INTRODUCING ALGEBRAIC STACKS

Contents

1. Why read this?

We give a very informal introduction to algebraic stacks aimed at graduate students and advanced undergraduates. The goal is to quickly introduce a simple language which you can use to think about local and global properties of your favorite moduli problem. Having done this it should be possible to ask yourself well-posed questions about moduli problems and to start solving them, whilst assuming a general theory exists. If you end up with an interesting result, you can go back to the general theory in the other parts of the stacks project and fill in the gaps as needed.

The point of view we take here is very close to the point of view taken in [KM85] and [Mum65].

2. Preliminary

Let S be a scheme. An *elliptic curve* over S is a triple $(E, f, 0)$ where E is a scheme and $f : E \to S$ and $0 : S \to E$ are morphisms of schemes such that

(1) $f : E \to S$ is proper, smooth of relative dimension 1,
(2) for every $s \in S$ the fibre E_s is a connected curve of genus 1, i.e., $H^0(E_s, \mathcal{O})$ and $H^1(E_s, \mathcal{O})$ both are 1-dimensional $\kappa(s)$-vector spaces, and
(3) 0 is a section of f.

Given elliptic curves $(E, f, 0)/S$ and $(E', f', 0')/S'$ a *morphism of elliptic curves over* $a : S \to S'$ is a morphism $\alpha : E \to E'$ such that the diagram

is commutative and the inner square is cartesian, in other words the morphism α induces an isomorphism $E \to S \times_{S'} E'$. We are going to define the stack of elliptic curves $\mathcal{M}_{1,1}$. In the rest of the stacks project we work out the method introduced in Deligne and Mumford's paper [DM69] which consists in presenting $\mathcal{M}_{1,1}$ as a category endowed with a functor

$$p : \mathcal{M}_{1,1} \longrightarrow Sch, \quad (E, f, 0)/S \longmapsto S$$

This means you work with fibred categories over the categories of schemes, topologies, stacks fibred in groupoids, coverings, etc, etc. In this chapter we throw all of that out of the window and we think about it a bit differently – probably closer to how the initiators of the theory started thinking about it themselves.

3. The moduli stack of elliptic curves

Here is what we are going to do:

(1) Start with your favorite category of schemes Sch.
(2) Add a new symbol $\mathcal{M}_{1,1}$.
(3) A morphism $S \to \mathcal{M}_{1,1}$ **is** an elliptic curve $(E, f, 0)$ over S.
(4) A diagram

is commutative if and only if there exists a morphism $\alpha : E \to E'$ of elliptic curves over $a : S \to S'$. We say α *witnesses* the commutativity of the diagram.
(5) Note that commutative diagrams glue as follows

because $\alpha' \circ \alpha$ witnesses the commutativity of the outer triangle if α and α' witness the commutativity of the left and right triangles.
(6) The composition

$$S \xrightarrow{a} S' \xrightarrow{(E', f', 0')} \mathcal{M}_{1,1}$$

is given by $(E' \times_{S'} S, f' \times_{S'} S, 0' \times_{S'} S)$.

At the end of this procedure we have enlarged the category Sch of schemes with exactly one object...

Except that we haven't defined what a morphism from $\mathcal{M}_{1,1}$ to a scheme T is. The answer is that it is the weakest possible notion such that compositions make sense. Thus a morphism $F : \mathcal{M}_{1,1} \to T$ is a rule which to every elliptic curve $(E, f, 0)/S$

associates a morphism $F(E, f, 0) : S \to T$ such that given any commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\quad a \quad} & S' \\
 & {\scriptstyle (E,f,0)} \searrow \quad \swarrow {\scriptstyle (E',F',0')} & \\
 & \mathcal{M}_{1,1} &
\end{array}
$$

the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\quad a \quad} & S' \\
 & {\scriptstyle F(E,f,0)} \searrow \quad \swarrow {\scriptstyle F(E',F',0')} & \\
 & T &
\end{array}
$$

is commutative also. An example is the j-invariant

$$ j : \mathcal{M}_{1,1} \longrightarrow \mathbf{A}^1_{\mathbf{Z}} $$

which you may have heard of. Aha, so now we're done...

Except, no we're not! We still have to define a notion of morphisms $\mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$. This we do in exactly the same way as before, i.e., a morphism $F : \mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$ is a rule which to every elliptic curve $(E, f, 0)/S$ associates another elliptic curve $F(E, f, 0)$ preserving commutativity of diagrams as above. However, since I don't know of a nontrivial example of such a functor, I'll just define the set of morphisms from $\mathcal{M}_{1,1}$ to itself to consist of the identity for now.

I hope you see how to add other objects to this enlarged category. Somehow it seems intuitively clear that given any "well-behaved" moduli problem we can perform the construction above and add an object to our category. In fact, much of modern day algebraic geometry takes place in such a universe where *Sch* is enlarged with countably many (explicitly constructed) moduli stacks.

You may object that the category we obtain isn't a category because there is a "vagueness" about when diagrams commute and which combinations of diagrams continue to commute as we have to produce a witness to the commutativity. However, it turns out that this, the idea of having witnesses to commutativity, is a valid approach to 2-categories! Thus we stick with it.

4. Fibre products

The question we pose here is what should be the fibre product

$$
\begin{array}{ccc}
 & ? & \\
 \swarrow & & \searrow \\
S & & S' \\
 {\scriptstyle (E,f,0)} \searrow & & \swarrow {\scriptstyle (E',f',0')} \\
 & \mathcal{M}_{1,1} &
\end{array}
$$

The answer: A morphism from a scheme T into ? should be a triple (a, a', α) where $a : T \to S$, $a' : T \to S'$ are morphisms of schemes and where $\alpha : E \times_{S,a} T \to E' \times_{S',a'} T$ is an isomorphism of elliptic curves over T. This makes sense because of our definition of composition and commutative diagrams earlier in the discussion.

Lemma 4.1 (Key fact). *The functor $Sch^{opp} \to Sets$, $T \mapsto \{(a, a', \alpha)$ as above$\}$ is representable by a scheme $S \times_{\mathcal{M}_{1,1}} S'$.*

Proof. Idea of proof. Relate this functor to

$$Isom_{S \times S'}(E \times S', S \times E')$$

and use Grothendieck's theory of Hilbert schemes. $\qquad\qquad\qquad\square$

Remark 4.2. We have the formula $S \times_{\mathcal{M}_{1,1}} S' = (S \times S') \times_{\mathcal{M}_{1,1} \times \mathcal{M}_{1,1}} \mathcal{M}_{1,1}$. Hence the key fact is a property of the diagonal $\Delta_{\mathcal{M}_{1,1}}$ of $\mathcal{M}_{1,1}$.

In any case the key fact allows us to make the following definition.

Definition 4.3. We say a morphism $S \to \mathcal{M}_{1,1}$ is *smooth* if for every morphism $S' \to \mathcal{M}_{1,1}$ the projection morphism

$$S \times_{\mathcal{M}_{1,1}} S' \longrightarrow S'$$

is smooth.

Note that this is compatible with the notion of a smooth morphism of schemes as the base change of a smooth morphism is smooth. Moreover, it is clear how to extend this definition to other properties of morphisms into $\mathcal{M}_{1,1}$ (or your own favorite moduli stack). In particular we will use it below for *surjective* morphisms.

5. The definition

We'll formulate it as a definition and not as a result since we expect the reader to try out other cases (not just the stack $\mathcal{M}_{1,1}$ and not just Sch the category of all schemes).

Definition 5.1. We say $\mathcal{M}_{1,1}$ is an *algebraic stack* if and only if

(1) We have descent for objects for the étale topology on Sch.
(2) The key fact holds.
(3) there exists a surjective and smooth morphism $S \to \mathcal{M}_{1,1}$.

The first condition is a "sheaf property". We're going to spell it out since there is a technical point we should make. Suppose given a scheme S and an étale covering $\{S_i \to S\}$ and morphisms $e_i : S_i \to \mathcal{M}_{1,1}$ such that the diagrams

commute. The sheaf condition does *not* guarantee the existence of a morphism $e : S \to \mathcal{M}_{1,1}$ in this situation. Namely, we need to pick witnesses α_{ij} for the diagrams above and require that

$$\mathrm{pr}_{02}^* \alpha_{ik} = \mathrm{pr}_{12} \alpha_{jk} \circ \mathrm{pr}_{01}^* \alpha_{ij}$$

as witnesses over $S_i \times_S S_j \times_S S_k$. I think it is clear what this means... If not, then I'm afraid you'll have to read some of the material on categories fibred in groupoids, etc. In any case, the displayed equation is often called the *cocycle condition*. A more precise statement of the "sheaf property" is: given $\{S_i \to S\}$, $e_i : S_i \to \mathcal{M}_{1,1}$

and witnesses α_{ij} satisfying the cocycle condition, there exists a unique (up to unique isomorphism) $e : S \to \mathcal{M}_{1,1}$ with $e_i \cong e|_{S_i}$ recovering the α_{ij}.

As you can see even formulating a precise statement takes a bit of work. The proof of this "sheaf property" relies on a fundamental technique in algebraic geometry, namely descent theory. My suggestion is to initially simply accept the "sheaf property" holds, and see what it implies in practice. In fact, a certain amount of mental agility is required to boil the "sheaf property" down to a manageable statement that you can fit on a napkin. Perhaps the simplest variant which is already a bit interesting is the following: Suppose we have a Galois extension $K \subset L$ of fields with Galois group $G = \mathrm{Gal}(L/K)$. Set $T = \mathrm{Spec}(L)$ and $S = \mathrm{Spec}(K)$. Then $\{T \to S\}$ is an étale covering. Let $(E, f, 0)$ be an elliptic curve over L. (Yes, this just means that $E \subset \mathbf{P}^2_L$ is given by a Weierstrass equation and 0 is the usual point at infinity.) Denote $E_\sigma = E \times_{T, \mathrm{Spec}(\sigma)} T$ the base change. (Yes, this corresponds to applying σ to the coefficients of the Weierstrass equation, or is it σ^{-1}?) Now, suppose moreover that for every $\sigma \in G$ we are given an isomorphism

$$\alpha_\sigma : E \longrightarrow E_\sigma$$

over T. The cocycle condition above means in this situation that

$$(\alpha_\tau)^\sigma \circ \alpha_\sigma = \alpha_{\tau\sigma}$$

for $\sigma, \tau \in G$. If you've ever done any group cohomology then this should be familiar. Anyway, the "glueing" condition on $\mathcal{M}_{1,1}$ says that if you have a solution to this set of equations, then there exists an elliptic curve E' over S such that $E \cong E \times_S T$ (it says a little bit more because it also tells you how to recover the α_σ).

Challenge: Can you prove this entirely using only elliptic curves defined in terms of Weierstrass equations?

6. A smooth cover

The last thing we have to do is find a smooth cover of $\mathcal{M}_{1,1}$. In fact, in some sense the existence of a smooth cover *implies*[1] the key fact! In the case of elliptic curves we use the Weierstrass equation to construct one.

Set

$$W = \mathrm{Spec}(\mathbf{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta])$$

where $\Delta \in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6]$ is a certain polynomial (see below). Set

$$\mathbf{P}^2_W \supset E_W : zy^2 + a_1 xyz + a_3 yz^2 = x^3 + a_2 x^2 z + a_4 xz^3 + a_6 z^3.$$

Denote $f_W : E_W \to W$ the projection. Finally, denote $0_W : W \to E_W$ the section of f_W given by $(0 : 1 : 0)$. It turns out that there is a degree 12 homogeneous polynomial Δ in a_i where $\deg(a_i) = i$ such that $E_W \to W$ is smooth. You can find

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[1]This is a bit of a cheat because in checking the smoothness you have to prove something very close to the key fact – after all smoothness is defined in terms of fibre products. The advantage is that you only have to prove the existence of these fibre products in the case that on one side you have the morphism that you are trying to show provides the smooth cover.

it explicitly by computing partials of the Weierstrass equation – of course you can also look it up. You can also use pari/gp to compute it for you. Here it is

$$\Delta = -a_6 a_1^6 + a_4 a_3 a_1^5 + ((-a_3^2 - 12a_6)a_2 + a_4^2)a_1^4 +$$
$$(8a_4 a_3 a_2 + (a_3^3 + 36a_6 a_3))a_1^3 +$$
$$((-8a_3^2 - 48a_6)a_2^2 + 8a_4^2 a_2 + (-30a_4 a_3^2 + 72a_6 a_4))a_1^2 +$$
$$(16a_4 a_3 a_2^2 + (36a_3^3 + 144a_6 a_3)a_2 - 96a_4^2 a_3)a_1 +$$
$$(-16a_3^2 - 64a_6)a_2^3 + 16a_4^2 a_2^2 + (72a_4 a_3^2 + 288a_6 a_4)a_2 +$$
$$- 27a_3^4 - 216a_6 a_3^2 - 64a_4^3 - 432a_6^2$$

You may recognize the last two terms from the case $y^2 = x^3 + Ax + B$ having discriminant $-64A^3 - 432B^2 = -16(4A^3 + 27B^2)$.

Lemma 6.1. *The morphism $W \xrightarrow{(E_W, f_W, 0_W)} \mathcal{M}_{1,1}$ is smooth and surjective.*

Proof. Surjectivity follows from the fact that every elliptic curve over a field has a Weierstrass equation. We give a very rough sketch of one way to prove smoothness. Consider the sub group scheme

$$H = \left\{ \begin{pmatrix} u^2 & s & 0 \\ 0 & u^3 & 0 \\ r & t & 1 \end{pmatrix} \middle| \begin{array}{l} u \text{ unit} \\ s, r, t \text{ arbitrary} \end{array} \right\} \subset \mathrm{GL}_{3,\mathbf{Z}}$$

There is an action $H \times W \to W$ of H on the Weierstrass scheme W. To find the equations for this action write out what a coordinate change given by a matrix in H does to the general Weierstrass equation. Then it turns out the following statements hold

(1) any elliptic curve $(E, f, 0)/S$ has Zariski locally on S a Weierstrass equation,
(2) any two Weierstrass equations for $(E, f, 0)$ differ (Zariski locally) by an element of H.

Considering the fibre product $S \times_{\mathcal{M}_{1,1}} W = Isom_{S \times W}(E \times W, S \times E_W)$ we conclude that this means that the morphism $W \to \mathcal{M}_{1,1}$ is an H-torsor. Since $H \to \mathrm{Spec}(\mathbf{Z})$ is smooth, and since torsors over smooth group schemes are smooth we win. $\square$

Remark 6.2. The argument sketched above actually shows that $\mathcal{M}_{1,1} = [W/H]$ is a global quotient stack. It is true about 50% of the time that an argument proving a moduli stack is algebraic will show that it is a global quotient stack.

7. Properties of algebraic stacks

Ok, so now we know that $\mathcal{M}_{1,1}$ is an algebraic stack. What can we do with this? Well, it isn't so much the fact that it is an algebraic stack that helps us here, but more the point of view that properties of $\mathcal{M}_{1,1}$ should be encoded in the properties of morphisms $S \to \mathcal{M}_{1,1}$, i.e., in families of elliptic curves. We list some examples

Local properties:

$$\mathcal{M}_{1,1} \to \mathrm{Spec}(\mathbf{Z}) \text{ is smooth} \Leftrightarrow W \to \mathrm{Spec}(\mathbf{Z}) \text{ is smooth}$$

Idea. Local properties of an algebraic stack are encoded in the local properties of its smooth cover.

Global properties:
$$\mathcal{M}_{1,1} \text{ is quasi-compact} \Leftarrow W \text{ is quasi-compact}$$
$$\mathcal{M}_{1,1} \text{ is irreducible} \Leftarrow W \text{ is irreducible}$$

Idea. Some global properties of an algebraic stack can be read off from the corresponding property of a *suitable*[2] smooth cover.

Quasi-coherent sheaves:
$$QCoh(\mathcal{O}_{\mathcal{M}_{1,1}}) = H\text{-equivariant quasi-coherent modules on } W$$

Idea. On the one hand a quasi-coherent module on $\mathcal{M}_{1,1}$ should correspond to a quasi-coherent sheaf $\mathcal{F}_{S,e}$ on S for each morphism $e : S \to \mathcal{M}_{1,1}$. In particular for the morphism $(E_W, f_W, 0_W) : W \to \mathcal{M}_{1,1}$. Since this morphism is H-equivariant we see the quasi-coherent module $\mathcal{F}_W$ we obtain is H-equivariant. Conversely, given an H-equivariant module we can recover the sheaves $\mathcal{F}_{S,e}$ by descent theory starting with the observation that $S \times_{e,\mathcal{M}_{1,1}} W$ is an H-torsor.

Picard group:
$$\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathrm{Pic}_H(W) = \mathbf{Z}/12\mathbf{Z}$$

Idea. We have seen the first equality above. Note that $\mathrm{Pic}(W) = 0$ because the ring $\mathbf{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta]$ has trivial class group. There is an exact sequence
$$\mathbf{Z}\Delta \to \mathrm{Pic}_H(\mathbf{A}_{\mathbf{Z}}^5) \to \mathrm{Pic}_H(W) \to 0$$

The middle group equals $\mathrm{Hom}(H, \mathbf{G}_m) = \mathbf{Z}$. The image Δ is 12 because Δ has degree 12. This argument is roughly correct, see [FO10].

Étale cohomology: Let Λ be a ring. There is a first quadrant spectral sequence converging to $H_{\acute{e}tale}^{p+q}(\mathcal{M}_{1,1}, \Lambda)$ with E_2-page
$$E_2^{p,q} = H_{\acute{e}tale}^q(W \times H \times \ldots \times H, \Lambda) \quad (p \text{ factors } H)$$

Idea. Note that
$$W \times_{\mathcal{M}_{1,1}} W \times_{\mathcal{M}_{1,1}} \ldots \times_{\mathcal{M}_{1,1}} W = W \times H \times \ldots \times H$$

because $W \to \mathcal{M}_{1,1}$ is a H-torsor. The spectral sequence is the Čech-to-cohomology spectral sequence for the smooth cover $\{W \to \mathcal{M}_{1,1}\}$. For example we see that $H_{\acute{e}tale}^0(\mathcal{M}_{1,1}, \Lambda) = \Lambda$ because W is connected, and $H_{\acute{e}tale}^1(\mathcal{M}_{1,1}, \Lambda) = 0$ because $H_{\acute{e}tale}^1(W, \Lambda) = 0$ (of course this requires a proof). Of course, the smooth covering $W \to \mathcal{M}_{1,1}$ may not be "optimal" for the computation of étale cohomology.

8. Other chapters

Preliminaries

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[2] I suppose that it is possible an irreducible algebraic stack exists which doesn't have an irreducible smooth cover – but if so it is going to be quite nasty!

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