MORPHISMS OF ALGEBRAIC STACKS

Contents

1. Introduction

In this chapter we introduce some types of morphisms of algebraic stacks. A reference in the case of quasi-separated algebraic stacks with representable diagonal is [LMB00].

The goal is to extend the definition of each of the types of morphisms of algebraic spaces to morphisms of algebraic stacks. Each case is slightly different and it seems best to treat them all separately.

For morphisms of algebraic stacks which are representable by algebraic spaces we have already defined a large number of types of morphisms, see Properties of Stacks, Section 3. For each corresponding case in this chapter we have to make sure the definition in the general case is compatible with the definition given there.

2. Conventions and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2.

3. Properties of diagonals

The diagonal of an algebraic stack is closely related to the *Isom*-sheaves, see Algebraic Stacks, Lemma 10.11. By the second defining property of an algebraic stack these *Isom*-sheaves are always algebraic spaces.

Lemma 3.1. *Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then the morphism $Isom_{\mathcal{X}}(x, y) \to T$ is locally of finite type.*

Proof. By Algebraic Stacks, Lemma 16.2 we may assume that $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces. By Descent on Spaces, Lemma 10.7 it suffices to check the property fppf locally on T. Thus we may assume that x, y come from morphisms $x', y' : T \to U$. By Groupoids in Spaces, Lemma 21.1 we see that in this case $Isom_{\mathcal{X}}(x, y) = T \times_{(y', x'), U \times_S U} R$. Hence it suffices to prove that $R \to U \times_S U$ is locally of finite type. This follows from the fact that the composition $s : R \to U \times_S U \to U$ is smooth (hence locally of finite type, see Morphisms of Spaces, Lemmas 34.5 and 27.5) and Morphisms of Spaces, Lemma 23.6. $\square$

Lemma 3.2. *Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then*

(1) *$Isom_{\mathcal{X}}(y, y)$ is a group algebraic space over T, and*
(2) *$Isom_{\mathcal{X}}(x, y)$ is a pseudo torsor for $Isom_{\mathcal{X}}(y, y)$ over T.*

Proof. See Groupoids in Spaces, Definitions 5.1 and 9.1. The lemma follows immediately from the fact that $\mathcal{X}$ is a stack in groupoids. $\square$

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The *diagonal of f* is the morphism

$$\Delta_f : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

Here are two properties that every diagonal morphism has.

Lemma 3.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *Δ_f is representable by algebraic spaces, and*
(2) *Δ_f is locally of finite type.*

Proof. Let T be a scheme and let $a : T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be a morphism. By definition of the fibre product and the 2-Yoneda lemma the morphism a is given by a triple $a = (x, x', \alpha)$ where x, x' are objects of $\mathcal{X}$ over T, and $\alpha : f(x) \to f(x')$ is a morphism in the fibre category of $\mathcal{Y}$ over T. By definition of an algebraic stack the sheaves $Isom_{\mathcal{X}}(x, x')$ and $Isom_{\mathcal{Y}}(f(x), f(x'))$ are algebraic spaces over T. In this language α defines a section of the morphism $Isom_{\mathcal{Y}}(f(x), f(x')) \to T$. A T'-valued point of $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T$ for $T' \to T$ a scheme over T is the same thing as an isomorphism $x|_{T'} \to x'|_{T'}$ whose image under f is $\alpha|_{T'}$. Thus we see that

(3.3.1)
$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T & \longrightarrow & Isom_{\mathcal{X}}(x, x') \\ \downarrow & & \downarrow \\ T & \xrightarrow{\alpha} & Isom_{\mathcal{Y}}(f(x), f(x')) \end{array}$$

is a fibre square of sheaves over T. In particular we see that $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T$ is an algebraic space which proves part (1) of the lemma.

To prove the second statement we have to show that the left vertical arrow of Diagram (3.3.1) is locally of finite type. By Lemma 3.1 the algebraic space $Isom_{\mathcal{X}}(x, x')$ and is locally of finite type over T. Hence the right vertical arrow of Diagram (3.3.1) is locally of finite type, see Morphisms of Spaces, Lemma 23.6. We conclude by Morphisms of Spaces, Lemma 23.3. $\square$

Lemma 3.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then*

  (1) Δ_f *is representable (by schemes),*
  (2) Δ_f *is locally of finite type,*
  (3) Δ_f *is a monomorphism,*
  (4) Δ_f *is separated, and*
  (5) Δ_f *is locally quasi-finite.*

Proof. We have already seen in Lemma 3.3 that Δ_f is representable by algebraic spaces. Hence the statements (2) – (5) make sense, see Properties of Stacks, Section 3. Also Lemma 3.3 guarantees (2) holds. Let $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be a morphism and contemplate Diagram (3.3.1). By Algebraic Stacks, Lemma 9.2 the right vertical arrow is injective as a map of sheaves, i.e., a monomorphism of algebraic spaces. Hence also the morphism $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is a monomorphism. Thus (3) holds. We already know that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is locally of finite type. Thus Morphisms of Spaces, Lemma 26.10 allows us to conclude that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is locally quasi-finite and separated. This proves (4) and (5). Finally, Morphisms of Spaces, Proposition 44.2 implies that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ is a scheme which proves (1). $\square$

Lemma 3.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

  (1) f *is separated,*
  (2) Δ_f *is a closed immersion,*
  (3) Δ_f *is proper, or*
  (4) Δ_f *is universally closed.*

Proof. The statements "f is separated", "Δ_f is a closed immersion", "Δ_f is universally closed", and "Δ_f is proper" refer to the notions defined in Properties of Stacks, Section 3. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_{\mathcal{Y}} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 29.14 and Properties of Stacks, Lemma 3.3 we see that for any property P (as in that lemma) we have: Δ_f has P if and only if $\Delta_{U/V} : U \to U \times_V U$ has P. Hence the equivalence of (2), (3) and (4) follows from Morphisms of Spaces, Lemma 37.9 applied to $U \to V$. Moreover, if (1) holds, then $U \to V$ is separated and we see that $\Delta_{U/V}$ is a closed immersion, i.e., (2) holds. Finally, assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_{\mathcal{Y}} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is separated. Using Categories, Lemma 29.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is a closed immersion, hence $T' \to T$ is separated as desired. $\square$

Lemma 3.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is quasi-separated,*
(2) *Δ_f is quasi-compact, or*
(3) *Δ_f is finite type.*

Proof. The statements "f is quasi-separated", "Δ_f is quasi-compact", and "Δ_f is finite type" refer to the notions defined in Properties of Stacks, Section 3. Note that (2) and (3) are equivalent in view of the fact that Δ_f is locally of finite type by Lemma 3.4 (and Algebraic Stacks, Lemma 10.9). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_{\mathcal{Y}} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 29.14 and Properties of Stacks, Lemma 3.3 we see that we have: Δ_f is quasi-compact if and only if $\Delta_{U/V} : U \to U \times_V U$ is quasi-compact. If (1) holds, then $U \to V$ is quasi-separated and we see that $\Delta_{U/V}$ is quasi-compact, i.e., (2) holds. Assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_{\mathcal{Y}} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is quasi-separated. Using Categories, Lemma 29.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is quasi-compact, hence $T' \to T$ is quasi-separated as desired. $\square$

Lemma 3.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is locally separated, and*
(2) *Δ_f is an immersion.*

Proof. The statements "f is locally separated", and "Δ_f is an immersion" refer to the notions defined in Properties of Stacks, Section 3. Proof omitted. Hint: Argue as in the proofs of Lemmas 3.5 and 3.6. $\square$

4. Separation axioms

Let $\mathcal{X} = [U/R]$ be a presentation of an algebraic stack. Then the properties of the diagonal of $\mathcal{X}$ over S, are the properties of the morphism $j : R \to U \times_S U$. For example, if $\mathcal{X} = [S/G]$ for some smooth group G in algebraic spaces over S then j is the structure morphism $G \to U$. Hence the diagonal is not automatically separated itself (contrary to what happens in the case of schemes and algebraic spaces). To say that $[S/G]$ is quasi-separated over S should certainly imply that $G \to S$ is quasi-compact, but we hesitate to say that $[S/G]$ is quasi-separated over S without also requiring the morphism $G \to S$ to be quasi-separated. In other words, requiring the diagonal morphism to be quasi-compact does not really agree with our intuition for a "quasi-separated algebraic stack", and we should also require the diagonal itself to be quasi-separated.

What about "separated algebraic stacks"? We have seen in Morphisms of Spaces, Lemma 37.9 that an algebraic space is separated if and only if the diagonal is proper. This is the condition that is usually used to define separated algebraic stacks too. In the example $[S/G] \to S$ above this means that $G \to S$ is a proper group scheme. This means algebraic stacks of the form $[\mathrm{Spec}(k)/E]$ are proper over k where E is an elliptic curve over k (insert future reference here). In certain situations it may be more natural to assume the diagonal is finite.

Definition 4.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *DM* if Δ_f is unramified[1].
(2) We say f is *quasi-DM* if Δ_f is locally quasi-finite[2].
(3) We say f is *separated* if Δ_f is proper.
(4) We say f is *quasi-separated* if Δ_f is quasi-compact and quasi-separated.

In this definition we are using that Δ_f is representable by algebraic spaces and we are using Properties of Stacks, Section 3 to make sense out of imposing conditions on Δ_f. We note that these definitions do not conflict with the already existing notions if f is representable by algebraic spaces, see Lemmas 3.6 and 3.5. There is an interesting way to characterize these conditions by looking at higher diagonals, see Lemma 6.4.

Definition 4.2. Let $\mathcal{X}$ be an algebraic stack over the base scheme S. Denote $p : \mathcal{X} \to S$ the structure morphism.

(1) We say $\mathcal{X}$ is *DM over S* if $p : \mathcal{X} \to S$ is DM.
(2) We say $\mathcal{X}$ is *quasi-DM over S* if $p : \mathcal{X} \to S$ is quasi-DM.
(3) We say $\mathcal{X}$ is *separated over S* if $p : \mathcal{X} \to S$ is separated.
(4) We say $\mathcal{X}$ is *quasi-separated over S* if $p : \mathcal{X} \to S$ is quasi-separated.
(5) We say $\mathcal{X}$ is *DM* if $\mathcal{X}$ is DM[3] over $\mathrm{Spec}(\mathbf{Z})$.
(6) We say $\mathcal{X}$ is *quasi-DM* if $\mathcal{X}$ is quasi-DM over $\mathrm{Spec}(\mathbf{Z})$.
(7) We say $\mathcal{X}$ is *separated* if $\mathcal{X}$ is separated over $\mathrm{Spec}(\mathbf{Z})$.
(8) We say $\mathcal{X}$ is *quasi-separated* if $\mathcal{X}$ is quasi-separated over $\mathrm{Spec}(\mathbf{Z})$.

In the last 4 definitions we view $\mathcal{X}$ as an algebraic stack over $\mathrm{Spec}(\mathbf{Z})$ via Algebraic Stacks, Definition 19.2.

Thus in each case we have an absolute notion and a notion relative to our given base scheme (mention of which is usually suppressed by our abuse of notation introduced in Properties of Stacks, Section 2). We will see that (1) $\Leftrightarrow$ (5) and (2) $\Leftrightarrow$ (6) in Lemma 4.13. We spend some time proving some standard results on these notions.

Lemma 4.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If f is separated, then f is quasi-separated.*
(2) *If f is DM, then f is quasi-DM.*
(3) *If f is representable by algebraic spaces, then f is DM.*

Proof. To see (1) note that a proper morphism of algebraic spaces is quasi-compact and quasi-separated, see Morphisms of Spaces, Definition 37.1. To see (2) note that an unramified morphism of algebraic spaces is locally quasi-finite, see Morphisms of Spaces, Lemma 35.7. Finally (3) follows from Lemma 3.4. □

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[1]The letters DM stand for Deligne-Mumford. If f is DM then given any scheme T and any morphism $T \to \mathcal{Y}$ the fibre product $\mathcal{X}_T = \mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic stack over T whose diagonal is unramified, i.e., $\mathcal{X}_T$ is DM. This implies $\mathcal{X}_T$ is a Deligne-Mumford stack, see Theorem 15.6. In other words a DM morphism is one whose "fibres" are Deligne-Mumford stacks. This hopefully at least motivates the terminology.

[2]If f is quasi-DM, then the "fibres" $\mathcal{X}_T$ of $\mathcal{X} \to \mathcal{Y}$ are quasi-DM. An algebraic stack $\mathcal{X}$ is quasi-DM exactly if there exists a scheme U and a surjective flat morphism $U \to \mathcal{X}$ of finite presentation which is locally quasi-finite, see Theorem 15.3. Note the similarity to being Deligne-Mumford, which is defined in terms of having an étale covering by a scheme.

[3]Theorem 15.6 shows that this is equivalent to $\mathcal{X}$ being a Deligne-Mumford stack.

Lemma 4.4. *All of the separation axioms listed in Definition 4.1 are stable under base change.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks. Let $f' : \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ be the base change of f by $\mathcal{Y}' \to \mathcal{Y}$. Then $\Delta_{f'}$ is the base change of Δ_f by the morphism $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}' \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, see Categories, Lemma 29.14. By the results of Properties of Stacks, Section 3 each of the properties of the diagonal used in Definition 4.1 is stable under base change. Hence the lemma is true. $\square$

Lemma 4.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ has one of the separation properties of Definition 4.1 then so does f.*

Proof. Denote $g : W \times_{\mathcal{Y}} \mathcal{X} \to W$ the base change. Then Δ_g is the base change of Δ_f by the morphism $q : W \times_{\mathcal{Y}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Since q is the base change of $W \to \mathcal{Y}$ we see that q is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Hence the result follows from Properties of Stacks, Lemma 3.4. $\square$

Lemma 4.6. *Let S be a scheme. The property of being quasi-DM over S, quasi-separated over S, or separated over S (see Definition 4.2) is stable under change of base scheme, see Algebraic Stacks, Definition 19.3.*

Proof. Follows immediately from Lemma 4.4. $\square$

Lemma 4.7. *Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ and $\mathcal{Z} \to \mathcal{T}$ be morphisms of algebraic stacks. Consider the induced morphism $i : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$. Then*

(1) *i is representable by algebraic spaces and locally of finite type,*
(2) *if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is quasi-separated, then i is quasi-separated,*
(3) *if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is separated, then i is separated,*
(4) *if $\mathcal{Z} \to \mathcal{T}$ is DM, then i is unramified,*
(5) *if $\mathcal{Z} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,*
(6) *if $\mathcal{Z} \to \mathcal{T}$ is separated, then i is proper, and*
(7) *if $\mathcal{Z} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.*

Proof. The following diagram

$$
\begin{array}{ccc}
\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} & \xrightarrow{\ i\ } & \mathcal{X} \times_{\mathcal{T}} \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{Z} & \xrightarrow{\ \Delta_{\mathcal{Z}/\mathcal{T}}\ } & \mathcal{Z} \times_{\mathcal{T}} \mathcal{Z}
\end{array}
$$

is a 2-fibre product diagram, see Categories, Lemma 29.13. Hence i is the base change of the diagonal morphism $\Delta_{\mathcal{Z}/\mathcal{T}}$. Thus the lemma follows from Lemma 3.3, and the material in Properties of Stacks, Section 3. $\square$

Lemma 4.8. *Let $\mathcal{T}$ be an algebraic stack. Let $g : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over $\mathcal{T}$. Consider the graph $i : \mathcal{X} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$ of g. Then*

(1) *i is representable by algebraic spaces and locally of finite type,*
(2) *if $\mathcal{Y} \to \mathcal{T}$ is DM, then i is unramified,*
(3) *if $\mathcal{Y} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,*

(4) *if $\mathcal{Y} \to \mathcal{T}$ is separated, then i is proper, and*
(5) *if $\mathcal{Y} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.*

Proof. This is a special case of Lemma 4.7 applied to the morphism $\mathcal{X} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$. $\square$

Lemma 4.9. *Let $f : \mathcal{X} \to \mathcal{T}$ be a morphism of algebraic stacks. Let $s : \mathcal{T} \to \mathcal{X}$ be a morphism such that $f \circ s$ is 2-isomorphic to $id_{\mathcal{T}}$. Then*

(1) *s is representable by algebraic spaces and locally of finite type,*
(2) *if f is DM, then s is unramified,*
(3) *if f is quasi-DM, then s is locally quasi-finite,*
(4) *if f is separated, then s is proper, and*
(5) *if f is quasi-separated, then s is quasi-compact and quasi-separated.*

Proof. This is a special case of Lemma 4.8 applied to $g = s$ and $\mathcal{Y} = \mathcal{T}$ in which case $i : \mathcal{T} \to \mathcal{T} \times_{\mathcal{T}} \mathcal{X}$ is 2-isomorphic to s. $\square$

Lemma 4.10. *All of the separation axioms listed in Definition 4.1 are stable under composition of morphisms.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks to which the axiom in question applies. The diagonal $\Delta_{\mathcal{X}/\mathcal{Z}}$ is the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 4.7 above we see that the second arrow also has this property. Hence the lemma follows since the composition of morphisms which are representable by algebraic spaces with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$, see our general discussion in Properties of Stacks, Section 3 and Morphisms of Spaces, Lemmas 35.3, 26.3, 37.4, 8.4, and 4.8. $\square$

Lemma 4.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.*

(1) *If $\mathcal{Y}$ is DM over S and f is DM, then $\mathcal{X}$ is DM over S.*
(2) *If $\mathcal{Y}$ is quasi-DM over S and f is quasi-DM, then $\mathcal{X}$ is quasi-DM over S.*
(3) *If $\mathcal{Y}$ is separated over S and f is separated, then $\mathcal{X}$ is separated over S.*
(4) *If $\mathcal{Y}$ is quasi-separated over S and f is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.*
(5) *If $\mathcal{Y}$ is DM and f is DM, then $\mathcal{X}$ is DM.*
(6) *If $\mathcal{Y}$ is quasi-DM and f is quasi-DM, then $\mathcal{X}$ is quasi-DM.*
(7) *If $\mathcal{Y}$ is separated and f is separated, then $\mathcal{X}$ is separated.*
(8) *If $\mathcal{Y}$ is quasi-separated and f is quasi-separated, then $\mathcal{X}$ is quasi-separated.*

Proof. Parts (1), (2), (3), and (4) follow immediately from Lemma 4.10 and Definition 4.2. For (5), (6), (7), and (8) think of $\mathcal{X}$ and $\mathcal{Y}$ as algebraic stacks over $\text{Spec}(\mathbf{Z})$ and apply Lemma 4.10. Details omitted. $\square$

The following lemma is a bit different to the analogue for algebraic spaces. To compare take a look at Morphisms of Spaces, Lemma 4.10.

Lemma 4.12. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks.*

(1) *If $g \circ f$ is DM then so is f.*
(2) *If $g \circ f$ is quasi-DM then so is f.*

(3) *If $g \circ f$ is separated and Δ_g is separated, then f is separated.*
(4) *If $g \circ f$ is quasi-separated and Δ_g is quasi-separated, then f is quasi-separated.*

Proof. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces, see Lemmas 3.3 and 4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T.$$

If $g \circ f$ is DM (resp. quasi-DM), then the composition $A \to T$ is unramified (resp. locally quasi-finite). Hence (1) (resp. (2)) follows on applying Morphisms of Spaces, Lemma 35.11 (resp. Morphisms of Spaces, Lemma 26.8). This proves (1) and (2).

Proof of (4). Assume $g \circ f$ is quasi-separated and Δ_g is quasi-separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is quasi-separated, see Lemmas 3.3 and 4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is quasi-separated. The composition $A \to T$ is quasi-compact and quasi-separated as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is quasi-separated by Morphisms of Spaces, Lemma 4.10. And $A \to B$ is quasi-compact by Morphisms of Spaces, Lemma 8.8. Thus f is quasi-separated.

Proof of (3). Assume $g \circ f$ is separated and Δ_g is separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is separated, see Lemmas 3.3 and 4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is separated. The composition $A \to T$ is proper as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is proper by Morphisms of Spaces, Lemma 37.6 which means that f is separated. $\square$

Lemma 4.13. *Let $\mathcal{X}$ be an algebraic stack over the base scheme S.*

(1) *$\mathcal{X}$ is DM $\Leftrightarrow$ $\mathcal{X}$ is DM over S.*
(2) *$\mathcal{X}$ is quasi-DM $\Leftrightarrow$ $\mathcal{X}$ is quasi-DM over S.*
(3) *If $\mathcal{X}$ is separated, then $\mathcal{X}$ is separated over S.*
(4) *If $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.*

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.

(5) *If $\mathcal{X}$ is DM over S, then f is DM.*
(6) *If $\mathcal{X}$ is quasi-DM over S, then f is quasi-DM.*
(7) *If $\mathcal{X}$ is separated over S and $\Delta_{\mathcal{Y}/S}$ is separated, then f is separated.*

(8) If $\mathcal{X}$ is quasi-separated over S and $\Delta_{\mathcal{Y}/S}$ is quasi-separated, then f is quasi-separated.

Proof. Parts (5), (6), (7), and (8) follow immediately from Lemma 4.12 and Spaces, Definition 13.2. To prove (3) and (4) think of X and Y as algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ and apply Lemma 4.12. Similarly, to prove (1) and (2), think of $\mathcal{X}$ as an algebraic stack over $\mathrm{Spec}(\mathbf{Z})$ consider the morphisms

$$\mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathrm{Spec}(\mathbf{Z})} \mathcal{X}$$

Both arrows are representable by algebraic spaces. The second arrow is unramified and locally quasi-finite as the base change of the immersion $\Delta_{S/\mathbf{Z}}$. Hence the composition is unramified (resp. locally quasi-finite) if and only if the first arrow is unramified (resp. locally quasi-finite), see Morphisms of Spaces, Lemmas 35.3 and 35.11 (resp. Morphisms of Spaces, Lemmas 26.3 and 26.8). $\square$

Lemma 4.14. *Let $\mathcal{X}$ be an algebraic stack. Let W be an algebraic space, and let $f : W \to \mathcal{X}$ be a surjective, flat, locally finitely presented morphism.*

(1) *If f is unramified (i.e., étale, i.e., $\mathcal{X}$ is Deligne-Mumford), then $\mathcal{X}$ is DM.*
(2) *If f is locally quasi-finite, then $\mathcal{X}$ is quasi-DM.*

Proof. Note that if f is unramified, then it is étale by Morphisms of Spaces, Lemma 36.12. This explains the parenthetical remark in (1). Assume f is unramified (resp. locally quasi-finite). We have to show that $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified (resp. locally quasi-finite). Note that $W \times W \to \mathcal{X} \times \mathcal{X}$ is also surjective, flat, and locally of finite presentation. Hence it suffices to show that

$$W \times_{\mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X} = W \times_{\mathcal{X}} W \longrightarrow W \times W$$

is unramified (resp. locally quasi-finite), see Properties of Stacks, Lemma 3.3. By assumption the morphism $\mathrm{pr}_i : W \times_{\mathcal{X}} W \to W$ is unramified (resp. locally quasi-finite). Hence the displayed arrow is unramified (resp. locally quasi-finite) by Morphisms of Spaces, Lemma 35.11 (resp. Morphisms of Spaces, Lemma 26.8). $\square$

Lemma 4.15. *A monomorphism of algebraic stacks is separated and DM. The same is true for immersions of algebraic stacks.*

Proof. If $f : \mathcal{X} \to \mathcal{Y}$ is a monomorphism of algebraic stacks, then Δ_f is an isomorphism, see Properties of Stacks, Lemma 8.4. Since an isomorphism of algebraic spaces is proper and unramified we see that f is separated and DM. The second assertion follows from the first as an immersion is a monomorphism, see Properties of Stacks, Lemma 9.5. $\square$

Lemma 4.16. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists. If $\mathcal{X}$ is DM, resp. quasi-DM, resp. separated, resp. quasi-separated, then so is $\mathcal{Z}_x$.*

Proof. This is true because $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism hence DM and separated by Lemma 4.15. Apply Lemma 4.11 to conclude. $\square$

5. Inertia stacks

The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 32.

Lemma 5.1. *Let $\mathcal{X}$ be an algebraic stack. Then the inertia stack $\mathcal{I}_\mathcal{X}$ is an algebraic stack as well. The morphism*

$$\mathcal{I}_\mathcal{X} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type. More generally, let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then the morphism

$$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type.

Proof. By Categories, Lemma 32.1 there are equivalences

$$\mathcal{I}_\mathcal{X} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_S \mathcal{X}, \Delta} \mathcal{X} \quad \text{and} \quad \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_\mathcal{Y} \mathcal{X}, \Delta} \mathcal{X}$$

which shows that the inertia stacks are algebraic stacks. Let $T \to \mathcal{X}$ be a morphism given by the object x of the fibre category of $\mathcal{X}$ over T. Then we get a 2-fibre product square

$$
\begin{array}{ccc}
\mathit{Isom}_\mathcal{X}(x, x) & \longrightarrow & \mathcal{I}_\mathcal{X} \\
\downarrow & & \downarrow \\
T & \xrightarrow{\quad x \quad} & \mathcal{X}
\end{array}
$$

This follows immediately from the definition of $\mathcal{I}_\mathcal{X}$. Since $\mathit{Isom}_\mathcal{X}(x, x)$ is always an algebraic space locally of finite type over T (see Lemma 3.1) we conclude that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is representable by algebraic spaces and locally of finite type. Finally, for the relative inertia we get

$$
\begin{array}{ccccc}
\mathit{Isom}_\mathcal{X}(x, x) & \longleftarrow & K & \longrightarrow & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathit{Isom}_\mathcal{Y}(f(x), f(x)) & \xleftarrow{\ e\ } & T & \xrightarrow{\ x\ } & \mathcal{X}
\end{array}
$$

with both squares 2-fibre products. This follows from Categories, Lemma 32.3. The left vertical arrow is a morphism of algebraic spaces locally of finite type over T, and hence is locally of finite type, see Morphisms of Spaces, Lemma 23.6. Thus K is an algebraic space and $K \to T$ is locally of finite type. This proves the assertion on the relative inertia. $\qquad\square$

Remark 5.2. Let $\mathcal{X}$ be an algebraic stack. In Properties of Stacks, Remark 3.7 we have seen that the 2-category of morphisms $\mathcal{X}' \to \mathcal{X}$ representable by algebraic spaces with target $\mathcal{X}$ forms a category. In this category the inertia stack of $\mathcal{X}$ is a *group object*. Recall that an object of $\mathcal{I}_\mathcal{X}$ is just a pair (x, α) where x is an object of $\mathcal{X}$ and α is an automorphism of x in the fibre category of $\mathcal{X}$ that x lives in. The composition

$$c : \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{I}_\mathcal{X} \longrightarrow \mathcal{I}_\mathcal{X}$$

is given by the rule on objects

$$((x, \alpha), (x', \alpha'), \beta) \mapsto (x, \alpha \circ \beta^{-1} \circ \alpha' \circ \beta)$$

which makes sense as $\beta : x \to x'$ is an isomorphism in the fibre category by our definition of fibre products. The neutral element $e : \mathcal{X} \to \mathcal{I}_\mathcal{X}$ is given by the functor $x \mapsto (x, \mathrm{id}_x)$. We omit the proof that the axioms of a group object hold. There is a variant of this remark for relative inertia stacks.

Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{I}_\mathcal{X}$ be its inertia stack. We have seen in the proof of Lemma 5.1 that for any scheme T and object x of $\mathcal{X}$ over T there is a canonical cartesian square

$$
\begin{array}{ccc}
Isom_\mathcal{X}(x,x) & \longrightarrow & \mathcal{I}_\mathcal{X} \\
\downarrow & & \downarrow \\
T & \xrightarrow{\ x\ } & \mathcal{X}
\end{array}
$$

The group structure on $\mathcal{I}_\mathcal{X}$ discussed in Remark 5.2 induces the group structure on $Isom_\mathcal{X}(x,x)$ of Lemma 3.2. This allows us to define the sheaf $Isom_\mathcal{X}$ also for morphisms from algebraic spaces to $\mathcal{X}$. We formalize this in the following definition.

Definition 5.3. Let $\mathcal{X}$ be an algebraic stack and let X be an algebraic space. Let $x : X \to \mathcal{X}$ be a morphism. We set

$$
Isom_\mathcal{X}(x,x) = X \times_{x,\mathcal{X}} \mathcal{I}_\mathcal{X}
$$

We endow it with the structure of a group algebraic space over X by pulling back the composition law discussed in Remark 5.2. We will sometimes refer to $Isom_\mathcal{X}(x,x)$ as the *sheaf of automorphisms of x*.

As a variant we may occasionally use the notation $Isom_\mathcal{X}(x,y)$ when given two morphisms $x, y : X \to \mathcal{X}$. This will mean simply the algebraic space

$$
(X \times_{x,\mathcal{X},y} X) \times_{X \times X, \Delta_X} X.
$$

Then it is true, as in Lemma 3.2, that $Isom_\mathcal{X}(x,y)$ is a pseudo torsor for $Isom_\mathcal{X}(x,x)$ over X. We omit the verification.

Lemma 5.4. *Let $\pi : \mathcal{X} \to X$ be a morphism from an algebraic stack to an algebraic space. Let $f : X' \to X$ be a morphism of algebraic spaces. Set $\mathcal{X}' = X' \times_X \mathcal{X}$. Then both squares in the diagram*

$$
\begin{array}{ccccc}
\mathcal{I}_{\mathcal{X}'} & \longrightarrow & \mathcal{X}' & \longrightarrow & X' \\
\Big\downarrow {\scriptstyle Categories,\ Equation\ (32.2.1)} & & \downarrow & & \downarrow \\
\mathcal{I}_\mathcal{X} & \longrightarrow & \mathcal{X} & \longrightarrow & X
\end{array}
$$

are fibre product squares.

Proof. The inertia stack $\mathcal{I}_{\mathcal{X}'}$ is the defined as the category of pairs (x', α') where x' is an object of $\mathcal{X}'$ and α' is an automorphism of x' in its fibre category over $(Sch/S)_{fppf}$, see Categories, Section 32. Suppose that x' lies over the scheme U and maps to the object x of $\mathcal{X}$. By the construction of the 2-fibre product in Categories, Lemma 30.3 we see that $x' = (U, a', x, 1)$ where $a' : U \to X'$ is a morphism and 1 indicates that $f \circ a' = \pi \circ x$ as morphisms $U \to X$. Moreover we have $Isom_{\mathcal{X}'}(x', x') = Isom_\mathcal{X}(x, x)$ as sheaves on U (by the very construction of the 2-fibre product). This implies that the left square is a fibre product square (details omitted). $\qquad\square$

Lemma 5.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Then the diagram*

$$\begin{array}{ccc} \mathcal{I}_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{I}_{\mathcal{Y}} & \longrightarrow & \mathcal{Y} \end{array}$$

is a fibre product square.

Proof. This follows immediately from the fact that f is fully faithful (see Properties of Stacks, Lemma 8.4) and the definition of the inertia in Categories, Section 32. Namely, an object of $\mathcal{I}_{\mathcal{X}}$ over a scheme T is the same thing as a pair (x, α) consisting of an object x of $\mathcal{X}$ over T and a morphism $\alpha : x \to x$ in the fibre category of $\mathcal{X}$ over T. As f is fully faithful we see that α is the same thing as a morphism $\beta : f(x) \to f(x)$ in the fibre category of $\mathcal{Y}$ over T. Hence we can think of objects of $\mathcal{I}_{\mathcal{X}}$ over T as triples $((y, \beta), x, \gamma)$ where y is an object of $\mathcal{Y}$ over T, $\beta : y \to y$ in $\mathcal{Y}_T$ and $\gamma : y \to f(x)$ is an isomorhism over T, i.e., an object of $\mathcal{I}_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ over T. $\qquad\square$

Lemma 5.6. *Let $\mathcal{X}$ be an algebraic stack. Let $[U/R] \to \mathcal{X}$ be a presentation. Let G/U be the stabilizer group algebraic space associated to the groupoid (U, R, s, t, c). Then*

$$\begin{array}{ccc} G & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{I}_{\mathcal{X}} & \longrightarrow & \mathcal{X} \end{array}$$

is a fibre product diagram.

Proof. Immediate from Groupoids in Spaces, Lemma 25.2. $\qquad\square$

6. Higher diagonals

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. In this situation it makes sense to consider not only the diagonal

$$\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

but also the diagonal of the diagonal, i.e., the morphism

$$\Delta_{\Delta_f} : \mathcal{X} \longrightarrow \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} \mathcal{X}$$

Because of this we sometimes use the following terminology. We denote $\Delta_{f,0} = f$ the *zeroth diagonal*, we denote $\Delta_{f,1} = \Delta_f$ the *first diagonal*, and we denote $\Delta_{f,2} = \Delta_{\Delta_f}$ the *second diagonal*. Note that $\Delta_{f,1}$ is representable by algebraic spaces and locally of finite type, see Lemma 3.3. Hence $\Delta_{f,2}$ is representable, a monomorphism, locally of finite type, separated, and locally quasi-finite, see Lemma 3.4.

We can describe the second diagonal using the relative inertia stack. Namely, the fibre product $\mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} \mathcal{X}$ is equivalent to the relative inertia stack $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ by Categories, Lemma 32.1. Moreover, via this identification the second diagonal becomes the *neutral section*

$$e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

of the relative inertia stack. Moreover, recall from the proof of Lemma 5.1 that given a morphism $x : T \to \mathcal{X}$ the fibre product $T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is given as the *kernel* K of the homomorphism of group algebraic spaces

$$Isom_{\mathcal{X}}(x, x) \longrightarrow Isom_{\mathcal{Y}}(f(x), f(x))$$

over T. The morphism e corresponds to the neutral section $e : T \to K$ in this situation.

Lemma 6.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then f is representable by algebraic spaces if and only if the second diagonal is an isomorphism.*

Proof. Namely, f is representable by algebraic spaces if and only if f is faithful, see Algebraic Stacks, Lemma 15.2. On the other hand, f is faithful if and only if for every object x of $\mathcal{X}$ over a scheme T the functor f induces an injection $Isom_{\mathcal{X}}(x, x) \to Isom_{\mathcal{Y}}(f(x), f(x))$, which happens if and only if the kernel K is trivial, which happens if and only if $e : T \to K$ is an isomorphism for every $x : T \to \mathcal{X}$. Since $K = T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ as discussed above, this proves the lemma. $\square$

This lemma leads to the following hierarchy for morphisms of algebraic stacks.

Lemma 6.2. *A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is*

(1) *a monomorphism if and only if $\Delta_{f,1}$ is an isomorphism,*
(2) *representable by algebraic spaces if and only if $\Delta_{f,1}$ is a monomorphism,*
(3) *the second diagonal $\Delta_{f,2}$ is always a monomorphism.*

Proof. Recall from Properties of Stacks, Lemma 8.4 that a morphism of algebraic stacks is a monomorphism if and only if its diagonal is an isomorphism of stacks. Thus Lemma 6.1 can be rephrased as saying that a morphism is representable by algebraic spaces if the diagonal is a monomorphism. In particular, it shows that condition (3) of Lemma 3.4 is actually an if and only if, i.e., a morphism of algebraic stacks is representable by algebraic spaces if and only if its diagonal is a monomorphism. $\square$

Lemma 6.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *$\Delta_{f,1}$ separated $\Leftrightarrow \Delta_{f,2}$ closed immersion $\Leftrightarrow \Delta_{f,2}$ proper $\Leftrightarrow \Delta_{f,2}$ universally closed,*
(2) *$\Delta_{f,1}$ quasi-separated $\Leftrightarrow \Delta_{f,2}$ finite type $\Leftrightarrow \Delta_{f,2}$ quasi-compact, and*
(3) *$\Delta_{f,1}$ locally separated $\Leftrightarrow \Delta_{f,2}$ immersion.*

Proof. Follows from Lemmas 3.5, 3.6, and 3.7 applied to $\Delta_{f,1}$. $\square$

The following lemma is kind of cute and it may suggest a generalization of these conditions to higher algebraic stacks.

Lemma 6.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *f is separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are universally closed, and*
(2) *f is quasi-separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are quasi-compact.*
(3) *f is quasi-DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are locally quasi-finite.*
(4) *f is DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are unramified.*

Proof. Proof of (1). Assume that $\Delta_{f,2}$ and $\Delta_{f,1}$ are universally closed. Then $\Delta_{f,1}$ is separated and universally closed by Lemma 6.3. By Morphisms of Spaces, Lemma 9.7 and Algebraic Stacks, Lemma 10.9 we see that $\Delta_{f,1}$ is quasi-compact.

Hence it is quasi-compact, separated, universally closed and locally of finite type (by Lemma 3.3) so proper. This proves "$\Leftarrow$" of (1). The proof of the implication in the other direction is omitted.

Proof of (2). This follows immediately from Lemma 6.3.

Proof of (3). This follows from the fact that $\Delta_{f,2}$ is always locally quasi-finite by Lemma 3.4 applied to $\Delta_f = \Delta_{f,1}$.

Proof of (4). This follows from the fact that $\Delta_{f,2}$ is always unramified as Lemma 3.4 applied to $\Delta_f = \Delta_{f,1}$ shows that $\Delta_{f,2}$ is locally of finite type and a monomorphism. See More on Morphisms of Spaces, Lemma 11.8. $\square$

7. Quasi-compact morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 3 we have defined what it means for f to be quasi-compact. Here is another characterization.

Lemma 7.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:*

   (1) *f is quasi-compact, and*
   (2) *for every quasi-compact algebraic stack $\mathcal{Z}$ and any morphism $\mathcal{Z} \to \mathcal{Y}$ the algebraic stack $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks with $\mathcal{Z}$ quasi-compact. By Properties of Stacks, Lemma 6.2 there exists a quasi-compact scheme U and a surjective smooth morphism $U \to \mathcal{Z}$. Since f is representable by algebraic spaces and quasi-compact we see by definition that $U \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and that $U \times_{\mathcal{Y}} \mathcal{X} \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact algebraic space. The morphism $U \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is smooth and surjective (as the base change of the smooth and surjective morphism $U \to \mathcal{Z}$). Hence $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact by another application of Properties of Stacks, Lemma 6.2

Assume (2). Let $Z \to \mathcal{Y}$ be a morphism, where Z is a scheme. We have to show that the morphism of algebraic spaces $p : Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_{\mathcal{Y}} \mathcal{Z}$ and the algebraic space $U \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Morphisms of Spaces, Lemma 8.7. $\square$

This motivates the following definition.

Definition 7.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *quasi-compact* if for every quasi-compact algebraic stack $\mathcal{Z}$ and morphism $\mathcal{Z} \to \mathcal{Y}$ the fibre product $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.

By Lemma 7.1 above this agrees with the already existing notion for morphisms of algebraic stacks representable by algebraic spaces. In particular this notion agrees with the notions already defined for morphisms between algebraic stacks and schemes.

Lemma 7.3. *The base change of a quasi-compact morphism of algebraic stacks by any morphism of algebraic stacks is quasi-compact.*

Proof. Omitted. $\square$

Lemma 7.4. *The composition of a pair of quasi-compact morphisms of algebraic stacks is quasi-compact.*

Proof. Omitted. $\square$

Lemma 7.5. *Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y} \\ & {}_{p}\searrow \quad \swarrow_{q} & \\ & \mathcal{Z} & \end{array}$$

be a 2-commutative diagram of morphisms of algebraic stacks. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $\mathcal{T}$ be a quasi-compact algebraic stack, and let $\mathcal{T} \to \mathcal{Z}$ be a morphism. By Properties of Stacks, Lemma 5.3 the morphism $\mathcal{T} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{T} \times_{\mathcal{Z}} \mathcal{Y}$ is surjective and by assumption $\mathcal{T} \times_{\mathcal{Z}} \mathcal{X}$ is quasi-compact. Hence $\mathcal{T} \times_{\mathcal{Z}} \mathcal{Y}$ is quasi-compact by Properties of Stacks, Lemma 6.2. $\square$

Lemma 7.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$. The first map is quasi-compact by Lemma 4.9 because it is a section of the quasi-separated morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X}$ (a base change of g, see Lemma 4.4). The second map is quasi-compact as it is the base change of f, see Lemma 7.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 7.4. $\square$

Lemma 7.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*
 (1) *If $\mathcal{X}$ is quasi-compact and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact.*
 (2) *If $\mathcal{X}$ is quasi-compact and quasi-separated and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact and quasi-separated.*
 (3) *A fibre product of quasi-compact and quasi-separated algebraic stacks is quasi-compact and quasi-separated.*

Proof. Part (1) follows from Lemma 7.6. Part (2) follows from (1) and Lemma 4.12. For (3) let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ be morphisms of quasi-compact and quasi-separated algebraic stacks. Then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is quasi-compact and quasi-separated as a base change of $\mathcal{X} \to \mathcal{Y}$ using (2) and Lemmas 7.3 and 4.4. Hence $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact and quasi-separated as an algebraic stack quasi-compact and quasi-separated over $\mathcal{Z}$, see Lemmas 4.11 and 7.4. $\square$

8. Noetherian algebraic stacks

We have already defined locally Noetherian algebraic stacks in Properties of Stacks, Section 7.

Definition 8.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *Noetherian* if $\mathcal{X}$ is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic stack $\mathcal{X}$ is not just quasi-compact and locally Noetherian, but also quasi-separated. In the language of Section 6 if we denote $p : \mathcal{X} \to \mathrm{Spec}(\mathbf{Z})$ the "absolute" structure morphism (i.e., the structure morphism of $\mathcal{X}$ viewed as an algebraic stack over $\mathbf{Z}$), then

$\mathcal{X}$ Noetherian $\Leftrightarrow \mathcal{X}$ locally Noetherian and $\Delta_{p,0}, \Delta_{p,1}, \Delta_{p,2}$ quasi-compact.

This will later mean that an algebraic stack of finite type over a Noetherian algebraic stack is not automatically Noetherian.

9. Open morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 3 we have defined what it means for f to be universally open. Here is another characterization.

Lemma 9.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

(1) *f is universally open, and*
(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$ is open.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_\mathcal{Y} \mathcal{X} \to V$ of algebraic spaces is universally open, in particular the map $|V \times_\mathcal{Y} \mathcal{X}| \to |V|$ is open. By Properties of Stacks, Section 4 in the commutative diagram

$$
\begin{array}{ccc}
|V \times_\mathcal{Y} \mathcal{X}| & \longrightarrow & |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \\
\downarrow & & \downarrow \\
|V| & \longrightarrow & |\mathcal{Z}|
\end{array}
$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_\mathcal{Y} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_\mathcal{Y} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. $\square$

Thus we may use the following natural definition.

Definition 9.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *open* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is open.
(2) We say f is *universally open* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces

$$|\mathcal{Z} \times_\mathcal{Y} \mathcal{X}| \to |\mathcal{Z}|$$

is open, i.e., the base change $\mathcal{Z} \times_\mathcal{Y} \mathcal{X} \to \mathcal{Z}$ is open.

Lemma 9.3. *The base change of a universally open morphism of algebraic stacks by any morphism of algebraic stacks is universally open.*

Proof. This is immediate from the definition. $\square$

Lemma 9.4. *The composition of a pair of (universally) open morphisms of algebraic stacks is (universally) open.*

Proof. Omitted. $\square$

10. Submersive morphisms

Definition 10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.
  (1) We say f is *submersive*[4] if the continuous map $|\mathcal{X}| \to |\mathcal{Y}|$ is submersive, see Topology, Definition 5.3.
  (2) We say f is *universally submersive* if for every morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ is submersive.

We note that a submersive morphism is in particular surjective.

11. Universally closed morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 3 we have defined what it means for f to be universally closed. Here is another characterization.

Lemma 11.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*
  (1) *f is universally closed, and*
  (2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is closed.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ of algebraic spaces is universally closed, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is closed. By Properties of Stacks, Section 4 in the commutative diagram

$$
\begin{array}{ccc}
|V \times_{\mathcal{Y}} \mathcal{X}| & \longrightarrow & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \\
\downarrow & & \downarrow \\
|V| & \longrightarrow & |\mathcal{Z}|
\end{array}
$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. $\square$

Thus we may use the following natural definition.

Definition 11.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.
  (1) We say f is *closed* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is closed.
  (2) We say f is *universally closed* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$

is closed, i.e., the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is closed.

Lemma 11.3. *The base change of a universally closed morphism of algebraic stacks by any morphism of algebraic stacks is universally closed.*

Proof. This is immediate from the definition. $\square$

---

[4]This is very different from the notion of a submersion of differential manifolds.

Lemma 11.4. *The composition of a pair of (universally) closed morphisms of algebraic stacks is (universally) closed.*

Proof. Omitted. $\square$

12. Types of morphisms smooth local on source-and-target

Given a property of morphisms of algebraic spaces which is *smooth local on the source-and-target*, see Descent on Spaces, Definition 18.1 we may use it to define a corresponding property of morphisms of algebraic stacks, namely by imposing either of the equivalent conditions of the lemma below.

Lemma 12.1. *Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider commutative diagrams*

$$
\begin{array}{ccc}
U & \xrightarrow{\ h\ } & V \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
\mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y}
\end{array}
$$

where U and V are algebraic spaces and the vertical arrows are smooth. The following are equivalent

(1) for any diagram as above such that in addition $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is smooth the morphism h has property $\mathcal{P}$, and

(2) for some diagram as above with $a : U \to \mathcal{X}$ surjective the morphism h has property $\mathcal{P}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces, then this is also equivalent to f (as a morphism of algebraic spaces) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for morphisms f which are representable by algebraic spaces this is also equivalent to f having property $\mathcal{P}$ in the sense of Properties of Stacks, Section 3.

Proof. Let us prove the implication (1) $\Rightarrow$ (2). Pick an algebraic space V and a surjective and smooth morphism $V \to \mathcal{Y}$. Pick an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Note that $U \to \mathcal{X}$ is surjective and smooth as well, as a composition of the base change $\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ and the chosen map $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Hence we obtain a diagram as in (1). Thus if (1) holds, then $h : U \to V$ has property $\mathcal{P}$, which means that (2) holds as $U \to \mathcal{X}$ is surjective.

Conversely, assume (2) holds and let U, V, a, b, h be as in (2). Next, let U', V', a', b', h' be any diagram as in (1). Picture

$$
\begin{array}{ccc}
U & \xrightarrow{\ h\ } & V \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y}
\end{array}
\qquad
\begin{array}{ccc}
U' & \xrightarrow{\ h'\ } & V' \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y}
\end{array}
$$

To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the commutative diagram

$$
\begin{array}{ccccc}
U & \longleftarrow & U \times_{\mathcal{X}} U' & \longrightarrow & U' \\
\downarrow{\scriptstyle h} & & \downarrow & & \downarrow{\scriptstyle h'} \\
& U \times_{\mathcal{Y}} V' & {\scriptstyle (h,h')} & & \\
\downarrow & & \downarrow & & \downarrow \\
V & \longleftarrow & V \times_{\mathcal{Y}} V' & \longrightarrow & V'
\end{array}
$$

of algebraic spaces. Note that the horizontal arrows are smooth as base changes of the smooth morphisms $V \to \mathcal{Y}$, $V' \to \mathcal{Y}$, $U \to \mathcal{X}$, and $U' \to \mathcal{X}$. Note that

$$
\begin{array}{ccc}
U \times_{\mathcal{X}} U' & \longrightarrow & U' \\
\downarrow & & \downarrow \\
U \times_{\mathcal{Y}} V' & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} V'
\end{array}
$$

is cartesian, hence the left vertical arrow is smooth as U', V', a', b', h' is as in (1). Since $\mathcal{P}$ is local on the target we see that the base change $U \times_{\mathcal{Y}} V' \to V \times_{\mathcal{Y}} V'$ has $\mathcal{P}$ and hence after precomposing by the smooth morphism $U \times_{\mathcal{X}} U' \to U \times_{\mathcal{Y}} V'$ the morphism we conclude (h, h') has $\mathcal{P}$. Finally, since $U \times_X U' \to U'$ is surjective this implies that h' has $\mathcal{P}$ as $\mathcal{P}$ is local on the source-and-target. This finishes the proof of the equivalence of (1) and (2).

If $\mathcal{X}$ and $\mathcal{Y}$ are representable, then Descent on Spaces, Lemma 18.3 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to \mathcal{X}$ the base change $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. Consider the diagram

$$
\begin{array}{ccc}
Z \times_{\mathcal{Y}} U & \longrightarrow & Z \times_{\mathcal{Y}} V \\
\downarrow & & \downarrow \\
Z \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & Z
\end{array}
$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is smooth and surjective and the right vertical arrow is smooth. Thus Descent on Spaces, Lemma 18.3 kicks in and shows that $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. $\square$

Definition 12.2. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. We say a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks *has property* $\mathcal{P}$ if the equivalent conditions of Lemma 12.1 hold.

Remark 12.3. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under composition. Then the property of morphisms of algebraic stacks defined in Definition 12.2 is stable under composition. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks having property $\mathcal{P}$. Choose an algebraic space W and a surjective smooth

morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Finally, choose an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Then the morphisms $V \to W$ and $U \to V$ have property $\mathcal{P}$ by definition. Whence $U \to W$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under composition. Thus, by definition again, we see that $g \circ f : \mathcal{X} \to \mathcal{Z}$ has property $\mathcal{P}$.

Remark 12.4. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under base change. Then the property of morphisms of algebraic stacks defined in Definition 12.2 is stable under base change. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks and assume f has property $\mathcal{P}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Finally, choose an algebraic space V' and a surjective and smooth morphism $V' \to \mathcal{Y}' \times_{\mathcal{Y}} V$. Then the morphism $U \to V$ has property $\mathcal{P}$ by definition. Whence $V' \times_V U \to V'$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under base change. Considering the diagram

$$
\begin{array}{ccccc}
V' \times_V U & \longrightarrow & \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
V' & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{Y}
\end{array}
$$

we see that the left top horizontal arrow is smooth and surjective, whence by definition we see that the projection $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property $\mathcal{P}$.

Remark 12.5. Let $\mathcal{P}, \mathcal{P}'$ be properties of morphisms of algebraic spaces which are smooth local on the source-and-target and stable under base change. Suppose that we have $\mathcal{P} \Rightarrow \mathcal{P}'$ for morphisms of algebraic spaces. Then we also have $\mathcal{P} \Rightarrow \mathcal{P}'$ for the properties of morphisms of algebraic stacks defined in Definition 12.2 using $\mathcal{P}$ and $\mathcal{P}'$. This is clear from the definition.

13. Morphisms of finite type

The property "locally of finite type" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 23.3 and Descent on Spaces, Lemma 10.7. Hence, by Lemma 12.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces.

Definition 13.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.
   (1) We say f *locally of finite type* if the equivalent conditions of Lemma 12.1 hold with $\mathcal{P} =$ locally of finite type.
   (2) We say f is *of finite type* if it is locally of finite type and quasi-compact.

Lemma 13.2. *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. Combine Remark 12.3 with Morphisms of Spaces, Lemma 23.2. $\square$

Lemma 13.3. *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. Combine Remark 12.4 with Morphisms of Spaces, Lemma 23.3. $\square$

Lemma 13.4. *An immersion is locally of finite type.*

Proof. Follows from Morphisms of Spaces, Lemma 23.7. $\square$

Lemma 13.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and $\mathcal{Y}$ is locally Noetherian, then $\mathcal{X}$ is locally Noetherian.*

Proof. Let

$$
\begin{array}{ccc}
U & \longrightarrow & V \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{Y}
\end{array}
$$

be a commutative diagram where U, V are schemes, $V \to \mathcal{Y}$ is surjective and smooth, and $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and smooth. Then $U \to V$ is locally of finite type. If $\mathcal{Y}$ is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 16.6 we see that U is locally Noetherian, which means that $\mathcal{X}$ is locally Noetherian. $\square$

The following two lemmas will be improved on later (after we have discussed morphisms of algebraic stacks which are locally of finite presentation).

Lemma 13.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is locally of finite type, then f is locally of finite type.*

Proof. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is locally of finite presentation. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of locally finite type morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is locally of finite type, we see that $U' \to V'$ is locally of finite type. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is locally of finite type by Descent on Spaces, Lemma 10.7 and we win. $\square$

Lemma 13.7. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume $\mathcal{X} \to \mathcal{Z}$ is locally of finite type and that $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite type.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Set $U = V \times_{\mathcal{Y}} \mathcal{X}$ which is an algebraic space. We know that $U \to V$ is surjective, flat, and locally of finite presentation and that $U \to W$ is locally of finite type. Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 14.2. $\square$

Lemma 13.8. *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f : \mathcal{X} \to \mathcal{Z}$ is locally of finite type, then $f : \mathcal{X} \to \mathcal{Y}$ is locally of finite type.*

Proof. We can find a diagram

$$
\begin{array}{ccccc}
U & \longrightarrow & V & \longrightarrow & W \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z}
\end{array}
$$

where U, V, W are schemes, the vertical arrow $W \to \mathcal{Z}$ is surjective and smooth, the arrow $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ is surjective and smooth, and the arrow $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is surjective and smooth. Then also $U \to \mathcal{X} \times_{\mathcal{Z}} V$ is surjective and smooth (as a composition of a surjective and smooth morphism with a base change of such). By definition we see that $U \to W$ is locally of finite type. Hence $U \to V$ is locally of finite type by Morphisms, Lemma 16.8 which in turn means (by definition) that $\mathcal{X} \to \mathcal{Y}$ is locally of finite type. $\square$

14. Points of finite type

Let $\mathcal{X}$ be an algebraic stack. A finite type point $x \in |\mathcal{X}|$ is a point which can be represented by a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ which is locally of finite type. Finite type points are a suitable replacement of closed points for algebraic spaces and algebraic stacks. There are always "enough of them" for example.

Lemma 14.1. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:*

    (1) *There exists a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ which is locally of finite type and represents x.*

    (2) *There exists a scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ such that $\varphi(u) = x$.*

Proof. Let $u \in U$ and $U \to \mathcal{X}$ be as in (2). Then $\mathrm{Spec}(\kappa(u)) \to U$ is of finite type, and $U \to \mathcal{X}$ is representable and locally of finite type (by Morphisms of Spaces, Lemmas 36.8 and 27.5). Hence we see (1) holds by Lemma 13.2.

Conversely, assume $\mathrm{Spec}(k) \to \mathcal{X}$ is locally of finite type and represents x. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. By assumption $U \times_{\mathcal{X}} \mathrm{Spec}(k) \to U$ is a morphism of algebraic spaces which is locally of finite type. Pick a finite type point v of $U \times_{\mathcal{X}} \mathrm{Spec}(k)$ (there exists at least one, see Morphisms of Spaces, Lemma 25.3). By Morphisms of Spaces, Lemma 25.4 the image $u \in U$ of v is a finite type point of U. Hence by Morphisms, Lemma 17.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. $\square$

Definition 14.2. Let $\mathcal{X}$ be an algebraic stack. We say a point $x \in |\mathcal{X}|$ is a *finite type point*[5] if the equivalent conditions of Lemma 14.1 are satisfied. We denote $\mathcal{X}_{\text{ft-pts}}$ the set of finite type points of $\mathcal{X}$.

We can describe the set of finite type points as follows.

---

[5]This is a slight abuse of language as it would perhaps be more correct to say "locally finite type point".

Lemma 14.3. *Let $\mathcal{X}$ be an algebraic stack. We have*

$$\mathcal{X}_{ft\text{-}pts} = \bigcup_{\varphi:U\to X \ smooth} |\varphi|(U_0)$$

where U_0 is the set of closed points of U. Here we may let U range over all schemes smooth over $\mathcal{X}$ or over all affine schemes smooth over $\mathcal{X}$.

Proof. Immediate from Lemma 14.1. $\qquad\square$

Lemma 14.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type, then $f(\mathcal{X}_{ft\text{-}pts}) \subset \mathcal{Y}_{ft\text{-}pts}$.*

Proof. Take $x \in \mathcal{X}_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$. Then $f \circ x$ is locally of finite type by Lemma 13.2. Hence $f(x) \in \mathcal{Y}_{\text{ft-pts}}$. $\qquad\square$

Lemma 14.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and surjective, then $f(\mathcal{X}_{ft\text{-}pts}) = \mathcal{Y}_{ft\text{-}pts}$.*

Proof. We have $f(\mathcal{X}_{\text{ft-pts}}) \subset \mathcal{Y}_{\text{ft-pts}}$ by Lemma 14.4. Let $y \in |\mathcal{Y}|$ be a finite type point. Represent y by a morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ which is locally of finite type. As f is surjective the algebraic stack $\mathcal{X}_k = \mathrm{Spec}(k) \times_\mathcal{Y} \mathcal{X}$ is nonempty, therefore has a finite type point $x \in |\mathcal{X}_k|$ by Lemma 14.3. Now $\mathcal{X}_k \to \mathcal{X}$ is a morphism which is locally of finite type as a base change of $\mathrm{Spec}(k) \to \mathcal{Y}$ (Lemma 13.3). Hence the image of x in $\mathcal{X}$ is a finite type point by Lemma 14.4 which maps to y by construction. $\qquad\square$

Lemma 14.6. *Let $\mathcal{X}$ be an algebraic stack. For any locally closed subset $T \subset |\mathcal{X}|$ we have*

$$T \neq \emptyset \Rightarrow T \cap \mathcal{X}_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset |\mathcal{X}|$ we see that $T \cap \mathcal{X}_{ft\text{-}pts}$ is dense in T.

Proof. Let $i : \mathcal{Z} \to \mathcal{X}$ be the reduced induced substack structure on T, see Properties of Stacks, Remark 10.5. An immersion is locally of finite type, see Lemma 13.4. Hence by Lemma 14.4 we see $\mathcal{Z}_{\text{ft-pts}} \subset \mathcal{X}_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with a smooth morphism towards $\mathcal{Z}$ has at least one closed point, hence $\mathcal{Z}$ has at least one finite type point by Lemma 14.3. The lemma follows. $\quad\square$

Here is another, more technical, characterization of a finite type point on an algebraic stack. It tells us in particular that the residual gerbe of $\mathcal{X}$ at x exists whenever x is a finite type point!

Lemma 14.7. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:*

(1) *x is a finite type point,*
(2) *there exists an algebraic stack $\mathcal{Z}$ whose underlying topological space $|\mathcal{Z}|$ is a singleton, and a morphism $f : \mathcal{Z} \to \mathcal{X}$ which is locally of finite type such that $\{x\} = |f|(|\mathcal{Z}|)$, and*
(3) *the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type.*

Proof. (All of the morphisms occurring in this paragraph are representable by algebraic spaces, hence the conventions and results of Properties of Stacks, Section 3 are applicable.) Assume x is a finite type point. Choose an affine scheme U, a

closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ with $\varphi(u) = x$, see Lemma 14.3. Set $u = \mathrm{Spec}(\kappa(u))$ as usual. Set $R = u \times_{\mathcal{X}} u$ so that we obtain a groupoid in algebraic spaces (u, R, s, t, c), see Algebraic Stacks, Lemma 16.1. The projection morphisms $R \to u$ are the compositions

$$R = u \times_{\mathcal{X}} u \to u \times_{\mathcal{X}} U \to u \times_{\mathcal{X}} X = u$$

where the first arrow is of finite type (a base change of the closed immersion of schemes $u \to U$) and the second arrow is smooth (a base change of the smooth morphism $U \to \mathcal{X}$). Hence $s, t : R \to u$ are locally of finite type (as compositions, see Morphisms of Spaces, Lemma 23.2). Since u is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 27.7). We see that $\mathcal{Z} = [u/R]$ is an algebraic stack by Criteria for Representability, Theorem 17.2. By Algebraic Stacks, Lemma 16.1 we obtain a canonical morphism

$$f : \mathcal{Z} \longrightarrow \mathcal{X}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 15.2 and a monomorphism, see Properties of Stacks, Lemma 8.4. It follows that the residual gerbe $\mathcal{Z}_x \subset \mathcal{X}$ of $\mathcal{X}$ at x exists and that f factors through an equivalence $\mathcal{Z} \to \mathcal{Z}_x$, see Properties of Stacks, Lemma 11.11. By construction the diagram

$$
\begin{array}{ccc}
u & \longrightarrow & U \\
\downarrow & & \downarrow \\
\mathcal{Z} & \overset{f}{\longrightarrow} & \mathcal{X}
\end{array}
$$

is commutative. By Criteria for Representability, Lemma 17.1 the left vertical arrow is surjective, flat, and locally of finite presentation. Consider

$$
\begin{array}{ccccc}
u \times_{\mathcal{X}} U & \longrightarrow & \mathcal{Z} \times_{\mathcal{X}} U & \longrightarrow & U \\
\downarrow & & \downarrow & & \downarrow \\
u & \longrightarrow & \mathcal{Z} & \overset{f}{\longrightarrow} & \mathcal{X}
\end{array}
$$

As $u \to \mathcal{X}$ is locally of finite type, we see that the base change $u \times_{\mathcal{X}} U \to U$ is locally of finite type. Moreover, $u \times_{\mathcal{X}} U \to \mathcal{Z} \times_{\mathcal{X}} U$ is surjective, flat, and locally of finite presentation as a base change of $u \to \mathcal{Z}$. Thus $\{u \times_{\mathcal{X}} U \to \mathcal{Z} \times_{\mathcal{X}} U\}$ is an fppf covering of algebraic spaces, and we conclude that $\mathcal{Z} \times_{\mathcal{X}} U \to U$ is locally of finite type by Descent on Spaces, Lemma 14.1. By definition this means that f is locally of finite type (because the vertical arrow $\mathcal{Z} \times_{\mathcal{X}} U \to \mathcal{Z}$ is smooth as a base change of $U \to \mathcal{X}$ and surjective as $\mathcal{Z}$ has only one point). Since $\mathcal{Z} = \mathcal{Z}_x$ we see that (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of $\mathcal{X}$ by Lemma 14.4 and Lemma 14.6 to see that $\mathcal{Z}_{\mathrm{ft\text{-}pts}}$ is nonempty, i.e., the unique point of $\mathcal{Z}$ is a finite type point of $\mathcal{Z}$. $\square$

15. Special presentations of algebraic stacks

The following lemma gives a criterion for when a "slice" of a presentation is still flat over the algebraic stack.

Lemma 15.1. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$
\begin{array}{ccc}
U & \xleftarrow{\ p\ } & F \\
\downarrow & & \downarrow \\
\mathcal{X} & \longleftarrow & \mathrm{Spec}(k)
\end{array}
$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $f_1, \ldots, f_r \in \Gamma(U, \mathcal{O}_U)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow \mathcal{X}$$

is flat and locally of finite presentation.

Proof. Choose a scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Choose an extension of fields $k \subset k'$ and a morphism $w : \mathrm{Spec}(k') \to W$ such that $\mathrm{Spec}(k') \to W \to \mathcal{X}$ is 2-isomorphic to $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \to \mathcal{X}$. This is possible as $W \to \mathcal{X}$ is surjective. Consider the commutative diagram

$$
\begin{array}{ccccc}
U & \xleftarrow{\ \mathrm{pr}_0\ } & U \times_{\mathcal{X}} W & \xleftarrow{\ p'\ } & F' \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X} & \longleftarrow & W & \longleftarrow & \mathrm{Spec}(k')
\end{array}
$$

both of whose squares are cartesian. By our choice of w we see that $F' = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$. Thus $F' \to F$ is surjective and we can choose a point $z' \in |F'|$ mapping to z. Since $F' \to F$ is flat we see that $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces, Lemma 28.8. Hence $f_1, \ldots, f_r$ map to a regular sequence in $\mathcal{O}_{F',\overline{z}'}$, see Algebra, Lemma 67.7. Note that $U \times_{\mathcal{X}} W \to W$ is a morphism of algebraic spaces which is flat and locally of finite presentation. Hence by More on Morphisms of Spaces, Lemma 22.1 we see that there exists an open subspace U' of $U \times_{\mathcal{X}} W$ containing $p(z')$ such that the intersection $U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W)$ is flat and locally of finite presentation over W. Note that $\mathrm{pr}_0(U')$ is an open subspace of U containing $p(z)$ as pr_0 is smooth hence open. Now we see that $U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W) \to \mathcal{X}$ is flat and locally of finite presentation as the composition

$$U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W) \to W \to \mathcal{X}.$$

Hence Properties of Stacks, Lemma 3.5 implies $\mathrm{pr}_0(U') \cap V(f_1, \ldots, f_r) \to \mathcal{X}$ is flat and locally of finite presentation as desired. $\square$

Lemma 15.2. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$
\begin{array}{ccc}
U & \xleftarrow{\ p\ } & F \\
\downarrow & & \downarrow \\
\mathcal{X} & \longleftarrow & \mathrm{Spec}(k)
\end{array}
$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is locally of finite type. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is locally quasi-finite.

Proof. Since $f : U \to \mathcal{X}$ is locally of finite type there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is locally quasi-finite, see Properties of Stacks, Remark 9.19 (2). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to \mathrm{Spec}(k)$ is locally quasi-finite at z. This follows immediately from Morphisms of Spaces, Lemma 32.6. $\qquad\square$

A quasi-DM stack has a locally quasi-finite "covering" by a scheme.

Theorem 15.3. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is quasi-DM, and*
(2) *there exists a scheme W and a surjective, flat, locally finitely presented, locally quasi-finite morphism $W \to \mathcal{X}$.*

Proof. The implication (2) $\Rightarrow$ (1) is Lemma 4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which "works" in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

Let U be an affine scheme, $U \to \mathcal{X}$ a smooth morphism, and $u \in U$ a closed point which maps to x, see Lemma 14.1. Denote $u = \mathrm{Spec}(\kappa(u))$ as usual. Consider the following commutative diagram

$$
\begin{array}{ccc}
u & \longleftarrow & R \\
\downarrow & & \downarrow \\
U & \xleftarrow{\ p\ } & F \\
\downarrow & & \downarrow \\
\mathcal{X} & \longleftarrow & u
\end{array}
$$

with both squares fibre product squares, in particular $R = u \times_\mathcal{X} u$. In the proof of Lemma 14.7 we have seen that (u, R, s, t, c) is a groupoid in algebraic spaces with s, t locally of finite type. Let $G \to u$ be the stabilizer group algebraic space (see Groupoids in Spaces, Definition 15.2). Note that

$$
G = R \times_{(u \times u)} u = (u \times_\mathcal{X} u) \times_{(u \times u)} u = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} u.
$$

As $\mathcal{X}$ is quasi-DM we see that G is locally quasi-finite over u. By More on Groupoids in Spaces, Lemma 7.11 we have $\dim(R) = 0$.

Let $e : u \to R$ be the identity of the groupoid. Thus both compositions $u \to R \to u$ are equal to the identity morphism of u. Note that $R \subset F$ is a closed subspace as $u \subset U$ is a closed subscheme. Hence we can also think of e as a point of F. Consider the maps of étale local rings

$$
\mathcal{O}_{U,u} \xrightarrow{p^\sharp} \mathcal{O}_{F,\bar{e}} \longrightarrow \mathcal{O}_{R,\bar{e}}
$$

Note that $\mathcal{O}_{R,\bar{e}}$ has dimension 0 by the result of the first paragraph. On the other hand, the kernel of the second arrow is $p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\bar{e}}$ as R is cut out in F by $\mathfrak{m}_u$.

Thus we see that

$$\mathfrak{m}_{\overline{z}} = \sqrt{p^{\sharp}(\mathfrak{m}_u)\mathcal{O}_{F,\overline{e}}}$$

On the other hand, as the morphism $U \to \mathcal{X}$ is smooth we see that $F \to u$ is a smooth morphism of algebraic spaces. This means that F is a regular algebraic space (Spaces over Fields, Lemma 9.1). Hence $\mathcal{O}_{F,\overline{e}}$ is a regular local ring (Properties of Spaces, Lemma 23.1). Note that a regular local ring is Cohen-Macaulay (Algebra, Lemma 102.3). Let $d = \dim(\mathcal{O}_{F,\overline{e}})$. By Algebra, Lemma 100.10 we can find $f_1, \dots, f_d \in \mathcal{O}_{U,u}$ whose images $\varphi(f_1), \dots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{F,\overline{z}}$. By Lemma 15.1 after shrinking U we may assume that $Z = V(f_1, \dots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation. Note that by construction $F_Z = Z \times_{\mathcal{X}} u$ is a closed subspace of $F = U \times_{\mathcal{X}} u$, that e is a point of this closed subspace, and that

$$\dim(\mathcal{O}_{F_Z,\overline{e}}) = 0.$$

By Morphisms of Spaces, Lemma 32.1 it follows that $\dim_e(F_Z) = 0$ because the transcendence degree of e relative to u is zero. Hence it follows from Lemma 15.2 that after possibly shrinking U the morphism $Z \to \mathcal{X}$ is locally quasi-finite.

We conclude that for every finite type point x of $\mathcal{X}$ there exists a locally quasi-finite, flat, locally finitely presented morphism $f_x : Z_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x Z_x$ and $f = \coprod f_x$. Then f is flat, locally of finite presentation, and locally quasi-finite. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 14.6 (and Properties of Stacks, Lemma 4.4). $\qquad\square$

Lemma 15.4. *Let $\mathcal{Z}$ be a DM, locally Noetherian, reduced algebraic stack with $|\mathcal{Z}|$ a singleton. Then there exists a field k and a surjective étale morphism $\mathrm{Spec}(k) \to \mathcal{Z}$.*

Proof. By Properties of Stacks, Lemma 11.3 there exists a field k and a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Set $U = \mathrm{Spec}(k)$ and $R = U \times_{\mathcal{Z}} U$ so we obtain a groupoid in algebraic spaces (U, R, s, t, c), see Algebraic Stacks, Lemma 9.2. Note that by Algebraic Stacks, Remark 16.3 we have an equivalence

$$f_{can} : [U/R] \longrightarrow \mathcal{Z}$$

The projections $s, t : R \to U$ are locally of finite presentation. As $\mathcal{Z}$ is DM we see that the stabilizer group algebraic space

$$G = U \times_{U \times U} R = U \times_{U \times U} (U \times_{\mathcal{Z}} U) = U \times_{\mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X}$$

is unramified over U. In particular $\dim(G) = 0$ and by More on Groupoids in Spaces, Lemma 7.11 we have $\dim(R) = 0$. This implies that R is a scheme, see Spaces over Fields, Lemma 7.1. By Varieties, Lemma 13.2 we see that R (and also G) is the disjoint union of spectra of Artinian local rings finite over k via either s or t. Let $P = \mathrm{Spec}(A) \subset R$ be the open and closed subscheme whose underlying point is the identity e of the groupoid scheme (U, R, s, t, c). As $s \circ e = t \circ e = \mathrm{id}_{\mathrm{Spec}(k)}$ we see that A is an Artinian local ring whose residue field is identified with k via either $s^{\sharp} : k \to A$ or $t^{\sharp} : k \to A$. Note that $s, t : \mathrm{Spec}(A) \to \mathrm{Spec}(k)$ are finite (by the lemma referenced above). Since $G \to \mathrm{Spec}(k)$ is unramified we see that

$$G \cap P = P \times_{U \times U} U = \mathrm{Spec}(A \otimes_{k \otimes k} k)$$

is unramified over k. On the other hand $A \otimes_{k \otimes k} k$ is local as a quotient of A and surjects onto k. We conclude that $A \otimes_{k \otimes k} k = k$. It follows that $P \to U \times U$

is universally injective (as P has only one point with residue field k, unramified (by the computation of the fibre over the unique image point above), and of finite type (because s, t are) hence a monomorphism (see Étale Morphisms, Lemma 7.1). Thus $s|_P, t|_P : P \to U$ define a finite flat equivalence relation. Thus we may apply Groupoids, Proposition 21.8 to conclude that U/P exists and is a scheme $\overline{U}$. Moreover, $U \to \overline{U}$ is finite locally free and $P = U \times_{\overline{U}} U$. In fact $\overline{U} = \mathrm{Spec}(k_0)$ where $k_0 \subset k$ is the ring of R-invariant functions. As k is a field it follows from the definition Groupoids, Equation (21.0.1) that k_0 is a field.

We claim that

$$(15.4.1) \qquad\qquad \mathrm{Spec}(k_0) = \overline{U} = U/P \to [U/R] = \mathcal{Z}$$

is the desired surjective étale morphism. It follows from Properties of Stacks, Lemma 11.1 that this morphism is surjective. Thus it suffices to show that (15.4.1) is étale[6]. Instead of proving the étaleness directly we first apply Bootstrap, Lemma 9.1 to see that there exists a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ such that (U, R, s, t, c) is the restriction of $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ via the quotient morphism $U \to \overline{U}$. (We verified all the hypothesis of the lemma above except for the assertion that $j : R \to U \times U$ is separated and locally quasi-finite which follows from the fact that R is a separated scheme locally quasi-finite over k.) Since $U \to \overline{U}$ is finite locally free we see that $[U/R] \to [\overline{U}/\overline{R}]$ is an equivalence, see Groupoids in Spaces, Lemma 24.2.

Note that s, t are the base changes of the morphisms $\overline{s}, \overline{t}$ by $U \to \overline{U}$. As $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}, \overline{t}$ are flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 19.13, 19.9, and 19.22. Consider the commutative diagram

$$
\begin{array}{ccc}
U \times_{\overline{U}} U = P & \longrightarrow & R \\
 & \searrow \quad \downarrow & \downarrow \\
 & \overline{U} \xrightarrow{\ \overline{e}\ } & \overline{R}
\end{array}
$$

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 19.14.

But of course, if $\overline{e}$ is an open immersion and $\overline{s}, \overline{t}$ are flat and locally of finite presentation then the morphisms $\overline{t}, \overline{s}$ are étale. For example you can see this by applying More on Groupoids, Lemma 4.1 which shows that $\Omega_{\overline{R}/\overline{U}} = 0$ implies that $\overline{s}, \overline{t} : \overline{R} \to \overline{U}$ is unramified (see Morphisms, Lemma 36.2), which in turn implies that $\overline{s}, \overline{t}$ are étale (see Morphisms, Lemma 37.16). Hence $\mathcal{Z} = [\overline{U}/\overline{R}]$ is an étale presentation of the algebraic stack $\mathcal{Z}$ and we conclude that $\overline{U} \to \mathcal{Z}$ is étale by Properties of Stacks, Lemma 3.3. $\square$

---

[6]We urge the reader to find his/her own proof of this fact. In fact the argument has a lot in common with the final argument of the proof of Bootstrap, Theorem 10.1 hence probably should be isolated into its own lemma somewhere.

Lemma 15.5. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$
\begin{array}{ccc}
U & \xleftarrow{\;p\;} & F \\
\downarrow & & \downarrow \\
\mathcal{X} & \longleftarrow & \mathrm{Spec}(k)
\end{array}
$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $z \in |F|$ be such that $F \to \mathrm{Spec}(k)$ is unramified at z. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is étale.

Proof. Since $f : U \to \mathcal{X}$ is flat and locally of finite presentation there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is étale, see Properties of Stacks, Remark 9.19 (5). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to \mathrm{Spec}(k)$ is étale at z. Since it is flat and locally of finite presentation as a base change of $U \to \mathcal{X}$ and since $F \to \mathrm{Spec}(k)$ is unramified at z by assumption, this follows from Morphisms of Spaces, Lemma 36.12. $\qquad\square$

A DM stack is a Deligne-Mumford stack.

Theorem 15.6. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

1. $\mathcal{X}$ *is DM,*
2. $\mathcal{X}$ *is Deligne-Mumford, and*
3. *there exists a scheme W and a surjective étale morphism $W \to \mathcal{X}$.*

Proof. Recall that (3) is the definition of (2), see Algebraic Stacks, Definition 12.2. The implication (3) $\Rightarrow$ (1) is Lemma 4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which "works" in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

By Lemma 14.7 the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type. By Lemma 4.16 the algebraic stack $\mathcal{Z}_x$ is DM. By Lemma 15.4 there exists a field k and a surjective étale morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}_x$. In particular the composition $x : \mathrm{Spec}(k) \to \mathcal{X}$ is locally of finite type (by Morphisms of Spaces, Lemmas 23.2 and 36.9).

Pick a scheme U and a smooth morphism $U \to \mathcal{X}$ such that x is in the image of $|U| \to |\mathcal{X}|$. Consider the following fibre square

$$
\begin{array}{ccc}
U & \longleftarrow & F \\
\downarrow & & \downarrow \\
\mathcal{X} & \xleftarrow{\;x\;} & \mathrm{Spec}(k)
\end{array}
$$

in other words $F = U \times_{\mathcal{X},x} \mathrm{Spec}(k)$. By Properties of Stacks, Lemma 4.3 we see that F is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we have

$$\mathrm{Spec}(k) \times_{z,\mathcal{Z}_x,z} \mathrm{Spec}(k) = \mathrm{Spec}(k) \times_{x,\mathcal{X},x} \mathrm{Spec}(k)$$

with étale projection maps to $\mathrm{Spec}(k)$ by construction of z. Since

$$F \times_U F = (\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k)) \times_{\mathrm{Spec}(k)} F$$

we see that the projections maps $F \times_U F \to F$ are étale as well. It follows that $\Delta_{F/U} : F \to F \times_U F$ is étale (see Morphisms of Spaces, Lemma 36.11). By Morphisms of Spaces, Lemma 45.2 this implies that $\Delta_{F/U}$ is an open immersion, which finally implies by Morphisms of Spaces, Lemma 35.9 that $F \to U$ is unramified.

Pick a nonempty affine scheme V and an étale morphism $V \to F$. (This could be avoided by working directly with F, but it seems easier to explain what's going on by doing so.) Picture

$$\begin{array}{ccccc} U & \longleftarrow & F & \longleftarrow & V \\ \downarrow & & \downarrow & \swarrow & \\ \mathcal{X} & \xleftarrow{\ x\ } & \mathrm{Spec}(k) & & \end{array}$$

Then $V \to \mathrm{Spec}(k)$ is a smooth morphism of schemes and $V \to U$ is an unramified morphism of schemes (see Morphisms of Spaces, Lemmas 34.2 and 35.3). Pick a closed point $v \in V$ with $k \subset \kappa(v)$ finite separable, see Varieties, Lemma 15.6. Let $u \in U$ be the image point. The local ring $\mathcal{O}_{V,v}$ is regular (see Varieties, Lemma 15.3) and the local ring homomorphism

$$\varphi : \mathcal{O}_{U,u} \longrightarrow \mathcal{O}_{V,v}$$

coming from the morphism $V \to U$ is such that $\varphi(\mathfrak{m}_u)\mathcal{O}_{V,v} = \mathfrak{m}_v$, see Morphisms, Lemma 36.14. Hence we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ such that the images $\varphi(f_1), \ldots, \varphi(f_d)$ form a basis for $\mathfrak{m}_v/\mathfrak{m}_v^2$ over $\kappa(v)$. Since $\mathcal{O}_{V,v}$ is a regular local ring this implies that $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{V,v}$ (see Algebra, Lemma 102.3). After replacing U by an open neighbourhood of u we may assume $f_1, \ldots, f_d \in \Gamma(U, \mathcal{O}_U)$. After replacing U by a possibly even smaller open neighbourhood of u we may assume that $V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation, see Lemma 15.1. By construction

$$V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k) \longleftarrow V(f_1, \ldots, f_d) \times_{\mathcal{X}} V$$

is étale and $V(f_1, \ldots, f_d) \times_{\mathcal{X}} V$ is the closed subscheme $T \subset V$ cut out by $f_1|_V, \ldots, f_d|_V$. Hence by construction $v \in T$ and

$$\mathcal{O}_{T,v} = \mathcal{O}_{V,v}/(\varphi(f_1), \ldots, \varphi(f_d)) = \kappa(v)$$

a finite separable extension of k. It follows that $T \to \mathrm{Spec}(k)$ is unramified at v, see Morphisms, Lemma 36.14. By definition of an unramified morphism of algebraic spaces this means that $V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is unramified at the image of v in $V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k)$. Applying Lemma 15.5 we see that on shrinking U to yet another open neighbourhood of u the morphism $V(f_1, \ldots, f_d) \to \mathcal{X}$ is étale.

We conclude that for every finite type point x of $\mathcal{X}$ there exists an étale morphism $f_x : W_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x W_x$ and $f = \coprod f_x$. Then f is étale. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 14.6 (and Properties of Stacks, Lemma 4.4). $\qquad\square$

16. Quasi-finite morphisms

The property "locally quasi-finite" of morphisms of algebraic spaces is not smooth local on the source-and-target so we cannot use the material in Section 12 to define locally quasi-finite morphisms of algebraic stacks. We do already know what it means for a morphism of algebraic stacks representable by algebraic spaces to be locally quasi-finite, see Properties of Stacks, Section 3. To find a condition suitable for general morphisms we make the following observation.

Lemma 16.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces. The following are equivalent*

(1) *f is locally quasi-finite, and*
(2) *f is locally of finite type and for every morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ where k is a field the space $|\mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}|$ is discrete.*

Proof. Assume (1). In this case the morphism of algebraic spaces $\mathcal{X}_k \to \mathrm{Spec}(k)$ is locally quasi-finite as a base change of f. Hence $|\mathcal{X}_k|$ is discrete by Morphisms of Spaces, Lemma 26.5. Conversely, assume (2). Pick a surjective smooth morphism $V \to \mathcal{Y}$ where V is a scheme. It suffices to show that the morphism of algebraic spaces $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally quasi-finite, see Properties of Stacks, Lemma 3.3. The morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally of finite type by assumption. For any morphism $\mathrm{Spec}(k) \to V$ where k is a field

$$\mathrm{Spec}(k) \times_V (V \times_{\mathcal{Y}} \mathcal{X}) = \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}$$

has a discrete space of points by assumption. Hence we conclude that $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally quasi-finite by Morphisms of Spaces, Lemma 26.5. $\square$

A morphism of algebraic stacks which is representable by algebraic spaces is quasi-DM, see Lemma 4.3. Combined with the lemma above we see that the following definition does not conflict with all of the already existing notion in the case of morphisms representable by algebraic spaces.

Definition 16.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *locally quasi-finite* if f is quasi-DM, locally of finite type, and for every morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ where k is a field the space $|\mathcal{X}_k|$ is discrete.

The condition that f be quasi-DM is natural. For example, let k be a field and consider the morphism $\pi : [\mathrm{Spec}(k)/\mathbf{G}_m] \to \mathrm{Spec}(k)$ which has singleton fibres and is locally of finite type. As we will see later this morphism is smooth of relative dimension -1, and we'd like our locally quasi-finite morphisms to have relative dimension 0. Also, note that the section $\mathrm{Spec}(k) \to [\mathrm{Spec}(k)/\mathbf{G}_m]$ does not have discrete fibres, hence is not locally quasi-finite, and we'd like to have the following permanence property for locally quasi-finite morphisms: If $f : \mathcal{X} \to \mathcal{X}'$ is a morphism of algebraic stacks locally quasi-finite over the algebraic stack $\mathcal{Y}$, then f is locally quasi-finite (in fact something a bit stronger holds, see Lemma 16.8).

Another justification for the definition above is Lemma 16.7 below which characterizes being locally quasi-finite in terms of the existence of suitable "presentations" or "coverings" of $\mathcal{X}$ and $\mathcal{Y}$.

Lemma 16.3. *A base change of a locally quasi-finite morphism is locally quasi-finite.*

Proof. We have seen this for quasi-DM morphisms in Lemma 4.4 and for locally finite type morphisms in Lemma 13.3. It is immediate that the condition on fibres is inherited by a base change. $\square$

Lemma 16.4. *Let $\mathcal{X} \to \mathrm{Spec}(k)$ be a locally quasi-finite morphism where $\mathcal{X}$ is an algebraic stack and k is a field. Let $f : V \to \mathcal{X}$ be a locally quasi-finite morphism where V is a scheme. Then $V \to \mathrm{Spec}(k)$ is locally quasi-finite.*

Proof. By Lemma 13.2 we see that $V \to \mathrm{Spec}(k)$ is locally of finite type. Assume, to get a contradiction, that $V \to \mathrm{Spec}(k)$ is not locally quasi-finite. Then there exists a nontrivial specialization $v \rightsquigarrow v'$ of points of V, see Morphisms, Lemma 21.6. In particular $\mathrm{trdeg}_k(\kappa(v)) > \mathrm{trdeg}_k(\kappa(v'))$, see Morphisms, Lemma 29.6. Because $|\mathcal{X}|$ is discrete we see that $|f|(v) = |f|(v')$. Consider $R = V \times_{\mathcal{X}} V$. Then R is an algebraic space and the projections $s, t : R \to V$ are locally quasi-finite as base changes of $V \to \mathcal{X}$ (which is representable by algebraic spaces so this follows from the discussion in Properties of Stacks, Section 3). By Properties of Stacks, Lemma 4.3 we see that there exists an $r \in |R|$ such that $s(r) = v$ and $t(r) = v'$. By Morphisms of Spaces, Lemma 31.3 we see that the transcendence degree of v/k is equal to the transcendence degree of r/k is equal to the transcendence degree of v'/k. This contradiction proves the lemma. $\square$

Lemma 16.5. *A composition of a locally quasi-finite morphisms is locally quasi-finite.*

Proof. We have seen this for quasi-DM morphisms in Lemma 4.10 and for locally finite type morphisms in Lemma 13.2. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be locally quasi-finite. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be a morphism. It suffices to show that $|\mathcal{X}_k|$ is discrete. By Lemma 16.3 the morphisms $\mathcal{X}_k \to \mathcal{Y}_k$ and $\mathcal{Y}_k \to \mathrm{Spec}(k)$ are locally quasi-finite. In particular we see that $\mathcal{Y}_k$ is a quasi-DM algebraic stack, see Lemma 4.13. By Theorem 15.3 we can find a scheme V and a surjective, flat, locally finitely presented, locally quasi-finite morphism $V \to \mathcal{Y}_k$. By Lemma 16.4 we see that V is locally quasi-finite over k, in particular $|V|$ is discrete. The morphism $V \times_{\mathcal{Y}_k} \mathcal{X}_k \to \mathcal{X}_k$ is surjective, flat, and locally of finite presentation hence $|V \times_{\mathcal{Y}_k} \mathcal{X}_k| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to show that $|V \times_{\mathcal{Y}_k} \mathcal{X}_k|$ is discrete. Note that V is a disjoint union of spectra of Artinian local k-algebras A_i with residue fields k_i, see Varieties, Lemma 13.2. Thus it suffices to show that each

$$|\mathrm{Spec}(A_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\mathrm{Spec}(k_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\mathrm{Spec}(k_i) \times_{\mathcal{Y}} \mathcal{X}|$$

is discrete, which follows from the assumption that $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite. $\square$

Before we characterize locally quasi-finite morphisms in terms of coverings we do it for quasi-DM morphisms.

Lemma 16.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is quasi-DM,*
(2) *for any morphism $V \to \mathcal{Y}$ with V an algebraic space there exists a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space, and*
(3) *there exist algebraic spaces U, V and a morphism $V \to \mathcal{Y}$ which is surjective, flat, and locally of finite presentation, and a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$*

which is surjective, flat, locally of finite presentation, and locally quasi-finite.

Proof. The implication $(2) \Rightarrow (3)$ is immediate.

Assume (1) and let $V \to \mathcal{Y}$ be as in (2). Then $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM, see Lemma 4.4. By Lemma 4.3 the algebraic space V is DM, hence quasi-DM. Thus $\mathcal{X} \times_{\mathcal{Y}} V$ is quasi-DM by Lemma 4.11. Hence we may apply Theorem 15.3 to get the morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ as in (2).

Assume (3). Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be as in (3). To prove that f is quasi-DM it suffices to show that $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM, see Lemma 4.5. By Lemma 4.14 we see that $\mathcal{X} \times_{\mathcal{Y}} V$ is quasi-DM. Hence $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM by Lemma 4.13 and (1) holds. This finishes the proof of the lemma. $\qquad\square$

Lemma 16.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *f is locally quasi-finite,*
(2) *f is quasi-DM and for any morphism $V \to \mathcal{Y}$ with V an algebraic space and any locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space the morphism $U \to V$ is locally quasi-finite,*
(3) *for any morphism $V \to \mathcal{Y}$ from an algebraic space V there exists a surjective, flat, locally finitely presented, and locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space such that $U \to V$ is locally quasi-finite,*
(4) *there exists algebraic spaces U, V, a surjective, flat, and locally of finite presentation morphism $V \to \mathcal{Y}$, and a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite such that $U \to V$ is locally quasi-finite.*

Proof. Assume (1). Then f is quasi-DM by assumption. Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be as in (2). By Lemma 16.5 the composition $U \to \mathcal{X} \times_{\mathcal{Y}} V \to V$ is locally quasi-finite. Thus (1) implies (2).

Assume (2). Let $V \to \mathcal{Y}$ be as in (3). By Lemma 16.6 we can find an algebraic space U and a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. By (2) the composition $U \to V$ is locally quasi-finite. Thus (2) implies (3).

It is immediate that (3) implies (4).

Assume (4). We will prove (1) holds, which finishes the proof. By Lemma 16.6 we see that f is quasi-DM. To prove that f is locally of finite type it suffices to prove that $g : \mathcal{X} \times_{\mathcal{Y}} V \to V$ is locally of finite type, see Lemma 13.6. Then it suffices to check that g precomposed with $h : U \to \mathcal{X} \times_{\mathcal{Y}} V$ is locally of finite type, see Lemma 13.7. Since $g \circ h : U \to V$ was assumed to be locally quasi-finite this holds, hence f is locally of finite type. Finally, let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Y}$ be a morphism. Then $V \times_{\mathcal{Y}} \mathrm{Spec}(k)$ is a nonempty algebraic space which is locally of finite presentation over k. Hence we can find a finite extension $k \subset k'$ and a

morphism $\mathrm{Spec}(k') \to V$ such that

$$
\begin{array}{ccc}
\mathrm{Spec}(k') & \longrightarrow & V \\
\downarrow & & \downarrow \\
\mathrm{Spec}(k) & \longrightarrow & \mathcal{Y}
\end{array}
$$

commutes (details omitted). Then $\mathcal{X}_{k'} \to \mathcal{X}_k$ is representable (by schemes), surjective, and finite locally free. In particular $|\mathcal{X}_{k'}| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to prove that $|\mathcal{X}_{k'}|$ is discrete. Since

$$U \times_V \mathrm{Spec}(k') = U \times_{\mathcal{X} \times_{\mathcal{Y}} V} \mathcal{X}_{k'}$$

we see that $U \times_V \mathrm{Spec}(k') \to \mathcal{X}_{k'}$ is surjective, flat, and locally of finite presentation (as a base change of $U \to \mathcal{X} \times_{\mathcal{Y}} V$). Hence $|U \times_V \mathrm{Spec}(k')| \to |\mathcal{X}_{k'}|$ is surjective and open. Thus it suffices to show that $|U \times_V \mathrm{Spec}(k')|$ is discrete. This follows from the fact that $U \to V$ is locally quasi-finite (either by our definition above or from the original definition for morphisms of algebraic spaces, via Morphisms of Spaces, Lemma 26.5). $\qquad\square$

Lemma 16.8. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume that $\mathcal{X} \to \mathcal{Z}$ is locally quasi-finite and $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Then $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite.*

Proof. Write $\mathcal{X} \to \mathcal{Y}$ as the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{Y}$$

The second arrow is locally quasi-finite as a base change of $\mathcal{X} \to \mathcal{Z}$, see Lemma 16.3. The first arrow is locally quasi-finite by Lemma 4.8 as $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Hence $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite by Lemma 16.5. $\qquad\square$

17. Flat morphisms

The property "being flat" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 28.4 and Descent on Spaces, Lemma 10.11. Hence, by Lemma 12.1 above, we may define what it means for a morphism of algebraic spaces to be flat as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces.

Definition 17.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *flat* if the equivalent conditions of Lemma 12.1 hold with $\mathcal{P} = $ flat.

Lemma 17.2. *The composition of flat morphisms is flat.*

Proof. Combine Remark 12.3 with Morphisms of Spaces, Lemma 28.3. $\qquad\square$

Lemma 17.3. *A base change of a flat morphism is flat.*

Proof. Combine Remark 12.4 with Morphisms of Spaces, Lemma 28.4. $\qquad\square$

Lemma 17.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective flat morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is flat, then f is flat.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Then $W \to \mathcal{Z}$ is surjective and flat (Morphisms of Spaces, Lemma 34.7) hence $W \to \mathcal{Y}$ is surjective and flat (by Properties of Stacks, Lemma 5.2 and Lemma 17.2). Since the base change of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ by $W \to \mathcal{Z}$ is a flat morphism (Lemma 17.3) we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is flat. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of flat morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is flat, we see that $U' \to V'$ is flat. Then, since $V' \to V$ is surjective as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is flat by Morphisms of Spaces, Lemma 29.3 (2) and we win. $\square$

Lemma 17.5. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is flat and $\mathcal{X} \to \mathcal{Y}$ is surjective and flat, then $\mathcal{Y} \to \mathcal{Z}$ is flat.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and that $U \to W$ is flat. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Morphisms of Spaces, Lemma 29.5. $\square$

18. Morphisms of finite presentation

The property "locally of finite presentation" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 27.3 and Descent on Spaces, Lemma 10.8. Hence, by Lemma 12.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite presentation as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces.

Definition 18.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.
  (1) We say f *locally of finite presentation* if the equivalent conditions of Lemma 12.1 hold with $\mathcal{P} = $ locally of finite presentation.
  (2) We say f is *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation.

Lemma 18.2. *The composition of finitely presented morphisms is of finite presentation. The same holds for morphisms which are locally of finite presentation.*

Proof. Combine Remark 12.3 with Morphisms of Spaces, Lemma 27.2. $\square$

Lemma 18.3. *A base change of a finitely presented morphism is of finite presentation. The same holds for morphisms which are locally of finite presentation.*

Proof. Combine Remark 12.4 with Morphisms of Spaces, Lemma 27.3. $\square$

Lemma 18.4. *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Combine Remark 12.5 with Morphisms of Spaces, Lemma 27.5. $\square$

Lemma 18.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks If $g \circ f$ is locally of finite presentation and g is locally of finite type, then f is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. The lemma follows upon applying Morphisms of Spaces, Lemma 27.9 to the morphisms $U \to V \to W$. $\square$

Lemma 18.6. *An open immersion is locally of finite presentation.*

Proof. Follows from Morphisms of Spaces, Lemma 27.11. $\square$

Lemma 18.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is locally of finite presentation, then f is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Then $W \to \mathcal{Z}$ is surjective, flat, and locally of finite presentation (Morphisms of Spaces, Lemmas 34.7 and 34.5) hence $W \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation (by Properties of Stacks, Lemma 5.2 and Lemmas 17.2 and 18.2). Since the base change of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ by $W \to \mathcal{Z}$ is locally of finite presentation (Lemma 17.3) we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is locally of finite presentation. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of locally finitely presented morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is locally of finite presentation, we see that $U' \to V'$ is locally of finite presentation. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is locally of finite presentation by Descent on Spaces, Lemma 10.8 and we win. $\square$

Lemma 18.8. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is locally of finite presentation and $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and locally of finite presentation and that $U \to W$ is locally of finite presentation. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces

to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 14.1. $\qquad\square$

Lemma 18.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Then for every scheme U and object y of $\mathcal{Y}$ over U there exists an fppf covering $\{U_i \to U\}$ and objects x_i of $\mathcal{X}$ over U_i such that $f(x_i) \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$.*

Proof. We may think of y as a morphism $U \to \mathcal{Y}$. By Properties of Stacks, Lemma 5.3 and Lemmas 18.3 and 17.3 we see that $\mathcal{X} \times_{\mathcal{Y}} U \to U$ is surjective, flat, and locally of finite presentation. Let V be a scheme and let $V \to \mathcal{X} \times_{\mathcal{Y}} U$ smooth and surjective. Then $V \to \mathcal{X} \times_{\mathcal{Y}} U$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 34.7 and 34.5). Hence also $V \to U$ is surjective, flat, and locally of finite presentation, see Properties of Stacks, Lemma 5.2 and Lemmas 18.2, and 17.2. Hence $\{V \to U\}$ is the desired fppf covering and $x : V \to \mathcal{X}$ is the desired object. $\qquad\square$

Lemma 18.10. *Let $f_j : \mathcal{X}_j \to \mathcal{X}$, $j \in J$ be a family of morphisms of algebraic stacks which are each flat and locally of finite presentation and which are jointly surjective, i.e., $|\mathcal{X}| = \bigcup |\mathcal{X}_i|$. Then for every scheme U and object x of $\mathcal{X}$ over U there exists an fppf covering $\{U_i \to U\}_{i \in I}$, a map $a : I \to J$, and objects x_i of $\mathcal{X}_{a(i)}$ over U_i such that $f_{a(i)}(x_i) \cong y|_{U_i}$ in $\mathcal{X}_{U_i}$.*

Proof. Apply Lemma 18.9 to the morphism $\coprod_{j \in J} \mathcal{X}_j \to \mathcal{X}$. (There is a slight set theoretic issue here – due to our setup of things – which we ignore.) To finish, note that a morphism $x_i : U_i \to \coprod_{j \in J} \mathcal{X}_j$ is given by a disjoint union decomposition $U_i = \coprod U_{i,j}$ and morphisms $U_{i,j} \to \mathcal{X}_j$. Then the fppf covering $\{U_{i,j} \to U\}$ and the morphisms $U_{i,j} \to \mathcal{X}_j$ do the job. $\qquad\square$

Lemma 18.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be flat and locally of finite presentation. Then $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is open.*

Proof. Choose a scheme V and a smooth surjective morphism $V \to \mathcal{Y}$. Choose a scheme U and a smooth surjective morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. By assumption the morphism of schemes $U \to V$ is flat and locally of finite presentation. Hence $U \to V$ is open by Morphisms, Lemma 26.9. By construction of the topology on $|\mathcal{Y}|$ the map $|V| \to |\mathcal{Y}|$ is open. The map $|U| \to |\mathcal{X}|$ is surjective. The result follows from these facts by elementary topology. $\qquad\square$

19. Gerbes

An important type of algebraic stack are the stacks of the form $[B/G]$ where B is an algebraic space and G is a flat and locally finitely presented group algebraic space over B (acting trivially on B), see Criteria for Representability, Lemma 18.3. It turns out that an algebraic stack is a gerbe when it locally in the fppf topology is of this form, see Lemma 19.8. In this section we briefly discuss this notion and the corresponding relative notion.

Definition 19.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say $\mathcal{X}$ is a *gerbe over* $\mathcal{Y}$ if $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ as stacks in groupoids over $(Sch/S)_{fppf}$, see Stacks, Definition 11.4. We say an algebraic stack $\mathcal{X}$ is a *gerbe* if there exists a morphism $\mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X.

The condition that $\mathcal{X}$ be a gerbe over $\mathcal{Y}$ is defined purely in terms of the topology and category theory underlying the given algebraic stacks; but as we will see later this condition has geometric consequences. For example it implies that $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, see Lemma 19.7. The absolute notion is trickier to parse, because it may not be at first clear that X is well determined. Actually, it is.

Lemma 19.2. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe, then the sheafification of the presheaf*

$$(Sch/S)^{opp}_{fppf} \to Sets, \quad U \mapsto \mathrm{Ob}(\mathcal{X}_U)/\cong$$

is an algebraic space and $\mathcal{X}$ is a gerbe over it.

Proof. (In this proof the abuse of language introduced in Section 2 really pays off.) Choose a morphism $\pi : \mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X. It suffices to prove that X is the sheafification of the presheaf $\mathcal{F}$ displayed in the lemma. It is clear that there is a map $c : \mathcal{F} \to X$. We will use Stacks, Lemma 11.3 properties (2)(a) and (2)(b) to see that the map $c^{\#} : \mathcal{F}^{\#} \to X$ is surjective and injective, hence an isomorphism, see Sites, Lemma 12.2. Surjective: Let T be a scheme and let $f : T \to X$. By property (2)(a) there exists an fppf covering $\{h_i : T_i \to T\}$ and morphisms $x_i : T_i \to \mathcal{X}$ such that $f \circ h_i$ corresponds to $\pi \circ x_i$. Hence we see that $f|_{T_i}$ is in the image of c. Injective: Let T be a scheme and let $x, y : T \to \mathcal{X}$ be morphisms such that $c \circ x = c \circ y$. By (2)(b) we can find a covering $\{T_i \to T\}$ and morphisms $x|_{T_i} \to y|_{T_i}$ in the fibre category $\mathcal{X}_{T_i}$. Hence the restrictions $x|_{T_i}, y|_{T_i}$ are equal in $\mathcal{F}(T_i)$. This proves that x, y give the same section of $\mathcal{F}^{\#}$ over T as desired. $\square$

Lemma 19.3. *Let*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

be a fibre product of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$.

Proof. Immediate from the definitions and Stacks, Lemma 11.5. $\square$

Lemma 19.4. *Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{Y}$ is a gerbe over $\mathcal{Z}$, then $\mathcal{X}$ is a gerbe over $\mathcal{Z}$.*

Proof. Immediate from Stacks, Lemma 11.6. $\square$

Lemma 19.5. *Let*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

be a fibre product of algebraic stacks. If $\mathcal{Y}' \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation and $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$, then $\mathcal{X}$ is a gerbe over $\mathcal{Y}$.

Proof. Follows immediately from Lemma 18.9 and Stacks, Lemma 11.7. $\square$

Lemma 19.6. *Let $\pi : \mathcal{X} \to U$ be a morphism from an algebraic stack to an algebraic space and let $x : U \to \mathcal{X}$ be a section of π. Set $G = Isom_\mathcal{X}(x,x)$, see Definition 5.3. If $\mathcal{X}$ is a gerbe over U, then*

(1) *there is a canonical equivalence of stacks in groupoids*

$$x_{can} : [U/G] \longrightarrow \mathcal{X}.$$

where $[U/G]$ is the quotient stack for the trivial action of G on U,

(2) *$G \to U$ is flat and locally of finite presentation, and*

(3) *$U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation.*

Proof. Set $R = U \times_{x,\mathcal{X},x} U$. The morphism $R \to U \times U$ factors through the diagonal $\Delta_U : U \to U \times U$ as it factors through $U \times_U U = U$. Hence $R = G$ because

$$\begin{aligned}
G &= Isom_\mathcal{X}(x,x) \\
&= U \times_{x,\mathcal{X}} \mathcal{I}_\mathcal{X} \\
&= U \times_{x,\mathcal{X}} (\mathcal{X} \times_{\Delta,\mathcal{X}\times_S\mathcal{X},\Delta} \mathcal{X}) \\
&= (U \times_{x,\mathcal{X},x} U) \times_{U\times U,\Delta_U} U \\
&= R \times_{U\times U,\Delta_U} U \\
&= R
\end{aligned}$$

for the fourth equality use Categories, Lemma 29.12. Let $t,s : R \to U$ be the projections. The composition law $c : R \times_{s,U,t} R \to R$ constructed on R in Algebraic Stacks, Lemma 16.1 agrees with the group law on G (proof omitted). Thus Algebraic Stacks, Lemma 16.1 shows we obtain a canonical fully faithful 1-morphism

$$x_{can} : [U/G] \longrightarrow \mathcal{X}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. To see that it is an equivalence it suffices to show that it is essentially surjective. To do this it suffices to show that any object of $\mathcal{X}$ over a scheme T comes fppf locally from x via a morphism $T \to U$, see Stacks, Lemma 4.8. However, this follows the condition that π turns $\mathcal{X}$ into a gerbe over X, see property (2)(a) of Stacks, Lemma 11.3.

By Criteria for Representability, Lemma 18.3 we conclude that $G \to U$ is flat and locally of finite presentation. Finally, $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 17.1. $\square$

Lemma 19.7. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe over $\mathcal{Y}$, and*

(2) *there exists an algebraic space U, a group algebraic space G flat and locally of finite presentation over U, and a surjective, flat, and locally finitely presented morphism $U \to \mathcal{Y}$ such that $\mathcal{X} \times_\mathcal{Y} U \cong [U/G]$ over U.*

Proof. Assume (2). By Lemma 19.5 to prove (1) it suffices to show that $[U/G]$ is a gerbe over U. This is immediate from Groupoids in Spaces, Lemma 26.2.

Assume (1). Any base change of π is a gerbe, see Lemma 19.3. As a first step we choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Thus we may assume that $\pi : \mathcal{X} \to V$ is a gerbe over a scheme. This means that there exists an fppf covering $\{V_i \to V\}$ such that the fibre category $\mathcal{X}_{V_i}$ is nonempty, see Stacks,

Lemma 11.3 (2)(a). Note that $U = \coprod V_i \to U$ is surjective, flat, and locally of finite presentation. Hence we may replace V by U and assume that $\pi : \mathcal{X} \to U$ is a gerbe over a scheme U and that there exists an object x of $\mathcal{X}$ over U. By Lemma 19.6 we see that $\mathcal{X} = [U/G]$ over U for some flat and locally finitely presented group algebraic space G over U. $\qquad\square$

Lemma 19.8. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then π is surjective, flat, and locally of finite presentation.*

Proof. By Properties of Stacks, Lemma 5.4 and Lemmas 17.4 and 18.7 it suffices to prove to the lemma after replacing π by a base change with a surjective, flat, locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$. By Lemma 19.7 we may assume $\mathcal{Y} = U$ is an algebraic space and $\mathcal{X} = [U/G]$ over U. Then $U \to [U/G]$ is surjective, flat, and locally of finite presentation, see Lemma 19.6. This implies that π is surjective, flat, and locally of finite presentation by Properties of Stacks, Lemma 5.5 and Lemmas 17.5 and 18.8. $\qquad\square$

Proposition 19.9. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe, and*
(2) *$\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is flat and locally of finite presentation.*

Proof. Assume (1). Choose a morphism $\mathcal{X} \to X$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $X' \to X$ is a surjective, flat, locally finitely presented morphism and set $\mathcal{X}' = X' \times_X \mathcal{X}$. Note that $\mathcal{X}'$ is a gerbe over X' by Lemma 19.3. Then both squares in

$$
\begin{array}{ccccc}
\mathcal{I}_{\mathcal{X}'} & \longrightarrow & \mathcal{X}' & \longrightarrow & X' \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{I}_\mathcal{X} & \longrightarrow & \mathcal{X} & \longrightarrow & X
\end{array}
$$

are fibre product squares, see Lemma 5.4. Hence to prove $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is flat and locally of finite presentation it suffices to do so after such a base change by Lemmas 17.4 and 18.7. Thus we can apply Lemma 19.7 to assume that $\mathcal{X} = [U/G]$. By Lemma 19.6 we see G is flat and locally of finite presentation over U and that $x : U \to [U/G]$ is surjective, flat, and locally of finite presentation. Moreover, the pullback of $\mathcal{I}_\mathcal{X}$ by x is G and we conclude that (2) holds by descent again, i.e., by Lemmas 17.4 and 18.7.

Conversely, assume (2). Choose a smooth presentation $\mathcal{X} = [U/R]$, see Algebraic Stacks, Section 16. Denote $G \to U$ the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Groupoids in Spaces, Definition 15.2. By Lemma 5.6 we see that $G \to U$ is flat and locally of finite presentation as a base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$, see Lemmas 17.3 and 18.3. Consider the following action

$$a : G \times_{U,t} R \to R, \quad (g, r) \mapsto c(g, r)$$

of G on R. This action is free on T-valued points for any scheme T as R is a groupoid. Hence $R' = R/G$ is an algebraic space and the quotient morphism $\pi : R \to R'$ is surjective, flat, and locally of finite presentation by Bootstrap, Lemma 11.7. The projections $s, t : R \to U$ are G-invariant, hence we obtain morphisms $s', t' : R' \to U$ such that $s = s' \circ \pi$ and $t = t' \circ \pi$. Since $s, t : R \to U$ are flat and locally of finite presentation we conclude that s', t' are flat and locally of

finite presentation, see Morphisms of Spaces, Lemmas 29.5 and Descent on Spaces, Lemma 14.1. Consider the morphism

$$j' = (t', s') : R' \longrightarrow U \times U.$$

We claim this is a monomorphism. Namely, suppose that T is a scheme and that $a, b : T \to R'$ are morphisms which have the same image in $U \times U$. By definition of the quotient $R' = R/G$ there exists an fppf covering $\{h_j : T_j \to T\}$ such that $a \circ h_j = \pi \circ a_j$ and $b \circ h_j = \pi \circ b_j$ for some morphisms $a_j, b_j : T_j \to R$. Since a_j, b_j have the same image in $U \times U$ we see that $g_j = c(a_j, i(b_j))$ is a T_j-valued point of G such that $c(g_j, b_j) = a_j$. In other words, a_j and b_j have the same image in R' and the claim is proved. Since $j : R \to U \times U$ is a pre-equivalence relation (see Groupoids in Spaces, Lemma 11.2) and $R \to R'$ is surjective (as a map of sheaves) we see that $j' : R' \to U \times U$ is an equivalence relation. Hence Bootstrap, Theorem 10.1 shows that $X = U/R'$ is an algebraic space. Finally, we claim that the morphism

$$\mathcal{X} = [U/R] \longrightarrow X = U/R'$$

turns $\mathcal{X}$ into a gerbe over X. This follows from Groupoids in Spaces, Lemma 26.1 as $R \to R'$ is surjective, flat, and locally of finite presentation (if needed use Bootstrap, Lemma 4.6 to see this implies the required hypothesis). $\square$

At this point we have developed enough machinery to prove that residual gerbes (when they exist) are gerbes.

Lemma 19.10. *Let $\mathcal{Z}$ be a reduced, locally Noetherian algebraic stack such that $|\mathcal{Z}|$ is a singleton. Then $\mathcal{Z}$ is a gerbe over a reduced, locally Noetherian algebraic space Z with $|Z|$ a singleton.*

Proof. By Properties of Stacks, Lemma 11.3 there exists a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field. Then $\mathcal{I}_Z \times_{\mathcal{Z}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is representable by algebraic spaces and locally of finite type (as a base change of $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$, see Lemmas 5.1 and 13.3). Therefore it is locally of finite presentation, see Morphisms of Spaces, Lemma 27.7. Of course it is also flat as k is a field. Hence we may apply Lemmas 17.4 and 18.7 to see that $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$ is flat and locally of finite presentation. We conclude that $\mathcal{Z}$ is a gerbe by Proposition 19.9. Let $\pi : \mathcal{Z} \to Z$ be a morphism to an algebraic space such that $\mathcal{Z}$ is a gerbe over Z. Then π is surjective, flat, and locally of finite presentation by Lemma 19.8. Hence $\mathrm{Spec}(k) \to Z$ is surjective, flat, and locally of finite presentation as a composition, see Properties of Stacks, Lemma 5.2 and Lemmas 17.2 and 18.2. Hence by Properties of Stacks, Lemma 11.3 we see that $|Z|$ is a singleton and that Z is locally Noetherian and reduced. $\square$

Lemma 19.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ then the map $|\mathcal{X}| \to |\mathcal{Y}|$ is a homeomorphism of topological spaces.*

Proof. Let k be a field and let y be an object of $\mathcal{Y}$ over $\mathrm{Spec}(k)$. By Stacks, Lemma 11.3 property (2)(a) there exists an fppf covering $\{T_i \to \mathrm{Spec}(k)\}$ and objects x_i of $\mathcal{X}$ over T_i with $f(x_i) \cong y|_{T_i}$. Choose an i such that $T_i \neq \emptyset$. Choose a morphism $\mathrm{Spec}(K) \to T_i$ for some field K. Then $k \subset K$ and $x_i|_K$ is an object of $\mathcal{X}$ lying over $y|_K$. Thus we see that $|\mathcal{Y}| \to |\mathcal{X}|$. is surjective. The map $|\mathcal{Y}| \to |\mathcal{X}|$ is also injective. Namely, if x, x' are objects of $\mathcal{X}$ over $\mathrm{Spec}(k)$ whose images $f(x), f(x')$ become isomorphic (over an extension) in $\mathcal{Y}$, then Stacks, Lemma 11.3 property

(2)(b) guarantees the existence of an extension of k over which x and x' become isomorphic (details omitted). Hence $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous and bijective and it suffices to show that it is also open. This follows from Lemmas 19.8 and 18.11. $\square$

The following lemma tells us that residual gerbes exist for all points on any algebraic stack which is a gerbe.

Lemma 19.12. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe then for every $x \in |\mathcal{X}|$ the residual gerbe of $\mathcal{X}$ at x exists.*

Proof. Let $\pi : \mathcal{X} \to X$ be a morphism from $\mathcal{X}$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $Z_x \to X$ be the residual space of X at x, see Decent Spaces, Definition 11.6. Let $\mathcal{Z} = \mathcal{X} \times_X Z_x$. By Lemma 19.3 the algebraic stack $\mathcal{Z}$ is a gerbe over Z_x. Hence $|\mathcal{Z}| = |Z_x|$ (Lemma 19.11) is a singleton. Since $\mathcal{Z} \to Z_x$ is locally of finite presentation as a base change of π (see Lemmas 19.8 and 18.3) we see that $\mathcal{Z}$ is locally Noetherian, see Lemma 13.5. Thus the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and is equal to $\mathcal{Z}_x = \mathcal{Z}_{red}$ the reduction of the algebraic stack $\mathcal{Z}$. Namely, we have seen above that $|\mathcal{Z}_{red}|$ is a singleton mapping to $x \in |\mathcal{X}|$, it is reduced by construction, and it is locally Noetherian (as the reduction of a locally Noetherian algebraic stack is locally Noetherian, details omitted). $\square$

20. Stratification by gerbes

The goal of this section is to show that many algebraic stacks $\mathcal{X}$ have a "stratification" by locally closed substacks $\mathcal{X}_i \subset \mathcal{X}$ such that each $\mathcal{X}_i$ is a gerbe. This shows that in some sense gerbes are the building blocks out of which any algebraic stack is constructed. Note that by stratification we only mean that

$$|\mathcal{X}| = \bigcup_i |\mathcal{X}_i|$$

is a stratification of the topological space associated to $\mathcal{X}$ and nothing more (in this section). Hence it is harmless to replace $\mathcal{X}$ by its reduction (see Properties of Stacks, Section 10) in order to study this stratification.

The following proposition tells us there is (almost always) a dense open substack of the reduction of $\mathcal{X}$

Proposition 20.1. *Let $\mathcal{X}$ be a reduced algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe.*

Proof. According to Proposition 19.9 it is enough to find a dense open substack $\mathcal{U}$ such that $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation. Note that $\mathcal{I}_\mathcal{U} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{U}$, see Lemma 5.4.

Choose a presentation $\mathcal{X} = [U/R]$. Let $G \to U$ be the stabilizer group algebraic space of the groupoid R. By Lemma 5.6 we see that $G \to U$ is the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ hence quasi-compact (by assumption) and locally of finite type (by Lemma 5.1). Let $W \subset U$ be the largest open (possibly empty) subscheme such that the restriction $G_W \to W$ is flat and locally of finite presentation (we omit the proof that W exists; hint: use that the properties are local). By Morphisms of Spaces, Proposition 30.1 we see that $W \subset U$ is dense. Note that $W \subset U$ is R-invariant by More on Groupoids in Spaces, Lemma 4.2. Hence W corresponds to an open substack $\mathcal{U} \subset \mathcal{X}$ by Properties of Stacks, Lemma 9.10. Since $|U| \to |\mathcal{X}|$ is open and $|W| \subset |U|$ is dense we conclude that $\mathcal{U}$ is dense in $\mathcal{X}$. Finally, the morphism

$\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation because the base change by the surjective smooth morphism $W \to \mathcal{U}$ is the morphism $G_W \to W$ which is flat and locally of finite presentation by construction. See Lemmas 17.4 and 18.7. $\qquad \square$

The above proposition immediately implies that any point has a residual gerbe on an algebraic stack with quasi-compact inertia, as we will show in Lemma 21.1. It turns out that there doesn't always exist a finite stratification by gerbes. Here is an example.

Example 20.2. Let k be a field. Take $U = \mathrm{Spec}(k[x_0, x_1, x_2, \ldots])$ and let $\mathbf{G}_m$ act by $t(x_0, x_1, x_2, \ldots) = (tx_0, t^p x_1, t^{p^2} x_2, \ldots)$ where p is a prime number. Let $\mathcal{X} = [U/\mathbf{G}_m]$. This is an algebraic stack. There is a stratification of $\mathcal{X}$ by strata

(1) $\mathcal{X}_0$ is where x_0 is not zero,
(2) $\mathcal{X}_1$ is where x_0 is zero but x_1 is not zero,
(3) $\mathcal{X}_2$ is where x_0, x_1 are zero, but x_2 is not zero,
(4) and so on, and
(5) $\mathcal{X}_\infty$ is where all the x_i are zero.

Each stratum is a gerbe over a scheme with group μ_{p^i} for $\mathcal{X}_i$ and $\mathbf{G}_m$ for $\mathcal{X}_\infty$. The strata are reduced locally closed substacks. There is no coarser stratification with the same properties.

Nonetheless, using transfinite induction we can use Proposition 20.1 find possibly infinite stratifications by gerbes...!

Lemma 20.3. *Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a well-ordered index set I and for every $i \in I$ a reduced locally closed substack $\mathcal{U}_i \subset \mathcal{X}$ such that*

(1) *each $\mathcal{U}_i$ is a gerbe,*
(2) *we have $|\mathcal{X}| = \bigcup_{i \in I} |\mathcal{U}_i|$,*
(3) *$T_i = |\mathcal{X}| \setminus \bigcup_{i' < i} |\mathcal{U}_{i'}|$ is closed in $|\mathcal{X}|$ for all $i \in I$, and*
(4) *$|\mathcal{U}_i|$ is open in T_i.*

We can moreover arrange it so that either (a) $|\mathcal{U}_i| \subset T_i$ is dense, or (b) $\mathcal{U}_i$ is quasi-compact. In case (a), if we choose $\mathcal{U}_i$ as large as possible (see proof for details), then the stratification is canonical.

Proof. Let $T \subset |\mathcal{X}|$ be a nonempty closed subset. We are going to find (resp. choose) for every such T a reduced locally closed substack $\mathcal{U}(T) \subset \mathcal{X}$ with $|\mathcal{U}(T)| \subset T$ open dense (resp. nonempty quasi-compact). Namely, by Properties of Stacks, Lemma 10.1 there exists a unique reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 5.5. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 7.3. Therefore Proposition 20.1 implies there exists a dense maximal (see proof proposition) open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. In case (a) we set $\mathcal{U}(T) = \mathcal{U}$ (this is canonical) and in case (b) we simply choose a nonempty quasi-compact open $\mathcal{U}(T) \subset \mathcal{U}$, see Properties of Stacks, Lemma 4.9 (we can do this for all T simultaneously by the axiom of choice).

By transfinite induction we construct for every ordinal α a closed subset $T_\alpha \subset |\mathcal{X}|$. For $\alpha = 0$ we set $T_0 = |\mathcal{X}|$. Given T_α set

$$T_{\alpha+1} = T_\alpha \setminus |\mathcal{U}(T_\alpha)|.$$

If β is a limit ordinal we set

$$T_\beta = \bigcap_{\alpha < \beta} T_\alpha.$$

We claim that $T_\alpha = \emptyset$ for all α large enough. Namely, assume that $T_\alpha \neq \emptyset$ for all α. Then we obtain an injective map from the class of ordinals into the set of subsets of $|\mathcal{X}|$ which is a contradiction.

The claim implies the lemma. Namely, let

$$I = \{\alpha \mid \mathcal{U}_\alpha \neq \emptyset\}.$$

This is a well-ordered set by the claim. For $i = \alpha \in I$ we set $\mathcal{U}_i = \mathcal{U}_\alpha$. So $\mathcal{U}_i$ is a reduced locally closed substack and a gerbe, i.e., (1) holds. By construction $T_i = T\alpha$ if $i = \alpha \in I$, hence (3) holds. Also, (4) and (a) or (b) hold by our choice of $\mathcal{U}(T)$ as well. Finally, to see (2) let $x \in |\mathcal{X}|$. There exists a smallest ordinal β with $x \notin T_\beta$ (because the ordinals are well-ordered). In this case β has to be a successor ordinal by the definition of T_β for limit ordinals. Hence $\beta = \alpha + 1$ and $x \in |\mathcal{U}(T_\alpha)|$ and we win. $\qquad\square$

Remark 20.4. We can wonder about the order type of the canonical stratifications which occur as output of the stratifications of type (a) constructed in Lemma 20.3. A natural guess is that the well-ordered set I has *cardinality* at most $\aleph_0$. We have no idea if this is true or false. If you do please email stacks.project@gmail.com.

21. Existence of residual gerbes

In this section we prove that residual gerbes (as defined in Properties of Stacks, Definition 11.8) exist on many algebraic stacks. First, here is the promised application of Proposition 20.1.

Lemma 21.1. *Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Let $T = \overline{\{x\}} \subset |\mathcal{X}|$ be the closure of x. By Properties of Stacks, Lemma 10.1 there exists a reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 5.5. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 7.3. Therefore Proposition 20.1 implies there exists a dense open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. Note that $x \in |\mathcal{U}|$ because $\{x\} \subset T$ is a dense subset too. Hence a residual gerbe $\mathcal{Z}_x \subset \mathcal{U}$ of $\mathcal{U}$ at x exists by Lemma 19.12. It is immediate from the definitions that $\mathcal{Z}_x \to \mathcal{X}$ is a residual gerbe of $\mathcal{X}$ at x. $\qquad\square$

If the stack is quasi-DM then residual gerbes exist too. In particular, residual gerbes always exist for Delinge-Mumford stacks.

Lemma 21.2. *Let $\mathcal{X}$ be a quasi-DM algebraic stack. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Choose a scheme U and a surjective, flat, locally finite presented, and locally quasi-finite morphism $U \to \mathcal{X}$, see Theorem 15.3. Set $R = U \times_\mathcal{X} U$. The projections $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite as base changes of the morphism $U \to \mathcal{X}$. There is a canonical morphism $[U/R] \to \mathcal{X}$ (see Algebraic Stacks, Lemma 16.1) which is an equivalence because $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Remark 16.3. Thus we may assume that $\mathcal{X} = [U/R]$ where (U, R, s, t, c) is

a groupoid in algebraic spaces such that $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite. Set

$$U' = \coprod_{u \in U \text{ lying over } x} \operatorname{Spec}(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_{\mathcal{X}} U' = R \times_{(U \times U)} (U' \times U')$$

Because $U' \to U$ is a monomorphism we see that both projections $s', t' : R' \to U'$ factor as a monomorphism followed by a locally quasi-finite morphism. Hence, as U' is a disjoint union of spectra of fields, using Spaces over Fields, Lemma 7.3 we conclude that the morphisms $s', t' : R' \to U'$ are locally quasi-finite. Again since U' is a disjoint union of spectra of fields, the morphisms s', t' are also flat. Finally, s', t' locally quasi-finite implies s', t' locally of finite type, hence s', t' locally of finite presentation (because U' is a disjoint union of spectra of fields in particular locally Noetherian, so that Morphisms of Spaces, Lemma 27.7 applies). Hence $\mathcal{Z} = [U'/R']$ is an algebraic stack by Criteria for Representability, Theorem 17.2. As R' is the restriction of R by $U' \to U$ we see $\mathcal{Z} \to \mathcal{X}$ is a monomorphism by Groupoids in Spaces, Lemma 24.1 and Properties of Stacks, Lemma 8.4. Since $\mathcal{Z} \to \mathcal{X}$ is a monomorphism we see that $|\mathcal{Z}| \to |\mathcal{X}|$ is injective, see Properties of Stacks, Lemma 8.5. By Properties of Stacks, Lemma 4.3 we see that

$$|U'| = |\mathcal{Z} \times_{\mathcal{X}} U'| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{X}|} |U'|$$

is surjective which implies (by our choice of U') that $|\mathcal{Z}| \to |\mathcal{X}|$ has image $\{x\}$. We conclude that $|\mathcal{Z}|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., $\mathcal{Z}$ is reduced and locally Noetherian. This means that the essential image of $\mathcal{Z} \to \mathcal{X}$ is the residual gerbe of $\mathcal{X}$ at x, see Properties of Stacks, Lemma 11.11. $\square$

22. Smooth morphisms

The property "being smooth" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 34.3 and Descent on Spaces, Lemma 10.24. Hence, by Lemma 12.1 above, we may define what it means for a morphism of algebraic spaces to be smooth as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 3 when the morphism is representable by algebraic spaces.

Definition 22.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *smooth* if the equivalent conditions of Lemma 12.1 hold with $\mathcal{P} = $ smooth.

Lemma 22.2. *The composition of smooth morphisms is smooth.*

Proof. Combine Remark 12.3 with Morphisms of Spaces, Lemma 34.2. $\square$

Lemma 22.3. *A base change of a smooth morphism is smooth.*

Proof. Combine Remark 12.4 with Morphisms of Spaces, Lemma 34.3. $\square$

23. Other chapters

References

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