LIMITS OF SCHEMES

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1. Introduction

In this chapter we put material related to limits of schemes. We mostly study limits of inverse systems over directed partially ordered sets with affine transition maps. We discuss absolute Noetherian approximation. We characterize schemes locally of finite presentation over a base as those whose associated functor of points is limit preserving. As an application of absolute Noetherian approximation we prove that the image of an affine under an integral morphism is affine. Moreover, we prove some very general variants of Chow's lemma. A basic reference is [DG67].

2. Directed limits of schemes with affine transition maps

In this section we construct the limit.

Lemma 2.1. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the schemes S_i are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. In fact S is affine and $S = \text{Spec}(\text{colim}_i R_i)$ with $R_i = \Gamma(S_i, \mathcal{O})$.*

Proof. Just define $S = \text{Spec}(\text{colim}_i R_i)$. It follows from Schemes, Lemma 6.4 that S is the limit even in the category of locally ringed spaces. $\square$

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Lemma 2.2. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. Moreover,*

1. *each of the morphisms $f_i : S \to S_i$ is affine,*
2. *for an element $0 \in I$ and any open subscheme $U_0 \subset S_0$ we have*
$$f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$$
in the category of schemes.

Proof. Choose an element $0 \in I$. Note that I is nonempty as the limit is directed. For every $i \geq 0$ consider the quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras $\mathcal{A}_i = f_{i0,*}\mathcal{O}_{S_i}$. Recall that $S_i = \underline{\mathrm{Spec}}_{S_0}(\mathcal{A}_i)$, see Morphisms, Lemma 13.3. Set $\mathcal{A} = \mathrm{colim}_{i \geq 0} \mathcal{A}_i$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras, see Schemes, Section 24. Set $S = \underline{\mathrm{Spec}}_{S_0}(\mathcal{A})$. By Morphisms, Lemma 13.5 we get for $i \geq 0$ morphisms $f_i : S \to S_i$ compatible with the transition morphisms. Note that the morphisms f_i are affine by Morphisms, Lemma 13.11 for example. By Lemma 2.1 above we see that for any affine open $U_0 \subset S_0$ the inverse image $U = f_0^{-1}(U_0) \subset S$ is the limit of the system of opens $U_i = f_{i0}^{-1}(U_0)$, $i \geq 0$ in the category of schemes.

Let T be a scheme. Let $g_i : T \to S_i$ be a compatible system of morphisms. To show that $S = \lim_i S_i$ we have to prove there is a unique morphism $g : T \to S$ with $g_i = f_i \circ g$ for all $i \in I$. For every $t \in T$ there exists an affine open $U_0 \subset S_0$ containing $g_0(t)$. Let $V \subset g_0^{-1}(U_0)$ be an affine open neighbourhood containing t. By the remarks above we obtain a unique morphism $g_V : V \to U = f_0^{-1}(U_0)$ such that $f_i \circ g_V = g_i|_{U_i}$ for all i. The open sets $V \subset T$ so constructed form a basis for the topology of T. The morphisms g_V glue to a morphism $g : T \to S$ because of the uniqueness property. This gives the desired morphism $g : T \to S$.

The final statement is clear from the construction of the limit above. $\square$

Lemma 2.3. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, Let $S = \lim_i S_i$. Let $0 \in I$. Suppose that T is a scheme over S_0. Then*
$$T \times_{S_0} S = \lim_{i \geq 0} T \times_{S_0} S_i$$

Proof. The right hand side is a scheme by Lemma 2.2. The equality is formal, see Categories, Lemma 14.9. $\square$

3. Descending properties

In this section we work in the following situation.

Situation 3.1. Let $S = \lim_{i \in I} S_i$ be the limit of a directed system of schemes with affine transition morphisms $f_{i'i} : S_{i'} \to S_i$ (Lemma 2.2). We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also choose an element $0 \in I$.

The type of result we are looking for is the following: If we have an object over S, then for some i there is a similar object over S_i.

Lemma 3.2. *In Situation 3.1.*

1. *We have $S_{set} = \lim_i S_{i,set}$ where S_{set} indicates the underlying set of the scheme S.*

(2) *We have* $S_{top} = \lim_i S_{i,top}$ *where* S_{top} *indicates the underlying topological space of the scheme* S.
(3) *If* $s, s' \in S$ *and* s' *is not a specialization of* s *then for some* $i \in I$ *the image* $s'_i \in S_i$ *of* s' *is not a specialization of the image* $s_i \in S_i$ *of* s.
(4) *Add more easy facts on topology of* S *here. (Requirement: whatever is added should be easy in the affine case.)*

Proof. Proof of (1). Pick $i \in I$. Take $U_i \subset S_i$ an affine open. Denote $U_{i'} = f_{i'i}^{-1}(U_i)$ and $U = f_i^{-1}(U_i)$. Suppose we can show that $U_{set} = \lim_{i' \geq i} U_{i',set}$. Then assertion (1) follows by a simple argument using an affine covering of S_i. Hence we may assume all S_i and S affine. This reduces us to the following algebra question: Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = \mathrm{colim}_i A_i$ with canonical maps $\varphi_i : A_i \to A$. Then

$$\mathrm{Spec}(A) = \lim_i \mathrm{Spec}(A_i)$$

Namely, suppose that we are given primes $\mathfrak{p}_i \subset A_i$ such that $\mathfrak{p}_i = \varphi_{ii'}^{-1}(\mathfrak{p}_{i'})$ for all $i' \geq i$. Then we simply set

$$\mathfrak{p} = \{x \in A \mid \exists i, x_i \in \mathfrak{p}_i \text{ with } \varphi_i(x_i) = x\}$$

It is clear that this is an ideal and has the property that $\varphi_i^{-1}(\mathfrak{p}) = \mathfrak{p}_i$. Then it follows easily that it is a prime ideal as well. This proves (1).

Proof of (2). Choose an i and a finite affine open covering $S_i = U_{1,i} \cup \ldots \cup U_{n,i}$. If we can show the topology on $f_i^{-1}(U_{k,i}) = \lim_{i' \geq i} f_{i'i}^{-1}(U_{k,i})$ is the limit topology, then the same is true for S. Hence we may assume that S and S_i are affine. Say $S_i = \mathrm{Spec}(A_i)$ and $S = \mathrm{Spec}(A)$ with $A = \mathrm{colim}\, A_i$. A basis for the topology of $\mathrm{Spec}(A)$ is given by the standard opens $D(g)$, $g \in A$. Since each $g \in A$ is the image of some $g_i \in A_i$ for some i we see that $D(g)$ is the inverse image of $D(g_i)$ by f_i. The deisred result now follows from the criterion of Topology, Lemma 13.3.

Proof of (3). Pick $i \in I$. Pick an affine open $U_i \subset S_i$ containing $f_i(s')$. If $f_i(s) \notin S_i$ then we are done. Hence reduce to the affine case by considering the inverse images of U_i as above. This reduces us to the following algebra question: Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = \mathrm{colim}_i A_i$ with canonical maps $\varphi_i : A_i \to A$. Suppose given primes $\mathfrak{p}$, $\mathfrak{p}'$ of A. Suppose that $\mathfrak{p} \not\subset \mathfrak{p}'$. Then for some i we have $\varphi_i^{-1}(\mathfrak{p}) \not\subset \varphi_i^{-1}(\mathfrak{p}')$. This is clear. □

Lemma 3.3. *In Situation 3.1. Suppose that* $\mathcal{F}_0$ *is a quasi-coherent sheaf on* S_0. *Set* $\mathcal{F}_i = f_{i0}^*\mathcal{F}_0$ *for* $i \geq 0$ *and set* $\mathcal{F} = f_0^*\mathcal{F}_0$. *Then*

$$\Gamma(S, \mathcal{F}) = \mathrm{colim}_{i \geq 0} \Gamma(S_i, \mathcal{F}_i)$$

Proof. Write $\mathcal{A}_j = f_{i0,*}\mathcal{O}_{S_i}$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras (see Morphisms, Lemma 13.5) and S_i is the relative spectrum of $\mathcal{A}_i$ over S_0. In the proof of Lemma 2.2 we constructed S as the relative spectrum of $\mathcal{A} = \mathrm{colim}_{i \geq 0} \mathcal{A}_i$ over S_0. Set

$$\mathcal{M}_i = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}_i$$

and

$$\mathcal{M} = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}.$$

Then we have $f_{i0,*}\mathcal{F}_i = \mathcal{M}_i$ and $f_{0,*}\mathcal{F} = \mathcal{M}$. Since $\mathcal{A}$ is the colimit of the sheaves $\mathcal{A}_i$ and since tensor product commutes with directed colimits, we conclude that $\mathcal{M} = \mathrm{colim}_{i \geq 0} \mathcal{M}_i$. Since S_0 is quasi-compact and quasi-separated we see that

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= \Gamma(S_0, \mathcal{M}) \\ &= \Gamma(S_0, \mathrm{colim}_{i \geq 0} \mathcal{M}_i) \\ &= \mathrm{colim}_{i \geq 0} \Gamma(S_0, \mathcal{M}_i) \\ &= \mathrm{colim}_{i \geq 0} \Gamma(S_i, \mathcal{F}_i) \end{aligned}$$

see Sheaves, Lemma 29.1 and Topology, Lemma 26.1 for the middle equality. □

Lemma 3.4. *In Situation 3.1. If all the schemes S_i are nonempty, then the limit $S = \lim_i S_i$ is nonempty.*

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience write $S_0 = S_{i_0}$ and $i_0 = 0$. Choose an affine open covering $S_0 = \bigcup_{j=1,\ldots,m} U_j$. Since I is directed there exists a $j \in \{1, \ldots, m\}$ such that $f_{i0}^{-1}(U_j) \neq \emptyset$ for all $i \geq 0$. Hence $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is not empty since a directed colimit of nonzero rings is nonzero (because $1 \neq 0$). As $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is an open subscheme of the limit we win. □

Lemma 3.5. *In Situation 3.1. Suppose for each i we are given a nonempty closed subset $Z_i \subset S_i$ with $f_{ii'}(Z_i) \subset Z_{i'}$. Then there exists a point $s \in S$ with $f_i(s) \in Z_i$ for all i.*

Proof. Let $Z_i \subset S_i$ also denote the reduced closed subscheme associated to Z_i, see Schemes, Definition 12.5. A closed immersion is affine, and a composition of affine morphisms is affine (see Morphisms, Lemmas 13.9 and 13.7), and hence $Z_i \to S_{i'}$ is affine when $i \geq i'$. We conclude that the morphism $f_{ii'} : Z_i \to Z_{i'}$ is affine by Morphisms, Lemma 13.11. Each of the schemes Z_i is quasi-compact as a closed subscheme of a quasi-compact scheme. Hence we may apply Lemma 3.4 to see that $Z = \lim_i Z_i$ is nonempty. Since there is a canonical morphism $Z \to S$ we win. □

Lemma 3.6. *In Situation 3.1. Suppose we are given an i and a morphism $T \to S_i$ such that*

1. *$T \times_{S_i} S = \emptyset$, and*
2. *T is quasi-compact.*

Then $T \times_{S_i} S_{i'} = \emptyset$ for all sufficiently large i'.

Proof. By Lemma 2.3 we see that $T \times_{S_i} S = \lim_{i' \geq i} T \times_{S_i} S_{i'}$. Hence the result follows from Lemma 3.4. □

Lemma 3.7. *In Situation 3.1. Suppose we are given an i and a locally constructible subset $E \subset S_i$ such that $f_i(S) \subset E$. Then $f_{ii'}(S_{i'}) \subset E$ for all sufficiently large i'.*

Proof. Writing S_i as a finite union of open affine subschemes reduces the question to the case that S_i is affine and E is constructible, see Lemma 2.2 and Properties, Lemma 2.1. In this case the complement $S_i \setminus E$ is constructible too. Hence there exists an affine scheme T and a morphism $T \to S_i$ whose image is $S_i \setminus E$, see Algebra, Lemma 28.3. By Lemma 3.6 we see that $T \times_{S_i} S_{i'}$ is empty for all sufficiently large i', and hence $f_{ii'}(S_{i'}) \subset E$ for all sufficiently large i'. □

Lemma 3.8. *In Situation 3.1 we have the following:*

(1) *Given any quasi-compact open* $V \subset S = \lim_i S_i$ *there exists an* $i \in I$ *and a quasi-compact open* $V_i \subset S_i$ *such that* $f_i^{-1}(V_i) = V$.
(2) *Given* $V_i \subset S_i$ *and* $V_{i'} \subset S_{i'}$ *quasi-compact opens such that* $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$ *there exists an index* $i'' \geq i, i'$ *such that* $f_{i''i}^{-1}(V_i) = f_{i''i'}^{-1}(V_{i'})$.
(3) *If* $V_{1,i}, \ldots, V_{n,i} \subset S_i$ *are quasi-compact opens and* $S = f_i^{-1}(V_{1,i}) \cup \ldots \cup f_i^{-1}(V_{n,i})$ *then* $S_{i'} = f_{i'i}^{-1}(V_{1,i}) \cup \ldots \cup f_{i'i}^{-1}(V_{n,i})$ *for some* $i' \geq i$.

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Choose an affine open covering $S_0 = U_{1,0} \cup \ldots \cup U_{m,0}$. Denote $U_{j,i} \subset S_i$ the inverse image of $U_{j,0}$ under the transition morphism for $i \geq 0$. Denote U_j the inverse image of $U_{j,0}$ in S. Note that $U_j = \lim_i U_{j,i}$ is a limit of affine schemes.

We first prove the uniqueness statement: Let $V_i \subset S_i$ and $V_{i'} \subset S_{i'}$ quasi-compact opens such that $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$. It suffices to show that $f_{i''i}^{-1}(V_i \cap U_{j,i''})$ and $f_{i''i'}^{-1}(V_{i'} \cap U_{j,i''})$ become equal for i'' large enough. Hence we reduce to the case of a limit of affine schemes. In this case write $S = \text{Spec}(R)$ and $S_i = \text{Spec}(R_i)$ for all $i \in I$. We may write $V_i = S_i \setminus V(h_1, \ldots, h_m)$ and $V_{i'} = S_{i'} \setminus V(g_1, \ldots, g_n)$. The assumption means that the ideals $\sum g_j R$ and $\sum h_j R$ have the same radical in R. This means that $g_j^N = \sum a_{jj'} h_{j'}$ and $h_j^N = \sum b_{jj'} g_{j'}$ for some $N \gg 0$ and $a_{jj'}$ and $b_{jj'}$ in R. Since $R = \text{colim}_i R_i$ we can chose an index $i'' \geq i$ such that the equations $g_j^N = \sum a_{jj'} h_{j'}$ and $h_j^N = \sum b_{jj'} g_{j'}$ hold in $R_{i''}$ for some $a_{jj'}$ and $b_{jj'}$ in $R_{i''}$. This implies that the ideals $\sum g_j R_{i''}$ and $\sum h_j R_{i''}$ have the same radical in $R_{i''}$ as desired.

We prove existence. We may apply the uniqueness statement to the limit of schemes $U_{j_1} \cap U_{j_2} = \lim_i U_{j_1,i} \cap U_{j_2,i}$ since these are still quasi-compact due to the fact that the S_i were assumed quasi-separated. Hence it is enough to prove existence in the affine case. In this case write $S = \text{Spec}(R)$ and $S_i = \text{Spec}(R_i)$ for all $i \in I$. Then $V = S \setminus V(g_1, \ldots, g_n)$ for some $g_1, \ldots, g_n \in R$. Choose any i large enough so that each of the g_j comes from an element $g_{j,i} \in R_i$ and take $V_i = S_i \setminus V(g_{1,i}, \ldots, g_{n,i})$.

The statement on coverings follows from the uniqueness statement for the opens $V_{1,i} \cup \ldots \cup V_{n,i}$ and S_i of S_i. $\square$

Lemma 3.9. *In Situation 3.1 if* S *is quasi-affine, then for some* $i_0 \in I$ *the schemes* S_i *for* $i \geq i_0$ *are quasi-affine.*

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Let $s \in S$. We may choose an affine open $U_0 \subset S_0$ containing $f_0(s)$. Since S is quasi-affine we may choose an element $a \in \Gamma(S, \mathcal{O}_S)$ such that $s \in D(a) \subset f_0^{-1}(U_0)$, and such that $D(a)$ is affine. By Lemma 3.3 there exists an $i \geq 0$ such that a comes from an element $a_i \in \Gamma(S_i, \mathcal{O}_{S_i})$. For any index $j \geq i$ we denote a_j the image of a_i in the global sections of the structure sheaf of S_j. Consider the opens $D(a_j) \subset S_j$ and $U_j = f_{j0}^{-1}(U_0)$. Note that U_j is affine and $D(a_j)$ is a quasi-compact open of S_j, see Properties, Lemma 24.4 for example. Hence we may apply Lemma 3.8 to the opens U_j and $U_j \cup D(a_j)$ to conclude that $D(a_j) \subset U_j$ for some $j \geq i$. For such an index j we see that $D(a_j) \subset S_j$ is an affine open (because $D(a_j)$ is a standard affine open of the affine open U_j) containing the image $f_j(s)$.

We conclude that for every $s \in S$ there exist an index $i \in I$, and a global section $a \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that $D(a) \subset S_i$ is an affine open containing $f_i(s)$. Because S is quasi-compact we may choose a single index $i \in I$ and global sections $a_1, \ldots, a_m \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that each $D(a_j) \subset S_i$ is affine open and such that $f_i : S \to S_i$ has image contained in the union $W_i = \bigcup_{j=1,\ldots,m} D(a_j)$. For $i' \geq i$ set $W_{i'} = f_{i'i}^{-1}(W_i)$. Since $f_i^{-1}(W_i)$ is all of S we see (by Lemma 3.8 again) that for a suitable $i' \geq i$ we have $S_{i'} = W_{i'}$. Thus we may replace i by i' and assume that $S_i = \bigcup_{j=1,\ldots,m} D(a_j)$. This implies that $\mathcal{O}_{S_i}$ is an ample invertible sheaf on S_i (see Properties, Definition 24.1) and hence that S_i is quasi-affine, see Properties, Lemma 25.1. Hence we win. $\square$

Lemma 3.10. *In Situation 3.1 if S is affine, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are affine.*

Proof. By Lemma 3.9 we may assume that S_0 is quasi-affine for some $0 \in I$. Set $R_0 = \Gamma(S_0, \mathcal{O}_{S_0})$. Then S_0 is a quasi-compact open of $T_0 = \text{Spec}(R_0)$. Denote $j_0 : S_0 \to T_0$ the corresponding quasi-compact open immersion. For $i \geq 0$ set $\mathcal{A}_i = f_{0i,*}\mathcal{O}_{S_i}$. Since f_{0i} is affine we see that $S_i = \underline{\text{Spec}}_{S_0}(\mathcal{A}_i)$. Set $T_i = \underline{\text{Spec}}_{T_0}(j_{0,*}\mathcal{A}_i)$. Then $T_i \to T_0$ is affine, hence T_i is affine. Thus T_i is the spectrum of

$$R_i = \Gamma(T_0, j_{0,*}\mathcal{A}_i) = \Gamma(S_0, \mathcal{A}_i) = \Gamma(S_i, \mathcal{O}_{S_i}).$$

Write $S = \text{Spec}(R)$. We have $R = \text{colim}_i R_i$ by Lemma 3.3. Hence also $S = \lim_i T_i$. As formation of the relative spectrum commutes with base change, the inverse image of the open $S_0 \subset T_0$ in T_i is S_i. Let $Z_0 = T_0 \setminus S_0$ and let $Z_i \subset T_i$ be the inverse image of Z_0. As $S_i = T_i \setminus Z_i$, it suffices to show that Z_i is empty for some i. Assume Z_i is nonempty for all i to get a contradiction. By Lemma 3.5 there exists a point s of $S = \lim T_i$ which maps to a point of Z_i for every i. But $S = \lim_i S_i$, and hence we arrive at a contradiction by Lemma 3.2. $\square$

Lemma 3.11. *In Situation 3.1 if S is separated, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are separated.*

Proof. Choose a finite affine open covering $S_0 = U_{0,1} \cup \ldots \cup U_{0,m}$. Set $U_{i,j} \subset S_i$ and $U_j \subset S$ equal to the inverse image of $U_{0,j}$. Note that $U_{i,j}$ and U_j are affine. As S is separated the intersections $U_{j_1} \cap U_{j_2}$ are affine. Since $U_{j_1} \cap U_{j_2} = \lim_{i \geq 0} U_{i,j_1} \cap U_{i,j_2}$ we see that $U_{i,j_1} \cap U_{i,j_2}$ is affine for large i by Lemma 3.10. To show that S_i is separated for large i it now suffices to show that

$$\mathcal{O}_{S_i}(V_{i,j_1}) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_{S_i}(V_{i,j_2}) \longrightarrow \mathcal{O}_{S_i}(V_{i,j_1} \cap V_{i,j_2})$$

is surjective for large i (Schemes, Lemma 21.8).

To get rid of the annoying indices, assume we have affine opens $U, V \subset S_0$ such that $U \cap V$ is affine too. Let $U_i, V_i \subset S_i$, resp. $U, V \subset S$ be the inverse images. We have to show that $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i) \to \mathcal{O}(U_i \cap V_i)$ is surjective for i large enough and we know that $\mathcal{O}(U_) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is surjective. Note that $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0) \to \mathcal{O}(U_0 \cap V_0)$ is of finite type, as the diagonal morphism $S_i \to S_i \times S_i$ is an immersion (Schemes, Lemma 21.2) hence locally of finite type (Morphisms, Lemmas 16.2 and 16.5). Thus we can choose elements $f_{0,1}, \ldots, f_{0,n} \in \mathcal{O}(U_0 \cap V_0)$ which generate $\mathcal{O}(U_0 \cap V_0)$ over

$\mathcal{O}(U_0) \otimes \mathcal{O}(V_0)$. Observe that for $i \geq 0$ the diagram of schemes

$$\begin{tikzcd} U_i \cap V_i \arrow[r] \arrow[d] & U_i \arrow[d] \\ U_0 \cap V_0 \arrow[r] & U_0 \end{tikzcd}$$

is cartesian. Thus we see that the images $f_{i,1}, \ldots, f_{i,n} \in \mathcal{O}(U_i \cap V_i)$ generate $\mathcal{O}(U_i \cap V_i)$ over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_0)$ and a fortiori over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$. By assumption the images $f_1, \ldots, f_n \in \mathcal{O}(U \otimes V)$ are in the image of the map $\mathcal{O}(U) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$. Since $\mathcal{O}(U) \otimes \mathcal{O}(V) = \operatorname{colim} \mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$ we see that they are in the image of the map at some finite level and the lemma is proved. $\square$

Lemma 3.12. *In Situation 3.1 let $\mathcal{L}_0$ be an invertible sheaf of modules on S_0. If the pullback $\mathcal{L}$ to S is ample, then for some $i \in I$ the pullback $\mathcal{L}_i$ to S_i is ample.*

Proof. The assumption means there are finitely many sections $s_1, \ldots, s_m \in \Gamma(S, \mathcal{L})$ such that S_{s_j} is affine and such that $S = \bigcup S_{s_j}$, see Properties, Definition 24.1. By Lemma 3.3 we can find an $i \in I$ and sections $s_{i,j} \in \Gamma(S_i, \mathcal{L}_i)$ mapping to s_j. By Lemma 3.10 we may, after increasing i, assume that $(S_i)_{s_{i,j}}$ is affine for $j = 1, \ldots, m$. By Lemma 3.8 we may, after increasing i a last time, assume that $S_i = \bigcup (S_i)_{s_{i,j}}$. Then $\mathcal{L}_i$ is ample by definition. $\square$

Lemma 3.13. *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Let $Y \to X$ be a morphism of schemes over S.*

1. *If $Y \to X$ is a closed immersion, X_i quasi-compact, and Y locally of finite type over S, then $Y \to X_i$ is a closed immersion for i large enough.*
2. *If $Y \to X$ is an immersion, X_i quasi-separated, $Y \to S$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is an immersion for i large enough.*

Proof. Proof of (1). Choose $0 \in I$ and a finite affine open covering $X_0 = U_{0,1} \cup \ldots \cup U_{0,m}$ with the property that $U_{0,j}$ maps into an affine open $W_j \subset S$. Let $V_j \subset Y$, resp. $U_{i,j} \subset X_i$, $i \geq 0$, resp. $U_j \subset X$ be the inverse image of $U_{0,j}$. It suffices to prove that $V_j \to U_{i,j}$ is a closed immersion for i sufficiently large and we know that $V_j \to U_j$ is a closed immersion. Thus we reduce to the following algebra fact: If $A = \operatorname{colim} A_i$ is a directed colimit of R-algebras, $A \to B$ is a surjection of R-algebras, and B is a finitely generated R-algebra, then $A_i \to B$ is surjective for i sufficiently large.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open $X'_0 \subset X_0$ such that $Y \to X_0$ factors through X'_0. After replacing X_i by the inverse image of X'_0 for $i \geq 0$ we may assume all X'_i are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ (U exists as Y is quasi-compact). By Lemma 3.8 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some i. Thus (2) holds. $\square$

Lemma 3.14. *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

1. *S quasi-separated,*
2. *X_i quasi-compact and quasi-separated,*

(3) *$X \to S$ separated.*

Then $X_i \to S$ is separated for all i large enough.

Proof. Let $0 \in I$. Note that I is nonempty as the limit is directed. As X_0 is quasi-compact we can find finitely many affine opens $U_1, \ldots, U_n \subset S$ such that $X_0 \to S$ maps into $U_1 \cup \ldots \cup U_n$. Denote $h_i : X_i \to S$ the structure morphism. It suffices to check that for some $i \geq 0$ the morphisms $h_i^{-1}(U_j) \to U_j$ are separated for $j = 1, \ldots, n$. Since S is quasi-separated the morphisms $U_j \to S$ are quasi-compact. Hence $h_i^{-1}(U_j)$ is quasi-compact and quasi-separated. In this way we reduce to the case S affine. In this case we have to show that X_i is separated and we know that X is separated. Thus the lemma follows from Lemma 3.11. □

Lemma 3.15. *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-compact and quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *$X \to S$ affine.*

Then $X_i \to S$ is affine for i large enough.

Proof. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. For each j the scheme $f^{-1}(V_j) = \lim_i f_i^{-1}(V_j)$ is affine (as a finite morphism is affine by definition). Hence by Lemma 3.10 there exists an $i \in I$ such that each $f_i^{-1}(V_j)$ is affine. In other words, $f_i : X_i \to S$ is affine for i large enough, see Morphisms, Lemma 13.3. □

Lemma 3.16. *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-compact and quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *the transition morphisms $X_{i'} \to X_i$ are finite,*
(4) *$X_i \to S$ locally of finite type*
(5) *$X \to S$ integral.*

Then $X_i \to S$ is finite for i large enough.

Proof. By Lemma 3.15 we may assume $X_i \to S$ is affine for all i. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. It suffices to show that there exists an i such that $f_i^{-1}(V_j)$ is finite over V_j for $j = 1, \ldots, m$ (Morphisms, Lemma 44.3). Namely, for $i' \geq i$ the composition $X_{i'} \to X_i \to S$ will be finite as a composition of finite morphisms (Morphisms, Lemma 44.5). This reduces us to the affine case: Let R be a ring and $A = \operatorname{colim} A_i$ with $R \to A$ integral and $A_i \to A_{i'}$ finite for all $i \leq i'$. Moreover $R \to A_i$ is of finite type for all i. Goal: Show that A_i is finite over R for some i. To prove this choose an $i \in I$ and pick generators $x_1, \ldots, x_m \in A_i$ of A_i as an R-algebra. Since A is integral over R we can find monic polynomials $P_j \in R[T]$ such that $P_j(x_j) = 0$ in A. Thus there exists an $i' \geq i$ such that $P_j(x_j) = 0$ in $A_{i'}$ for $j = 1, \ldots, m$. Then the image A'_i of A_i in $A_{i'}$ is finite over R by Algebra, Lemma 35.5. Since $A'_i \subset A_{i'}$ is finite too we conclude that $A_{i'}$ is finite over R by Algebra, Lemma 7.3. □

Lemma 3.17. *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

1. *S quasi-compact and quasi-separated,*
2. *X_i quasi-compact and quasi-separated,*
3. *the transition morphisms $X_{i'} \to X_i$ are closed immersions,*
4. *$X_i \to S$ locally of finite type*
5. *$X \to S$ a closed immersion.*

Then $X_i \to S$ is a closed immersion for i large enough.

Proof. By Lemma 3.15 we may assume $X_i \to S$ is affine for all i. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. It suffices to show that there exists an i such that $f_i^{-1}(V_j)$ is a closed subscheme of V_j for $j = 1, \ldots, m$ (Morphisms, Lemma 2.1). This reduces us to the affine case: Let R be a ring and $A = \operatorname{colim} A_i$ with $R \to A$ surjective and $A_i \to A_{i'}$ surjective for all $i \leq i'$. Moreover $R \to A_i$ is of finite type for all i. Goal: Show that $R \to A_i$ is surjective for some i. To prove this choose an $i \in I$ and pick generators $x_1, \ldots, x_m \in A_i$ of A_i as an R-algebra. Since $R \to A$ is surjective we can find $r_j \in R$ such that r_j maps to x_j in A. Thus there exists an $i' \geq i$ such that r_j maps to the image of x_j in $A_{i'}$ for $j = 1, \ldots, m$. Since $A_i \to A_{i'}$ is surjective this implies that $R \to A_{i'}$ is surjective. □

4. Absolute Noetherian Approximation

A nice reference for this section is Appendix C of the article by Thomason and Trobaugh [TT90]. See Categories, Section 21 for our conventions regarding directed systems. We will use the existence result and properties of the limit from Section 2 without further mention.

Lemma 4.1. *Let W be a quasi-affine scheme of finite type over $\mathbf{Z}$. Suppose $W \to \operatorname{Spec}(R)$ is an open immersion into an affine scheme. There exists a finite type $\mathbf{Z}$-algebra $A \subset R$ which induces an open immersion $W \to \operatorname{Spec}(A)$. Moreover, R is the directed colimit of such subalgebras.*

Proof. Choose an affine open covering $W = \bigcup_{i=1,\ldots,n} W_i$ such that each W_i is a standard affine open in $\operatorname{Spec}(R)$. In other words, if we write $W_i = \operatorname{Spec}(R_i)$ then $R_i = R_{f_i}$ for some $f_i \in R$. Choose finitely many $x_{ij} \in R_i$ which generate R_i over $\mathbf{Z}$. Pick an $N \gg 0$ such that each $f_i^N x_{ij}$ comes from an element of R, say $y_{ij} \in R$. Set A equal to the $\mathbf{Z}$-algebra generated by the f_i and the y_{ij} and (optionally) finitely many additional elements of R. Then A works. Details omitted. □

Lemma 4.2. *Suppose given a cartesian diagram of rings*

$$\begin{tikzcd} B \arrow[r, "s"] & R \\ B' \arrow[u] \arrow[r] & R' \arrow[u, "t"'] \end{tikzcd}$$

Let $W' \subset \operatorname{Spec}(R')$ be an open of the form $W' = D(f_1) \cup \ldots \cup D(f_n)$ such that $t(f_i) = s(g_i)$ for some $g_i \in B$ and $B_{g_i} \cong R_{s(g_i)}$. Then $B' \to R'$ induces an open immersion of W' into $\operatorname{Spec}(B')$.

Proof. Set $h_i = (g_i, f_i) \in B'$. More on Algebra, Lemma 4.3 shows that $(B')_{h_i} \cong (R')_{f_i}$ as desired. □

The following lemma is a precise statement of Noetherian approximation.

Lemma 4.3. *Let S be a quasi-compact and quasi-separated scheme. Let $V \subset S$ be a quasi-compact open. Let I be a directed partially ordered set and let $(V_i, f_{ii'})$ be an inverse system of schemes over I with affine transition maps, with each V_i of finite type over $\mathbf{Z}$, and with $V = \lim V_i$. Then there exist*

1. *a directed partially ordered set J,*
2. *an inverse system of schemes $(S_j, g_{jj'})$ over J,*
3. *an order preserving map $\alpha : J \to I$,*
4. *open subschemes $V'_j \subset S_j$, and*
5. *isomorphisms $V'_j \to V_{\alpha(j)}$*

such that

1. *the transition morphisms $g_{jj'} : S_j \to S_{j'}$ are affine,*
2. *each S_j is of finite type over $\mathbf{Z}$,*
3. *$g_{jj'}^{-1}(V_{j'}) = V_j$,*
4. *$S = \lim S_j$ and $V = \lim V_j$, and*
5. *the diagrams*

$$\begin{tikzcd} V \arrow[d] \arrow[dr] & \\ V'_j \arrow[r] & V_{\alpha(j)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} V_j \arrow[r] \arrow[d] & V_{\alpha(j)} \arrow[d] \\ V_{j'} \arrow[r] & V_{\alpha(j')} \end{tikzcd}$$

   are commutative.

Proof. Set $Z = S \setminus V$. Choose affine opens $U_1, \ldots, U_m \subset S$ such that $Z \subset \bigcup_{l=1,\ldots,m} U_l$. Consider the opens

$$V \subset V \cup U_1 \subset V \cup U_1 \cup U_2 \subset \ldots \subset V \cup \bigcup\nolimits_{l=1,\ldots,m} U_l = S$$

If we can prove the lemma successively for each of the cases

$$V \cup U_1 \cup \ldots \cup U_l \subset V \cup U_1 \cup \ldots \cup U_{l+1}$$

then the lemma will follow for $V \subset S$. In each case we are adding one affine open. Thus we may assume

1. $S = U \cup V$,
2. U affine open in S,
3. V quasi-compact open in S, and
4. $V = \lim_i V_i$ with $(V_i, f_{ii'})$ an inverse system over a directed set I, each $f_{ii'}$ affine and each V_i of finite type over $\mathbf{Z}$.

Set $W = U \cap V$. As S is quasi-separated, this is a quasi-compact open of V. By Lemma 3.8 (and after shrinking I) we may assume that there exist opens $W_i \subset V_i$ such that $f_{ij}^{-1}(W_j) = W_i$ and such that $f_i^{-1}(W_i) = W$. Since W is a quasi-compact open of U it is quasi-affine. Hence we may assume (after shrinking I again) that W_i is quasi-affine for all i, see Lemma 3.9.

Write $U = \mathrm{Spec}(B)$. Set $R = \Gamma(W, \mathcal{O}_W)$, and $R_i = \Gamma(W_i, \mathcal{O}_{W_i})$. By Lemma 3.3 we have $R = \mathrm{colim}_i R_i$. Now we have the maps of rings

$$\begin{tikzcd} B \arrow[r, "s"'] & R \\ & R_i \arrow[u, "t_i"'] \end{tikzcd}$$

We set $B_i = \{(b, r) \in B \times R_i \mid s(b) = t_i(t)\}$ so that we have a cartesian diagram

$$\begin{tikzcd} B \arrow[r, "s"'] & R \\ B_i \arrow[u] \arrow[r] & R_i \arrow[u, "t_i"'] \end{tikzcd}$$

for each i. The transition maps $R_i \to R_{i'}$ induce maps $B_i \to B_{i'}$. It is clear that $B = \mathrm{colim}_i B_i$. In the next paragraph we show that for all sufficiently large i the composition $W_i \to \mathrm{Spec}(R_i) \to \mathrm{Spec}(B_i)$ is an open immersion.

As W is a quasi-compact open of $U = \mathrm{Spec}(B)$ we can find a finitely many elements $g_l \in B$, $l = 1, \ldots, m$ such that $D(g_l) \subset W$ and such that $W = \bigcup_{l=1,\ldots,m} D(g_l)$. Note that this implies $D(g_l) = W_{s(g_l)}$ as open subsets of U, where $W_{s(g_l)}$ denotes the largest open subset of W on which $s(g_l)$ is invertible. Hence

$$B_{g_l} = \Gamma(D(g_l), \mathcal{O}_U) = \Gamma(W_{s(g_l)}, \mathcal{O}_W) = R_{s(g_l)},$$

where the last equality is Properties, Lemma 15.2. Since $W_{s(g_l)}$ is affine this also implies that $D(s(g_l)) = W_{s(g_l)}$ as open subsets of $\mathrm{Spec}(R)$. Since $R = \mathrm{colim}_i R_i$ we can (after shrinking I) assume there exist $g_{l,i} \in R_i$ for all $i \in I$ such that $s(g_l) = t_i(g_{l,i})$. Of course we choose the $g_{l,i}$ such that $g_{l,i}$ maps to $g_{l,i'}$ under the transition maps $R_i \to R_{i'}$. Then, by Lemma 3.8 we can (after shrinking I again) assume the corresponding opens $D(g_{l,i}) \subset \mathrm{Spec}(R_i)$ are contained in W_i, $j = 1, \ldots, m$ and cover W_i. We conclude that the morphism $W_i \to \mathrm{Spec}(R_i) \to \mathrm{Spec}(B_i)$ is an open immersion, see Lemma 4.2

By Lemma 4.1 we can write B_i as a directed colimit of subalgebras $A_{i,p} \subset B_i$, $p \in P_i$ each of finite type over $\mathbf{Z}$ and such that W_i is identified with an open subscheme of $\mathrm{Spec}(A_{i,p})$. Let $S_{i,p}$ be the scheme obtained by glueing V_i and $\mathrm{Spec}(A_{i,p})$ along the open W_i, see Schemes, Section 14. Here is the resulting commutative diagram of schemes:

$$\begin{tikzcd} & & V \arrow[dll] \arrow[d] & W \arrow[l] \arrow[dl] \arrow[d] \\ V_i \arrow[d] & W_i \arrow[l] \arrow[d] & S \arrow[dll] & U \arrow[l] \arrow[dll] \\ S_{i,p} & \mathrm{Spec}(A_{i,p}) \arrow[l] & & \end{tikzcd}$$

The morphism $S \to S_{i,p}$ arises because the upper right square is a pushout in the category of schemes. Note that $S_{i,p}$ is of finite type over $\mathbf{Z}$ since it has a finite affine open covering whose members are spectra of finite type $\mathbf{Z}$-algebras. We define a partial ordering on $J = \coprod_{i \in I} P_i$ by the rule $(i', p') \geq (i, p)$ if and only if $i' \geq i$ and the map $B_i \to B_{i'}$ maps $A_{i,p}$ into $A_{i',p'}$. This is exactly the condition needed to define a morphism $S_{i',p'} \to S_{i,p}$: namely make a commutative diagram as above using the transition morphisms $V_{i'} \to V_i$ and $W_{i'} \to W_i$ and the morphism $\mathrm{Spec}(A_{i',p'}) \to \mathrm{Spec}(A_{i,p})$ induced by the ring map $A_{i,p} \to A_{i',p'}$. The relevant commutativities have been built into the constructions. We claim that S is the directed limit of the schemes $S_{i,p}$. Since by construction the schemes V_i have limit V this boils down to the fact that B is the limit of the rings $A_{i,p}$ which is true by construction. The map $\alpha : J \to I$ is given by the rule $j = (i,p) \mapsto i$. The open subscheme V'_j is just the image of $V_i \to S_{i,p}$ above. The commutativity of the diagrams in (5) is clear from the construction. This finishes the proof of the lemma. □

Proposition 4.4. *Let S be a quasi-compact and quasi-separated scheme. There exist a directed partially ordered set I and an inverse system of schemes $(S_i, f_{ii'})$ over I such that*

1. *the transition morphisms $f_{ii'}$ are affine*
2. *each S_i is of finite type over $\mathbf{Z}$, and*
3. *$S = \lim_i S_i$.*

Proof. This is a special case of Lemma 4.3 with $V = \emptyset$. □

5. Limits and morphisms of finite presentation

The following is a generalization of Algebra, Lemma 123.2.

Proposition 5.1. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is locally of finite presentation.*
2. *For any directed partially ordered set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each T_i affine, we have*
$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \, \mathrm{Mor}_S(T_i, X)$$
3. *For any directed partially ordered set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each $f_{ii'}$ affine and every T_i quasi-compact and quasi-separated as a scheme, we have*
$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \, \mathrm{Mor}_S(T_i, X)$$

Proof. It is clear that (3) implies (2).

Let us prove that (2) implies (1). Assume (2). Choose any affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$. We have to show that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation. Let $(A_i, \varphi_{ii'})$ be a directed system of $\mathcal{O}_S(V)$-algebras. Set $A = \mathrm{colim}_i A_i$. According to Algebra, Lemma 123.2 we have to show that

$$\mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A) = \mathrm{colim}_i \, \mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i)$$

Consider the schemes $T_i = \mathrm{Spec}(A_i)$. They form an inverse system of V-schemes over I with transition morphisms $f_{ii'} : T_i \to T_{i'}$ induced by the $\mathcal{O}_S(V)$-algebra maps $\varphi_{i'i}$. Set $T := \mathrm{Spec}(A) = \lim_i T_i$. The formula above becomes in terms of morphism sets of schemes

$$\mathrm{Mor}_V(\lim_i T_i, U) = \mathrm{colim}_i \, \mathrm{Mor}_V(T_i, U).$$

We first observe that $\mathrm{Mor}_V(T_i, U) = \mathrm{Mor}_S(T_i, U)$ and $\mathrm{Mor}_V(T, U) = \mathrm{Mor}_S(T, U)$. Hence we have to show that

$$\mathrm{Mor}_S(\lim_i T_i, U) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, U)$$

and we are given that

$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X).$$

Hence it suffices to prove that given a morphism $g_i : T_i \to X$ over S such that the composition $T \to T_i \to X$ ends up in U there exists some $i' \geq i$ such that the composition $g_{i'} : T_{i'} \to T_i \to X$ ends up in U. Denote $Z_{i'} = g_{i'}^{-1}(X \setminus U)$. Assume each $Z_{i'}$ is nonempty to get a contradiction. By Lemma 3.5 there exists a point t of T which is mapped into $Z_{i'}$ for all $i' \geq i$. Such a point is not mapped into U. A contradiction.

Finally, let us prove that (1) implies (3). Assume (1). Let an inverse directed system $(T_i, f_{ii'})$ of S-schemes be given. Assume the morphisms $f_{ii'}$ are affine and each T_i is quasi-compact and quasi-separated as a scheme. Let $T = \lim_i T_i$. Denote $f_i : T \to T_i$ the projection morphisms. We have to show:

(a) Given morphisms $g_i, g'_i : T_i \to X$ over S such that $g_i \circ f_i = g'_i \circ f_i$, then there exists an $i' \geq i$ such that $g_i \circ f_{i'i} = g'_i \circ f_{i'i}$.

(b) Given any morphism $g : T \to X$ over S there exists an $i \in I$ and a morphism $g_i : T_i \to X$ such that $g = f_i \circ g_i$.

First let us prove the uniqueness part (a). Let $g_i, g'_i : T_i \to X$ be morphisms such that $g_i \circ f_i = g'_i \circ f_i$. For any $i' \geq i$ we set $g_{i'} = g_i \circ f_{i'i}$ and $g'_{i'} = g'_i \circ f_{i'i}$. We also set $g = g_i \circ f_i = g'_i \circ f_i$. Consider the morphism $(g_i, g'_i) : T_i \to X \times_S X$. Set

$$W = \bigcup\nolimits_{U \subset X \text{ affine open}, V \subset S \text{ affine open}, f(U) \subset V} U \times_V U.$$

This is an open in $X \times_S X$, with the property that the morphism $\Delta_{X/S}$ factors through a closed immersion into W, see the proof of Schemes, Lemma 21.2. Note that the composition $(g_i, g'_i) \circ f_i : T \to X \times_S X$ is a morphism into W because it factors through the diagonal by assumption. Set $Z_{i'} = (g_{i'}, g'_{i'})^{-1}(X \times_S X \setminus W)$. If each $Z_{i'}$ is nonempty, then by Lemma 3.5 there exists a point $t \in T$ which maps to $Z_{i'}$ for all $i' \geq i$. This is a contradiction with the fact that T maps into W. Hence we may increase i and assume that $(g_i, g'_i) : T_i \to X \times_S X$ is a morphism into W. By construction of W, and since T_i is quasi-compact we can find a finite affine open covering $T_i = T_{1,i} \cup \ldots \cup T_{n,i}$ such that $(g_i, g'_i)|_{T_{j,i}}$ is a morphism into $U \times_V U$ for some pair (U, V) as in the definition of W above. Since it suffices to prove that $g_{i'}$ and $g'_{i'}$ agree on each of the $f_{i'i}^{-1}(T_{j,i})$ this reduces us to the affine case. The affine case follows from Algebra, Lemma 123.2 and the fact that the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation (see Morphisms, Lemma 22.2).

Finally, we prove the existence part (b). Let $g : T \to X$ be a morphism of schemes over S. We can find a finite affine open covering $T = W_1 \cup \ldots \cup W_n$ such that for each $j \in \{1, \ldots, n\}$ there exist affine opens $U_j \subset X$ and $V_j \subset S$ with $f(U_j) \subset V_j$ and $g(W_j) \subset U_j$. By Lemmas 3.8 and 3.10 (after possibly shrinking I) we may assume that there exist affine open coverings $T_i = W_{1,i} \cup \ldots \cup W_{n,i}$ compatible with transition maps such that $W_j = \lim_i W_{j,i}$. We apply Algebra, Lemma 123.2 to the rings corresponding to the affine schemes U_j, V_j, $W_{j,i}$ and W_j using that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_j)$ is of finite presentation (see Morphisms, Lemma 22.2). Thus we can find for each j an index $i_j \in I$ and a morphism $g_{j,i_j} : W_{j,i_j} \to X$ such that

$g_{j,i_j} \circ f_i|_{W_j} : W_j \to W_{j,i} \to X$ equals $g|_{W_j}$. By part (a) proved above, using the quasi-compactness of $W_{j_1,i} \cap W_{j_2,i}$ which follows as T_i is quasi-separated, we can find an index $i' \in I$ larger than all i_j such that

$$g_{j_1,i_{j_1}} \circ f_{i'i_{j_1}}|_{W_{j_1,i'} \cap W_{j_2,i'}} = g_{j_2,i_{j_2}} \circ f_{i'i_{j_2}}|_{W_{j_1,i'} \cap W_{j_2,i'}}$$

for all $j_1, j_2 \in \{1, \ldots, n\}$. Hence the morphisms $g_{j,i_j} \circ f_{i'i_j}|_{W_{j,i'}}$ glue to given the desired morphism $T_{i'} \to X$. $\square$

Remark 5.2. Let S be a scheme. Let us say that a functor $F : (Sch/S)^{opp} \to Sets$ is *limit preserving* if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes with limit T we have $F(T) = \mathrm{colim}_i F(T_i)$. Let X be a scheme over S, and let $h_X : (Sch/S)^{opp} \to Sets$ be its functor of points, see Schemes, Section 15. In this terminology Proposition 5.1 says that a scheme X is locally of finite presentation over S if and only if h_X is limit preserving.

6. Relative approximation

The title of this section refers to results of the following type.

Lemma 6.1. *Let $f : X \to S$ be a morphism of schemes. Assume that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *S is quasi-separated.*

Then $X = \lim X_i$ is a limit of a directed system of schemes X_i of finite presentation over S with affine transition morphisms over S.

Proof. Since $f(X)$ is quasi-compact we may replace S by a quasi-compact open containing $f(X)$. Hence we may assume S is quasi-compact as well. Write $X = \lim X_a$ and $S = \lim S_b$ as in Proposition 4.4, i.e., with X_a and S_b of finite type over $\mathbf{Z}$ and with affine transition morphisms. By Proposition 5.1 we find that for each b there exists an a and a morphism $f_{a,b} : X_a \to S_b$ making the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S \arrow[d] \\ X_a \arrow[r] & S_b \end{tikzcd}$$

commute. Moreover the same proposition implies that, given a second triple $(a', b', f_{a',b'})$, there exists an $a'' \geq a'$ such that the compositions $X_{a''} \to X_a \to X_b$ and $X_{a''} \to X_{a'} \to X_{b'} \to X_b$ are equal. Consider the set of triples $(a, b, f_{a,b})$ endowed with the partial ordering

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a',\ b' \geq b, \text{ and } f_{a',b'} \circ h_{a,a'} = g_{b',b} \circ f_{a,b}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b',b} : S_{b'} \to S_b$ are the transition morphisms. The remarks above show that this system is directed. It follows formally from the equalities $X = \lim X_a$ and $S = \lim S_b$ that

$$X = \lim\nolimits_{(a,b,f_{a,b})} X_a \times_{f_{a,b},S_b} S.$$

where the limit is over our directed system above. The transition morphisms $X_a \times_{S_b} S \to X_{a'} \times_{S_{b'}} S$ are affine as the composition

$$X_a \times_{S_b} S \to X_a \times_{S_{b'}} S \to X_{a'} \times_{S_{b'}} S$$

where the first morphism is a closed immersion (by Schemes, Lemma 21.10) and the second is a base change of an affine morphism (Morphisms, Lemma 13.8) and the composition of affine morphisms is affine (Morphisms, Lemma 13.7). The morphisms $f_{a,b}$ are of finite presentation (Morphisms, Lemmas 22.9 and 22.11) and hence the base changes $X_a \times_{f_{a,b}, S_b} S \to S$ are of finite presentation (Morphisms, Lemma 22.4). $\square$

Lemma 6.2. *Let $X \to S$ be an integral morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ with $X_i \to S$ finite and of finite presentation.*

Proof. Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 24.1. Combining Properties, Lemma 20.13 we can write $\mathcal{A} = \operatorname{colim}_i \mathcal{A}_i$ as a filtered colimit of finite and finitely presented $\mathcal{O}_S$-algebras. Then

$$X_i = \underline{\mathrm{Spec}}_S(\mathcal{A}_i) \longrightarrow S$$

is a finite and finitely presented morphism of schemes. By construction $X = \lim_i X_i$ which proves the lemma. $\square$

7. Descending properties of morphisms

This section is the analogue of Section 3 for properties of morphisms over S. We will work in the following situation.

Situation 7.1. Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms (Lemma 2.2). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S.

Lemma 7.2. *Notation and assumptions as in Situation 7.1. If f is affine, then there exists an index $i \geq 0$ such that f_i is affine.*

Proof. Let $Y_0 = \bigcup_{j=1,\ldots,m} V_{j,0}$ be a finite affine open covering. Set $U_{j,0} = f_0^{-1}(V_{j,0})$. For $i \geq 0$ we denote $V_{j,i}$ the inverse image of $V_{j,0}$ in Y_i and $U_{j,i} = f_i^{-1}(V_{j,i})$. Similarly we have $U_j = f^{-1}(V_j)$. Then $U_j = \lim_{i \geq 0} U_{j,i}$ (see Lemma 2.2). Since U_j is affine by assumption we see that each $U_{j,i}$ is affine for i large enough, see Lemma 3.10. As there are finitely many j we can pick an i which works for all j. Thus f_i is affine for i large enough, see Morphisms, Lemma 13.3. $\square$

Lemma 7.3. *Notation and assumptions as in Situation 7.1. If*

1. *f is a finite morphism, and*
2. *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is finite.

Proof. A finite morphism is affine, see Morphisms, Definition 44.1. Hence by Lemma 7.2 above after increasing 0 we may assume that f_0 is affine. By writing Y_0 as a finite union of affines we reduce to proving the result when X_0 and Y_0 are affine and map into a common affine $W \subset S_0$. The corresponding algebra statement follows from Algebra, Lemma 156.3. $\square$

Lemma 7.4. *Notation and assumptions as in Situation 7.1. If*

1. *f is a closed immersion, and*
2. *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is a closed immersion.

Proof. A closed immersion is affine, see Morphisms, Lemma 13.9. Hence by Lemma 7.2 above after increasing 0 we may assume that f_0 is affine. By writing Y_0 as a finite union of affines we reduce to proving the result when X_0 and Y_0 are affine and map into a common affine $W \subset S_0$. The corresponding algebra statement is a consequence of Algebra, Lemma 156.4. □

Lemma 7.5. *Notation and assumptions as in Situation 7.1. If f is separated, then f_i is separated for some $i \geq 0$.*

Proof. Apply Lemma 7.4 to the diagonal morphism $\Delta_{X_0/S_0} : X_0 \to X_0 \times_{S_0} X_0$. (This is permissible as diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{S_0} X_0$ is quasi-compact and quasi-separated, see Schemes, Lemma 21.2, Morphisms, Lemma 16.5, and Schemes, Remark 21.18. □

Lemma 7.6. *Notation and assumptions as in Situation 7.1. If*

1. *f is flat,*
2. *f_0 is locally of finite presentation,*

then f_i is flat for some $i \geq 0$.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being flat is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \text{colim}_{i \in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite presentation. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is flat, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is flat. This follows from Algebra, Lemma 156.1 part (3). □

Lemma 7.7. *Notation and assumptions as in Situation 7.1. If*

1. *f is finite locally free (of degree d),*
2. *f_0 is locally of finite presentation,*

then f_i is finite locally free (of degree d) for some $i \geq 0$.

Proof. By Lemmas 7.6 and 7.3 we find an i such that f_i is flat and finite. On the other hand, f_i is locally of finite presentation. Hence f_i is finite locally free by Morphisms, Lemma 46.2. If moreover f is finite locally free of degree d, then the image of $Y \to Y_i$ is contained in the open and closed locus $W_d \subset Y_i$ over which f_i has degree d. By Lemma 3.7 we see that for some $i' \geq i$ the image of $Y_{i'} \to Y_i$ is contained in W_d. Then $f_{i'}$ will be finite locally free of degree d. □

Lemma 7.8. *Notation and assumptions as in Situation 7.1. If*

1. *f is étale,*
2. *f_0 is locally of finite presentation,*

then f_i is étale for some $i \geq 0$.

Proof. Being étale is local on the source and the target (Morphisms, Lemma 37.2) hence we may assume S_0, X_0, Y_0 affine (details omitted). The corresponding algebra fact is Algebra, Lemma 156.5. □

Lemma 7.9. *Notation and assumptions as in Situation 7.1. If*

(1) *f is an isomorphism, and*
(2) *f_0 is locally of finite presentation,*

then f_i is an isomorphism for some $i \geq 0$.

Proof. By Lemmas 7.8 and 7.4 we can find an i such that f_i is flat and a closed immersion. Then f_i identifies X_i with an open and closed subscheme of Y_i, see Morphisms, Lemma 27.2. By assumption the image of $Y \to Y_i$ maps into $f_i(X_i)$. Thus by Lemma 3.7 we find that $Y_{i'}$ maps into $f_i(X_i)$ for some $i' \geq i$. It follows that $X_{i'} \to Y_{i'}$ is surjective and we win. $\square$

Lemma 7.10. *Notation and assumptions as in Situation 7.1. If*

(1) *f is a monomorphism, and*
(2) *f_0 is locally of finite type,*

then f_i is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism of schemes $V \to W$ is a monomorphism if and only if the diagonal $V \to V \times_W V$ is an isomorphism (Schemes, Lemma 23.2). The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms, Lemma 22.12. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated (Schemes, Remark 21.18) we conclude from Lemma 7.9 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. $\square$

Lemma 7.11. *Notation and assumptions as in Situation 7.1. If*

(1) *f is surjective, and*
(2) *f_0 is locally of finite presentation,*

then there exists an $i \geq 0$ such that f_i is surjective.

Proof. The morphism f_0 is of finite presentation. Hence $E = f_0(X_0)$ is a constructible subset of Y_0, see Morphisms, Lemma 23.2. Since f_i is the base change of f_0 by $Y_i \to Y_0$ we see that the image of f_i is the inverse image of E in Y_i. Moreover, we know that $Y \to Y_0$ maps into E. Hence we win by Lemma 3.7. $\square$

8. Finite type closed in finite presentation

A result of this type is [Kie72, Satz 2.10]. Another reference is [Con07].

Lemma 8.1. *Let $f : X \to S$ be a morphism of schemes. Assume:*

(1) *The morphism f is locally of finite type.*
(2) *The scheme X is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to S$ and an immersion $X \to X'$ of schemes over S.

Proof. By Proposition 4.4 we can write $X = \lim_i X_i$ with each X_i of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X_{i,S} \arrow[r] \arrow[d] & X_i \arrow[d] \\ & S \arrow[r] & \mathrm{Spec}(\mathbf{Z}) \end{tikzcd}$$

Note that X_i is of finite presentation over $\mathrm{Spec}(\mathbf{Z})$, see Morphisms, Lemma 22.9. Hence the base change $X_{i,S} \to S$ is of finite presentation by Morphisms, Lemma 22.4. Thus it suffices to show that the arrow $X \to X_{i,S}$ is an immersion for i sufficiently large.

To do this we choose a finite affine open covering $X = V_1 \cup \ldots \cup V_n$ such that f maps each V_j into an affine open $U_j \subset S$. Let $h_{j,a} \in \mathcal{O}_X(V_j)$ be a finite set of elements which generate $\mathcal{O}_X(V_j)$ as an $\mathcal{O}_S(U_j)$-algebra, see Morphisms, Lemma 16.2. By Lemmas 3.8 and 3.10 (after possibly shrinking I) we may assume that there exist affine open coverings $X_i = V_{1,i} \cup \ldots \cup V_{n,i}$ compatible with transition maps such that $V_j = \lim_i V_{j,i}$. By Lemma 3.3 we can choose i so large that each $h_{j,a}$ comes from an element $h_{j,a,i} \in \mathcal{O}_{X_i}(V_{j,i})$. Thus the arrow in

$$V_j \longrightarrow U_j \times_{\mathrm{Spec}(\mathbf{Z})} V_{j,i} = (V_{j,i})_{U_j} \subset (V_{j,i})_S \subset X_{i,S}$$

is a closed immersion. Since $\bigcup (V_{j,i})_{U_j}$ forms an open of $X_{i,S}$ and since the inverse image of $(V_{j,i})_{U_j}$ in X is V_j it follows that $X \to X_{i,S}$ is an immersion. $\square$

Remark 8.2. We cannot do better than this if we do not assume more on S and the morphism $f : X \to S$. For example, in general it will not be possible to find a *closed* immersion $X \to X'$ as in the lemma. The reason is that this would imply that f is quasi-compact which may not be the case. An example is to take S to be infinite dimensional affine space with 0 doubled and X to be one of the two infinite dimensional affine spaces.

Lemma 8.3. *Let $f : X \to S$ be a morphism of schemes. Assume:*

1. *The morphism f is of locally of finite type.*
2. *The scheme X is quasi-compact and quasi-separated, and*
3. *The scheme S is quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. By Lemma 8.1 above there exists a morphism $Y \to S$ of finite presentation and an immersion $i : X \to Y$ of schemes over S. For every point $x \in X$, there exists an affine open $V_x \subset Y$ such that $i^{-1}(V_x) \to V_x$ is a closed immersion. Since X is quasi-compact we can find finitely may affine opens $V_1, \ldots, V_n \subset Y$ such that $i(X) \subset V_1 \cup \ldots \cup V_n$ and $i^{-1}(V_j) \to V_j$ is a closed immersion. In other words such that $i : X \to X' = V_1 \cup \ldots \cup V_n$ is a closed immersion of schemes over S. Since S is quasi-separated and Y is quasi-separated over S we deduce that Y is quasi-separated, see Schemes, Lemma 21.13. Hence the open immersion $X' = V_1 \cup \ldots \cup V_n \to Y$ is quasi-compact. This implies that $X' \to Y$ is of finite presentation, see Morphisms, Lemma 22.6. We conclude since then $X' \to Y \to S$ is a composition of morphisms of finite presentation, and hence of finite presentation (see Morphisms, Lemma 22.3). $\square$

Lemma 8.4. *Let $X \to Y$ be a closed immersion of schemes. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ of schemes over Y where $X_i \to Y$ is a closed immersion of finite presentation.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Properties, Lemma 20.3 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = \mathrm{colim}_{i \in I} \mathcal{I}_i$ of its quasi-coherent sheaves of ideals of finite type. Let $X_i \subset Y$ be the closed subscheme defined by $\mathcal{I}_i$. These form an inverse system of schemes indexed by I. The transition morphisms $X_i \to X_{i'}$ are affine because they are closed immersions. Each X_i is quasi-compact and quasi-separated since it is a closed subscheme of Y and Y is quasi-compact and quasi-separated by our assumptions. We have $X = \lim_i X_i$ as follows directly from the fact that $\mathcal{I} = \mathrm{colim}_{i \in I} \mathcal{I}_a$. Each of the morphisms $X_i \to Y$ is of finite presentation, see Morphisms, Lemma 22.7. □

Lemma 8.5. *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *The morphism f is of locally of finite type.*
(2) *The scheme X is quasi-compact and quasi-separated, and*
(3) *The scheme S is quasi-separated.*

Then $X = \lim X_i$ where the $X_i \to S$ are of finite presentation, the X_i are quasi-compact and quasi-separated, and the transition morphisms $X_{i'} \to X_i$ are closed immersions (which implies that $X \to X_i$ are closed immersions for all i).

Proof. By Lemma 8.3 there is a closed immersion $X \to Y$ with $Y \to S$ of finite presentation. Then Y is quasi-separated by Schemes, Lemma 21.13. Since X is quasi-compact, we may assume Y is quasi-compact by replacing Y with a quasi-compact open containing X. We see that $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation by Lemma 8.4. The morphisms $X_i \to S$ are of finite presentation by Morphisms, Lemma 22.3. □

Proposition 8.6. *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is of finite type and separated, and*
(2) *S is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. Apply Lemma 8.5 and note that $X_i \to S$ is separated for large i by Lemma 3.14 as we have assumed that $X \to S$ is separated. □

Lemma 8.7. *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is finite, and*
(2) *S is quasi-compact and quasi-separated.*

Then there exists a morphism which is finite and of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. We may write $X = \lim X_i$ as in Lemma 8.5. Applying Lemma 3.16 we see that $X_i \to S$ is finite for large enough i. □

Lemma 8.8. *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is finite, and*
(2) *S quasi-compact and quasi-separated.*

Then X is a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over S.

Proof. We may write $X = \lim X_i$ as in Lemma 8.5. Applying Lemma 3.16 we see that $X_i \to S$ is finite for large enough i. □

9. Descending relative objects

The following lemma is typical of the type of results in this section. We write out the "standard" proof completely. It may be faster to convince yourself that the result is true than to read this proof.

Lemma 9.1. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
(2) *the schemes S_i are quasi-compact and quasi-separated.*

Let $S = \lim_i S_i$. Then we have the following:

(1) *For any morphism of finite presentation $X \to S$ there exists an index $i \in I$ and a morphism of finite presentation $X_i \to S_i$ such that $X \cong X_{i,S}$ as schemes over S.*
(2) *Given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i, and a morphism $\varphi : X_{i,S} \to Y_{i,S}$ over S, there exists an index $i' \geq i$ and a morphism $\varphi_{i'} : X_{i,S_{i'}} \to Y_{i,S_{i'}}$ whose base change to S is φ.*
(3) *Given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$ whose base changes $\varphi_{i,S} = \psi_{i,S}$ are equal, there exists an index $i' \geq i$ such that $\varphi_{i,S_{i'}} = \psi_{i,S_{i'}}$.*

In other words, the category of schemes of finite presentation over S is the colimit over I of the categories of schemes of finite presentation over S_i.

Proof. In case each of the schemes S_i is affine, and we consider only affine schemes of finite presentation over S_i, resp. S this lemma is equivalent to Algebra, Lemma 123.6. We claim that the affine case implies the lemma in general.

Let us prove (3). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$. Assume that the base changes are equal: $\varphi_{i,S} = \psi_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 2.3 we have $X = \lim_{i' \geq i} X_{i'}$ and similarly for Y. Additionally we denote $\varphi_{i'}$ and $\psi_{i'}$ (resp. φ and ψ) the base change of φ_i and ψ_i to $S_{i'}$ (resp. S). So our assumption means that $\varphi = \psi$. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 22.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_{j,i'} \to Y_{i'}$ are quasi-compact, and the inverse images $U_{j,i'} = \varphi_i^{-1}(V_{j,i'})$ and $U'_{j,i'} = \psi_i^{-1}(V_{j,i'})$ are quasi-compact opens of $X_{i'}$. By assumption the inverse images of V_j under φ and ψ in X are equal. Hence by Lemma 3.8 there exists an index $i' \geq i$ such that of $U_{j,i'} = U'_{j,i'}$ in $X_{i'}$. Choose an finite affine open covering $U_{j,i'} = U'_{j,i'} = \bigcup W_{j,k,i'}$ which induce coverings $U_{j,i''} = U'_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$. By the affine case there exists an index i'' such that $\varphi_{i''}|_{W_{j,k,i''}} = \psi_{i''}|_{W_{j,k,i''}}$ for all j, k. Then i'' is an index such that $\varphi_{i''} = \psi_{i''}$ and (3) is proved.

Let us prove (2). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a morphism $\varphi : X_{i,S} \to Y_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 2.3 we have $X = \lim_{i' \geq i} X_{i'}$ and similarly for Y. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 22.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_j \to Y$ are quasi-compact, and the inverse images $U_j = \varphi^{-1}(V_j)$ are quasi-compact opens of X. Hence by Lemma 3.8 there exists an index $i' \geq i$ and quasi-compact opens $U_{j,i'}$ of $X_{i'}$ whose inverse image in X is U_j. Choose an finite affine open covering $U_{j,i'} = \bigcup W_{j,k,i'}$ which induce affine open coverings $U_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$ and an affine open covering $U_j = \bigcup W_{j,k}$. By the affine case there exists an index i'' and morphisms $\varphi_{j,k,i''} : W_{j,k,i''} \to V_{j,i''}$ such that $\varphi|_{W_{j,k}} = \varphi_{j,k,i'',S}$ for all j, k. By part (3) proved above, there is a further index $i''' \geq i''$ such that

$$\varphi_{j_1,k_1,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}} = \varphi_{j_2,k_2,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}}$$

for all j_1, j_2, k_1, k_2. Then i''' is an index such that there exists a morphism $\varphi_{i'''} : X_{i'''} \to Y_{i'''}$ whose base change to S gives φ. Hence (2) holds.

Let us prove (1). Suppose given a scheme X of finite presentation over S. Since X is of finite presentation over S, and since S is quasi-compact and quasi-separated, also X is quasi-compact and quasi-separated (see Morphisms, Lemma 22.10). Choose a finite affine open covering $X = \bigcup U_j$ such that each U_j maps into an affine open $V_j \subset S$. Denote $U_{j_1 j_2} = U_{j_1} \cap U_{j_2}$ and $U_{j_1 j_2 j_3} = U_{j_1} \cap U_{j_2} \cap U_{j_3}$. By Lemmas 3.8 and 3.10 we can find an index i_1 and affine opens $V_{j,i_1} \subset S_{i_1}$ such that each V_j is the inverse of this in S. Let $V_{j,i}$ be the inverse image of V_{j,i_1} in S_i for $i \geq i_1$. By the affine case we may find an index $i_2 \geq i_1$ and affine schemes $U_{j,i_2} \to V_{j,i_2}$ such that $U_j = S \times_{S_{i_2}} U_{j,i_2}$ is the base change. Denote $U_{j,i} = S_i \times_{S_{i_2}} U_{j,i_2}$ for $i \geq i_2$. By Lemma 3.8 there exists an index $i_3 \geq i_2$ and open subschemes $W_{j_1,j_2,i_3} \subset U_{j_1,i_3}$ whose base change to S is equal to $U_{j_1 j_2}$. Denote $W_{j_1,j_2,i} = S_i \times_{S_{i_3}} W_{j_1,j_2,i_3}$ for $i \geq i_3$. By part (2) shown above there exists an index $i_4 \geq i_3$ and morphisms $\varphi_{j_1,j_2,i_4} : W_{j_1,j_2,i_4} \to W_{j_2,j_1,i_4}$ whose base change to S gives the identity morphism $U_{j_1 j_2} = U_{j_2 j_1}$ for all j_1, j_2. For all $i \geq i_4$ denote $\varphi_{j_1,j_2,i} = \mathrm{id}_S \times \varphi_{j_1,j_2,i_4}$ the base change. We claim that for some $i_5 \geq i_4$ the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ forms a glueing datum as in Schemes, Section 14. In order to see this we have to verify that for i large enough we have

$$\varphi_{j_1,j_2,i}^{-1}(W_{j_1,j_2,i} \cap W_{j_1,j_3,i}) = W_{j_1,j_2,i} \cap W_{j_1,j_3,i}$$

and that for large enough i the cocycle condition holds. The first condition follows from Lemma 3.8 and the fact that $U_{j_2 j_1 j_3} = U_{j_1 j_2 j_3}$. The second from part (1) of the lemma proved above and the fact that the cocycle condition holds for the maps $\mathrm{id} : U_{j_1 j_2} \to U_{j_2 j_1}$. Ok, so now we can use Schemes, Lemma 14.2 to glue the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ to get a scheme $X_{i_5} \to S_{i_5}$. By construction the base change of X_{i_5} to S is formed by glueing the open affines U_j along the opens $U_{j_1} \leftarrow U_{j_1 j_2} \to U_{j_2}$. Hence $S \times_{S_{i_5}} X_{i_5} \cong X$ as desired. $\square$

Lemma 9.2. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

1. *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
2. *all the schemes S_i are quasi-compact and quasi-separated.*

Let $S = \lim_i S_i$. Then we have the following:

(1) *For any sheaf of $\mathcal{O}_S$-modules $\mathcal{F}$ of finite presentation there exists an index $i \in I$ and a sheaf of $\mathcal{O}_{S_i}$-modules of finite presentation $\mathcal{F}_i$ such that $\mathcal{F} \cong f_i^*\mathcal{I}_i$.*
(2) *Suppose given an index $i \in I$, sheaves of $\mathcal{O}_{S_i}$-modules $\mathcal{F}_i$, $\mathcal{G}_i$ of finite presentation and a morphism $\varphi : f_i^*\mathcal{F}_i \to f_i^*\mathcal{G}_i$ over S. Then there exists an index $i' \geq i$ and a morphism $\varphi_{i'} : f_{i'i}^*\mathcal{F}_i \to f_{i'i}^*\mathcal{G}_i$ whose base change to S is φ.*
(3) *Suppose given an index $i \in I$, sheaves of $\mathcal{O}_{S_i}$-modules $\mathcal{F}_i$, $\mathcal{G}_i$ of finite presentation and a pair of morphisms $\varphi_i, \psi_i : \mathcal{F}_i \to \mathcal{G}_i$. Assume that the base changes are equal: $f_i^*\varphi_i = f_i^*\psi_i$. Then there exists an index $i' \geq i$ such that $f_{i'i}^*\varphi_i = f_{i'i}^*\psi_i$.*

In other words, the category of modules of finite presentation over S is the colimit over I of the categories modules of finite presentation over S_i.

Proof. Omitted. Since we have written out completely the proof of Lemma 9.1 above it seems wise to use this here and not completely write this proof out also. For example we can use:

(1) there is an equivalence of categories between quasi-coherent $\mathcal{O}_S$-modules and vector bundles over S, see Constructions, Section 6.
(2) a vector bundle $\mathbf{V}(\mathcal{F}) \to S$ is of finite presentation over S if and only if $\mathcal{F}$ is an $\mathcal{O}_S$-module of finite presentation.

Then you can descend morphisms in terms of morphisms of the associated vector-bundles. Similarly for objects. $\square$

Lemma 9.3. *With notation and assumptions as in Lemma 9.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over S_i and that $\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_{X_i}$-module of finite presentation. If the pullback of $\mathcal{F}_i$ to $X_i \times_{S_i} S$ is flat over $Y_i \times_{S_i} S$, then there exists an index $i' \geq i$ such that the pullback of $\mathcal{F}_i$ to $X_i \times_{S_i} S_{i'}$ is flat over $Y_i \times_{S_i} S_{i'}$.*

Proof. (This lemma is the analogue of Lemma 7.6 for modules.) For $i' \geq i$ denote $X_{i'} = S_{i'} \times_{S_i} X_i$, $\mathcal{F}_{i'} = (X_{i'} \to X_i)^*\mathcal{F}_i$ and similarly for $Y_{i'}$. Denote $\varphi_{i'}$ the base change of φ_i to $S_{i'}$. Also set $X = S \times_{S_i} X_i$, $Y = S \times_{S_i} X_i$, $\mathcal{F} = (X \to X_i)^*\mathcal{F}_i$ and φ the base change of φ_i to S. Let $Y_i = \bigcup_{j=1,\ldots,m} V_{j,i}$ be a finite affine open covering such that each $V_{j,i}$ maps into some affine open of S_i. For each $j = 1, \ldots m$ let $\varphi_i^{-1}(V_{j,i}) = \bigcup_{k=1,\ldots,m(j)} U_{k,j,i}$ be a finite affine open covering. For $i' \geq i$ we denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and $U_{k,j,i'}$ the inverse image of $U_{k,j,i}$ in $X_{i'}$. Similarly we have $U_{k,j} \subset X$ and $V_j \subset Y$. Then $U_{k,j} = \lim_{i' \geq i} U_{k,j,i'}$ and $V_j = \lim_{i' \geq i} V_j$ (see Lemma 2.2). Since $X_{i'} = \bigcup_{k,j} U_{k,j,i'}$ is a finite open covering it suffices to prove the lemma for each of the morphisms $U_{k,j,i} \to V_{j,i}$ and the sheaf $\mathcal{F}_i|_{U_{k,j,i}}$. Hence we see that the lemma reduces to the case that X_i and Y_i are affine and map into an affine open of S_i, i.e., we may also assume that S is affine.

In the affine case we reduce to the following algebra result. Suppose that $R = \text{colim}_{i \in I} R_i$. For some $i \in I$ suppose given a map $A_i \to B_i$ of finitely presented R_i-algebras. Let N_i be a finitely presented B_i-module. Then, if $R \otimes_{R_i} N_i$ is flat over $R \otimes_{R_i} A_i$, then for some $i' \geq i$ the module $R_{i'} \otimes_{R_i} N_i$ is flat over $R_{i'} \otimes_{R_i} A$. This is exactly the result proved in Algebra, Lemma 156.1 part (3). $\square$

10. Characterizing affine schemes

If $f : X \to S$ is a surjective integral morphism of schemes such that X is an affine scheme then S is affine too. See [Con07, A.2]. Our proof relies on the Noetherian case which we stated and proved in Cohomology of Schemes, Lemma 13.3. See also [DG67, II 6.7.1].

Lemma 10.1. *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and finite, and assume that X is affine. Then S is affine.*

Proof. Since f is surjective and X is quasi-compact we see that S is quasi-compact. Since X is separated and f is surjective and universally closed (Morphisms, Lemma 44.7), we see that S is separated (Morphisms, Lemma 42.11).

By Lemma 8.8 we can write $X = \lim_a X_a$ with $X_a \to S$ finite and of finite presentation. By Lemma 3.10 we see that X_a is affine for some $a \in A$. Replacing X by X_a we may assume that $X \to S$ is surjective, finite, of finite presentation and that X is affine.

By Proposition 4.4 we may write $S = \lim_{i \in I} S_i$ as a directed limits as schemes of finite type over $\mathbf{Z}$. By Lemma 9.1 we can after shrinking I assume there exist schemes $X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_S S_{i'}$ for $i' \geq i$ and such that $X = \lim_i X_i$. By Lemma 7.3 we may assume that $X_i \to S_i$ is finite for all $i \in I$ as well. By Lemma 3.10 once again we may assume that X_i is affine for all $i \in I$. Hence the result follows from the Noetherian case, see Cohomology of Schemes, Lemma 13.3. $\square$

Proposition 10.2. *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and integral, and assume that X is affine. Then S is affine.*

Proof. Since f is surjective and X is quasi-compact we see that S is quasi-compact. Since X is separated and f is surjective and universally closed (Morphisms, Lemma 44.7), we see that S is separated (Morphisms, Lemma 42.11).

By Lemma 6.2 we can write $X = \lim_i X_i$ with $X_i \to S$ finite. By Lemma 3.10 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to S$ factors through each X_i we see that $X_i \to S$ is surjective. Hence we conclude that S is affine by Lemma 10.1. $\square$

Lemma 10.3. *Let X be a scheme which is set theoretically the union of finitely many affine closed subschemes. Then X is affine.*

Proof. Let $Z_i \subset X$, $i = 1, \ldots, n$ be affine closed subschemes such that $X = \bigcup Z_i$ set theoretically. Then $\coprod Z_i \to X$ is surjective and integral with affine source. Hence X is affine by Proposition 10.2. $\square$

Lemma 10.4. *Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Let $\mathcal{L}$ be an invertible sheaf on X. If $i^*\mathcal{L}$ is ample on Z, then $\mathcal{L}$ is ample on X.*

Proof. Since $i^*\mathcal{L}$ is ample we see that Z is quasi-compact (Properties, Definition 24.1) and separated (Properties, Lemma 24.9). Since i is surjective, we see that X is quasi-compact. Since i is universally closed and surjective, we see that X is separated (Morphisms, Lemma 42.11).

By Proposition 4.4 we can write $X = \lim X_i$ as a directed limit of finite type schemes over $\mathbf{Z}$ with affine transition morphisms. We can find an i and an invertible sheaf $\mathcal{L}_i$ on X_i whose pullback to X is isomorphic to $\mathcal{L}$, see Lemma 9.2.

For each i let $Z_i \subset X_i$ be the scheme theoretic image of the morphism $Z \to X$. If $\mathrm{Spec}(A_i) \subset X_i$ is an affine open subscheme with inverse image of $\mathrm{Spec}(A)$ in X and if $Z \cap \mathrm{Spec}(A)$ is defined by the ideal $I \subset A$, then $Z_i \cap \mathrm{Spec}(A_i)$ is defined by the ideal $I_i \subset A_i$ which is the inverse image of I in A_i under the ring map $A_i \to A$, see Morphisms, Example 6.4. Since $\mathrm{colim}\, A_i/I_i = A/I$ it follows that $\lim Z_i = Z$. By Lemma 3.12 we see that $\mathcal{L}_i|_{Z_i}$ is ample for some i. Since Z and hence X maps into Z_i set theoretically, we see that $X_{i'} \to X_i$ maps into Z_i set theoretically for some $i' \geq i$, see Lemma 3.7. (Observe that since X_i is Noetherian, every closed subset of X_i is constructible.) Let $T \subset X_{i'}$ be the scheme theoretic inverse image of Z_i in $X_{i'}$. Observe that $\mathcal{L}_{i'}|_T$ is the pullback of $\mathcal{L}_i|_{Z_i}$ and hence ample by Morphisms, Lemma 38.7 and the fact that $T \to Z_i$ is an affine morphism. Thus we see that $\mathcal{L}_{i'}$ is ample on $X_{i'}$ by Cohomology of Schemes, Lemma 14.5. Pulling back to X (using the same lemma as above) we find that $\mathcal{L}$ is ample. $\square$

11. Variants of Chow's Lemma

In this section we prove a number of variants of Chow's lemma. The most interesting version is probably just the Noetherian case, which we stated and proved in Cohomology of Schemes, Section 16.

Lemma 11.1. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exists an $n \geq 0$ and a diagram*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective.

Proof. By Proposition 8.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Clearly, if we prove the assertion for Y, then the result follows for X. Hence we may assume that X is of finite presentation over S.

Write $S = \lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 4.4. By Lemma 9.1 we can find an index $i \in I$ and a scheme $X_i \to S_i$ of finite presentation so that $X = S \times_{S_i} X_i$. By Lemma 7.5 we may assume that $X_i \to S_i$ is separated. Clearly, if we prove the assertion for X_i over S_i, then the assertion holds for X. The case $X_i \to S_i$ is treated by Cohomology of Schemes, Lemma 16.1. $\square$

Here is a variant of Chow's lemma where we assume the scheme on top has finitely many irreducible components.

Lemma 11.2. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Assume that X has finitely many* *irreducible components. Then there exists an $n \geq 0$ and a diagram*

$$
\begin{tikzcd}
X \arrow[rd] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[ld] \\
& S &
\end{tikzcd}
$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, there exists an open dense subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism of schemes.

Proof. Let $X = Z_1 \cup \ldots \cup Z_n$ be the decomposition of X into irreducible components. Let $\eta_j \in Z_j$ be the generic point.

There are (at least) two ways to proceed with the proof. The first is to redo the proof of Cohomology of Schemes, Lemma 16.1 using the general Properties, Lemma 27.4 to find suitable affine opens in X. (This is the "standard" proof.) The second is to use absolute Noetherian approximation as in the proof of Lemma 11.1 above. This is what we will do here.

By Proposition 8.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Write $S = \lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 4.4. By Lemma 9.1 we can find an index $i \in I$ and a scheme $Y_i \to S_i$ of finite presentation so that $Y = S \times_{S_i} Y_i$. By Lemma 7.5 we may assume that $Y_i \to S_i$ is separated. We have the following diagram

$$
\begin{tikzcd}
\eta_j \in Z_j \arrow[r] & X \arrow[r] \arrow[rd] & Y \arrow[r] \arrow[d] & Y_i \arrow[d] \\
& & S \arrow[r] & S_i
\end{tikzcd}
$$

Denote $h : X \to Y_i$ the composition.

For $i' \geq i$ write $Y_{i'} = S_{i'} \times_{S_i} Y_i$. Then $Y = \lim_{i' \geq i} Y_{i'}$, see Lemma 2.3. Choose $j, j' \in \{1, \ldots, n\}$, $j \neq j'$. Note that η_j is not a specialization of $\eta_{j'}$. By Lemma 3.2 we can replace i by a bigger index and assume that $h(\eta_j)$ is not a specialization of $h(\eta_{j'})$ for all pairs (j, j') as above. For such an index, let $Y' \subset Y_i$ be the scheme theoretic image of $h : X \to Y_i$, see Morphisms, Definition 6.2. The morphism h is quasi-compact as the composition of the quasi-compact morphisms $X \to Y$ and $Y \to Y_i$ (which is affine). Hence by Morphisms, Lemma 6.3 the morphism $X \to Y'$ is dominant. Thus the generic points of Y' are all contained in the set $\{h(\eta_1), \ldots, h(\eta_n)\}$, see Morphisms, Lemma 8.3. Since none of the $h(\eta_j)$ is the specialization of another we see that the points $h(\eta_1), \ldots, h(\eta_n)$ are pairwise distinct and are each a generic point of Y'.

We apply Cohomology of Schemes, Lemma 16.1 above to the morphism $Y' \to S_i$. This gives a diagram

$$
\begin{tikzcd}
Y' \arrow[rd] & Y^* \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{S_i} \arrow[ld] \\
& S_i &
\end{tikzcd}
$$

such that π is proper and surjective and an isomorphism over a dense open subscheme $V \subset Y'$. By our choice of i above we know that $h(\eta_1), \ldots, h(\eta_n) \in V$. Consider the commutative diagram

$$\begin{tikzcd} X' \arrow[r, equal] & X \times_{Y'} Y^* \arrow[r] \arrow[d] & Y^* \arrow[r] \arrow[d] & \mathbf{P}^n_{S_i} \arrow[ddl] \\ & X \arrow[r] \arrow[d] & Y' \arrow[d] & \\ & S \arrow[r] & S_i & \end{tikzcd}$$

Note that $X' \to X$ is an isomorphism over the open subscheme $U = h^{-1}(V)$ which contains each of the η_j and hence is dense in X. We conclude $X \leftarrow X' \to \mathbf{P}^n_S$ is a solution to the problem posed in the lemma. □

12. Applications of Chow's lemma

We can use Chow's lemma to investigate the notions of proper and separated morphisms. As a first application we have the following.

Lemma 12.1. *Let S be a scheme. Let $f : X \to S$ be a separated morphism of finite type. The following are equivalent:*

1. *The morphism f is proper.*
2. *For any morphism $S' \to S$ which is locally of finite type the base change $X_{S'} \to S'$ is closed.*
3. *For every $n \geq 0$ the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.*

Proof. Clearly (1) implies (2), and (2) implies (3), so we just need to show (3) implies (1). First we reduce to the case when S is affine. Assume that (3) implies (1) when the base is affine. Now let $f : X \to S$ be a separated morphism of finite type. Being proper is local on the base (see Morphisms, Lemma 42.3), so if $S = \bigcup_\alpha S_\alpha$ is an open affine cover, and if we denote $X_\alpha := f^{-1}(S_\alpha)$, then it is enough to show that $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is proper for all α. Since S_α is affine, if the map $f|_{X_\alpha}$ satisfies (3), then it will satisfy (1) by assumption, and will be proper. To finish the reduction to the case S is affine, we must show that if $f : X \to S$ is separated of finite type satisfying (3), then $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is separated of finite type satisfying (3). Separatedness and finite type are clear. To see (3), notice that $\mathbf{A}^n \times X_\alpha$ is the open preimage of $\mathbf{A}^n \times S_\alpha$ under the map $1 \times f$. Fix a closed set $Z \subset \mathbf{A}^n \times X_\alpha$. Let $\bar{Z}$ denote the closure of Z in $\mathbf{A}^n \times X$. Then for topological reasons,

$$1 \times f(\bar{Z}) \cap \mathbf{A}^n \times S_\alpha = 1 \times f(Z).$$

Hence $1 \times f(Z)$ is closed, and we have reduced the proof of (3) $\Rightarrow$ (1) to the affine case.

Assume S affine, and $f : X \to S$ separated of finite type. We can apply Chow's Lemma 11.1 to get $\pi : X' \to X$ proper surjective and $X' \to \mathbf{P}^n_S$ an immersion. If X is proper over S, then $X' \to S$ is proper (Morphisms, Lemma 42.4). Since $\mathbf{P}^n_S \to S$ is separated, we conclude that $X' \to \mathbf{P}^n_S$ is proper (Morphisms, Lemma 42.7) and hence a closed immersion (Schemes, Lemma 10.4). Conversely, assume $X' \to \mathbf{P}^n_S$ is a closed immersion. Consider the diagram:

(12.1.1)
$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\pi"'] & \mathbf{P}^n_S \arrow[d] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

All maps are a priori proper except for $X \to S$. Hence we conclude that $X \to S$ is proper by Morphisms, Lemma 42.8. Therefore, we have shown that $X \to S$ is proper if and only if $X' \to \mathbf{P}^n_S$ is a closed immersion.

Assume S is affine and (3) holds, and let n, X', π be as above. Since being a closed morphism is local on the base, the map $X \times \mathbf{P}^n \to S \times \mathbf{P}^n$ is closed since by (3) $X \times \mathbf{A}^n \to S \times \mathbf{A}^n$ is closed and since projective space is covered by copies of affine n-space, see Constructions, Lemma 13.3. By Morphisms, Lemma 42.5 the morphism

$$X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n$$

is proper. Since $\mathbf{P}^n$ is separated, the projection

$$X' \times_S \mathbf{P}^n_S = \mathbf{P}^n_{X'} \to X'$$

will be separated as it is just a base change of a separated morphism. Therefore, the map $X' \to X' \times_S \mathbf{P}^n_S$ is proper, since it is a section to a separated map (see Schemes, Lemma 21.12). Composing all these proper morphisms

$$X' \to X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n \to S \times \mathbf{P}^n = \mathbf{P}^n_S$$

we see that the map $X' \to \mathbf{P}^n_S$ is proper, and hence a closed immersion. □

If the base is Noetherian we can show that the valuative criterion holds using only discrete valuation rings. First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of schemes having the following shape

(12.1.2)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

with A a valuation ring and K its field of fractions.

Lemma 12.2. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. The following are equivalent:*

1. *The morphism f is separated.*
2. *For any diagram (12.1.2) there is at most one dotted arrow.*
3. *For all diagrams (12.1.2) with A a discrete valuation ring there is at most one dotted arrow.*
4. *For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (12.1.2) such that the morphism $\mathrm{Spec}(K) \to X$ is the canonical one (see Schemes, Section 13) there is at most one dotted arrow.*

Proof. Clearly (1) implies (2), (2) implies (3), and (3) implies (4). It remains to show (4) implies (1). Assume (4). We begin by reducing to S affine. Being separated is a local on the base (see Schemes, Lemma 21.8). Hence, as in the proof of Lemma 12.1, if we can show that whenever $X \to S$ has (4) that the restriction $X_\alpha \to S_\alpha$ has (4) where $S_\alpha \subset S$ is an (affine) open subset and $X_\alpha := f^{-1}(S_\alpha)$, then we will be done. The generic points of the irreducible components of X_α will be the generic points of irreducible components of X, since X_α is open in X. Therefore, any two distinct dotted arrows in the diagram

(12.2.1)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & S_\alpha \end{tikzcd}$$

would then give two distinct arrows in diagram (12.1.2) via the maps $X_\alpha \to X$ and $S_\alpha \to S$, which is a contradiction. Thus we have reduced to the case S is affine. We remark that in the course of this reduction, we prove that if $X \to S$ has (4) then the restriction $U \to V$ has (4) for opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$.

We next wish to reduce to the case $X \to S$ is finite type. Assume that we know (4) implies (1) when X is finite type. Since S is Noetherian and X is locally of finite type over S we see X is locally Noetherian as well (see Morphisms, Lemma 16.6). Thus, $X \to S$ is quasi-separated (see Properties, Lemma 5.4), and therefore we may apply the valuative criterion to check whether X is separated (see Schemes, Lemma 22.2). Let $X = \bigcup_\alpha X_\alpha$ be an affine open cover of X. Given any two dotted arrows, in a diagram (12.1.2), the image of the closed points of Spec A will fall in two sets X_α and X_β. Since $X_\alpha \cup X_\beta$ is open, for topological reasons it must contain the image of Spec(A) under both maps. Therefore, the two dotted arrows factor through $X_\alpha \cup X_\beta \to X$, which is a scheme of finite type over S. Since $X_\alpha \cup X_\beta$ is an open subset of X, by our previous remark, $X_\alpha \cup X_\beta$ satisfies (4), so by assumption, is separated. This implies the two given dotted arrows are the same. Therefore, we have reduced to $X \to S$ is finite type.

Assume $X \to S$ of finite type and assume (4). Since $X \to S$ is finite type, and S is an affine Noetherian scheme, X is also Noetherian (see Morphisms, Lemma 16.6). Therefore, $X \to X \times_S X$ will be a quasi-compact immersion of Noetherian schemes. We proceed by contradiction. Assume that $X \to X \times_S X$ is not closed. Then, there is some $y \in X \times_S X$ in the closure of the image that is not in the image. As X is Noetherian it has finitely many irreducible components. Therefore, y is in the closure of the image of one of the irreducible components $X_0 \subset X$. Give X_0 the reduced induced structure. The composition $X_0 \to X \to X \times_S X$ factors through the closed subscheme $X_0 \times_S X_0 \subset X \times_S X$. Denote the closure of $\Delta(X_0)$ in $X_0 \times_S X_0$ by $\bar{X}_0$ (again as a reduced closed subscheme). Thus $y \in \bar{X}_0$. Since $X_0 \to X_0 \times_S X_0$ is an immersion, the image of X_0 will be open in $\bar{X}_0$. Hence X_0 and $\bar{X}_0$ are birational. Since $\bar{X}_0$ is a closed subscheme of a Noetherian scheme, it is Noetherian. Thus, the local ring $\mathcal{O}_{\bar{X}_0,y}$ is a local Noetherian domain with fraction field K equal to the function field of X_0. By the Krull-Akizuki theorem (see Algebra, Lemma 115.12), there exists a discrete valuation ring A dominating

$\mathcal{O}_{\bar{X}_0,y}$ with fraction field K. This allows to to construct a diagram:

(12.2.2)
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X_0 \arrow[d, "\Delta"] \\ A \arrow[r] \arrow[ur, dashed] & X_0 \times_S X_0 \end{tikzcd}$$

which sends Spec K to the generic point of $\Delta(X_0)$ and the closed point of A to $y \in X_0 \times_S X_0$ (use the material in Schemes, Section 13 to construct the arrows). There cannot even exist a set theoretic dotted arrow, since y is not in the image of Δ by our choice of y. By categorical means, the existence of the dotted arrow in the above diagram is equivalent to the uniqueness of the dotted arrow in the following diagram:

(12.2.3)
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X_0 \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

Therefore, we have non-uniqueness in this latter diagram by the nonexistence in the first. Therefore, X_0 does not satisfy uniqueness for discrete valuation rings, and since X_0 is an irreducible component of X, we have that $X \to S$ does not satisfy (4). Therefore, we have shown (4) implies (1). □

Lemma 12.3. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of finite type. The following are equivalent:*

1. *The morphism f is proper.*
2. *For any diagram (12.1.2) there exists exactly one dotted arrow.*
3. *For all diagrams (12.1.2) with A a discrete valuation ring there exists exactly one dotted arrow.*
4. *For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (12.1.2) such that the morphism $\operatorname{Spec}(K) \to X$ is the canonical one (see Schemes, Section 13) there exists exactly one dotted arrow.*

Proof. (1) implies (2) implies (3) implies (4). We will now show (4) implies (1). As in the proof of Lemma 12.2, we can reduce to the case S is affine, since properness is local on the base, and if $X \to S$ satisfies (4), then $X_\alpha \to S_\alpha$ does as well for open $S_\alpha \subset S$ and $X_\alpha = f^{-1}(S_\alpha)$.

Now S is a Noetherian scheme, and so X is as well, since $X \to S$ is of finite type. Now we may use Chow's lemma (Cohomology of Schemes, Lemma 16.1) to get a surjective, proper, birational $X' \to X$ and an immersion $X' \to \mathbf{P}^n_S$. We wish to show $X \to S$ is universally closed. As in the proof of Lemma 12.1, it is enough to check that $X' \to \mathbf{P}^n_S$ is a closed immersion. For the sake of contradiction, assume that $X' \to \mathbf{P}^n_S$ is not a closed immersion. Then there is some $y \in \mathbf{P}^n_S$ that is in the closure of the image of X', but is not in the image. So y is in the closure of the image of an irreducible component X'_0 of X', but not in the image. Let $\bar{X}'_0 \subset \mathbf{P}^n_S$ be the closure of the image of X'_0. As $X' \to \mathbf{P}^n_S$ is an immersion of Noetherian schemes, the morphism $X'_0 \to \bar{X}'_0$ is open and dense. By Algebra, Lemma 115.12 or Properties, Lemma 5.9 we can find a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}'_0,y}$

and with identical field of fractions K. It is clear that K is the residue field at the generic point of X'_0. Thus the solid commutative diagram

(12.3.1)
$$\begin{tikzcd} \mathrm{Spec}\ K \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[d] \\ \mathrm{Spec}\ A \arrow[r, dashed] \arrow[ru, dashed] \arrow[rru] & X \arrow[r] & S \end{tikzcd}$$

Note that the closed point of A maps to $y \in \mathbf{P}^n_S$. By construction, there does not exist a set theoretic lift to X'. As $X' \to X$ is birational, the image of X'_0 in X is an irreducible component X_0 of X and K is also identified with the function field of X_0. Hence, as $X \to S$ is assumed to satisfy (4), the dotted arrow $\mathrm{Spec}(A) \to X$ exists. Since $X' \to X$ is proper, the dotted arrow lifts to the dotted arrow $\mathrm{Spec}(A) \to X'$ (use Schemes, Proposition 20.6). We can compose this with the immersion $X' \to \mathbf{P}^n_S$ to obtain another morphism (not depicted in the diagram) from $\mathrm{Spec}(A) \to \mathbf{P}^n_S$. Since $\mathbf{P}^n_S$ is proper over S, it satisfies (2), and so these two morphisms agree. This is a contradiction, for we have constructed the forbidden lift of our original map $\mathrm{Spec}(A) \to \mathbf{P}^n_S$ to X'. □

Here is an application of Chow's lemma which goes in a slightly different direction.

Lemma 12.4. *Assumptions and notation as in Situation 7.1. If*

1. *f is proper, and*
2. *f_0 is locally of finite type,*

then there exists an i such that f_i is proper.

Proof. By Lemma 7.5 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact, see Schemes, Lemma 21.15. By Lemma 11.1 we can choose a diagram

$$\begin{tikzcd} X_0 \arrow[rd] & X'_0 \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{Y_0} \arrow[ld] \\ & Y_0 & \end{tikzcd}$$

where $X'_0 \to \mathbf{P}^n_{Y_0}$ is an immersion, and $\pi : X'_0 \to X_0$ is proper and surjective. Introduce $X' = X'_0 \times_{Y_0} Y$ and $X'_i = X'_0 \times_{Y_0} Y_i$. By Morphisms, Lemmas 42.4 and 42.5 we see that $X' \to Y$ is proper. Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms, Lemma 42.7). By Morphisms, Lemma 42.8 it suffices to prove that $X'_i \to Y_i$ is proper for some i. By Lemma 7.4 we find that $X'_i \to \mathbf{P}^n_{Y_i}$ is a closed immersion for i large enough. Then $X'_i \to Y_i$ is proper and we win. □

Lemma 12.5. *Let $f : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ with $X_i \to S$ proper and of finite presentation.*

Proof. By Proposition 8.6 we can find a closed immersion $X \to Y$ with Y separated and of finite presentation over S. By Lemma 11.1 we can find a diagram

$$\begin{tikzcd} Y \arrow[rd] & Y' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[ld] \\ & S & \end{tikzcd}$$

where $Y' \to \mathbf{P}^n_S$ is an immersion, and $\pi : Y' \to Y$ is proper and surjective. By Lemma 8.4 we can write $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation. Denote $X'_i \subset Y'$, resp. $X' \subset Y'$ the scheme theoretic inverse image of $X_i \subset Y$, resp. $X \subset Y$. Then $\lim X'_i = X'$. Since $X' \to S$ is proper (Morphisms, Lemmas 42.4), we see that $X' \to \mathbf{P}^n_S$ is a closed immersion (Morphisms, Lemma 42.7). Hence for i large enough we find that $X'_i \to \mathbf{P}^n_S$ is a closed immersion by Lemma 3.17. Thus X'_i is proper over S. For such i the morphism $X_i \to S$ is proper by Morphisms, Lemma 42.8. $\square$

Lemma 12.6. *Let $f : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Then $(X \to S) = \lim(X_i \to S_i)$ with S_i of finite type over $\mathbf{Z}$ and $X_i \to S_i$ proper and of finite presentation.*

Proof. By Lemma 12.5 we can write $X = \lim_{k \in K} X_k$ with $X_k \to S$ proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 4.4) we can write $S = \lim_{j \in J} S_j$ with S_j of finite type over $\mathbf{Z}$. For each k there exists a j and a morphism $X_{k,j} \to S_j$ of finite presentation with $X_k \cong S \times_{S_j} X_{k,j}$ as schemes over S, see Lemma 9.1. After increasing j we may assume $X_{k,j} \to S_j$ is proper, see Lemma 12.4. The set I will be consist of these pairs (k, j) and the corresponding morphism is $X_{k,j} \to S_j$. For every $k' \geq k$ we can find a $j' \geq j$ and a morphism $X_{j',k'} \to X_{j,k}$ over $S_{j'} \to S_j$ whose base change to S gives the morphism $X_{k'} \to X_k$ (follows again from Lemma 9.1). These morphisms form the transition morphisms of the system. Some details omitted. $\square$

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms, Definition 5.5.

Lemma 12.7. *Assumptions and notation as in Situation 7.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module. Denote $\mathcal{F}$ and $\mathcal{F}_i$ the pullbacks of $\mathcal{F}_0$ to X and X_i. Assume*

1. *f_0 is locally of finite type,*
2. *$\mathcal{F}_0$ is of finite type,*
3. *the scheme theoretic support of $\mathcal{F}$ is proper over Y.*

Then the scheme theoretic support of $\mathcal{F}_i$ is proper over Y_i for some i.

Proof. We may replace X_0 by the scheme theoretic support of $\mathcal{F}_0$. By Morphisms, Lemma 5.3 this guarantees that X_i is the support of $\mathcal{F}_i$ and X is the support of $\mathcal{F}$. Then, if $Z \subset X$ denotes the scheme theoretic support of $\mathcal{F}$, we see that $Z \to X$ is a universal homeomorphism. We conclude that $X \to Y$ is proper as this is true for $Z \to Y$ by assumption, see Morphisms, Lemma 42.8. By Lemma 12.4 we see that $X_i \to Y$ is proper for some i. Then it follows that the scheme theoretic support Z_i of $\mathcal{F}_i$ is proper over Y by Morphisms, Lemmas 42.6 and 42.4. $\square$

13. Universally closed morphisms

In this section we discuss when a quasi-compact but not necessarily separated morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

Lemma 13.1. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $g : T \to S$ be a morphism of schemes. Let $t \in T$ be a point and $Z \subset X_T$ be a closed* *subscheme such that $Z \cap X_t = \emptyset$. Then there exists an open neighbourhood $V \subset T$ of t, a commutative diagram*

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) *the morphism $b : T' \to S$ is locally of finite presentation,*
(2) *with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and*
(3) *$Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.*

Proof. Let $s = g(t)$. During the proof we may always replace T by an open neighbourhood of t. Hence we may also replace S by an open neighbourhood of s. Thus we may and do assume that T and S are affine. Say $S = \text{Spec}(A)$, $T = \text{Spec}(B)$, g is given by the ring map $A \to B$, and t correspond to the prime ideal $\mathfrak{q} \subset B$.

As $X \to S$ is quasi-compact and S is affine we may write $X = \bigcup_{i=1,\ldots,n} U_i$ as a finite union of affine opens. Write $U_i = \text{Spec}(C_i)$. In particular we have $X_T = \bigcup_{i=1,\ldots,n} U_{i,T} = \bigcup_{i=1,\ldots n} \text{Spec}(C_i \otimes_A B)$. Let $I_i \subset C_i \otimes_A B$ be the ideal corresponding to the closed subscheme $Z \cap U_{i,T}$. The condition that $Z \cap X_t = \emptyset$ signifies that I_i generates the unit ideal in the ring

$$C_i \otimes_A \kappa(\mathfrak{q}) = (B \setminus \mathfrak{q})^{-1} (C_i \otimes_A B/\mathfrak{q}C_i \otimes_A B)$$

Since $I_i(B \setminus \mathfrak{q})^{-1}(C_i \otimes_A B) = (B \setminus \mathfrak{q})^{-1} I_i$ this means that $1 = x_i/g_i$ for some $x_i \in I_i$ and $g_i \in B$, $g_i \notin \mathfrak{q}$. Thus, clearing denominators we can find a relation of the form

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = g_i$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$, and $g_i \in B$, $g_i \notin \mathfrak{q}$. After replacing B by $B_{g_1 \ldots g_n}$, i.e., after replacing T by a smaller affine neighbourhood of t, we may assume the equations read

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = 1$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$.

To finish the argument write B as a colimit of finitely presented A-algebras B_λ over a directed partially ordered set Λ. For each λ set $\mathfrak{q}_\lambda = (B_\lambda \to B)^{-1}(\mathfrak{q})$. For sufficiently large $\lambda \in \Lambda$ we can find

(1) an element $x_{i,\lambda} \in C_i \otimes_A B_\lambda$ which maps to x_i,
(2) elements $f_{i,j,\lambda} \in \mathfrak{q}_{i,\lambda}$ mapping to $f_{i,j}$, and
(3) elements $c_{i,j,\lambda} \in C_i \otimes_A B_\lambda$ mapping to $c_{i,j}$.

After increasing λ a bit more the equation

$$x_{i,\lambda} + \sum\nolimits_j f_{i,j,\lambda} c_{i,j,\lambda} = 1$$

will hold. Fix such a λ and set $T' = \text{Spec}(B_\lambda)$. Then $t' \in T'$ is the point corresponding to the prime $\mathfrak{q}_\lambda$. Finally, let $Z' \subset X_{T'}$ be the scheme theoretic closure of $Z \to X_T \to X_{T'}$. As $X_T \to X_{T'}$ is affine, we can compute Z' on the affine open pieces $U_{i,T'}$ as the closed subscheme associated to $\text{Ker}(C_i \otimes_A B_\lambda \to C_i \otimes_A B/I_i)$, see Morphisms, Example 6.4. Hence $x_{i,\lambda}$ is in the ideal defining Z'. Thus the last displayed equation shows that $Z' \cap X_{t'}$ is empty. $\square$

Lemma 13.2. *Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent*

1. *f is universally closed,*
2. *for every morphism $S' \to S$ which is locally of finite presentation the base change $X_{S'} \to S'$ is closed, and*
3. *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.*

Proof. It is clear that (1) implies (2). Let us prove that (2) implies (1). Suppose that the base change $X_T \to T$ is not closed for some scheme T over S. By Schemes, Lemma 19.8 this means that there exists some specialization $t_1 \leadsto t$ in T and a point $\xi \in X_T$ mapping to t_1 such that ξ does not specialize to a point in the fibre over t. Set $Z = \overline{\{\xi\}} \subset X_T$. Then $Z \cap X_t = \emptyset$. Apply Lemma 13.1. We find an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

1. the morphism $b : T' \to S$ is locally of finite presentation,
2. with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and
3. $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

Clearly this means that $X_{T'} \to T'$ maps the closed subset Z' to a subset of T' which contains $a(t_1)$ but not $t' = a(t)$. Since $a(t_1) \leadsto a(t) = t'$ we conclude that $X_{T'} \to T'$ is not closed. Hence we have shown that $X \to S$ not universally closed implies that $X_{T'} \to T'$ is not closed for some $T' \to S$ which is locally of finite presentation. In order words (2) implies (1).

Assume that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed for every integer n. We want to prove that $X_T \to T$ is closed for every scheme T which is locally of finite presentation over S. We may of course assume that T is affine and maps into an affine open V of S (since $X_T \to T$ being a closed is local on T). In this case there exists a closed immersion $T \to \mathbf{A}^n \times V$ because $\mathcal{O}_T(T)$ is a finitely presented $\mathcal{O}_S(V)$-algebra, see Morphisms, Lemma 22.2. Then $T \to \mathbf{A}^n \times S$ is a locally closed immersion. Hence we get a cartesian diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d, "f_T"'] & \mathbf{A}^n \times X \arrow[d, "f_n"] \\ T \arrow[r] & \mathbf{A}^n \times S \end{tikzcd}$$

of schemes where the horizontal arrows are locally closed immersions. Hence any closed subset $Z \subset X_T$ can be written as $X_T \cap Z'$ for some closed subset $Z' \subset \mathbf{A}^n \times X$. Then $f_T(Z) = T \cap f_n(Z')$ and we see that if f_n is closed, then also f_T is closed. $\square$

Lemma 13.3. *Let $f : X \to S$ be a finite type morphism of schemes. Assume S is locally Noetherian. Then the following are equivalent*

1. *f is universally closed,*
2. *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed,*
3. *for any diagram (12.1.2) there exists some dotted arrow,*

(4) *for all diagrams (12.1.2) with A a discrete valuation ring there exists some dotted arrow.*

Proof. The equivalence of (1) and (2) is a special case of Lemma 13.2. The equivalence of (1) and (3) is a special case of Schemes, Proposition 20.6. Trivially (3) implies (4). Thus all we have to do is prove that (4) implies (2). We will prove that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed by the criterion of Schemes, Lemma 19.8. Pick n and a specialization $z \rightsquigarrow z'$ of points in $\mathbf{A}^n \times S$ and a point $y \in \mathbf{A}^n \times X$ lying over z. Note that $\kappa(y)$ is a finitely generated field extension of $\kappa(z)$ as $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is of finite type. Hence by Properties, Lemma 5.9 or Algebra, Lemma 115.12 implies that there exists a discrete valuation ring $A \subset \kappa(y)$ with fraction field $\kappa(z)$ dominating the image of $\mathcal{O}_{\mathbf{A}^n \times S, z'}$ in $\kappa(z)$. This gives a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\kappa(y)) \arrow[r] \arrow[d] & \mathbf{A}^n \times X \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathbf{A}^n \times S \arrow[r] & S \end{tikzcd}$$

Now property (4) implies that there exists a morphism $\mathrm{Spec}(A) \to X$ which fits into this diagram. Since we already have the morphism $\mathrm{Spec}(A) \to \mathbf{A}^n$ from the left lower horizontal arrow we also get a morphism $\mathrm{Spec}(A) \to \mathbf{A}^n \times X$ fitting into the left square. Thus the image $y' \in \mathbf{A}^n \times X$ of the closed point is a specialization of y lying over z'. This proves that specializations lift along $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ and we win. $\square$

14. Limits and dimensions of fibres

The following lemma is most often used in the situation of Lemma 9.1 to assure that if the fibres of the limit have dimension $\leq d$, then the fibres at some finite stage have dimension $\leq d$.

Lemma 14.1. *Let I be a directed partially ordered set. Let $(f_i : X_i \to S_i)$ be an inverse system of morphisms of schemes over I. Assume*

(1) *all the morphisms $S_{i'} \to S_i$ are affine,*
(2) *all the schemes S_i are quasi-compact and quasi-separated,*
(3) *the morphisms f_i are of finite type, and*
(4) *the morphisms $X_{i'} \to X_i \times_{S_i} S_{i'}$ are closed immersions.*

Let $f : X = \lim_i X_i \to S = \lim_i S_i$ be the limit. Let $d \geq 0$. If every fibre of f has dimension $\leq d$, then for some i every fibre of f_i has dimension $\leq d$.

Proof. For each i let $U_i = \{x \in X_i \mid \dim_x((X_i)_{f_i(x)}) \leq d\}$. This is an open subset of X_i, see Morphisms, Lemma 29.4. Set $Z_i = X_i \setminus U_i$ (with reduced induced scheme structure). We have to show that $Z_i = \emptyset$ for some i. If not, then $Z = \lim Z_i \neq \emptyset$, see Lemma 3.4. Say $z \in Z$ is a point. Note that $Z \subset X$ is a closed subscheme. Set $s = f(z)$. For each i let $s_i \in S_i$ be the image of s. We remark that Z_s is the limit of the schemes $(Z_i)_{s_i}$ and Z_s is also the limit of the schemes $(Z_i)_{s_i}$ base changed to $\kappa(s)$. Moreover, all the morphisms

$$Z_s \longrightarrow (Z_{i'})_{s_{i'}} \times_{\mathrm{Spec}(\kappa(s_{i'}))} \mathrm{Spec}(\kappa(s)) \longrightarrow (Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s)) \longrightarrow X_s$$

are closed immersions by assumption (4). Hence Z_s is the scheme theoretic intersection of the closed subschemes $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$ in X_s. Since all the irreducible components of the schemes $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$ have dimension $> d$ and contain z we conclude that Z_s contains an irreducible component of dimension $> d$ passing through z which contradicts the fact that $Z_s \subset X_s$ and $\dim(X_s) \leq d$. □

Lemma 14.2. *Notation and assumptions as in Situation 7.1. If*

(1) *f is a quasi-finite morphism, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is quasi-finite.

Proof. Follows immediately from Lemma 14.1. □

Lemma 14.3. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of finite presentation. Let $d \geq 0$ be an integer. If $Z \subset X$ be a closed subscheme such that $\dim(Z_s) \leq d$ for all $s \in S$, then there exists a closed subscheme $Z' \subset X$ such that*

(1) *$Z \subset Z'$,*
(2) *$Z' \to X$ is of finite presentation, and*
(3) *$\dim(Z'_s) \leq d$ for all $s \in S$.*

Proof. By Proposition 4.4 we can write $S = \lim S_i$ as the limit of a directed inverse system of Noetherian schemes with affine transition maps. By Lemma 9.1 we may assume that there exist a system of morphisms $f_i : X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_{S_i} S_{i'}$ for all $i' \geq i$ and such that $X = X_i \times_{S_i} S$. Let $Z_i \subset X_i$ be the scheme theoretic image of $Z \to X \to X_i$. Then for $i' \geq i$ the morphism $X_{i'} \to X_i$ maps $Z_{i'}$ into Z_i and the induced morphism $Z_{i'} \to Z_i \times_{S_i} S_{i'}$ is a closed immersion. By Lemma 14.1 we see that the dimension of the fibres of $Z_i \to S_i$ all have dimension $\leq d$ for a suitable $i \in I$. Fix such an i and set $Z' = Z_i \times_{S_i} S \subset X$. Since S_i is Noetherian, we see that X_i is Noetherian, and hence the morphism $Z_i \to X_i$ is of finite presentation. Therefore also the base change $Z' \to X$ is of finite presentation. Moreover, the fibres of $Z' \to S$ are base changes of the fibres of $Z_i \to S_i$ and hence have dimension $\leq d$. □

15. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
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