

DERIVED CATEGORIES

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1. Introduction

We first discuss triangulated categories and localization in triangulated categories. Next, we prove that the homotopy category of complexes in an additive category is a triangulated category. Once this is done we define the derived category of an abelian category as the localization of the homotopy category with respect to quasi-isomorphisms. A good reference is Verdier's thesis [Ver96].

2. Triangulated categories

Triangulated categories are a convenient tool to describe the type of structure inherent in the derived category of an abelian category. Some references are [Ver96], [KS06], and [Nee01].

3. The definition of a triangulated category

In this section we collect most of the definitions concerning triangulated and pre-triangulated categories.

Definition 3.1. Let \mathcal{D} be an additive category. Let $[n] : \mathcal{D} \rightarrow \mathcal{D}$, $E \mapsto E[n]$ be a collection of additive functors indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n + m]$ and $[0] = \text{id}$ (equality as functors). In this situation we call *triangle* a sextuple (X, Y, Z, f, g, h) where $X, Y, Z \in \text{Ob}(\mathcal{D})$ and $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : Z \rightarrow X[1]$ are morphisms of \mathcal{D} . A *morphism of triangles* $(X, Y, Z, f, g, h) \rightarrow (X', Y', Z', f', g', h')$ is given by morphisms $a : X \rightarrow X'$, $b : Y \rightarrow Y'$ and $c : Z \rightarrow Z'$ of \mathcal{D} such that $b \circ f = f' \circ a$, $c \circ g = g' \circ b$ and $a[1] \circ h = h' \circ c$.

A morphism of triangles is visualized by the following commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow a[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

Here is the definition of a triangulated category as given in Verdier's thesis.

Definition 3.2. A *triangulated category* consists of a triple $(\mathcal{D}, \{[n]\}_{n \in \mathbf{Z}}, \mathcal{T})$ where

- (1) \mathcal{D} is an additive category,
- (2) $[n] : \mathcal{D} \rightarrow \mathcal{D}$, $E \mapsto E[n]$ be a collection of additive functors indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n + m]$ and $[0] = \text{id}$ (equality as functors), and
- (3) \mathcal{T} is a set of triangles called the *distinguished triangles*

subject to the following conditions

- TR1 Any triangle isomorphic to a distinguished triangle is a distinguished triangle. Any triangle of the form $(X, X, 0, \text{id}, 0, 0)$ is distinguished. For any morphism $f : X \rightarrow Y$ of \mathcal{D} there exists a distinguished triangle of the form (X, Y, Z, f, g, h) .
- TR2 The triangle (X, Y, Z, f, g, h) is distinguished if and only if the triangle $(Y, Z, X[1], g, h, -f[1])$ is.

TR3 Given a solid commutative square

$$\begin{array}{ccccccc}
 X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
 \downarrow a & & \downarrow b & & \downarrow & & \downarrow a[1] \\
 X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
 \end{array}$$

whose rows are distinguished triangles there exists a morphism $c : Z \rightarrow Z'$ such that (a, b, c) is a morphism of triangles.

TR4 Given objects X, Y, Z of \mathcal{D} , and morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$, and distinguished triangles (X, Y, Q_1, f, p_1, d_1) , $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3) , there exist morphisms $a : Q_1 \rightarrow Q_2$ and $b : Q_2 \rightarrow Q_3$ such that

- (a) $(Q_1, Q_2, Q_3, a, b, p_1[1] \circ d_3)$ is a distinguished triangle,
- (b) the triple (id_X, g, a) is a morphism of triangles $(X, Y, Q_1, f, p_1, d_1) \rightarrow (X, Z, Q_2, g \circ f, p_2, d_2)$, and
- (c) the triple (f, id_Z, b) is a morphism of triangles $(X, Z, Q_2, g \circ f, p_2, d_2) \rightarrow (Y, Z, Q_3, g, p_3, d_3)$.

We will call $(\mathcal{D}, [\], \mathcal{T})$ a *pre-triangulated category* if TR1, TR2 and TR3 hold.

The explanation of TR4 is that if you think of Q_1 as Y/X , Q_2 as Z/X and Q_3 as Z/Y , then TR4(a) expresses the isomorphism $(Z/X)/(Y/X) \cong Z/Y$ and TR(b) and TR(c) express that we can compare the triangles $X \rightarrow Y \rightarrow Q_1 \rightarrow X[1]$ etc with morphisms of triangles. For a more precise reformulation of this idea see the proof of Lemma 10.2.

The sign in TR2 means that if (X, Y, Z, f, g, h) is a distinguished triangle then in the long sequence

$$(3.2.1) \quad \dots \rightarrow Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \rightarrow \dots$$

each four term sequence gives a distinguished triangle.

As usual we abuse notation and we simply speak of a (pre-)triangulated category \mathcal{D} without explicitly introducing notation for the additional data. The notion of a pre-triangulated category is useful in finding statements equivalent to TR4.

We have the following definition of a triangulated functor.

Definition 3.3. Let $\mathcal{D}, \mathcal{D}'$ be pre-triangulated categories. An *exact functor*, or a *triangulated functor* from \mathcal{D} to \mathcal{D}' is a functor $F : \mathcal{D} \rightarrow \mathcal{D}'$ together with given functorial isomorphisms $\xi_X : F(X[1]) \rightarrow F(X)[1]$ such that for every distinguished triangle (X, Y, Z, f, g, h) of \mathcal{D} the triangle $(F(X), F(Y), F(Z), F(f), F(g), \xi_X \circ F(h))$ is a distinguished triangle of \mathcal{D}' .

An exact functor is additive, see Lemma 4.15. When we say two triangulated categories are equivalent we mean that they are equivalent in the 2-category of triangulated categories. A 2-morphism $a : (F, \xi) \rightarrow (F', \xi')$ in this 2-category is simply a transformation of functors $a : F \rightarrow F'$ which is compatible with ξ and ξ' ,

i.e.,

$$\begin{array}{ccc} F \circ [1] & \xrightarrow{\xi} & [1] \circ F \\ a \star 1 \downarrow & & \downarrow 1 \star a \\ F' \circ [1] & \xrightarrow{\xi'} & [1] \circ F' \end{array}$$

commutes.

Definition 3.4. Let $(\mathcal{D}, [\], \mathcal{T})$ be a pre-triangulated category. A *pre-triangulated subcategory*¹ is a pair $(\mathcal{D}', \mathcal{T}')$ such that

- (1) \mathcal{D}' is an additive subcategory of \mathcal{D} which is preserved under $[1]$ and $[-1]$,
- (2) $\mathcal{T}' \subset \mathcal{T}$ is a subset such that for every $(X, Y, Z, f, g, h) \in \mathcal{T}'$ we have $X, Y, Z \in \text{Ob}(\mathcal{D}')$ and $f, g, h \in \text{Arrows}(\mathcal{D}')$, and
- (3) $(\mathcal{D}', [\], \mathcal{T}')$ is a pre-triangulated category.

If \mathcal{D} is a triangulated category, then we say $(\mathcal{D}', \mathcal{T}')$ is a *triangulated subcategory* if it is a pre-triangulated subcategory and $(\mathcal{D}', [\], \mathcal{T}')$ is a triangulated category.

In this situation the inclusion functor $\mathcal{D}' \rightarrow \mathcal{D}$ is an exact functor with $\xi_X : X[1] \rightarrow X[1]$ given by the identity on $X[1]$.

We will see in Lemma 4.1 that for a distinguished triangle (X, Y, Z, f, g, h) in a pre-triangulated category the composition $g \circ f : X \rightarrow Z$ is zero. Thus the sequence (3.2.1) is a complex. A homological functor is one that turns this complex into a long exact sequence.

Definition 3.5. Let \mathcal{D} be a pre-triangulated category. Let \mathcal{A} be an abelian category. An additive functor $H : \mathcal{D} \rightarrow \mathcal{A}$ is called *homological* if for every distinguished triangle (X, Y, Z, f, g, h) the sequence

$$H(X) \rightarrow H(Y) \rightarrow H(Z)$$

is exact in the abelian category \mathcal{A} . An additive functor $H : \mathcal{D}^{opp} \rightarrow \mathcal{A}$ is called *cohomological* if the corresponding functor $\mathcal{D} \rightarrow \mathcal{A}^{opp}$ is homological.

If $H : \mathcal{D} \rightarrow \mathcal{A}$ is a homological functor we often write $H^n(X) = H(X[n])$ so that $H(X) = H^0(X)$. Our discussion of TR2 above implies that a distinguished triangle (X, Y, Z, f, g, h) determines a long exact sequence

$$(3.5.1) \quad H^{-1}(Z) \xrightarrow{h[-1]} H^0(X) \xrightarrow{f} H^0(Y) \xrightarrow{g} H^0(Z) \xrightarrow{h} H^1(X)$$

This will be called the *long exact sequence* associated to the distinguished triangle and the homological functor. As indicated we will not use any signs for the morphisms in the long exact sequence. This has the side effect that maps in the long exact sequence associated to the rotation (TR2) of a distinguished triangle differ from the maps in the sequence above by some signs.

Definition 3.6. Let \mathcal{A} be an abelian category. Let \mathcal{D} be a triangulated category. A *δ -functor from \mathcal{A} to \mathcal{D}* is given by a functor $G : \mathcal{A} \rightarrow \mathcal{D}$ and a rule which assigns to every short exact sequence

$$0 \rightarrow A \xrightarrow{a} B \xrightarrow{b} C \rightarrow 0$$

a morphism $\delta = \delta_{A \rightarrow B \rightarrow C} : G(C) \rightarrow G(A)[1]$ such that

¹This definition may be nonstandard. If \mathcal{D}' is a full subcategory then \mathcal{T}' is the intersection of the set of triangles in \mathcal{D}' with \mathcal{T} , see Lemma 4.14. In this case we drop \mathcal{T}' from the notation.

- (1) the triangle $(G(A), G(B), G(C), G(a), G(b), \delta_{A \rightarrow B \rightarrow C})$ is a distinguished triangle of \mathcal{D} for any short exact sequence as above, and
- (2) for every morphism $(A \rightarrow B \rightarrow C) \rightarrow (A' \rightarrow B' \rightarrow C')$ of short exact sequences the diagram

$$\begin{array}{ccc} G(C) & \xrightarrow{\delta_{A \rightarrow B \rightarrow C}} & G(A)[1] \\ \downarrow & & \downarrow \\ G(C') & \xrightarrow{\delta_{A' \rightarrow B' \rightarrow C'}} & G(A')[1] \end{array}$$

is commutative.

In this situation we call $(G(A), G(B), G(C), G(a), G(b), \delta_{A \rightarrow B \rightarrow C})$ the *image of the short exact sequence under the given δ -functor*.

Note how a δ -functor comes equipped with additional structure. Strictly speaking it does not make sense to say that a given functor $\mathcal{A} \rightarrow \mathcal{D}$ is a δ -functor, but we will often do so anyway.

4. Elementary results on triangulated categories

Most of the results in this section are proved for pre-triangulated categories and a fortiori hold in any triangulated category.

Lemma 4.1. *Let \mathcal{D} be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. Then $g \circ f = 0$, $h \circ g = 0$ and $f[1] \circ h = 0$.*

Proof. By TR1 we know $(X, X, 0, 1, 0, 0)$ is a distinguished triangle. Apply TR3 to

$$\begin{array}{ccccccc} X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \downarrow 1 & & \downarrow f & & \downarrow \text{ } & & \downarrow 1[1] \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \end{array}$$

Of course the dotted arrow is the zero map. Hence the commutativity of the diagram implies that $g \circ f = 0$. For the other cases rotate the triangle, i.e., apply TR2. \square

Lemma 4.2. *Let \mathcal{D} be a pre-triangulated category. For any object W of \mathcal{D} the functor $\text{Hom}_{\mathcal{D}}(W, -)$ is homological, and the functor $\text{Hom}_{\mathcal{D}}(-, W)$ is cohomological.*

Proof. Consider a distinguished triangle (X, Y, Z, f, g, h) . We have already seen that $g \circ f = 0$, see Lemma 4.1. Suppose $a : W \rightarrow Y$ is a morphism such that $g \circ a = 0$. Then we get a commutative diagram

$$\begin{array}{ccccccc} W & \xrightarrow{1} & W & \longrightarrow & 0 & \longrightarrow & W[1] \\ \vdots \downarrow b & & \downarrow a & & \downarrow 0 & & \downarrow b[1] \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

Both rows are distinguished triangles (use TR1 for the top row). Hence we can fill the dotted arrow b (first rotate using TR2, then apply TR3, and then rotate back). This proves the lemma. \square

Lemma 4.3. *Let \mathcal{D} be a pre-triangulated category. Let*

$$(a, b, c) : (X, Y, Z, f, g, h) \rightarrow (X', Y', Z', f', g', h')$$

be a morphism of distinguished triangles. If two among a, b, c are isomorphisms so is the third.

Proof. Assume that a and c are isomorphisms. For any object W of \mathcal{D} write $H_W(-) = \text{Hom}_{\mathcal{D}}(W, -)$. Then we get a commutative diagram of abelian groups

$$\begin{array}{ccccccccc} H_W(Z[-1]) & \longrightarrow & H_W(X) & \longrightarrow & H_W(Y) & \longrightarrow & H_W(Z) & \longrightarrow & H_W(X[1]) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_W(Z'[-1]) & \longrightarrow & H_W(X') & \longrightarrow & H_W(Y') & \longrightarrow & H_W(Z') & \longrightarrow & H_W(X'[1]) \end{array}$$

By assumption the right two and left two vertical arrows are bijective. As H_W is homological by Lemma 4.2 and the five lemma (Homology, Lemma 5.20) it follows that the middle vertical arrow is an isomorphism. Hence by Yoneda's lemma, see Categories, Lemma 3.5 we see that b is an isomorphism. This implies the other cases by rotating (using TR2). \square

Remark 4.4. Let \mathcal{D} be an additive category with translation functors $[n]$ as in Definition 3.1. Let us call a triangle (X, Y, Z, f, g, h) *special*² if for every object W of \mathcal{D} the long sequence of abelian groups

$$\dots \rightarrow \text{Hom}_{\mathcal{D}}(W, X) \rightarrow \text{Hom}_{\mathcal{D}}(W, Y) \rightarrow \text{Hom}_{\mathcal{D}}(W, Z) \rightarrow \text{Hom}_{\mathcal{D}}(W, X[1]) \rightarrow \dots$$

is exact. The proof of Lemma 4.3 shows that if

$$(a, b, c) : (X, Y, Z, f, g, h) \rightarrow (X', Y', Z', f', g', h')$$

is a morphism of special triangles and if two among a, b, c are isomorphisms so is the third. There is a dual statement for *co-special* triangles, i.e., triangles which turn into long exact sequences on applying the functor $\text{Hom}_{\mathcal{D}}(-, W)$. Thus distinguished triangles are special and co-special, but in general there are many more (co-)special triangles, then there are distinguished triangles.

Lemma 4.5. *Let \mathcal{D} be a pre-triangulated category. Let*

$$(0, b, 0), (0, b', 0) : (X, Y, Z, f, g, h) \rightarrow (X, Y, Z, f, g, h)$$

be endomorphisms of a distinguished triangle. Then $bb' = 0$.

Proof. Picture

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow 0 & \nearrow \alpha & \downarrow b, b' & \nearrow \beta & \downarrow 0 & & \downarrow 0 \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

Applying Lemma 4.2 we find dotted arrows α and β such that $b' = f \circ \alpha$ and $b = \beta \circ g$. Then $bb' = \beta \circ g \circ f \circ \alpha = 0$ as $g \circ f = 0$ by Lemma 4.1. \square

²This is nonstandard notation.

Lemma 4.6. *Let \mathcal{D} be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. If*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X[1] \\ c \downarrow & & \downarrow a[1] \\ Z & \xrightarrow{h} & X[1] \end{array}$$

is commutative and $a^2 = a$, $c^2 = c$, then there exists a morphism $b : Y \rightarrow Y$ with $b^2 = b$ such that (a, b, c) is an endomorphism of the triangle (X, Y, Z, f, g, h) .

Proof. By TR3 there exists a morphism b' such that (a, b', c) is an endomorphism of (X, Y, Z, f, g, h) . Then $(0, (b')^2 - b', 0)$ is also an endomorphism. By Lemma 4.5 we see that $(b')^2 - b'$ has square zero. Set $b = b' - (2b' - 1)((b')^2 - b') = 3(b')^2 - 2(b')^3$. A computation shows that (a, b, c) is an endomorphism and that $b^2 - b = (4(b')^2 - 4b' - 3)((b')^2 - b')^2 = 0$. \square

Lemma 4.7. *Let \mathcal{D} be a pre-triangulated category. Let $f : X \rightarrow Y$ be a morphism of \mathcal{D} . There exists a distinguished triangle (X, Y, Z, f, g, h) which is unique up to (nonunique) isomorphism of triangles. More precisely, given a second such distinguished triangle (X, Y, Z', f, g', h') there exists an isomorphism*

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Z', f, g', h')$$

Proof. Existence by TR1. Uniqueness up to isomorphism by TR3 and Lemma 4.3. \square

Lemma 4.8. *Let \mathcal{D} be a pre-triangulated category. Let $f : X \rightarrow Y$ be a morphism of \mathcal{D} . The following are equivalent*

- (1) *f is an isomorphism,*
- (2) *$(X, Y, 0, f, 0, 0)$ is a distinguished triangle, and*
- (3) *for any distinguished triangle (X, Y, Z, f, g, h) we have $Z = 0$.*

Proof. By TR1 the triangle $(X, X, 0, 1, 0, 0)$ is distinguished. Let (X, Y, Z, f, g, h) be a distinguished triangle. By TR3 there is a map of distinguished triangles $(1, f, 0) : (X, X, 0) \rightarrow (X, Y, Z)$. If f is an isomorphism, then $(1, f, 0)$ is an isomorphism of triangles by Lemma 4.3 and $Z = 0$. Conversely, if $Z = 0$, then $(1, f, 0)$ is an isomorphism of triangles as well, hence f is an isomorphism. \square

Lemma 4.9. *Let \mathcal{D} be a pre-triangulated category. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be triangles. The following are equivalent*

- (1) *$(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$ is a distinguished triangle,*
- (2) *both (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') are distinguished triangles.*

Proof. Assume (2). By TR1 we may choose a distinguished triangle $(X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. By TR3 we can find morphisms of distinguished triangles $(X, Y, Z, f, g, h) \rightarrow (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$ and $(X', Y', Z', f', g', h') \rightarrow (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. Taking the direct sum of these morphisms we obtain a morphism of triangles

$$\begin{array}{c} (X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h') \\ \downarrow (1, 1, c) \\ (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h''). \end{array}$$

In the terminology of Remark 4.4 this is a map of special triangles (because a direct sum of special triangles is special) and we conclude that c is an isomorphism. Thus (1) holds.

Assume (1). We will show that (X, Y, Z, f, g, h) is a distinguished triangle. First observe that (X, Y, Z, f, g, h) is a special triangle (terminology from Remark 4.4) as a direct summand of the distinguished hence special triangle $(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$. Using TR1 let (X, Y, Q, f, g'', h'') be a distinguished triangle. By TR3 there exists a morphism of distinguished triangles $(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h') \rightarrow (X, Y, Q, f, g'', h'')$. Composing this with the inclusion map we get a morphism of triangles

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Q, f, g'', h'')$$

By Remark 4.4 we find that c is an isomorphism and we conclude that (2) holds. \square

Lemma 4.10. *Let \mathcal{D} be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle.*

- (1) *If $h = 0$, then there exists a right inverse $s : Z \rightarrow Y$ to g .*
- (2) *For any right inverse $s : Z \rightarrow Y$ of g the map $f \oplus s : X \oplus Z \rightarrow Y$ is an isomorphism.*
- (3) *For any objects X', Z' of \mathcal{D} the triangle $(X', X' \oplus Z', Z', (1, 0), (0, 1), 0)$ is distinguished.*

Proof. To see (1) use that $\text{Hom}_{\mathcal{D}}(Z, Y) \rightarrow \text{Hom}_{\mathcal{D}}(Z, Z) \rightarrow \text{Hom}_{\mathcal{D}}(Z, X[1])$ is exact by Lemma 4.2. By the same token, if s is as in (2), then $h = 0$ and the sequence

$$0 \rightarrow \text{Hom}_{\mathcal{D}}(W, X) \rightarrow \text{Hom}_{\mathcal{D}}(W, Y) \rightarrow \text{Hom}_{\mathcal{D}}(W, Z) \rightarrow 0$$

is split exact (split by $s : Z \rightarrow Y$). Hence by Yoneda's lemma we see that $X \oplus Z \rightarrow Y$ is an isomorphism. The last assertion follows from TR1 and Lemma 4.9. \square

Lemma 4.11. *Let \mathcal{D} be a pre-triangulated category. Let $f : X \rightarrow Y$ be a morphism of \mathcal{D} . The following are equivalent*

- (1) *f has a kernel,*
- (2) *f has a cokernel,*
- (3) *f is isomorphic to a map $K \oplus Z \rightarrow Z \oplus Q$ induced by id_Z .*

Proof. Any morphism isomorphic to a map of the form $X' \oplus Z \rightarrow Z \oplus Y'$ has both a kernel and a cokernel. Hence (3) \Rightarrow (1), (2). Next we prove (1) \Rightarrow (3). Suppose first that $f : X \rightarrow Y$ is a monomorphism, i.e., its kernel is zero. By TR1 there exists a distinguished triangle (X, Y, Z, f, g, h) . By Lemma 4.1 the composition $h[-1] \circ f = 0$. As f is a monomorphism we see that $h[-1] = 0$ and hence $h = 0$. Then Lemma 4.10 implies that $Y = X \oplus Z$, i.e., we see that (3) holds. Next, assume f has a kernel K . As $K \rightarrow X$ is a monomorphism we conclude $X = K \oplus X'$ and $f|_{X'} : X' \rightarrow Y$ is a monomorphism. Hence $Y = X' \oplus Y'$ and we win. The implication (2) \Rightarrow (3) is dual to this. \square

Lemma 4.12. *Let \mathcal{D} be a pre-triangulated category. If \mathcal{D} has countable products, then \mathcal{D} is Karoubian. If \mathcal{D} has countable coproducts, then \mathcal{D} is Karoubian.*

Proof. Assume \mathcal{D} has countable products. By Homology, Lemma 4.3 it suffices to check that morphisms which have a right inverse have kernels. Any morphism which has a right inverse is an epimorphism, hence has a kernel by Lemma 4.11. The second statement is dual to the first. \square

The following lemma makes it slightly easier to prove that a pre-triangulated category is triangulated.

Lemma 4.13. *Let \mathcal{D} be a pre-triangulated category. In order to prove TR4 it suffices to show that given any pair of composable morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ there exist*

- (1) *isomorphisms $i : X' \rightarrow X$, $j : Y' \rightarrow Y$ and $k : Z' \rightarrow Z$, and then setting $f' = j^{-1}fi : X' \rightarrow Y'$ and $g' = k^{-1}gj : Y' \rightarrow Z'$ there exist*
- (2) *distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$, such that the assertion of TR4 holds.*

Proof. The replacement of X, Y, Z by X', Y', Z' is harmless by our definition of distinguished triangles and their isomorphisms. The lemma follows from the fact that the distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$ are unique up to isomorphism by Lemma 4.7. \square

Lemma 4.14. *Let \mathcal{D} be a pre-triangulated category. Assume that \mathcal{D}' is an additive full subcategory of \mathcal{D} . The following are equivalent*

- (1) *there exists a set of triangles \mathcal{T}' such that $(\mathcal{D}', \mathcal{T}')$ is a pre-triangulated subcategory of \mathcal{D} ,*
- (2) *\mathcal{D}' is preserved under $[1], [-1]$ and given any morphism $f : X \rightarrow Y$ in \mathcal{D}' there exists a distinguished triangle (X, Y, Z, f, g, h) in \mathcal{D} such that Z is isomorphic to an object of \mathcal{D}' .*

In this case \mathcal{T}' is the set of distinguished triangles (X, Y, Z, f, g, h) of \mathcal{D} such that $X, Y, Z \in \text{Ob}(\mathcal{D}')$ and $f, g, h \in \text{Arrows}(\mathcal{D}')$. Finally, if \mathcal{D} is a triangulated category, then (1) and (2) are also equivalent to

- (3) *\mathcal{D}' is a triangulated subcategory.*

Proof. Omitted. \square

Lemma 4.15. *An exact functor of pre-triangulated categories is additive.*

Proof. Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor of pre-triangulated categories. Since $(0, 0, 0, 1_0, 1_0, 0)$ is a distinguished triangle of \mathcal{D} the triangle

$$(F(0), F(0), F(0), 1_{F(0)}, 1_{F(0)}, F(0))$$

is distinguished in \mathcal{D}' . This implies that $1_{F(0)} \circ 1_{F(0)}$ is zero, see Lemma 4.1. Hence $F(0)$ is the zero object of \mathcal{D}' . This also implies that F applied to any zero morphism is zero (since a morphism in an additive category is zero if and only if it factors through the zero object). Next, using that $(X, X \oplus Y, Y, (1, 0), (0, 1), 0)$ is a distinguished triangle, we see that $(F(X), F(X \oplus Y), F(Y), F(1, 0), F(0, 1), 0)$ is one too. This implies that the map $F(1, 0) \oplus F(0, 1) : F(X) \oplus F(Y) \rightarrow F(X \oplus Y)$ is an isomorphism, see Lemma 4.10. We omit the rest of the argument. \square

Lemma 4.16. *Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be a fully faithful exact functor of pre-triangulated categories. Then a triangle (X, Y, Z, f, g, h) of \mathcal{D} is distinguished if and only if $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished in \mathcal{D}' .*

Proof. The “only if” part is clear. Assume $(F(X), F(Y), F(Z))$ is distinguished in \mathcal{D}' . Pick a distinguished triangle (X, Y, Z', f, g', h') in \mathcal{D} . By Lemma 4.7 there exists an isomorphism of triangles

$$(1, 1, c') : (F(X), F(Y), F(Z)) \longrightarrow (F(X), F(Y), F(Z')).$$

Since F is fully faithful, there exists a morphism $c : Z \rightarrow Z'$ such that $F(c) = c'$. Then $(1, 1, c)$ is an isomorphism between (X, Y, Z) and (X, Y, Z') . Hence (X, Y, Z) is distinguished by TR1. \square

Lemma 4.17. *Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be pre-triangulated categories. Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ and $F' : \mathcal{D}' \rightarrow \mathcal{D}''$ be exact functors. Then $F' \circ F$ is an exact functor.*

Proof. Omitted. \square

Lemma 4.18. *Let \mathcal{D} be a pre-triangulated category. Let \mathcal{A} be an abelian category. Let $H : \mathcal{D} \rightarrow \mathcal{A}$ be a homological functor.*

- (1) *Let \mathcal{D}' be a pre-triangulated category. Let $F : \mathcal{D}' \rightarrow \mathcal{D}$ be an exact functor. Then the composition $G \circ F$ is a homological functor as well.*
- (2) *Let \mathcal{A}' be an abelian category. Hence $G : \mathcal{A} \rightarrow \mathcal{A}'$ be an exact functor. Hence $G \circ H$ is a homological functor as well.*

Proof. Omitted. \square

Lemma 4.19. *Let \mathcal{D} be a triangulated category. Let \mathcal{A} be an abelian category. Let $G : \mathcal{A} \rightarrow \mathcal{D}$ be a δ -functor.*

- (1) *Let \mathcal{D}' be a triangulated category. Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor. Then the composition $F \circ G$ is a δ -functor as well.*
- (2) *Let \mathcal{A}' be an abelian category. Hence $H : \mathcal{A}' \rightarrow \mathcal{A}$ be an exact functor. Hence $G \circ H$ is a δ -functor as well.*

Proof. Omitted. \square

Lemma 4.20. *Let \mathcal{D} be a triangulated category. Let \mathcal{A} be an abelian category. Let $G : \mathcal{A} \rightarrow \mathcal{D}$ be a δ -functor. Let $H : \mathcal{D} \rightarrow \mathcal{B}$ be a homological functor. Assume that $H^{-1}(G(A)) = 0$ for all A in \mathcal{A} . Then the collection*

$$\{H^n \circ G, H^n(\delta_{A \rightarrow B \rightarrow C})\}_{n \geq 0}$$

is a δ -functor from $\mathcal{A} \rightarrow \mathcal{B}$, see Homology, Definition 11.1.

Proof. The notation signifies the following. If $0 \rightarrow A \xrightarrow{a} B \xrightarrow{b} C \rightarrow 0$ is a short exact sequence in \mathcal{A} , then

$$\delta = \delta_{A \rightarrow B \rightarrow C} : G(C) \rightarrow G(A)[1]$$

is a morphism in \mathcal{D} such that $(G(A), G(B), G(C), a, b, \delta)$ is a distinguished triangle, see Definition 3.6. Then $H^n(\delta) : H^n(G(C)) \rightarrow H^n(G(A)[1]) = H^{n+1}(G(A))$ is clearly functorial in the short exact sequence. Finally, the long exact cohomology sequence (3.5.1) combined with the vanishing of $H^{-1}(G(C))$ gives a long exact sequence

$$0 \rightarrow H^0(G(A)) \rightarrow H^0(G(B)) \rightarrow H^0(G(C)) \xrightarrow{H^0(\delta)} H^1(G(A)) \rightarrow \dots$$

in \mathcal{B} as desired. \square

The proof of the following result uses TR4.

Proposition 4.21. *Let \mathcal{D} be a triangulated category. Any commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

can be extended to a diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X[1] & \longrightarrow & Y[1] & \longrightarrow & Z[1] & \longrightarrow & X[2] \end{array}$$

where all the squares are commutative, except for the lower right square which is anticommutative. Moreover, each of the rows and columns are distinguished triangles. Finally, the morphisms on the bottom row (resp. right column) are obtained from the morphisms of the top row (resp. left column) by applying $[1]$.

Proof. During this proof we avoid writing the arrows in order to make the proof legible. Choose distinguished triangles (X, Y, Z) , (X', Y', Z') , (X, X', X'') , (Y, Y', Y'') , and (X, Y', A) . Note that the morphism $X \rightarrow Y'$ is both equal to the composition $X \rightarrow Y \rightarrow Y'$ and equal to the composition $X \rightarrow X' \rightarrow Y'$. Hence, we can find morphisms

- (1) $a : Z \rightarrow A$ and $b : A \rightarrow Y''$, and
- (2) $a' : X'' \rightarrow A$ and $b' : A \rightarrow Z'$

as in TR4. Denote $c : Y'' \rightarrow Z[1]$ the composition $Y'' \rightarrow Y[1] \rightarrow Z[1]$ and denote $c' : Z' \rightarrow X''[1]$ the composition $Z' \rightarrow X'[1] \rightarrow X''[1]$. The conclusion of our application TR4 are that

- (1) (Z, A, Y'', a, b, c) , (X'', A, Z', a', b', c') are distinguished triangles,
- (2) $(X, Y, Z) \rightarrow (X, Y', A)$, $(X, Y', A) \rightarrow (Y, Y', Y'')$, $(X, X', X'') \rightarrow (X, Y', A)$, $(X, Y', A) \rightarrow (X', Y', Z')$ are morphisms of triangles.

First using that $(X, X', X'') \rightarrow (X, Y', A)$ and $(X, Y', A) \rightarrow (Y, Y', Y'')$ are morphisms of triangles we see the first of the commutative diagrams

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X'' & \xrightarrow{b \circ a'} & Y'' \\ \downarrow & & \downarrow \\ X[1] & \longrightarrow & Y[1] \end{array} \quad \begin{array}{ccc} Y & \longrightarrow & Z \longrightarrow X[1] \\ \downarrow & & \downarrow b' \circ a \quad \downarrow \\ Y' & \longrightarrow & Z' \longrightarrow X'[1] \end{array}$$

is commutative. The second is commutative too using that $(X, Y, Z) \rightarrow (X, Y', A)$ and $(X, Y', A) \rightarrow (X', Y', Z')$ are morphisms of triangles. At this point we choose a distinguished triangle (X'', Y'', Z'') starting with the map $b \circ a' : X'' \rightarrow Y''$.

Next we apply TR4 one more time to the morphisms $X'' \rightarrow A \rightarrow Y''$ and the triangles (X'', A, Z', a', b', c') , (X'', Y'', Z'') , and $(A, Y'', Z[1], b, c, -a[1])$ to get morphisms $a'' : Z' \rightarrow Z''$ and $b'' : Z'' \rightarrow Z[1]$. Then $(Z', Z'', Z[1], a'', b'', -b'[1] \circ a[1])$ is a distinguished triangle, hence also $(Z, Z', Z'', -b' \circ a, a'', -b'')$ and hence also $(Z, Z', Z'', b' \circ a, a'', b'')$. Moreover, $(X'', A, Z') \rightarrow (X'', Y'', Z'')$ and $(X'', Y'', Z'') \rightarrow (A, Y'', Z[1], b, c, -a[1])$ are morphisms of triangles. At this point we have defined all the distinguished triangles and all the morphisms, and all that's left is to verify some commutativity relations.

To see that the middle square in the diagram commutes, note that the arrow $Y' \rightarrow Z'$ factors as $Y' \rightarrow A \rightarrow Z'$ because $(X, Y', A) \rightarrow (X', Y', Z')$ is a morphism of triangles. Similarly, the morphism $Y' \rightarrow Y''$ factors as $Y' \rightarrow A \rightarrow Y''$ because $(X, Y', A) \rightarrow (Y, Y', Y'')$ is a morphism of triangles. Hence the middle square commutes because the square with sides (A, Z', Z'', Y'') commutes as $(X'', A, Z') \rightarrow (X'', Y'', Z'')$ is a morphism of triangles (by TR4). The square with sides $(Y'', Z'', Y[1], Z[1])$ commutes because $(X'', Y'', Z'') \rightarrow (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and $c : Y'' \rightarrow Z[1]$ is the composition $Y'' \rightarrow Y[1] \rightarrow Z[1]$. The square with sides $(Z', X'[1], X''[1], Z'')$ is commutative because $(X'', A, Z') \rightarrow (X'', Y'', Z'')$ is a morphism of triangles and $c' : Z' \rightarrow X''[1]$ is the composition $Z' \rightarrow X'[1] \rightarrow X''[1]$. Finally, we have to show that the square with sides $(Z'', X''[1], Z[1], X[2])$ anticommutes. This holds because $(X'', Y'', Z'') \rightarrow (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and we're done. \square

5. Localization of triangulated categories

In order to construct the derived category starting from the homotopy category of complexes, we will use a localization process.

Definition 5.1. Let \mathcal{D} be a pre-triangulated category. We say a multiplicative system S is *compatible with the triangulated structure* if the following two conditions hold:

MS5 For $s \in S$ we have $s[n] \in S$ for all $n \in \mathbf{Z}$.

MS6 Given a solid commutative square

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow s & & \downarrow s' & & \downarrow & & \downarrow s[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

whose rows are distinguished triangles with $s, s' \in S$ there exists a morphism $s'' : Z \rightarrow Z'$ in S such that (s, s', s'') is a morphism of triangles.

It turns out that these axioms are not independent of the axioms defining multiplicative systems.

Lemma 5.2. Let \mathcal{D} be a pre-triangulated category. Let S be a set of morphisms of \mathcal{D} and assume that axioms MS1, MS5, MS6 hold (see Categories, Definition 25.1 and Definition 5.1). Then MS2 holds.

Proof. Suppose that $f : X \rightarrow Y$ is a morphism of \mathcal{D} and $t : X \rightarrow X'$ an element of S . Choose a distinguished triangle (X, Y, Z, f, g, h) . Next, choose a distinguished triangle $(X', Y', Z, f', g', t[1] \circ h)$ (here we use TR1 and TR2). By MS5, MS6 (and TR2 to rotate) we can find the dotted arrow in the commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow t & & \downarrow \text{ } & & \downarrow 1 & & \downarrow t[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & X'[1] \end{array}$$

with moreover $s' \in S$. This proves LMS2. The proof of RMS2 is dual. \square

Lemma 5.3. *Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let*

$$S = \{f \in \text{Arrows}(\mathcal{D}) \mid F(f) \text{ is an isomorphism}\}$$

Then S is a saturated (see Categories, Definition 25.17) multiplicative system compatible with the triangulated structure on \mathcal{D} .

Proof. We have to prove axioms MS1 – MS6, see Categories, Definitions 25.1 and 25.17 and Definition 5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and Lemma 4.3. By Lemma 5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \rightarrow Y$ be morphisms of \mathcal{D} , and let $t : Z \rightarrow X$ be an element of S such that $f \circ t = g \circ t$. As \mathcal{D} is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, d, h) using TR1. Since $a \circ t = 0$ we see by Lemma 4.2 there exists a morphism $i : Q \rightarrow Y$ such that $i \circ d = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k) . Here is a picture

$$\begin{array}{ccccccc} Z & \xrightarrow{t} & X & \xrightarrow{d} & Q & \longrightarrow & Z[1] \\ & & \downarrow 1 & & \downarrow i & & \\ & & X & \xrightarrow{a} & Y & & \\ & & & & \downarrow j & & \\ & & & & W & & \end{array}$$

OK, and now we apply the functor F to this diagram. Since $t \in S$ we see that $F(Q) = 0$, see Lemma 4.8. Hence $F(j)$ is an isomorphism by the same lemma, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ d = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. \square

Lemma 5.4. *Let $H : \mathcal{D} \rightarrow \mathcal{A}$ be a homological functor between a pre-triangulated category and an abelian category. Let*

$$S = \{f \in \text{Arrows}(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z}\}$$

Then S is a saturated (see Categories, Definition 25.17) multiplicative system compatible with the triangulated structure on \mathcal{D} .

Proof. We have to prove axioms MS1 – MS6, see Categories, Definitions 25.1 and 25.17 and Definition 5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and the long exact cohomology sequence (3.5.1). By Lemma 5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds.

To do this let $f, g : X \rightarrow Y$ be morphisms of \mathcal{D} , and let $t : Z \rightarrow X$ be an element of S such that $f \circ t = g \circ t$. As \mathcal{D} is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 4.2 there exists a morphism $i : Q \rightarrow Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k) . Here is a picture

$$\begin{array}{ccccccc}
 Z & \xrightarrow{t} & X & \xrightarrow{g} & Q & \longrightarrow & Z[1] \\
 & & \downarrow 1 & & \downarrow i & & \\
 & & X & \xrightarrow{a} & Y & & \\
 & & & & \downarrow j & & \\
 & & & & W & &
 \end{array}$$

OK, and now we apply the functors H^i to this diagram. Since $t \in S$ we see that $H^i(Q) = 0$ by the long exact cohomology sequence (3.5.1). Hence $H^i(j)$ is an isomorphism for all i by the same argument, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ g = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. \square

Proposition 5.5. *Let \mathcal{D} be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Then there exists a unique structure of a pre-triangulated category on $S^{-1}\mathcal{D}$ such that the localization functor $Q : \mathcal{D} \rightarrow S^{-1}\mathcal{D}$ is exact. Moreover, if \mathcal{D} is a triangulated category, so is $S^{-1}\mathcal{D}$.*

Proof. We have seen that $S^{-1}\mathcal{D}$ is an additive category and that the localization functor Q is additive in Homology, Lemma 8.2. It is clear that we may define $Q(X)[n] = Q(X[n])$ since S is preserved under the shift functors $[n]$ by MS5. Finally, we say a triangle of $S^{-1}\mathcal{D}$ is distinguished if it is isomorphic to the image of a distinguished triangle under the localization functor Q .

Proof of TR1. The only thing to prove here is that if $a : Q(X) \rightarrow Q(Y)$ is a morphism of $S^{-1}\mathcal{D}$, then a fits into a distinguished triangle. Write $a = Q(s)^{-1} \circ Q(f)$ for some $s : Y \rightarrow Y'$ in S and $f : X \rightarrow Y'$. Choose a distinguished triangle (X, Y', Z, f, g, h) in \mathcal{D} . Then we see that $(Q(X), Q(Y), Q(Z), a, Q(g) \circ Q(s), Q(h))$ is a distinguished triangle of $S^{-1}\mathcal{D}$.

Proof of TR2. This is immediate from the definitions.

Proof of TR3. Note that the existence of the dotted arrow which is required to exist may be proven after replacing the two triangles by isomorphic triangles. Hence we may assume given distinguished triangles (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') of \mathcal{D} and a commutative diagram

$$\begin{array}{ccc}
 Q(X) & \xrightarrow{Q(f)} & Q(Y) \\
 a \downarrow & & \downarrow b \\
 Q(X') & \xrightarrow{Q(f')} & Q(Y')
 \end{array}$$

in $S^{-1}\mathcal{D}$. Now we apply Categories, Lemma 25.8 to find a morphism $f'' : X'' \rightarrow Y''$ in \mathcal{D} and a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & X'' & \xleftarrow{\quad} & X' \\ f \downarrow & & \downarrow f'' & & \downarrow f' \\ Y & \xrightarrow{\quad} & Y'' & \xleftarrow{\quad} & Y' \end{array}$$

in \mathcal{D} with $s, t \in S$ and $a = s^{-1}k$, $b = t^{-1}l$. At this point we can use TR3 for \mathcal{D} and MS6 to find a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow k & & \downarrow l & & \downarrow m & & \downarrow g[1] \\ X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\ \uparrow s & & \uparrow t & & \uparrow r & & \uparrow s[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

with $r \in S$. It follows that setting $c = Q(r)^{-1}Q(m)$ we obtain the desired morphism of triangles

$$\begin{array}{c} (Q(X), Q(Y), Q(Z), Q(f), Q(g), Q(h)) \\ \downarrow (a, b, c) \\ (Q(X'), Q(Y'), Q(Z'), Q(f'), Q(g'), Q(h')) \end{array}$$

This proves the first statement of the lemma. If \mathcal{D} is also a triangulated category, then we still have to prove TR4 in order to show that $S^{-1}\mathcal{D}$ is triangulated as well. To do this we reduce by Lemma 4.13 to the following statement: Given composable morphisms $a : Q(X) \rightarrow Q(Y)$ and $b : Q(Y) \rightarrow Q(Z)$ we have to produce an octahedron after possibly replacing $Q(X), Q(Y), Q(Z)$ by isomorphic objects. To do this we may first replace Y by an object such that $a = Q(f)$ for some morphism $f : X \rightarrow Y$ in \mathcal{D} . (More precisely, write $a = s^{-1}f$ with $s : Y \rightarrow Y'$ in S and $f : X \rightarrow Y'$. Then replace Y by Y' .) After this we similarly replace Z by an object such that $b = Q(g)$ for some morphism $g : Y \rightarrow Z$. Now we can find distinguished triangles (X, Y, Q_1, f, p_1, d_1) , $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3) in \mathcal{D} (by TR1), and morphisms $a : Q_1 \rightarrow Q_2$ and $b : Q_2 \rightarrow Q_3$ as in TR4. Then it is immediately verified that applying the functor Q to all these data gives a corresponding structure in $S^{-1}\mathcal{D}$ \square

The universal property of the localization of a triangulated category is as follows (we formulate this for pre-triangulated categories, hence it holds a fortiori for triangulated categories).

Lemma 5.6. *Let \mathcal{D} be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated category. Let $Q : \mathcal{D} \rightarrow S^{-1}\mathcal{D}$ be the localization functor, see Proposition 5.5.*

- (1) *If $H : \mathcal{D} \rightarrow \mathcal{A}$ is a homological functor into an abelian category \mathcal{A} such that $H(s)$ is an isomorphism for all $s \in S$, then the unique factorization $H' : S^{-1}\mathcal{D} \rightarrow \mathcal{A}$ such that $H = H' \circ Q$ (see Categories, Lemma 25.6) is a homological functor too.*

- (2) If $F : \mathcal{D} \rightarrow \mathcal{D}'$ is an exact functor into a pre-triangulated category \mathcal{D}' such that $F(s)$ is an isomorphism for all $s \in S$, then the unique factorization $F' : S^{-1}\mathcal{D} \rightarrow \mathcal{D}'$ such that $F = F' \circ Q$ (see Categories, Lemma 25.6) is an exact functor too.

Proof. This lemma proves itself. Details omitted. \square

The following lemma describes the kernel (see Definition 6.5) of the localization functor.

Lemma 5.7. *Let \mathcal{D} be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Let Z be an object of \mathcal{D} . The following are equivalent*

- (1) $Q(Z) = 0$ in $S^{-1}\mathcal{D}$,
- (2) there exists $Z' \in \text{Ob}(\mathcal{D})$ such that $0 : Z \rightarrow Z'$ is an element of S ,
- (3) there exists $Z' \in \text{Ob}(\mathcal{D})$ such that $0 : Z' \rightarrow Z$ is an element of S , and
- (4) there exists an object Z' and a distinguished triangle $(X, Y, Z \oplus Z', f, g, h)$ such that $f \in S$.

If S is saturated, then these are also equivalent to

- (4) the morphism $0 \rightarrow Z$ is an element of S ,
- (5) the morphism $Z \rightarrow 0$ is an element of S ,
- (6) there exists a distinguished triangle (X, Y, Z, f, g, h) such that $f \in S$.

Proof. The equivalence of (1), (2), and (3) is Homology, Lemma 8.3. If (2) holds, then $(Z'[-1], Z'[-1] \oplus Z, Z, (1, 0), (0, 1), 0)$ is a distinguished triangle (see Lemma 4.10) with “ $0 \in S$ ”. By rotating we conclude that (4) holds. If $(X, Y, Z \oplus Z', f, g, h)$ is a distinguished triangle with $f \in S$ then $Q(f)$ is an isomorphism hence $Q(Z \oplus Z') = 0$ hence $Q(Z) = 0$. Thus (1) – (4) are all equivalent.

Next, assume that S is saturated. Note that each of (4), (5), (6) implies one of the equivalent conditions (1) – (4). Suppose that $Q(Z) = 0$. Then $0 \rightarrow Z$ is a morphism of \mathcal{D} which becomes an isomorphism in $S^{-1}\mathcal{D}$. According to Categories, Lemma 25.18 the fact that S is saturated implies that $0 \rightarrow Z$ is in S . Hence (1) \Rightarrow (4). Dually (1) \Rightarrow (5). Finally, if $0 \rightarrow Z$ is in S , then the triangle $(0, Z, Z, 0, \text{id}_Z, 0)$ is distinguished by TR1 and TR2 and is a triangle as in (4). \square

Lemma 5.8. *Let \mathcal{D} be a triangulated category. Let S be a saturated multiplicative system in \mathcal{D} that is compatible with the triangulated structure. Let (X, Y, Z, f, g, h) be a distinguished triangle in \mathcal{D} . Consider the category of morphisms of triangles*

$$\mathcal{I} = \{(s, s', s'') : (X, Y, Z, f, g, h) \rightarrow (X', Y', Z', f', g', h') \mid s, s', s'' \in S\}$$

Then \mathcal{I} is a filtered category and the functors $\mathcal{I} \rightarrow X/S$, $\mathcal{I} \rightarrow Y/S$, and $\mathcal{I} \rightarrow Z/S$ are cofinal.

Proof. We strongly suggest the reader skip the proof of this lemma and instead works it out on a napkin.

The first remark is that using rotation of distinguished triangles (TR2) gives an equivalence of categories between \mathcal{I} and the corresponding category for the distinguished triangle $(Y, Z, X[1], g, h, -f[1])$. Using this we see for example that if we prove the functor $\mathcal{I} \rightarrow X/S$ is cofinal, then the same thing is true for the functors $\mathcal{I} \rightarrow Y/S$ and $\mathcal{I} \rightarrow Z/S$.

Note that if $s : X \rightarrow X'$ is a morphism of S , then using MS2 we can find $s' : Y \rightarrow Y'$ and $f' : X' \rightarrow Y'$ such that $f' \circ s = s' \circ f$, whereupon we can use MS6 to complete this into an object of \mathcal{I} . Hence the functor $\mathcal{I} \rightarrow X/S$ is surjective on objects. Using rotation as above this implies the same thing is true for the functors $\mathcal{I} \rightarrow Y/S$ and $\mathcal{I} \rightarrow Z/S$.

Suppose given objects $s_1 : X \rightarrow X_1$ and $s_2 : X \rightarrow X_2$ in X/S and a morphism $a : X_1 \rightarrow X_2$ in X/S . Since S is saturated, we see that $a \in S$, see Categories, Lemma 25.18. By the argument of the previous paragraph we can complete $s_1 : X \rightarrow X_1$ to an object $(s_1, s'_1, s''_1) : (X, Y, Z, f, g, h) \rightarrow (X_1, Y_1, Z_1, f_1, g_1, h_1)$ in \mathcal{I} . Then we can repeat and find $(a, b, c) : (X_1, Y_1, Z_1, f_1, g_1, h_1) \rightarrow (X_2, Y_2, Z_2, f_2, g_2, h_2)$ with $a, b, c \in S$ completing the given $a : X_1 \rightarrow X_2$. But then (a, b, c) is a morphism in \mathcal{I} . In this way we conclude that the functor $\mathcal{I} \rightarrow X/S$ is also surjective on arrows. Using rotation as above, this implies the same thing is true for the functors $\mathcal{I} \rightarrow Y/S$ and $\mathcal{I} \rightarrow Z/S$.

The category \mathcal{I} is nonempty as the identity provides an object. This proves the condition (1) of the definition of a filtered category, see Categories, Definition 19.1.

We check condition (2) of Categories, Definition 19.1 for the category \mathcal{I} . Suppose given objects $(s_1, s'_1, s''_1) : (X, Y, Z, f, g, h) \rightarrow (X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(s_2, s'_2, s''_2) : (X, Y, Z, f, g, h) \rightarrow (X_2, Y_2, Z_2, f_2, g_2, h_2)$ in \mathcal{I} . We want to find an object of \mathcal{I} which is the target of an arrow from both $(X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(X_2, Y_2, Z_2, f_2, g_2, h_2)$. By Categories, Remark 25.5 the categories $X/S, Y/S, Z/S$ are filtered. Thus we can find $X \rightarrow X_3$ in X/S and morphisms $s : X_2 \rightarrow X_3$ and $a : X_1 \rightarrow X_3$. By the above we can find a morphism $(s, s', s'') : (X_2, Y_2, Z_2, f_2, g_2, h_2) \rightarrow (X_3, Y_3, Z_3, f_3, g_3, h_3)$ with $s', s'' \in S$. After replacing (X_2, Y_2, Z_2) by (X_3, Y_3, Z_3) we may assume that there exists a morphism $a : X_1 \rightarrow X_2$ in X/S . Repeating the argument for Y and Z (by rotating as above) we may assume there is a morphism $a : X_1 \rightarrow X_2$ in X/S , $b : Y_1 \rightarrow Y_2$ in Y/S , and $c : Z_1 \rightarrow Z_2$ in Z/S . However, these morphisms do not necessarily give rise to a morphism of distinguished triangles. On the other hand, the necessary diagrams do commute in $S^{-1}\mathcal{D}$. Hence we see (for example) that there exists a morphism $s'_2 : Y_2 \rightarrow Y_3$ in S such that $s'_2 \circ f_2 \circ a = s'_2 \circ b \circ f_1$. Another replacement of (X_2, Y_2, Z_2) as above then gets us to the situation where $f_2 \circ a = b \circ f_1$. Rotating and applying the same argument two more times we see that we may assume (a, b, c) is a morphism of triangles. This proves condition (2).

Next we check condition (3) of Categories, Definition 19.1. Suppose $(s_1, s'_1, s''_1) : (X, Y, Z) \rightarrow (X_1, Y_1, Z_1)$ and $(s_2, s'_2, s''_2) : (X, Y, Z) \rightarrow (X_2, Y_2, Z_2)$ are objects of \mathcal{I} , and suppose $(a, b, c), (a', b', c')$ are two morphisms between them. Since $a \circ s_1 = a' \circ s_1$ there exists a morphism $s_3 : X_2 \rightarrow X_3$ such that $s_3 \circ a = s_3 \circ a'$. Using the surjectivity statement we can complete this to a morphism of triangles $(s_3, s'_3, s''_3) : (X_2, Y_2, Z_2) \rightarrow (X_3, Y_3, Z_3)$ with $s_3, s'_3, s''_3 \in S$. Thus $(s_3 \circ s_2, s'_3 \circ s'_2, s''_3 \circ s''_2) : (X, Y, Z) \rightarrow (X_3, Y_3, Z_3)$ is also an object of \mathcal{I} and after composing the maps $(a, b, c), (a', b', c')$ with (s_3, s'_3, s''_3) we obtain $a = a'$. By rotating we may do the same to get $b = b'$ and $c = c'$.

Finally, we check that $\mathcal{I} \rightarrow X/S$ is cofinal, see Categories, Definition 17.1. The first condition is true as the functor is surjective. Suppose that we have an object $s : X \rightarrow X'$ in X/S and two objects $(s_1, s'_1, s''_1) : (X, Y, Z, f, g, h) \rightarrow (X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(s_2, s'_2, s''_2) : (X, Y, Z, f, g, h) \rightarrow (X_2, Y_2, Z_2, f_2, g_2, h_2)$ in \mathcal{I} as well as morphisms $t_1 : X' \rightarrow X_1$ and $t_2 : X' \rightarrow X_2$ in X/S . By property (2) of \mathcal{I} proved above we

can find morphisms $(s_3, s'_3, s''_3) : (X_1, Y_1, Z_1, f_1, g_1, h_1) \rightarrow (X_3, Y_3, Z_3, f_3, g_3, h_3)$ and $(s_4, s'_4, s''_4) : (X_2, Y_2, Z_2, f_2, g_2, h_2) \rightarrow (X_3, Y_3, Z_3, f_3, g_3, h_3)$ in \mathcal{I} . We would be done if the compositions $X' \rightarrow X_1 \rightarrow X_3$ and $X' \rightarrow X_2 \rightarrow X_3$ were equal (see displayed equation in Categories, Definition 17.1). If not, then, because X/S is filtered, we can choose a morphism $X_3 \rightarrow X_4$ in S such that the compositions $X' \rightarrow X_1 \rightarrow X_3 \rightarrow X_4$ and $X' \rightarrow X_2 \rightarrow X_3 \rightarrow X_4$ are equal. Then we finally complete $X_3 \rightarrow X_4$ to a morphism $(X_3, Y_3, Z_3) \rightarrow (X_4, Y_4, Z_4)$ in \mathcal{I} and compose with that morphism to see that the result is true. \square

6. Quotients of triangulated categories

Given a triangulated category and a triangulated subcategory we can construct another triangulated category by taking the “quotient”. The construction uses a localization. This is similar to the quotient of an abelian category by a Serre subcategory, see Homology, Section 9. Before we do the actual construction we briefly discuss kernels of exact functors.

Definition 6.1. Let \mathcal{D} be a pre-triangulated category. We say a full pre-triangulated subcategory \mathcal{D}' of \mathcal{D} is *saturated* if whenever $X \oplus Y$ is isomorphic to an object of \mathcal{D}' then both X and Y are isomorphic to objects of \mathcal{D}' .

A saturated triangulated subcategory is sometimes called a *thick triangulated subcategory*. In some references, this is only used for strictly full triangulated subcategories (and sometimes the definition is written such that it implies strictness). There is another notion, that of an *épaisse triangulated subcategory*. The definition is that given a commutative diagram

$$\begin{array}{ccccc} & & S & & \\ & \nearrow & & \searrow & \\ X & \longrightarrow & Y & \longrightarrow & T \longrightarrow X[1] \end{array}$$

where the second line is a distinguished triangle and S and T isomorphic to objects of \mathcal{D}' , then also X and Y are isomorphic to objects of \mathcal{D} . It turns out that this is equivalent to being saturated (this is elementary and can be found in [Ric89]) and the notion of a saturated category is easier to work with.

Lemma 6.2. Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let \mathcal{D}'' be the full subcategory of \mathcal{D} with objects

$$\text{Ob}(\mathcal{D}'') = \{X \in \text{Ob}(\mathcal{D}) \mid F(X) = 0\}$$

Then \mathcal{D}'' is a strictly full saturated pre-triangulated subcategory of \mathcal{D} . If \mathcal{D} is a triangulated category, then \mathcal{D}'' is a triangulated subcategory.

Proof. It is clear that \mathcal{D}'' is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of \mathcal{D} and $F(X) = F(Y) = 0$, then also $F(Z) = 0$ as $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished. Hence we may apply Lemma 4.14 to see that \mathcal{D}'' is a pre-triangulated subcategory (respectively a triangulated subcategory if \mathcal{D} is a triangulated category). The final assertion of being saturated follows from $F(X) \oplus F(Y) = 0 \Rightarrow F(X) = F(Y) = 0$. \square

Lemma 6.3. *Let $H : \mathcal{D} \rightarrow \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let \mathcal{D}' be the full subcategory of \mathcal{D} with objects*

$$\text{Ob}(\mathcal{D}') = \{X \in \text{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \in \mathbf{Z}\}$$

Then \mathcal{D}' is a strictly full saturated pre-triangulated subcategory of \mathcal{D} . If \mathcal{D} is a triangulated category, then \mathcal{D}' is a triangulated subcategory.

Proof. It is clear that \mathcal{D}' is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of \mathcal{D} and $H(X[n]) = H(Y[n]) = 0$ for all n , then also $H(Z[n]) = 0$ for all n by the long exact sequence (3.5.1). Hence we may apply Lemma 4.14 to see that \mathcal{D}' is a pre-triangulated subcategory (respectively a triangulated subcategory if \mathcal{D} is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

Lemma 6.4. *Let $H : \mathcal{D} \rightarrow \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}_H^+, \mathcal{D}_H^-, \mathcal{D}_H^b$ be the full subcategory of \mathcal{D} with objects*

$$\begin{aligned} \text{Ob}(\mathcal{D}_H^+) &= \{X \in \text{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \ll 0\} \\ \text{Ob}(\mathcal{D}_H^-) &= \{X \in \text{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \gg 0\} \\ \text{Ob}(\mathcal{D}_H^b) &= \{X \in \text{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } |n| \gg 0\} \end{aligned}$$

Each of these is a strictly full saturated pre-triangulated subcategory of \mathcal{D} . If \mathcal{D} is a triangulated category, then each is a triangulated subcategory.

Proof. Let us prove this for \mathcal{D}_H^+ . It is clear that it is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of \mathcal{D} and $H(X[n]) = H(Y[n]) = 0$ for all $n \ll 0$, then also $H(Z[n]) = 0$ for all $n \ll 0$ by the long exact sequence (3.5.1). Hence we may apply Lemma 4.14 to see that \mathcal{D}_H^+ is a pre-triangulated subcategory (respectively a triangulated subcategory if \mathcal{D} is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

Definition 6.5. Let \mathcal{D} be a (pre-)triangulated category.

- (1) Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor. The *kernel of F* is the strictly full saturated (pre-)triangulated subcategory described in Lemma 6.2.
- (2) Let $H : \mathcal{D} \rightarrow \mathcal{A}$ be a homological functor. The *kernel of H* is the strictly full saturated (pre-)triangulated subcategory described in Lemma 6.3.

These are sometimes denoted $\text{Ker}(F)$ or $\text{Ker}(H)$.

The proof of the following lemma uses TR4.

Lemma 6.6. *Let \mathcal{D} be a triangulated category. Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory. Set*

$$(6.6.1) \quad S = \left\{ f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle } (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{D}' \right\}$$

Then S is a multiplicative system compatible with the triangulated structure on \mathcal{D} . In this situation the following are equivalent

- (1) *S is a saturated multiplicative system,*
- (2) *\mathcal{D}' is a saturated triangulated subcategory.*

Proof. To prove the first assertion we have to prove that MS1, MS2, MS3 and MS5, MS6 hold.

Proof of MS1. It is clear that identities are in S because $(X, X, 0, 1, 0, 0)$ is distinguished for every object X of \mathcal{D} and because 0 is an object of \mathcal{D}' . Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be composable morphisms contained in S . Choose distinguished triangles (X, Y, Q_1, f, p_1, d_1) , $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3) . By assumption we know that Q_1 and Q_3 are isomorphic to objects of \mathcal{D}' . By TR4 we know there exists a distinguished triangle (Q_1, Q_2, Q_3, a, b, c) . Since \mathcal{D}' is a triangulated subcategory we conclude that Q_2 is isomorphic to an object of \mathcal{D}' . Hence $g \circ f \in S$.

Proof of MS3. Let $a : X \rightarrow Y$ be a morphism and let $t : Z \rightarrow X$ be an element of S such that $a \circ t = 0$. To prove LMS3 it suffices to find an $s \in S$ such that $s \circ a = 0$, compare with the proof of Lemma 5.3. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 4.2 there exists a morphism $i : Q \rightarrow Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, s, k) . Here is a picture

$$\begin{array}{ccccc} Z & \xrightarrow{t} & X & \xrightarrow{g} & Q & \longrightarrow & Z[1] \\ & & \downarrow 1 & & \downarrow i & & \\ & & X & \xrightarrow{a} & Y & & \\ & & & & \downarrow s & & \\ & & & & W & & \end{array}$$

Since $t \in S$ we see that Q is isomorphic to an object of \mathcal{D}' . Hence $s \in S$. Finally, $s \circ a = s \circ i \circ g = 0$ as $s \circ i = 0$ by Lemma 4.1. We conclude that LMS3 holds. The proof of RMS3 is dual.

Proof of MS5. Follows as distinguished triangles and \mathcal{D}' are stable under translations

Proof of MS6. Suppose given a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow s & & \downarrow s' \\ X' & \longrightarrow & Y' \end{array}$$

with $s, s' \in S$. By Proposition 4.21 we can extend this to a nine square diagram. As s, s' are elements of S we see that X'', Y'' are isomorphic to objects of \mathcal{D}' . Since

\mathcal{D}' is a full triangulated subcategory we see that Z'' is also isomorphic to an object of \mathcal{D}' . Whence the morphism $Z \rightarrow Z'$ is an element of S . This proves MS6.

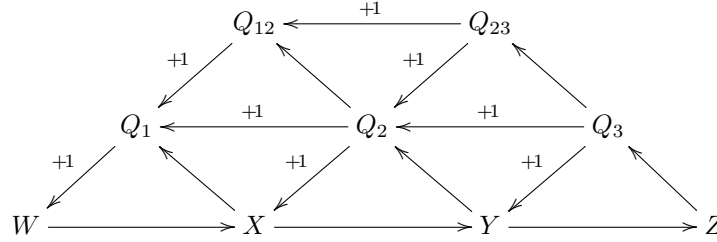
MS2 is a formal consequence of MS1, MS5, and MS6, see Lemma 5.2. This finishes the proof of the first assertion of the lemma.

Let's assume that S is saturated. (In the following we will use rotation of distinguished triangles without further mention.) Let $X \oplus Y$ be an object isomorphic to an object of \mathcal{D}' . Consider the morphism $f : 0 \rightarrow X$. The composition $0 \rightarrow X \rightarrow X \oplus Y$ is an element of S as $(0, X \oplus Y, X \oplus Y, 0, 1, 0)$ is a distinguished triangle. The composition $Y[-1] \rightarrow 0 \rightarrow X$ is an element of S as $(X, X \oplus Y, Y, (1, 0), (0, 1), 0)$ is a distinguished triangle, see Lemma 4.10. Hence $0 \rightarrow X$ is an element of S (as S is saturated). Thus X is isomorphic to an object of \mathcal{D}' as desired.

Finally, assume \mathcal{D}' is a saturated triangulated subcategory. Let

$$W \xrightarrow{h} X \xrightarrow{g} Y \xrightarrow{f} Z$$

be composable morphisms of \mathcal{D} such that $fg, gh \in S$. We will build up a picture of objects as in the diagram below.



First choose distinguished triangles (W, X, Q_1) , (X, Y, Q_2) , (Y, Z, Q_3) , (W, Y, Q_{12}) , and (X, Z, Q_{23}) . Denote $s : Q_2 \rightarrow Q_1[1]$ the composition $Q_2 \rightarrow X[1] \rightarrow Q_1[1]$. Denote $t : Q_3 \rightarrow Q_2[1]$ the composition $Q_3 \rightarrow Y[1] \rightarrow Q_2[1]$. By TR4 applied to the composition $W \rightarrow X \rightarrow Y$ and the composition $X \rightarrow Y \rightarrow Z$ there exist distinguished triangles (Q_1, Q_{12}, Q_2) and (Q_2, Q_{23}, Q_3) which use the morphisms s and t . The objects Q_{12} and Q_{23} are isomorphic to objects of \mathcal{D}' as $W \rightarrow Y$ and $X \rightarrow Z$ are assumed in S . Hence also $s[1]t$ is an element of S as S is closed under compositions and shifts. Note that $s[1]t = 0$ as $Y[1] \rightarrow Q_2[1] \rightarrow X[2]$ is zero, see Lemma 4.1. Hence $Q_3 \oplus Q_1[2]$ is isomorphic to an object of \mathcal{D}' , see Lemma 4.10. By assumption on \mathcal{D}' we conclude that Q_3, Q_1 are isomorphic to objects of \mathcal{D}' . Looking at the distinguished triangle (Q_1, Q_{12}, Q_2) we conclude that Q_2 is also isomorphic to an object of \mathcal{D}' . Looking at the distinguished triangle (X, Y, Q_2) we finally conclude that $g \in S$. (It is also follows that $h, f \in S$, but we don't need this.) \square

Definition 6.7. Let \mathcal{D} be a triangulated category. Let \mathcal{B} be a full triangulated subcategory. We define the *quotient category* \mathcal{D}/\mathcal{B} by the formula $\mathcal{D}/\mathcal{B} = S^{-1}\mathcal{D}$, where S is the multiplicative system of \mathcal{D} associated to \mathcal{B} via Lemma 6.6. The localization functor $Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{B}$ is called the *quotient functor* in this case.

Note that the quotient functor $Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{B}$ is an exact functor of triangulated categories, see Proposition 5.5. The universal property of this construction is the following.

Lemma 6.8. *Let \mathcal{D} be a triangulated category. Let \mathcal{B} be a full triangulated subcategory of \mathcal{D} . Let $Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{B}$ be the quotient functor.*

- (1) *If $H : \mathcal{D} \rightarrow \mathcal{A}$ is a homological functor into an abelian category \mathcal{A} such that $\mathcal{B} \subset \text{Ker}(H)$ then there exists a unique factorization $H' : \mathcal{D}/\mathcal{B} \rightarrow \mathcal{A}$ such that $H = H' \circ Q$ and H' is a homological functor too.*
- (2) *If $F : \mathcal{D} \rightarrow \mathcal{D}'$ is an exact functor into a pre-triangulated category \mathcal{D}' such that $\mathcal{B} \subset \text{Ker}(F)$ then there exists a unique factorization $F' : \mathcal{D}/\mathcal{B} \rightarrow \mathcal{D}'$ such that $F = F' \circ Q$ and F' is an exact functor too.*

Proof. This lemma follows from Lemma 5.6. Namely, if $f : X \rightarrow Y$ is a morphism of \mathcal{D} such that for some distinguished triangle (X, Y, Z, f, g, h) the object Z is isomorphic to an object of \mathcal{B} , then $H(f)$, resp. $F(f)$ is an isomorphism under the assumptions of (1), resp. (2). Details omitted. \square

The kernel of the quotient functor can be described as follows.

Lemma 6.9. *Let \mathcal{D} be a triangulated category. Let \mathcal{B} be a full triangulated subcategory. The kernel of the quotient functor $Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{B}$ is the strictly full subcategory of \mathcal{D} whose objects are*

$$\text{Ob}(\text{Ker}(Q)) = \left\{ \begin{array}{l} Z \in \text{Ob}(\mathcal{D}) \text{ such that there exists a } Z' \in \text{Ob}(\mathcal{D}) \\ \text{such that } Z \oplus Z' \text{ is isomorphic to an object of } \mathcal{B} \end{array} \right\}$$

In other words it is the smallest strictly full saturated triangulated subcategory of \mathcal{D} containing \mathcal{B} .

Proof. First note that the kernel is automatically a strictly full triangulated subcategory containing summands of any of its objects, see Lemma 6.2. The description of its objects follows from the definitions and Lemma 5.7 part (4). \square

Let \mathcal{D} be a triangulated category. At this point we have constructions which induce order preserving maps between

- (1) the partially ordered set of multiplicative systems S in \mathcal{D} compatible with the triangulated structure, and
- (2) the partially ordered set of full triangulated subcategories $\mathcal{B} \subset \mathcal{D}$.

Namely, the constructions are given by $S \mapsto \mathcal{B}(S) = \text{Ker}(Q : \mathcal{D} \rightarrow S^{-1}\mathcal{D})$ and $\mathcal{B} \mapsto S(\mathcal{B})$ where $S(\mathcal{B})$ is the multiplicative set of (6.6.1), i.e.,

$$S(\mathcal{B}) = \left\{ \begin{array}{l} f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{B} \end{array} \right\}$$

Note that it is not the case that these operations are mutually inverse.

Lemma 6.10. *Let \mathcal{D} be a triangulated category. The operations described above have the following properties*

- (1) *$S(\mathcal{B}(S))$ is the “saturation” of S , i.e., it is the smallest saturated multiplicative system in \mathcal{D} containing S , and*
- (2) *$\mathcal{B}(S(\mathcal{B}))$ is the “saturation” of \mathcal{B} , i.e., it is the smallest strictly full saturated triangulated subcategory of \mathcal{D} containing \mathcal{B} .*

In particular, the constructions define mutually inverse maps between the (partially ordered) set of saturated multiplicative systems in \mathcal{D} compatible with the triangulated structure on \mathcal{D} and the (partially ordered) set of strictly full saturated triangulated subcategories of \mathcal{D} .

Proof. First, let's start with a full triangulated subcategory \mathcal{B} . Then $\mathcal{B}(S(\mathcal{B})) = \text{Ker}(Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{B})$ and hence (2) is the content of Lemma 6.9.

Next, suppose that S is multiplicative system in \mathcal{D} compatible with the triangulation on \mathcal{D} . Then $\mathcal{B}(S) = \text{Ker}(Q : \mathcal{D} \rightarrow S^{-1}\mathcal{D})$. Hence (using Lemma 4.8 in the localized category)

$$\begin{aligned} S(\mathcal{B}(S)) &= \left\{ f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle } (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Q(Z) = 0 \right\} \\ &= \{f \in \text{Arrows}(\mathcal{D}) \mid Q(f) \text{ is an isomorphism}\} \\ &= \hat{S} = S' \end{aligned}$$

in the notation of Categories, Lemma 25.18. The final statement of that lemma finishes the proof. \square

Lemma 6.11. *Let $H : \mathcal{D} \rightarrow \mathcal{A}$ be a homological functor from a triangulated category \mathcal{D} to an abelian category \mathcal{A} , see Definition 3.5. The subcategory $\text{Ker}(H)$ of \mathcal{D} is a strictly full saturated triangulated subcategory of \mathcal{D} whose corresponding saturated multiplicative system (see Lemma 6.10) is the set*

$$S = \{f \in \text{Arrows}(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z}\}.$$

The functor H factors through the quotient functor $Q : \mathcal{D} \rightarrow \mathcal{D}/\text{Ker}(H)$.

Proof. The category $\text{Ker}(H)$ is a strictly full saturated triangulated subcategory of \mathcal{D} by Lemma 6.3. The set S is a saturated multiplicative system compatible with the triangulated structure by Lemma 5.4. Recall that the multiplicative system corresponding to $\text{Ker}(H)$ is the set

$$\left\{ f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle } (X, Y, Z, f, g, h) \text{ with } H^i(Z) = 0 \text{ for all } i \right\}$$

By the long exact cohomology sequence, see (3.5.1), it is clear that f is an element of this set if and only if f is an element of S . Finally, the factorization of H through Q is a consequence of Lemma 6.8. \square

It is clear that in the lemma above the factorization of H through $\mathcal{D}/\text{Ker}(H)$ is the universal factorization. Namely, if $F : \mathcal{D} \rightarrow \mathcal{D}'$ is an exact functor of triangulated categories and if there exists a homological functor $H' : \mathcal{D}' \rightarrow \mathcal{A}$ such that $H \cong H' \circ F$, then F factors through the quotient functor $Q : \mathcal{D} \rightarrow \mathcal{D}/\text{Ker}(H)$.

7. Adjoints for exact functors

Results on adjoint functors between triangulated categories.

Lemma 7.1. *Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor between triangulated categories. If F has a right adjoint, then it is an exact functor.*

Proof. Let G be a right adjoint. Let X be an object of \mathcal{D} and A an object of \mathcal{D}' . Since F is an exact functor we see that

$$\begin{aligned} \text{Mor}_{\mathcal{D}}(X, G(A[1])) &= \text{Mor}_{\mathcal{D}'}(F(X), A[1]) \\ &= \text{Mor}_{\mathcal{D}'}(F(X)[-1], A) \\ &= \text{Mor}_{\mathcal{D}'}(F(X[-1]), A) \\ &= \text{Mor}_{\mathcal{D}}(X[-1], G(A)) \\ &= \text{Mor}_{\mathcal{D}}(X, G(A)[1]) \end{aligned}$$

By Yoneda's lemma (Categories, Lemma 3.5) we obtain a canonical isomorphism $G(A)[1] = G(A[1])$. Let $A \rightarrow B \rightarrow C \rightarrow A[1]$ be a distinguished triangle in \mathcal{D}' . Choose a distinguished triangle

$$G(A) \rightarrow G(B) \rightarrow X \rightarrow G(A)[1]$$

in \mathcal{D} . Then $F(G(A)) \rightarrow F(G(B)) \rightarrow F(X) \rightarrow F(G(A))[1]$ is a distinguished triangle in \mathcal{D}' . By TR3 we can choose a morphism of distinguished triangles

$$\begin{array}{ccccccc} F(G(A)) & \longrightarrow & F(G(B)) & \longrightarrow & F(X) & \longrightarrow & F(G(A))[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \end{array}$$

Since G is the adjoint the new morphism determines a morphism $X \rightarrow G(C)$ such that the diagram

$$\begin{array}{ccccccc} G(A) & \longrightarrow & G(B) & \longrightarrow & X & \longrightarrow & G(A)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G(A) & \longrightarrow & G(B) & \longrightarrow & G(C) & \longrightarrow & G(A)[1] \end{array}$$

commutes. Applying the cohomological functor $\text{Hom}_{\mathcal{D}'}(W, -)$ for an object W of \mathcal{D}' we deduce from the 5 lemma that

$$\text{Hom}_{\mathcal{D}'}(W, X) \rightarrow \text{Hom}_{\mathcal{D}'}(W, G(C))$$

is a bijection and using the Yoneda lemma once more we conclude that $X \rightarrow G(C)$ is an isomorphism. Hence we conclude that $G(A) \rightarrow G(B) \rightarrow G(C) \rightarrow G(A)[1]$ is a distinguished triangle which is what we wanted to show. \square

Lemma 7.2. *Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories. Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ and $G : \mathcal{D}' \rightarrow \mathcal{D}$ be functors. Assume that*

- (1) *F and G are exact functors,*
- (2) *F is fully faithful,*
- (3) *G is a right adjoint to F , and*
- (4) *the kernel of G is zero.*

Then F is an equivalence of categories.

Proof. Since F is fully faithful the adjunction map $\text{id} \rightarrow G \circ F$ is an isomorphism (Categories, Lemma 24.3). Let X be an object of \mathcal{D}' . Choose a distinguished triangle

$$F(G(X)) \rightarrow X \rightarrow Y \rightarrow F(G(X))[1]$$

in \mathcal{D}' . Applying G and using that $G(F(G(X))) = G(X)$ we find a distinguished triangle

$$G(X) \rightarrow G(X) \rightarrow G(Y) \rightarrow G(X)[1]$$

Hence $G(Y) = 0$. Thus $Y = 0$. Thus $F(G(X)) \rightarrow X$ is an isomorphism. \square

8. The homotopy category

Let \mathcal{A} be an additive category. The homotopy category $K(\mathcal{A})$ of \mathcal{A} is the category of complexes of \mathcal{A} with morphisms given by morphisms of complexes up to homotopy. Here is the formal definition.

Definition 8.1. Let \mathcal{A} be an additive category.

- (1) We set $\text{Comp}(\mathcal{A}) = \text{CoCh}(\mathcal{A})$ be the *category of (cochain) complexes*.
- (2) A complex K^\bullet is said to be *bounded below* if $K^n = 0$ for all $n \ll 0$.
- (3) A complex K^\bullet is said to be *bounded above* if $K^n = 0$ for all $n \gg 0$.
- (4) A complex K^\bullet is said to be *bounded* if $K^n = 0$ for all $|n| \gg 0$.
- (5) We let $\text{Comp}^+(\mathcal{A})$, $\text{Comp}^-(\mathcal{A})$, resp. $\text{Comp}^b(\mathcal{A})$ be the full subcategory of $\text{Comp}(\mathcal{A})$ whose objects are the complexes which are bounded below, bounded above, resp. bounded.
- (6) We let $K(\mathcal{A})$ be the category with the same objects as $\text{Comp}(\mathcal{A})$ but as morphisms homotopy classes of maps of complexes (see Homology, Lemma 12.7).
- (7) We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ be the full subcategory of $K(\mathcal{A})$ whose objects are bounded below, bounded above, resp. bounded complexes of \mathcal{A} .

It will turn out that the categories $K(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. To prove this we first develop some machinery related to cones and split exact sequences.

9. Cones and termwise split sequences

Let \mathcal{A} be an additive category, and let $K(\mathcal{A})$ denote the category of complexes of \mathcal{A} with morphisms given by morphisms of complexes up to homotopy. Note that the shift functors $[n]$ on complexes, see Homology, Definition 14.7, give rise to functors $[n] : K(\mathcal{A}) \rightarrow K(\mathcal{A})$ such that $[n] \circ [m] = [n + m]$ and $[0] = \text{id}$.

Definition 9.1. Let \mathcal{A} be an additive category. Let $f : K^\bullet \rightarrow L^\bullet$ be a morphism of complexes of \mathcal{A} . The *cone* of f is the complex $C(f)^\bullet$ given by $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d_{C(f)}^n = \begin{pmatrix} d_L^n & f^{n+1} \\ 0 & -d_K^{n+1} \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L^\bullet \rightarrow C(f)^\bullet$ and $p : C(f)^\bullet \rightarrow K^\bullet[1]$ induced by the obvious maps $L^n \rightarrow C(f)^n \rightarrow K^{n+1}$.

In other words $(K, L, C(f), f, i, p)$ forms a triangle:

$$K^\bullet \rightarrow L^\bullet \rightarrow C(f)^\bullet \rightarrow K^\bullet[1]$$

The formation of this triangle is functorial in the following sense.

Lemma 9.2. *Suppose that*

$$\begin{array}{ccc} K_1^\bullet & \xrightarrow{f_1} & L_1^\bullet \\ a \downarrow & & \downarrow b \\ K_2^\bullet & \xrightarrow{f_2} & L_2^\bullet \end{array}$$

is a diagram of morphisms of complexes which is commutative up to homotopy. Then there exists a morphism $c : C(f_1)^\bullet \rightarrow C(f_2)^\bullet$ which gives rise to a morphism of triangles $(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \rightarrow (K_2^\bullet, L_2^\bullet, C(f_2)^\bullet, f_2, i_2, p_2)$ of $K(\mathcal{A})$.

Proof. Let $h^n : K_1^n \rightarrow L_2^{n-1}$ be a family of morphisms such that $b \circ f_1 - f_2 \circ a = d \circ h + h \circ d$. Define c^n by the matrix

$$c^n = \begin{pmatrix} b^n & h^{n+1} \\ 0 & a^{n+1} \end{pmatrix} : L_1^n \oplus K_1^{n+1} \rightarrow L_2^n \oplus K_2^{n+1}$$

A matrix computation show that c is a morphism of complexes. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. \square

Note that the morphism $c : C(f_1)^\bullet \rightarrow C(f_2)^\bullet$ constructed in the proof of Lemma 9.2 in general depends on the chosen homotopy h between $f_2 \circ a$ and $b \circ f_1$.

Lemma 9.3. *Suppose that $f : K^\bullet \rightarrow L^\bullet$ and $g : L^\bullet \rightarrow M^\bullet$ are morphisms of complexes such that $g \circ f$ is homotopic to zero. Then g factors through a morphism $C(f)^\bullet \rightarrow M^\bullet$ of $K(\mathcal{A})$.*

Proof. The assumptions say that the diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{f} & L^\bullet \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & M^\bullet \end{array}$$

commutes up to homotopy. Since the cone on $0 \rightarrow M^\bullet$ is M^\bullet the map $C(f)^\bullet \rightarrow C(0 \rightarrow M^\bullet) = M^\bullet$ of Lemma 9.2 is the desired map. \square

Note that the morphism $C(f)^\bullet \rightarrow M^\bullet$ constructed in the proof of Lemma 9.3 in general depends on the chosen homotopy.

Definition 9.4. Let \mathcal{A} be an additive category. A *termwise split injection* $\alpha : A^\bullet \rightarrow B^\bullet$ is a morphism of complexes such that each $A^n \rightarrow B^n$ is isomorphic to the inclusion of a direct summand. A *termwise split surjection* $\beta : B^\bullet \rightarrow C^\bullet$ is a morphism of complexes such that each $B^n \rightarrow C^n$ is isomorphic to the projection onto a direct summand.

Lemma 9.5. *Let \mathcal{A} be an additive category. Let*

$$\begin{array}{ccc} A^\bullet & \xrightarrow{f} & B^\bullet \\ a \downarrow & & \downarrow b \\ C^\bullet & \xrightarrow{g} & D^\bullet \end{array}$$

be a diagram of morphisms of complexes commuting up to homotopy. If f is a split injection, then b is homotopic to a morphism which makes the diagram commute. If g is a split surjection, then a is homotopic to a morphism which makes the diagram commute.

Proof. Let $h^n : A^n \rightarrow D^{n-1}$ be a collection of morphisms such that $bf - ga = dh + hd$. Suppose that $\pi^n : B^n \rightarrow A^n$ are morphisms splitting the morphisms f^n . Take $b' = b - dh\pi - h\pi d$. Suppose $s^n : D^n \rightarrow C^n$ are morphisms splitting the morphisms $g^n : C^n \rightarrow D^n$. Take $a' = a + dsh + shd$. Computations omitted. \square

The following lemma can be used to replace a morphism of complexes by a morphism where in each degree the map is the injection of a direct summand.

Lemma 9.6. *Let \mathcal{A} be an additive category. Let $\alpha : K^\bullet \rightarrow L^\bullet$ be a morphism of complexes of \mathcal{A} . There exists a factorization*

$$K^\bullet \xrightarrow{\tilde{\alpha}} \tilde{L}^\bullet \xrightarrow{\pi} L^\bullet$$

α

such that

- (1) $\tilde{\alpha}$ is a termwise split injection (see Definition 9.4),
- (2) there is a map of complexes $s : L^\bullet \rightarrow \tilde{L}^\bullet$ such that $\pi \circ s = \text{id}_{L^\bullet}$ and such that $s \circ \pi$ is homotopic to $\text{id}_{\tilde{L}^\bullet}$.

Moreover, if both K^\bullet and L^\bullet are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is \tilde{L}^\bullet .

Proof. We set

$$\tilde{L}^n = L^n \oplus K^n \oplus K^{n+1}$$

and we define

$$d_{\tilde{L}}^n = \begin{pmatrix} d_L^n & 0 & 0 \\ 0 & d_K^n & \text{id}_{K^{n+1}} \\ 0 & 0 & -d_K^{n+1} \end{pmatrix}$$

In other words, $\tilde{L}^\bullet = L^\bullet \oplus C(1_{K^\bullet})$. Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha \\ \text{id}_{K^n} \\ 0 \end{pmatrix}$$

which is clearly a split injection. It is also clear that it defines a morphism of complexes. We define

$$\pi = (\text{id}_{L^n} \quad 0 \quad 0)$$

so that clearly $\pi \circ \tilde{\alpha} = \alpha$. We set

$$s = \begin{pmatrix} \text{id}_{L^n} \\ 0 \\ 0 \end{pmatrix}$$

so that $\pi \circ s = \text{id}_{L^\bullet}$. Finally, let $h^n : \tilde{L}^n \rightarrow \tilde{L}^{n-1}$ be the map which maps the summand K^n of \tilde{L}^n via the identity morphism to the summand K^n of \tilde{L}^{n-1} . Then it is a trivial matter (see computations in remark below) to prove that

$$\text{id}_{\tilde{L}^\bullet} - s \circ \pi = d \circ h + h \circ d$$

which finishes the proof of the lemma. \square

Remark 9.7. To see the last displayed equality in the proof above we can argue with elements as follows. We have $s\pi(l, k, k^+) = (l, 0, 0)$. Hence the morphism of the left hand side maps (l, k, k^+) to $(0, k, k^+)$. On the other hand $h(l, k, k^+) = (0, 0, k)$ and $d(l, k, k^+) = (dl, dk + k^+, -dk^+)$. Hence $(dh + hd)(l, k, k^+) = d(0, 0, k) + h(dl, dk + k^+, -dk^+) = (0, k, -dk) + (0, 0, dk + k^+) = (0, k, k^+)$ as desired.

Lemma 9.8. *Let \mathcal{A} be an additive category. Let $\alpha : K^\bullet \rightarrow L^\bullet$ be a morphism of complexes of \mathcal{A} . There exists a factorization*

$$K^\bullet \xrightarrow{i} \tilde{K}^\bullet \xrightarrow{\tilde{\alpha}} L^\bullet$$

α

such that

- (1) $\tilde{\alpha}$ is a termwise split surjection (see Definition 9.4),
- (2) there is a map of complexes $s : \tilde{K}^\bullet \rightarrow K^\bullet$ such that $s \circ i = \text{id}_{K^\bullet}$ and such that $i \circ s$ is homotopic to $\text{id}_{\tilde{K}^\bullet}$.

Moreover, if both K^\bullet and L^\bullet are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is \tilde{K}^\bullet .

Proof. Dual to Lemma 9.6. Take

$$\tilde{K}^n = K^n \oplus L^{n-1} \oplus L^n$$

and we define

$$d_{\tilde{K}}^n = \begin{pmatrix} d_K^n & 0 & 0 \\ 0 & -d_L^{n-1} & \text{id}_{L^n} \\ 0 & 0 & d_L^n \end{pmatrix}$$

in other words $\tilde{K}^\bullet = K^\bullet \oplus C(1_{L^\bullet[-1]})$. Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha & 0 & \text{id}_{L^n} \end{pmatrix}$$

which is clearly a split surjection. It is also clear that it defines a morphism of complexes. We define

$$i = \begin{pmatrix} \text{id}_{K^n} \\ 0 \\ 0 \end{pmatrix}$$

so that clearly $\tilde{\alpha} \circ i = \alpha$. We set

$$s = \begin{pmatrix} \text{id}_{K^n} & 0 & 0 \end{pmatrix}$$

so that $s \circ i = \text{id}_{K^\bullet}$. Finally, let $h^n : \tilde{K}^n \rightarrow \tilde{K}^{n-1}$ be the map which maps the summand L^{n-1} of \tilde{K}^n via the identity morphism to the summand L^{n-1} of \tilde{K}^{n-1} . Then it is a trivial matter to prove that

$$\text{id}_{\tilde{K}^\bullet} - i \circ s = d \circ h + h \circ d$$

which finishes the proof of the lemma. \square

Definition 9.9. Let \mathcal{A} be an additive category. A *termwise split sequence of complexes of \mathcal{A}* is a complex of complexes

$$0 \rightarrow A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \rightarrow 0$$

together with given direct sum decompositions $B^n = A^n \oplus C^n$ compatible with α^n and β^n . We often write $s^n : C^n \rightarrow B^n$ and $\pi^n : B^n \rightarrow A^n$ for the maps induced by the direct sum decompositions. According to Homology, Lemma 14.10 we get an associated morphism of complexes

$$\delta : C^\bullet \rightarrow A^\bullet[1]$$

which in degree n is the map $\pi^{n+1} \circ d_B^n \circ s^n$. In other words $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ forms a triangle

$$A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1]$$

This will be the *triangle associated to the termwise split sequence of complexes*.

Lemma 9.10. *Let \mathcal{A} be an additive category. Let $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ be termwise split exact sequences as in Definition 9.9. Let $(\pi')^n, (s')^n$ be a second collection of splittings. Denote $\delta' : C^\bullet \rightarrow A^\bullet[1]$ the morphism associated to this second set of splittings. Then*

$$(1, 1, 1) : (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta) \rightarrow (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta')$$

is an isomorphism of triangles in $K(\mathcal{A})$.

Proof. The statement simply means that δ and δ' are homotopic maps of complexes. This is Homology, Lemma 14.12. \square

Remark 9.11. Let \mathcal{A} be an additive category. Let $0 \rightarrow A_i^\bullet \rightarrow B_i^\bullet \rightarrow C_i^\bullet \rightarrow 0$, $i = 1, 2$ be termwise split exact sequences. Suppose that $a : A_1^\bullet \rightarrow A_2^\bullet$, $b : B_1^\bullet \rightarrow B_2^\bullet$, and $c : C_1^\bullet \rightarrow C_2^\bullet$ are morphisms of complexes such that

$$\begin{array}{ccccc} A_1^\bullet & \longrightarrow & B_1^\bullet & \longrightarrow & C_1^\bullet \\ a \downarrow & & b \downarrow & & c \downarrow \\ A_2^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & C_2^\bullet \end{array}$$

commutes in $K(\mathcal{A})$. In general, there does **not** exist a morphism $b' : B_1^\bullet \rightarrow B_2^\bullet$ which is homotopic to b such that the diagram above commutes in the category of complexes. Namely, consider Examples, Equation (54.0.1). If we could replace the middle map there by a homotopic one such that the diagram commutes, then we would have additivity of traces which we do not.

Lemma 9.12. *Let \mathcal{A} be an additive category. Let $0 \rightarrow A_i^\bullet \rightarrow B_i^\bullet \rightarrow C_i^\bullet \rightarrow 0$, $i = 1, 2, 3$ be termwise split exact sequences of complexes. Let $b : B_1^\bullet \rightarrow B_2^\bullet$ and $b' : B_2^\bullet \rightarrow B_3^\bullet$ be morphisms of complexes such that*

$$\begin{array}{ccccc} A_1^\bullet & \longrightarrow & B_1^\bullet & \longrightarrow & C_1^\bullet \\ 0 \downarrow & & b \downarrow & & 0 \downarrow \\ A_2^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & C_2^\bullet \end{array} \quad \text{and} \quad \begin{array}{ccccc} A_2^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & C_2^\bullet \\ 0 \downarrow & & b' \downarrow & & 0 \downarrow \\ A_3^\bullet & \longrightarrow & B_3^\bullet & \longrightarrow & C_3^\bullet \end{array}$$

commute in $K(\mathcal{A})$. Then $b' \circ b = 0$ in $K(\mathcal{A})$.

Proof. By Lemma 9.5 we can replace b and b' by homotopic maps such that the right square of the left diagram commutes and the left square of the right diagram commutes. In other words, we have $\text{Im}(b^n) \subset \text{Im}(A_2^n \rightarrow B_2^n)$ and $\text{Ker}((b')^n) \supset \text{Im}(A_2^n \rightarrow B_2^n)$. Then $b \circ b' = 0$ as a map of complexes. \square

Lemma 9.13. *Let \mathcal{A} be an additive category. Let $f_1 : K_1^\bullet \rightarrow L_1^\bullet$ and $f_2 : K_2^\bullet \rightarrow L_2^\bullet$ be morphisms of complexes. Let*

$$(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \rightarrow (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_2, i_2, p_2)$$

be any morphism of triangles of $K(\mathcal{A})$. If a and b are homotopy equivalences then so is c .

Proof. Let $a^{-1} : K_2^\bullet \rightarrow K_1^\bullet$ be a morphism of complexes which is inverse to a in $K(\mathcal{A})$. Let $b^{-1} : L_2^\bullet \rightarrow L_1^\bullet$ be a morphism of complexes which is inverse to b in $K(\mathcal{A})$. Let $c' : C(f_2)^\bullet \rightarrow C(f_1)^\bullet$ be the morphism from Lemma 9.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\mathcal{A})$ then we win. Hence it suffices to prove the following: Given a morphism of

triangles $(1, 1, c) : (K^\bullet, L^\bullet, C(f)^\bullet, f, i, p)$ in $K(\mathcal{A})$ the morphism c is an isomorphism in $K(\mathcal{A})$. By assumption the two squares in the diagram

$$\begin{array}{ccccc} L^\bullet & \longrightarrow & C(f)^\bullet & \longrightarrow & K^\bullet[1] \\ \downarrow 1 & & \downarrow c & & \downarrow 1 \\ L^\bullet & \longrightarrow & C(f)^\bullet & \longrightarrow & K^\bullet[1] \end{array}$$

commute up to homotopy. By construction of $C(f)^\bullet$ the rows form termwise split sequences of complexes. Thus we see that $(c - 1)^2 = 0$ in $K(\mathcal{A})$ by Lemma 9.12. Hence c is an isomorphism in $K(\mathcal{A})$ with inverse $2 - c$. \square

Hence if a and b are homotopy equivalences then the resulting morphism of triangles is an isomorphism of triangles in $K(\mathcal{A})$. It turns out that the collection of triangles of $K(\mathcal{A})$ given by cones and the collection of triangles of $K(\mathcal{A})$ given by termwise split sequences of complexes are the same up to isomorphisms, at least up to sign!

Lemma 9.14. *Let \mathcal{A} be an additive category.*

- (1) *Given a termwise split sequence of complexes $(\alpha : A^\bullet \rightarrow B^\bullet, \beta : B^\bullet \rightarrow C^\bullet, s^n, \pi^n)$ there exists a homotopy equivalence $C(\alpha)^\bullet \rightarrow C^\bullet$ such that the diagram*

$$\begin{array}{ccccccc} A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C(\alpha)^\bullet & \xrightarrow{-p} & A^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \xrightarrow{\delta} & A^\bullet[1] \end{array}$$

defines an isomorphism of triangles in $K(\mathcal{A})$.

- (2) *Given a morphism of complexes $f : K^\bullet \rightarrow L^\bullet$ there exists an isomorphism of triangles*

$$\begin{array}{ccccccc} K^\bullet & \longrightarrow & \tilde{L}^\bullet & \longrightarrow & M^\bullet & \xrightarrow{\delta} & K^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & C(f)^\bullet & \xrightarrow{-p} & K^\bullet[1] \end{array}$$

where the upper triangle is the triangle associated to a termwise split exact sequence $K^\bullet \rightarrow \tilde{L}^\bullet \rightarrow M^\bullet$.

Proof. Proof of (1). We have $C(\alpha)^n = B^n \oplus A^{n+1}$ and we simply define $C(\alpha)^n \rightarrow C^n$ via the projection onto B^n followed by β^n . This defines a morphism of complexes because the compositions $A^{n+1} \rightarrow B^{n+1} \rightarrow C^{n+1}$ are zero. To get a homotopy inverse we take $C^\bullet \rightarrow C(\alpha)^\bullet$ given by $(s^n, -\delta^n)$ in degree n . This is a morphism of complexes because the morphism δ^n can be characterized as the unique morphism $C^n \rightarrow A^{n+1}$ such that $d \circ s^n - s^{n+1} \circ d = \alpha \circ \delta^n$, see proof of Homology, Lemma 14.10. The composition $C^\bullet \rightarrow C(f)^\bullet \rightarrow C^\bullet$ is the identity. The composition $C(f)^\bullet \rightarrow C^\bullet \rightarrow C(f)^\bullet$ is equal to the morphism

$$\begin{pmatrix} s^n \circ \beta^n & 0 \\ -\delta^n \circ \beta^n & 0 \end{pmatrix}$$

To see that this is homotopic to the identity map use the homotopy $h^n : C(\alpha)^n \rightarrow C(\alpha)^{n-1}$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} : C(\alpha)^n = B^n \oplus A^{n+1} \rightarrow B^{n-1} \oplus A^n = C(\alpha)^{n-1}$$

It is trivial to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s^n & \\ -\delta^n & \end{pmatrix} (\beta^n \quad 0) = \begin{pmatrix} d & \alpha^n \\ 0 & -d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi^{n+1} & 0 \end{pmatrix} \begin{pmatrix} d & \alpha^{n+1} \\ 0 & -d \end{pmatrix}$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha)^\bullet \rightarrow A^\bullet[1]$ (see Definition 9.1) and $C(\alpha)^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : C^\bullet \rightarrow C(\alpha)^\bullet$ and check instead that the two maps $C^\bullet \rightarrow A^\bullet[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K^\bullet \rightarrow \tilde{L}^\bullet$, $s : L^\bullet \rightarrow \tilde{L}^\bullet$ and $\pi : L^\bullet \rightarrow L^\bullet$ be as in Lemma 9.6. By Lemmas 9.2 and 9.13 the triangles $(K^\bullet, L^\bullet, C(f), i, p)$ and $(K^\bullet, \tilde{L}^\bullet, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may replace L^\bullet by \tilde{L}^\bullet and f by \tilde{f} . In other words we may assume that f is a termwise split injection. In this case the result follows from part (1). \square

Lemma 9.15. *Let \mathcal{A} be an additive category. Let $A_1^\bullet \rightarrow A_2^\bullet \rightarrow \dots \rightarrow A_n^\bullet$ be a sequence of composable morphisms of complexes. There exists a commutative diagram*

$$\begin{array}{ccccccc} A_1^\bullet & \longrightarrow & A_2^\bullet & \longrightarrow & \dots & \longrightarrow & A_n^\bullet \\ \uparrow & & \uparrow & & & & \uparrow \\ B_1^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & \dots & \longrightarrow & B_n^\bullet \end{array}$$

such that each morphism $B_i^\bullet \rightarrow B_{i+1}^\bullet$ is a split injection and each $B_i^\bullet \rightarrow A_i^\bullet$ is a homotopy equivalence. Moreover, if all A_i^\bullet are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so are the B_i^\bullet .

Proof. The case $n = 1$ is without content. Lemma 9.6 is the case $n = 2$. Suppose we have constructed the diagram except for B_n^\bullet . Apply Lemma 9.6 to the composition $B_{n-1}^\bullet \rightarrow A_{n-1}^\bullet \rightarrow A_n^\bullet$. The result is a factorization $B_{n-1}^\bullet \rightarrow B_n^\bullet \rightarrow A_n^\bullet$ as desired. \square

Lemma 9.16. *Let \mathcal{A} be an additive category. Let $(\alpha : A^\bullet \rightarrow B^\bullet, \beta : B^\bullet \rightarrow C^\bullet, s^n, \pi^n)$ be a termwise split sequence of complexes. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the associated triangle. Then the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to the triangle $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$.*

Proof. We write $B^n = A^n \oplus C^n$ and we identify α^n and β^n with the natural inclusion and projection maps. By construction of δ we have

$$d_B^n = \begin{pmatrix} d_A^n & \delta^n \\ 0 & d_C^n \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : C^\bullet[-1] \rightarrow A^\bullet$ is given as $C(\delta[-1])^n = A^n \oplus C^n$ with differential identical with the matrix above! Whence the lemma. \square

Lemma 9.17. *Let \mathcal{A} be an additive category. Let $f : K^\bullet \rightarrow L^\bullet$ be a morphism of complexes. The triangle $(L^\bullet, C(f)^\bullet, K^\bullet[1], i, p, f[1])$ is the triangle associated to the termwise split sequence*

$$0 \rightarrow L^\bullet \rightarrow C(f)^\bullet \rightarrow K^\bullet[1] \rightarrow 0$$

coming from the definition of the cone of f .

Proof. Immediate from the definitions. \square

10. Distinguished triangles in the homotopy category

Since we want our boundary maps in long exact sequences of cohomology to be given by the maps in the snake lemma without signs we define distinguished triangles in the homotopy category as follows.

Definition 10.1. Let \mathcal{A} be an additive category. A triangle (X, Y, Z, f, g, h) of $K(\mathcal{A})$ is called a *distinguished triangle* of $K(\mathcal{A})$ if it is isomorphic to the triangle associated to a termwise split exact sequence of complexes, see Definition 9.9. Same definition for $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$.

Note that according to Lemma 9.14 a triangle of the form $(K^\bullet, L^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle. This does indeed lead to a triangulated category, see Proposition 10.3. Before we can prove the proposition we need one more lemma in order to be able to prove TR4.

Lemma 10.2. *Let \mathcal{A} be an additive category. Suppose that $\alpha : A^\bullet \rightarrow B^\bullet$ and $\beta : B^\bullet \rightarrow C^\bullet$ are split injections of complexes. Then there exist distinguished triangles $(A^\bullet, B^\bullet, Q_1^\bullet, \alpha, p_1, d_1)$, $(A^\bullet, C^\bullet, Q_2^\bullet, \beta \circ \alpha, p_2, d_2)$ and $(B^\bullet, C^\bullet, Q_3^\bullet, \beta, p_3, d_3)$ for which TR4 holds.*

Proof. Say $\pi_1^n : B^n \rightarrow A^n$, and $\pi_3^n : C^n \rightarrow B^n$ are the splittings. Then also $A^\bullet \rightarrow C^\bullet$ is a split injection with splittings $\pi_2^n = \pi_1^n \circ \pi_3^n$. Let us write Q_1^\bullet , Q_2^\bullet and Q_3^\bullet for the “quotient” complexes. In other words, $Q_1^n = \text{Ker}(\pi_1^n)$, $Q_3^n = \text{Ker}(\pi_3^n)$ and $Q_2^n = \text{Ker}(\pi_2^n)$. Note that the kernels exist. Then $B^n = A^n \oplus Q_1^n$ and $C^n = B^n \oplus Q_3^n$, where we think of A^n as a subobject of B^n and so on. This implies $C^n = A^n \oplus Q_1^n \oplus Q_3^n$. Note that $\pi_2^n = \pi_1^n \circ \pi_3^n$ is zero on both Q_1^n and Q_3^n . Hence $Q_2^n = Q_1^n \oplus Q_3^n$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & A^\bullet & \rightarrow & B^\bullet & \rightarrow & Q_1^\bullet & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & A^\bullet & \rightarrow & C^\bullet & \rightarrow & Q_2^\bullet & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & B^\bullet & \rightarrow & C^\bullet & \rightarrow & Q_3^\bullet & \rightarrow & 0 \end{array}$$

The rows of this diagram are termwise split exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of triangles

$$(A^\bullet \rightarrow B^\bullet \rightarrow Q_1^\bullet \rightarrow A^\bullet[1]) \longrightarrow (A^\bullet \rightarrow C^\bullet \rightarrow Q_2^\bullet \rightarrow A^\bullet[1])$$

and

$$(A^\bullet \rightarrow C^\bullet \rightarrow Q_2^\bullet \rightarrow A^\bullet[1]) \longrightarrow (B^\bullet \rightarrow C^\bullet \rightarrow Q_3^\bullet \rightarrow B^\bullet[1]).$$

Note that the splittings $Q_3^n \rightarrow C^n$ of the bottom split sequence in the diagram provides a splitting for the split sequence $0 \rightarrow Q_1^\bullet \rightarrow Q_2^\bullet \rightarrow Q_3^\bullet \rightarrow 0$ upon composing

with $C^n \rightarrow Q_2^n$. It follows easily from this that the morphism $\delta : Q_3^\bullet \rightarrow Q_1^\bullet[1]$ in the corresponding distinguished triangle

$$(Q_1^\bullet \rightarrow Q_2^\bullet \rightarrow Q_3^\bullet \rightarrow Q_1^\bullet[1])$$

is equal to the composition $Q_3^\bullet \rightarrow B^\bullet[1] \rightarrow Q_1^\bullet[1]$. Hence we get a structure as in the conclusion of axiom TR4. \square

Proposition 10.3. *Let \mathcal{A} be an additive category. The category $K(\mathcal{A})$ of complexes up to homotopy with its natural translation functors and distinguished triangles as defined above is a triangulated category.*

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(A^\bullet, A^\bullet, 0, 1, 0, 0)$ is distinguished since $0 \rightarrow A^\bullet \rightarrow A^\bullet \rightarrow 0 \rightarrow 0$ is a termwise split sequence of complexes. Finally, given any morphism of complexes $f : K^\bullet \rightarrow L^\bullet$ the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 9.14.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists a termwise split sequence of complexes $A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet$ such that the associated triangle $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, B^\bullet, -\delta[-1], \alpha, \beta)$. It follows from Lemma 9.16 that the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$. Precomposing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, -p)$. Hence it is distinguished by Lemma 9.14. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 9.14 this means that it is isomorphic to a triangle of the form $(K^\bullet, L^\bullet, C(f), f, i, -p)$ for some morphism of complexes f . Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L^\bullet, C(f), K^\bullet[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L^\bullet, C(f), K^\bullet[1], i, p, f[1])$. By Lemma 9.17 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \rightarrow X'$ and $b : Y \rightarrow Y'$ be morphisms such that $f' \circ a = b \circ f$. By Lemma 9.14 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, -p)$ and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', -p')$. At this point we simply apply Lemma 9.2 to the commutative diagram given by f, f', a, b .

Proof of TR4. At this point we know that $K(\mathcal{A})$ is a pre-triangulated category. Hence we can use Lemma 4.13. Let $A^\bullet \rightarrow B^\bullet$ and $B^\bullet \rightarrow C^\bullet$ be composable morphisms of $K(\mathcal{A})$. By Lemma 9.15 we may assume that $A^\bullet \rightarrow B^\bullet$ and $B^\bullet \rightarrow C^\bullet$ are split injective morphisms. In this case the result follows from Lemma 10.2. \square

Remark 10.4. Let \mathcal{A} be an additive category. Exactly the same proof as the proof of Proposition 10.3 shows that the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. Namely, the cone of a morphisms between bounded (above, below) is bounded (above, below). But we prove below that these are triangulated subcategories of $K(\mathcal{A})$ which gives another proof.

Lemma 10.5. *Let \mathcal{A} be an additive subcategory. The categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are full triangulated subcategories of $K(\mathcal{A})$.*

Proof. Each of the categories mentioned is a full additive subcategory. We use the criterion of Lemma 4.14 to show that they are triangulated subcategories. It is clear that each of the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ is preserved under the shift functors $[1], [-1]$. Finally, suppose that $f : A^\bullet \rightarrow B^\bullet$ is a morphism in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$. Then $(A^\bullet, B^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle of $K(\mathcal{A})$ with $C(f)^\bullet \in K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$ as is clear from the construction of the cone. Thus the lemma is proved. (Alternatively, $K^\bullet \rightarrow L^\bullet$ is isomorphic to an termwise split injection of complexes in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, see Lemma 9.6 and then one can directly take the associated distinguished triangle.) \square

Lemma 10.6. *Let \mathcal{A}, \mathcal{B} be additive categories. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. The induced functors*

$$\begin{aligned} F : K(\mathcal{A}) &\longrightarrow K(\mathcal{B}) \\ F : K^+(\mathcal{A}) &\longrightarrow K^+(\mathcal{B}) \\ F : K^-(\mathcal{A}) &\longrightarrow K^-(\mathcal{B}) \\ F : K^b(\mathcal{A}) &\longrightarrow K^b(\mathcal{B}) \end{aligned}$$

are exact functors of triangulated categories.

Proof. Suppose $A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet$ is a termwise split sequence of complexes of \mathcal{A} with splittings (s^n, π^n) and associated morphism $\delta : C^\bullet \rightarrow A^\bullet[1]$, see Definition 9.9. Then $F(A^\bullet) \rightarrow F(B^\bullet) \rightarrow F(C^\bullet)$ is a termwise split sequence of complexes with splittings $(F(s^n), F(\pi^n))$ and associated morphism $F(\delta) : F(C^\bullet) \rightarrow F(A^\bullet)[1]$. Thus F transforms distinguished triangles into distinguished triangles. \square

11. Derived categories

In this section we construct the derived category of an abelian category \mathcal{A} by inverting the quasi-isomorphisms in $K(\mathcal{A})$. Before we do this recall that the functors $H^i : \text{Comp}(\mathcal{A}) \rightarrow \mathcal{A}$ factor through $K(\mathcal{A})$, see Homology, Lemma 12.11. Moreover, in Homology, Definition 14.8 we have defined identifications $H^i(K^\bullet[n]) = H^{i+n}(K^\bullet)$. At this point it makes sense to redefine

$$H^i(K^\bullet) = H^0(K^\bullet[i])$$

in order to avoid confusion and possible sign errors.

Lemma 11.1. *Let \mathcal{A} be an abelian category. The functor*

$$H^0 : K(\mathcal{A}) \longrightarrow \mathcal{A}$$

is homological.

Proof. Because H^0 is a functor, and by our definition of distinguished triangles it suffices to prove that given a termwise split short exact sequence of complexes $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ the sequence $H^0(A^\bullet) \rightarrow H^0(B^\bullet) \rightarrow H^0(C^\bullet)$ is exact. This follows from Homology, Lemma 12.12. \square

In particular, this lemma implies that a distinguished triangle (X, Y, Z, f, g, h) in $K(\mathcal{A})$ gives rise to a long exact cohomology sequence

$$(11.1.1) \quad \dots \longrightarrow H^i(X) \xrightarrow{H^i(f)} H^i(Y) \xrightarrow{H^i(g)} H^i(Z) \xrightarrow{H^i(h)} H^{i+1}(X) \longrightarrow \dots$$

see (3.5.1). Moreover, there is a compatibility with the long exact sequence of cohomology associated to a short exact sequence of complexes (insert future reference

here). For example, if $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is the distinguished triangle associated to a termwise split exact sequence of complexes (see Definition 9.9), then the cohomology sequence above agrees with the one defined using the snake lemma, see Homology, Lemma 12.12 and for agreement of sequences, see Homology, Lemma 14.11.

Recall that a complex K^\bullet is *acyclic* if $H^i(K^\bullet) = 0$ for all $i \in \mathbf{Z}$. Moreover, recall that a morphism of complexes $f : K^\bullet \rightarrow L^\bullet$ is a *quasi-isomorphism* if and only if $H^i(f)$ is an isomorphism for all i . See Homology, Definition 12.10.

Lemma 11.2. *Let \mathcal{A} be an abelian category. The full subcategory $\text{Ac}(\mathcal{A})$ of $K(\mathcal{A})$ consisting of acyclic complexes is a strictly full saturated triangulated subcategory of $K(\mathcal{A})$. The corresponding saturated multiplicative system (see Lemma 6.10) of $K(\mathcal{A})$ is the set $\text{Qis}(\mathcal{A})$ of quasi-isomorphisms. In particular, the kernel of the localization functor $Q : K(\mathcal{A}) \rightarrow \text{Qis}(\mathcal{A})^{-1}K(\mathcal{A})$ is $\text{Ac}(\mathcal{A})$ and the functor H^0 factors through Q .*

Proof. We know that H^0 is a homological functor by Lemma 11.1. Thus this lemma is a special case of Lemma 6.11. \square

Definition 11.3. Let \mathcal{A} be an abelian category. Let $\text{Ac}(\mathcal{A})$ and $\text{Qis}(\mathcal{A})$ be as in Lemma 11.2. The *derived category of \mathcal{A}* is the triangulated category

$$D(\mathcal{A}) = K(\mathcal{A})/\text{Ac}(\mathcal{A}) = \text{Qis}(\mathcal{A})^{-1}K(\mathcal{A}).$$

We denote $H^0 : D(\mathcal{A}) \rightarrow \mathcal{A}$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above. Using Lemma 6.4 we introduce the strictly full saturated triangulated subcategories $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ whose sets of objects are

$$\begin{aligned} \text{Ob}(D^+(\mathcal{A})) &= \{X \in \text{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \ll 0\} \\ \text{Ob}(D^-(\mathcal{A})) &= \{X \in \text{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \gg 0\} \\ \text{Ob}(D^b(\mathcal{A})) &= \{X \in \text{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } |n| \gg 0\} \end{aligned}$$

The category $D^b(\mathcal{A})$ is called the *bounded derived category* of \mathcal{A} .

If K^\bullet and L^\bullet are complexes of \mathcal{A} then we sometimes say “ K^\bullet is *quasi-isomorphic* to L^\bullet ” to indicate that K^\bullet and L^\bullet are isomorphic objects of $D(\mathcal{A})$.

Remark 11.4. In this chapter, we consistently work with “small” abelian categories (as is the convention in the Stacks project). For a “big” abelian category \mathcal{A} , it isn’t clear that the derived category $D(\mathcal{A})$ exists, because it isn’t clear that morphisms in the derived category are sets. In fact, in general they aren’t, see Examples, Lemma 52.1. However, if \mathcal{A} is a Grothendieck abelian category, and given K^\bullet, L^\bullet in $K(\mathcal{A})$, then by Injectives, Theorem 12.6 there exists a quasi-isomorphism $L^\bullet \rightarrow I^\bullet$ to a K -injective complex I^\bullet and Lemma 29.2 shows that

$$\text{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \text{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$$

which is a set. Some examples of Grothendieck abelian categories are the category of modules over a ring, or more generally the category of sheaves of modules on a ringed site.

Each of the variants $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ can be constructed as a localization of the corresponding homotopy category. This relies on the following simple lemma.

Lemma 11.5. *Let \mathcal{A} be an abelian category. Let K^\bullet be a complex.*

- (1) If $H^n(K^\bullet) = 0$ for all $n \ll 0$, then there exists a quasi-isomorphism $K^\bullet \rightarrow L^\bullet$ with L^\bullet bounded below.
- (2) If $H^n(K^\bullet) = 0$ for all $n \gg 0$, then there exists a quasi-isomorphism $M^\bullet \rightarrow K^\bullet$ with M^\bullet bounded above.
- (3) If $H^n(K^\bullet) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \uparrow & & \uparrow \\ M^\bullet & \longrightarrow & N^\bullet \end{array}$$

where all the arrows are quasi-isomorphisms, L^\bullet bounded below, M^\bullet bounded above, and N^\bullet a bounded complex.

Proof. Pick $a \ll 0 \ll b$ and set $M^\bullet = \tau_{\leq a} K^\bullet$, $L^\bullet = K^\bullet / \tau_{\leq b} K^\bullet$, and $N^\bullet = L^\bullet / M^\bullet$. See Homology, Section 13 for the truncation functors. \square

To state the following lemma denote $\text{Ac}^+(\mathcal{A})$, $\text{Ac}^-(\mathcal{A})$, resp. $\text{Ac}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\text{Ac}(\mathcal{A})$. Denote $\text{Qis}^+(\mathcal{A})$, $\text{Qis}^-(\mathcal{A})$, resp. $\text{Qis}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\text{Qis}(\mathcal{A})$.

Lemma 11.6. *Let \mathcal{A} be an abelian category. The subcategories $\text{Ac}^+(\mathcal{A})$, $\text{Ac}^-(\mathcal{A})$, resp. $\text{Ac}^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 6.10) are the sets $\text{Qis}^+(\mathcal{A})$, $\text{Qis}^-(\mathcal{A})$, resp. $\text{Qis}^b(\mathcal{A})$.*

- (1) The kernel of the functor $K^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$ is $\text{Ac}^+(\mathcal{A})$ and this induces an equivalence of triangulated categories

$$K^+(\mathcal{A}) / \text{Ac}^+(\mathcal{A}) = \text{Qis}^+(\mathcal{A})^{-1} K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$$

- (2) The kernel of the functor $K^-(\mathcal{A}) \rightarrow D^-(\mathcal{A})$ is $\text{Ac}^-(\mathcal{A})$ and this induces an equivalence of triangulated categories

$$K^-(\mathcal{A}) / \text{Ac}^-(\mathcal{A}) = \text{Qis}^-(\mathcal{A})^{-1} K^-(\mathcal{A}) \longrightarrow D^-(\mathcal{A})$$

- (3) The kernel of the functor $K^b(\mathcal{A}) \rightarrow D^b(\mathcal{A})$ is $\text{Ac}^b(\mathcal{A})$ and this induces an equivalence of triangulated categories

$$K^b(\mathcal{A}) / \text{Ac}^b(\mathcal{A}) = \text{Qis}^b(\mathcal{A})^{-1} K^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$$

Proof. The initial statements follow from Lemma 6.11 by considering the restriction of the homological functor H^0 . The statement on kernels in (1), (2), (3) is a consequence of the definitions in each case. Each of the functors is essentially surjective by Lemma 11.5. To finish the proof we have to show the functors are fully faithful. We first do this for the bounded below version.

Suppose that K^\bullet, L^\bullet are bounded above complexes. A morphism between these in $D(\mathcal{A})$ is of the form $s^{-1}f$ for a pair $f : K^\bullet \rightarrow (L')^\bullet$, $s : L^\bullet \rightarrow (L')^\bullet$ where s is a quasi-isomorphism. This implies that $(L')^\bullet$ has cohomology bounded below. Hence by Lemma 11.5 we can choose a quasi-isomorphism $s' : (L')^\bullet \rightarrow (L'')^\bullet$ with $(L'')^\bullet$ bounded below. Then the pair $(s' \circ f, s' \circ s)$ defines a morphism in $\text{Qis}^+(\mathcal{A})^{-1} K^+(\mathcal{A})$. Hence the functor is “full”. Finally, suppose that the pair $f : K^\bullet \rightarrow (L')^\bullet$, $s : L^\bullet \rightarrow (L')^\bullet$ defines a morphism in $\text{Qis}^+(\mathcal{A})^{-1} K^+(\mathcal{A})$ which is zero in $D(\mathcal{A})$. This means that there exists a quasi-isomorphism $s' : (L')^\bullet \rightarrow (L'')^\bullet$

such that $s' \circ f = 0$. Using Lemma 11.5 once more we obtain a quasi-isomorphism $s'' : (L'')^\bullet \rightarrow (L''')^\bullet$ with $(L''')^\bullet$ bounded below. Thus we see that $s'' \circ s' \circ f = 0$ which implies that $s^{-1}f$ is zero in $\mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. This finishes the proof that the functor in (1) is an equivalence.

The proof of (2) is dual to the proof of (1). To prove (3) we may use the result of (2). Hence it suffices to prove that the functor $\mathrm{Qis}^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \rightarrow \mathrm{Qis}^-(\mathcal{A})^{-1}K^-(\mathcal{A})$ is fully faithful. The argument given in the previous paragraph applies directly to show this where we consistently work with complexes which are already bounded above. \square

12. The canonical delta-functor

The derived category should be the receptacle for the universal cohomology functor. In order to state the result we use the notion of a δ -functor from an abelian category into a triangulated category, see Definition 3.6.

Consider the functor $\mathrm{Comp}(\mathcal{A}) \rightarrow K(\mathcal{A})$. This functor is **not** a δ -functor in general. The easiest way to see this is to consider a nonsplit short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of objects of \mathcal{A} . Since $\mathrm{Hom}_{K(\mathcal{A})}(C[0], A[1]) = 0$ we see that any distinguished triangle arising from this short exact sequence would look like $(A[0], B[0], C[0], a, b, 0)$. But the existence of such a distinguished triangle in $K(\mathcal{A})$ implies that the extension is split. A contradiction.

It turns out that the functor $\mathrm{Comp}(\mathcal{A}) \rightarrow D(\mathcal{A})$ is a δ -functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \rightarrow A^\bullet \xrightarrow{a} B^\bullet \xrightarrow{b} C^\bullet \rightarrow 0$$

of complexes in the abelian category \mathcal{A} . Consider the cone $C(a)^\bullet$ of the morphism a . We have $C(a)^n = B^n \oplus A^{n+1}$ and we define $q^n : C(a)^n \rightarrow C^n$ via the projection to B^n followed by b^n . Hence a morphism of complexes

$$q : C(a)^\bullet \rightarrow C^\bullet.$$

It is clear that $q \circ i = b$ where i is as in Definition 9.1. Note that, as a^\bullet is injective in each degree, the kernel of q is identified with the cone of id_{A^\bullet} which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 9.14 the triangle

$$(A, B, C(a), a, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$. As the localization functor $K(\mathcal{A}) \rightarrow D(\mathcal{A})$ is exact we see that $(A, B, C(a), a, i, -p)$ is a distinguished triangle in $D(\mathcal{A})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(\mathcal{A})$. Hence we deduce that

$$(A, B, C, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(\mathcal{A})$. This suggests the following lemma.

Lemma 12.1. *Let \mathcal{A} be an abelian category. The functor $\mathrm{Comp}(\mathcal{A}) \rightarrow D(\mathcal{A})$ defined has the natural structure of a δ -functor, with*

$$\delta_{A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet} = -p \circ q^{-1}$$

with p and q as explained above. The same construction turns the functors $\mathrm{Comp}^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$, $\mathrm{Comp}^-(\mathcal{A}) \rightarrow D^-(\mathcal{A})$, and $\mathrm{Comp}^b(\mathcal{A}) \rightarrow D^b(\mathcal{A})$ into δ -functors.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show that given a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^\bullet & \xrightarrow{a} & B^\bullet & \xrightarrow{b} & C^\bullet \longrightarrow 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & (A')^\bullet & \xrightarrow{a'} & (B')^\bullet & \xrightarrow{b'} & (C')^\bullet \longrightarrow 0 \end{array}$$

we get the desired commutative diagram of Definition 3.6 (2). By Lemma 9.2 the pair (f, g) induces a canonical morphism $c : C(a)^\bullet \rightarrow C(a')^\bullet$. It is a simple computation to show that $q' \circ c = h \circ q$ and $f[1] \circ p = p' \circ c$. From this the result follows directly. \square

Lemma 12.2. *Let \mathcal{A} be an abelian category. Let*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & D^\bullet & \longrightarrow & E^\bullet & \longrightarrow & F^\bullet \longrightarrow 0 \end{array}$$

be a commutative diagram of morphisms of complexes such that the rows are short exact sequences of complexes, and the vertical arrows are quasi-isomorphisms. The δ -functor of Lemma 12.1 above maps the short exact sequences $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ and $0 \rightarrow D^\bullet \rightarrow E^\bullet \rightarrow F^\bullet \rightarrow 0$ to isomorphic distinguished triangles.

Proof. Trivial from the fact that $K(\mathcal{A}) \rightarrow D(\mathcal{A})$ transforms quasi-isomorphisms into isomorphisms and that the associated distinguished triangles are functorial. \square

Lemma 12.3. *Let \mathcal{A} be an abelian category. Let*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

be a short exact sequences of complexes. Assume this short exact sequence is termwise split. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the distinguished triangle of $K(\mathcal{A})$ associated to the sequence. The δ -functor of Lemma 12.1 above maps the short exact sequences $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ to a triangle isomorphic to the distinguished triangle

$$(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta).$$

Proof. Follows from Lemma 9.14. \square

Remark 12.4. Let \mathcal{A} be an abelian category. Let K^\bullet be a complex of \mathcal{A} . Let $a \in \mathbb{Z}$. We claim there is a canonical distinguished triangle

$$\tau_{\leq a} K^\bullet \rightarrow K^\bullet \rightarrow \tau_{\geq a+1} K^\bullet \rightarrow (\tau_{\leq a} K^\bullet)[1]$$

in $D(\mathcal{A})$. Here we have used the canonical truncation functors τ from Homology, Section 13. Namely, we first take the distinguished triangle associated by our δ -functor (Lemma 12.1) to the short exact sequence of complexes

$$0 \rightarrow \tau_{\leq a} K^\bullet \rightarrow K^\bullet \rightarrow K^\bullet / \tau_{\leq a} K^\bullet \rightarrow 0$$

Next, we use that the map $K^\bullet \rightarrow \tau_{\geq a+1} K^\bullet$ factors through a quasi-isomorphism $K^\bullet / \tau_{\leq a} K^\bullet \rightarrow \tau_{\geq a+1} K^\bullet$ by the description of cohomology groups in Homology, Section 13. In a similar way we obtain canonical distinguished triangles

$$\tau_{\leq a} K^\bullet \rightarrow \tau_{\leq a+1} K^\bullet \rightarrow H^{a+1}(K^\bullet)[-a-1] \rightarrow (\tau_{\leq a} K^\bullet)[1]$$

and

$$H^a(K^\bullet)[-a] \rightarrow \tau_{\geq a} K^\bullet \rightarrow \tau_{\geq a+1} K^\bullet \rightarrow H^a(K^\bullet)[-a+1]$$

Lemma 12.5. *Let \mathcal{A} be an abelian category. Let*

$$K_0^\bullet \rightarrow K_1^\bullet \rightarrow \dots \rightarrow K_n^\bullet$$

be maps of complexes such that

- (1) $H^i(K_0^\bullet) = 0$ for $i > 0$,
- (2) $H^{-j}(K_j^\bullet) \rightarrow H^{-j}(K_{j+1}^\bullet)$ is zero.

Then the composition $K_0^\bullet \rightarrow K_n^\bullet$ factors through $\tau_{\leq -n} K_n^\bullet \rightarrow K_n^\bullet$ in $D(\mathcal{A})$.

Proof. The case $n = 1$. Since $\tau_{\leq 0} K_0^\bullet = K_0^\bullet$ in $D(\mathcal{A})$ we can replace K_0^\bullet by $\tau_{\leq 0} K_0^\bullet$ and K_1^\bullet by $\tau_{\leq 0} K_1^\bullet$. Consider the distinguished triangle

$$\tau_{\leq -1} K_1^\bullet \rightarrow K_1^\bullet \rightarrow H^0(K_1^\bullet)[0] \rightarrow (\tau_{\leq -1} K_1^\bullet)[1]$$

(Remark 12.4). The composition $K_0^\bullet \rightarrow K_1^\bullet \rightarrow H^0(K_1^\bullet)[0]$ is zero as it is equal to $K_0^\bullet \rightarrow H^0(K_0^\bullet)[0] \rightarrow H^0(K_1^\bullet)[0]$ which is zero by assumption. The fact that $\text{Hom}_{D(\mathcal{A})}(K_0^\bullet, -)$ is a homological functor (Lemma 4.2), allows us to find the desired factorization. For $n = 2$ we get a factorization $K_0^\bullet \rightarrow \tau_{\leq -1} K_1^\bullet$ by the case $n = 1$ and we can apply the case $n = 1$ to the map of complexes $\tau_{\leq -1} K_1^\bullet \rightarrow \tau_{\leq -1} K_2^\bullet$ to get a factorization $\tau_{\leq -1} K_1^\bullet \rightarrow \tau_{\leq -2} K_2^\bullet$. The general case is proved in exactly the same manner. \square

13. Triangulated subcategories of the derived category

Let \mathcal{A} be an abelian category. In this section we are going to look for strictly full saturated triangulated subcategories $\mathcal{D}' \subset D(\mathcal{A})$ and in the bounded versions.

Here is a simple construction. Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory, see Homology, Section 9. We let $D_{\mathcal{B}}(\mathcal{A})$ the full subcategory of $D(\mathcal{A})$ whose objects are

$$\text{Ob}(D_{\mathcal{B}}(\mathcal{A})) = \{X \in \text{Ob}(D(\mathcal{A})) \mid H^n(X) \text{ is an object of } \mathcal{B} \text{ for all } n\}$$

We also define $D_{\mathcal{B}}^+(\mathcal{A}) = D^+(\mathcal{A}) \cap D_{\mathcal{B}}(\mathcal{A})$ and similarly for the other bounded versions.

Lemma 13.1. *Let \mathcal{A} be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory. The category $D_{\mathcal{B}}(\mathcal{A})$ is a strictly full saturated triangulated subcategory of $D(\mathcal{A})$. Similarly for the bounded versions.*

Proof. It is clear that $D_{\mathcal{B}}(\mathcal{A})$ is an additive subcategory preserved under the translation functors. If $X \oplus Y$ is in $D_{\mathcal{B}}(\mathcal{A})$, then both $H^n(X)$ and $H^n(Y)$ are kernels of maps between maps of objects of \mathcal{B} as $H^n(X \oplus Y) = H^n(X) \oplus H^n(Y)$. Hence both X and Y are in $D_{\mathcal{B}}(\mathcal{A})$. By Lemma 4.14 it therefore suffices to show that given a distinguished triangle (X, Y, Z, f, g, h) such that X and Y are in $D_{\mathcal{B}}(\mathcal{A})$ then Z is an object of $D_{\mathcal{B}}(\mathcal{A})$. The long exact cohomology sequence (11.1.1) and the definition of a weak Serre subcategory (see Homology, Definition 9.1) show that $H^n(Z)$ is an object of \mathcal{B} for all n . Thus Z is an object of $D_{\mathcal{B}}(\mathcal{A})$. \square

An interesting feature of the situation of the lemma is that the functor $D(\mathcal{B}) \rightarrow D(\mathcal{A})$ factors through a canonical exact functor

$$(13.1.1) \quad D(\mathcal{B}) \longrightarrow D_{\mathcal{B}}(\mathcal{A})$$

After all a complex made from objects of \mathcal{B} certainly gives rise to an object of $D_{\mathcal{B}}(\mathcal{A})$ and as distinguished triangles in $D_{\mathcal{B}}(\mathcal{A})$ are exactly the distinguished triangles of $D(\mathcal{A})$ whose vertices are in $D_{\mathcal{B}}(\mathcal{A})$ we see that the functor is exact since $D(\mathcal{B}) \rightarrow D(\mathcal{A})$ is exact. Similarly we obtain functors $D^+(\mathcal{B}) \rightarrow D_{\mathcal{B}}^+(\mathcal{A})$ etc for the bounded versions. A key question in many cases is whether the displayed functor is an equivalence.

Now, suppose that \mathcal{B} is a Serre subcategory of \mathcal{A} . In this case we have the quotient functor $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$, see Homology, Lemma 9.6. In this case $D_{\mathcal{B}}(\mathcal{A})$ is the kernel of the functor $D(\mathcal{A}) \rightarrow D(\mathcal{A}/\mathcal{B})$. Thus we obtain a canonical functor

$$D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) \longrightarrow D(\mathcal{A}/\mathcal{B})$$

by Lemma 6.8. Similarly for the bounded versions.

Lemma 13.2. *Let \mathcal{A} be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Then $D(\mathcal{A}) \rightarrow D(\mathcal{A}/\mathcal{B})$ is essentially surjective.*

Proof. We will use the description of the category \mathcal{A}/\mathcal{B} in the proof of Homology, Lemma 9.6. Let (X^\bullet, d^\bullet) be a complex of \mathcal{A}/\mathcal{B} . For each i we have an object X^i of \mathcal{A} and $d^i = (s^i, f^i)$ where $s^i : Y^i \rightarrow X^i$ is a morphism of \mathcal{A} whose kernel and cokernel are in \mathcal{B} and $f^i : Y^i \rightarrow X^{i+1}$ is an arbitrary morphism of \mathcal{A} . Next, consider the complex

$$\dots \rightarrow X^i \oplus Y^i \oplus Y^{i+1} \rightarrow X^{i+1} \oplus Y^{i+1} \oplus Y^{i+2} \rightarrow \dots$$

in \mathcal{A} with differential given by

$$\begin{pmatrix} 0 & f^i & s^{i+1} \\ 0 & 0 & -\text{id}_{Y^{i+1}} \\ 0 & 0 & 0 \end{pmatrix}.$$

This complex becomes quasi-isomorphic to the complex (X^\bullet, d^\bullet) in \mathcal{A}/\mathcal{B} by the maps

$$(\text{id}_{X^i}, s^i, 0) : X^i \oplus Y^i \oplus Y^{i+1} \rightarrow X^i$$

Calculation omitted. \square

Lemma 13.3. *Let \mathcal{A} be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Suppose that the functor $v : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$ has a left adjoint $u : \mathcal{A}/\mathcal{B} \rightarrow \mathcal{A}$ such that $vu \cong \text{id}$. Then*

$$D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) = D(\mathcal{A}/\mathcal{B})$$

and similarly for the bounded versions.

Proof. The functor $D(v) : D(\mathcal{A}) \rightarrow D(\mathcal{A}/\mathcal{B})$ is essentially surjective by Lemma 13.2. For an object X of $D(\mathcal{A})$ the adjunction mapping $c_X : uvX \rightarrow X$ maps to an isomorphism in $D(\mathcal{A}/\mathcal{B})$ because $vuv \cong v$ by the assumption that $vu \cong \text{id}$. Thus in a distinguished triangle (uvX, X, Z, c_X, g, h) the object Z is an object of $D_{\mathcal{B}}(\mathcal{A})$ as we see by looking at the long exact cohomology sequence. Hence c_X is an element of the multiplicative system used to define the quotient category $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. Thus $uvX \cong X$ in $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. For $X, Y \in \text{Ob}(\mathcal{A})$ the map

$$\text{Hom}_{D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})}(X, Y) \longrightarrow \text{Hom}_{D(\mathcal{A}/\mathcal{B})}(vX, vY)$$

is bijective because u gives an inverse (by the remarks above). \square

14. Filtered derived categories

A reference for this section is [Ill72, I, Chapter V]. Let \mathcal{A} be an abelian category. In this section we will define the filtered derived category $DF(\mathcal{A})$ of \mathcal{A} . In short, we will define it as the derived category of the exact category of objects of \mathcal{A} endowed with a finite filtration. (Thus our construction is a special case of a more general construction of the derived category of an exact category, see for example [Büh10], [Kel90].) Illusie's filtered derived category is the full subcategory of ours consisting of those objects whose filtration is finite. (In our category the filtration is still finite in each degree, but may not be uniformly bounded.) The rationale for our choice is that it is not harder and it allows us to apply the discussion to the spectral sequences of Lemma 21.3, see also Remark 21.4.

We will use the notation regarding filtered objects introduced in Homology, Section 16. The category of filtered objects of \mathcal{A} is denoted $\text{Fil}(\mathcal{A})$. All filtrations will be decreasing by fiat.

Definition 14.1. Let \mathcal{A} be an abelian category. The *category of finite filtered objects of \mathcal{A}* is the category of filtered objects (A, F) of \mathcal{A} whose filtration F is finite. We denote it $\text{Fil}^f(\mathcal{A})$.

Thus $\text{Fil}^f(\mathcal{A})$ is a full subcategory of $\text{Fil}(\mathcal{A})$. For each $p \in \mathbf{Z}$ there is a functor $\text{gr}^p : \text{Fil}^f(\mathcal{A}) \rightarrow \mathcal{A}$. There is a functor

$$\text{gr} = \bigoplus_{p \in \mathbf{Z}} \text{gr}^p : \text{Fil}^f(\mathcal{A}) \rightarrow \text{Gr}(\mathcal{A})$$

where $\text{Gr}(\mathcal{A})$ is the category of graded objects of \mathcal{A} , see Homology, Definition 15.1. Finally, there is a functor

$$(\text{forget } F) : \text{Fil}^f(\mathcal{A}) \longrightarrow \mathcal{A}$$

which associates to the filtered object (A, F) the underlying object of \mathcal{A} . The category $\text{Fil}^f(\mathcal{A})$ is an additive category, but not abelian in general, see Homology, Example 3.12.

Because the functors gr^p , gr , $(\text{forget } F)$ are additive they induce exact functors of triangulated categories

$$\text{gr}^p, (\text{forget } F) : K(\text{Fil}^f(\mathcal{A})) \rightarrow K(\mathcal{A}) \quad \text{and} \quad \text{gr} : K(\text{Fil}^f(\mathcal{A})) \rightarrow K(\text{Gr}(\mathcal{A}))$$

by Lemma 10.6. By analogy with the case of the homotopy category of an abelian category we make the following definitions.

Definition 14.2. Let \mathcal{A} be an abelian category.

- (1) Let $\alpha : K^\bullet \rightarrow L^\bullet$ be a morphism of $K(\text{Fil}^f(\mathcal{A}))$. We say that α is a *filtered quasi-isomorphism* if the morphism $\text{gr}(\alpha)$ is a quasi-isomorphism.
- (2) Let K^\bullet be an object of $K(\text{Fil}^f(\mathcal{A}))$. We say that K^\bullet is *filtered acyclic* if the complex $\text{gr}(K^\bullet)$ is acyclic.

Note that $\alpha : K^\bullet \rightarrow L^\bullet$ is a filtered quasi-isomorphism if and only if each $\text{gr}^p(\alpha)$ is a quasi-isomorphism. Similarly a complex K^\bullet is filtered acyclic if and only if each $\text{gr}^p(K^\bullet)$ is acyclic.

Lemma 14.3. *Let \mathcal{A} be an abelian category.*

- (1) *The functor $K(\text{Fil}^f(\mathcal{A})) \longrightarrow \text{Gr}(\mathcal{A})$, $K^\bullet \mapsto H^0(\text{gr}(K^\bullet))$ is homological.*

- (2) The functor $K(\text{Fil}^f(\mathcal{A})) \rightarrow \mathcal{A}$, $K^\bullet \mapsto H^0(\text{gr}^p(K^\bullet))$ is homological.
- (3) The functor $K(\text{Fil}^f(\mathcal{A})) \rightarrow \mathcal{A}$, $K^\bullet \mapsto H^0((\text{forget } F)K^\bullet)$ is homological.

Proof. This follows from the fact that $H^0 : K(\mathcal{A}) \rightarrow \mathcal{A}$ is homological, see Lemma 11.1 and the fact that the functors gr , gr^p , $(\text{forget } F)$ are exact functors of triangulated categories. See Lemma 4.18. \square

Lemma 14.4. *Let \mathcal{A} be an abelian category. The full subcategory $\text{FAC}(\mathcal{A})$ of $K(\text{Fil}^f(\mathcal{A}))$ consisting of filtered acyclic complexes is a strictly full saturated triangulated subcategory of $K(\text{Fil}^f(\mathcal{A}))$. The corresponding saturated multiplicative system (see Lemma 6.10) of $K(\text{Fil}^f(\mathcal{A}))$ is the set $\text{FQis}(\mathcal{A})$ of filtered quasi-isomorphisms. In particular, the kernel of the localization functor*

$$Q : K(\text{Fil}^f(\mathcal{A})) \longrightarrow \text{FQis}(\mathcal{A})^{-1}K(\text{Fil}^f(\mathcal{A}))$$

is $\text{FAC}(\mathcal{A})$ and the functor $H^0 \circ \text{gr}$ factors through Q .

Proof. We know that $H^0 \circ \text{gr}$ is a homological functor by Lemma 14.3. Thus this lemma is a special case of Lemma 6.11. \square

Definition 14.5. Let \mathcal{A} be an abelian category. Let $\text{FAC}(\mathcal{A})$ and $\text{FQis}(\mathcal{A})$ be as in Lemma 14.4. The *filtered derived category of \mathcal{A}* is the triangulated category

$$DF(\mathcal{A}) = K(\text{Fil}^f(\mathcal{A})) / \text{FAC}(\mathcal{A}) = \text{FQis}(\mathcal{A})^{-1}K(\text{Fil}^f(\mathcal{A})).$$

Lemma 14.6. *The functors gr^p , gr , $(\text{forget } F)$ induce canonical exact functors*

$$\text{gr}^p, \text{gr}, (\text{forget } F) : DF(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

which commute with the localization functors.

Proof. This follows from the universal property of localization, see Lemma 5.6, provided we can show that a filtered quasi-isomorphism is turned into a quasi-isomorphism by each of the functors gr^p , gr , $(\text{forget } F)$. This is true by definition for the first two. For the last one the statement we have to do a little bit of work. Let $f : K^\bullet \rightarrow L^\bullet$ be a filtered quasi-isomorphism in $K(\text{Fil}^f(\mathcal{A}))$. Choose a distinguished triangle $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ which contains f . Then M^\bullet is filtered acyclic, see Lemma 14.4. Hence by the corresponding lemma for $K(\mathcal{A})$ it suffices to show that a filtered acyclic complex is an acyclic complex if we forget the filtration. This follows from Homology, Lemma 16.15. \square

Definition 14.7. Let \mathcal{A} be an abelian category. The *bounded filtered derived category* $DF^b(\mathcal{A})$ is the full subcategory of $DF(\mathcal{A})$ with objects those X such that $\text{gr}(X) \in D^b(\mathcal{A})$. Similarly for the bounded below filtered derived category $DF^+(\mathcal{A})$ and the bounded above filtered derived category $DF^-(\mathcal{A})$.

Lemma 14.8. *Let \mathcal{A} be an abelian category. Let $K^\bullet \in K(\text{Fil}^f(\mathcal{A}))$.*

- (1) *If $H^n(\text{gr}(K^\bullet)) = 0$ for all $n < a$, then there exists a filtered quasi-isomorphism $K^\bullet \rightarrow L^\bullet$ with $L^n = 0$ for all $n < a$.*
- (2) *If $H^n(\text{gr}(K^\bullet)) = 0$ for all $n > b$, then there exists a filtered quasi-isomorphism $M^\bullet \rightarrow K^\bullet$ with $M^n = 0$ for all $n > b$.*

- (3) If $H^n(\text{gr}(K^\bullet)) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \uparrow & & \uparrow \\ M^\bullet & \longrightarrow & N^\bullet \end{array}$$

where all the arrows are filtered quasi-isomorphisms, L^\bullet bounded below, M^\bullet bounded above, and N^\bullet a bounded complex.

Proof. Suppose that $H^n(\text{gr}(K^\bullet)) = 0$ for all $n < a$. By Homology, Lemma 16.15 the sequence

$$K^{a-1} \xrightarrow{d^{a-2}} K^{a-1} \xrightarrow{d^{a-1}} K^a$$

is an exact sequence of objects of \mathcal{A} and the morphisms d^{a-2} and d^{a-1} are strict. Hence $\text{Coim}(d^{a-1}) = \text{Im}(d^{a-1})$ in $\text{Fil}^f(\mathcal{A})$ and the map $\text{gr}(\text{Im}(d^{a-1})) \rightarrow \text{gr}(K^a)$ is injective with image equal to the image of $\text{gr}(K^{a-1}) \rightarrow \text{gr}(K^a)$, see Homology, Lemma 16.13. This means that the map $K^\bullet \rightarrow \tau_{\geq a} K^\bullet$ into the truncation

$$\tau_{\geq a} K^\bullet = (\dots \rightarrow 0 \rightarrow K^a / \text{Im}(d^{a-1}) \rightarrow K^{a+1} \rightarrow \dots)$$

is a filtered quasi-isomorphism. This proves (1). The proof of (2) is dual to the proof of (1). Part (3) follows formally from (1) and (2). \square

To state the following lemma denote $\text{FAC}^+(\mathcal{A})$, $\text{FAC}^-(\mathcal{A})$, resp. $\text{FAC}^b(\mathcal{A})$ the intersection of $K^+(\text{Fil}^f \mathcal{A})$, $K^-(\text{Fil}^f \mathcal{A})$, resp. $K^b(\text{Fil}^f \mathcal{A})$ with $\text{FAC}(\mathcal{A})$. Denote $\text{FQis}^+(\mathcal{A})$, $\text{FQis}^-(\mathcal{A})$, resp. $\text{FQis}^b(\mathcal{A})$ the intersection of $K^+(\text{Fil}^f \mathcal{A})$, $K^-(\text{Fil}^f \mathcal{A})$, resp. $K^b(\text{Fil}^f \mathcal{A})$ with $\text{FQis}(\mathcal{A})$.

Lemma 14.9. *Let \mathcal{A} be an abelian category. The subcategories $\text{FAC}^+(\mathcal{A})$, $\text{FAC}^-(\mathcal{A})$, resp. $\text{FAC}^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of $K^+(\text{Fil}^f \mathcal{A})$, $K^-(\text{Fil}^f \mathcal{A})$, resp. $K^b(\text{Fil}^f \mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 6.10) are the sets $\text{FQis}^+(\mathcal{A})$, $\text{FQis}^-(\mathcal{A})$, resp. $\text{FQis}^b(\mathcal{A})$.*

- (1) *The kernel of the functor $K^+(\text{Fil}^f \mathcal{A}) \rightarrow DF^+(\mathcal{A})$ is $\text{FAC}^+(\mathcal{A})$ and this induces an equivalence of triangulated categories*

$$K^+(\text{Fil}^f \mathcal{A}) / \text{FAC}^+(\mathcal{A}) = \text{FQis}^+(\mathcal{A})^{-1} K^+(\text{Fil}^f \mathcal{A}) \longrightarrow DF^+(\mathcal{A})$$

- (2) *The kernel of the functor $K^-(\text{Fil}^f \mathcal{A}) \rightarrow DF^-(\mathcal{A})$ is $\text{FAC}^-(\mathcal{A})$ and this induces an equivalence of triangulated categories*

$$K^-(\text{Fil}^f \mathcal{A}) / \text{FAC}^-(\mathcal{A}) = \text{FQis}^-(\mathcal{A})^{-1} K^-(\text{Fil}^f \mathcal{A}) \longrightarrow DF^-(\mathcal{A})$$

- (3) *The kernel of the functor $K^b(\text{Fil}^f \mathcal{A}) \rightarrow DF^b(\mathcal{A})$ is $\text{FAC}^b(\mathcal{A})$ and this induces an equivalence of triangulated categories*

$$K^b(\text{Fil}^f \mathcal{A}) / \text{FAC}^b(\mathcal{A}) = \text{FQis}^b(\mathcal{A})^{-1} K^b(\text{Fil}^f \mathcal{A}) \longrightarrow DF^b(\mathcal{A})$$

Proof. This follows from the results above, in particular Lemma 14.8, by exactly the same arguments as used in the proof of Lemma 11.6. \square

15. Derived functors in general

A reference for this section is Deligne’s exposé XVII in [AGV71]. A very general notion of right and left derived functors exists where we have an exact functor between triangulated categories, a multiplicative system in the source category and we want to find the “correct” extension of the exact functor to the localized category.

Situation 15.1. Here $F : \mathcal{D} \rightarrow \mathcal{D}'$ is an exact functor of triangulated categories and S is a saturated multiplicative system in \mathcal{D} compatible with the structure of triangulated category on \mathcal{D} .

Let $X \in \text{Ob}(\mathcal{D})$. Recall from Categories, Remark 25.5 the filtered category X/S of arrows $s : X \rightarrow X'$ in S with source X . Dually, in Categories, Remark 25.12 we defined the cofiltered category S/X of arrows $s : X' \rightarrow X$ in S with target X .

Definition 15.2. Assumptions and notation as in Situation 15.1. Let $X \in \text{Ob}(\mathcal{D})$.

- (1) we say the *right derived functor* RF is defined at X if the ind-object

$$(X/S) \longrightarrow \mathcal{D}', \quad (s : X \rightarrow X') \mapsto F(X')$$

is essentially constant³; in this case the value Y in \mathcal{D}' is called the *value of* RF at X .

- (2) we say the *left derived functor* LF is defined at X if the pro-object

$$(S/X) \longrightarrow \mathcal{D}', \quad (s : X' \rightarrow X) \mapsto F(X')$$

is essentially constant; in this case the value Y in \mathcal{D}' is called the *value of* LF at X .

By abuse of notation we often denote the values simply $RF(X)$ or $LF(X)$.

It will turn out that the full subcategory of \mathcal{D} consisting of objects where RF is defined is a triangulated subcategory, and RF will define a functor on this subcategory which transforms morphisms of S into isomorphisms.

Lemma 15.3. Assumptions and notation as in Situation 15.1. Let $f : X \rightarrow Y$ be a morphism of \mathcal{D} .

- (1) If RF is defined at X and Y then there exists a unique morphism $RF(f) : RF(X) \rightarrow RF(Y)$ between the values such that for any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{s'} & Y' \end{array}$$

with $s, s' \in S$ the diagram

$$\begin{array}{ccccc} F(X) & \longrightarrow & F(X') & \longrightarrow & RF(X) \\ \downarrow & & \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F(Y') & \longrightarrow & RF(Y) \end{array}$$

commutes.

³For a discussion of when an ind-object or pro-object of a category is essentially constant we refer to Categories, Section 22.

- (2) If LF is defined at X and Y then there exists a unique morphism $LF(f) : LF(X) \rightarrow LF(Y)$ between the values such that for any commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{s} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{s'} & Y \end{array}$$

with s, s' in S the diagram

$$\begin{array}{ccccc} LF(X) & \longrightarrow & F(X') & \longrightarrow & F(X) \\ \downarrow & & \downarrow & & \downarrow \\ LF(Y) & \longrightarrow & F(Y') & \longrightarrow & F(Y) \end{array}$$

commutes.

Proof. Part (1) holds if we only assume that the colimits

$$RF(X) = \operatorname{colim}_{s: X \rightarrow X'} F(X') \quad \text{and} \quad RF(Y) = \operatorname{colim}_{s': Y \rightarrow Y'} F(Y')$$

exist. Namely, to give a morphism $RF(X) \rightarrow RF(Y)$ between the colimits is the same thing as giving for each $s : X \rightarrow X'$ in $\operatorname{Ob}(X/S)$ a morphism $F(X') \rightarrow RF(Y)$ compatible with morphisms in the category X/S . To get the morphism we choose a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{s'} & Y' \end{array}$$

with s, s' in S as is possible by MS2 and we set $F(X') \rightarrow RF(Y)$ equal to the composition $F(X') \rightarrow F(Y') \rightarrow RF(Y)$. To see that this is independent of the choice of the diagram above use MS3. Details omitted. The proof of (2) is dual. \square

Lemma 15.4. *Assumptions and notation as in Situation 15.1. Let $s : X \rightarrow Y$ be an element of S .*

- (1) *RF is defined at X if and only if it is defined at Y . In this case the map $RF(s) : RF(X) \rightarrow RF(Y)$ between values is an isomorphism.*
- (2) *LF is defined at X if and only if it is defined at Y . In this case the map $LF(s) : LF(X) \rightarrow LF(Y)$ between values is an isomorphism.*

Proof. Omitted. \square

Lemma 15.5. *Assumptions and notation as in Situation 15.1. Let X be an object of \mathcal{D} and $n \in \mathbf{Z}$.*

- (1) *RF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $RF(X)[n] = RF(X[n])$ between values.*
- (2) *LF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $LF(X)[n] \rightarrow LF(X[n])$ between values.*

Proof. Omitted. \square

Lemma 15.6. *Assumptions and notation as in Situation 15.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of \mathcal{D} . If RF is defined at two out of three of X, Y, Z , then it is defined at the third. Moreover, in this case*

$$(RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h))$$

is a distinguished triangle in \mathcal{D}' . Similarly for LF .

Proof. Say RF is defined at X, Y with values A, B . Let $RF(f) : A \rightarrow B$ be the induced morphism, see Lemma 15.3. We may choose a distinguished triangle $(A, B, C, RF(f), b, c)$ in \mathcal{D}' . We claim that C is a value of RF at Z .

To see this pick $s : X \rightarrow X'$ in S such that there exists a morphism $\alpha : A \rightarrow F(X')$ as in Categories, Definition 22.1. We may choose a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{s'} & Y' \end{array}$$

with $s' \in S$ by MS2. Using that Y/S is filtered we can (after replacing s' by some $s'' : Y \rightarrow Y''$ in S) assume that there exists a morphism $\beta : B \rightarrow F(Y')$ as in Categories, Definition 22.1. Picture

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & F(X') & \longrightarrow & A \\ RF(f) \downarrow & & \downarrow F(f') & & \downarrow RF(f) \\ B & \xrightarrow{\beta} & F(Y') & \longrightarrow & B \end{array}$$

It may not be true that the left square commutes, but the outer and right squares commute. The assumption that the ind-object $\{F(Y')\}_{s': Y' \rightarrow Y}$ is essentially constant means that there exists a $s'' : Y \rightarrow Y''$ in S and a morphism $h : Y' \rightarrow Y''$ such that $s'' = h \circ s'$ and such that $F(h)$ equal to $F(Y') \rightarrow B \rightarrow F(Y') \rightarrow F(Y'')$. Hence after replacing Y' by Y'' and β by $F(h) \circ \beta$ the diagram will commute (by direct computation with arrows).

Using MS6 choose a morphism of triangles

$$(s, s', s'') : (X, Y, Z, f, g, h) \longrightarrow (X', Y', Z', f', g', h')$$

with $s'' \in S$. By TR3 choose a morphism of triangles

$$(\alpha, \beta, \gamma) : (A, B, C, RF(f), b, c) \longrightarrow (F(X'), F(Y'), F(Z'), F(f'), F(g'), F(h'))$$

By Lemma 15.4 it suffices to prove that $RF(Z')$ is defined and has value C . Consider the category \mathcal{I} of Lemma 5.8 of triangles

$$\mathcal{I} = \{(t, t', t'') : (X', Y', Z', f', g', h') \rightarrow (X'', Y'', Z'', f'', g'', h'') \mid (t, t', t'') \in S\}$$

To show that the system $F(Z'')$ is essentially constant over the category Z'/S is equivalent to showing that the system of $F(Z'')$ is essentially constant over \mathcal{I} because $\mathcal{I} \rightarrow Z'/S$ is cofinal, see Categories, Lemma 22.8 (cofinality is proven in

Lemma 5.8). For any object W in \mathcal{D}' we consider the diagram

$$\begin{array}{ccc}
\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(X'')) & \longleftarrow & \mathrm{Mor}_{\mathcal{D}'}(W, A) \\
\uparrow & & \uparrow \\
\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(Y'')) & \longleftarrow & \mathrm{Mor}_{\mathcal{D}'}(W, B) \\
\uparrow & & \uparrow \\
\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(Z'')) & \longleftarrow & \mathrm{Mor}_{\mathcal{D}'}(W, C) \\
\uparrow & & \uparrow \\
\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(X''[1])) & \longleftarrow & \mathrm{Mor}_{\mathcal{D}'}(W, A[1]) \\
\uparrow & & \uparrow \\
\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(Y''[1])) & \longleftarrow & \mathrm{Mor}_{\mathcal{D}'}(W, B[1])
\end{array}$$

where the horizontal arrows are given by composing with (α, β, γ) . Since filtered colimits are exact (Algebra, Lemma 8.9) the left column is an exact sequence. Thus the 5 lemma (Homology, Lemma 5.20) tells us the

$$\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(Z'')) \longrightarrow \mathrm{Mor}_{\mathcal{D}'}(W, C)$$

is bijective. Choose an object $(t, t', t'') : (X', Y', Z') \rightarrow (X'', Y'', Z'')$ of \mathcal{I} . Applying what we just showed to $W = F(Z'')$ and the element $\mathrm{id}_{F(X'')}$ of the colimit we find a unique morphism $c_{(X'', Y'', Z'')} : F(Z'') \rightarrow C$ such that for some $(X''', Y''', Z'') \rightarrow (X''', Y''', Z'')$ in \mathcal{I}

$$F(Z'') \xrightarrow{c_{(X'', Y'', Z'')}} C \xrightarrow{\gamma} F(Z') \rightarrow F(Z'') \rightarrow F(Z''') \quad \text{equals} \quad F(Z'') \rightarrow F(Z''')$$

The family of morphisms $c_{(X'', Y'', Z'')}$ form an element c of $\lim_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(F(Z''), C)$ by uniqueness (computation omitted). Finally, we show that $\mathrm{colim}_{\mathcal{I}} F(Z'') = C$ via the morphisms $c_{(X'', Y'', Z'')}$ which will finish the proof by Categories, Lemma 22.6. Namely, let W be an object of \mathcal{D}' and let $d_{(X'', Y'', Z'')} : F(Z'') \rightarrow W$ be a family of maps corresponding to an element of $\lim_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(F(Z''), W)$. If $d_{(X', Y', Z')} \circ \gamma = 0$, then for every object (X'', Y'', Z'') of \mathcal{I} the morphism $d_{(X'', Y'', Z'')}$ is zero by the existence of $c_{(X'', Y'', Z'')}$ and the morphism $(X'', Y'', Z'') \rightarrow (X''', Y''', Z'')$ in \mathcal{I} satisfying the displayed equality above. Hence the map

$$\lim_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(F(Z''), W) \longrightarrow \mathrm{Mor}_{\mathcal{D}'}(C, W)$$

(coming from precomposing by γ) is injective. However, it is also surjective because the element c gives a left inverse. We conclude that C is the colimit by Categories, Remark 14.4. \square

Lemma 15.7. *Assumptions and notation as in Situation 15.1. Let X, Y be objects of \mathcal{D} .*

- (1) *If RF is defined at X and Y , then RF is defined at $X \oplus Y$.*
- (2) *If \mathcal{D}' is Karoubian and RF is defined at $X \oplus Y$, then RF is defined at both X and Y .*

In either case we have $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. Similarly for LF .

Proof. If RF is defined at X and Y , then the distinguished triangle $X \rightarrow X \oplus Y \rightarrow Y \rightarrow X[1]$ (Lemma 4.10) and Lemma 15.6 shows that RF is defined at $X \oplus Y$ and that we have a distinguished triangle $RF(X) \rightarrow RF(X \oplus Y) \rightarrow RF(Y) \rightarrow RF(X)[1]$. Applying Lemma 4.10 to this once more we find that $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. This proves (1) and the final assertion.

Conversely, assume that RF is defined at $X \oplus Y$ and that \mathcal{D}' is Karoubian. Since S is a saturated system S is the set of arrows which become invertible under the additive localization functor $Q : \mathcal{D} \rightarrow S^{-1}\mathcal{D}$, see Categories, Lemma 25.18. Thus for any $s : X \rightarrow X'$ and $s' : Y \rightarrow Y'$ in S the morphism $s \oplus s' : X \oplus Y \rightarrow X' \oplus Y'$ is an element of S . In this way we obtain a functor

$$X/S \times Y/S \longrightarrow (X \oplus Y)/S$$

Recall that the categories $X/S, Y/S, (X \oplus Y)/S$ are filtered (Categories, Remark 25.5). By Categories, Lemma 22.9 $X/S \times Y/S$ is filtered and $F|_{X/S} : X/S \rightarrow \mathcal{D}'$ (resp. $G|_{Y/S} : Y/S \rightarrow \mathcal{D}'$) is essentially constant if and only if $F|_{X/S} \circ \text{pr}_1 : X/S \times Y/S \rightarrow \mathcal{D}'$ (resp. $G|_{Y/S} \circ \text{pr}_2 : X/S \times Y/S \rightarrow \mathcal{D}'$) is essentially constant. Below we will show that the displayed functor is cofinal, hence by Categories, Lemma 22.8. we see that $F|_{(X \oplus Y)/S}$ is essentially constant implies that $F|_{X/S} \circ \text{pr}_1 \oplus F|_{Y/S} \circ \text{pr}_2 : X/S \times Y/S \rightarrow \mathcal{D}'$ is essentially constant. By Homology, Lemma 26.3 (and this is where we use that \mathcal{D}' is Karoubian) we see that $F|_{X/S} \circ \text{pr}_1 \oplus F|_{Y/S} \circ \text{pr}_2$ being essentially constant implies $F|_{X/S} \circ \text{pr}_1$ and $F|_{Y/S} \circ \text{pr}_2$ are essentially constant proving that RF is defined at X and Y .

Proof that the displayed functor is cofinal. To do this pick any $t : X \oplus Y \rightarrow Z$ in S . Using MS2 we can find morphisms $Z \rightarrow X', Z \rightarrow Y'$ and $s : X \rightarrow X', s' : Y \rightarrow Y'$ in S such that

$$\begin{array}{ccccc} X & \longleftarrow & X \oplus Y & \longrightarrow & Y \\ \downarrow s & & \downarrow & & \downarrow s' \\ X' & \longleftarrow & Z & \longrightarrow & Y' \end{array}$$

commutes. This proves there is a map $Z \rightarrow X' \oplus Y'$ in $(X \oplus Y)/S$, i.e., we get part (1) of Categories, Definition 17.1. To prove part (2) it suffices to prove that given $t : X \oplus Y \rightarrow Z$ and morphisms $s_i \oplus s'_i : Z \rightarrow X'_i \oplus Y'_i$, $i = 1, 2$ in $(X \oplus Y)/S$ we can find morphisms $a : X'_1 \rightarrow X', b : X'_2 \rightarrow X', c : Y'_1 \rightarrow Y', d : Y'_2 \rightarrow Y'$ in S such that $a \circ s_1 = b \circ s_2$ and $c \circ s'_1 = d \circ s'_2$. To do this we first choose any X' and Y' and maps a, b, c, d in S ; this is possible as X/S and Y/S are filtered. Then the two maps $a \circ s_1, b \circ s_2 : Z \rightarrow X'$ become equal in $S^{-1}\mathcal{D}$. Hence we can find a morphism $X' \rightarrow X''$ in S equalizing them. Similarly we find $Y' \rightarrow Y''$ in S equalizing $c \circ s'_1$ and $d \circ s'_2$. Replacing X' by X'' and Y' by Y'' we get $a \circ s_1 = b \circ s_2$ and $c \circ s'_1 = d \circ s'_2$.

The proof of the corresponding statements for LF are dual. \square

Proposition 15.8. *Assumptions and notation as in Situation 15.1. The full subcategory \mathcal{E} of \mathcal{D} consisting of objects at which RF is defined is a strictly full triangulated subcategory of \mathcal{D} . Choosing values using the axiom of choice gives rise to an exact functor*

$$RF : \mathcal{E} \longrightarrow \mathcal{D}'$$

of triangulated categories. Elements of S with either source or target in \mathcal{E} are morphisms of \mathcal{E} . Any element of $S_{\mathcal{E}} = \text{Arrows}(\mathcal{E}) \cap S$ is transformed into an isomorphism by RF . Hence an exact functor

$$RF : S_{\mathcal{E}}^{-1}\mathcal{E} \longrightarrow \mathcal{D}'.$$

If \mathcal{D}' is Karoubian, then \mathcal{E} is a saturated triangulated subcategory of \mathcal{D} . A similar result holds for LF .

Proof. This is just a summary of the results obtained in Lemmas 15.3, 15.4, 15.5, 15.6, and 15.7. \square

Definition 15.9. In Situation 15.1. We say F is *right deriveable*, or that RF *everywhere defined* if RF is defined at every object of \mathcal{D} . We say F is *left deriveable*, or that LF *everywhere defined* if LF is defined at every object of \mathcal{D} .

In this case we obtain a right (resp. left) derived functor

$$(15.9.1) \quad RF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}', \quad (\text{resp. } LF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}'),$$

see Proposition 15.8. In most interesting situations it is not the case that $RF \circ Q$ is equal to F . In fact, it might happen that the canonical map $F(X) \rightarrow RF(X)$ is never an isomorphism. In practice this does not happen, because in practice we only know how to prove F is right deriveable by showing that RF can be computed by evaluating F at judiciously chosen objects of the triangulated category \mathcal{D} . This warrants a definition.

Definition 15.10. In Situation 15.1.

- (1) An object X of \mathcal{D} *computes* RF if RF is defined at X and the canonical map $F(X) \rightarrow RF(X)$ is an isomorphism.
- (2) An object X of \mathcal{D} *computes* LF if LF is defined at X and the canonical map $LF(X) \rightarrow F(X)$ is an isomorphism.

Lemma 15.11. *Assumptions and notation as in Situation 15.1. Let X be an object of \mathcal{D} and $n \in \mathbf{Z}$.*

- (1) X *computes* RF if and only if $X[n]$ *computes* RF .
- (2) X *computes* LF if and only if $X[n]$ *computes* LF .

Proof. Omitted. \square

Lemma 15.12. *Assumptions and notation as in Situation 15.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of \mathcal{D} . If X, Y compute RF then so does Z . Similar for LF .*

Proof. By Lemma 15.6 we know that RF is defined at Z and that RF applied to the triangle produces a distinguished triangle. Consider the morphism of distinguished triangles

$$\begin{array}{c} (F(X), F(Y), F(Z), F(f), F(g), F(h)) \\ \downarrow \\ (RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h)) \end{array}$$

Two out of three maps are isomorphisms, hence so is the third. \square

Lemma 15.13. *Assumptions and notation as in Situation 15.1. Let X, Y be objects of \mathcal{D} . If $X \oplus Y$ computes RF , then X and Y compute RF . Similarly for LF .*

Proof. If $X \oplus Y$ computes RF , then $RF(X \oplus Y) = F(X) \oplus F(Y)$. In the proof of Lemma 15.7 we have seen that the functor $X/S \times Y/S \rightarrow (X \oplus Y)/S$, $(s, s') \mapsto s \oplus s'$ is cofinal. We will use this without further mention. Let $s : X \rightarrow X'$ be an element of S . Then $F(X) \rightarrow F(X')$ has a section, namely,

$$F(X') \rightarrow F(X' \oplus Y) \rightarrow RF(X' \oplus Y) = RF(X \oplus Y) = F(X) \oplus F(Y) \rightarrow F(X).$$

where we have used Lemma 15.4. Hence $F(X') = F(X) \oplus E$ for some object E of \mathcal{D}' such that $E \rightarrow F(X' \oplus Y) \rightarrow RF(X' \oplus Y) = RF(X \oplus Y)$ is zero (Lemma 4.11). Because RF is defined at $X' \oplus Y$ with value $F(X) \oplus F(Y)$ we can find a morphism $t : X' \oplus Y \rightarrow Z$ of S such that $F(t)$ annihilates E . We may assume $Z = X'' \oplus Y''$ and $t = t' \oplus t''$ with $t', t'' \in S$. Then $F(t')$ annihilates E . It follows that F is essentially constant on X/S with value $F(X)$ as desired. \square

Lemma 15.14. *Assumptions and notation as in Situation 15.1.*

- (1) *If for every object $X \in \text{Ob}(\mathcal{D})$ there exists an arrow $s : X \rightarrow X'$ in S such that X' computes RF , then RF is everywhere defined.*
- (2) *If for every object $X \in \text{Ob}(\mathcal{D})$ there exists an arrow $s : X' \rightarrow X$ in S such that X' computes LF , then LF is everywhere defined.*

Proof. This is clear from the definitions. \square

Lemma 15.15. *Assumptions and notation as in Situation 15.1. If there exists a subset $\mathcal{I} \subset \text{Ob}(\mathcal{D})$ such that*

- (1) *for all $X \in \text{Ob}(\mathcal{D})$ there exists $s : X \rightarrow X'$ in S with $X' \in \mathcal{I}$, and*
- (2) *for every arrow $s : X \rightarrow X'$ in S with $X, X' \in \mathcal{I}$ the map $F(s) : F(X) \rightarrow F(X')$ is an isomorphism,*

then RF is everywhere defined and every $X \in \mathcal{I}$ computes RF . Dually, if there exists a subset $\mathcal{P} \subset \text{Ob}(\mathcal{D})$ such that

- (1) *for all $X \in \text{Ob}(\mathcal{D})$ there exists $s : X' \rightarrow X$ in S with $X' \in \mathcal{P}$, and*
- (2) *for every arrow $s : X \rightarrow X'$ in S with $X, X' \in \mathcal{P}$ the map $F(s) : F(X) \rightarrow F(X')$ is an isomorphism,*

then LF is everywhere defined and every $X \in \mathcal{P}$ computes LF .

Proof. Let X be an object of \mathcal{D} . Assumption (1) implies that the arrows $s : X \rightarrow X'$ in S with $X' \in \mathcal{I}$ are cofinal in the category X/S . Assumption (2) implies that F is constant on this cofinal subcategory. Clearly this implies that $F : (X/S) \rightarrow \mathcal{D}'$ is essentially constant with value $F(X')$ for any $s : X \rightarrow X'$ in S with $X' \in \mathcal{I}$. \square

Lemma 15.16. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be triangulated categories. Let S , resp. S' be a saturated multiplicative system in \mathcal{A} , resp. \mathcal{B} compatible with the triangulated structure. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ be exact functors. Denote $F' : \mathcal{A} \rightarrow (S')^{-1}\mathcal{B}$ the composition of F with the localization functor.*

- (1) *If RF' , RG , $R(G \circ F)$ are everywhere defined, then there is a canonical transformation of functors $t : R(G \circ F) \rightarrow RG \circ RF'$.*
- (2) *If LF' , LG , $L(G \circ F)$ are everywhere defined, then there is a canonical transformation of functors $t : LG \circ LF' \rightarrow L(G \circ F)$.*

Proof. In this proof we try to be careful. Hence let us think of the derived functors as the functors

$$RF' : S^{-1}\mathcal{A} \rightarrow (S')^{-1}\mathcal{B}, \quad R(G \circ F) : S^{-1}\mathcal{A} \rightarrow \mathcal{C}, \quad RG : (S')^{-1}\mathcal{B} \rightarrow \mathcal{C}.$$

Let us denote $Q_A : \mathcal{A} \rightarrow S^{-1}\mathcal{A}$ and $Q_B : \mathcal{B} \rightarrow (S')^{-1}\mathcal{B}$ the localization functors. Then $F' = Q_B \circ F$. Note that for every object Y of \mathcal{B} there is a canonical map

$$G(Y) \longrightarrow RG(Q_B(Y))$$

in other words, there is a transformation of functors $t' : G \rightarrow RG \circ Q_B$. Let X be an object of \mathcal{A} . We have

$$\begin{aligned} R(G \circ F)(Q_A(X)) &= \operatorname{colim}_{s: X \rightarrow X' \in S} G(F(X')) \\ &\xrightarrow{t'} \operatorname{colim}_{s: X \rightarrow X' \in S} RG(Q_B(F(X'))) \\ &= \operatorname{colim}_{s: X \rightarrow X' \in S} RG(F'(X')) \\ &= RG(\operatorname{colim}_{s: X \rightarrow X' \in S} F'(X')) \\ &= RG(RF'(X)). \end{aligned}$$

The system $F'(X')$ is essentially constant in the category $(S')^{-1}\mathcal{B}$. Hence we may pull the colimit inside the functor RG in the third equality of the diagram above, see Categories, Lemma 22.5 and its proof. We omit the proof this this defines a transformation of functors. The case of left derived functors is similar. \square

16. Derived functors on derived categories

In practice derived functors come about most often when given an additive functor between abelian categories.

Situation 16.1. Here $F : \mathcal{A} \rightarrow \mathcal{B}$ is an additive functor between abelian categories. This induces exact functors

$$F : K(\mathcal{A}) \rightarrow K(\mathcal{B}), \quad K^+(\mathcal{A}) \rightarrow K^+(\mathcal{B}), \quad K^-(\mathcal{A}) \rightarrow K^-(\mathcal{B}).$$

We also denote F the composition $K(\mathcal{A}) \rightarrow D(\mathcal{B})$, $K^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$, and $K^-(\mathcal{A}) \rightarrow D^-(\mathcal{B})$ of F with the localization functor $K(\mathcal{B}) \rightarrow D(\mathcal{B})$, etc. This situation leads to four derived functors we will consider in the following.

- (1) The right derived functor of $F : K(\mathcal{A}) \rightarrow D(\mathcal{B})$ relative to the multiplicative system $\operatorname{Qis}(\mathcal{A})$.
- (2) The right derived functor of $F : K^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ relative to the multiplicative system $\operatorname{Qis}^+(\mathcal{A})$.
- (3) The left derived functor of $F : K(\mathcal{A}) \rightarrow D(\mathcal{B})$ relative to the multiplicative system $\operatorname{Qis}(\mathcal{A})$.
- (4) The left derived functor of $F : K^-(\mathcal{A}) \rightarrow D^-(\mathcal{B})$ relative to the multiplicative system $\operatorname{Qis}^-(\mathcal{A})$.

Each of these cases is an example of Situation 15.1.

Some of the ambiguity that may arise is alleviated by the following.

Lemma 16.2. *In Situation 16.1.*

- (1) *Let X be an object of $K^+(\mathcal{A})$. The right derived functor of $K(\mathcal{A}) \rightarrow D(\mathcal{B})$ is defined at X if and only if the right derived functor of $K^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is defined at X . Moreover, the values are canonically isomorphic.*

- (2) Let X be an object of $K^+(\mathcal{A})$. Then X computes the right derived functor of $K(\mathcal{A}) \rightarrow D(\mathcal{B})$ if and only if X computes the right derived functor of $K^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$.
- (3) Let X be an object of $K^-(\mathcal{A})$. The left derived functor of $K(\mathcal{A}) \rightarrow D(\mathcal{B})$ is defined at X if and only if the left derived functor of $K^-(\mathcal{A}) \rightarrow D^-(\mathcal{B})$ is defined at X . Moreover, the values are canonically isomorphic.
- (4) Let X be an object of $K^-(\mathcal{A})$. Then X computes the left derived functor of $K(\mathcal{A}) \rightarrow D(\mathcal{B})$ if and only if X computes the left derived functor of $K^-(\mathcal{A}) \rightarrow D^-(\mathcal{B})$.

Proof. Let X be an object of $K^+(\mathcal{A})$. Consider a quasi-isomorphism $s : X \rightarrow X'$ in $K(\mathcal{A})$. By Lemma 11.5 there exists quasi-isomorphism $X' \rightarrow X''$ with X'' bounded below. Hence we see that $X/\text{Qis}^+(\mathcal{A})$ is cofinal in $X/\text{Qis}(\mathcal{A})$. Thus it is clear that (1) holds. Part (2) follows directly from part (1). Parts (3) and (4) are dual to parts (1) and (2). \square

Given an object A of an abelian category \mathcal{A} we get a complex

$$A[0] = (\dots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \dots)$$

where A is placed in degree zero. Hence a functor $\mathcal{A} \rightarrow K(\mathcal{A})$, $A \mapsto A[0]$. Let us temporarily say that a partial functor is one that is defined on a subcategory.

Definition 16.3. In Situation 16.1.

- (1) The *right derived functors of F* are the partial functors RF associated to cases (1) and (2) of Situation 16.1.
- (2) The *left derived functors of F* are the partial functors LF associated to cases (3) and (4) of Situation 16.1.
- (3) An object A of \mathcal{A} is said to be *right acyclic for F* , or *acyclic for RF* if $A[0]$ computes RF .
- (4) An object A of \mathcal{A} is said to be *left acyclic for F* , or *acyclic for LF* if $A[0]$ computes RF .

The following few lemmas give some criteria for the existence of enough acyclics.

Lemma 16.4. Let \mathcal{A} be an abelian category. Let $\mathcal{I} \subset \text{Ob}(\mathcal{A})$ be a subset containing 0 such that every object of \mathcal{A} is a subobject of an element of \mathcal{I} . Let $a \in \mathbf{Z}$.

- (1) Given K^\bullet with $K^n = 0$ for $n < a$ there exists a quasi-isomorphism $K^\bullet \rightarrow I^\bullet$ with $K^n \rightarrow I^n$ injective and $I^n \in \mathcal{I}$ for all n and $I^n = 0$ for $n < a$,
- (2) Given K^\bullet with $H^n(K^\bullet) = 0$ for $n < a$ there exists a quasi-isomorphism $K^\bullet \rightarrow I^\bullet$ with $I^n \in \mathcal{I}$ and $I^n = 0$ for $n < a$.

Proof. Proof of part (1). Consider the following induction hypothesis IH_n : There are $I^j \in \mathcal{I}$, $j \leq n$ almost all zero, maps $d^j : I^j \rightarrow I^{j+1}$ for $j < n$ and injective maps $\alpha^j : K^j \rightarrow I^j$ for $j \leq n$ such that the diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} \longrightarrow \dots \\ & & \downarrow \alpha & & \downarrow \alpha & & \\ \dots & \longrightarrow & I^{n-1} & \longrightarrow & I^n & & \end{array}$$

is commutative, such that $d^j \circ d^{j-1} = 0$ for $j < n$ and such that α induces isomorphisms $H^j(K^\bullet) \rightarrow \text{Ker}(d^j)/\text{Im}(d^{j-1})$ for $j < n$. Note that this implies

$$(16.4.1) \quad \alpha(\text{Im}(d_K^{n-1})) \subset \alpha(\text{Ker}(d_K^n)) \cap \text{Im}(d^{n-1}) \subset \alpha(K^n) \cap \text{Im}(d^{n-1}).$$

If these inclusions are not equalities, then choose an injection

$$I^n \oplus K^n / \text{Im}(d_K^{n-1}) \longrightarrow I$$

with $I \in \mathcal{I}$. Denote $\alpha' : K^n \rightarrow I$ the map obtained by composing $\alpha \oplus 1 : K^n \rightarrow I^n \oplus K^n / \text{Im}(d_K^{n-1})$ with the displayed injection. Denote $d' : I^{n-1} \rightarrow I$ the composition $I^{n-1} \rightarrow I^n \rightarrow I$ of d^{n-1} by the inclusion of the first summand. Then $\alpha'(K^n) \cap \text{Im}(d') = \alpha'(\text{Im}(d_K^{n-1}))$ simply because the intersection of $\alpha'(K^n)$ with the first summand of $I^n \oplus K^n / \text{Im}(d_K^{n-1})$ is equal to $\alpha'(\text{Im}(d_K^{n-1}))$. Hence, after replacing I^n by I , α by α' and d^{n-1} by d' we may assume that we have equality in Equation (16.4.1). Once this is the case consider the solid diagram

$$\begin{array}{ccc} K^n / \text{Ker}(d_K^n) & \xrightarrow{\quad} & K^{n+1} \\ \downarrow & & \downarrow \\ I^n / (\text{Im}(d^{n-1}) + \alpha(\text{Ker}(d_K^n))) & \xrightarrow{\quad} & M \end{array}$$

The horizontal arrow is injective by fiat and the vertical arrow is injective as we have equality in (16.4.1). Hence the push-out M of this diagram contains both K^{n+1} and $I^n / (\text{Im}(d^{n-1}) + \alpha(\text{Ker}(d_K^n)))$ as subobjects. Choose an injection $M \rightarrow I^{n+1}$ with $I^{n+1} \in \mathcal{I}$. By construction we get $d^n : I^n \rightarrow I^{n+1}$ and an injective map $\alpha^{n+1} : K^{n+1} \rightarrow I^{n+1}$. The equality in Equation (16.4.1) and the construction of d^n guarantee that $\alpha : H^n(K^\bullet) \rightarrow \text{Ker}(d^n) / \text{Im}(d^{n-1})$ is an isomorphism. In other words IH_{n+1} holds.

We finish the proof of by the following observations. First we note that IH_n is true for $n = a$ since we can just take $I^j = 0$ for $j < a$ and $K^a \rightarrow I^a$ an injection of K^a into an element of \mathcal{I} . Next, we note that in the proof of $IH_n \Rightarrow IH_{n+1}$ we only modified the object I^n , the map d^{n-1} and the map α^n . Hence we see that proceeding by induction we produce a complex I^\bullet with $I^n = 0$ for $n < a$ consisting of objects from \mathcal{I} , and a termwise injective quasi-isomorphism $\alpha : K^\bullet \rightarrow I^\bullet$ as desired.

Proof of part (2). The assumption implies that the morphism $K^\bullet \rightarrow \tau_{\geq a} K^\bullet$ (Homology, Section 13) is a quasi-isomorphism. Apply part (1) to find $\tau_{\geq a} K^\bullet \rightarrow I^\bullet$. The composition $K^\bullet \rightarrow I^\bullet$ is the desired quasi-isomorphism. \square

Lemma 16.5. *Let \mathcal{A} be an abelian category. Let $\mathcal{P} \subset \text{Ob}(\mathcal{A})$ be a subset containing 0 such that every object of \mathcal{A} is a quotient of an element of \mathcal{P} . Let $a \in \mathbf{Z}$.*

- (1) *Given K^\bullet with $K^n = 0$ for $n > a$ there exists a quasi-isomorphism $P^\bullet \rightarrow K^\bullet$ with $P^n \in \mathcal{P}$ and $P^n \rightarrow K^n$ surjective for all n and $P^n = 0$ for $n > a$.*
- (2) *Given K^\bullet with $H^n(K^\bullet) = 0$ for $n > a$ there exists a quasi-isomorphism $P^\bullet \rightarrow K^\bullet$ with $P^n \in \mathcal{P}$ for all n and $P^n = 0$ for $n > a$.*

Proof. This lemma is dual to Lemma 16.4. \square

Lemma 16.6. *In Situation 16.1. Let $\mathcal{I} \subset \text{Ob}(\mathcal{A})$ be a subset with the following properties:*

- (1) *every object of \mathcal{A} is a subobject of an element of \mathcal{I} ,*
- (2) *for any short exact sequence $0 \rightarrow P \rightarrow Q \rightarrow R \rightarrow 0$ of \mathcal{A} with $P, Q \in \mathcal{I}$, then $R \in \mathcal{I}$, and $0 \rightarrow F(P) \rightarrow F(Q) \rightarrow F(R) \rightarrow 0$ is exact.*

Then every object of \mathcal{I} is acyclic for RF .

Proof. We may add 0 to \mathcal{I} if necessary. Pick $A \in \mathcal{I}$. Let $A[0] \rightarrow K^\bullet$ be a quasi-isomorphism with K^\bullet bounded below. Then we can find a quasi-isomorphism $K^\bullet \rightarrow I^\bullet$ with I^\bullet bounded below and each $I^n \in \mathcal{I}$, see Lemma 16.4. Hence we see that these resolutions are cofinal in the category $A[0]/\text{Qis}^+(\mathcal{A})$. To finish the proof it therefore suffices to show that for any quasi-isomorphism $A[0] \rightarrow I^\bullet$ with I^\bullet bounded above and $I^n \in \mathcal{I}$ we have $F(A)[0] \rightarrow F(I^\bullet)$ is a quasi-isomorphism. To see this suppose that $I^n = 0$ for $n < n_0$. Of course we may assume that $n_0 < 0$. Starting with $n = n_0$ we prove inductively that $\text{Im}(d^{n-1}) = \text{Ker}(d^n)$ and $\text{Im}(d^{-1})$ are elements of \mathcal{I} using property (2) and the exact sequences

$$0 \rightarrow \text{Ker}(d^n) \rightarrow I^n \rightarrow \text{Im}(d^n) \rightarrow 0.$$

Moreover, property (2) also guarantees that the complex

$$0 \rightarrow F(I^{n_0}) \rightarrow F(I^{n_0+1}) \rightarrow \dots \rightarrow F(I^{-1}) \rightarrow F(\text{Im}(d^{-1})) \rightarrow 0$$

is exact. The exact sequence $0 \rightarrow \text{Im}(d^{-1}) \rightarrow I^0 \rightarrow I^0/\text{Im}(d^{-1}) \rightarrow 0$ implies that $I^0/\text{Im}(d^{-1})$ is an element of \mathcal{I} . The exact sequence $0 \rightarrow A \rightarrow I^0/\text{Im}(d^{-1}) \rightarrow \text{Im}(d^0) \rightarrow 0$ then implies that $\text{Im}(d^0) = \text{Ker}(d^1)$ is an element of \mathcal{I} and from then on one continues as before to show that $\text{Im}(d^{n-1}) = \text{Ker}(d^n)$ is an element of \mathcal{I} for all $n > 0$. Applying F to each of the short exact sequences mentioned above and using (2) we observe that $F(A)[0] \rightarrow F(I^\bullet)$ is an isomorphism as desired. \square

Lemma 16.7. *In Situation 16.1. Let $\mathcal{P} \subset \text{Ob}(\mathcal{A})$ be a subset with the following properties:*

- (1) *every object of \mathcal{A} is a quotient of an element of \mathcal{P} ,*
- (2) *for any short exact sequence $0 \rightarrow P \rightarrow Q \rightarrow R \rightarrow 0$ of \mathcal{A} with $Q, R \in \mathcal{P}$, then $P \in \mathcal{P}$, and $0 \rightarrow F(P) \rightarrow F(Q) \rightarrow F(R) \rightarrow 0$ is exact.*

Then every object of \mathcal{P} is acyclic for LF .

Proof. Dual to the proof of Lemma 16.6. \square

Proposition 16.8. *In Situation 16.1.*

- (1) *If every object of \mathcal{A} injects into an object acyclic for RF , then RF is defined on all of $K^+(\mathcal{A})$ and we obtain an exact functor*

$$RF : D^+(\mathcal{B}) \longrightarrow D^+(\mathcal{A})$$

see (15.9.1). Moreover, any bounded below complex K^\bullet whose terms are acyclic for RF computes RF .

- (2) *If every object of \mathcal{A} is quotient of an object acyclic for LF , then LF is defined on all of $K^-(\mathcal{A})$ and we obtain an exact functor*

$$LF : D^-(\mathcal{B}) \longrightarrow D^-(\mathcal{A})$$

see (15.9.1). Moreover, any bounded above complex K^\bullet whose terms are acyclic for LF computes LF .

Proof. Suppose every object of \mathcal{A} injects into an object acyclic for RF . Let \mathcal{I} be the set of objects acyclic for RF . Let K^\bullet be a bounded below complex with $K^n \in \mathcal{I}$. By Lemma 16.4 the quasi-isomorphisms $\alpha : K^\bullet \rightarrow I^\bullet$ with I^\bullet bounded below and $I^n \in \mathcal{I}$ are cofinal in the category $K^\bullet/\text{Qis}^+(\mathcal{A})$. Hence in order to show that K^\bullet computes RF it suffices to show that $F(K^\bullet) \rightarrow F(I^\bullet)$ is an isomorphism. Note that $C(\alpha)^\bullet$ is an acyclic bounded below complex all of whose terms are in \mathcal{I} .

Hence it suffices to show: given an acyclic bounded below complex I^\bullet all of whose terms are in \mathcal{I} the complex $F(I^\bullet)$ is acyclic.

Say $I^n = 0$ for $n < n_0$. Then we break I^\bullet into short exact sequences $0 \rightarrow \text{Im}(d^n) \rightarrow I^{n+1} \rightarrow \text{Im}(d^{n+1}) \rightarrow 0$ for $n \geq n_0$. These sequences induce distinguished triangles $(\text{Im}(d^n), I^{n+1}, \text{Im}(d^{n+1}))$ by Lemma 12.1. This implies inductively that each $\text{Im}(d^n)$ is acyclic for RF by Lemma 15.12. Moreover, the long exact cohomology sequences (11.1.1) associated to the distinguished triangles $(F(\text{Im}(d^n)), F(I^{n+1}), F(\text{Im}(d^{n+1})))$ of $D^+(\mathcal{B})$ imply that

$$0 \rightarrow F(\text{Im}(d^n)) \rightarrow F(I^{n+1}) \rightarrow F(\text{Im}(d^{n+1})) \rightarrow 0$$

is short exact, and this in turn proves that $F(I^\bullet)$ is exact.

Finally, since by Lemma 16.4 every object of $K^+(\mathcal{A})$ is quasi-isomorphic to such a bounded below complex with terms in \mathcal{I} we see that RF is everywhere defined, see Lemma 15.14. The proof in the case of LF is dual. \square

17. Higher derived functors

The following simple lemma shows that right derived functors “move to the right”.

Lemma 17.1. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories. Let $K^\bullet \in K^+(\mathcal{A})$ and $a \in \mathbf{Z}$.*

- (1) *If $H^i(K^\bullet) = 0$ for all $i < a$ and RF is defined at K^\bullet , then $H^i(RF(K^\bullet)) = 0$ for all $i < a$.*
- (2) *If RF is defined at K^\bullet and $\tau_{\leq a} K^\bullet$, then $H^i(RF(\tau_{\leq a} K^\bullet)) = H^i(RF(K^\bullet))$ for all $i \leq a$.*

Proof. Assume K^\bullet satisfies the assumptions of (1). Let $K^\bullet \rightarrow L^\bullet$ be any quasi-isomorphism. Then it is also true that $K^\bullet \rightarrow \tau_{\geq a} L^\bullet$ is a quasi-isomorphism by our assumption on K^\bullet . Hence in the category $K^\bullet/\text{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : K^\bullet \rightarrow L^\bullet$ with $L^n = 0$ for $n < a$ are cofinal. Thus RF is the value of the essentially constant ind-object $F(L^\bullet)$ for these s it follows that $H^i(RF(K^\bullet)) = 0$ for $i < 0$.

To prove (2) we use the distinguished triangle

$$\tau_{\leq a} K^\bullet \rightarrow K^\bullet \rightarrow \tau_{\geq a+1} K^\bullet \rightarrow (\tau_{\leq a} K^\bullet)[1]$$

of Remark 12.4 to conclude via Lemma 15.6 that RF is defined at $\tau_{\geq a+1} K^\bullet$ as well and that we have a distinguished triangle

$$RF(\tau_{\leq a} K^\bullet) \rightarrow RF(K^\bullet) \rightarrow RF(\tau_{\geq a+1} K^\bullet) \rightarrow RF(\tau_{\leq a} K^\bullet)[1]$$

in $D(\mathcal{B})$. By part (1) we see that $RF(\tau_{\geq a+1} K^\bullet)$ has vanishing cohomology in degrees $< a+1$. The long exact cohomology sequence of this distinguished triangle then shows what we want. \square

Definition 17.2. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories. Assume $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is everywhere defined. Let $i \in \mathbf{Z}$. The i th right derived functor $R^i F$ of F is the functor

$$R^i F = H^i \circ RF : \mathcal{A} \longrightarrow \mathcal{B}$$

The following lemma shows that it really does not make a lot of sense to take the right derived functor unless the functor is left exact.

Lemma 17.3. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is everywhere defined.*

- (1) *We have $R^i F = 0$ for $i < 0$,*
- (2) *$R^0 F$ is left exact,*
- (3) *the map $F \rightarrow R^0 F$ is an isomorphism if and only if F is left exact.*

Proof. Let A be an object of \mathcal{A} . Let $A[0] \rightarrow K^\bullet$ be any quasi-isomorphism. Then it is also true that $A[0] \rightarrow \tau_{\geq 0} K^\bullet$ is a quasi-isomorphism. Hence in the category $A[0]/\text{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : A[0] \rightarrow K^\bullet$ with $K^n = 0$ for $n < 0$ are cofinal. Thus it is clear that $H^i(RF(A[0])) = 0$ for $i < 0$. Moreover, for such an s the sequence

$$0 \rightarrow A \rightarrow K^0 \rightarrow K^1$$

is exact. Hence if F is left exact, then $0 \rightarrow F(A) \rightarrow F(K^0) \rightarrow F(K^1)$ is exact as well, and we see that $F(A) \rightarrow H^0(F(K^\bullet))$ is an isomorphism for every $s : A[0] \rightarrow K^\bullet$ as above which implies that $H^0(RF(A[0])) = F(A)$.

Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of \mathcal{A} . By Lemma 12.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $K^+(\mathcal{A})$. From the long exact cohomology sequence (and the vanishing for $i < 0$ proved above) we deduce that $0 \rightarrow R^0 F(A) \rightarrow R^0 F(B) \rightarrow R^0 F(C)$ is exact. Hence $R^0 F$ is left exact. Of course this also proves that if $F \rightarrow R^0 F$ is an isomorphism, then F is left exact. \square

Lemma 17.4. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is everywhere defined. Let A be an object of \mathcal{A} .*

- (1) *A is right acyclic for F if and only if $F(A) \rightarrow R^0 F(A)$ is an isomorphism and $R^i F(A) = 0$ for all $i > 0$,*
- (2) *if F is left exact, then A is right acyclic for F if and only if $R^i F(A) = 0$ for all $i > 0$.*

Proof. If A is right acyclic for F , then $RF(A[0]) = F(A)[0]$ and in particular $F(A) \rightarrow R^0 F(A)$ is an isomorphism and $R^i F(A) = 0$ for $i \neq 0$. Conversely, if $F(A) \rightarrow R^0 F(A)$ is an isomorphism and $R^i F(A) = 0$ for all $i > 0$ then $F(A[0]) \rightarrow RF(A[0])$ is a quasi-isomorphism by Lemma 17.3 part (1) and hence A is acyclic. If F is left exact then $F = R^0 F$, see Lemma 17.3. \square

Lemma 17.5. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor between abelian categories and assume $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is everywhere defined. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of \mathcal{A} .*

- (1) *If A and C are right acyclic for F then so is B .*
- (2) *If A and B are right acyclic for F then so is C .*
- (3) *If B and C are right acyclic for F and $F(B) \rightarrow F(C)$ is surjective then A is right acyclic for F .*

In each of the three cases

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$$

is a short exact sequence of \mathcal{B} .

Proof. By Lemma 12.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $K^+(\mathcal{A})$. As RF is an exact functor and since $R^i F = 0$ for $i < 0$ and $R^0 F = F$ (Lemma 17.3) we obtain an exact cohomology sequence

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow R^1 F(A) \rightarrow \dots$$

in the abelian category \mathcal{B} . Thus the lemma follows from the characterization of acyclic objects in Lemma 17.4. \square

Lemma 17.6. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is everywhere defined.*

- (1) *The functors $R^i F$, $i \geq 0$ come equipped with a canonical structure of a δ -functor from $\mathcal{A} \rightarrow \mathcal{B}$, see Homology, Definition 11.1.*
- (2) *If every object of \mathcal{A} is a subobject of a right acyclic object for F , then $\{R^i F, \delta\}_{i \geq 0}$ is a universal δ -functor, see Homology, Definition 11.3.*

Proof. The functor $\mathcal{A} \rightarrow \text{Comp}^+(\mathcal{A})$, $A \mapsto A[0]$ is exact. The functor $\text{Comp}^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$ is a δ -functor, see Lemma 12.1. The functor $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is exact. Finally, the functor $H^0 : D^+(\mathcal{B}) \rightarrow \mathcal{B}$ is a homological functor, see Definition 11.3. Hence we get the structure of a δ -functor from Lemma 4.20 and Lemma 4.19. Part (2) follows from Homology, Lemma 11.4 and the description of acyclics in Lemma 17.4. \square

Lemma 17.7 (Leray's acyclicity lemma). *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is everywhere defined. Let A^\bullet be a bounded below complex of F -acyclic objects. The canonical map*

$$F(A^\bullet) \longrightarrow RF(A^\bullet)$$

is an isomorphism in $D^+(\mathcal{B})$, i.e., A^\bullet computes RF .

Proof. First we claim the lemma holds for a bounded complex of acyclic objects. Namely, it holds for complexes with at most one nonzero object by definition. Suppose that A^\bullet is a complex with $A^n = 0$ for $n \notin [a, b]$. Using the “stupid” truncations we obtain a termwise split short exact sequence of complexes

$$0 \rightarrow \sigma_{\geq a+1} A^\bullet \rightarrow A^\bullet \rightarrow \sigma_{\leq a} A^\bullet \rightarrow 0$$

see Homology, Section 13. Thus a distinguished triangle $(\sigma_{\geq a+1} A^\bullet, A^\bullet, \sigma_{\leq a} A^\bullet)$. By induction hypothesis the two outer complexes compute RF . Then the middle one does too by Lemma 15.12.

Suppose that A^\bullet is a bounded below complex of acyclic objects. To show that $F(A) \rightarrow RF(A)$ is an isomorphism in $D^+(\mathcal{B})$ it suffices to show that $H^i(F(A)) \rightarrow H^i(RF(A))$ is an isomorphism for all i . Pick i . Consider the termwise split short exact sequence of complexes

$$0 \rightarrow \sigma_{\geq i+2} A^\bullet \rightarrow A^\bullet \rightarrow \sigma_{\leq i+1} A^\bullet \rightarrow 0.$$

Note that this induces a termwise split short exact sequence

$$0 \rightarrow \sigma_{\geq i+2} F(A^\bullet) \rightarrow F(A^\bullet) \rightarrow \sigma_{\leq i+1} F(A^\bullet) \rightarrow 0.$$

Hence we get distinguished triangles

$$\begin{aligned} & (\sigma_{\geq i+2} A^\bullet, A^\bullet, \sigma_{\leq i+1} A^\bullet) \\ & (\sigma_{\geq a+1} F(A^\bullet), F(A^\bullet), \sigma_{\leq a} F(A^\bullet)) \\ & (RF(\sigma_{\geq a+1} A^\bullet), RF(A^\bullet), RF(\sigma_{\leq a} A^\bullet)) \end{aligned}$$

Using the last two we obtain a map of exact sequences

$$\begin{array}{ccccccc}
 H^i(\sigma_{\geq i+2}F(A^\bullet)) & \longrightarrow & H^i(F(A^\bullet)) & \longrightarrow & H^i(\sigma_{\leq i+1}F(A^\bullet)) & \longrightarrow & H^{i+1}(\sigma_{\geq i+2}F(A^\bullet)) \\
 \downarrow & & \downarrow \alpha & & \downarrow \beta & & \downarrow \\
 R^iF(\sigma_{\geq i+2}A^\bullet) & \longrightarrow & R^iF(A^\bullet) & \longrightarrow & R^iF(\sigma_{\leq i+1}A^\bullet) & \longrightarrow & R^{i+1}F(\sigma_{\geq i+2}A^\bullet)
 \end{array}$$

By the results of the first paragraph the map β is an isomorphism. By inspection the objects on the upper left and the upper right are zero. Hence to finish the proof we have to show that $R^iF(\sigma_{\geq i+2}A^\bullet) = 0$ and $R^{i+1}F(\sigma_{\geq i+2}A^\bullet) = 0$. This follows immediately from Lemma 17.1. \square

Lemma 17.8. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor of abelian categories. Then*

- (1) *every object of \mathcal{A} is right acyclic for F ,*
- (2) *$RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is everywhere defined,*
- (3) *$RF : D(\mathcal{A}) \rightarrow D(\mathcal{B})$ is everywhere defined,*
- (4) *every complex computes RF , in other words, the canonical map $F(K^\bullet) \rightarrow RF(K^\bullet)$ is an isomorphism for all complexes, and*
- (5) *$R^iF = 0$ for $i \neq 0$.*

Proof. This is true because F transforms acyclic complexes into acyclic complexes and quasi-isomorphisms into quasi-isomorphisms. Details omitted. \square

18. Injective resolutions

In this section we prove some lemmas regarding the existence of injective resolutions in abelian categories having enough injectives.

Definition 18.1. Let \mathcal{A} be an abelian category. Let $A \in \text{Ob}(\mathcal{A})$. An *injective resolution* of A is a complex I^\bullet together with a map $A \rightarrow I^0$ such that:

- (1) We have $I^n = 0$ for $n < 0$.
- (2) Each I^n is an injective object of \mathcal{A} .
- (3) The map $A \rightarrow I^0$ is an isomorphism onto $\text{Ker}(d^0)$.
- (4) We have $H^i(I^\bullet) = 0$ for $i > 0$.

Hence $A[0] \rightarrow I^\bullet$ is a quasi-isomorphism. In other words the complex

$$\dots \rightarrow 0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

is acyclic. Let K^\bullet be a complex in \mathcal{A} . An *injective resolution* of K^\bullet is a complex I^\bullet together with a map $\alpha : K^\bullet \rightarrow I^\bullet$ of complexes such that

- (1) We have $I^n = 0$ for $n \ll 0$, i.e., I^\bullet is bounded below.
- (2) Each I^n is an injective object of \mathcal{A} .
- (3) The map $\alpha : K^\bullet \rightarrow I^\bullet$ is a quasi-isomorphism.

In other words an injective resolution $K^\bullet \rightarrow I^\bullet$ gives rise to a diagram

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & I^{n-1} & \longrightarrow & I^n & \longrightarrow & I^{n+1} \longrightarrow \dots
 \end{array}$$

which induces an isomorphism on cohomology objects in each degree. An injective resolution of an object A of \mathcal{A} is almost the same thing as an injective resolution of the complex $A[0]$.

Lemma 18.2. *Let \mathcal{A} be an abelian category. Let K^\bullet be a complex of \mathcal{A} .*

- (1) *If K^\bullet has an injective resolution then $H^n(K^\bullet) = 0$ for $n \ll 0$.*
- (2) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then there exists a quasi-isomorphism $K^\bullet \rightarrow L^\bullet$ with L^\bullet bounded below.*

Proof. Omitted. For the second statement use $L^\bullet = \tau_{\geq n} K^\bullet$ for some $n \ll 0$. See Homology, Section 13 for the definition of the truncation $\tau_{\geq n}$. \square

Lemma 18.3. *Let \mathcal{A} be an abelian category. Assume \mathcal{A} has enough injectives.*

- (1) *Any object of \mathcal{A} has an injective resolution.*
- (2) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then K^\bullet has an injective resolution.*
- (3) *If K^\bullet is a complex with $K^n = 0$ for $n < a$, then there exists an injective resolution $\alpha : K^\bullet \rightarrow I^\bullet$ with $I^n = 0$ for $n < a$ such that each $\alpha^n : K^n \rightarrow I^n$ is injective.*

Proof. Proof of (1). First choose an injection $A \rightarrow I^0$ of A into an injective object of \mathcal{A} . Next, choose an injection $I_0/A \rightarrow I^1$ into an injective object of \mathcal{A} . Denote d^0 the induced map $I^0 \rightarrow I^1$. Next, choose an injection $I^1/\text{Im}(d^0) \rightarrow I^2$ into an injective object of \mathcal{A} . Denote d^1 the induced map $I^1 \rightarrow I^2$. And so on. By Lemma 18.2 part (2) follows from part (3). Part (3) is a special case of Lemma 16.4. \square

Lemma 18.4. *Let \mathcal{A} be an abelian category. Let K^\bullet be an acyclic complex. Let I^\bullet be bounded below and consisting of injective objects. Any morphism $K^\bullet \rightarrow I^\bullet$ is homotopic to zero.*

Proof. Let $\alpha : K^\bullet \rightarrow I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \rightarrow I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if } j \leq n \\ \alpha^{n+1} - d \circ h & \text{if } j = n+1 \\ \alpha^j & \text{if } j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n|_{d^{n-1}(K^{n-1})} = 0$ since $\alpha^{n-1} = 0$. Since K^\bullet is acyclic we have $d^{n-1}(K^{n-1}) = \text{Ker}(K^n \rightarrow K^{n+1})$. Hence we can think of α^n as a map into I^n defined on the subobject $\text{Im}(K^n \rightarrow K^{n+1})$ of K^{n+1} . By injectivity of the object I^n we can extend this to a map $h : K^{n+1} \rightarrow I^n$ as desired. \square

Remark 18.5. Let \mathcal{A} be an abelian category. Using the fact that $K(\mathcal{A})$ is a triangulated category we may use Lemma 18.4 to obtain proofs of some of the lemmas below which are usually proved by chasing through diagrams. Namely, suppose that $\alpha : K^\bullet \rightarrow L^\bullet$ is a quasi-isomorphism of complexes. Then

$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$ (Lemma 9.14) and $C(f)^\bullet$ is an acyclic complex (Lemma 11.2). Next, let I^\bullet be a bounded below complex of injective objects. Then

$$\begin{array}{ccccc} \text{Hom}_{K(\mathcal{A})}(C(\alpha)^\bullet, I^\bullet) & \longrightarrow & \text{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) & \longrightarrow & \text{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet) \\ & & & \swarrow & \\ & & & \text{Hom}_{K(\mathcal{A})}(C(\alpha)^\bullet[-1], I^\bullet) & \end{array}$$

is an exact sequence of abelian groups, see Lemma 4.2. At this point Lemma 18.4 guarantees that the outer two groups are zero and hence $\text{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \text{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

Lemma 18.6. *Let \mathcal{A} be an abelian category. Consider a solid diagram*

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ \gamma \downarrow & \nearrow \beta & \\ I^\bullet & & \end{array}$$

where I^\bullet is bounded below and consists of injective objects, and α is a quasi-isomorphism.

- (1) *There exists a map of complexes β making the diagram commute up to homotopy.*
- (2) *If α is injective in every degree then we can find a β which makes the diagram commute.*

Proof. The “correct” proof of part (1) is explained in Remark 18.5. We also give a direct proof here.

We first show that (2) implies (1). Namely, let $\tilde{\alpha} : K \rightarrow \tilde{L}^\bullet$, π, s be as in Lemma 9.6. Since $\tilde{\alpha}$ is injective by (2) there exists a morphism $\tilde{\beta} : \tilde{L}^\bullet \rightarrow I^\bullet$ such that $\gamma = \tilde{\beta} \circ \tilde{\alpha}$. Set $\beta = \tilde{\beta} \circ s$. Then we have

$$\beta \circ \alpha = \tilde{\beta} \circ s \circ \pi \circ \tilde{\alpha} \sim \tilde{\beta} \circ \tilde{\alpha} = \gamma$$

as desired.

Assume that $\alpha : K^\bullet \rightarrow L^\bullet$ is injective. Suppose we have already defined β in all degrees $\leq n-1$ compatible with differentials and such that $\gamma^j = \beta^j \circ \alpha^j$ for all $j \leq n-1$. Consider the commutative solid diagram

$$\begin{array}{ccc} K^{n-1} & \xrightarrow{\quad} & K^n \\ \downarrow \alpha & & \downarrow \alpha \\ L^{n-1} & \xrightarrow{\quad} & L^n \\ \downarrow \beta & & \downarrow \beta \\ I^{n-1} & \xrightarrow{\quad} & I^n \end{array} \quad \begin{array}{c} \gamma \curvearrowright \\ \gamma \end{array}$$

Thus we see that the dotted arrow is prescribed on the subobjects $\alpha(K^n)$ and $d^{n-1}(L^{n-1})$. Moreover, these two arrows agree on $\alpha(d^{n-1}(K^{n-1}))$. Hence if

$$(18.6.1) \quad \alpha(d^{n-1}(K^{n-1})) = \alpha(K^n) \cap d^{n-1}(L^{n-1})$$

then these morphisms glue to a morphism $\alpha(K^n) + d^{n-1}(L^{n-1}) \rightarrow I^n$ and, using the injectivity of I^n , we can extend this to a morphism from all of L^n into I^n . After this by induction we get the morphism β for all n simultaneously (note that we can set $\beta^n = 0$ for all $n \ll 0$ since I^\bullet is bounded below – in this way starting the induction).

It remains to prove the equality (18.6.1). The reader is encouraged to argue this for themselves with a suitable diagram chase. Nonetheless here is our argument. Note that the inclusion $\alpha(d^{n-1}(K^{n-1})) \subset \alpha(K^n) \cap d^{n-1}(L^{n-1})$ is obvious. Take an object T of \mathcal{A} and a morphism $x : T \rightarrow L^n$ whose image is contained in the

subobject $\alpha(K^n) \cap d^{n-1}(L^{n-1})$. Since α is injective we see that $x = \alpha \circ x'$ for some $x' : T \rightarrow K^n$. Moreover, since x lies in $d^{n-1}(L^{n-1})$ we see that $d^n \circ x = 0$. Hence using injectivity of α again we see that $d^n \circ x' = 0$. Thus x' gives a morphism $[x'] : T \rightarrow H^n(K^\bullet)$. On the other hand the corresponding map $[x] : T \rightarrow H^n(L^\bullet)$ induced by x is zero by assumption. Since α is a quasi-isomorphism we conclude that $[x'] = 0$. This of course means exactly that the image of x' is contained in $d^{n-1}(K^{n-1})$ and we win. \square

Lemma 18.7. *Let \mathcal{A} be an abelian category. Consider a solid diagram*

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ \gamma \downarrow & \nearrow \beta_i & \\ I^\bullet & & \end{array}$$

where I^\bullet is bounded below and consists of injective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. This follows from Remark 18.5. We also give a direct argument here.

Let $\tilde{\alpha} : K \rightarrow \tilde{L}^\bullet$, π, s be as in Lemma 9.6. If we can show that $\beta_1 \circ \pi$ is homotopic to $\beta_2 \circ \pi$, then we deduce that $\beta_1 \sim \beta_2$ because $\pi \circ s$ is the identity. Hence we may assume $\alpha^n : K^n \rightarrow L^n$ is the inclusion of a direct summand for all n . Thus we get a short exact sequence of complexes

$$0 \rightarrow K^\bullet \rightarrow L^\bullet \rightarrow M^\bullet \rightarrow 0$$

which is termwise split and such that M^\bullet is acyclic. We choose splittings $L^n = K^n \oplus M^n$, so we have $\beta_i^n : K^n \oplus M^n \rightarrow I^n$ and $\gamma^n : K^n \rightarrow I^n$. In this case the condition on β_i is that there are morphisms $h_i^n : K^n \rightarrow I^{n-1}$ such that

$$\gamma^n - \beta_i^n|_{K^n} = d \circ h_i^n + h_i^{n+1} \circ d$$

Thus we see that

$$\beta_1^n|_{K^n} - \beta_2^n|_{K^n} = d \circ (h_1^n - h_2^n) + (h_1^{n+1} - h_2^{n+1}) \circ d$$

Consider the map $h^n : K^n \oplus M^n \rightarrow I^{n-1}$ which equals $h_1^n - h_2^n$ on the first summand and zero on the second. Then we see that

$$\beta_1^n - \beta_2^n - (d \circ h^n + h^{n+1} \circ d)$$

is a morphism of complexes $L^\bullet \rightarrow I^\bullet$ which is identically zero on the subcomplex K^\bullet . Hence it factors as $L^\bullet \rightarrow M^\bullet \rightarrow I^\bullet$. Thus the result of the lemma follows from Lemma 18.4. \square

Lemma 18.8. *Let \mathcal{A} be an abelian category. Let I^\bullet be bounded below complex consisting of injective objects. Let $L^\bullet \in K(\mathcal{A})$. Then*

$$\text{Mor}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \text{Mor}_{D(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \rightarrow L^\bullet$ is a quasi-isomorphism and $\gamma : K^\bullet \rightarrow I^\bullet$ is a map of complexes. By Lemma 18.6 we can find a morphism $\beta : L^\bullet \rightarrow I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ . This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \rightarrow I^\bullet$ such that there exists a quasi-isomorphism $\alpha : K^\bullet \rightarrow L^\bullet$

with $\beta \circ \alpha$ homotopic to zero. Then Lemma 18.7 shows that β is homotopic to zero also, i.e., $b = 0$. \square

Lemma 18.9. *Let \mathcal{A} be an abelian category. Assume \mathcal{A} has enough injectives. For any short exact sequence $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ of $\text{Comp}^+(\mathcal{A})$ there exists a commutative diagram in $\text{Comp}^+(\mathcal{A})$*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_1^\bullet & \longrightarrow & I_2^\bullet & \longrightarrow & I_3^\bullet \longrightarrow 0 \end{array}$$

where the vertical arrows are injective resolutions and the rows are short exact sequences of complexes. In fact, given any injective resolution $A^\bullet \rightarrow I^\bullet$ we may assume $I_1^\bullet = I^\bullet$.

Proof. Step 1. Choose an injective resolution $A^\bullet \rightarrow I^\bullet$ (see Lemma 18.3) or use the given one. Recall that $\text{Comp}^+(\mathcal{A})$ is an abelian category, see Homology, Lemma 12.9. Hence we may form the pushout along the injective map $A^\bullet \rightarrow I^\bullet$ to get

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I^\bullet & \longrightarrow & E^\bullet & \longrightarrow & C^\bullet \longrightarrow 0 \end{array}$$

Note that the lower short exact sequence is termwise split, see Homology, Lemma 23.2. Hence it suffices to prove the lemma when $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ is termwise split.

Step 2. Choose splittings. In other words, write $B^n = A^n \oplus C^n$. Denote $\delta : C^\bullet \rightarrow A^\bullet[1]$ the morphism as in Homology, Lemma 14.10. Choose injective resolutions $f_1 : A^\bullet \rightarrow I_1^\bullet$ and $f_3 : C^\bullet \rightarrow I_3^\bullet$. (If A^\bullet is a complex of injectives, then use $I_1^\bullet = A^\bullet$.) We may assume f_3 is injective in every degree. By Lemma 18.6 we may find a morphism $\delta' : I_3^\bullet \rightarrow I_1^\bullet[1]$ such that $\delta' \circ f_3 = f_1[1] \circ \delta$ (equality of morphisms of complexes). Set $I_2^n = I_1^n \oplus I_3^n$. Define

$$d_{I_2}^n = \begin{pmatrix} d_{I_1}^n & (\delta')^n \\ 0 & d_{I_3}^n \end{pmatrix}$$

and define the maps $B^n \rightarrow I_2^n$ to be given as the sum of the maps $A^n \rightarrow I_1^n$ and $C^n \rightarrow I_3^n$. Everything is clear. \square

19. Projective resolutions

This section is dual to Section 18. We give definitions and state results, but we do not reprove the lemmas.

Definition 19.1. Let \mathcal{A} be an abelian category. Let $A \in \text{Ob}(\mathcal{A})$. An *projective resolution of A* is a complex P^\bullet together with a map $P^0 \rightarrow A$ such that:

- (1) We have $P^n = 0$ for $n > 0$.
- (2) Each P^n is a projective object of \mathcal{A} .
- (3) The map $P^0 \rightarrow A$ induces an isomorphism $\text{Coker}(d^{-1}) \rightarrow A$.
- (4) We have $H^i(P^\bullet) = 0$ for $i < 0$.

Hence $P^\bullet \rightarrow A[0]$ is a quasi-isomorphism. In other words the complex

$$\dots \rightarrow P^{-1} \rightarrow P^0 \rightarrow A \rightarrow 0 \rightarrow \dots$$

is acyclic. Let K^\bullet be a complex in \mathcal{A} . An *projective resolution* of K^\bullet is a complex P^\bullet together with a map $\alpha : P^\bullet \rightarrow K^\bullet$ of complexes such that

- (1) We have $P^n = 0$ for $n \gg 0$, i.e., P^\bullet is bounded above.
- (2) Each P^n is a projective object of \mathcal{A} .
- (3) The map $\alpha : P^\bullet \rightarrow K^\bullet$ is a quasi-isomorphism.

Lemma 19.2. *Let \mathcal{A} be an abelian category. Let K^\bullet be a complex of \mathcal{A} .*

- (1) *If K^\bullet has a projective resolution then $H^n(K^\bullet) = 0$ for $n \gg 0$.*
- (2) *If $H^n(K^\bullet) = 0$ for $n \gg 0$ then there exists a quasi-isomorphism $L^\bullet \rightarrow K^\bullet$ with L^\bullet bounded above.*

Proof. Dual to Lemma 18.2. □

Lemma 19.3. *Let \mathcal{A} be an abelian category. Assume \mathcal{A} has enough projectives.*

- (1) *Any object of \mathcal{A} has a projective resolution.*
- (2) *If $H^n(K^\bullet) = 0$ for all $n \gg 0$ then K^\bullet has a projective resolution.*
- (3) *If K^\bullet is a complex with $K^n = 0$ for $n > a$, then there exists a projective resolution $\alpha : P^\bullet \rightarrow K^\bullet$ with $P^n = 0$ for $n > a$ such that each $\alpha^n : P^n \rightarrow K^n$ is surjective.*

Proof. Dual to Lemma 18.3. □

Lemma 19.4. *Let \mathcal{A} be an abelian category. Let K^\bullet be an acyclic complex. Let P^\bullet be bounded above and consisting of projective objects. Any morphism $P^\bullet \rightarrow K^\bullet$ is homotopic to zero.*

Proof. Dual to Lemma 18.4. □

Remark 19.5. Let \mathcal{A} be an abelian category. Suppose that $\alpha : K^\bullet \rightarrow L^\bullet$ is a quasi-isomorphism of complexes. Let P^\bullet be a bounded above complex of projectives. Then

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, L^\bullet)$$

is an isomorphism. This is dual to Remark 18.5.

Lemma 19.6. *Let \mathcal{A} be an abelian category. Consider a solid diagram*

$$\begin{array}{ccc} K^\bullet & \xleftarrow{\alpha} & L^\bullet \\ \uparrow & \nearrow \beta & \\ P^\bullet & & \end{array}$$

where P^\bullet is bounded above and consists of projective objects, and α is a quasi-isomorphism.

- (1) *There exists a map of complexes β making the diagram commute up to homotopy.*
- (2) *If α is surjective in every degree then we can find a β which makes the diagram commute.*

Proof. Dual to Lemma 18.6. □

Lemma 19.7. *Let \mathcal{A} be an abelian category. Consider a solid diagram*

$$\begin{array}{ccc} K^\bullet & \xleftarrow{\alpha} & L^\bullet \\ \uparrow & \nearrow \beta_i & \\ P^\bullet & & \end{array}$$

where P^\bullet is bounded above and consists of projective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. Dual to Lemma 18.7. \square

Lemma 19.8. *Let \mathcal{A} be an abelian category. Let P^\bullet be bounded above complex consisting of projective objects. Let $L^\bullet \in K(\mathcal{A})$. Then*

$$\text{Mor}_{K(\mathcal{A})}(P^\bullet, L^\bullet) = \text{Mor}_{D(\mathcal{A})}(P^\bullet, L^\bullet).$$

Proof. Dual to Lemma 18.8. \square

Lemma 19.9. *Let \mathcal{A} be an abelian category. Assume \mathcal{A} has enough projectives. For any short exact sequence $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ of $\text{Comp}^+(\mathcal{A})$ there exists a commutative diagram in $\text{Comp}^+(\mathcal{A})$*

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_1^\bullet & \longrightarrow & P_2^\bullet & \longrightarrow & P_3^\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet \longrightarrow 0 \end{array}$$

where the vertical arrows are projective resolutions and the rows are short exact sequences of complexes. In fact, given any projective resolution $P^\bullet \rightarrow C^\bullet$ we may assume $P_3^\bullet = P^\bullet$.

Proof. Dual to Lemma 18.9. \square

Lemma 19.10. *Let \mathcal{A} be an abelian category. Let P^\bullet, K^\bullet be complexes. Let $n \in \mathbf{Z}$. Assume that*

- (1) P^\bullet is a bounded complex consisting of projective objects,
- (2) $P^i = 0$ for $i < n$, and
- (3) $H^i(K^\bullet) = 0$ for $i \geq n$.

Then $\text{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet) = \text{Hom}_{D(\mathcal{A})}(P^\bullet, K^\bullet) = 0$.

Proof. The first equality follows from Lemma 19.8. Note that there is a distinguished triangle

$$(\tau_{\leq n-1} K^\bullet, K^\bullet, \tau_{\geq n} K^\bullet, f, g, h)$$

by Remark 12.4. Hence, by Lemma 4.2 it suffices to prove $\text{Hom}_{K(\mathcal{A})}(P^\bullet, \tau_{\leq n-1} K^\bullet) = 0$ and $\text{Hom}_{K(\mathcal{A})}(P^\bullet, \tau_{\geq n} K^\bullet) = 0$. The first vanishing is trivial and the second is Lemma 19.4. \square

Lemma 19.11. *Let \mathcal{A} be an abelian category. Let $\beta : P^\bullet \rightarrow L^\bullet$ and $\alpha : E^\bullet \rightarrow L^\bullet$ be maps of complexes. Let $n \in \mathbf{Z}$. Assume*

- (1) P^\bullet is a bounded complex of projectives and $P^i = 0$ for $i < n$,
- (2) $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$.

Then there exists a map of complexes $\gamma : P^\bullet \rightarrow E^\bullet$ such that $\alpha \circ \gamma$ and β are homotopic.

Proof. Consider the cone $C^\bullet = C(\alpha)^\bullet$ with map $i : L^\bullet \rightarrow C^\bullet$. Note that $i \circ \beta$ is zero by Lemma 19.10. Hence we can lift β to E^\bullet by Lemma 4.2. \square

20. Right derived functors and injective resolutions

At this point we can use the material above to define the right derived functors of an additive functor between an abelian category having enough injectives and a general abelian category.

Lemma 20.1. *Let \mathcal{A} be an abelian category. Let $I \in \text{Ob}(\mathcal{A})$ be an injective object. Let I^\bullet be a bounded below complex of injectives in \mathcal{A} .*

- (1) *I^\bullet computes RF relative to $\text{Qis}^+(\mathcal{A})$ for any exact functor $F : K^+(\mathcal{A}) \rightarrow \mathcal{D}$ into any triangulated category \mathcal{D} .*
- (2) *I is right acyclic for any additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$ into any abelian category \mathcal{B} .*

Proof. Part (2) is a direct consequences of part (1) and Definition 16.3. To prove (1) let $\alpha : I^\bullet \rightarrow K^\bullet$ be a quasi-isomorphism into a complex. By Lemma 18.7 we see that α has a left inverse. Hence the category $I^\bullet/\text{Qis}^+(\mathcal{A})$ is essentially constant with value $\text{id} : I^\bullet \rightarrow I^\bullet$. Thus also the ind-object

$$I^\bullet/\text{Qis}^+(\mathcal{A}) \longrightarrow \mathcal{D}, \quad (I^\bullet \rightarrow K^\bullet) \longmapsto F(K^\bullet)$$

is essentially constant with value $F(I^\bullet)$. This proves (1), see Definitions 15.2 and 15.10. \square

Lemma 20.2. *Let \mathcal{A} be an abelian category with enough injectives.*

- (1) *For any exact functor $F : K^+(\mathcal{A}) \rightarrow \mathcal{D}$ into a triangulated category \mathcal{D} the right derived functor*

$$RF : D^+(\mathcal{A}) \longrightarrow \mathcal{D}$$

is everywhere defined.

- (2) *For any additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$ into an abelian category \mathcal{B} the right derived functor*

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$

is everywhere defined.

Proof. Combine Lemma 20.1 and Proposition 16.8 for the second assertion. To see the first assertion combine Lemma 18.3, Lemma 20.1, Lemma 15.14, and Equation (15.9.1). \square

Lemma 20.3. *Let \mathcal{A} be an abelian category with enough injectives. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor.*

- (1) *The functor RF is an exact functor $D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$.*
- (2) *The functor RF induces an exact functor $K^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$.*
- (3) *The functor RF induces a δ -functor $\text{Comp}^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$.*
- (4) *The functor RF induces a δ -functor $\mathcal{A} \rightarrow D^+(\mathcal{B})$.*

Proof. This lemma simply reviews some of the results obtained so far. Note that by Lemma 20.2 RF is everywhere defined. Here are some references:

- (1) The derived functor is exact: This boils down to Lemma 15.6.
- (2) This is true because $K^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$ is exact and compositions of exact functors are exact.

- (3) This is true because $\text{Comp}^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$ is a δ -functor, see Lemma 12.1.
- (4) This is true because $\mathcal{A} \rightarrow \text{Comp}^+(\mathcal{A})$ is exact and precomposing a δ -functor by an exact functor gives a δ -functor.

□

Lemma 20.4. *Let \mathcal{A} be an abelian category with enough injectives. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor.*

- (1) *For any short exact sequence $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ of complexes in $\text{Comp}^+(\mathcal{A})$ there is an associated long exact sequence*
 $\dots \rightarrow H^i(RF(A^\bullet)) \rightarrow H^i(RF(B^\bullet)) \rightarrow H^i(RF(C^\bullet)) \rightarrow H^{i+1}(RF(A^\bullet)) \rightarrow \dots$
- (2) *The functors $R^iF : \mathcal{A} \rightarrow \mathcal{B}$ are zero for $i < 0$. Also $R^0F = F : \mathcal{A} \rightarrow \mathcal{B}$.*
- (3) *We have $R^iF(I) = 0$ for $i > 0$ and I injective.*
- (4) *The sequence (R^iF, δ) forms a universal δ -functor (see Homology, Definition 11.3) from \mathcal{A} to \mathcal{B} .*

Proof. This lemma simply reviews some of the results obtained so far. Note that by Lemma 20.2 RF is everywhere defined. Here are some references:

- (1) This follows from Lemma 20.3 part (3) combined with the long exact cohomology sequence (11.1.1) for $D^+(\mathcal{B})$.
- (2) This is Lemma 17.3.
- (3) This is the fact that injective objects are acyclic.
- (4) This is Lemma 17.6.

□

21. Cartan-Eilenberg resolutions

This section can be expanded. The material can be generalized and applied in more cases. Resolutions need not use injectives and the method also works in the unbounded case in some situations.

Definition 21.1. Let \mathcal{A} be an abelian category. Let K^\bullet be a bounded below complex. A *Cartan-Eilenberg resolution* of K^\bullet is given by a double complex $I^{\bullet, \bullet}$ and a morphism of complexes $\epsilon : K^\bullet \rightarrow I^{\bullet, 0}$ with the following properties:

- (1) There exists a $i \ll 0$ such that $I^{p, q} = 0$ for all $p < i$ and all q .
- (2) We have $I^{p, q} = 0$ if $q < 0$.
- (3) The complex $I^{p, \bullet}$ is an injective resolution of K^p .
- (4) The complex $\text{Ker}(d_1^{p, \bullet})$ is an injective resolution of $\text{Ker}(d_K^p)$.
- (5) The complex $\text{Im}(d_1^{p, \bullet})$ is an injective resolution of $\text{Im}(d_K^p)$.
- (6) The complex $H_I^p(I^{\bullet, \bullet})$ is an injective resolution of $H^p(K^\bullet)$.

Lemma 21.2. *Let \mathcal{A} be an abelian category with enough injectives. Let K^\bullet be a bounded below complex. There exists a Cartan-Eilenberg resolution of K^\bullet .*

Proof. Suppose that $K^p = 0$ for $p < n$. Decompose K^\bullet into short exact sequences as follows: Set $Z^p = \text{Ker}(d^p)$, $B^p = \text{Im}(d^{p-1})$, $H^p = Z^p/B^p$, and consider

$$\begin{aligned}
 0 &\rightarrow Z^n \rightarrow K^n \rightarrow B^{n+1} \rightarrow 0 \\
 0 &\rightarrow B^{n+1} \rightarrow Z^{n+1} \rightarrow H^{n+1} \rightarrow 0 \\
 0 &\rightarrow Z^{n+1} \rightarrow K^{n+1} \rightarrow B^{n+2} \rightarrow 0 \\
 0 &\rightarrow B^{n+2} \rightarrow Z^{n+2} \rightarrow H^{n+2} \rightarrow 0 \\
 &\dots
 \end{aligned}$$

Set $I^{p,q} = 0$ for $p < n$. Inductively we choose injective resolutions as follows:

- (1) Choose an injective resolution $Z^n \rightarrow J_Z^{n,\bullet}$.
- (2) Using Lemma 18.9 choose injective resolutions $K^n \rightarrow I^{n,\bullet}$, $B^{n+1} \rightarrow J_B^{n+1,\bullet}$, and an exact sequence of complexes $0 \rightarrow J_Z^{n,\bullet} \rightarrow I^{n,\bullet} \rightarrow J_B^{n+1,\bullet} \rightarrow 0$ compatible with the short exact sequence $0 \rightarrow Z^n \rightarrow K^n \rightarrow B^{n+1} \rightarrow 0$.
- (3) Using Lemma 18.9 choose injective resolutions $Z^{n+1} \rightarrow J_Z^{n+1,\bullet}$, $H^{n+1} \rightarrow J_H^{n+1,\bullet}$, and an exact sequence of complexes $0 \rightarrow J_B^{n+1,\bullet} \rightarrow J_Z^{n+1,\bullet} \rightarrow J_H^{n+1,\bullet} \rightarrow 0$ compatible with the short exact sequence $0 \rightarrow B^{n+1} \rightarrow Z^{n+1} \rightarrow H^{n+1} \rightarrow 0$.
- (4) Etc.

Taking as maps $d_1^\bullet : I^{p,\bullet} \rightarrow I^{p+1,\bullet}$ the compositions $I^{p,\bullet} \rightarrow J_B^{p+1,\bullet} \rightarrow J_Z^{p+1,\bullet} \rightarrow I^{p+1,\bullet}$ everything is clear. \square

Lemma 21.3. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor of abelian categories. Let K^\bullet be a bounded below complex of \mathcal{A} . Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution for K^\bullet . The spectral sequences $({}^iE_r, {}^i d_r)_{r \geq 0}$ and $({}^{\prime\prime}E_r, {}^{\prime\prime} d_r)_{r \geq 0}$ associated to the double complex $F(I^{\bullet,\bullet})$ satisfy the relations⁴*

$${}^iE_2^{p,q} = H^p(R^q F(K^\bullet)) \quad \text{and} \quad {}^{\prime\prime}E_2^{p,q} = R^p F(H^q(K^\bullet))$$

Moreover, these spectral sequences converge to $H^{p+q}(RF(K^\bullet))$ and the associated induced filtrations on $H^{p+q}(RF(K^\bullet))$ are finite.

Proof. We will use the following remarks without further mention:

- (1) As $I^{p,\bullet}$ is an injective resolution of K^p we see that RF is defined at $K^p[0]$ with value $F(I^{p,\bullet})$.
- (2) As $H_I^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$ the derived functor RF is defined at $H^p(K^\bullet)[0]$ with value $F(H_I^p(I^{\bullet,\bullet}))$.
- (3) By Homology, Lemma 22.7 the total complex sI^\bullet is an injective resolution of K^\bullet . Hence RF is defined at K^\bullet with value $F(sI^\bullet)$.

Consider the spectral sequences associated to the double complex $K^{\bullet,\bullet} = F(I^{\bullet,\bullet})$, see Homology, Lemma 22.4. These both converge, see Homology, Lemma 22.6, to the cohomology groups of the associated total complex $s(F(I^{\bullet,\bullet})) = F(sI^\bullet)$ which computes $H^n(RF(K^\bullet))$.

Computation of the first spectral sequence. We have ${}^iE_1^{p,q} = H^q(K^{p,\bullet})$ in other words

$${}^iE_1^{p,q} = H^q(F(I^{p,\bullet})) = R^q F(K^p)$$

and the maps ${}^iE_1^{p,q} \rightarrow {}^iE_1^{p+1,q}$ are the maps $R^q F(K^p) \rightarrow R^q F(K^{p+1})$ as desired.

Computation of the second spectral sequence. We have ${}^{\prime\prime}E_1^{p,q} = H^q(K^{\bullet,p}) = H^q(F(I^{\bullet,p}))$. Note that the complex $I^{\bullet,p}$ is bounded below, consists of injectives, and moreover each kernel, image, and cohomology group of the differentials is an injective object of \mathcal{A} . Hence we can split the differentials, i.e., each differential is a split surjection onto a direct summand. It follows that the same is true after applying F . Hence ${}^{\prime\prime}E_1^{p,q} = F(H^q(I^{\bullet,p})) = F(H_I^q(I^{\bullet,p}))$. The differentials on this are $(-1)^q$ times F applied to the differential of the complex $H_I^p(I^{\bullet,\bullet})$ which is an injective resolution of $H^p(K^\bullet)$. Hence the description of the E_2 terms. \square

⁴The notation $H^p(R^q F(K^\bullet))$ really means the p th cohomology group of the complex with terms $R^q F(K^n)$. Not the p th cohomology of the q th derived functor of F applied to K^\bullet .

Remark 21.4. The spectral sequences of Lemma 21.3 are functorial in the complex K^\bullet . This follows from functoriality properties of Cartan-Eilenberg resolutions. On the other hand, they are both examples of a more general spectral sequence which may be associated to a filtered complex of \mathcal{A} . The functoriality will follow from its construction. We will return to this in the section on the filtered derived category, see Remark 26.15.

22. Composition of right derived functors

Sometimes we can compute the right derived functor of a composition. Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ be left exact functors. Assume that the right derived functors $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$, $RG : D^+(\mathcal{B}) \rightarrow D^+(\mathcal{C})$, and $R(G \circ F) : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{C})$ are everywhere defined. Then there exists a canonical transformation

$$t : R(G \circ F) \longrightarrow RG \circ RF$$

of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$, see Lemma 15.16. This transformation need not always be an isomorphism.

Lemma 22.1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ be left exact functors. Assume \mathcal{A}, \mathcal{B} have enough injectives. If $F(I)$ is right acyclic for G for each injective object I of \mathcal{A} , then we have an isomorphism of functors*

$$t : R(G \circ F) \longrightarrow RG \circ RF.$$

of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$.

Proof. Let A^\bullet be a bounded below complex of \mathcal{A} . Choose an injective resolution $A^\bullet \rightarrow I^\bullet$. The map t is given (see proof of Lemma 15.16) by the maps

$$R(G \circ F)(A^\bullet) = (G \circ F)(I^\bullet) = G(F(I^\bullet)) \rightarrow RG(F(I^\bullet)) = RG(RF(A^\bullet))$$

where the arrow is an isomorphism by Lemma 17.7. \square

Lemma 22.2 (Grothendieck spectral sequence). *With assumptions as in Lemma 22.1. Let A be an object of \mathcal{A} . There exists a spectral sequence $(E_r^{p,q}, d_r^{p,q})_{r \geq 0}$ associated to a filtered complex with*

$$E_2^{p,q} = R^p G(R^q F(A))$$

converging to $R^{p+q}(G \circ F)(A)$. Moreover, the induced filtration on each $R^n(G \circ F)(A)$ is finite.

Proof. Choose an injective resolution $A \rightarrow I^\bullet$. Choose a Cartan-Eilenberg resolution $F(I^\bullet) \rightarrow I^{\bullet,\bullet}$ using Lemma 21.2. Apply Lemma 21.3 (use the second spectral sequence). Details omitted. \square

23. Resolution functors

Let \mathcal{A} be an abelian category with enough injectives. Denote \mathcal{I} the full additive subcategory of \mathcal{A} whose objects are the injective objects of \mathcal{A} . It turns out that $K^+(\mathcal{I})$ and $D^+(\mathcal{A})$ are equivalent in this case (see Proposition 23.1). For many purposes it therefore makes sense to think of $D^+(\mathcal{A})$ as the (easier to grok) category $K^+(\mathcal{I})$ in this case.

Proposition 23.1. *Let \mathcal{A} be an abelian category. Assume \mathcal{A} has enough injectives. Denote $\mathcal{I} \subset \mathcal{A}$ the strictly full additive subcategory whose objects are the injective objects of \mathcal{A} . The functor*

$$K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$$

is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories.

Proof. It is clear that the functor is exact. It is essentially surjective by Lemma 18.3. Fully faithfulness is a consequence of Lemma 18.8. \square

Proposition 23.1 implies that we can find resolution functors. It turns out that we can prove resolution functors exist even in some cases where the abelian category \mathcal{A} is a “big” category, i.e., has a class of objects.

Definition 23.2. Let \mathcal{A} be an abelian category with enough injectives. A *resolution functor*⁵ for \mathcal{A} is given by the following data:

- (1) for all $K^\bullet \in \text{Ob}(K^+(\mathcal{A}))$ a bounded below complex of injectives $j(K^\bullet)$, and
- (2) for all $K^\bullet \in \text{Ob}(K^+(\mathcal{A}))$ a quasi-isomorphism $i_{K^\bullet} : K^\bullet \rightarrow j(K^\bullet)$.

Lemma 23.3. *Let \mathcal{A} be an abelian category with enough injectives. Given a resolution functor (j, i) there is a unique way to turn j into a functor and i into a 2-isomorphism producing a 2-commutative diagram*

$$\begin{array}{ccc} K^+(\mathcal{A}) & \xrightarrow{j} & K^+(\mathcal{I}) \\ & \searrow & \swarrow \\ & D^+(\mathcal{A}) & \end{array}$$

where \mathcal{I} is the full additive subcategory of \mathcal{A} consisting of injective objects.

Proof. For every morphism $\alpha : K^\bullet \rightarrow L^\bullet$ of $K^+(\mathcal{A})$ there is a unique morphism $j(\alpha) : j(K^\bullet) \rightarrow j(L^\bullet)$ in $K^+(\mathcal{I})$ such that

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ i_{K^\bullet} \downarrow & & \downarrow i_{L^\bullet} \\ j(K^\bullet) & \xrightarrow{j(\alpha)} & j(L^\bullet) \end{array}$$

is commutative in $K^+(\mathcal{A})$. To see this either use Lemmas 18.6 and 18.7 or the equivalent Lemma 18.8. The uniqueness implies that j is a functor, and the commutativity of the diagram implies that i gives a 2-morphism which witnesses the 2-commutativity of the diagram of categories in the statement of the lemma. \square

Lemma 23.4. *Let \mathcal{A} be an abelian category. Assume \mathcal{A} has enough injectives. Then a resolution functor j exists and is unique up to unique isomorphism of functors.*

Proof. Consider the set of all objects K^\bullet of $K^+(\mathcal{A})$. (Recall that by our conventions any category has a set of objects unless mentioned otherwise.) By Lemma 18.3 every object has an injective resolution. By the axiom of choice we can choose for each K^\bullet an injective resolution $i_{K^\bullet} : K^\bullet \rightarrow j(K^\bullet)$. \square

⁵This is likely nonstandard terminology.

Lemma 23.5. *Let \mathcal{A} be an abelian category with enough injectives. Any resolution functor $j : K^+(\mathcal{A}) \rightarrow K^+(\mathcal{I})$ is exact.*

Proof. Denote $i_{K^\bullet} : K^\bullet \rightarrow j(K^\bullet)$ the canonical maps of Definition 23.2. First we discuss the existence of the functorial isomorphism $j(K^\bullet[1]) \rightarrow j(K^\bullet)[1]$. Consider the diagram

$$\begin{array}{ccc} K^\bullet[1] & \xlongequal{\quad} & K^\bullet[1] \\ \downarrow i_{K^\bullet[1]} & & \downarrow i_{K^\bullet[1]} \\ j(K^\bullet[1]) & \xrightarrow{\quad \xi_{K^\bullet} \quad} & j(K^\bullet)[1] \end{array}$$

By Lemmas 18.6 and 18.7 there exists a unique dotted arrow ξ_{K^\bullet} in $K^+(\mathcal{I})$ making the diagram commute in $K^+(\mathcal{A})$. We omit the verification that this gives a functorial isomorphism. (Hint: use Lemma 18.7 again.)

Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle of $K^+(\mathcal{A})$. We have to show that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle of $K^+(\mathcal{I})$. Note that we have a commutative diagram

$$\begin{array}{ccccccc} K^\bullet & \xrightarrow{f} & L^\bullet & \xrightarrow{g} & M^\bullet & \xrightarrow{h} & K^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ j(K^\bullet) & \xrightarrow{j(f)} & j(L^\bullet) & \xrightarrow{j(g)} & j(M^\bullet) & \xrightarrow{\xi_{K^\bullet} \circ j(h)} & j(K^\bullet)[1] \end{array}$$

in $K^+(\mathcal{A})$ whose vertical arrows are the quasi-isomorphisms i_K, i_L, i_M . Hence we see that the image of $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ in $D^+(\mathcal{A})$ is isomorphic to a distinguished triangle and hence a distinguished triangle by TR1. Thus we see from Lemma 4.16 that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle in $K^+(\mathcal{I})$. \square

Lemma 23.6. *Let \mathcal{A} be an abelian category which has enough injectives. Let j be a resolution functor. Write $Q : K^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$ for the natural functor. Then $j = j' \circ Q$ for a unique functor $j' : D^+(\mathcal{A}) \rightarrow K^+(\mathcal{I})$ which is quasi-inverse to the canonical functor $K^+(\mathcal{I}) \rightarrow D^+(\mathcal{A})$.*

Proof. By Lemma 11.6 Q is a localization functor. To prove the existence of j' it suffices to show that any element of $\text{Qis}^+(\mathcal{A})$ is mapped to an isomorphism under the functor j , see Lemma 5.6. This is true by the remarks following Definition 23.2. \square

Remark 23.7. Suppose that \mathcal{A} is a “big” abelian category with enough injectives such as the category of abelian groups. In this case we have to be slightly more careful in constructing our resolution functor since we cannot use the axiom of choice with a quantifier ranging over a class. But note that the proof of the lemma does show that any two localization functors are canonically isomorphic. Namely, given quasi-isomorphisms $i : K^\bullet \rightarrow I^\bullet$ and $i' : K^\bullet \rightarrow J^\bullet$ of a bounded below complex K^\bullet into bounded below complexes of injectives there exists a unique(!) morphism $a : I^\bullet \rightarrow J^\bullet$ in $K^+(\mathcal{I})$ such that $i' = i \circ a$ as morphisms in $K^+(\mathcal{I})$. Hence the only issue is existence, and we will see how to deal with this in the next section.

24. Functorial injective embeddings and resolution functors

In this section we redo the construction of a resolution functor $K^+(\mathcal{A}) \rightarrow K^+(\mathcal{I})$ in case the category \mathcal{A} has functorial injective embeddings. There are two reasons for this: (1) the proof is easier and (2) the construction also works if \mathcal{A} is a “big” abelian category. See Remark 24.3 below.

Let \mathcal{A} be an abelian category. As before denote \mathcal{I} the additive full subcategory of \mathcal{A} consisting of injective objects. Consider the category $\text{InjRes}(\mathcal{A})$ of arrows $\alpha : K^\bullet \rightarrow I^\bullet$ where K^\bullet is a bounded below complex of \mathcal{A} , I^\bullet is a bounded below complex of injectives of \mathcal{A} and α is a quasi-isomorphism. In other words, α is an injective resolution and K^\bullet is bounded below. There is an obvious functor

$$s : \text{InjRes}(\mathcal{A}) \longrightarrow \text{Comp}^+(\mathcal{A})$$

defined by $(\alpha : K^\bullet \rightarrow I^\bullet) \mapsto K^\bullet$. There is also a functor

$$t : \text{InjRes}(\mathcal{A}) \longrightarrow K^+(\mathcal{I})$$

defined by $(\alpha : K^\bullet \rightarrow I^\bullet) \mapsto I^\bullet$.

Lemma 24.1. *Let \mathcal{A} be an abelian category. Assume \mathcal{A} has functorial injective embeddings, see Homology, Definition 23.5.*

- (1) *There exists a functor $\text{inj} : \text{Comp}^+(\mathcal{A}) \rightarrow \text{InjRes}(\mathcal{A})$ such that $s \circ \text{inj} = \text{id}$.*
- (2) *For any functor $\text{inj} : \text{Comp}^+(\mathcal{A}) \rightarrow \text{InjRes}(\mathcal{A})$ such that $s \circ \text{inj} = \text{id}$ we obtain a resolution functor, see Definition 23.2.*

Proof. Let $A \mapsto (A \rightarrow J(A))$ be a functorial injective embedding, see Homology, Definition 23.5. We first note that we may assume $J(0) = 0$. Namely, if not then for any object A we have $0 \rightarrow A \rightarrow 0$ which gives a direct sum decomposition $J(A) = J(0) \oplus \text{Ker}(J(A) \rightarrow J(0))$. Note that the functorial morphism $A \rightarrow J(A)$ has to map into the second summand. Hence we can replace our functor by $J'(A) = \text{Ker}(J(A) \rightarrow J(0))$ if needed.

Let K^\bullet be a bounded below complex of \mathcal{A} . Say $K^p = 0$ if $p < B$. We are going to construct a double complex $I^{\bullet, \bullet}$ of injectives, together with a map $\alpha : K^\bullet \rightarrow I^{\bullet, 0}$ such that α induces a quasi-isomorphism of K^\bullet with the associated total complex of $I^{\bullet, \bullet}$. First we set $I^{p, q} = 0$ whenever $q < 0$. Next, we set $I^{p, 0} = J(K^p)$ and $\alpha^p : K^p \rightarrow I^{p, 0}$ the functorial embedding. Since J is a functor we see that $I^{\bullet, 0}$ is a complex and that α is a morphism of complexes. Each α^p is injective. And $I^{p, 0} = 0$ for $p < B$ because $J(0) = 0$. Next, we set $I^{p, 1} = J(\text{Coker}(K^p \rightarrow I^{p, 0}))$. Again by functoriality we see that $I^{\bullet, 1}$ is a complex. And again we get that $I^{p, 1} = 0$ for $p < B$. It is also clear that K^p maps isomorphically onto $\text{Ker}(I^{p, 0} \rightarrow I^{p, 1})$. As our third step we take $I^{p, 2} = J(\text{Coker}(I^{p, 0} \rightarrow I^{p, 1}))$. And so on and so forth.

At this point we can apply Homology, Lemma 22.7 to get that the map

$$\alpha : K^\bullet \rightarrow sI^\bullet$$

is a quasi-isomorphism. To prove we get a functor inj it rests to show that the construction above is functorial. This verification is omitted.

Suppose we have a functor inj such that $s \circ \text{inj} = \text{id}$. For every object K^\bullet of $\text{Comp}^+(\mathcal{A})$ we can write

$$\text{inj}(K^\bullet) = (i_{K^\bullet} : K^\bullet \rightarrow j(K^\bullet))$$

This provides us with a resolution functor as in Definition 23.2. \square

Remark 24.2. Suppose inj is a functor such that $s \circ inj = \text{id}$ as in part (2) of Lemma 24.1. Write $inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \rightarrow j(K^\bullet))$ as in the proof of that lemma. Suppose $\alpha : K^\bullet \rightarrow L^\bullet$ is a map of bounded below complexes. Consider the map $inj(\alpha)$ in the category $\text{InjRes}(\mathcal{A})$. It induces a commutative diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ i_K \downarrow & & \downarrow i_L \\ j(K)^\bullet & \xrightarrow{inj(\alpha)} & j(L)^\bullet \end{array}$$

of morphisms of complexes. Hence, looking at the proof of Lemma 23.3 we see that the functor $j : K^+(\mathcal{A}) \rightarrow K^+(\mathcal{I})$ is given by the rule

$$j(\alpha \text{ up to homotopy}) = inj(\alpha) \text{ up to homotopy} \in \text{Hom}_{K^+(\mathcal{I})}(j(K^\bullet), j(L^\bullet))$$

Hence we see that j matches $t \circ inj$ in this case, i.e., the diagram

$$\begin{array}{ccc} \text{Comp}^+(\mathcal{A}) & \xrightarrow[t \circ inj]{} & K^+(\mathcal{I}) \\ & \searrow & \nearrow j \\ & K^+(\mathcal{A}) & \end{array}$$

is commutative.

Remark 24.3. Let $\text{Mod}(\mathcal{O}_X)$ be the category of \mathcal{O}_X -modules on a ringed space (X, \mathcal{O}_X) (or more generally on a ringed site). We will see later that $\text{Mod}(\mathcal{O}_X)$ has enough injectives and in fact functorial injective embeddings, see Injectives, Theorem 8.4. Note that the proof of Lemma 23.4 does not apply to $\text{Mod}(\mathcal{O}_X)$. But the proof of Lemma 24.1 does apply to $\text{Mod}(\mathcal{O}_X)$. Thus we obtain

$$j : K^+(\text{Mod}(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

which is a resolution functor where \mathcal{I} is the additive category of injective \mathcal{O}_X -modules. This argument also works in the following cases:

- (1) The category Mod_R of R -modules over a ring R .
- (2) The category $\text{PMod}(\mathcal{O})$ of presheaves of \mathcal{O} -modules on a site endowed with a presheaf of rings.
- (3) The category $\text{Mod}(\mathcal{O})$ of sheaves of \mathcal{O} -modules on a ringed site.
- (4) Add more here as needed.

25. Right derived functors via resolution functors

The content of the following lemma is that we can simply define $RF(K^\bullet) = F(j(K^\bullet))$ if we are given a resolution functor j .

Lemma 25.1. *Let \mathcal{A} be an abelian category with enough injectives. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor into an abelian category. Let (i, j) be a resolution functor, see Definition 23.2. The right derived functor RF of F fits into the following 2-commutative diagram*

$$\begin{array}{ccc} D^+(\mathcal{A}) & \xrightarrow{j'} & K^+(\mathcal{I}) \\ & \searrow RF \quad \swarrow F & \\ & D^+(\mathcal{B}) & \end{array}$$

where j' is the functor from Lemma 23.6.

Proof. By Lemma 20.1 we have $RF(K^\bullet) = F(j(K^\bullet))$. \square

Remark 25.2. In the situation of Lemma 25.1 we see that we have actually lifted the right derived functor to an exact functor $F \circ j' : D^+(\mathcal{A}) \rightarrow K^+(\mathcal{B})$. It is occasionally useful to use such a factorization.

26. Filtered derived category and injective resolutions

Let \mathcal{A} be an abelian category. In this section we will show that if \mathcal{A} has enough injectives, then so does the category $\text{Fil}^f(\mathcal{A})$ in some sense. One can use this observation to compute in the filtered derived category of \mathcal{A} .

The category $\text{Fil}^f(\mathcal{A})$ is an example of an exact category, see Injectives, Remark 9.6. A special role is played by the strict morphisms, see Homology, Definition 16.3, i.e., the morphisms f such that $\text{Coim}(f) = \text{Im}(f)$. We will say that a complex $A \rightarrow B \rightarrow C$ in $\text{Fil}^f(\mathcal{A})$ is *exact* if the sequence $\text{gr}(A) \rightarrow \text{gr}(B) \rightarrow \text{gr}(C)$ is exact in \mathcal{A} . This implies that $A \rightarrow B$ and $B \rightarrow C$ are strict morphisms, see Homology, Lemma 16.15.

Definition 26.1. Let \mathcal{A} be an abelian category. We say an object I of $\text{Fil}^f(\mathcal{A})$ is *filtered injective* if each $\text{gr}^p(I)$ is an injective object of \mathcal{A} .

Lemma 26.2. Let \mathcal{A} be an abelian category. An object I of $\text{Fil}^f(\mathcal{A})$ is filtered injective if and only if there exist $a \leq b$, injective objects I_n , $a \leq n \leq b$ of \mathcal{A} and an isomorphism $I \cong \bigoplus_{a \leq n \leq b} I_n$ such that $F^p I = \bigoplus_{n \geq p} I_n$.

Proof. Follows from the fact that any injection $J \rightarrow M$ of \mathcal{A} is split if J is an injective object. Details omitted. \square

Lemma 26.3. Let \mathcal{A} be an abelian category. Any strict monomorphism $u : I \rightarrow A$ of $\text{Fil}^f(\mathcal{A})$ where I is a filtered injective object is a split injection.

Proof. Let p be the largest integer such that $F^p I \neq 0$. In particular $\text{gr}^p(I) = F^p I$. Let I' be the object of $\text{Fil}^f(\mathcal{A})$ whose underlying object of \mathcal{A} is $F^p I$ and with filtration given by $F^n I' = 0$ for $n > p$ and $F^n I' = I' = F^p I$ for $n \leq p$. Note that $I' \rightarrow I$ is a strict monomorphism too. The fact that u is a strict monomorphism implies that $F^p I \rightarrow A/F^{p+1}(A)$ is injective, see Homology, Lemma 16.13. Choose a splitting $s : A/F^{p+1}(A) \rightarrow F^p I$ in \mathcal{A} . The induced morphism $s' : A \rightarrow I'$ is a strict morphism of filtered objects splitting the composition $I' \rightarrow I \rightarrow A$. Hence we can write $A = I' \oplus \text{Ker}(s')$ and $I = I' \oplus \text{Ker}(s'|_I)$. Note that $\text{Ker}(s'|_I) \rightarrow \text{Ker}(s')$ is a strict monomorphism and that $\text{Ker}(s'|_I)$ is a filtered injective object. By induction on the length of the filtration on I the map $\text{Ker}(s'|_I) \rightarrow \text{Ker}(s')$ is a split injection. Thus we win. \square

Lemma 26.4. Let \mathcal{A} be an abelian category. Let $u : A \rightarrow B$ be a strict monomorphism of $\text{Fil}^f(\mathcal{A})$ and $f : A \rightarrow I$ a morphism from A into a filtered injective object in $\text{Fil}^f(\mathcal{A})$. Then there exists a morphism $g : B \rightarrow I$ such that $f = g \circ u$.

Proof. The pushout $f' : I \rightarrow I \amalg_A B$ of f by u is a strict monomorphism, see Homology, Lemma 16.10. Hence the result follows formally from Lemma 26.3. \square

Lemma 26.5. *Let \mathcal{A} be an abelian category with enough injectives. For any object A of $\text{Fil}^f(\mathcal{A})$ there exists a strict monomorphism $A \rightarrow I$ where I is a filtered injective object.*

Proof. Pick $a \leq b$ such that $\text{gr}^p(A) = 0$ unless $p \in \{a, a+1, \dots, b\}$. For each $n \in \{a, a+1, \dots, b\}$ choose an injection $u_n : A/F^n A \rightarrow I_n$ with I_n and injective object. Set $I = \bigoplus_{a \leq n \leq b} I_p$ with filtration $F^p I = \bigoplus_{n \geq p} I_n$ and set $u : A \rightarrow I$ equal to the direct sum of the maps u_n . \square

Lemma 26.6. *Let \mathcal{A} be an abelian category with enough injectives. For any object A of $\text{Fil}^f(\mathcal{A})$ there exists a filtered quasi-isomorphism $A[0] \rightarrow I^\bullet$ where I^\bullet is a complex of filtered injective objects with $I^n = 0$ for $n < 0$.*

Proof. First choose a strict monomorphism $u_0 : A \rightarrow I^0$ of A into a filtered injective object, see Lemma 26.5. Next, choose a strict monomorphism $u_1 : \text{Coker}(u_0) \rightarrow I^1$ into a filtered injective object of \mathcal{A} . Denote d^0 the induced map $I^0 \rightarrow I^1$. Next, choose a strict monomorphism $u_2 : \text{Coker}(u_1) \rightarrow I^2$ into a filtered injective object of \mathcal{A} . Denote d^1 the induced map $I^1 \rightarrow I^2$. And so on. This works because each of the sequences

$$0 \rightarrow \text{Coker}(u_n) \rightarrow I^{n+1} \rightarrow \text{Coker}(u_{n+1}) \rightarrow 0$$

is short exact, i.e., induces a short exact sequence on applying gr . To see this use Homology, Lemma 16.13. \square

Lemma 26.7. *Let \mathcal{A} be an abelian category with enough injectives. Let $f : A \rightarrow B$ be a morphism of $\text{Fil}^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \rightarrow I^\bullet$ and $B[0] \rightarrow J^\bullet$ where I^\bullet, J^\bullet are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram*

$$\begin{array}{ccc} A[0] & \longrightarrow & B[0] \\ \downarrow & & \downarrow \\ I^\bullet & \longrightarrow & J^\bullet \end{array}$$

Proof. As $A[0] \rightarrow I^\bullet$ and $C[0] \rightarrow J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \rightarrow I^0$, $b : B \rightarrow J^0$ and all the morphisms d_I^n , d_J^n are strict, see Homology, Lemma 14.4. We will inductively construct the maps f^n in the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{a} & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 \longrightarrow \dots \\ f \downarrow & & \downarrow f^0 & & \downarrow f^1 & & \downarrow f^2 \\ B & \xrightarrow{b} & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 \longrightarrow \dots \end{array}$$

Because $A \rightarrow I^0$ is a strict monomorphism and because J^0 is filtered injective, we can find a morphism $f^0 : I^0 \rightarrow J^0$ such that $f^0 \circ a = b \circ f$, see Lemma 26.4. The composition $d_J^0 \circ b \circ f$ is zero, hence $d_J^0 \circ f^0 \circ a = 0$, hence $d_J^0 \circ f^0$ factors through a unique morphism

$$\text{Coker}(a) = \text{Coim}(d_I^0) = \text{Im}(d_I^0) \longrightarrow J^1.$$

As $\text{Im}(d_I^0) \rightarrow I^1$ is a strict monomorphism we can extend the displayed arrow to a morphism $f^1 : I^1 \rightarrow J^1$ by Lemma 26.4 again. And so on. \square

Lemma 26.8. *Let \mathcal{A} be an abelian category with enough injectives. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence in $\text{Fil}^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \rightarrow I^\bullet$ and $C[0] \rightarrow J^\bullet$ where I^\bullet, J^\bullet are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A[0] & \longrightarrow & B[0] & \longrightarrow & C[0] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I^\bullet & \longrightarrow & M^\bullet & \longrightarrow & J^\bullet \longrightarrow 0 \end{array}$$

where the lower row is a termwise split sequence of complexes.

Proof. As $A[0] \rightarrow I^\bullet$ and $C[0] \rightarrow J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \rightarrow I^0$, $c : C \rightarrow J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 14.4. We are going to step by step construct the south-east and the south arrows in the following commutative diagram

$$\begin{array}{ccccccc} B & \xrightarrow{\quad} & C & \xrightarrow{c} & J^0 & \longrightarrow & J^1 \longrightarrow \dots \\ & \nearrow \beta & & \searrow \bar{c} & \downarrow \delta^0 & & \downarrow \delta^1 \\ A & \xrightarrow{a} & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 \longrightarrow \dots \end{array}$$

As $A \rightarrow B$ is a strict monomorphism, we can find a morphism $b : B \rightarrow I^0$ such that $b \circ \alpha = a$, see Lemma 26.4. As A is the kernel of the strict morphism $I^0 \rightarrow I^1$ and $\beta = \text{Coker}(\alpha)$ we obtain a unique morphism $\bar{b} : C \rightarrow I^1$ fitting into the diagram. As c is a strict monomorphism and I^1 is filtered injective we can find $\delta^0 : J^0 \rightarrow I^1$, see Lemma 26.4. Because $B \rightarrow C$ is a strict epimorphism and because $B \rightarrow I^0 \rightarrow I^1 \rightarrow I^2$ is zero, we see that $C \rightarrow I^1 \rightarrow I^2$ is zero. Hence $d_I^1 \circ \delta^0$ is zero on $C \cong \text{Im}(c)$. Hence $d_I^1 \circ \delta^0$ factors through a unique morphism

$$\text{Coker}(c) = \text{Coim}(d_J^0) = \text{Im}(d_J^0) \longrightarrow I^2.$$

As I^2 is filtered injective and $\text{Im}(d_J^0) \rightarrow J^1$ is a strict monomorphism we can extend the displayed morphism to a morphism $\delta^1 : J^1 \rightarrow I^2$, see Lemma 26.4. And so on. We set $M^\bullet = I^\bullet \oplus J^\bullet$ with differential

$$d_M^n = \begin{pmatrix} d_I^n & (-1)^{n+1} \delta^n \\ 0 & d_J^n \end{pmatrix}$$

Finally, the map $B[0] \rightarrow M^\bullet$ is given by $b \oplus c \circ \beta : M \rightarrow I^0 \oplus J^0$. \square

Lemma 26.9. *Let \mathcal{A} be an abelian category with enough injectives. For every $K^\bullet \in K^+(\text{Fil}^f(\mathcal{A}))$ there exists a filtered quasi-isomorphism $K^\bullet \rightarrow I^\bullet$ with I^\bullet bounded below, each I^n a filtered injective object, and each $K^n \rightarrow I^n$ a strict monomorphism.*

Proof. After replacing K^\bullet by a shift (which is harmless for the proof) we may assume that $K^n = 0$ for $n < 0$. Consider the short exact sequences

$$\begin{array}{l} 0 \rightarrow \text{Ker}(d_K^0) \rightarrow K^0 \rightarrow \text{Coim}(d_K^0) \rightarrow 0 \\ 0 \rightarrow \text{Ker}(d_K^1) \rightarrow K^1 \rightarrow \text{Coim}(d_K^1) \rightarrow 0 \\ 0 \rightarrow \text{Ker}(d_K^2) \rightarrow K^2 \rightarrow \text{Coim}(d_K^2) \rightarrow 0 \\ \dots \end{array}$$

of the exact category $\text{Fil}^f(\mathcal{A})$ and the maps $u_i : \text{Coim}(d_K^i) \rightarrow \text{Ker}(d_K^{i+1})$. For each $i \geq 0$ we may choose filtered quasi-isomorphisms

$$\begin{aligned} \text{Ker}(d_K^i)[0] &\rightarrow I_{\text{ker},i}^\bullet \\ \text{Coim}(d_K^i)[0] &\rightarrow I_{\text{coim},i}^\bullet \end{aligned}$$

with $I_{\text{ker},i}^n, I_{\text{coim},i}^n$ filtered injective and zero for $n < 0$, see Lemma 26.6. By Lemma 26.7 we may lift u_i to a morphism of complexes $u_i^\bullet : I_{\text{coim},i}^\bullet \rightarrow I_{\text{ker},i+1}^\bullet$. Finally, for each $i \geq 0$ we may complete the diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(d_K^i)[0] & \longrightarrow & K^i[0] & \longrightarrow & \text{Coim}(d_K^i)[0] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_{\text{ker},i}^\bullet & \xrightarrow{\alpha_i} & I_i^\bullet & \xrightarrow{\beta_i} & I_{\text{coim},i}^\bullet \longrightarrow 0 \end{array}$$

with the lower sequence a termwise split exact sequence, see Lemma 26.8. For $i \geq 0$ set $d_i : I_i^\bullet \rightarrow I_{i+1}^\bullet$ equal to $d_i = \alpha_{i+1} \circ u_i^\bullet \circ \beta_i$. Note that $d_i \circ d_{i-1} = 0$ because $\beta_i \circ \alpha_i = 0$. Hence we have constructed a commutative diagram

$$\begin{array}{ccccccc} I_0^\bullet & \longrightarrow & I_1^\bullet & \longrightarrow & I_2^\bullet & \longrightarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ K^0[0] & \longrightarrow & K^1[0] & \longrightarrow & K^2[0] & \longrightarrow & \dots \end{array}$$

Here the vertical arrows are filtered quasi-isomorphisms. The upper row is a complex of complexes and each complex consists of filtered injective objects with no nonzero objects in degree < 0 . Thus we obtain a double complex by setting $I^{a,b} = I_a^b$ and using

$$d_1^{a,b} : I^{a,b} = I_a^b \rightarrow I_{a+1}^b = I^{a+1,b}$$

the map d_a^b and using for

$$d_2^{a,b} : I^{a,b} = I_a^b \rightarrow I_a^{b+1} = I^{a,b+1}$$

the map $d_{I_a}^b$. Denote $\text{Tot}(I^{\bullet,\bullet})$ the total complex associated to this double complex, see Homology, Definition 22.3. Observe that the maps $K^n[0] \rightarrow I_n^\bullet$ come from maps $K^n \rightarrow I^{n,0}$ which give rise to a map of complexes

$$K^\bullet \longrightarrow \text{Tot}(I^{\bullet,\bullet})$$

We claim this is a filtered quasi-isomorphism. As $\text{gr}(-)$ is an additive functor, we see that $\text{gr}(\text{Tot}(I^{\bullet,\bullet})) = \text{Tot}(\text{gr}(I^{\bullet,\bullet}))$. Thus we can use Homology, Lemma 22.7 to conclude that $\text{gr}(K^\bullet) \rightarrow \text{gr}(\text{Tot}(I^{\bullet,\bullet}))$ is a quasi-isomorphism as desired. \square

Lemma 26.10. *Let \mathcal{A} be an abelian category. Let $K^\bullet, I^\bullet \in K(\text{Fil}^f(\mathcal{A}))$. Assume K^\bullet is filtered acyclic and I^\bullet bounded below and consisting of filtered injective objects. Any morphism $K^\bullet \rightarrow I^\bullet$ is homotopic to zero: $\text{Hom}_{K(\text{Fil}^f(\mathcal{A}))}(K^\bullet, I^\bullet) = 0$.*

Proof. Let $\alpha : K^\bullet \rightarrow I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \rightarrow I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if } j \leq n \\ \alpha^{n+1} - d \circ h & \text{if } j = n+1 \\ \alpha^j & \text{if } j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n \circ d_K^{n-1} = 0$ since $\alpha^{n-1} = 0$. Since K^\bullet is filtered acyclic we see that d_K^{n-1} and d_K^n are strict and that

$$0 \rightarrow \text{Im}(d_K^{n-1}) \rightarrow K^n \rightarrow \text{Im}(d_K^n) \rightarrow 0$$

is an exact sequence of the exact category $\text{Fil}^f(\mathcal{A})$, see Homology, Lemma 16.15. Hence we can think of α^n as a map into I^n defined on $\text{Im}(d_K^n)$. Using that $\text{Im}(d_K^n) \rightarrow K^{n+1}$ is a strict monomorphism and that I^n is filtered injective we may lift this map to a map $h : K^{n+1} \rightarrow I^n$ as desired, see Lemma 26.4. \square

Lemma 26.11. *Let \mathcal{A} be an abelian category. Let $I^\bullet \in K(\text{Fil}^f(\mathcal{A}))$ be a bounded below complex consisting of filtered injective objects.*

- (1) *Let $\alpha : K^\bullet \rightarrow L^\bullet$ in $K(\text{Fil}^f(\mathcal{A}))$ be a filtered quasi-isomorphism. Then the map*

$$\text{Hom}_{K(\text{Fil}^f(\mathcal{A}))}(L^\bullet, I^\bullet) \rightarrow \text{Hom}_{K(\text{Fil}^f(\mathcal{A}))}(K^\bullet, I^\bullet)$$

is bijective.

- (2) *Let $L^\bullet \in K(\mathcal{A})$. Then*

$$\text{Hom}_{K(\text{Fil}^f(\mathcal{A}))}(L^\bullet, I^\bullet) = \text{Hom}_{DF(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Proof of (1). Note that

$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$

is a distinguished triangle in $K(\text{Fil}^f(\mathcal{A}))$ (Lemma 9.14) and $C(f)^\bullet$ is a filtered acyclic complex (Lemma 14.4). Then

$$\begin{array}{ccccc} \text{Hom}_{K(\text{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet, I^\bullet) & \longrightarrow & \text{Hom}_{K(\text{Fil}^f(\mathcal{A}))}(L^\bullet, I^\bullet) & \longrightarrow & \text{Hom}_{K(\text{Fil}^f(\mathcal{A}))}(K^\bullet, I^\bullet) \\ & & & \swarrow & \\ & & & \text{Hom}_{K(\text{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet[-1], I^\bullet) & \end{array}$$

is an exact sequence of abelian groups, see Lemma 4.2. At this point Lemma 26.10 guarantees that the outer two groups are zero and hence $\text{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \text{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

Proof of (2). Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \rightarrow L^\bullet$ is a filtered quasi-isomorphism and $\gamma : K^\bullet \rightarrow I^\bullet$ is a map of complexes. By part (1) we can find a morphism $\beta : L^\bullet \rightarrow I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ . This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \rightarrow I^\bullet$ such that there exists a filtered quasi-isomorphism $\alpha : K^\bullet \rightarrow L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then part (1) shows that β is homotopic to zero also, i.e., $b = 0$. \square

Lemma 26.12. *Let \mathcal{A} be an abelian category with enough injectives. Let $\mathcal{I}^f \subset \text{Fil}^f(\mathcal{A})$ denote the strictly full additive subcategory whose objects are the filtered injective objects. The canonical functor*

$$K^+(\mathcal{I}^f) \longrightarrow DF^+(\mathcal{A})$$

is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories. Furthermore the diagrams

$$\begin{array}{ccc} K^+(\mathcal{I}^f) & \longrightarrow & DF^+(\mathcal{A}) \\ \text{gr}^p \downarrow & & \downarrow \text{gr}^p \\ K^+(\mathcal{I}) & \longrightarrow & D^+(\mathcal{A}) \end{array} \quad \begin{array}{ccc} K^+(\mathcal{I}^f) & \longrightarrow & DF^+(\mathcal{A}) \\ \downarrow \text{forget } F & & \downarrow \text{forget } F \\ K^+(\mathcal{I}) & \longrightarrow & D^+(\mathcal{A}) \end{array}$$

are commutative, where $\mathcal{I} \subset \mathcal{A}$ is the strictly full additive subcategory whose objects are the injective objects.

Proof. The functor $K^+(\mathcal{I}^f) \rightarrow DF^+(\mathcal{A})$ is essentially surjective by Lemma 26.9. It is fully faithful by Lemma 26.11. It is an exact functor by our definitions regarding distinguished triangles. The commutativity of the squares is immediate. \square

Remark 26.13. We can invert the arrow of the lemma only if \mathcal{A} is a category in our sense, namely if it has a set of objects. However, suppose given a big abelian category \mathcal{A} with enough injectives, such as $\text{Mod}(\mathcal{O}_X)$ for example. Then for any given set of objects $\{A_i\}_{i \in I}$ there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ containing all of them and having enough injectives, see Sets, Lemma 12.1. Thus we may use the lemma above for \mathcal{A}' . This essentially means that if we use a set worth of diagrams, etc then we will never run into trouble using the lemma.

Let \mathcal{A}, \mathcal{B} be abelian categories. Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor. (We cannot use the letter F for the functor since this would conflict too much with our use of the letter F to indicate filtrations.) Note that T induces an additive functor

$$T : \text{Fil}^f(\mathcal{A}) \rightarrow \text{Fil}^f(\mathcal{B})$$

by the rule $T(A, F) = (T(A), F)$ where $F^p T(A) = T(F^p A)$ which makes sense as T is left exact. (Warning: It may not be the case that $\text{gr}(T(A)) = T(\text{gr}(A))$.) This induces functors of triangulated categories

$$(26.13.1) \quad T : K^+(\text{Fil}^f(\mathcal{A})) \longrightarrow K^+(\text{Fil}^f(\mathcal{B}))$$

The filtered right derived functor of T is the right derived functor of Definition 15.2 for this exact functor composed with the exact functor $K^+(\text{Fil}^f(\mathcal{B})) \rightarrow DF^+(\mathcal{B})$ and the multiplicative set $\text{FQis}^+(\mathcal{A})$. Assume \mathcal{A} has enough injectives. At this point we can redo the discussion of Section 20 to define the *filtered right derived functors*

$$(26.13.2) \quad RT : DF^+(\mathcal{A}) \longrightarrow DF^+(\mathcal{B})$$

of our functor T .

However, instead we will proceed as in Section 25, and it will turn out that we can define RT even if T is just additive. Namely, we first choose a quasi-inverse $j' : DF^+(\mathcal{A}) \rightarrow K^+(\mathcal{I}^f)$ of the equivalence of Lemma 26.12. By Lemma 4.16 we see that j' is an exact functor of triangulated categories. Next, we note that for a filtered injective object I we have a (noncanonical) decomposition

$$(26.13.3) \quad I \cong \bigoplus_{p \in \mathbb{Z}} I_p, \quad \text{with} \quad F^p I = \bigoplus_{q \geq p} I_q$$

by Lemma 26.2. Hence if T is any additive functor $T : \mathcal{A} \rightarrow \mathcal{B}$ then we get an additive functor

$$(26.13.4) \quad T_{ext} : \mathcal{I}^f \rightarrow \text{Fil}^f(\mathcal{B})$$

by setting $T_{ext}(I) = \bigoplus T(I_p)$ with $F^p T_{ext}(I) = \bigoplus_{q \geq p} T(I_q)$. Note that we have the property $\text{gr}(T_{ext}(I)) = T(\text{gr}(I))$ by construction. Hence we obtain a functor

$$(26.13.5) \quad T_{ext} : K^+(\mathcal{I}^f) \rightarrow K^+(\text{Fil}^f(\mathcal{B}))$$

which commutes with gr . Then we define (26.13.2) by the composition

$$(26.13.6) \quad RT = T_{ext} \circ j'.$$

Since $RT : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is computed by injective resolutions as well, see Lemmas 20.1, the commutation of T with gr , and the commutative diagrams of Lemma 26.12 imply that

$$(26.13.7) \quad \text{gr}^p \circ RT \cong RT \circ \text{gr}^p$$

and

$$(26.13.8) \quad (\text{forget } F) \circ RT \cong RT \circ (\text{forget } F)$$

as functors $DF^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$.

The filtered derived functor RT (26.13.2) induces functors

$$\begin{aligned} RT : \text{Fil}^f(\mathcal{A}) &\rightarrow DF^+(\mathcal{B}), \\ RT : \text{Comp}^+(\text{Fil}^f(\mathcal{A})) &\rightarrow DF^+(\mathcal{B}), \\ RT : KF^+(\mathcal{A}) &\rightarrow DF^+(\mathcal{B}). \end{aligned}$$

Note that since $\text{Fil}^f(\mathcal{A})$, and $\text{Comp}^+(\text{Fil}^f(\mathcal{A}))$ are no longer abelian it does not make sense to say that RT restricts to a δ -functor on them. (This can be repaired by thinking of these categories as exact categories and formulating the notion of a δ -functor from an exact category into a triangulated category.) But it does make sense, and it is true by construction, that RT is an exact functor on the triangulated category $KF^+(\mathcal{A})$.

Lemma 26.14. *Let \mathcal{A}, \mathcal{B} be abelian categories. Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor. Assume \mathcal{A} has enough injectives. Let (K^\bullet, F) be an object of $\text{Comp}^+(\text{Fil}^f(\mathcal{A}))$. There exists a spectral sequence $(E_r^{p,q}, d_r)_{r \geq 0}$ which is the spectral sequence associated to an object of $\text{Comp}^+(\text{Fil}^f(\mathcal{B}))$ with*

$$E_1^{p,q} = R^{p+q}T(\text{gr}^p(K^\bullet))$$

which converges to $R^{p+q}T(K^\bullet)$ inducing a finite filtration on each $R^n T(K^\bullet)$. Moreover the construction of this spectral sequence is functorial in the object K^\bullet of $\text{Comp}^+(\text{Fil}^f(\mathcal{A}))$. In fact the terms (E_r, d_r) for $r \geq 2$ do not depend on any choices.

Proof. Choose a filtered quasi-isomorphism $K^\bullet \rightarrow I^\bullet$ with I^\bullet a bounded below complex of filtered injective objects, see Lemma 26.9. Consider the complex $RT(K^\bullet) = T_{ext}(I^\bullet)$, see (26.13.6). Thus we can consider the spectral sequence $(E_r, d_r)_{r \geq 0}$ associated to this as a filtered complex in \mathcal{B} , see Homology, Section 21. By Homology, Lemma 21.2 we have $E_1^{p,q} = H^{p+q}(\text{gr}^p(T(I^\bullet)))$. By Equation (26.13.3) we have $E_1^{p,q} = H^{p+q}(T(\text{gr}^p(I^\bullet)))$, and by definition of a filtered injective resolution the map $\text{gr}^p(K^\bullet) \rightarrow \text{gr}^p(I^\bullet)$ is an injective resolution. Hence $E_1^{p,q} = R^{p+q}T(\text{gr}^p(K^\bullet))$.

On the other hand, each I^n has a finite filtration and hence each $T(I^n)$ has a finite filtration. Thus we may apply Homology, Lemma 21.9 to conclude that the spectral sequence converges to $H^n(T(I^\bullet)) = R^n T(K^\bullet)$ moreover inducing finite filtrations on each of the terms.

Suppose that $K^\bullet \rightarrow L^\bullet$ is a morphism of $\text{Comp}^+(\text{Fil}^f(\mathcal{A}))$. Choose a filtered quasi-isomorphism $L^\bullet \rightarrow J^\bullet$ with J^\bullet a bounded below complex of filtered injective objects, see Lemma 26.9. By our results above, for example Lemma 26.11, there exists a diagram

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \downarrow & & \downarrow \\ I^\bullet & \longrightarrow & J^\bullet \end{array}$$

which commutes up to homotopy. Hence we get a morphism of filtered complexes $T(I^\bullet) \rightarrow T(J^\bullet)$ which gives rise to the morphism of spectral sequences, see Homology, Lemma 21.4. The last statement follows from this. \square

Remark 26.15. As promised in Remark 21.4 we discuss the connection of the lemma above with the constructions using Cartan-Eilenberg resolutions. Namely, assume the notations of Lemma 21.3. In particular K^\bullet is a bounded below complex of \mathcal{A} and $T : \mathcal{A} \rightarrow \mathcal{B}$ is a left exact functor. We give an alternative construction of the spectral sequences $'E$ and $''E$

First spectral sequence. Consider the “stupid” filtration on K^\bullet obtained by setting $F^p(K^\bullet) = \sigma_{\geq p}(K^\bullet)$, see Homology, Section 13. Note that this stupid in the sense that $d(F^p(K^\bullet)) \subset F^{p+1}(K^\bullet)$, compare Homology, Lemma 21.3. Note that $\text{gr}^p(K^\bullet) = K^p[p]$ with this filtration. According to the above there is a spectral sequence with E_1 term $E_1^{p,q} = R^{p+q}T(K^p[p])$. Then the E_2 term is clearly $E_2^{p,q} = H^p(R^{p+q}T(K^\bullet))$ as in the spectral sequence $'E_r$.

Second spectral sequence. Consider the filtration on the complex K^\bullet obtained by setting $F^p(K^\bullet) = \tau_{\leq -p}(K^\bullet)$, see Homology, Section 13. The minus sign is necessary to get a decreasing filtration. Note that $\text{gr}^p(K^\bullet)$ is quasi-isomorphic to $H^{-p}(K^\bullet)[-p]$ with this filtration. According to the above there is a spectral sequence with E_1 term

$$E_1^{p,q} = R^{p+q}T(H^{-p}(K^\bullet)[-p]) = R^{2p+q}T(H^{-p}(K^\bullet)) = ''E_2^{i,j}$$

with $i = 2p+q$ and $j = -p$. (This looks unnatural, but note that we could just have well developed the whole theory of filtered complexes using increasing filtrations, with the end result that this then looks natural, but the other one doesn't.) We leave it to the reader to see that the differentials match up.

Actually, given a Cartan-Eilenberg resolution $K^\bullet \rightarrow I^{\bullet,\bullet}$ the induced morphism $K^\bullet \rightarrow sI^\bullet$ into the associated simple complex will be a filtered injective resolution for either filtration using suitable filtrations on sI^\bullet . This can be used to match up the spectral sequences exactly.

27. Ext groups

In this section we start describing the ext groups of objects of an abelian category. First we have the following very general definition.

Definition 27.1. Let \mathcal{A} be an abelian category. Let $i \in \mathbf{Z}$. Let X, Y be objects of $D(\mathcal{A})$. The i th extension group of X by Y is the group

$$\text{Ext}_{\mathcal{A}}^i(X, Y) = \text{Hom}_{D(\mathcal{A})}(X, Y[i]) = \text{Hom}_{D(\mathcal{A})}(X[-i], Y).$$

If $A, B \in \text{Ob}(\mathcal{A})$ we set $\text{Ext}_{\mathcal{A}}^i(A, B) = \text{Ext}_{\mathcal{A}}^i(A[0], B[0])$.

Since $\text{Hom}_{D(\mathcal{A})}(X, -)$, resp. $\text{Hom}_{D(\mathcal{A})}(-, Y)$ is a homological, resp. cohomological functor, see Lemma 4.2, we see that a distinguished triangle (Y, Y', Y'') , resp. (X, X', X'') leads to a long exact sequence

$$\dots \rightarrow \text{Ext}_{\mathcal{A}}^i(X, Y) \rightarrow \text{Ext}_{\mathcal{A}}^i(X, Y') \rightarrow \text{Ext}_{\mathcal{A}}^i(X, Y'') \rightarrow \text{Ext}_{\mathcal{A}}^{i+1}(X, Y) \rightarrow \dots$$

respectively

$$\dots \rightarrow \text{Ext}_{\mathcal{A}}^i(X'', Y) \rightarrow \text{Ext}_{\mathcal{A}}^i(X', Y) \rightarrow \text{Ext}_{\mathcal{A}}^i(X, Y) \rightarrow \text{Ext}_{\mathcal{A}}^{i+1}(X'', Y) \rightarrow \dots$$

Note that since $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ are full subcategories we may compute the ext groups by Hom groups in these categories provided X, Y are contained in them. In case the category \mathcal{A} has enough injectives or enough projectives we can compute the Ext groups using injective or projective resolutions. To avoid confusion, recall that having an injective (resp. projective) resolution implies vanishing of homology in all low (resp. high) degrees, see Lemmas 18.2 and 19.2.

Lemma 27.2. *Let \mathcal{A} be an abelian category. Let $X^\bullet, Y^\bullet \in \text{Ob}(K(\mathcal{A}))$.*

(1) *Let $Y^\bullet \rightarrow I^\bullet$ be an injective resolution (Definition 18.1). Then*

$$\text{Ext}_{\mathcal{A}}^i(X^\bullet, Y^\bullet) = \text{Hom}_{K(\mathcal{A})}(X^\bullet, I^\bullet[i]).$$

(2) *Let $P^\bullet \rightarrow X^\bullet$ be a projective resolution (Definition 19.1). Then*

$$\text{Ext}_{\mathcal{A}}^i(X^\bullet, Y^\bullet) = \text{Hom}_{K(\mathcal{A})}(P^\bullet[-i], Y^\bullet).$$

Proof. Follows immediately from Lemma 18.8 and Lemma 19.8. \square

In the rest of this section we discuss extensions of objects of the abelian category itself. First we observe the following.

Lemma 27.3. *Let \mathcal{A} be an abelian category.*

(1) *Let X, Y be objects of $D(\mathcal{A})$. Given $a, b \in \mathbf{Z}$ such that $H^i(X) = 0$ for $i > a$ and $H^j(Y) = 0$ for $j < b$, we have $\text{Ext}_{\mathcal{A}}^n(X, Y) = 0$ for $n < b - a$ and*

$$\text{Ext}_{\mathcal{A}}^{b-a}(X, Y) = \text{Hom}_{\mathcal{A}}(H^a(X), H^b(Y))$$

(2) *Let $A, B \in \text{Ob}(\mathcal{A})$. For $i < 0$ we have $\text{Ext}_{\mathcal{A}}^i(B, A) = 0$. We have $\text{Ext}_{\mathcal{A}}^0(B, A) = \text{Hom}_{\mathcal{A}}(B, A)$.*

Proof. Choose complexes X^\bullet and Y^\bullet representing X and Y . Since $Y^\bullet \rightarrow \tau_{\geq b} Y^\bullet$ is a quasi-isomorphism, we may assume that $Y^j = 0$ for $j < b$. Let $L^\bullet \rightarrow X^\bullet$ be any quasi-isomorphism. Then $\tau_{\leq a} L^\bullet \rightarrow X^\bullet$ is a quasi-isomorphism. Hence a morphism $X \rightarrow Y[n]$ in $D(\mathcal{A})$ can be represented as fs^{-1} where $s : L^\bullet \rightarrow X^\bullet$ is a quasi-isomorphism, $f : L^\bullet \rightarrow Y^\bullet[n]$ a morphism, and $L^i = 0$ for $i < a$. Note that f maps L^i to Y^{i+n} . Thus $f = 0$ if $n < b - a$ because always either L^i or Y^{i+n} is zero. If $n = b - a$, then f corresponds exactly to a morphism $H^a(X) \rightarrow H^b(Y)$. Part (2) is a special case of (1). \square

Let \mathcal{A} be an abelian category. Suppose that $0 \rightarrow A \rightarrow A' \rightarrow A'' \rightarrow 0$ is a short exact sequence of objects of \mathcal{A} . Then $0 \rightarrow A[0] \rightarrow A'[0] \rightarrow A''[0] \rightarrow 0$ leads to a distinguished triangle in $D(\mathcal{A})$ (see Lemma 12.1) hence a long exact sequence of Ext groups

$$0 \rightarrow \text{Ext}_{\mathcal{A}}^0(B, A) \rightarrow \text{Ext}_{\mathcal{A}}^0(B, A') \rightarrow \text{Ext}_{\mathcal{A}}^0(B, A'') \rightarrow \text{Ext}_{\mathcal{A}}^1(B, A) \rightarrow \dots$$

Similarly, given a short exact sequence $0 \rightarrow B \rightarrow B' \rightarrow B'' \rightarrow 0$ we obtain a long exact sequence of Ext groups

$$0 \rightarrow \text{Ext}_{\mathcal{A}}^0(B'', A) \rightarrow \text{Ext}_{\mathcal{A}}^0(B', A) \rightarrow \text{Ext}_{\mathcal{A}}^0(B, A) \rightarrow \text{Ext}_{\mathcal{A}}^1(B'', A) \rightarrow \dots$$

We may view these Ext groups as an application of the construction of the derived category. It shows one can define Ext groups and construct the long exact sequence of Ext groups without needing the existence of enough injectives or projectives. There is an alternative construction of the Ext groups due to Yoneda which avoids the use of the derived category, see [Yon60].

Definition 27.4. Let \mathcal{A} be an abelian category. Let $A, B \in \text{Ob}(\mathcal{A})$. A degree i Yoneda extension of B by A is an exact sequence

$$E : 0 \rightarrow A \rightarrow Z_{i-1} \rightarrow Z_{i-2} \rightarrow \dots \rightarrow Z_0 \rightarrow B \rightarrow 0$$

in \mathcal{A} . We say two Yoneda extensions E and E' of the same degree are *equivalent* if there exists a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & Z_{i-1} & \longrightarrow & \dots \longrightarrow Z_0 \longrightarrow B \longrightarrow 0 \\ & & \uparrow \text{id} & & \uparrow & & \uparrow \text{id} \\ 0 & \longrightarrow & A & \longrightarrow & Z''_{i-1} & \longrightarrow & \dots \longrightarrow Z''_0 \longrightarrow B \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \longrightarrow & Z'_{i-1} & \longrightarrow & \dots \longrightarrow Z'_0 \longrightarrow B \longrightarrow 0 \end{array}$$

where the middle row is a Yoneda extension as well.

It is not immediately clear that the equivalence of the definition is an equivalence relation. Although it is instructive to prove this directly this will also follow from Lemma 27.5 below.

Let \mathcal{A} be an abelian category with objects A, B . Given a Yoneda extension $E : 0 \rightarrow A \rightarrow Z_{i-1} \rightarrow Z_{i-2} \rightarrow \dots \rightarrow Z_0 \rightarrow B \rightarrow 0$ we define an associated element $\delta(E) \in \text{Ext}^i(B, A)$ as the morphism $\delta(E) = fs^{-1} : B[0] \rightarrow A[i]$ where s is the quasi-isomorphism

$$(\dots \rightarrow 0 \rightarrow A \rightarrow Z_{i-1} \rightarrow \dots \rightarrow Z_0 \rightarrow 0 \rightarrow \dots) \longrightarrow B[0]$$

and f is the morphism of complexes

$$(\dots \rightarrow 0 \rightarrow A \rightarrow Z_{i-1} \rightarrow \dots \rightarrow Z_0 \rightarrow 0 \rightarrow \dots) \longrightarrow A[i]$$

We call $\delta(E) = fs^{-1}$ the *class* of the Yoneda extension. It turns out that this class characterizes the equivalence class of the Yoneda extension.

Lemma 27.5. Let \mathcal{A} be an abelian category with objects A, B . Any element in $\text{Ext}_{\mathcal{A}}^i(B, A)$ is $\delta(E)$ for some degree i Yoneda extension of B by A . Given two Yoneda extensions E, E' of the same degree then E is equivalent to E' if and only if $\delta(E) = \delta(E')$.

Proof. Let $\xi : B[0] \rightarrow A[i]$ be an element of $\text{Ext}_{\mathcal{A}}^i(B, A)$. We may write $\xi = fs^{-1}$ for some quasi-isomorphism $s : L^\bullet \rightarrow B[0]$ and map $f : L^\bullet \rightarrow A[i]$. After replacing

L^\bullet by $\tau_{\leq 0}L^\bullet$ we may assume that $L^i = 0$ for $i > 0$. Picture

$$\begin{array}{ccccccc} L^{-i-1} & \longrightarrow & L^{-i} & \longrightarrow & \dots & \longrightarrow & L^0 \longrightarrow B \longrightarrow 0 \\ & & \downarrow & & & & \\ & & A & & & & \end{array}$$

Then setting $Z_{i-1} = (L^{-i+1} \oplus A)/L^{-i}$ and $Z_j = L^{-j}$ for $j = i-2, \dots, 0$ we see that we obtain a degree i extension E of B by A whose class $\delta(E)$ equals ξ .

It is immediate from the definitions that equivalent Yoneda extensions have the same class. Suppose that $E : 0 \rightarrow A \rightarrow Z_{i-1} \rightarrow Z_{i-2} \rightarrow \dots \rightarrow Z_0 \rightarrow B \rightarrow 0$ and $E' : 0 \rightarrow A \rightarrow Z'_{i-1} \rightarrow Z'_{i-2} \rightarrow \dots \rightarrow Z'_0 \rightarrow B \rightarrow 0$ are Yoneda extensions with the same class. By construction of $D(\mathcal{A})$ as the localization of $K(\mathcal{A})$ at the set of quasi-isomorphisms, this means there exists a complex L^\bullet and quasi-isomorphisms

$$t : L^\bullet \rightarrow (\dots \rightarrow 0 \rightarrow A \rightarrow Z_{i-1} \rightarrow \dots \rightarrow Z_0 \rightarrow 0 \rightarrow \dots)$$

and

$$t' : L^\bullet \rightarrow (\dots \rightarrow 0 \rightarrow A \rightarrow Z'_{i-1} \rightarrow \dots \rightarrow Z'_0 \rightarrow 0 \rightarrow \dots)$$

such that $s \circ t = s' \circ t'$ and $f \circ t = f' \circ t'$, see Categories, Section 25. Let E'' be the degree i extension of B by A constructed from the pair $L^\bullet \rightarrow B[0]$ and $L^\bullet \rightarrow A[i]$ in the first paragraph of the proof. Then the reader sees readily that there exists “morphisms” of degree i Yoneda extensions $E'' \rightarrow E$ and $E'' \rightarrow E'$ as in the definition of equivalent Yoneda extensions (details omitted). This finishes the proof. \square

Lemma 27.6. *Let \mathcal{A} be an abelian category. Let A, B be objects of \mathcal{A} . Then $\text{Ext}_{\mathcal{A}}^1(B, A)$ is the group $\text{Ext}_{\mathcal{A}}(B, A)$ constructed in Homology, Definition 6.2.*

Proof. This is the case $i = 1$ of Lemma 27.5. \square

28. Unbounded complexes

A reference for the material in this section is [Spa88]. The following lemma is useful to find “good” left resolutions of unbounded complexes.

Lemma 28.1. *Let \mathcal{A} be an abelian category. Let $\mathcal{P} \subset \text{Ob}(\mathcal{A})$ be a subset. Assume that every object of \mathcal{A} is a quotient of an element of \mathcal{P} . Let K^\bullet be a complex. There exists a commutative diagram*

$$\begin{array}{ccccc} P_1^\bullet & \longrightarrow & P_2^\bullet & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}K^\bullet & \longrightarrow & \tau_{\leq 2}K^\bullet & \longrightarrow & \dots \end{array}$$

in the category of complexes such that

- (1) the vertical arrows are quasi-isomorphisms,
- (2) P_1^\bullet is a bounded above complex with terms in \mathcal{P} ,
- (3) the arrows $P_n^\bullet \rightarrow P_{n+1}^\bullet$ are termwise split injections and each cokernel P_{n+1}^i/P_n^i is an element of \mathcal{P} .

Proof. By Lemma 16.5 any bounded above complex has a resolution by a bounded above complex whose terms are in \mathcal{P} . Thus we obtain the first complex P_1^\bullet . By induction it suffices, given $P_1^\bullet, \dots, P_n^\bullet$ to construct P_{n+1}^\bullet and the maps $P_n^\bullet \rightarrow P_{n+1}^\bullet$ and $P_n^\bullet \rightarrow \tau_{\leq n+1} K^\bullet$. Consider the cone C_1^\bullet of the composition $P_n^\bullet \rightarrow \tau_{\leq n} K^\bullet \rightarrow \tau_{\leq n+1} K^\bullet$. This fits into the distinguished triangle

$$P_n^\bullet \rightarrow \tau_{\leq n+1} K^\bullet \rightarrow C_1^\bullet \rightarrow P_n^\bullet[1]$$

Note that C_1^\bullet is bounded above, hence we can choose a quasi-isomorphism $Q^\bullet \rightarrow C_1^\bullet$ where Q^\bullet is a bounded above complex whose terms are elements of \mathcal{P} . Take the cone C_2^\bullet of the map of complexes $Q^\bullet \rightarrow P_n^\bullet[1]$ to get the distinguished triangle

$$Q^\bullet \rightarrow P_n^\bullet[1] \rightarrow C_2^\bullet \rightarrow Q^\bullet[1]$$

By the axioms of triangulated categories we obtain a map of distinguished triangles

$$\begin{array}{ccccccc} P_n^\bullet & \longrightarrow & C_2^\bullet[-1] & \longrightarrow & Q^\bullet & \longrightarrow & P_n^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_n^\bullet & \longrightarrow & \tau_{\leq n+1} K^\bullet & \longrightarrow & C_1^\bullet & \longrightarrow & P_n^\bullet[1] \end{array}$$

in the triangulated category $K(\mathcal{A})$. Set $P_{n+1}^\bullet = C_2^\bullet[-1]$. Note that (3) holds by construction. Choose an actual morphism of complexes $f : P_{n+1}^\bullet \rightarrow \tau_{\leq n+1} K^\bullet$. The left square of the diagram above commutes up to homotopy, but as $P_n^\bullet \rightarrow P_{n+1}^\bullet$ is a termwise split injection we can lift the homotopy and modify our choice of f to make it commute. Finally, f is a quasi-isomorphism, because both $P_n^\bullet \rightarrow P_n^\bullet$ and $Q^\bullet \rightarrow C_1^\bullet$ are. \square

In some cases we can use the lemma above to show that a left derived functor is everywhere defined.

Proposition 28.2. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor of abelian categories. Let $\mathcal{P} \subset \text{Ob}(\mathcal{A})$ be a subset. Assume*

- (1) *every object of \mathcal{A} is a quotient of an element of \mathcal{P} ,*
- (2) *for any bounded above acyclic complex P^\bullet of \mathcal{A} with $P^n \in \mathcal{P}$ for all n the complex $F(P^\bullet)$ is exact,*
- (3) *\mathcal{A} and \mathcal{B} have colimits of systems over \mathbf{N} ,*
- (4) *colimits over \mathbf{N} are exact in both \mathcal{A} and \mathcal{B} , and*
- (5) *F commutes with colimits over \mathbf{N} .*

Then LF is defined on all of $D(\mathcal{A})$.

Proof. By (1) and Lemma 16.5 for any bounded above complex K^\bullet there exists a quasi-isomorphism $P^\bullet \rightarrow K^\bullet$ with P^\bullet bounded above and $P^n \in \mathcal{P}$ for all n . Suppose that $s : P^\bullet \rightarrow (P')^\bullet$ is a quasi-isomorphism of bounded above complexes consisting of objects of \mathcal{P} . Then $F(P^\bullet) \rightarrow F((P')^\bullet)$ is a quasi-isomorphism because $F(C(s)^\bullet)$ is acyclic by assumption (2). This already shows that LF is defined on $D^-(\mathcal{A})$ and that a bounded above complex consisting of objects of \mathcal{P} computes LF , see Lemma 15.15.

Next, let K^\bullet be an arbitrary complex of \mathcal{A} . Choose a diagram

$$\begin{array}{ccccc} P_1^\bullet & \longrightarrow & P_2^\bullet & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1} K^\bullet & \longrightarrow & \tau_{\leq 2} K^\bullet & \longrightarrow & \dots \end{array}$$

as in Lemma 28.1. Note that the map $\text{colim } P_n^\bullet \rightarrow K^\bullet$ is a quasi-isomorphism because colimits over \mathbf{N} in \mathcal{A} are exact and $H^i(P_n^\bullet) = H^i(K^\bullet)$ for $n > i$. We claim that

$$F(\text{colim } P_n^\bullet) = \text{colim } F(P_n^\bullet)$$

(termwise colimits) is $LF(K^\bullet)$, i.e., that $\text{colim } P_n^\bullet$ computes LF . To see this, by Lemma 15.15, it suffices to prove the following claim. Suppose that

$$\text{colim } Q_n^\bullet = Q^\bullet \xrightarrow{\alpha} P^\bullet = \text{colim } P_n^\bullet$$

is a quasi-isomorphism of complexes, such that each P_n^\bullet, Q_n^\bullet is a bounded above complex whose terms are in \mathcal{P} and the maps $P_n^\bullet \rightarrow \tau_{\leq n} P^\bullet$ and $Q_n^\bullet \rightarrow \tau_{\leq n} Q^\bullet$ are quasi-isomorphisms. Claim: $F(\alpha)$ is a quasi-isomorphism.

The problem is that we do not assume that α is given as a colimit of maps between the complexes P_n^\bullet and Q_n^\bullet . However, for each n we know that the solid arrows in the diagram

$$\begin{array}{ccccc} & & R^\bullet & & \\ & & \vdots & & \\ & & L^\bullet & & \\ P_n^\bullet & \xleftarrow{\dots\dots\dots} & L^\bullet & \xrightarrow{\dots\dots\dots} & Q_n^\bullet \\ \downarrow & & & & \downarrow \\ \tau_{\leq n} P^\bullet & \xrightarrow{\tau_{\leq n} \alpha} & & & \tau_{\leq n} Q^\bullet \end{array}$$

are quasi-isomorphisms. Because quasi-isomorphisms form a multiplicative system in $K(\mathcal{A})$ (see Lemma 11.2) we can find a quasi-isomorphism $L^\bullet \rightarrow P_n^\bullet$ and map of complexes $L^\bullet \rightarrow Q_n^\bullet$ such that the diagram above commutes up to homotopy. Then $\tau_{\leq n} L^\bullet \rightarrow L^\bullet$ is a quasi-isomorphism. Hence (by the first part of the proof) we can find a bounded above complex R^\bullet whose terms are in \mathcal{P} and a quasi-isomorphism $R^\bullet \rightarrow L^\bullet$ (as indicated in the diagram). Using the result of the first paragraph of the proof we see that $F(R^\bullet) \rightarrow F(P_n^\bullet)$ and $F(R^\bullet) \rightarrow F(Q_n^\bullet)$ are quasi-isomorphisms. Thus we obtain a isomorphisms $H^i(F(P_n^\bullet)) \rightarrow H^i(F(Q_n^\bullet))$ fitting into the commutative diagram

$$\begin{array}{ccc} H^i(F(P_n^\bullet)) & \longrightarrow & H^i(F(Q_n^\bullet)) \\ \downarrow & & \downarrow \\ H^i(F(P^\bullet)) & \longrightarrow & H^i(F(Q^\bullet)) \end{array}$$

The exact same argument shows that these maps are also compatible as n varies. Since by (4) and (5) we have

$$H^i(F(P^\bullet)) = H^i(F(\text{colim } P_n^\bullet)) = H^i(\text{colim } F(P_n^\bullet)) = \text{colim } H^i(F(P_n^\bullet))$$

and similarly for Q^\bullet we conclude that $H^i(\alpha) : H^i(F(P^\bullet)) \rightarrow H^i(F(Q^\bullet))$ is an isomorphism and the claim follows. \square

Lemma 28.3. *Let \mathcal{A} be an abelian category. Let $\mathcal{I} \subset \text{Ob}(\mathcal{A})$ be a subset. Assume that every object of \mathcal{A} is a subobject of an element of \mathcal{I} . Let K^\bullet be a complex. There exists a commutative diagram*

$$\begin{array}{ccccc} \dots & \longrightarrow & \tau_{\geq -2} K^\bullet & \longrightarrow & \tau_{\geq -1} K^\bullet \\ & & \downarrow & & \downarrow \\ \dots & \longrightarrow & I_2^\bullet & \longrightarrow & I_1^\bullet \end{array}$$

in the category of complexes such that

- (1) the vertical arrows are quasi-isomorphisms,
- (2) I_1^\bullet is a bounded above complex with terms in \mathcal{I} ,
- (3) the arrows $I_{n+1}^\bullet \rightarrow I_n^\bullet$ are termwise split surjections and $\text{Ker}(I_{n+1}^\bullet \rightarrow I_n^\bullet)$ is an element of \mathcal{I} .

Proof. This lemma is dual to Lemma 28.1. □

The following lemma is an example of why it is easier to work with unbounded derived categories. Namely, without having the unbounded derived functors, the lemma could not even be stated.

Lemma 28.4. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$ be functors such that F is a right adjoint to G . If the derived functors $RF : D(\mathcal{A}) \rightarrow D(\mathcal{B})$ and $LG : D(\mathcal{B}) \rightarrow D(\mathcal{A})$ exist, then RF is a right adjoint to LG .*

Proof. Let K^\bullet be a complex of \mathcal{A} and let M^\bullet be a complex of \mathcal{B} . Since RF is defined at K^\bullet , we see that the rule which assigns to a quasi-isomorphism $s : K^\bullet \rightarrow I^\bullet$ the object $F(I^\bullet)$ is essentially constant as an ind-object of $D(\mathcal{B})$ with value $RF(K^\bullet)$. Similarly, the rule which assigns to a quasi-isomorphism $t : P^\bullet \rightarrow M^\bullet$ the object $G(P^\bullet)$ is essentially constant as a pro-object of $D(\mathcal{A})$ with value $LG(M^\bullet)$. Thus we have

$$\begin{aligned} \text{Hom}_{D(\mathcal{B})}(M^\bullet, RF(K^\bullet)) &= \text{colim}_{s:K^\bullet \rightarrow I^\bullet} \text{Hom}_{D(\mathcal{B})}(M^\bullet, F(I^\bullet)) \\ &= \text{colim}_{s:K^\bullet \rightarrow I^\bullet} \text{colim}_{t:P^\bullet \rightarrow M^\bullet} \text{Hom}_{K(\mathcal{B})}(P^\bullet, F(I^\bullet)) \\ &= \text{colim}_{t:P^\bullet \rightarrow M^\bullet} \text{colim}_{s:K^\bullet \rightarrow I^\bullet} \text{Hom}_{K(\mathcal{B})}(P^\bullet, F(I^\bullet)) \\ &= \text{colim}_{t:P^\bullet \rightarrow M^\bullet} \text{colim}_{s:K^\bullet \rightarrow I^\bullet} \text{Hom}_{K(\mathcal{A})}(G(P^\bullet), I^\bullet) \\ &= \text{colim}_{s:K^\bullet \rightarrow I^\bullet} \text{Hom}_{D(\mathcal{A})}(G(P^\bullet), K^\bullet) \\ &= \text{Hom}_{D(\mathcal{A})}(LG(M^\bullet), K^\bullet) \end{aligned}$$

The first equality holds by Categories, Lemma 22.6. The second equality holds by the definition of morphisms in $D(\mathcal{B})$. The third equality holds by Categories, Lemma 14.9. The fourth equality holds because F and G are adjoint. The fifth equality holds by definition of morphism in $D(\mathcal{A})$. The sixth equality holds by Categories, Lemma 22.7. □

29. K-injective complexes

The following types of complexes can be used to compute right derived functors on the unbounded derived category.

Definition 29.1. Let \mathcal{A} be an abelian category. A complex I^\bullet is *K-injective* if for every acyclic complex M^\bullet we have $\text{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.

In the situation of the definition we have in fact $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet[i], I^\bullet) = 0$ for all i as the translate of an acyclic complex is acyclic.

Lemma 29.2. *Let \mathcal{A} be an abelian category. Let I^\bullet be a complex. The following are equivalent*

- (1) I^\bullet is K -injective,
- (2) for every quasi-isomorphism $M^\bullet \rightarrow N^\bullet$ the map

$$\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \rightarrow \mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$$

is bijective, and

- (3) for every complex N^\bullet the map

$$\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \rightarrow \mathrm{Hom}_{D(\mathcal{A})}(N^\bullet, I^\bullet)$$

is an isomorphism.

Proof. Assume (1). Then (2) holds because the functor $\mathrm{Hom}_{K(\mathcal{A})}(-, I^\bullet)$ is cohomological and the cone on a quasi-isomorphism is acyclic.

Assume (2). A morphism $N^\bullet \rightarrow I^\bullet$ in $D(\mathcal{A})$ is of the form $fs^{-1} : N^\bullet \rightarrow I^\bullet$ where $s : M^\bullet \rightarrow N^\bullet$ is a quasi-isomorphism and $f : M^\bullet \rightarrow I^\bullet$ is a map. By (2) this corresponds to a unique morphism $N^\bullet \rightarrow I^\bullet$ in $K(\mathcal{A})$, i.e., (3) holds.

Assume (3). If M^\bullet is acyclic then M^\bullet is isomorphic to the zero complex in $D(\mathcal{A})$ hence $\mathrm{Hom}_{D(\mathcal{A})}(N^\bullet, I^\bullet) = 0$, whence $\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) = 0$ by (3), i.e., (1) holds. \square

Lemma 29.3. *Let \mathcal{A} be an abelian category. Let (K, L, M, f, g, h) be a distinguished triangle of $K(\mathcal{A})$. If two out of K, L, M are K -injective complexes, then the third is too.*

Proof. Follows from the definition, Lemma 4.2, and the fact that $K(\mathcal{A})$ is a triangulated category (Proposition 10.3). \square

Lemma 29.4. *Let \mathcal{A} be an abelian category. A bounded below complex of injectives is K -injective.*

Proof. Follows from Lemmas 29.2 and 18.8. \square

Lemma 29.5. *Let \mathcal{A} be an abelian category. Let $F : K(\mathcal{A}) \rightarrow \mathcal{D}'$ be an exact functor of triangulated categories. Then RF is defined at every complex in $K(\mathcal{A})$ which is quasi-isomorphic to a K -injective complex. In fact, every K -injective complex computes RF .*

Proof. By Lemma 15.4 it suffices to show that RF is defined at a K -injective complex, i.e., it suffices to show a K -injective complex I^\bullet computes RF . Any quasi-isomorphism $I^\bullet \rightarrow N^\bullet$ is a homotopy equivalence as it has an inverse by Lemma 29.2. Thus $I^\bullet \rightarrow I^\bullet$ is a final object of $I^\bullet/\mathrm{Qis}(\mathcal{A})$ and we win. \square

Lemma 29.6. *Let \mathcal{A} be an abelian category. Assume every complex has a quasi-isomorphism towards a K -injective complex. Then any exact functor $F : K(\mathcal{A}) \rightarrow \mathcal{D}'$ of triangulated categories has a right derived functor*

$$RF : D(\mathcal{A}) \longrightarrow \mathcal{D}'$$

and $RF(I^\bullet) = F(I^\bullet)$ for K -injective complexes I^\bullet .

Proof. To see this we apply Lemma 15.15 with \mathcal{I} the collection of K-injective complexes. Since (1) holds by assumption, it suffices to prove that if $I^\bullet \rightarrow J^\bullet$ is a quasi-isomorphism of K-injective complexes, then $F(I^\bullet) \rightarrow F(J^\bullet)$ is an isomorphism. This is clear because $I^\bullet \rightarrow J^\bullet$ is a homotopy equivalence, i.e., an isomorphism in $K(\mathcal{A})$, by Lemma 29.2. \square

The following lemma can be generalized to limits over bigger ordinals.

Lemma 29.7. *Let \mathcal{A} be an abelian category. Let*

$$\dots \rightarrow I_3^\bullet \rightarrow I_2^\bullet \rightarrow I_1^\bullet$$

be an inverse system of K-injective complexes. Assume

- (1) *each I_n^\bullet is K-injective,*
- (2) *each map $I_{n+1}^m \rightarrow I_n^m$ is a split surjection,*
- (3) *the limits $I^m = \lim I_n^m$ exist.*

Then the complex I^\bullet is K-injective.

Proof. Let M^\bullet be an acyclic complex. Let us abbreviate $H_n(a, b) = \text{Hom}_{\mathcal{A}}(M^a, I_n^b)$. With this notation $\text{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$ is the cohomology of the complex

$$\prod_m \lim_n H_n(m, m-2) \rightarrow \prod_m \lim_n H_n(m, m-1) \rightarrow \prod_m \lim_n H_n(m, m) \rightarrow \prod_m \lim_n H_n(m, m+1)$$

in the third spot from the left. We may exchange the order of \prod and \lim and each of the complexes

$$\prod_m H_n(m, m-2) \rightarrow \prod_m H_n(m, m-1) \rightarrow \prod_m H_n(m, m) \rightarrow \prod_m H_n(m, m+1)$$

is exact by assumption (1). By assumption (2) the maps in the systems

$$\dots \rightarrow \prod_m H_3(m, m-2) \rightarrow \prod_m H_2(m, m-2) \rightarrow \prod_m H_1(m, m-2)$$

are surjective. Thus the lemma follows from Homology, Lemma 27.4. \square

Remark 29.8. It appears that a combination of Lemmas 28.3, 29.4, and 29.7 produces “enough K-injectives” for any abelian category with enough injectives and countable limits. Actually, this may not work! Namely, suppose that K^\bullet is a complex and I_n^\bullet is the system of bounded above complexes of injectives produced by Lemma 28.3. Each complex I_n^\bullet is K-injective by Lemma 29.4. Let $I^\bullet = \lim I_n^\bullet$ be the termwise limit which is K-injective by Lemma 29.7. The problem is that the map $K^\bullet \rightarrow I^\bullet$ may not be a quasi-isomorphism. The following lemma shows that it does work if \mathcal{A} has countable exact products.

Lemma 29.9. *Let \mathcal{A} be an abelian category having enough injectives and exact countable products. Then for every complex there is a quasi-isomorphism to a K-injective complex.*

Proof. Let K^\bullet be a complex. Let I_n^\bullet be the system of bounded above complexes of injectives produced by Lemma 28.3. Choose direct sum decompositions $I_{n+1}^p = C_{n+1}^p \oplus I_n^p$ for all $n \geq 1$. Set $C_1^p = I_1^p$. Each complex I_n^\bullet is K-injective by Lemma 29.4. Then $I^\bullet = \lim I_n^\bullet$ exists because $I^p = \prod_{n \geq 1} C_n^p$. The I^\bullet is K-injective by Lemma 29.7. Fix $p \in \mathbf{Z}$. We claim there is a short exact sequence

$$0 \rightarrow I^p \rightarrow \prod I_n^p \rightarrow \prod I_n^p \rightarrow 0$$

of objects of \mathcal{A} . Here the first map is given by the projection maps $I^p \rightarrow I_n^p$ and the second map by $(x_n) \mapsto (x_n - f_{n+1}^p(x_{n+1}))$ where $f_n^p : I_n^p \rightarrow I_{n-1}^p$ are the transition maps. In fact, the sequence is split exact as $I_n^p = \prod_{m \leq n} C_m^p$ and the f_n are the projection maps (details omitted). We obtain a short exact sequence of complexes

$$0 \rightarrow I^\bullet \rightarrow \prod I_n^\bullet \rightarrow \prod I_n^\bullet \rightarrow 0$$

Now, using that countable products are exact, we see that we get a long exact cohomology sequence

$$\dots \rightarrow H^p(I^\bullet) \rightarrow \prod H^p(I_m^\bullet) \rightarrow \prod H^p(I_m^\bullet) \rightarrow \dots$$

Since I_m^\bullet is quasi-isomorphic to $\tau_{\geq -m} K^\bullet$ for each m we see that the system of objects $H^p(I_m^\bullet)$ (fixed p varying m) is eventually constant with value $H^p(K)$. Thus the map $\prod H^p(I_m^\bullet) \rightarrow \prod H^p(I_m^\bullet)$ is surjective with kernel $H^p(K^\bullet)$ (details omitted). This proves that $K^\bullet \rightarrow I^\bullet$ is a quasi-isomorphism as desired. \square

Lemma 29.10. *Let \mathcal{A} and \mathcal{B} be abelian categories. Let $u : \mathcal{A} \rightarrow \mathcal{B}$ and $v : \mathcal{B} \rightarrow \mathcal{A}$ be additive functors. Assume*

- (1) *u is right adjoint to v , and*
- (2) *v is exact.*

Then u transforms K -injective complexes into K -injective complexes.

Proof. Let I^\bullet be a K -injective complex of \mathcal{A} . Let M^\bullet be a acyclic complex of \mathcal{B} . As v is exact we see that $v(M^\bullet)$ is an acyclic complex. By adjointness we get

$$0 = \text{Hom}_{K(\mathcal{A})}(v(M^\bullet), I^\bullet) = \text{Hom}_{K(\mathcal{B})}(M^\bullet, u(I^\bullet))$$

hence the lemma follows. \square

30. Bounded cohomological dimension

There is another case where the unbounded derived functor exists. Namely, when the functor has bounded cohomological dimension.

Lemma 30.1. *Let \mathcal{A} be an abelian category. Let $d : \text{Ob}(\mathcal{A}) \rightarrow \{0, 1, 2, \dots, \infty\}$ be a function. Assume that*

- (1) *every object of \mathcal{A} is a subobject of an object A with $d(A) = 0$,*
- (2) *if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence then $d(C) \leq \max\{d(A) - 1, d(B)\}$.*

Let K^\bullet be a complex such that $n + d(K^n)$ tends to $-\infty$ as $n \rightarrow -\infty$. Then there exists a quasi-isomorphism $K^\bullet \rightarrow L^\bullet$ with $d(L^n) = 0$ for all $n \in \mathbf{Z}$.

Proof. By Lemma 16.4 we can find a quasi-isomorphism $\sigma_{\geq 0} K^\bullet \rightarrow M^\bullet$ with $M^n = 0$ for $n < 0$ and $d(M^n) = 0$ for $n \geq 0$. Then K^\bullet is quasi-isomorphic to the complex

$$\dots \rightarrow K^{-2} \rightarrow K^{-1} \rightarrow M^0 \rightarrow M^1 \rightarrow \dots$$

Hence we may assume that $d(K^n) = 0$ for $n \gg 0$. Note that the condition $n + d(K^n) \rightarrow -\infty$ as $n \rightarrow -\infty$ is not violated by this replacement.

We are going to improve K^\bullet by an (infinite) sequence of elementary replacements. An *elementary replacement* is the following. Choose an index n such that $d(K^n) >$

0. Choose an injection $K^n \rightarrow M$ where $d(M) = 0$. Set $M' = \text{Coker}(K^n \rightarrow M \oplus K^{n+1})$. Consider the map of complexes

$$\begin{array}{ccccccc} K^\bullet : & & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & K^{n+2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (K')^\bullet : & & K^{n-1} & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & K^{n+2} \end{array}$$

It is clear that $K^\bullet \rightarrow (K')^\bullet$ is a quasi-isomorphism. Moreover, it is clear that $d((K')^n) = 0$ and $d((K')^{n+1}) \leq \max\{d(K^{n+1}), d(K^n) - 1\}$ and the other values are unchanged.

To finish the proof we carefully choose the order in which to do the elementary replacements so that for every integer m the complex $\sigma_{\geq m} K^\bullet$ is changed only a finite number of times. To do this set

$$\xi(K^\bullet) = \max\{n + d(K^n) \mid d(K^n) > 0\}$$

and

$$I = \{n \in \mathbf{Z} \mid \xi(K^\bullet) = n + d(K^n) \wedge d(K^n) > 0\}$$

Our assumption that $n + d(K^n)$ tends to $-\infty$ as $n \rightarrow -\infty$ and the fact that $d(K^n) = 0$ for $n \gg 0$ implies $\xi(K^\bullet) < +\infty$ and that I is a finite set. It is clear that $\xi((K')^\bullet) \leq \xi(K^\bullet)$ for an elementary transformation as above. An elementary transformation changes the complex in degrees $\leq \xi(K^\bullet) + 1$. Hence if we can find finite sequence of elementary transformations which decrease $\xi(K^\bullet)$, then we win. However, note that if we do an elementary transformation starting with the smallest element $n \in I$, then we either decrease the size of I , or we increase $\min I$. Since every element of I is $\leq \xi(K^\bullet)$ we see that we win after a finite number of steps. \square

Lemma 30.2. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor of abelian categories. If*

- (1) *every object of \mathcal{A} is a subobject of an object which is right acyclic for F ,*
- (2) *there exists an integer n such that $R^n F = 0$,*

then $RF : D(\mathcal{A}) \rightarrow D(\mathcal{B})$ exists. Any complex consisting of right acyclic objects for F computes RF and any complex is the source of a quasi-isomorphism into such a complex.

Proof. Note that the first condition implies that $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ exists, see Proposition 16.8. Let A be an object of \mathcal{A} . Choose an injection $A \rightarrow A'$ with A' acyclic. Then we see that $R^{n+1}F(A) = R^n F(A'/A) = 0$ by the long exact cohomology sequence. Hence we conclude that $R^{n+1}F = 0$. Continuing like this using induction we find that $R^m F = 0$ for all $m \geq n$.

We are going to use Lemma 30.1 with the function $d : \text{Ob}(\mathcal{A}) \rightarrow \{0, 1, 2, \dots\}$ given by $d(A) = \min\{0\} \cup \{i \mid R^i F(A) \neq 0\}$. The first assumption of Lemma 30.1 is our assumption (1) and the second assumption of Lemma 30.1 follows from the long exact cohomology sequence. Hence for every complex K^\bullet there exists a quasi-isomorphism $K^\bullet \rightarrow L^\bullet$ with L^n right acyclic for F . We claim that if $L^\bullet \rightarrow M^\bullet$ is a quasi-isomorphism of complexes of right acyclic objects for F , then $F(L^\bullet) \rightarrow F(M^\bullet)$ is a quasi-isomorphism. If we prove this claim then we are done by Lemma 15.15. To prove the claim pick an integer $i \in \mathbf{Z}$. Consider the distinguished triangle

$$\sigma_{\geq i-n-1} L^\bullet \rightarrow \sigma_{\geq i-n-1} M^\bullet \rightarrow Q^\bullet,$$

i.e., let Q^\bullet be the cone of the first map. Note that Q^\bullet is bounded below and that $H^j(Q^\bullet)$ is zero except possibly for $j = i - n - 1$ or $j = i - n - 2$. We may apply RF to Q^\bullet . Using the spectral sequence of Lemma 21.3 and the assumed vanishing of cohomology (2) we conclude that $R^j F(Q^\bullet)$ is zero except possibly for $j \in \{i - n - 2, \dots, i - 1\}$. Hence we see that $RF(\sigma_{\geq i-n-1} L^\bullet) \rightarrow RF(\sigma_{\geq i-n-1} M^\bullet)$ induces an isomorphism of cohomology objects in degrees $\geq i$. By Proposition 16.8 we know that $RF(\sigma_{\geq i-n-1} L^\bullet) = \sigma_{\geq i-n-1} F(L^\bullet)$ and $RF(\sigma_{\geq i-n-1} M^\bullet) = \sigma_{\geq i-n-1} F(M^\bullet)$. We conclude that $F(L^\bullet) \rightarrow F(M^\bullet)$ is an isomorphism in degree i as desired. \square

Lemma 30.3. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor of abelian categories. If*

- (1) *every object of \mathcal{A} is a quotient of an object which is left acyclic for F ,*
- (2) *there exists an integer n such that $L^n F = 0$,*

then $LF : D(\mathcal{A}) \rightarrow D(\mathcal{B})$ exists. Any complex consisting of left acyclic objects for F computes LF and any complex is the target of a quasi-isomorphism into such a complex.

Proof. This is dual to Lemma 30.2. \square

31. Derived colimits

In a triangulated category there is a notion of derived colimit.

Definition 31.1. Let \mathcal{D} be a triangulated category. Let (K_n, f_n) be a system of objects of \mathcal{D} . We say an object K is a *derived colimit*, or a *homotopy colimit* of the system (K_n) if the direct sum $\bigoplus K_n$ exists and there is a distinguished triangle

$$\bigoplus K_n \rightarrow \bigoplus K_n \rightarrow K \rightarrow \bigoplus K_n[1]$$

where the map $\bigoplus K_n \rightarrow \bigoplus K_n$ is given by $1 - f_n$ in degree n . If this is the case, then we sometimes indicate this by the notation $K = \text{hocolim } K_n$.

By TR3 a derived colimit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived colimit of K_n exists as soon as $\bigoplus K_n$ exists. The derived category $D(\text{Ab})$ of the category of abelian groups is an example. More generally we have the following lemma.

Lemma 31.2. *Let \mathcal{A} be an abelian category. If \mathcal{A} has exact countable direct sums, then $D(\mathcal{A})$ has countable direct sums. In fact given a collection of complexes K_i^\bullet indexed by a countable index set I the termwise direct sum $\bigoplus K_i^\bullet$ is the direct sum of K_i^\bullet in $D(\mathcal{A})$.*

Proof. Let L^\bullet be a complex. Suppose given maps $\alpha_i : K_i^\bullet \rightarrow L^\bullet$ in $D(\mathcal{A})$. This means there exist quasi-isomorphisms $s_i : M_i^\bullet \rightarrow K_i^\bullet$ of complexes and maps of complexes $f_i : M_i^\bullet \rightarrow L^\bullet$ such that $\alpha_i = f_i s_i^{-1}$. By assumption the map of complexes

$$s : \bigoplus M_i^\bullet \longrightarrow \bigoplus K_i^\bullet$$

is a quasi-isomorphism. Hence setting $f = \bigoplus f_i$ we see that $\alpha = f s^{-1}$ is a map in $D(\mathcal{A})$ whose composition with the coprojection $K_i^\bullet \rightarrow \bigoplus K_i^\bullet$ is α_i . We omit the verification that α is unique. \square

Lemma 31.3. *Let \mathcal{A} be an abelian category. Assume colimits over \mathbf{N} exist and are exact. Then countable direct sums exist and are exact. Moreover, if (A_n, f_n) is a system over \mathbf{N} , then there is a short exact sequence*

$$0 \rightarrow \bigoplus A_n \rightarrow \bigoplus A_n \rightarrow \operatorname{colim} A_n \rightarrow 0$$

where the first map in degree n is given by $1 - f_n$.

Proof. The first statement follows from $\bigoplus A_n = \operatorname{colim}(A_1 \oplus \dots \oplus A_n)$. For the second, note that for each n we have the short exact sequence

$$0 \rightarrow A_1 \oplus \dots \oplus A_{n-1} \rightarrow A_1 \oplus \dots \oplus A_n \rightarrow A_n \rightarrow 0$$

where the first map is given by the maps $1 - f_i$ and the second map is the sum of the transition maps. Take the colimit to get the sequence of the lemma. \square

Lemma 31.4. *Let \mathcal{A} be an abelian category. Let L_n^\bullet be a system of complexes of \mathcal{A} . Assume colimits over \mathbf{N} exist and are exact in \mathcal{A} . Then the termwise colimit $L^\bullet = \operatorname{colim} L_n^\bullet$ is a homotopy colimit of the system in $D(\mathcal{A})$.*

Proof. We have an exact sequence of complexes

$$0 \rightarrow \bigoplus L_n^\bullet \rightarrow \bigoplus L_n^\bullet \rightarrow L^\bullet \rightarrow 0$$

by Lemma 31.3. The direct sums are direct sums in $D(\mathcal{A})$ by Lemma 31.2. Thus the result follows from the definition of derived colimits in Definition 31.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Lemma 12.1). \square

The following lemma tells us that taking maps out of a compact object (to be defined later) commutes with derived colimits.

Lemma 31.5. *Let \mathcal{D} be a triangulated category with countable direct sums. Let $K \in \mathcal{D}$ be an object such that for every countable set of objects $E_n \in \mathcal{D}$ the canonical map*

$$\bigoplus \operatorname{Hom}_{\mathcal{D}}(K, E_n) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(K, \bigoplus E_n)$$

is a bijection. Then, given any system L_n of \mathcal{D} over \mathbf{N} whose derived colimit $L = \operatorname{hocolim} L_n$ exists we have that

$$\operatorname{colim} \operatorname{Hom}_{\mathcal{D}}(K, L_n) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(K, L)$$

is a bijection.

Proof. Consider the defining distinguished triangle

$$\bigoplus L_n \rightarrow \bigoplus L_n \rightarrow L \rightarrow \bigoplus L_n[1]$$

Apply the cohomological functor $\operatorname{Hom}_{\mathcal{D}}(K, -)$ (see Lemma 4.2). By elementary considerations concerning colimits of abelian groups we get the result. \square

32. Derived limits

In a triangulated category there is a notion of derived limit.

Definition 32.1. Let \mathcal{D} be a triangulated category. Let (K_n, f_n) be an inverse system of objects of \mathcal{D} . We say an object K is a *derived limit*, or a *homotopy limit* of the system (K_n) if the product $\prod K_n$ exists and there is a distinguished triangle

$$K \rightarrow \prod K_n \rightarrow \prod K_n \rightarrow K[1]$$

where the map $\prod K_n \rightarrow \prod K_n$ is given by $(k_n) \mapsto (k_n - f_{n+1}(k_{n+1}))$. If this is the case, then we sometimes indicate this by the notation $K = R\lim K_n$.

By TR3 a derived limit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived limit $R\lim K_n$ exists as soon as $\prod K_n$ exists. The derived category $D(\mathcal{A})$ of the category of abelian groups is an example. More generally, we have the following lemma.

Lemma 32.2. *Let \mathcal{A} be an abelian category. If \mathcal{A} has exact countable products, then $D(\mathcal{A})$ has countable products. In fact given a collection of complexes K_i^\bullet indexed by a countable index set I the termwise product $\prod K_i^\bullet$ is the product of K_i^\bullet in $D(\mathcal{A})$.*

Proof. Let L^\bullet be a complex. Suppose given maps $\alpha_i : L^\bullet \rightarrow K_i^\bullet$ in $D(\mathcal{A})$. This means there exist quasi-isomorphisms $s_i : K_i^\bullet \rightarrow M_i^\bullet$ of complexes and maps of complexes $f_i : L^\bullet \rightarrow M_i^\bullet$ such that $\alpha_i = s_i^{-1}f_i$. By assumption the map of complexes

$$s : \prod K_i^\bullet \rightarrow \prod M_i^\bullet$$

is a quasi-isomorphism. Hence setting $f = \prod f_i$ we see that $\alpha = s^{-1}f$ is a map in $D(\mathcal{A})$ whose composition with the projection $\prod K_i^\bullet \rightarrow K_i^\bullet$ is α_i . We omit the verification that α is unique. \square

The duals of Lemmas 31.3, 31.4, and 31.5 should be stated here and proved. However, we do not know any applications of these lemmas for now.

33. Generators of triangulated categories

In this section we briefly introduce a few of the different notions of a generator for a triangulated category. Our terminology is taken from [BV03] (except that we use “saturated” for what they call “épaisse”, see Definition 6.1).

Let \mathcal{D} be a triangulated category. Let E be an object of \mathcal{D} . Denote $\langle E \rangle_1$ the strictly full subcategory of \mathcal{D} consisting of objects in \mathcal{D} isomorphic to direct summands of finite direct sums

$$\bigoplus E[n_i]$$

of shifts of E . For $n > 1$ let $\langle E \rangle_n$ denote the full subcategory of \mathcal{D} consisting of objects of \mathcal{D} isomorphic to direct summands of objects X which fit into a distinguished triangle

$$A \rightarrow X \rightarrow B \rightarrow A[1]$$

where A and B are objects of $\langle E \rangle_{n-1}$. Each of the categories $\langle E \rangle_n$ is a strictly full additive subcategory of \mathcal{D} preserved under shifts and under taking summands. But, $\langle E \rangle_n$ is not necessarily closed under “taking cones”, hence not necessarily a triangulated subcategory. By the same token, the subcategory

$$\langle E \rangle = \bigcup_n \langle E \rangle_n$$

is a strictly full, saturated, triangulated subcategory of \mathcal{D} and it is the smallest such subcategory of \mathcal{D} containing the object E .

Definition 33.1. Let \mathcal{D} be a triangulated category. Let E be an object of \mathcal{D} .

- (1) We say E is a *classical generator* of \mathcal{D} if the smallest strictly full, saturated, triangulated subcategory of \mathcal{D} containing E is equal to \mathcal{D} , in other words, if $\langle E \rangle = \mathcal{D}$.
- (2) We say E is a *strong generator* of \mathcal{D} if $\langle E \rangle_n = \mathcal{D}$ for some $n \geq 1$.
- (3) We say E is a *weak generator* or a *generator* of \mathcal{D} if for any nonzero object K of \mathcal{D} there exists an integer n and a nonzero map $E \rightarrow K[n]$.

This definition can be generalized to the case of a family of objects.

Lemma 33.2. Let \mathcal{D} be a triangulated category. Let E, K be objects of \mathcal{D} . The following are equivalent

- (1) $\text{Hom}(E, K[i]) = 0$ for all $i \in \mathbf{Z}$,
- (2) $\text{Hom}(E', K) = 0$ for all $E' \in \langle E \rangle$.

Proof. The implication (2) \Rightarrow (1) is immediate. Conversely, assume (1). Then $\text{Hom}(X, K) = 0$ for all X in $\langle E \rangle_1$. Arguing by induction on n and using Lemma 4.2 we see that $\text{Hom}(X, K) = 0$ for all X in $\langle E \rangle_n$. \square

Lemma 33.3. Let \mathcal{D} be a triangulated category. Let E be an object of \mathcal{D} . If E is a classical generator of \mathcal{D} , then E is a generator.

Proof. Assume E is a classical generator. Let K be an object of \mathcal{D} such that $\text{Hom}(E, K[i]) = 0$ for all $i \in \mathbf{Z}$. By Lemma 33.2 $\text{Hom}(E', K) = 0$ for all E' in $\langle E \rangle$. However, since $\mathcal{D} = \langle E \rangle$ we conclude that $\text{id}_K = 0$, i.e., $K = 0$. \square

34. Compact objects

Here is the definition.

Definition 34.1. Let \mathcal{D} be an additive category with arbitrary direct sums. A *compact object* of \mathcal{D} is an object K such that the map

$$\bigoplus_{i \in I} \text{Hom}_{\mathcal{D}}(K, E_i) \longrightarrow \text{Hom}_{\mathcal{D}}(K, \bigoplus_{i \in I} E_i)$$

is bijective for any set I and objects $E_i \in \text{Ob}(\mathcal{D})$ parametrized by $i \in I$.

This notion turns out to be very useful in algebraic geometry. It is an intrinsic condition on objects that forces the objects to be, well, compact.

Lemma 34.2. Let \mathcal{D} be a (pre-)triangulated category with direct sums. Then the compact objects of \mathcal{D} form the objects of a Karoubian, saturated, strictly full, (pre-)triangulated subcategory \mathcal{D}_c of \mathcal{D} .

Proof. Let (X, Y, Z, f, g, h) be a distinguished triangle of \mathcal{D} with X and Y compact. Then it follows from Lemma 4.2 and the five lemma (Homology, Lemma 5.20) that Z is a compact object too. It is clear that if $X \oplus Y$ is compact, then X, Y are compact objects too. Hence \mathcal{D}_c is a saturated triangulated subcategory. Since \mathcal{D} is Karoubian by Lemma 4.12 we conclude that the same is true for \mathcal{D}_c . \square

Lemma 34.3. *Let \mathcal{D} be a triangulated category with direct sums. Let E_i , $i \in I$ be a family of compact objects of \mathcal{D} such that $\bigoplus E_i$ generates \mathcal{D} . Then every object X of \mathcal{D} can be written as*

$$X = \operatorname{hocolim} X_n$$

where X_1 is a direct sum of shifts of the E_i and each transition morphism fits into a distinguished triangle $Y_n \rightarrow X_n \rightarrow X_{n+1} \rightarrow Y_n[1]$ where Y_n is a direct sum of shifts of the E_i .

Proof. Set $X_1 = \bigoplus_{(i,m,\varphi)} E_i[m]$ where the direct sum is over all triples (i, m, φ) such that $i \in I$, $m \in \mathbf{Z}$ and $\varphi : E_i[m] \rightarrow X$. Then X_1 comes equipped with a canonical morphism $X_1 \rightarrow X$. Given $X_n \rightarrow X$ we set $Y_n = \bigoplus_{(i,m,\varphi)} E_i[m]$ where the direct sum is over all triples (i, m, φ) such that $i \in I$, $m \in \mathbf{Z}$, and $\varphi : E_i[m] \rightarrow X_n$ is a morphism such that $E_i[m] \rightarrow X_n \rightarrow X$ is zero. Choose a distinguished triangle $Y_n \rightarrow X_n \rightarrow X_{n+1} \rightarrow Y_n[1]$ and let $X_{n+1} \rightarrow X$ be any morphism such that $X_n \rightarrow X_{n+1} \rightarrow X$ is the given one; such a morphism exists by our choice of Y_n . We obtain a morphism $\operatorname{hocolim} X_n \rightarrow X$ by the construction of our maps $X_n \rightarrow X$. Choose a distinguished triangle

$$C \rightarrow \operatorname{hocolim} X_n \rightarrow X \rightarrow C[1]$$

Let $E_i[m] \rightarrow C$ be a morphism. Since E_i is compact, the composition $E_i[m] \rightarrow X \rightarrow \operatorname{hocolim} X_n$ factors through X_n for some n , say by $E_i[m] \rightarrow X_n$. Then the construction of Y_n shows that the composition $E_i[m] \rightarrow X_n \rightarrow X_{n+1}$ is zero. In other words, the composition $E_i[m] \rightarrow C \rightarrow \operatorname{hocolim} X_n$ is zero. This means that our morphism $E_i[m] \rightarrow C$ comes from a morphism $E_i[m] \rightarrow X[-1]$. The construction of X_1 then shows that such morphism lifts to $\operatorname{hocolim} X_n$ and we conclude that our morphism $E_i[m] \rightarrow C$ is zero. The assumption that $\bigoplus E_i$ generates \mathcal{D} implies that C is zero and the proof is done. \square

Lemma 34.4. *With assumptions and notation as in Lemma 34.3. If C is a compact object and $C \rightarrow X_n$ is a morphism, then there is a factorization $C \rightarrow E \rightarrow X_n$ where E is an object of $\langle E_{i_1} \oplus \dots \oplus E_{i_t} \rangle$ for some $i_1, \dots, i_t \in I$.*

Proof. We prove this by induction on n . The base case $n = 1$ is clear. If $n > 1$ consider the composition $C \rightarrow X_n \rightarrow Y_{n-1}[1]$. This can be factored through some $E'[1] \rightarrow Y_{n-1}[1]$ where E' is a finite direct sum of shifts of the E_i . Let $I' \subset I$ be the finite set of indices that occur in this direct sum. Thus we obtain

$$\begin{array}{ccccccc} E' & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & E'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_{n-1} & \longrightarrow & X_{n-1} & \longrightarrow & X_n & \longrightarrow & Y_{n-1}[1] \end{array}$$

By induction the morphism $C' \rightarrow X_{n-1}$ factors through $E'' \rightarrow X_{n-1}$ with E'' an object of $\langle \bigoplus_{i \in I''} E_i \rangle$ for some finite subset $I'' \subset I$. Choose a distinguished triangle

$$E' \rightarrow E'' \rightarrow E \rightarrow E'[1]$$

then E is an object of $\langle \bigoplus_{i \in I' \cup I''} E_i \rangle$. By construction and the axioms of a triangulated category we can choose morphisms $C \rightarrow E$ and a morphism $E \rightarrow X_n$ fitting into morphisms of triangles $(E', C', C) \rightarrow (E', E'', E)$ and $(E', E'', E) \rightarrow (Y_{n-1}, X_{n-1}, X_n)$. The composition $C \rightarrow E \rightarrow X_n$ may not equal the given morphism $C \rightarrow X_n$, but the compositions into Y_{n-1} are equal. Let $C \rightarrow X_{n-1}$ be

a morphism that lifts the difference. By induction assumption we can factor this through a morphism $E''' \rightarrow X_{n-1}$ with E'' an object of $\langle \bigoplus_{i \in I'''} E_i \rangle$ for some finite subset $I' \subset I$. Thus we see that we get a solution on considering $E \oplus E''' \rightarrow X_n$ because $E \oplus E'''$ is an object of $\langle \bigoplus_{i \in I' \cup I'' \cup I'''} E_i \rangle$. \square

Definition 34.5. Let \mathcal{D} be a triangulated category with arbitrary direct sums. We say \mathcal{D} is *compactly generated* if there exists a set E_i , $i \in I$ of compact objects such that $\bigoplus E_i$ generates \mathcal{D} .

The following proposition clarifies the relationship between classical generators and weak generators.

Proposition 34.6. *Let \mathcal{D} be a triangulated category with direct sums. Let E be a compact object of \mathcal{D} . The following are equivalent*

- (1) *E is a classical generator for \mathcal{D}_c and \mathcal{D} is compactly generated, and*
- (2) *E is a generator for \mathcal{D} .*

Proof. If E is a classical generator for \mathcal{D}_c , then $\mathcal{D}_c = \langle E \rangle$. It follows formally from the assumption that \mathcal{D} is compactly generated and Lemma 33.2 that E is a generator for \mathcal{D} .

The converse is more interesting. Assume that E is a generator for \mathcal{D} . Let X be a compact object of \mathcal{D} . Apply Lemma 34.3 with $I = \{1\}$ and $E_1 = E$ to write

$$X = \text{hocolim} X_n$$

as in the lemma. Since X is compact we find that $X \rightarrow \text{hocolim} X_n$ factors through X_n for some n (Lemma 31.5). Thus X is a direct summand of X_n . By Lemma 34.4 we see that X is an object of $\langle E \rangle$ and the lemma is proven. \square

35. Brown representability

A reference for the material in this section is [Nee96].

Lemma 35.1. *Let \mathcal{D} be a triangulated with direct sums which is compactly generated. Let $H : \mathcal{D} \rightarrow \text{Ab}$ be a contravariant cohomological functor which transforms direct sums into products. Then H is representable.*

Proof. Let E_i , $i \in I$ be a set of compact objects such that $\bigoplus_{i \in I} E_i$ generates \mathcal{D} . We may and do assume that the set of objects $\{E_i\}$ is preserved under shifts. Consider pairs (i, a) where $i \in I$ and $a \in H(E_i)$ and set

$$X_1 = \bigoplus_{(i,a)} E_i$$

Since $H(X_1) = \prod_{(i,a)} H(E_i)$ we see that $(a)_{(i,a)}$ defines an element $a_1 \in H(X_1)$. Set $H_1 = \text{Hom}_{\mathcal{D}}(-, X_1)$. By Yoneda's lemma (Categories, Lemma 3.5) the element a_1 defines a natural transformation $H_1 \rightarrow H$.

We are going to inductively construct X_n and transformations $a_n : H_n \rightarrow H$ where $H_n = \text{Hom}_{\mathcal{D}}(-, X_n)$. Namely, we apply the procedure above to the functor $\text{Ker}(H_n \rightarrow H)$ to get an object

$$K_{n+1} = \bigoplus_{(i,k), k \in \text{Ker}(H_n(E_i) \rightarrow H(E_i))} E_i$$

and a transformation $\mathrm{Hom}_{\mathcal{D}}(-, K_{n+1}) \rightarrow \mathrm{Ker}(H_n \rightarrow H)$. By Yoneda's lemma the composition $\mathrm{Hom}_{\mathcal{D}}(-, K_{n+1}) \rightarrow H_n$ gives a morphism $K_{n+1} \rightarrow X_n$. We choose a distinguished triangle

$$K_{n+1} \rightarrow X_n \rightarrow X_{n+1} \rightarrow K_{n+1}[1]$$

in \mathcal{D} . The element $a_n \in H(X_n)$ maps to zero in $H(K_{n+1})$ by construction. Since H is cohomological we can lift it to an element $a_{n+1} \in H(X_{n+1})$.

We claim that $X = \mathrm{hocolim} X_n$ represents H . First of all, by our definition of derived colimits and the fact that H transforms direct sums into products, we see that $H(X) = \lim H(X_n)$. Thus $a = (a_n)$ gives an element in $H(X)$ and hence a natural transformation $\mathrm{Hom}_{\mathcal{D}}(-, X) \rightarrow H$ such that

$$\mathrm{Hom}_{\mathcal{D}}(-, X_1) \rightarrow \mathrm{Hom}_{\mathcal{D}}(-, X_2) \rightarrow \mathrm{Hom}_{\mathcal{D}}(-, X_3) \rightarrow \dots \rightarrow \mathrm{Hom}_{\mathcal{D}}(-, X) \rightarrow H$$

commutes. For each i the map $\mathrm{Hom}_{\mathcal{D}}(E_i, X) \rightarrow H(E_i)$ is surjective, by construction of X_1 . On the other hand, by construction of $X_n \rightarrow X_{n+1}$ the kernel of $\mathrm{Hom}_{\mathcal{D}}(E_i, X_n) \rightarrow H(E_i)$ is killed by the map $\mathrm{Hom}_{\mathcal{D}}(E_i, X_n) \rightarrow \mathrm{Hom}_{\mathcal{D}}(E_i, X_{n+1})$. Since

$$\mathrm{Hom}_{\mathcal{D}}(E_i, X) = \mathrm{colim} \mathrm{Hom}_{\mathcal{D}}(E_i, X_n)$$

by Lemma 31.5 we see that $\mathrm{Hom}_{\mathcal{D}}(E_i, X) \rightarrow H(E_i)$ is injective.

To finish the proof, consider the subcategory

$$\mathcal{D}' = \{Y \in \mathrm{Ob}(\mathcal{D}) \mid \mathrm{Hom}_{\mathcal{D}}(Y[n], X) \rightarrow H(Y[n]) \text{ is an isomorphism for all } n\}$$

As $\mathrm{Hom}_{\mathcal{D}}(-, X) \rightarrow H$ is a transformation between cohomological functors, the subcategory \mathcal{D}' is a strictly full, saturated, triangulated subcategory of \mathcal{D} (details omitted; see proof of Lemma 6.3). Moreover, as both H and $\mathrm{Hom}_{\mathcal{D}}(-, X)$ transform direct sums into products, we see that direct sums of objects of \mathcal{D}' are in \mathcal{D}' . Thus derived colimits of objects of \mathcal{D}' are in \mathcal{D}' . Since $\{E_i\}$ is preserved under shifts, we see that E_i is an object of \mathcal{D}' for all i . It follows from Lemma 34.3 that $\mathcal{D}' = \mathcal{D}$ and the proof is complete. \square

Proposition 35.2. *Let \mathcal{D} be a triangulated with direct sums which is compactly generated. Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an additive functor of triangulated categories which transforms direct sums into direct sums. Then F has a right adjoint. If F is an exact functor, then so is the adjoint.*

Proof. For an object Y of \mathcal{D}' consider the contravariant functor

$$\mathcal{D} \rightarrow \mathrm{Ab}, \quad W \mapsto \mathrm{Hom}_{\mathcal{D}'}(F(W), Y)$$

This is a cohomological functors as F is exact and transforms direct sums into products as F transforms direct sums into direct sums. Thus by Lemma 35.1 we find an object X of \mathcal{D} such that $\mathrm{Hom}_{\mathcal{D}}(W, X) = \mathrm{Hom}_{\mathcal{D}'}(F(W), Y)$. The existence of the adjoint follows from Categories, Lemma 24.2. Exactness if F is exact, follows from Lemma 7.1. \square

36. Other chapters

Preliminaries

- | | |
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| (2) Conventions | (5) Topology |
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