beta : y \rightarrow y'$ be morphisms of $(\mathcal{T}_1)_U$ such that $G(\alpha) = G(\beta)$. Our object is to show that $\alpha = \beta$. Considering instead $\gamma = \alpha^{-1} \circ \beta$ we see that $G(\gamma) = \text{id}_{G(y)}$ and we have to show that $\gamma = \text{id}_y$. By assumption we can find a covering $\{U_i \rightarrow U\}$ such that $G(y)|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \rightarrow (\mathcal{S}_1)_{U_i}$. Since it suffices to show that $\gamma|_{U_i} = \text{id}$ for each i , we may therefore assume that we have $f : F(x) \rightarrow G(y)$ for some object x of \mathcal{S}_2 over U and morphisms f of $(\mathcal{S}_1)_U$. In this case we get a morphism

$$(1, \gamma) : (U, x, y, f) \longrightarrow (U, x, y, f)$$

in the fibre category of $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ over U whose image under G' in \mathcal{S}_1 is id_x . As G' is faithful we conclude that $\gamma = \text{id}_y$ and we win. \square

Lemma 6.10. *Let \mathcal{C} be a site. Let*

$$\begin{array}{ccc} \mathcal{T}_2 & \longrightarrow & \mathcal{T}_1 \\ \downarrow & & \downarrow G \\ \mathcal{S}_2 & \xrightarrow{F} & \mathcal{S}_1 \end{array}$$

be a 2-cartesian diagram of stacks in groupoids over \mathcal{C} . If

- (1) $F : \mathcal{S}_2 \rightarrow \mathcal{S}_1$ is fully faithful,
- (2) for every $U \in \text{Ob}(\mathcal{C})$ and $x \in \text{Ob}((\mathcal{S}_1)_U)$ there exists a covering $\{U_i \rightarrow U\}$ such that $x|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \rightarrow (\mathcal{S}_1)_{U_i}$, and
- (3) \mathcal{T}_2 is a stack in setoids.

then \mathcal{T}_1 is a stack in setoids.

Proof. We may assume that \mathcal{T}_2 is the category $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ described in Categories, Lemma 30.3. Pick $U \in \text{Ob}(\mathcal{C})$ and $y \in \text{Ob}((\mathcal{T}_1)_U)$. We have to show that the sheaf $\text{Aut}(y)$ on \mathcal{C}/U is trivial. To do this we may replace U by the members of a covering of U . Hence by assumption (2) we may assume that there exists an object $x \in \text{Ob}((\mathcal{S}_2)_U)$ and an isomorphism $f : F(x) \rightarrow G(y)$. Then $y' = (U, x, y, f)$ is an object of \mathcal{T}_2 over U which is mapped to y under the projection $\mathcal{T}_2 \rightarrow \mathcal{T}_1$. Because F is fully faithful by (1) the map $\text{Aut}(y') \rightarrow \text{Aut}(y)$ is surjective, use the explicit description of morphisms in \mathcal{T}_2 in Categories, Lemma 30.3. Since by (3) the sheaf $\text{Aut}(y')$ is trivial we get the result of the lemma. \square

7. The inertia stack

Let $p : \mathcal{S} \rightarrow \mathcal{C}$ and $p' : \mathcal{S}' \rightarrow \mathcal{C}$ be fibred categories over the category \mathcal{C} . Let $F : \mathcal{S} \rightarrow \mathcal{S}'$ be a 1-morphism of fibred categories over \mathcal{C} . Recall that we have defined in Categories, Definition 32.2 an *relative inertia fibred category* $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \rightarrow \mathcal{C}$ as the category whose objects are pairs (x, α) where $x \in \text{Ob}(\mathcal{S})$ and $\alpha : x \rightarrow x$ with $F(\alpha) = \text{id}_{F(x)}$. There is also an absolute version, namely the *inertia* $\mathcal{I}_{\mathcal{S}}$ of \mathcal{S} . These inertia categories are actually stacks over \mathcal{C} provided that \mathcal{S} and \mathcal{S}' are stacks.

Lemma 7.1. *Let \mathcal{C} be a site. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ and $p' : \mathcal{S}' \rightarrow \mathcal{C}$ be stacks over the site \mathcal{C} . Let $F : \mathcal{S} \rightarrow \mathcal{S}'$ be a 1-morphism of stacks over \mathcal{C} .*

- (1) *The inertia $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$ are stacks over \mathcal{C} .*
- (2) *If $\mathcal{S}, \mathcal{S}'$ are stacks in groupoids over \mathcal{S} , then so are $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$.*
- (3) *If $\mathcal{S}, \mathcal{S}'$ are stacks in setoids over \mathcal{S} , then so are $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$.*

Proof. The first three assertions follow from Lemmas 4.6, 5.6, and 6.6 and the equivalence in Categories, Lemma 32.1 part (1). \square

Lemma 7.2. *Let \mathcal{C} be a site. If \mathcal{S} is a stack in groupoids, then the canonical 1-morphism $\mathcal{I}_{\mathcal{S}} \rightarrow \mathcal{S}$ is an equivalence if and only if \mathcal{S} is a stack in setoids.*

Proof. Follows directly from Categories, Lemma 37.7. \square

8. Stackification of fibred categories

Here is the result.

Lemma 8.1. *Let \mathcal{C} be a site. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category over \mathcal{C} . There exists a stack $p' : \mathcal{S}' \rightarrow \mathcal{C}$ and a 1-morphism $G : \mathcal{S} \rightarrow \mathcal{S}'$ of fibred categories over \mathcal{C} (see Categories, Definition 31.9) such that*

- (1) *for every $U \in \text{Ob}(\mathcal{C})$, and any $x, y \in \text{Ob}(\mathcal{S}_U)$ the map*

$$\text{Mor}(x, y) \longrightarrow \text{Mor}(G(x), G(y))$$

induced by G identifies the right hand side with the sheafification of the left hand side, and

- (2) *for every $U \in \text{Ob}(\mathcal{C})$, and any $x' \in \text{Ob}(\mathcal{S}'_U)$ there exists a covering $\{U_i \rightarrow U\}_{i \in I}$ such that for every $i \in I$ the object $x'|_{U_i}$ is in the essential image of the functor $G : \mathcal{S}_U \rightarrow \mathcal{S}'_U$.*

Moreover the stack \mathcal{S}' is determined up to unique 2-isomorphism by these conditions.

Proof by naive method. In this proof method we proceed in stages:

First, given x lying over U and any object y of \mathcal{S} , we say that two morphisms $a, b : x \rightarrow y$ of \mathcal{S} lying over the same arrow of \mathcal{C} are *locally equal* if there exists a covering $\{f_i : U_i \rightarrow U\}$ of \mathcal{C} such that the compositions

$$f_i^* x \rightarrow x \xrightarrow{a} y, \quad f_i^* x \rightarrow x \xrightarrow{b} y$$

are equal. This gives an equivalence relation \sim on arrows of \mathcal{S} . If $b \sim b'$ then $a \circ b \circ c \sim a \circ b' \circ c$ (verification omitted). Hence we can quotient out by this equivalence relation to obtain a new category \mathcal{S}^1 over \mathcal{C} together with a morphism $G^1 : \mathcal{S} \rightarrow \mathcal{S}^1$.

One checks that G^1 preserves strongly cartesian morphisms and that \mathcal{S}^1 is a fibred category over \mathcal{C} . Checks omitted. Thus we reduce to the case where locally equal morphisms are equal.

Next, we add morphisms as follows. Given x lying over U and any object y of lying over V a *locally defined morphism* from x to y is given by

- (1) a morphism $f : U \rightarrow V$,
- (2) a covering $\{f_i : U_i \rightarrow U\}$ of U , and
- (3) morphisms $a_i : f_i^* x \rightarrow Y$ with $p(a_i) = h \circ f_i$

with the property that the compositions

$$(f_i \times f_j)^* x \rightarrow f_i^* x \xrightarrow{a_i} y, \quad (f_i \times f_j)^* x \rightarrow f_j^* x \xrightarrow{a_j} y$$

are equal. Note that a usual morphism $a : x \rightarrow y$ gives a locally defined morphism $(p(a) : U \rightarrow V, \{\text{id}_U\}, a)$. We say two locally defined morphisms $(f, \{f_i : U_i \rightarrow U\}, a_i)$ and $(g, \{g_j : U_j \rightarrow U\}, b_j)$ are *equal* if $f = g$ and the compositions

$$(f_i \times g_j)^* x \rightarrow f_i^* x \xrightarrow{a_i} y, \quad (f_i \times g_j)^* x \rightarrow g_j^* x \xrightarrow{b_j} y$$

are equal (this is the right condition since we are in the situation where locally equal morphisms are equal). To compose locally defined morphisms $(f, \{f_i : U_i \rightarrow U\}, a_i)$ from x to y and $(g, \{g_j : V_j \rightarrow V\}, b_j)$ from y to z lying over W , just take $g \circ f : U \rightarrow W$, the covering $\{U_i \times_V V_j \rightarrow U\}$, and as maps the compositions

$$x|_{U_i \times_V V_j} \xrightarrow{\text{pr}_0^* a_i} y|_{V_j} \xrightarrow{b_j} z$$

We omit the verification that this is a locally defined morphism.

One checks that \mathcal{S}^2 with the same objects as \mathcal{S} and with locally defined morphisms as morphisms is a category over \mathcal{C} , that there is a functor $G^2 : \mathcal{S} \rightarrow \mathcal{S}^2$ over \mathcal{C} , that this functor preserves strongly cartesian objects, and that \mathcal{S}^2 is a fibred category over \mathcal{C} . Checks omitted. This reduces one to the case where the morphism presheaves of \mathcal{S} are all sheaves, by checking that the effect of using locally defined morphisms is to take the sheafification of the (separated) morphisms presheaves.

Finally, in the case where the morphism presheaves are all sheaves we have to add objects in order to make sure descent conditions are effective in the end result. The simplest way to do this is to consider the category \mathcal{S}' whose objects are pairs (\mathcal{U}, ξ) where $\mathcal{U} = \{U_i \rightarrow U\}$ is a covering of \mathcal{C} and $\xi = (X_i, \varphi_{ii'})$ is a descent datum relative \mathcal{U} . Suppose given two such data $(\mathcal{U}, \xi) = (\{f_i : U_i \rightarrow U\}, x_i, \varphi_{ii'})$ and $(\mathcal{V}, \eta) = (\{g_j : V_j \rightarrow V\}, y_j, \psi_{jj'})$. We define

$$\text{Mor}_{\mathcal{S}'}((\mathcal{U}, \xi), (\mathcal{V}, \eta))$$

as the set of (f, a_{ij}) , where $f : U \rightarrow V$ and

$$a_{ij} : x_i|_{U_i \times_V V_j} \longrightarrow y_j$$

are morphisms of \mathcal{S} lying over $U_i \times_V V_j \rightarrow V_j$. These have to satisfy the following condition: for any $i, i' \in I$ and $j, j' \in J$ set $W = (U_i \times_U U_{i'}) \times_V (V_j \times_V V_{j'})$. Then

$$\begin{array}{ccc} x_i|_W & \xrightarrow{a_{ij}|_W} & y_j|_W \\ \varphi_{ii'}|_W \downarrow & & \downarrow \psi_{jj'}|_W \\ x_{i'}|_W & \xrightarrow{a_{i'j'}|_W} & y_{j'}|_W \end{array}$$

commutes. At this point you have to verify the following things:

- (1) there is a well defined composition on morphisms as above,
- (2) this turns \mathcal{S}' into a category over \mathcal{C} ,
- (3) there is a functor $G : \mathcal{S} \rightarrow \mathcal{S}'$ over \mathcal{C} ,
- (4) for x, y objects of \mathcal{S} we have $\text{Mor}_{\mathcal{S}}(x, y) = \text{Mor}_{\mathcal{S}'}(G(x), G(y))$,
- (5) any object of \mathcal{S}' locally comes from an object of \mathcal{S} , i.e., part (2) of the lemma holds,
- (6) G preserves strongly cartesian morphisms,
- (7) \mathcal{S}' is a fibred category over \mathcal{C} , and

(8) \mathcal{S}' is a stack over \mathcal{C} .

This is all not hard but there is a lot of it. Details omitted. \square

Less naive proof. Here is a less naive proof. By Categories, Lemma 34.4 there exists an equivalence of fibred categories $\mathcal{S} \rightarrow \mathcal{S}'$ where \mathcal{S}' is a split fibred category, i.e., one in which the pullback functors compose on the nose. Obviously the lemma for \mathcal{S}' implies the lemma for \mathcal{S} . Hence we may think of \mathcal{S} as a presheaf in categories.

Consider the 2-category Cat temporarily as a category by forgetting about 2-morphisms. Let us think of a category as a quintuple $(\text{Ob}, \text{Arrows}, s, t, \circ)$ as in Categories, Section 2. Consider the forgetful functor

$$\text{forget} : Cat \rightarrow \text{Sets}, \quad (\text{Ob}, \text{Arrows}, s, t, \circ) \mapsto \text{Ob} \coprod \text{Arrows}.$$

Then forget is faithful, Cat has limits and forget commutes with them, Cat has directed colimits and forget commutes with them, and forget reflects isomorphisms. Hence, according to the first part of Sites, Section 43 we can sheafify presheaves with values in Cat , and the result commutes with forget . Applying this to \mathcal{S} we obtain a sheafification $\mathcal{S}^\#$ which has a sheaf of objects and a sheaf of morphisms both of which are the sheafifications of the corresponding presheaves for \mathcal{S} . In this case it is quite easy to see that the map $\mathcal{S} \rightarrow \mathcal{S}^\#$ has the properties (1) and (2) of the lemma.

However, the category $\mathcal{S}^\#$ may not yet be a stack since, although the presheaf of objects is a sheaf, the descent condition may not yet be satisfied. To remedy this we have to add more objects. But the argument above does reduce us to the case where $\mathcal{S} = \mathcal{S}_F$ for some sheaf(!) $F : \mathcal{C}^{opp} \rightarrow Cat$ of categories. In this case consider the functor $F' : \mathcal{C}^{opp} \rightarrow Cat$ defined by

- (1) The set $\text{Ob}(F'(U))$ is the set of pairs (\mathcal{U}, ξ) where $\mathcal{U} = \{U_i \rightarrow U\}$ is a covering of U and $\xi = (x_i, \varphi_{ii'})$ is a descent datum relative to \mathcal{U} .
- (2) A morphism in $F'(U)$ from (\mathcal{U}, ξ) to (\mathcal{V}, η) is an element of

$$\text{colim Mor}_{DD(\mathcal{W})}(a^*\xi, b^*\eta)$$

where the colimit is over all common refinements $a : \mathcal{W} \rightarrow \mathcal{U}$, $b : \mathcal{W} \rightarrow \mathcal{V}$. This colimit is filtered (verification omitted). Hence composition of morphisms in $F(U)$ is defined by finding a common refinement and composing in $DD(\mathcal{W})$.

- (3) Given $h : V \rightarrow U$ and an object (\mathcal{U}, ξ) of $F'(U)$ we set $F'(h)(\mathcal{U}, \xi)$ equal to $(V \times_U \mathcal{U}, \text{pr}_1^*\xi)$. More precisely, if $\mathcal{U} = \{U_i \rightarrow U\}$ and $\xi = (x_i, \varphi_{ii'})$, then $V \times_U \mathcal{U} = \{V \times_U U_i \rightarrow V\}$ which comes with a canonical morphism $\text{pr}_1 : V \times_U \mathcal{U} \rightarrow \mathcal{U}$ and $\text{pr}_1^*\xi$ is the pullback of ξ with respect to this morphism (see Definition 3.4).
- (4) Given $h : V \rightarrow U$, objects (\mathcal{U}, ξ) and (\mathcal{V}, η) and a morphism between them, represented by $a : \mathcal{W} \rightarrow \mathcal{U}$, $b : \mathcal{W} \rightarrow \mathcal{V}$, and $\alpha : a^*\xi \rightarrow b^*\eta$, then $F'(h)(\alpha)$ is represented by $a' : V \times_U \mathcal{W} \rightarrow V \times_U \mathcal{U}$, $b' : V \times_U \mathcal{W} \rightarrow V \times_U \mathcal{V}$, and the pullback α' of the morphism α via the map $V \times_U \mathcal{W} \rightarrow \mathcal{W}$. This works since pullbacks in \mathcal{S}_F commute on the nose.

There is a map $F \rightarrow F'$ given by associating to an object x of $F(U)$ the object $(\{U \rightarrow U\}, (x, \text{triv}))$ of $F'(U)$. At this point you have to check that the corresponding functor $\mathcal{S}_F \rightarrow \mathcal{S}_{F'}$ has properties (1) and (2) of the lemma, and finally that $\mathcal{S}_{F'}$ is a stack. Details omitted. \square

Lemma 8.2. *Let \mathcal{C} be a site. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category over \mathcal{C} . Let $p' : \mathcal{S}' \rightarrow \mathcal{C}$ and $G : \mathcal{S} \rightarrow \mathcal{S}'$ the stack and 1-morphism constructed in Lemma 8.1. This construction has the following universal property: Given a stack $q : \mathcal{X} \rightarrow \mathcal{C}$ and a 1-morphism $F : \mathcal{S} \rightarrow \mathcal{S}'$ of fibred categories over \mathcal{C} there exists a 1-morphism $H : \mathcal{S}' \rightarrow \mathcal{X}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{F} & \mathcal{X} \\ & \searrow G & \nearrow H \\ & \mathcal{S}' & \end{array}$$

is 2-commutative.

Proof. Omitted. Hint: Suppose that $x' \in \text{Ob}(\mathcal{S}'_U)$. By the result of Lemma 8.1 there exists a covering $\{U_i \rightarrow U\}_{i \in I}$ such that $x'|_{U_i} = G(x_i)$ for some $x_i \in \text{Ob}(\mathcal{S}_{U_i})$. Moreover, there exist coverings $\{U_{ijk} \rightarrow U_i \times_U U_j\}$ and isomorphisms $\alpha_{ijk} : x_i|_{U_{ijk}} \rightarrow x_j|_{U_{ijk}}$ with $G(\alpha_{ijk}) = \text{id}_{x'|_{U_{ijk}}}$. Set $y_i = F(x_i)$. Then you can check that

$$F(\alpha_{ijk}) : y_i|_{U_{ijk}} \rightarrow y_j|_{U_{ijk}}$$

agree on overlaps and therefore (as \mathcal{X} is a stack) define a morphism $\beta_{ij} : y_i|_{U_i \times_U U_j} \rightarrow y_j|_{U_i \times_U U_j}$. Next, you check that the β_{ij} define a descent datum. Since \mathcal{X} is a stack these descent data are effective and we find an object y of \mathcal{X}_U agreeing with $G(x_i)$ over U_i . The hint is to set $H(x') = y$. \square

Lemma 8.3. *Notation and assumptions as in Lemma 8.2. There is a canonical equivalence of categories*

$$\text{Mor}_{\text{Fib}/\mathcal{C}}(\mathcal{S}, \mathcal{X}) = \text{Mor}_{\text{Stacks}/\mathcal{C}}(\mathcal{S}', \mathcal{X})$$

given by the constructions in the proof of the aforementioned lemma.

Proof. Omitted. \square

Lemma 8.4. *Let \mathcal{C} be a site. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $g : \mathcal{Z} \rightarrow \mathcal{Y}$ be morphisms of fibred categories over \mathcal{C} . In this case the stackification of the 2-fibre product is the 2-fibre product of the stackifications.*

Proof. Let us denote $\mathcal{X}', \mathcal{Y}', \mathcal{Z}'$ the stackifications and \mathcal{W} the stackification of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$. By construction of 2-fibre products there is a canonical 1-morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$. As the second 2-fibre product is a stack (see Lemma 4.6) this 1-morphism induces a 1-morphism $h : \mathcal{W} \rightarrow \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ by the universal property of stackification, see Lemma 8.2. Now h is a morphism of stacks, and we may check that it is an equivalence using Lemmas 4.7 and 4.8.

Thus we first prove that h induces isomorphisms of *Mor*-sheaves. Let ξ, ξ' be objects of \mathcal{W} over $U \in \text{Ob}(\mathcal{C})$. We want to show that

$$h : \text{Mor}(\xi, \xi') \longrightarrow \text{Mor}(h(\xi), h(\xi'))$$

is an isomorphism. To do this we may work locally on U (see Sites, Section 25). Hence by construction of \mathcal{W} (see Lemma 8.1) we may assume that ξ, ξ' actually come from objects (x, y, α) and (x', y', α') of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U . By the same lemma once more we see that in this case $\text{Mor}(\xi, \xi')$ is the sheafification of

$$V/U \longmapsto \text{Mor}_{\mathcal{X}_V}(x|_V, x'|_V) \times_{\text{Mor}_{\mathcal{Z}_V}(f(x)|_V, f(x')|_V)} \text{Mor}_{\mathcal{Y}_V}(y|_V, y'|_V)$$

and that $Mor(h(\xi), h(\xi'))$ is equal to the fibre product

$$Mor(i(x), i(x')) \times_{Mor(k(f(x)), k(f(x'))} Mor(j(x), j(x'))$$

where $i : \mathcal{X} \rightarrow \mathcal{X}'$, $j : \mathcal{Y} \rightarrow \mathcal{Y}'$, and $k : \mathcal{Z} \rightarrow \mathcal{Z}'$ are the canonical functors. Thus the first displayed map of this paragraph is an isomorphism as sheafification is exact (and hence the sheafification of a fibre product of presheaves is the fibre product of the sheafifications).

Finally, we have to check that any object of $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ over U is locally on U in the essential image of h . Write such an object as a triple (x', y', α) . Then x' locally comes from an object of \mathcal{X} , y' locally comes from an object of \mathcal{Y} , and having made suitable replacements for x' , y' the morphism α of \mathcal{Z}'_U locally comes from a morphism of \mathcal{Z} . In other words, we have shown that any object of $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ over U is locally on U in the essential image of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$, hence a fortiori it is locally in the essential image of h . \square

Lemma 8.5. *Let \mathcal{C} be a site. Let \mathcal{X} be a fibred category over \mathcal{C} . The stackification of the inertia fibred category $\mathcal{I}_{\mathcal{X}}$ is inertia of the stackification of \mathcal{X} .*

Proof. This follows from the fact that stackification is compatible with 2-fibre products by Lemma 8.4 and the fact that there is a formula for the inertia in terms of 2-fibre products of categories over \mathcal{C} , see Categories, Lemma 32.1. \square

9. Stackification of categories fibred in groupoids

Here is the result.

Lemma 9.1. *Let \mathcal{C} be a site. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a category fibred in groupoids over \mathcal{C} . There exists a stack in groupoids $p' : \mathcal{S}' \rightarrow \mathcal{C}$ and a 1-morphism $G : \mathcal{S} \rightarrow \mathcal{S}'$ of categories fibred in groupoids over \mathcal{C} (see Categories, Definition 33.6) such that*

- (1) *for every $U \in \text{Ob}(\mathcal{C})$, and any $x, y \in \text{Ob}(\mathcal{S}_U)$ the map*

$$Mor(x, y) \longrightarrow Mor(G(x), G(y))$$

induced by G identifies the right hand side with the sheafification of the left hand side, and

- (2) *for every $U \in \text{Ob}(\mathcal{C})$, and any $x' \in \text{Ob}(\mathcal{S}'_U)$ there exists a covering $\{U_i \rightarrow U\}_{i \in I}$ such that for every $i \in I$ the object $x'|_{U_i}$ is in the essential image of the functor $G : \mathcal{S}_{U_i} \rightarrow \mathcal{S}'_{U_i}$.*

Moreover the stack in groupoids \mathcal{S}' is determined up to unique 2-isomorphism by these conditions.

Proof. Apply Lemma 8.1. The result will be a stack in groupoids by applying Lemma 5.2. \square

Lemma 9.2. *Let \mathcal{C} be a site. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a category fibred in groupoids over \mathcal{C} . Let $p' : \mathcal{S}' \rightarrow \mathcal{C}$ and $G : \mathcal{S} \rightarrow \mathcal{S}'$ the stack in groupoids and 1-morphism constructed in Lemma 9.1. This construction has the following universal property: Given a stack in groupoids $q : \mathcal{X} \rightarrow \mathcal{C}$ and a 1-morphism $F : \mathcal{S} \rightarrow \mathcal{X}$ of categories over \mathcal{C} there exists a 1-morphism $H : \mathcal{S}' \rightarrow \mathcal{X}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\quad F \quad} & \mathcal{X} \\ & \searrow G \quad \nearrow H & \\ & \mathcal{S}' & \end{array}$$

is 2-commutative.

Proof. This is a special case of Lemma 8.2. \square

Lemma 9.3. *Let \mathcal{C} be a site. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $g : \mathcal{Y} \rightarrow \mathcal{Z}$ be morphisms categories fibred in groupoids over \mathcal{C} . In this case the stackification of the 2-fibre product is the 2-fibre product of the stackifications.*

Proof. This is a special case of Lemma 8.4. \square

10. Inherited topologies

It turns out that a fibred category over a site inherits a canonical topology from the underlying site.

Lemma 10.1. *Let \mathcal{C} be a site. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category. Let $\text{Cov}(\mathcal{S})$ be the set of families $\{x_i \rightarrow x\}_{i \in I}$ of morphisms in \mathcal{S} with fixed target such that (a) each $x_i \rightarrow x$ is strongly cartesian, and (b) $\{p(x_i) \rightarrow p(x)\}_{i \in I}$ is a covering of \mathcal{C} . Then $(\mathcal{S}, \text{Cov}(\mathcal{S}))$ is a site.*

Proof. We have to check the three conditions of Sites, Definition 6.2.

- (1) If $x \rightarrow y$ is an isomorphism of \mathcal{S} , then it is strongly cartesian by Categories, Lemma 31.2 and $p(x) \rightarrow p(y)$ is an isomorphism of \mathcal{C} . Thus $\{p(x) \rightarrow p(y)\}$ is a covering of \mathcal{C} whence $\{x \rightarrow y\} \in \text{Cov}(\mathcal{S})$.
- (2) If $\{x_i \rightarrow x\}_{i \in I} \in \text{Cov}(\mathcal{S})$ and for each i we have $\{y_{ij} \rightarrow x_i\}_{j \in J_i} \in \text{Cov}(\mathcal{S})$, then each composition $p(y_{ij}) \rightarrow p(x)$ is strongly cartesian by Categories, Lemma 31.2 and $\{p(y_{ij}) \rightarrow p(x)\}_{i \in I, j \in J_i} \in \text{Cov}(\mathcal{C})$. Hence also $\{y_{ij} \rightarrow x\}_{i \in I, j \in J_i} \in \text{Cov}(\mathcal{S})$.
- (3) Suppose $\{x_i \rightarrow x\}_{i \in I} \in \text{Cov}(\mathcal{S})$ and $y \rightarrow x$ is a morphism of \mathcal{S} . As $\{p(x_i) \rightarrow p(x)\}$ is a covering of \mathcal{C} we see that $p(x_i) \times_{p(x)} p(y)$ exists. Hence Categories, Lemma 31.13 implies that $x_i \times_x y$ exists, that $p(x_i \times_x y) = p(x_i) \times_{p(x)} p(y)$, and that $x_i \times_x y \rightarrow y$ is strongly cartesian. Since also $\{p(x_i) \times_{p(x)} p(y) \rightarrow p(y)\}_{i \in I} \in \text{Cov}(\mathcal{C})$ we conclude that $\{x_i \times_x y \rightarrow y\}_{i \in I} \in \text{Cov}(\mathcal{S})$.

This finishes the proof. \square

Note that if $p : \mathcal{S} \rightarrow \mathcal{C}$ is fibred in groupoids, then the coverings of the site \mathcal{S} in Lemma 10.1 are characterized by

$$\{x_i \rightarrow x\} \in \text{Cov}(\mathcal{S}) \Leftrightarrow \{p(x_i) \rightarrow p(x)\} \in \text{Cov}(\mathcal{C})$$

because every morphism of \mathcal{S} is strongly cartesian.

Definition 10.2. Let \mathcal{C} be a site. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a fibred category. We say $(\mathcal{S}, \text{Cov}(\mathcal{S}))$ as in Lemma 10.1 is the *structure of site on \mathcal{S} inherited from \mathcal{C}* . We sometimes indicate this by saying that \mathcal{S} is *endowed with the topology inherited from \mathcal{C}* .

In particular we obtain a topos of sheaves $\text{Sh}(\mathcal{S})$ in this situation. It turns out that this topos is functorial with respect to 1-morphisms of fibred categories.

Lemma 10.3. *Let \mathcal{C} be a site. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of fibred categories over \mathcal{C} . Then F is a continuous and cocontinuous functor between the structure of sites inherited from \mathcal{C} . Hence F induces a morphism of topoi $f : \text{Sh}(\mathcal{X}) \rightarrow \text{Sh}(\mathcal{Y})$ with $f_* = {}_s F = {}_p F$ and $f^{-1} = F^s = F^p$. In particular $f^{-1}(\mathcal{G})(x) = \mathcal{G}(F(x))$ for a sheaf \mathcal{G} on \mathcal{Y} and object x of \mathcal{X} .*

Proof. We first prove that F is continuous. Let $\{x_i \rightarrow x\}_{i \in I}$ be a covering of \mathcal{X} . By Categories, Definition 31.9 the functor F transforms strongly cartesian morphisms into strongly cartesian morphisms, hence $\{F(x_i) \rightarrow F(x)\}_{i \in I}$ is a covering of \mathcal{Y} . This proves part (1) of Sites, Definition 14.1. Moreover, let $x' \rightarrow x$ be a morphism of \mathcal{X} . By Categories, Lemma 31.13 the fibre product $x_i \times_x x'$ exists and $x_i \times_x x' \rightarrow x'$ is strongly cartesian. Hence $F(x_i \times_x x') \rightarrow F(x')$ is strongly cartesian. By Categories, Lemma 31.13 applied to \mathcal{Y} this means that $F(x_i \times_x x') = F(x_i) \times_{F(x)} F(x')$. This proves part (2) of Sites, Definition 14.1 and we conclude that F is continuous.

Next we prove that F is cocontinuous. Let $x \in \text{Ob}(\mathcal{X})$ and let $\{y_i \rightarrow F(x)\}_{i \in I}$ be a covering in \mathcal{Y} . Denote $\{U_i \rightarrow U\}_{i \in I}$ the corresponding covering of \mathcal{C} . For each i choose a strongly cartesian morphism $x_i \rightarrow x$ in \mathcal{X} lying over $U_i \rightarrow U$. Then $F(x_i) \rightarrow F(x)$ and $y_i \rightarrow F(x)$ are both a strongly cartesian morphisms in \mathcal{Y} lying over $U_i \rightarrow U$. Hence there exists a unique isomorphism $F(x_i) \rightarrow y_i$ in \mathcal{Y}_{U_i} compatible with the maps to $F(x)$. Thus $\{x_i \rightarrow x\}_{i \in I}$ is a covering of \mathcal{X} such that $\{F(x_i) \rightarrow F(x)\}_{i \in I}$ is isomorphic to $\{y_i \rightarrow F(x)\}_{i \in I}$. Hence F is cocontinuous, see Sites, Definition 19.1.

The final assertion follows from the first two, see Sites, Lemmas 20.1, 19.2, and 20.5. \square

Lemma 10.4. *Let \mathcal{C} be a site. Let $p : \mathcal{X} \rightarrow \mathcal{C}$ and $q : \mathcal{Y} \rightarrow \mathcal{C}$ be stacks in groupoids. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories over \mathcal{C} . If F turns \mathcal{X} into a category fibred in groupoids over \mathcal{Y} , then \mathcal{X} is a stack in groupoids over \mathcal{Y} (with topology inherited from \mathcal{C}).*

Proof. Let us prove descent for objects. Let $\{y_i \rightarrow y\}$ be a covering of \mathcal{Y} . Let (x_i, φ_{ij}) be a descent datum in \mathcal{X} with respect to this covering. Then (x_i, φ_{ij}) is also a descent datum with respect to the covering $\{q(y_i) \rightarrow q(y)\}$ of \mathcal{C} . As \mathcal{X} is a stack in groupoids we obtain an object x over $q(y)$ and isomorphisms $\psi_i : x|_{q(y_i)} \rightarrow x_i$ over $q(y_i)$ compatible with the φ_{ij} , i.e., such that

$$\varphi_{ij} = \psi_j|_{q(y_i) \times_{q(y)} q(y_j)} \circ \psi_i^{-1}|_{q(y_i) \times_{q(y)} q(y_j)}.$$

Consider the sheaf $I = \text{Isom}_{\mathcal{Y}}(F(x), y)$ on $\mathcal{C}/p(x)$. Note that $s_i = F(\psi_i) \in I(q(x_i))$ because $F(x_i) = y_i$. Because $F(\varphi_{ij}) = \text{id}$ (as we started with a descent datum over $\{y_i \rightarrow y\}$) the displayed formula shows that $s_i|_{q(y_i) \times_{q(y)} q(y_j)} = s_j|_{q(y_i) \times_{q(y)} q(y_j)}$. Hence the local sections s_i glue to $s : F(x) \rightarrow y$. As F is fibred in groupoids we see that x is isomorphic to an object x' with $F(x') = y$. We omit the verification that x' in the fibre category of \mathcal{X} over y is a solution to the problem of descent posed by the descent datum (x_i, φ_{ij}) . We also omit the proof of the sheaf property of the Isom -presheaves of \mathcal{X}/\mathcal{Y} . \square

Lemma 10.5. *Let \mathcal{C} be a site. Let $p : \mathcal{X} \rightarrow \mathcal{C}$ be a stack. Endow \mathcal{X} with the topology inherited from \mathcal{C} and let $q : \mathcal{Y} \rightarrow \mathcal{X}$ be a stack. Then \mathcal{Y} is a stack over \mathcal{C} . If p and q define stacks in groupoids, then \mathcal{Y} is a stack in groupoids over \mathcal{C} .*

Proof. We check the three conditions in Definition 4.1 to prove that \mathcal{Y} is a stack over \mathcal{C} . By Categories, Lemma 31.12 we find that \mathcal{Y} is a fibred category over \mathcal{C} . Thus condition (1) holds.

Let U be an object of \mathcal{C} and let y_1, y_2 be objects of \mathcal{Y} over U . Denote $x_i = q(y_i)$ in \mathcal{X} . Consider the map of presheaves

$$q : \text{Mor}_{\mathcal{Y}/\mathcal{C}}(y_1, y_2) \longrightarrow \text{Mor}_{\mathcal{X}/\mathcal{C}}(x_1, x_2)$$

on \mathcal{C}/U , see Lemma 2.3. Let $\{U_i \rightarrow U\}$ be a covering and let φ_i be a section of the presheaf on the left over U_i such that φ_i and φ_j restrict to the same section over $U_i \times_U U_j$. We have to find a morphism $\varphi : x_1 \rightarrow x_2$ restricting to φ_i . Note that $q(\varphi_i) = \psi|_{U_i}$ for some morphism $\psi : x_1 \rightarrow x_2$ over U because the second presheaf is a sheaf (by assumption). Let $y_{12} \rightarrow y_2$ be the strongly \mathcal{X} -cartesian morphism of \mathcal{Y} lying over ψ . Then φ_i corresponds to a morphism $\varphi'_i : y_1|_{U_i} \rightarrow y_{12}|_{U_i}$ over $x_1|_{U_i}$. In other words, φ'_i now define local sections of the presheaf

$$\text{Mor}_{\mathcal{Y}/\mathcal{X}}(y_1, y_{12})$$

over the members of the covering $\{x_1|_{U_i} \rightarrow x_1\}$. By assumption these glue to a unique morphism $y_1 \rightarrow y_{12}$ which composed with the given morphism $y_{12} \rightarrow y_2$ produces the desired morphism $y_1 \rightarrow y_2$.

Finally, we show that descent data are effective. Let $\{f_i : U_i \rightarrow U\}$ be a covering of \mathcal{C} and let (y_i, φ_{ij}) be a descent datum relative to this covering (Definition 3.1). Setting $x_i = q(y_i)$ and $\psi_{ij} = q(\varphi_{ij})$ we obtain a descent datum (x_i, ψ_{ij}) for the covering in \mathcal{X} . By assumption on \mathcal{X} we may assume $x_i = x|_{U_i}$ and the ψ_{ij} equal to the canonical descent datum (Definition 3.5). In this case $\{x|_{U_i} \rightarrow x\}$ is a covering and we can view (y_i, φ_{ij}) as a descent datum relative to this covering. By our assumption that \mathcal{Y} is a stack over \mathcal{C} we see that it is effective which finishes the proof of condition (3).

The final assertion follows because \mathcal{Y} is a stack over \mathcal{C} and is fibred in groupoids by Categories, Lemma 33.13. \square

11. Gerbes

Gerbes are a special kind of stacks in groupoids.

Definition 11.1. A *gerbe* over a site \mathcal{C} is a category $p : \mathcal{S} \rightarrow \mathcal{C}$ over \mathcal{C} such that

- (1) $p : \mathcal{S} \rightarrow \mathcal{C}$ is a stack in groupoids over \mathcal{C} (see Definition 5.1),
- (2) for $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \rightarrow U\}$ in \mathcal{C} such that \mathcal{S}_{U_i} is nonempty, and
- (3) for $U \in \text{Ob}(\mathcal{C})$ and $x, y \in \text{Ob}(\mathcal{S}_U)$ there exists a covering $\{U_i \rightarrow U\}$ in \mathcal{C} such that $x|_{U_i} \cong y|_{U_i}$ in \mathcal{S}_{U_i} .

In other words, a gerbe is a stack in groupoids such that any two objects are locally isomorphic and such that objects exist locally.

Lemma 11.2. Let \mathcal{C} be a site. Let $\mathcal{S}_1, \mathcal{S}_2$ be categories over \mathcal{C} . Suppose that \mathcal{S}_1 and \mathcal{S}_2 are equivalent as categories over \mathcal{C} . Then \mathcal{S}_1 is a gerbe over \mathcal{C} if and only if \mathcal{S}_2 is a gerbe over \mathcal{C} .

Proof. Assume \mathcal{S}_1 is a gerbe over \mathcal{C} . By Lemma 5.4 we see \mathcal{S}_2 is a stack in groupoids over \mathcal{C} . Let $F : \mathcal{S}_1 \rightarrow \mathcal{S}_2$, $G : \mathcal{S}_2 \rightarrow \mathcal{S}_1$ be equivalences of categories over \mathcal{C} . Given $U \in \text{Ob}(\mathcal{C})$ we see that there exists a covering $\{U_i \rightarrow U\}$ such that $(\mathcal{S}_1)_{U_i}$ is nonempty. Applying F we see that $(\mathcal{S}_2)_{U_i}$ is nonempty. Given $U \in \text{Ob}(\mathcal{C})$ and $x, y \in \text{Ob}((\mathcal{S}_2)_U)$ there exists a covering $\{U_i \rightarrow U\}$ in \mathcal{C} such that $G(x)|_{U_i} \cong G(y)|_{U_i}$ in $(\mathcal{S}_1)_{U_i}$. By Categories, Lemma 33.8 this implies $x|_{U_i} \cong y|_{U_i}$ in $(\mathcal{S}_2)_{U_i}$. \square

We want to generalize the definition of gerbes a bit. Namely, let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks in groupoids over a site \mathcal{C} . We want to say what it means for \mathcal{X} to be a gerbe over \mathcal{Y} . By Section 10 the category \mathcal{Y} inherits the structure of a site from \mathcal{C} . A naive guess is: Just require that $\mathcal{X} \rightarrow \mathcal{Y}$ is a gerbe in the sense above. Except the notion so obtained is not invariants under replacing \mathcal{X} by an equivalent stack in groupoids over \mathcal{C} ; this is even the case for the property of being fibred in groupoids over \mathcal{Y} . However, it turns out that we can replace \mathcal{X} by an equivalent stack in groupoids over \mathcal{Y} which is fibred in groupoids over \mathcal{Y} , and then the property of being a gerbe over \mathcal{Y} is independent of this choice. Here is the precise formulation.

Lemma 11.3. *Let \mathcal{C} be a site. Let $p : \mathcal{X} \rightarrow \mathcal{C}$ and $q : \mathcal{Y} \rightarrow \mathcal{C}$ be stacks in groupoids. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories over \mathcal{C} . The following are equivalent*

- (1) *For some (equivalently any) factorization $F = F' \circ a$ where $a : \mathcal{X} \rightarrow \mathcal{X}'$ is an equivalence of categories over \mathcal{C} and F' is fibred in groupoids, the map $F' : \mathcal{X}' \rightarrow \mathcal{Y}$ is a gerbe (with the topology on \mathcal{Y} inherited from \mathcal{C}).*
- (2) *The following two conditions are satisfied*
 - (a) *for $y \in \text{Ob}(\mathcal{Y})$ lying over $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \rightarrow U\}$ in \mathcal{C} and objects x_i of \mathcal{X} over U_i such that $F(x_i) \cong y|_{U_i}$ in \mathcal{Y}_{U_i} , and*
 - (b) *for $U \in \text{Ob}(\mathcal{C})$, $x, x' \in \text{Ob}(\mathcal{X}_U)$, and $b : F(x) \rightarrow F(x')$ in \mathcal{Y}_U there exists a covering $\{U_i \rightarrow U\}$ in \mathcal{C} and morphisms $a_i : x|_{U_i} \rightarrow x'|_{U_i}$ in \mathcal{X}_{U_i} with $F(a_i) = b|_{U_i}$.*

Proof. By Categories, Lemma 33.15 there exists a factorization $F = F' \circ a$ where $a : \mathcal{X} \rightarrow \mathcal{X}'$ is an equivalence of categories over \mathcal{C} and F' is fibred in groupoids. By Categories, Lemma 33.16 given any two such factorizations $F = F' \circ a = F'' \circ b$ we have that \mathcal{X}' is equivalent to \mathcal{X}'' as categories over \mathcal{Y} . Hence Lemma 11.2 guarantees that the condition (1) is independent of the choice of the factorization. Moreover, this means that we may assume $\mathcal{X}' = \mathcal{X} \times_{F, \mathcal{Y}, \text{id}} \mathcal{Y}$ as in the proof of Categories, Lemma 33.15

Let us prove that (a) and (b) imply that $\mathcal{X}' \rightarrow \mathcal{Y}$ is a gerbe. First of all, by Lemma 10.4 we see that $\mathcal{X}' \rightarrow \mathcal{Y}$ is a stack in groupoids. Next, let y be an object of \mathcal{Y} lying over $U \in \text{Ob}(\mathcal{C})$. By (a) we can find a covering $\{U_i \rightarrow U\}$ in \mathcal{C} and objects x_i of \mathcal{X} over U_i and isomorphisms $f_i : F(x_i) \rightarrow y|_{U_i}$ in \mathcal{Y}_{U_i} . Then $(U_i, x_i, y|_{U_i}, f_i)$ are objects of \mathcal{X}'_{U_i} , i.e., the second condition of Definition 11.1 holds. Finally, let (U, x, y, f) and (U, x', y, f') be objects of \mathcal{X}' lying over the same object $y \in \text{Ob}(\mathcal{Y})$. Set $b = (f')^{-1} \circ f$. By condition (b) we can find a covering $\{U_i \rightarrow U\}$ and isomorphisms $a_i : x|_{U_i} \rightarrow x'|_{U_i}$ in \mathcal{X}_{U_i} with $F(a_i) = b|_{U_i}$. Then

$$(a_i, \text{id}) : (U, x, y, f)|_{U_i} \rightarrow (U, x', y, f')|_{U_i}$$

is a morphism in \mathcal{X}'_{U_i} as desired. This proves that (2) implies (1).

To prove that (1) implies (2) one reads the arguments in the preceding paragraph backwards. Details omitted. \square

Definition 11.4. Let \mathcal{C} be a site. Let \mathcal{X} and \mathcal{Y} be stacks in groupoids over \mathcal{C} . Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of categories over \mathcal{C} . We say \mathcal{X} is a *gerbe over \mathcal{Y}* if the equivalent conditions of Lemma 11.3 are satisfied.

This definition does not conflict with Definition 11.1 when $\mathcal{Y} = \mathcal{C}$ because in this case we may take $\mathcal{X}' = \mathcal{X}$ in part (1) of Lemma 11.3. Note that conditions (2)(a)

and (2)(b) of Lemma 11.3 are quite close in spirit to conditions (2) and (3) of Definition 11.1. Namely, (2)(a) says that the map of presheaves of isomorphism classes of objects becomes a surjection after sheafification. Moreover, (2)(b) says that

$$\mathrm{Isom}_{\mathcal{X}}(x, x') \longrightarrow \mathrm{Isom}_{\mathcal{Y}}(F(x), F(x'))$$

is a surjection of sheaves on \mathcal{C}/U for any U and $x, x' \in \mathrm{Ob}(\mathcal{X}_U)$.

Lemma 11.5. *Let \mathcal{C} be a site. Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{G'} & \mathcal{X} \\ F' \downarrow & & \downarrow F \\ \mathcal{Y}' & \xrightarrow{G} & \mathcal{Y} \end{array}$$

be a 2-fibre product of stacks in groupoids over \mathcal{C} . If \mathcal{X} is a gerbe over \mathcal{Y} , then \mathcal{X}' is a gerbe over \mathcal{Y}' .

Proof. By the uniqueness property of a 2-fibre product may assume that $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as in Categories, Lemma 30.3. Let us prove properties (2)(a) and (2)(b) of Lemma 11.3 for $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \rightarrow \mathcal{Y}'$.

Let y' be an object of \mathcal{Y}' lying over the object U of \mathcal{C} . By assumption there exists a covering $\{U_i \rightarrow U\}$ of U and objects $x_i \in \mathcal{X}_{U_i}$ with isomorphisms $\alpha_i : G(y')|_{U_i} \rightarrow F(x_i)$. Then $(U_i, y'|_{U_i}, x_i, \alpha_i)$ is an object of $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ over U_i whose image in \mathcal{Y}' is $y'|_{U_i}$. Thus (2)(a) holds.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x'_1, x'_2 be objects of $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ over U , and let $b' : F'(x'_1) \rightarrow F'(x'_2)$ be a morphism in \mathcal{Y}'_U . Write $x'_i = (U, y'_i, x_i, \alpha_i)$. Note that $F'(x'_1) = x_i$ and $G'(x'_1) = y'_i$. By assumption there exists a covering $\{U_i \rightarrow U\}$ in \mathcal{C} and morphisms $a_i : x_1|_{U_i} \rightarrow x_2|_{U_i}$ in \mathcal{X}_{U_i} with $F(a_i) = G(b')|_{U_i}$. Then $(b'|_{U_i}, a_i)$ is a morphism $x'_1|_{U_i} \rightarrow x'_2|_{U_i}$ as required in (2)(b). \square

Lemma 11.6. *Let \mathcal{C} be a site. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ and $G : \mathcal{Y} \rightarrow \mathcal{Z}$ be 1-morphisms of stacks in groupoids over \mathcal{C} . If \mathcal{X} is a gerbe over \mathcal{Y} and \mathcal{Y} is a gerbe over \mathcal{Z} , then \mathcal{X} is a gerbe over \mathcal{Z} .*

Proof. Let us prove properties (2)(a) and (2)(b) of Lemma 11.3 for $\mathcal{X} \rightarrow \mathcal{Z}$.

Let z be an object of \mathcal{Z} lying over the object U of \mathcal{C} . By assumption on G there exists a covering $\{U_i \rightarrow U\}$ of U and objects $y_i \in \mathcal{Y}_{U_i}$ such that $G(y_i) \cong z|_{U_i}$. By assumption on F there exist coverings $\{U_{ij} \rightarrow U_i\}$ and objects $x_{ij} \in \mathcal{X}_{U_{ij}}$ such that $F(x_{ij}) \cong y_i|_{U_{ij}}$. Then $\{U_{ij} \rightarrow U\}$ is a covering of \mathcal{C} and $(G \circ F)(x_{ij}) \cong z|_{U_{ij}}$. Thus (2)(a) holds.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x_1, x_2 be objects of \mathcal{X} over U , and let $c : (G \circ F)(x_1) \rightarrow (G \circ F)(x_2)$ be a morphism in \mathcal{Z}_U . By assumption on G there exists a covering $\{U_i \rightarrow U\}$ of U and morphisms $b_i : F(x_1)|_{U_i} \rightarrow F(x_2)|_{U_i}$ in \mathcal{Y}_{U_i} such that $G(b_i) = c|_{U_i}$. By assumption on F there exist coverings $\{U_{ij} \rightarrow U_i\}$ and morphisms $a_{ij} : x_1|_{U_{ij}} \rightarrow x_2|_{U_{ij}}$ in $\mathcal{X}_{U_{ij}}$ such that $F(a_{ij}) = b_i|_{U_{ij}}$. Then $\{U_{ij} \rightarrow U\}$ is a covering of \mathcal{C} and $(G \circ F)(a_{ij}) = c|_{U_{ij}}$ as required in (2)(b). \square

Lemma 11.7. *Let \mathcal{C} be a site. Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{G'} & \mathcal{X} \\ F' \downarrow & & \downarrow F \\ \mathcal{Y}' & \xrightarrow{G} & \mathcal{Y} \end{array}$$

be a 2-cartesian diagram of stacks in groupoids over \mathcal{C} . If for every $U \in \text{Ob}(\mathcal{C})$ and $x \in \text{Ob}(\mathcal{Y}_U)$ there exists a covering $\{U_i \rightarrow U\}$ such that $x|_{U_i}$ is in the essential image of $G : \mathcal{Y}'_{U_i} \rightarrow \mathcal{Y}_{U_i}$ and \mathcal{X}' is a gerbe over \mathcal{Y}' , then \mathcal{X} is a gerbe over \mathcal{Y} .

Proof. By the uniqueness property of a 2-fibre product may assume that $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as in Categories, Lemma 30.3. Let us prove properties (2)(a) and (2)(b) of Lemma 11.3 for $\mathcal{X} \rightarrow \mathcal{Y}$.

Let y be an object of \mathcal{Y} lying over the object U of \mathcal{C} . By assumption there exists a covering $\{U_i \rightarrow U\}$ of U and objects $y'_i \in \mathcal{Y}'_{U_i}$ with $G(y'_i) \cong y|_{U_i}$. By (2)(a) for $\mathcal{X}' \rightarrow \mathcal{Y}'$ there exist coverings $\{U_{ij} \rightarrow U_i\}$ and objects x'_{ij} of \mathcal{X}' over U_{ij} with $F'(x'_{ij})$ isomorphic to the restriction of y'_i to U_{ij} . Then $\{U_{ij} \rightarrow U\}$ is a covering of \mathcal{C} and $G'(x'_{ij})$ are objects of \mathcal{X} over U_{ij} whose images in \mathcal{Y} are isomorphic to the restrictions $y|_{U_{ij}}$. This proves (2)(a) for $\mathcal{X} \rightarrow \mathcal{Y}$.

Let $U \in \text{Ob}(\mathcal{C})$, let x_1, x_2 be objects of \mathcal{X} over U , and let $b : F(x_1) \rightarrow F(x_2)$ be a morphism in \mathcal{Y}_U . By assumption we may choose a covering $\{U_i \rightarrow U\}$ and objects y'_{1i}, y'_{2i} of \mathcal{Y}' over U_i such that there exist isomorphisms $\alpha_{1i} : G(y'_{1i}) \cong F(x_1)|_{U_i}$ and $\alpha_{2i} : G(y'_{2i}) \cong F(x_2)|_{U_i}$. Then we get objects $x'_{1i} = (U_i, y'_{1i}, x_1|_{U_i}, \alpha_{1i})$ and $x'_{2i} = (U_i, y'_{2i}, x_2|_{U_i}, \alpha_{2i})$ of \mathcal{X}' over U_i . The restriction $b|_{U_i}$ is a morphism $F'(x'_{1i}) \rightarrow F'(x'_{2i})$. By (2)(b) for $\mathcal{X}' \rightarrow \mathcal{Y}'$ there exist coverings $\{U_{ij} \rightarrow U_i\}$ and morphisms $a'_{ij} : x'_{1i}|_{U_{ij}} \rightarrow x'_{2i}|_{U_{ij}}$ such that $F'(a'_{ij}) = b|_{U_{ij}}$. Unwinding the definition of morphisms in $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ we see that $G'(a'_{ij}) : x_1|_{U_{ij}} \rightarrow x_2|_{U_{ij}}$ are the morphism we're looking for, i.e., (2)(b) holds for $\mathcal{X} \rightarrow \mathcal{Y}$. \square

12. Functoriality for stacks

In this section we study what happens if we want to change the base site of a stack. This section can be skipped on a first reading.

Let $u : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between categories. Let $p : \mathcal{S} \rightarrow \mathcal{D}$ be a category over \mathcal{D} . In this situation we denote $u^p\mathcal{S}$ the category over \mathcal{C} defined as follows

- (1) An object of $u^p\mathcal{S}$ is a pair (U, y) consisting of an object U of \mathcal{C} and an object y of $\mathcal{S}_{u(U)}$.
- (2) A morphism $(a, \beta) : (U, y) \rightarrow (U', y')$ is given by a morphism $a : U \rightarrow U'$ of \mathcal{C} and a morphism $\beta : y \rightarrow y'$ of \mathcal{S} such that $p(\beta) = u(a)$.

Note that with these definitions the fibre category of $u^p\mathcal{S}$ over U is equal to the fibre category of \mathcal{S} over $u(U)$.

Lemma 12.1. *In the situation above, if \mathcal{S} is a fibred category over \mathcal{D} then $u^p\mathcal{S}$ is a fibred category over \mathcal{C} .*

Proof. Please take a look at the discussion surrounding Categories, Definitions 31.1 and 31.5 before reading this proof. Let $(a, \beta) : (U, y) \rightarrow (U', y')$ be a morphism of $u^p\mathcal{S}$. We claim that (a, β) is strongly cartesian if and only if β is strongly

cartesian. First, assume β is strongly cartesian. Consider any second morphism $(a_1, \beta_1) : (U_1, y_1) \rightarrow (U', y')$ of $u^p\mathcal{S}$. Then

$$\begin{aligned} & \text{Mor}_{u^p\mathcal{S}}((U_1, y_1), (U, y)) \\ &= \text{Mor}_{\mathcal{C}}(U_1, U) \times_{\text{Mor}_{\mathcal{D}}(u(U_1), u(U))} \text{Mor}_{\mathcal{S}}(y_1, y) \\ &= \text{Mor}_{\mathcal{C}}(U_1, U) \times_{\text{Mor}_{\mathcal{D}}(u(U_1), u(U))} \text{Mor}_{\mathcal{S}}(y_1, y') \times_{\text{Mor}_{\mathcal{D}}(u(U_1), u(U'))} \text{Mor}_{\mathcal{D}}(u(U_1), u(U)) \\ &= \text{Mor}_{\mathcal{S}}(y_1, y') \times_{\text{Mor}_{\mathcal{D}}(u(U_1), u(U'))} \text{Mor}_{\mathcal{C}}(U_1, U) \\ &= \text{Mor}_{u^p\mathcal{S}}((U_1, y_1), (U', y')) \times_{\text{Mor}_{\mathcal{C}}(U_1, U')} \text{Mor}_{\mathcal{C}}(U_1, U) \end{aligned}$$

the second equality as β is strongly cartesian. Hence we see that indeed (a, β) is strongly cartesian. Conversely, suppose that (a, β) is strongly cartesian. Choose a strongly cartesian morphism $\beta' : y'' \rightarrow y'$ in \mathcal{S} with $p(\beta') = u(a)$. Then both $(a, \beta) : (U, y) \rightarrow (U, y')$ and $(a, \beta') : (U, y'') \rightarrow (U, y)$ are strongly cartesian and lift a . Hence, by the uniqueness of strongly cartesian morphisms (see discussion in Categories, Section 31) there exists an isomorphism $\iota : y \rightarrow y''$ in $\mathcal{S}_{u(U)}$ such that $\beta = \beta' \circ \iota$, which implies that β is strongly cartesian in \mathcal{S} by Categories, Lemma 31.2.

Finally, we have to show that given (U', y') and $U \rightarrow U'$ we can find a strongly cartesian morphism $(U, y) \rightarrow (U', y')$ in $u^p\mathcal{S}$ lifting the morphism $U \rightarrow U'$. This follows from the above as by assumption we can find a strongly cartesian morphism $y \rightarrow y'$ lifting the morphism $u(U) \rightarrow u(U')$. \square

Lemma 12.2. *Let $u : \mathcal{C} \rightarrow \mathcal{D}$ be a continuous functor of sites. Let $p : \mathcal{S} \rightarrow \mathcal{D}$ be a stack over \mathcal{D} . Then $u^p\mathcal{S}$ is a stack over \mathcal{S} .*

Proof. We have seen in Lemma 12.1 that $u^p\mathcal{S}$ is a fibred category over \mathcal{S} . Moreover, in the proof of that lemma we have seen that a morphism (a, β) of $u^p\mathcal{S}$ is strongly cartesian if and only if β is strongly cartesian in \mathcal{S} . Hence, given a morphism $a : U \rightarrow U'$ of \mathcal{C} , not only do we have the equalities $(u^p\mathcal{S})_U = \mathcal{S}_U$ and $(u^p\mathcal{S})_{U'} = \mathcal{S}_{U'}$, but via these equalities the pullback functors agree; in a formula $a^*(U', y') = (U, u(a)^*y')$.

Having said this, let $\mathcal{U} = \{U_i \rightarrow U\}$ be a covering of \mathcal{C} . As u is continuous we see that $\mathcal{V} = \{u(U_i) \rightarrow u(U)\}$ is a covering of \mathcal{D} , and that $u(U_i \times_U U_j) = u(U_i) \times_{u(U)} u(U_j)$ and similarly for the triple fibre products $U_i \times_U U_j \times_U U_k$. As we have the identifications of fibre categories and pullbacks we see that descend data relative to \mathcal{U} are identical to descend data relative to \mathcal{V} . Since by assumption we have effective descent in \mathcal{S} we conclude the same holds for $u^p\mathcal{S}$. \square

Lemma 12.3. *Let $u : \mathcal{C} \rightarrow \mathcal{D}$ be a continuous functor of sites. Let $p : \mathcal{S} \rightarrow \mathcal{D}$ be a stack in groupoids over \mathcal{D} . Then $u^p\mathcal{S}$ is a stack in groupoids over \mathcal{S} .*

Proof. This follows immediately from Lemma 12.2 and the fact that all fibre categories are groupoids. \square

Definition 12.4. Let $f : \mathcal{D} \rightarrow \mathcal{C}$ be a morphism of sites given by the continuous functor $u : \mathcal{C} \rightarrow \mathcal{D}$. Let \mathcal{S} be a fibred category over \mathcal{D} . In this setting we write $f_*\mathcal{S}$ for the fibred category $u^p\mathcal{S}$ defined above. We say that $f_*\mathcal{S}$ is the *pushforward of \mathcal{S} along f* .

By the results above we know that $f_*\mathcal{S}$ is a stack (in groupoids) if \mathcal{S} is a stack (in groupoids). It is harder to define the pullback of a stack (and we'll need additional

assumptions for our particular construction – feel free to write up and submit a more general construction). We do this in several steps.

Let $u : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between categories. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ be a category over \mathcal{C} . In this setting we define a category $u_{pp}\mathcal{S}$ as follows:

- (1) An object of $u_{pp}\mathcal{S}$ is a triple $(U, \phi : V \rightarrow u(U), x)$ where $U \in \text{Ob}(\mathcal{C})$, the map $\phi : V \rightarrow u(U)$ is a morphism in \mathcal{D} , and $x \in \text{Ob}(\mathcal{S}_U)$.
- (2) A morphism

$$(U_1, \phi_1 : V_1 \rightarrow u(U_1), x_1) \longrightarrow (U_2, \phi_2 : V_2 \rightarrow u(U_2), x_2)$$

of $u_{pp}\mathcal{S}$ is given by a (a, b, α) where $a : U_1 \rightarrow U_2$ is a morphism of \mathcal{C} , $b : V_1 \rightarrow V_2$ is a morphism of \mathcal{D} , and $\alpha : x_1 \rightarrow x_2$ is morphism of \mathcal{S} , such that $p(\alpha) = a$ and the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\quad b \quad} & V_2 \\ \phi_1 \downarrow & & \downarrow \phi_2 \\ u(U_1) & \xrightarrow{u(a)} & u(U_2) \end{array}$$

commutes in \mathcal{D} .

We think of $u_{pp}\mathcal{S}$ as a category over \mathcal{D} via

$$p_{pp} : u_{pp}\mathcal{S} \longrightarrow \mathcal{D}, \quad (U, \phi : V \rightarrow u(U), x) \longmapsto V.$$

The fibre category of $u_{pp}\mathcal{S}$ over an object V of \mathcal{D} does not have a simple description. Moreover, it is in general not the case that $u_{pp}\mathcal{S}$ is a fibred category over \mathcal{D} if \mathcal{S} is a fibred category over \mathcal{C} .

Lemma 12.5. *In the situation above assume*

- (1) $p : \mathcal{S} \rightarrow \mathcal{C}$ is a fibred category,
- (2) \mathcal{C} has nonempty finite limits, and
- (3) $u : \mathcal{C} \rightarrow \mathcal{D}$ commutes with nonempty finite limits.

Consider the set $R \subset \text{Arrows}(u_{pp}\mathcal{S})$ of morphisms of the form

$$(a, \text{id}_V, \alpha) : (U', \phi' : V \rightarrow u(U'), x') \longrightarrow (U, \phi : V \rightarrow u(U), x)$$

with α strongly cartesian. Then R is a right multiplicative system.

Proof. According to Categories, Definition 25.1 we have to check RMS1, RMS2, RMS3. Condition RMS1 holds as a composition of strongly cartesian morphisms is strongly cartesian, see Categories, Lemma 31.2.

To check RMS2 suppose we have a morphism

$$(a, b, \alpha) : (U_1, \phi_1 : V_1 \rightarrow u(U_1), x_1) \longrightarrow (U, \phi : V \rightarrow u(U), x)$$

of $u_{pp}\mathcal{S}$ and a morphism

$$(c, \text{id}_V, \gamma) : (U', \phi' : V \rightarrow u(U'), x') \longrightarrow (U, \phi : V \rightarrow u(U), x)$$

with γ strongly cartesian from R . In this situation set $U'_1 = U_1 \times_U U'$, and denote $a' : U'_1 \rightarrow U'$ and $c' : U'_1 \rightarrow U_1$ the projections. As $u(U'_1) = u(U_1) \times_{u(U)} u(U')$ we see that $\phi'_1 = (\phi_1, \phi') : V_1 \rightarrow u(U'_1)$ is a morphism in \mathcal{D} . Let $\gamma_1 : x'_1 \rightarrow x_1$ be a strongly cartesian morphism of \mathcal{S} with $p(\gamma_1) = \phi'_1$ (which exists because \mathcal{S} is a

fibred category over \mathcal{C}). Then as $\gamma : x' \rightarrow x$ is strongly cartesian there exists a unique morphism $\alpha' : x'_1 \rightarrow x'$ with $p(\alpha') = a'$. At this point we see that

$$(a', b, \alpha') : (U_1, \phi_1 : V_1 \rightarrow u(U'_1), x'_1) \longrightarrow (U, \phi : V \rightarrow u(U'), x')$$

is a morphism and that

$$(c', \text{id}_{V_1}, \gamma_1) : (U'_1, \phi'_1 : V_1 \rightarrow u(U'_1), x'_1) \longrightarrow (U_1, \phi : V_1 \rightarrow u(U_1), x_1)$$

is an element of R which form a solution of the existence problem posed by RMS2.

Finally, suppose that

$$(a, b, \alpha), (a', b', \alpha') : (U_1, \phi_1 : V_1 \rightarrow u(U_1), x_1) \longrightarrow (U, \phi : V \rightarrow u(U), x)$$

are two morphisms of $u_{pp}\mathcal{S}$ and suppose that

$$(c, \text{id}_V, \gamma) : (U, \phi : V \rightarrow u(U), x) \longrightarrow (U', \phi : V \rightarrow u(U'), x')$$

is an element of R which equalizes the morphisms (a, b, α) and (a', b', α') . This implies in particular that $b = b'$. Let $d : U_2 \rightarrow U_1$ be the equalizer of a, a' which exists (see Categories, Lemma 18.3). Moreover, $u(d) : u(U_2) \rightarrow u(U_1)$ is the equalizer of $u(a), u(a')$ hence (as $b = b'$) there is a morphism $\phi_2 : V_1 \rightarrow u(U_2)$ such that $\phi_1 = u(d) \circ \phi_2$. Let $\delta : x_2 \rightarrow x_1$ be a strongly cartesian morphism of \mathcal{S} with $p(\delta) = u(d)$. Now we claim that $\alpha \circ \delta = \alpha' \circ \delta$. This is true because γ is strongly cartesian, $\gamma \circ \alpha \circ \delta = \gamma \circ \alpha' \circ \delta$, and $p(\alpha \circ \delta) = p(\alpha' \circ \delta)$. Hence the arrow

$$(d, \text{id}_{V_1}, \delta) : (U_2, \phi_2 : V_1 \rightarrow u(U_2), x_2) \longrightarrow (U_1, \phi_1 : V_1 \rightarrow u(U_1), x_1)$$

is an element of R and equalizes (a, b, α) and (a', b', α') . Hence R satisfies RMS3 as well. \square

Lemma 12.6. *With notation and assumptions as in Lemma 12.5. Set $u_p\mathcal{S} = R^{-1}u_{pp}\mathcal{S}$, see Categories, Section 25. Then $u_p\mathcal{S}$ is a fibred category over \mathcal{D} .*

Proof. We use the description of $u_p\mathcal{S}$ given just above Categories, Lemma 25.9. Note that the functor $p_{pp} : u_{pp}\mathcal{S} \rightarrow \mathcal{D}$ transforms every element of R to an identity morphism. Hence by Categories, Lemma 25.13 we obtain a canonical functor $p_p : u_p\mathcal{S} \rightarrow \mathcal{D}$ extending the given functor. This is how we think of $u_p\mathcal{S}$ as a category over \mathcal{D} .

First we want to characterize the \mathcal{D} -strongly cartesian morphisms in $u_p\mathcal{S}$. A morphism $f : X \rightarrow Y$ of $u_p\mathcal{S}$ is the equivalence class of a pair $(f' : X' \rightarrow Y, r : X' \rightarrow X)$ with $r \in R$. In fact, in $u_p\mathcal{S}$ we have $f = (f', 1) \circ (r, 1)^{-1}$ with obvious notation. Note that an isomorphism is always strongly cartesian, as are compositions of strongly cartesian morphisms, see Categories, Lemma 31.2. Hence f is strongly cartesian if and only if $(f', 1)$ is so. Thus the following claim completely characterizes strongly cartesian morphisms. Claim: A morphism

$$(a, b, \alpha) : X_1 = (U_1, \phi_1 : V_1 \rightarrow u(U_1), x_1) \longrightarrow (U_2, \phi_2 : V_2 \rightarrow u(U_2), x_2) = X_2$$

of $u_{pp}\mathcal{S}$ has image $f = ((a, b, \alpha), 1)$ strongly cartesian in $u_p\mathcal{S}$ if and only if α is a strongly cartesian morphism of \mathcal{S} .

Assume α strongly cartesian. Let $X = (U, \phi : V \rightarrow u(U), x)$ be another object, and let $f_2 : X \rightarrow X_2$ be a morphism of $u_p\mathcal{S}$ such that $p_p(f_2) = b \circ b_1$ for some $b_1 : U \rightarrow U_1$. To show that f is strongly cartesian we have to show that there exists a unique morphism $f_1 : X \rightarrow X_1$ in $u_p\mathcal{S}$ such that $p_p(f_1) = b_1$ and $f_2 = f \circ f_1$ in $u_p\mathcal{S}$. Write $f_2 = (f'_2 : X' \rightarrow X_2, r : X' \rightarrow X)$. Again we can write $f_2 = (f'_2, 1) \circ (r, 1)^{-1}$

in $u_p\mathcal{S}$. Since $(r, 1)$ is an isomorphism whose image in \mathcal{D} is an identity we see that finding a morphism $f_1 : X \rightarrow X_1$ with the required properties is the same thing as finding a morphism $f'_1 : X' \rightarrow X_1$ in $u_p\mathcal{S}$ with $p(f'_1) = b_1$ and $f'_2 = f \circ f'_1$. Hence we may assume that f_2 is of the form $f_2 = ((a_2, b_2, \alpha_2), 1)$ with $b_2 = b \circ b_1$. Here is a picture

$$\begin{array}{ccc} & (U_1, V_1 \rightarrow u(U_1), x_1) & \\ & \downarrow (a, b, \alpha) & \\ (U, V \rightarrow u(U), x) & \xrightarrow{(a_2, b_2, \alpha_2)} & (U_2, V_2 \rightarrow u(U_2), x_2) \end{array}$$

Now it is clear how to construct the morphism f_1 . Namely, set $U' = U \times_{U_2} U_1$ with projections $c : U' \rightarrow U$ and $a_1 : U' \rightarrow U_1$. Pick a strongly cartesian morphism $\gamma : x' \rightarrow x$ lifting the morphism c . Since $b_2 = b \circ b_1$, and since $u(U') = u(U) \times_{u(U_2)} u(U_1)$ we see that $\phi' = (\phi, \phi_1 \circ b_1) : V \rightarrow u(U')$. Since α is strongly cartesian, and $a \circ a_1 = a_2 \circ c = p(\alpha_2 \circ \gamma)$ there exists a morphism $\alpha_1 : x' \rightarrow x_1$ lifting a_1 such that $\alpha \circ \alpha_1 = \alpha_2 \circ \gamma$. Set $X' = (U', \phi' : V \rightarrow u(U'), x')$. Thus we see that

$$f_1 = ((a_1, b_1, \alpha_1) : X' \rightarrow X_1, (c, \text{id}_V, \gamma) : X' \rightarrow X) : X \longrightarrow X_1$$

works, in fact the diagram

$$\begin{array}{ccc} (U', \phi' : V \rightarrow u(U'), x') & \xrightarrow{(a_1, b_1, \alpha_1)} & (U_1, V_1 \rightarrow u(U_1), x_1) \\ \downarrow (c, \text{id}_V, \gamma) & & \downarrow (a, b, \alpha) \\ (U, V \rightarrow u(U), x) & \xrightarrow{(a_2, b_2, \alpha_2)} & (U_2, V_2 \rightarrow u(U_2), x_2) \end{array}$$

is commutative by construction. This proves existence.

Next we prove uniqueness, still in the special case $f = ((a, b, \alpha), 1)$ and $f_2 = ((a_2, b_2, \alpha_2), 1)$. We strongly advise the reader to skip this part. Suppose that $g_1, g'_1 : X \rightarrow X_1$ are two morphisms of $u_p\mathcal{S}$ such that $p_p(g_1) = p_p(g'_1) = b_1$ and $f_2 = f \circ g_1 = f \circ g'_1$. Our goal is to show that $g_1 = g'_1$. By Categories, Lemma 25.10 we may represent g_1 and g'_1 as the equivalence classes of $(f_1 : X' \rightarrow X_1, r : X' \rightarrow X)$ and $(f'_1 : X' \rightarrow X_1, r' : X' \rightarrow X)$ for some $r \in R$. By Categories, Lemma 25.11 we see that $f_2 = f \circ g_1 = f \circ g'_1$ means that there exists a morphism $r' : X'' \rightarrow X'$ in $u_{pp}\mathcal{S}$ such that $r' \circ r \in R$ and

$$(a, b, \alpha) \circ f_1 \circ r' = (a, b, \alpha) \circ f'_1 \circ r' = (a_2, b_2, \alpha_2) \circ r'$$

in $u_{pp}\mathcal{S}$. Note that now g_1 is represented by $(f_1 \circ r', r \circ r')$ and similarly for g'_1 . Hence we may assume that

$$(a, b, \alpha) \circ f_1 = (a, b, \alpha) \circ f'_1 = (a_2, b_2, \alpha_2).$$

Write $r = (c, \text{id}_V, \gamma) : (U', \phi' : V \rightarrow u(U'), x')$, $f_1 = (a_1, b_1, \alpha_1)$, and $f'_1 = (a'_1, b_1, \alpha'_1)$. Here we have used the condition that $p_p(g_1) = p_p(g'_1)$. The equalities above are now equivalent to $a \circ a_1 = a \circ a'_1 = a_2 \circ c$ and $\alpha \circ \alpha_1 = \alpha \circ \alpha'_1 = \alpha_2 \circ \gamma$. It need not be the case that $a_1 = a'_1$ in this situation. Thus we have to precompose by one more morphism from R . Namely, let $U'' = \text{Eq}(a_1, a'_1)$ be the equalizer of a_1 and a'_1 which is a subobject of U' . Denote $c' : U'' \rightarrow U'$ the canonical monomorphism. Because of the relations among the morphisms above we see that $V \rightarrow u(U')$ maps into $u(U'') = u(\text{Eq}(a_1, a'_1)) = \text{Eq}(u(a_1), u(a'_1))$. Hence we get a new object $(U'', \phi'' : V \rightarrow u(U''), x'')$, where $\gamma' : x'' \rightarrow x'$ is a strongly cartesian

morphism lifting γ . Then we see that we may precompose f_1 and f'_1 with the element $(c', \text{id}_V, \gamma')$ of R . After doing this, i.e., replacing $(U', \phi' : V \rightarrow u(U'), x')$ with $(U'', \phi'' : V \rightarrow u(U''), x'')$, we get back to the previous situation where in addition we now have that $a_1 = a'_1$. In this case it follows formally from the fact that α is strongly cartesian (!) that $\alpha_1 = \alpha'_1$. This shows that $g_1 = g'_1$ as desired.

We omit the proof of the fact that for any strongly cartesian morphism of $u_p\mathcal{S}$ of the form $((a, b, \alpha), 1)$ the morphism α is strongly cartesian in \mathcal{S} . (We do not need the characterization of strongly cartesian morphisms in the rest of the proof, although we do use it later in this section.)

Let $(U, \phi : V \rightarrow u(U), x)$ be an object of $u_p\mathcal{S}$. Let $b : V' \rightarrow V$ be a morphism of \mathcal{D} . Then the morphism

$$(\text{id}_U, b, \text{id}_x) : (U, \phi \circ b : V' \rightarrow u(U), x) \longrightarrow (U, \phi : V \rightarrow u(U), x)$$

is strongly cartesian by the result of the preceding paragraphs and we win. \square

Lemma 12.7. *With notation and assumptions as in Lemma 12.6. If \mathcal{S} is fibred in groupoids, then $u_p\mathcal{S}$ is fibred in groupoids.*

Proof. By Lemma 12.6 we know that $u_p\mathcal{S}$ is a fibred category. Let $f : X \rightarrow Y$ be a morphism of $u_p\mathcal{S}$ with $p_p(f) = \text{id}_V$. We are done if we can show that f is invertible, see Categories, Lemma 33.2. Write f as the equivalence class of a pair $((a, b, \alpha), r)$ with $r \in R$. Then $p_p(r) = \text{id}_V$, hence $p_{pp}((a, b, \alpha)) = \text{id}_V$. Hence $b = \text{id}_V$. But any morphism of \mathcal{S} is strongly cartesian, see Categories, Lemma 33.2 hence we see that $(a, b, \alpha) \in R$ is invertible in $u_p\mathcal{S}$ as desired. \square

Lemma 12.8. *Let $u : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ and $q : \mathcal{T} \rightarrow \mathcal{D}$ be categories over \mathcal{C} and \mathcal{D} . Assume that*

- (1) $p : \mathcal{S} \rightarrow \mathcal{C}$ is a fibred category,
- (2) $q : \mathcal{T} \rightarrow \mathcal{D}$ is a fibred category,
- (3) \mathcal{C} has nonempty finite limits, and
- (4) $u : \mathcal{C} \rightarrow \mathcal{D}$ commutes with nonempty finite limits.

Then we have a canonical equivalence of categories

$$\text{Mor}_{\text{Fib}/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) = \text{Mor}_{\text{Fib}/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

of morphism categories.

Proof. In this proof we use the notation x/U to denote an object x of \mathcal{S} which lies over U in \mathcal{C} . Similarly y/V denotes an object y of \mathcal{T} which lies over V in \mathcal{D} . In the same vein $\alpha/a : x/U \rightarrow x'/U'$ denotes the morphism $\alpha : x \rightarrow x'$ with image $a : U \rightarrow U'$ in \mathcal{C} .

Let $G : u_p\mathcal{S} \rightarrow \mathcal{T}$ be a 1-morphism of fibred categories over \mathcal{D} . Denote $G' : u_{pp}\mathcal{S} \rightarrow \mathcal{T}$ the composition of G with the canonical (localization) functor $u_{pp}\mathcal{S} \rightarrow u_p\mathcal{S}$. Then consider the functor $H : \mathcal{S} \rightarrow u^p\mathcal{T}$ given by

$$H(x/U) = (U, G'(U, \text{id}_{u(U)} : u(U) \rightarrow u(U), x))$$

on objects and by

$$H((\alpha, a) : x/U \rightarrow x'/U') = G'(a, u(a), \alpha)$$

on morphisms. Since G transforms strongly cartesian morphisms into strongly cartesian morphisms, we see that if α is strongly cartesian, then $H(\alpha)$ is strongly

cartesian. Namely, we've seen in the proof of Lemma 12.6 that in this case the map $(a, u(a), \alpha)$ becomes strongly cartesian in $u_p\mathcal{S}$. Clearly this construction is functorial in G and we obtain a functor

$$A : \text{Mor}_{\text{Fib}/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T}) \longrightarrow \text{Mor}_{\text{Fib}/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T})$$

Conversely, let $H : \mathcal{S} \rightarrow u^p\mathcal{T}$ be a 1-morphism of fibred categories. Recall that an object of $u^p\mathcal{T}$ is a pair (U, y) with $y \in \text{Ob}(\mathcal{T}_{u(U)})$. We denote $\text{pr} : u^p\mathcal{T} \rightarrow \mathcal{T}$ the functor $(U, y) \mapsto y$. In this case we define a functor $G' : u_{pp}\mathcal{S} \rightarrow \mathcal{T}$ by the rules

$$G'(U, \phi : V \rightarrow u(U), x) = \phi^*\text{pr}(H(x))$$

on objects and we let

$$G'((a, b, \alpha) : (U, \phi : V \rightarrow u(U), x) \rightarrow (U', \phi' : V' \rightarrow u(U'), x')) = \beta$$

be the unique morphism $\beta : \phi^*\text{pr}(H(x)) \rightarrow (\phi')^*\text{pr}(H(x'))$ such that $q(\beta) = b$ and the diagram

$$\begin{array}{ccc} \phi^*\text{pr}(H(x)) & \xrightarrow[\beta]{} & (\phi')^*\text{pr}(H(x')) \\ \downarrow & & \downarrow \\ \text{pr}(H(x)) & \xrightarrow{\text{pr}(H(a, \alpha))} & \text{pr}(H(x')) \end{array}$$

Such a morphism exists and is unique because \mathcal{T} is a fibred category.

We check that $G'(r)$ is an isomorphism if $r \in R$. Namely, if

$$(a, \text{id}_V, \alpha) : (U', \phi' : V \rightarrow u(U'), x') \longrightarrow (U, \phi : V \rightarrow u(U), x)$$

with α strongly cartesian is an element of the right multiplicative system R of Lemma 12.5 then $H(\alpha)$ is strongly cartesian, and $\text{pr}(H(\alpha))$ is strongly cartesian, see proof of Lemma 12.1. Hence in this case the morphism β has $q(\beta) = \text{id}_V$ and is strongly cartesian. Hence β is an isomorphism by Categories, Lemma 31.2. Thus by Categories, Lemma 25.13 we obtain a canonical extension $G : u_p\mathcal{S} \rightarrow \mathcal{T}$.

Next, let us prove that G transforms strongly cartesian morphisms into strongly cartesian morphisms. Suppose that $f : X \rightarrow Y$ is a strongly cartesian. By the characterization of strongly cartesian morphisms in $u_p\mathcal{S}$ we can write f as $((a, b, \alpha) : X' \rightarrow Y, r : X' \rightarrow Y)$ where $r \in R$ and α strongly cartesian in \mathcal{S} . By the above it suffices to show that $G'(a, b\alpha)$ is strongly cartesian. As before the condition that α is strongly cartesian implies that $\text{pr}(H(a, \alpha)) : \text{pr}(H(x)) \rightarrow \text{pr}(H(x'))$ is strongly cartesian in \mathcal{T} . Since in the commutative square above now all arrows except possibly β is strongly cartesian it follows that also β is strongly cartesian as desired. Clearly the construction $H \mapsto G$ is functorial in H and we obtain a functor

$$B : \text{Mor}_{\text{Fib}/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) \longrightarrow \text{Mor}_{\text{Fib}/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

To finish the proof of the lemma we have to show that the functors A and B are mutually quasi-inverse. We omit the verifications. \square

Definition 12.9. Let $f : \mathcal{D} \rightarrow \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \rightarrow \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 15.6. Let \mathcal{S} be a stack over \mathcal{C} . In this setting we write $f^{-1}\mathcal{S}$ for the stackification of the fibred category $u_p\mathcal{S}$ over \mathcal{D} constructed above. We say that $f^{-1}\mathcal{S}$ is the *pullback of \mathcal{S} along f* .

Of course, if \mathcal{S} is a stack in groupoids, then $f^{-1}\mathcal{S}$ is a stack in groupoids by Lemmas 9.1 and 12.7.

Lemma 12.10. *Let $f : \mathcal{D} \rightarrow \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \rightarrow \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 15.6. Let $p : \mathcal{S} \rightarrow \mathcal{C}$ and $q : \mathcal{T} \rightarrow \mathcal{D}$ be stacks. Then we have a canonical equivalence of categories*

$$\mathrm{Mor}_{\mathrm{Stacks}/\mathcal{C}}(\mathcal{S}, f_*\mathcal{T}) = \mathrm{Mor}_{\mathrm{Stacks}/\mathcal{D}}(f^{-1}\mathcal{S}, \mathcal{T})$$

of morphism categories.

Proof. For $i = 1, 2$ an i -morphism of stacks is the same thing as a i -morphism of fibred categories, see Definition 4.5. By Lemma 12.8 we have already

$$\mathrm{Mor}_{\mathrm{Fib}/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) = \mathrm{Mor}_{\mathrm{Fib}/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

Hence the result follows from Lemma 8.3 as $u^p\mathcal{T} = f_*\mathcal{T}$ and $f^{-1}\mathcal{S}$ is the stackification of $u_p\mathcal{S}$. \square

Lemma 12.11. *Let $f : \mathcal{D} \rightarrow \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \rightarrow \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 15.6. Let $\mathcal{S} \rightarrow \mathcal{C}$ be a fibred category, and let $\mathcal{S} \rightarrow \mathcal{S}'$ be the stackification of \mathcal{S} . Then $f^{-1}\mathcal{S}'$ is the stackification of $u_p\mathcal{S}$.*

Proof. Omitted. Hint: This is the analogue of Sites, Lemma 14.4. \square

The following lemma tells us that the 2-category of stacks over Sch_{fppf} is a “full 2-sub category” of the 2-category of stacks over Sch'_{fppf} provided that Sch'_{fppf} contains Sch_{fppf} (see Topologies, Section 10).

Lemma 12.12. *Let \mathcal{C} and \mathcal{D} be sites. Let $u : \mathcal{C} \rightarrow \mathcal{D}$ be a functor satisfying the assumptions of Sites, Lemma 20.8. Let $f : \mathcal{D} \rightarrow \mathcal{C}$ be the corresponding morphism of sites. Then*

- (1) *for every stack $p : \mathcal{S} \rightarrow \mathcal{C}$ the canonical functor $\mathcal{S} \rightarrow f_*f^{-1}\mathcal{S}$ is an equivalence of stacks,*
- (2) *given stacks $\mathcal{S}, \mathcal{S}'$ over \mathcal{C} the construction f^{-1} induces an equivalence*

$$\mathrm{Mor}_{\mathrm{Stacks}/\mathcal{C}}(\mathcal{S}, \mathcal{S}') \longrightarrow \mathrm{Mor}_{\mathrm{Stacks}/\mathcal{D}}(f^{-1}\mathcal{S}, f^{-1}\mathcal{S}')$$

of morphism categories.

Proof. Note that by Lemma 12.10 we have an equivalence of categories

$$\mathrm{Mor}_{\mathrm{Stacks}/\mathcal{D}}(f^{-1}\mathcal{S}, f^{-1}\mathcal{S}') = \mathrm{Mor}_{\mathrm{Stacks}/\mathcal{C}}(\mathcal{S}, f_*f^{-1}\mathcal{S}')$$

Hence (2) follows from (1).

To prove (1) we are going to use Lemma 4.8. This lemma tells us that we have to show that $can : \mathcal{S} \rightarrow f_*f^{-1}\mathcal{S}$ is fully faithful and that all objects of $f_*f^{-1}\mathcal{S}$ are locally in the essential image.

We quickly describe the functor can , see proof of Lemma 12.8. To do this we introduce the functor $c'' : \mathcal{S} \rightarrow u_{pp}\mathcal{S}$ defined by $c''(x/U) = (U, \mathrm{id} : u(U) \rightarrow u(U), x)$, and $c''(\alpha/a) = (a, u(a), \alpha)$. We set $c' : \mathcal{S} \rightarrow u_p\mathcal{S}$ equal to the composition of c'' and the canonical functor $u_{pp}\mathcal{S} \rightarrow u_p\mathcal{S}$. We set $c : \mathcal{S} \rightarrow f^{-1}\mathcal{S}$ equal to the composition of c' and the canonical functor $u_p\mathcal{S} \rightarrow f^{-1}\mathcal{S}$. Then $can : \mathcal{S} \rightarrow f_*f^{-1}\mathcal{S}$ is the functor which to x/U associates the pair $(U, c(x))$ and to α/a the morphism $(a, c(\alpha))$.

Fully faithfulness. To prove this we are going to use Lemma 4.7. Let $U \in \text{Ob}(\mathcal{C})$. Let $x, y \in \mathcal{S}_U$. First off, as u is fully faithful, we have

$$\text{Mor}_{(f_* f^{-1} \mathcal{S})_U}(\text{can}(x), \text{can}(y)) = \text{Mor}_{(f^{-1} \mathcal{S})_{u(U)}}(c(x), c(y))$$

directly from the definition of f_* . Similar holds after pulling back to any U'/U . Because $f^{-1} \mathcal{S}$ is the stackification of $u_p \mathcal{S}$, and since u is continuous and cocontinuous the presheaf

$$U'/U \mapsto \text{Mor}_{(f^{-1} \mathcal{S})_{u(U')}}(c(x|_{U'}), c(y|_{U'}))$$

is the sheafification of the presheaf

$$U'/U \mapsto \text{Mor}_{(u_p \mathcal{S})_{u(U')}}(c'(x|_{U'}), c'(y|_{U'}))$$

Hence to finish the proof of fully faithfulness it suffices to show that for any U and x, y the map

$$\text{Mor}_{\mathcal{S}_U}(x, y) \longrightarrow \text{Mor}_{(u_p \mathcal{S})_U}(c'(x), c'(y))$$

is bijective. A morphism $f : x \rightarrow y$ in $u_p \mathcal{S}$ over $u(U)$ is given by an equivalence class of diagrams

$$\begin{array}{ccc} (U', \phi : u(U) \rightarrow u(U'), x') & \xrightarrow{(a, b, \alpha)} & (U, \text{id} : u(U) \rightarrow u(U), y) \\ (c, \text{id}_{u(U)}, \gamma) \downarrow & & \\ (U, \text{id} : u(U) \rightarrow u(U), x) & & \end{array}$$

with γ strongly cartesian and $b = \text{id}_{u(U)}$. But since u is fully faithful we can write $\phi = u(c')$ for some morphism $c' : U \rightarrow U'$ and then we see that $a \circ c' = \text{id}_U$ and $c \circ c' = \text{id}_{U'}$. Because γ is strongly cartesian we can find a morphism $\gamma' : x \rightarrow x'$ lifting c' such that $\gamma \circ \gamma' = \text{id}_x$. By definition of the equivalence classes defining morphisms in $u_p \mathcal{S}$ it follows that the morphism

$$(U, \text{id} : u(U) \rightarrow u(U), x) \xrightarrow{(\text{id}, \text{id}, \alpha \circ \gamma')} (U, \text{id} : u(U) \rightarrow u(U), y)$$

of $u_{pp} \mathcal{S}$ induces the morphism f in $u_p \mathcal{S}$. This proves that the map is surjective. We omit the proof that it is injective.

Finally, we have to show that any object of $f_* f^{-1} \mathcal{S}$ locally comes from an object of \mathcal{S} . This is clear from the constructions (details omitted). \square

13. Stacks and localization

Let \mathcal{C} be a site. Let U be an object of \mathcal{C} . We want to understand stacks over \mathcal{C}/U as stacks over \mathcal{C} together with a morphism towards U . The following lemma is the reason why this is easier to do when the presheaf h_U is a sheaf.

Lemma 13.1. *Let \mathcal{C} be a site. Let $U \in \text{Ob}(\mathcal{C})$. Then $j_U : \mathcal{C}/U \rightarrow \mathcal{C}$ is a stack over \mathcal{C} if and only if h_U is a sheaf.*

Proof. Combine Lemma 6.3 with Categories, Example 36.7. \square

Assume that \mathcal{C} is a site, and U is an object of \mathcal{C} whose associated representable presheaf is a sheaf. We denote $j : \mathcal{C}/U \rightarrow \mathcal{C}$ the localization functor.

Construction A. Let $p : \mathcal{S} \rightarrow \mathcal{C}/U$ be a stack over the site \mathcal{C}/U . We define a stack $j_! p : j_! \mathcal{S} \rightarrow \mathcal{C}$ as follows:

- (1) As a category $j_! \mathcal{S} = \mathcal{S}$, and

(2) the functor $j_!p : j_!\mathcal{S} \rightarrow \mathcal{C}$ is just the composition $j \circ p$.

We omit the verification that this is a stack (hint: Use that h_U is a sheaf to glue morphisms to U). There is a canonical functor

$$j_!\mathcal{S} \longrightarrow \mathcal{C}/U$$

namely the functor p which is a 1-morphism of stacks over \mathcal{C} .

Construction B. Let $q : \mathcal{T} \rightarrow \mathcal{C}$ be a stack over \mathcal{C} which is endowed with a morphism of stacks $p : \mathcal{T} \rightarrow \mathcal{C}/U$ over \mathcal{C} . In this case it is automatically the case that $p : \mathcal{T} \rightarrow \mathcal{C}/U$ is a stack over \mathcal{C}/U .

Lemma 13.2. *Assume that \mathcal{C} is a site, and U is an object of \mathcal{C} whose associated representable presheaf is a sheaf. Constructions A and B above define mutually inverse (!) functors of 2-categories*

$$\left\{ \begin{array}{l} \text{2-category of} \\ \text{stacks over } \mathcal{C}/U \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{2-category of pairs } (\mathcal{T}, p) \text{ consisting} \\ \text{of a stack } \mathcal{T} \text{ over } \mathcal{C} \text{ and a morphism} \\ p : \mathcal{T} \rightarrow \mathcal{C}/U \text{ of stacks over } \mathcal{C} \end{array} \right\}$$

Proof. This is clear. □

14. Other chapters

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