# ALGEBRAIC SPACES OVER FIELDS

## Contents

## 1. Introduction

This chapter is the analogue of the chapter on varieties in the setting of algebraic spaces. A reference for algebraic spaces is [Knu71].

## 2. Conventions

The standing assumption is that all schemes are contained in a big fppf site $Sch_{fppf}$. And all rings $A$ considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let $S$ be a scheme and let $X$ be an algebraic space over $S$. In this chapter and the following we will write $X \times_S X$ for the product of $X$ with itself (in the category of algebraic spaces over $S$), instead of $X \times X$.

## 3. Geometric components

**Lemma 3.1.** *Let $k$ be an algebraically closed field. Let $A$, $B$ be strictly henselian local $k$-algebras with residue field equal to $k$. Let $C$ be the strict henselization of $A \otimes_k B$ at the maximal ideal $\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B$. Then the minimal primes of $C$ correspond 1-to-1 to pairs of minimal primes of $A$ and $B$.*

**Proof.** First note that a minimal prime $\mathfrak{r}$ of $C$ maps to a minimal prime $\mathfrak{p}$ in $A$ and to a minimal prime $\mathfrak{q}$ of $B$ because the ring maps $A \to C$ and $B \to C$ are flat (by going down for flat ring map Algebra, Lemma 38.17). Hence it suffices to show that the strict henselization of $(A/\mathfrak{p} \otimes_k B/\mathfrak{q})_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$ has a unique minimal prime ideal. By Algebra, Lemma 145.30 the rings $A/\mathfrak{p}$, $B/\mathfrak{q}$ are strictly henselian. Hence we may assume that $A$ and $B$ are strictly henselian local domains and our goal is

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This is a chapter of the Stacks Project, version 714e994, compiled on Oct 28, 2014.

1

to show that $C$ has a unique minimal prime. By Properties of Spaces, Lemma 21.3. we see that the integral closure $A'$ of $A$ in its fraction field is a normal local domain with residue field $k$ and similarly for the integral closure $B'$ of $B$ into its fraction field. By Algebra, Lemma 153.4 we see that $A' \otimes_k B'$ is a normal ring. Hence its localization

$$R = (A' \otimes_k B')_{\mathfrak{m}_{A'} \otimes_k B' + A' \otimes_k \mathfrak{m}_{B'}}$$

is a normal local domain. Note that $A \otimes_k B \to A' \otimes_k B'$ is integral (hence gong up holds – Algebra, Lemma 35.20) and that $\mathfrak{m}_{A'} \otimes_k B' + A' \otimes_k \mathfrak{m}_{B'}$ is the unique maximal ideal of $A' \otimes_k B'$ lying over $\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B$. Hence we see that

$$R = (A' \otimes_k B')_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$$

by Algebra, Lemma 40.11. It follows that

$$(A \otimes_k B)_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B} \longrightarrow R$$

is integral. We conclude that $R$ is the integral closure of $(A \otimes_k B)_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$ in its fraction field, and by Properties of Spaces, Lemma 21.3 once again we conclude that $C$ has a unique prime ideal. $\square$

## 4. Generically finite morphisms

This section discusses for morphisms of algebraic spaces the material discussed in Morphisms, Section 47 and Varieties, Section 24 for morphisms of schemes.

**Lemma 4.1.** *Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume that $f$ is quasi-separated of finite type and $Y$ is decent. Let $y \in |Y|$ be a generic point of an irreducible component of $|Y|$. The following are equivalent:*

(1) *the set $f^{-1}(\{y\})$ is finite,*
(2) *$X \to Y$ is quasi-finite at all points of $|X|$ over $y$,*
(3) *there exists an open subspace $Y' \subset Y$ with $y \in |Y'|$ such that $Y' \times_Y X \to Y'$ is finite.*

**Proof.** Since $Y$ is decent and $f$ is quasi-separated, we see that $X$ is decent too; to see this use Decent Spaces, Lemmas 15.2 and 15.5. Hence Decent Spaces, Lemma 16.10 applies and we see that (1) implies (2). On the other hand, we see that (2) implies (1) by Morphisms of Spaces, Lemma 26.9. The same lemma also shows that (3) implies (1).

Assume the equivalent conditions of (1) and (2). Choose an affine scheme $V$ and an étale morphism $V \to Y$ mapping a point $v \in V$ to $y$. Then $v$ is a generic point of an irreducible component of $V$ by Decent Spaces, Lemma 10.8. Choose an affine scheme $U$ and a surjective étale morphism $U \to V \times_Y X$. Then $U \to V$ is of finite type. The morphism $U \to V$ is quasi-finite at every point lying over $v$ by (2). It follows that the fibre of $U \to V$ over $v$ is finite (Morphisms, Lemma 21.14). By Morphisms, Lemma 47.1 after shrinking $V$ we may assume that $U \to V$ is finite. Let

$$R = U \times_{V \times_Y X} U$$

Since $f$ is quasi-separated, we see that $V \times_Y X$ is quasi-separated and hence $R$ is a quasi-compact scheme. Moreover the morphisms $R \to V$ is quasi-finite as the composition of an étale morphism $R \to U$ and a finite morphism $U \to V$. Hence we may apply Morphisms, Lemma 47.1 once more and after shrinking $V$ we may assume that $R \to V$ is finite as well. This of course implies that the two projections $R \to V$

are finite étale. It follows that $V/R = V \times_Y X$ is an affine scheme, see Groupoids, Proposition 21.8. By Morphisms, Lemma 42.8 we conclude that $V \times_Y X \to V$ is proper and by Morphisms, Lemma 44.10 we conclude that $V \times_Y X \to V$ is finite. Finally, we let $Y' \subset Y$ be the open subspace of $Y$ corresponding to the image of $|V| \to |Y|$. By Morphisms of Spaces, Lemma 41.3 we conclude that $Y' \times_Y X \to Y'$ is finite as the base change to $V$ is finite and as $V \to Y'$ is a surjective étale morphism. $\square$

**Lemma 4.2.** *Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume that $f$ is quasi-separated and locally of finite type and $Y$ is quasi-separated. Let $y \in |Y|$ be a generic point of an irreducible component of $|Y|$. The following are equivalent:*

(1) *the set $f^{-1}(\{y\})$ is finite,*
(2) *there exist open subspaces $X' \subset X$ and $Y' \subset Y$ with $f(X') \subset Y'$, $y \in |Y'|$, and $f^{-1}(\{y\}) \subset |X'|$ such that $f|_{X'} : X' \to Y'$ is finite.*

**Proof.** This is just an application of Lemma 4.1. we may first replace $Y$ by a quasi-compact open subspace containing $y$. If (1) holds, then we can find a quasi-compact open subspace $X' \subset X$ containing $f^{-1}(\{y\})$. Since $Y$ is quasi-separated, the morphism $f|_{X'} : X' \to Y$ is quasi-compact and quasi-separated (Morphisms of Spaces, Lemma 8.9). Applying Lemma 4.1 to $f|_{X'} : X' \to Y$ we see that (2) holds. We omit the proof that (2) implies (1). $\square$

**Lemma 4.3.** *Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume $f$ is locally of finite type, $Y$ is locally Noetherian, and $X$ is a decent algebraic space. Let $y \in Y$ be a point such that the dimension of the local ring at $y$ is $\leq 1$. Assume in addition one of the following conditions is satisfied*

(1) *for every generic point $x$ of an irreducible component of $|X|$ the transcendence degree of $x/f(x)$ is $0$,*
(2) *for every generic point $x$ of an irreducible component of $|X|$ such that $f(x) \rightsquigarrow y$ the transcendence degree of $x/f(x)$ is $0$,*
(3) *$f$ is quasi-finite at every generic point of $|X|$,*
(4) *$f$ is quasi-finite at a dense set of points of $|X|$,*
(5) *add more here.*

*Then $f$ is quasi-finite at every point of $X$ lying over $y$.*

**Proof.** Observe that $X$ is locally Noetherian (Morphisms of Spaces, Lemma 23.5) and hence $|X|$ is locally Noetherian (Properties of Spaces, Lemma 22.2). Since $X$ is decent $|X|$ is also a sober topological space (Decent Spaces, Proposition 10.6). The set of points at which morphism is quasi-finite is open (Morphisms of Spaces, Lemma 26.2). A dense open of a sober locally Noetherian topological space contains all generic point of irreducible components, hence (4) implies (3). Condition (3) implies condition (1) for example by Morphisms of Spaces, Lemma 31.3 applied to $X \to Y \to Y$. Condition (1) implies condition (2). Thus it suffices to prove the lemma in case (2) holds.

We want to reduce the proof to the case of schemes. To do this we choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array}$$

where $U$, $V$ are schemes and where the horizontal arrows are étale. Say $v \in V$ maps to $y$. Let $u \in U$ be a generic point of an irreducible component of $U$. Then $\dim(\mathcal{O}_{U,u}) = 0$ which implies that $x = f(u)$ is a generic point of an irreducible component of $|X|$ by Decent Spaces, Lemma 10.8. Moreover, if $g(u) \rightsquigarrow v$, then of course $f(x) \rightsquigarrow y$. Thus we see that $\kappa(u)/\kappa(g(u))$ is a field extension of transcendence degree 0. In other words, assumption (2) of Varieties, Lemma 24.1 is satisfied for $g : U \to V$ and $v \in V$. We conclude that $g$ is quasi-finite at all points of $U$ lying over $v$ as desired. $\square$

**Lemma 4.4.** *Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over $S$. Assume $f$ is proper, $Y$ is locally Noetherian, and $X$ is a decent algebraic space. Let $y \in Y$ be a point such that the dimension of the local ring at $y$ is $\leq 1$. Assume in addition one of the following conditions is satisfied*

1. *for every generic point $x$ of an irreducible component of $|X|$ the transcendence degree of $x/f(x)$ is 0,*
2. *for every generic point $x$ of an irreducible component of $|X|$ such that $f(x) \rightsquigarrow y$ the transcendence degree of $x/f(x)$ is 0,*
3. *$f$ is quasi-finite at every generic point of $|X|$,*
4. *$f$ is quasi-finite at a dense set of points of $|X|$,*
5. *add more here.*

*Then there exists an open subspace $Y' \subset Y$ containing $y$ such that $Y' \times_Y X \to Y'$ is finite.*

**Proof.** By Lemma 4.3 the morphism $f$ is quasi-finite at every point lying over $y$. Let $\mathrm{Spec}(k) \to Y$ be any morphism from the spectrum of a field in the equivalence class of $y$. Then $|X_k|$ is a discrete space (Decent Spaces, Lemma 16.10). Since $X_k$ is quasi-compact as $f$ is proper we conclude that $|X_k|$ is finite. Thus we can apply Cohomology of Spaces, Lemma 21.2 to conclude. $\square$

## 5. Integral algebraic spaces

We have not yet defined the notion of an integral algebraic space. The problem is that begin integral is not an étale local property of schemes. We could use the property, that $X$ is reduced and $|X|$ is irreducible, given in Properties, Lemma 3.4 to define integral algebraic spaces. In this case the algebraic space described in Spaces, Example 14.9 would be integral which does not seem right. To avoid this type of patholopgy we will in addition assume that $X$ is a decent algebraic space, although perhaps a weaker alternative exists.

**Definition 5.1.** Let $S$ be a scheme. We say an algebraic space $X$ over $S$ is *integral* if it is reduced, decent, and $|X|$ is irreducible.

In this case the irreducible topological space $|X|$ is sober (Decent Spaces, Proposition 10.6). Hence it has a unique generic point $x$. Then $x$ is contained in the schematic locus of $X$ (Decent Spaces, Theorem 9.2) and we can look at its residue

field as a substitute for the function field of $X$ (not yet defined; insert future reference here).

The following lemma characterizes dominant morphisms of finite degree between integral algebraic spaces.

**Lemma 5.2.** *Let $S$ be a scheme. Let $X$, $Y$ be integral algebraic spaces over $S$ Let $x \in |X|$ and $y \in |Y|$ be the generic points. Let $f : X \to Y$ be locally of finite type. Assume $f$ is dominant (Morphisms of Spaces, Definition 18.1). The following are equivalent:*

1. *the transcendence degree of $x/y$ is $0$,*
2. *the extension $\kappa(x) \supset \kappa(y)$ (see proof) is finite,*
3. *there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is finite,*
4. *$f$ is quasi-finite at $x$, and*
5. *$x$ is the only point of $|X|$ mapping to $y$.*

*If $f$ is separated, or if $f$ is quasi-compact, then these are also equivalent to*

6. *there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.*

**Proof.** By elementary topology, we see that $f(x) = y$ as $f$ is dominant. Let $Y' \subset Y$ be the schematic locus of $Y$ and let $X' \subset f^{-1}(Y')$ be the schematic locus of $X$. By the discussion above, using Decent Spaces, Proposition 10.6 and Decent Spaces, Theorem 9.2, we see that $x \in |X'|$ and $y \in |Y|$. Then $f|_{X'} : X' \to Y'$ is a morphism of integral schemes which is locally of finite type. Thus we see that (1), (2), (3) are equivalent by Morphisms, Lemma 47.4.

Condition (4) implies condition (1) by Morphisms of Spaces, Lemma 31.3 applied that $X \to Y \to Y$. On the other hand, condition (3) implies condition (4) as a finite morphism is quasi-finite and as $x \in U$ because $x$ is the generic point. Thus (1) – (4) are equivalent.

Assume the equivalent conditions (1) – (4). Suppose that $x' \mapsto y$. Then $x \rightsquigarrow x'$ is a specialization in the fibre of $|X| \to |Y|$ over $y$. If $x' \neq x$, then $f$ is not quasi-finite at $x$ by Decent Spaces, Lemma 16.9. Hence $x = x'$ and (5) holds. Conversely, if (5) holds, then (5) holds for the morphism of schemes $X' \to Y'$ (see above) and we can use Morphisms, Lemma 47.4 to see that (1) holds.

Observe that (6) implies the equivalent conditions (1) – (5) without any further assumptions on $f$. Assume (1) – (5) hold. To prove (6) we may shrink $Y$ and assume that $Y$ is an affine scheme and that there exists an affine open $U \subset X$ such that $U \to Y$ is finite.

Assume $f$ is quasi-compact. Then $Z = X \setminus U$ is a quasi-compact closed subspace of $X$ such that $y \notin f(Z)$. Then there exists an open neighbourhood of $y$ which is disjoint from $f(Z)$ (details omitted; hint: use a variant of Morphisms, Lemma 8.3). After shrinking $Y$ we obtain $X = U$.

Assume $f$ separated. Then $U \to X$ has closed image by Morphisms of Spaces, Lemma 37.6. Since $|X|$ is irreducible we get $U = X$. $\qquad\square$

**Definition 5.3.** Let $S$ be a scheme. Let $X$ and $Y$ be integral algebraic spaces over $S$. Let $f : X \to Y$ be locally of finite type and dominant. Assume any of

the equivalent conditions (1) – (5) of Lemma 5.2. Let $x \in |X|$ and $y \in |Y|$ be the generic points. Then the positive integer

$$\deg(X/Y) = [\kappa(x) : \kappa(y)]$$

is called the *degree of $X$ over $Y$*.

## 6. Modifications and alterations

Using our notion of an integral algebraic space we can define a modification as follows.

**Definition 6.1.** Let $S$ be a scheme. Let $X$ be an integral algebraic space. A *modification of $X$* is a birational proper morphism $f : X' \to X$ of algebraic spaces over $S$ with $X'$ integral.

For birational morphisms of algebraic spaces, see Decent Spaces, Definition 18.1.

**Lemma 6.2.** *Let $f : X' \to X$ be a modification as in Definition 6.1. There exists a nonempty open $U \subset X$ such that $f^{-1}(U) \to U$ is an isomorphism.*

**Proof.** By Lemma 5.2 there exists a nonempty $U \subset X$ such that $f^{-1}(U) \to U$ is finite. By generic flatness (Morphisms of Spaces, Proposition 30.1) we may assume $f^{-1}(U) \to U$ is flat and of finite presentation. So $f^{-1}(U) \to U$ is finite locally free (Morphisms of Spaces, Lemma 42.6). Since $f$ is birational, the degree of $X'$ over $X$ is 1. Hence $f^{-1}(U) \to U$ is finite locally free of degree 1, in other words it is an isomorphism. $\square$

**Definition 6.3.** Let $S$ be a scheme. Let $X$ be an integral algebraic space over $S$. An *alteration of $X$* is a proper dominant morphism $f : Y \to X$ of algebraic spaces over $S$ with $Y$ integral such that $f^{-1}(U) \to U$ is finite for some nonempty open $U \subset X$.

If $f : Y \to X$ is a dominant and proper morphism between integral algebraic spaces, then it is an alteration as soon as the induced extension of residue fields in generic points is finite. Here is the precise statement.

**Lemma 6.4.** *Let $S$ be a scheme. Let $f : X \to Y$ be a proper dominant morphism of integral algebraic spaces over $S$. Then $f$ is an alteration if and only if any of the equivalent conditions (1) – (6) of Lemma 5.2 hold.*

**Proof.** Immediate consequence of the lemma referenced in the statement. $\square$

## 7. Schematic locus

We have already proven a number of results on the schematic locus of an algebraic space in Properties of Spaces, Sections 10 and 11 and Decent Spaces, Section 9.

**Lemma 7.1.** *Let $X$ be an algebraic space over some base scheme $S$. In each of the following cases $X$ is a scheme:*

(1) *$X$ is quasi-compact and quasi-separated and $\dim(X) = 0$,*
(2) *$X$ is locally of finite type over a field $k$ and $\dim(X) = 0$,*
(3) *$X$ is Noetherian and $\dim(X) = 0$, and*
(4) *add more here.*

**Proof.** Cases (2) and (3) follow immediately from case (1) but we will give a separate proofs of (2) and (3) as these proofs use significantly less theory.

Proof of (3). Let $U$ be an affine scheme and let $U \to X$ be an étale morphism. Set $R = U \times_X U$. The two projection morphisms $s, t : R \to U$ are étale morphisms of schemes. By Properties of Spaces, Definition 8.2 we see that $\dim(U) = 0$ and $\dim(R) = 0$. Since $R$ is a locally Noetherian scheme of dimension 0, we see that $R$ is a disjoint union of spectra of Artinian local rings (Properties, Lemma 10.3). Since we assumed that $X$ is Noetherian (so quasi-separated) we conclude that $R$ is quasi-compact. Hence $R$ is an affine scheme (use Schemes, Lemma 6.8). The étale morphisms $s, t : R \to U$ induce finite residue field extensions. Hence $s$ and $t$ are finite by Algebra, Lemma 52.3 (small detail omitted). Thus Groupoids, Proposition 21.8 shows that $X = U/R$ is an affine scheme.

Proof of (2) – almost identical to the proof of (4). Let $U$ be an affine scheme and let $U \to X$ be an étale morphism. Set $R = U \times_X U$. The two projection morphisms $s, t : R \to U$ are étale morphisms of schemes. By Properties of Spaces, Definition 8.2 we see that $\dim(U) = 0$ and similarly $\dim(R) = 0$. On the other hand, the morphism $U \to \mathrm{Spec}(k)$ is locally of finite type as the composition of the étale morphism $U \to X$ and $X \to \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemmas 23.2 and 36.9. Similarly, $R \to \mathrm{Spec}(k)$ is locally of finite type. Hence by Varieties, Lemma 13.2 we see that $U$ and $R$ are disjoint unions of spectra of local Artinian $k$-algebras finite over $k$. The same thing is therefore true of $U \times_{\mathrm{Spec}(k)} U$. As

$$R = U \times_X U \longrightarrow U \times_{\mathrm{Spec}(k)} U$$

is a monomorphism, we see that $R$ is a finite(!) union of spectra of finite $k$-algebras. It follows that $R$ is affine, see Schemes, Lemma 6.8. Applying Varieties, Lemma 13.2 once more we see that $R$ is finite over $k$. Hence $s, t$ are finite, see Morphisms, Lemma 44.12. Thus Groupoids, Proposition 21.8 shows that the open subspace $U/R$ of $X$ is an affine scheme. Since the schematic locus of $X$ is an open subspace (see Properties of Spaces, Lemma 10.1), and since $U \to X$ was an arbitrary étale morphism from an affine scheme we conclude that $X$ is a scheme.

Proof of (1). By Cohomology of Spaces, Lemma 9.1 we have vanishing of higher cohomology groups for all quasi-coherent sheaves $\mathcal{F}$ on $X$. Hence $X$ is affine (in particular a scheme) by Cohomology of Spaces, Proposition 15.9. □

Please compare the following lemma to Decent Spaces, Lemma 16.8.

**Lemma 7.2.** *Let $k$ be a field. Let $X$ be an algebraic space over $k$. The following are equivalent*

(1) *$X$ is locally quasi-finite over $k$,*
(2) *$X$ is locally of finite type over $k$ and has dimension $0$,*
(3) *$X$ is a scheme and is locally quasi-finite over $k$,*
(4) *$X$ is a scheme and is locally of finite type over $k$ and has dimension $0$, and*
(5) *$X$ is a disjoint union of spectra of Artinian local $k$-algebras $A$ over $k$ with $\dim_k(A) < \infty$.*

**Proof.** Because we are over a field relative dimension of $X/k$ is the same as the dimension of $X$. Hence by Morphisms of Spaces, Lemma 32.6 we see that (1) and (2) are equivalent. Hence it follows from Lemma 7.1 (and trivial implications) that

(1) – (4) are equivalent. Finally, Varieties, Lemma 13.2 shows that (1) – (4) are equivalent with (5). $\square$

**Lemma 7.3.** *Let $k$ be a field. Let $f : X \to Y$ be a monomorphism of algebraic spaces over $k$. If $Y$ is locally quasi-finite over $k$ so is $X$.*

**Proof.** Assume $Y$ is locally quasi-finite over $k$. By Lemma 7.2 we see that $Y = \coprod \mathrm{Spec}(A_i)$ where each $A_i$ is an Artinian local ring finite over $k$. By Decent Spaces, Lemma 17.1 we see that $X$ is a scheme. Consider $X_i = f^{-1}(\mathrm{Spec}(A_i))$. Then $X_i$ has either one or zero points. If $X_i$ has zero points there is nothing to prove. If $X_i$ has one point, then $X_i = \mathrm{Spec}(B_i)$ with $B_i$ a zero dimensional local ring and $A_i \to B_i$ is an epimorphism of rings. In particular $A_i/\mathfrak{m}_{A_i} = B_i/\mathfrak{m}_{A_i}B_i$ and we see that $A_i \to B_i$ is surjective by Nakayama's lemma, Algebra, Lemma 19.1 (because $\mathfrak{m}_{A_i}$ is a nilpotent ideal!). Thus $B_i$ is a finite local $k$-algebra, and we conclude by Lemma 7.2 that $X \to \mathrm{Spec}(k)$ is locally quasi-finite. $\square$

The following lemma tells us that a quasi-separated algebraic space is a scheme away from codimension 1.

**Lemma 7.4.** *Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $x \in |X|$. If $X$ is quasi-separated and $x$ is a generic point of an irreducible component of $|X|$, then there exists an open subspace of $X$ containing $x$ which is a scheme.*

**Proof.** We can replace $X$ by an quasi-compact neighbourhood of $x$, hence we may assume $X$ is quasi-compact and quasi-separated. Choose a stratification

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and $f_p : V_p \to U_p$ and $T_p = U_p \setminus U_{p+1}$ as in Decent Spaces, Lemma 8.5. Then $x \in T_p$ for a unique $p$. Let $v \in f_p^{-1}(T_p)$ be the corresponding point. Note that $v$ is a generic point of an irreducible component of $V_p$ by Decent Spaces, Lemma 10.8. Since $U_{p+1}$ is quasi-compact and $f_p : V_p \to U_p$ is a quasi-compact morphism (Morphisms of Spaces, Lemma 8.9), we see that $f_p^{-1}(T_p) = V_p \setminus f_p^{-1}(U_{p+1})$ is a constructible closed subset of $V_p$. Hence an open neighbourhood $W$ of $v \in V_p$ is contained in $f_p^{-1}(T_p)$, see Properties, Lemma 2.2. Then $f_p(W) \subset X$ is an open neighbourhood of $x$ and $f_p|_W : W \to f_p(W)$ is an étale morphism which induces an isomorphism on the reductions (by our choice of the stratification). It follows that $W \to f_p(W)$ is an isomorphism (Morphisms of Spaces, Lemma 45.2). This concludes the proof. $\square$

The following lemma says that a separated locally Noetherian algebraic space is a scheme in codimension 1, i.e., away from codimension 2.

**Lemma 7.5.** *Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $x \in |X|$. If $X$ is separated, locally Noetherian, and the dimension of the local ring of $X$ at $x$ is $\le 1$ (Properties of Spaces, Definition 20.2), then there exists an open subspace of $X$ containing $x$ which is a scheme.*

**Proof.** (Please see the remark below for a different approach avoiding the material on finite groupoids.) We can replace $X$ by an quasi-compact neighbourhood of $x$, hence we may assume $X$ is quasi-compact, separated, and Noetherian. There exists a scheme $U$ and a finite surjective morphism $U \to X$, see Limits of Spaces, Proposition 16.2. Let $R = U \times_X U$. Then $j : R \to U \times_S U$ is an equivalence relation and we obtain a groupoid scheme $(U, R, s, t, c)$ over $S$ with $s, t$ finite and

$U$ Noetherian and separated. Let $\{u_1, \ldots, u_n\} \subset U$ be the set of points mapping to $x$. Then $\dim(\mathcal{O}_{U,u_i}) \leq 1$ by Decent Spaces, Lemma 10.9.

By More on Groupoids, Lemma 13.10 there exists an $R$-invariant affine open $W \subset U$ containing the orbit $\{u_1, \ldots, u_n\}$. Since $U \to X$ is finite surjective the continuous map $|U| \to |X|$ is closed surjective, hence submersive by Topology, Lemma 5.5. Thus $f(W)$ is open and there is an open subspace $X' \subset X$ with $f : W \to X'$ a surjective finite morphism. Then $X'$ is an affine scheme by Cohomology of Spaces, Lemma 16.1 and the proof is finished. $\qquad\square$

**Remark 7.6.** Here is a sketch of a proof of Lemma 7.5 which avoids using More on Groupoids, Lemma 13.10.

Step 1. We may assume $X$ is a reduced Noetherian separated algebraic space (for example by Cohomology of Spaces, Lemma 16.1 or by Limits of Spaces, Lemma 15.3) and we may choose a finite surjective morphism $Y \to X$ where $Y$ is a Noetherian scheme (by Limits of Spaces, Proposition 16.2).

Step 2. After replacing $X$ by an open neighbourhood of $x$, there exists a birational finite morphism $X' \to X$ and a closed subscheme $Y' \subset X' \times_X Y$ such that $Y' \to X'$ is surjective finite locally free. Namely, because $X$ is reduced there is a dense open subspace $U \subset X$ over which $Y$ is flat (Morphisms of Spaces, Proposition 30.1). Then we can choose a $U$-admissible blow up $b : \tilde{X} \to X$ such that the strict transform $\tilde{Y}$ of $Y$ is flat over $\tilde{X}$, see More on Morphisms of Spaces, Lemma 28.1. (An alternative is to use Hilbert schemes if one wants to avoid using the result on blow ups). Then we let $X' \subset \tilde{X}$ be the scheme theoretic closure of $b^{-1}(U)$ and $Y' = X' \times_{\tilde{X}} \tilde{Y}$. Since $x$ is a codimension 1 point, we see that $X' \to X$ is finite over a neighbourhood of $x$ (Lemma 4.4).

Step 3. After shrinking $X$ to a smaller neighbourhood of $x$ we get that $X'$ is a scheme. This holds because $Y'$ is a scheme and $Y' \to X'$ being finite locally free and because every finite set of codimension 1 points of $Y'$ is contained in an affine open. Use Properties of Spaces, Proposition 11.1 and Varieties, Proposition 22.7.

Step 4. There exists an affine open $W' \subset X'$ containing all points lying over $x$ which is the inverse image of an open subspace of $X$. To prove this let $Z \subset X$ be the closure of the set of points where $X' \to X$ is not an isomorphism. We may assume $x \in Z$ otherwise we are already done. Then $x$ is a generic point of an irreducible component of $Z$ and after shrinking $X$ we may assume $Z$ is an affine scheme (Lemma 7.4). Then the inverse image $Z' \subset X'$ is an affine scheme as well. Say $x_1, \ldots, x_n \in Z'$ are the points mapping to $x$. Then we can find an affine open $W'$ in $X'$ whose intersection with $Z'$ is the inverse image of a principal open of $Z$ containing $x$. Namely, we first pick an affine open $W' \subset X'$ containing $x_1, \ldots, x_n$ using Varieties, Proposition 22.7. Then we pick a principal open $D(f) \subset Z$ containing $x$ whose inverse image $D(f|_{Z'})$ is contained in $W' \cap Z'$. Then we pick $f' \in \Gamma(W', \mathcal{O}_{W'})$ restricting to $f|_{Z'}$ and we replace $W'$ by $D(f') \subset W'$. Since $X' \to X$ is an isomorphism away from $Z' \to Z$ the choice of $W'$ guarantees that the image $W \subset X$ of $W'$ is open with inverse image $W'$ in $X'$.

Step 5. Then $W' \to W$ is a finite surjective morphism and $W$ is a scheme by Cohomology of Spaces, Lemma 16.1 and the proof is complete.

## 8. Geometrically connected algebraic spaces

If $X$ is a connected algebraic space over a field, then it can happen that $X$ becomes disconnected after extending the ground field. This does not happen for geometrically connected schemes.

**Definition 8.1.** Let $X$ be an algebraic space over the field $k$. We say $X$ is *geometrically connected* over $k$ if the base change $X_{k'}$ is connected for every field extension $k'$ of $k$.

By convention a connected topological space is nonempty; hence a fortiori geometrically connected algebraic spaces are nonempty.

**Lemma 8.2.** *Let $X$ be an algebraic space over the field $k$. Let $k \subset k'$ be a field extension. Then $X$ is geometrically connected over $k$ if and only if $X_{k'}$ is geometrically connected over $k'$.*

**Proof.** If $X$ is geometrically connected over $k$, then it is clear that $X_{k'}$ is geometrically connected over $k'$. For the converse, note that for any field extension $k \subset k''$ there exists a common field extension $k' \subset k'''$ and $k'' \subset k'''$. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over $k'$ then $X$ is geometrically connected over $k$. $\square$

**Lemma 8.3.** *Let $k$ be a field. Let $X$, $Y$ be algebraic spaces over $k$. Assume $X$ is geometrically connected over $k$. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

*induces a bijection between connected components.*

**Proof.** Let $y \in |Y|$ be represented by a morphism $\mathrm{Spec}(K) \to Y$ be a morphism where $K$ is a field. The fibre of $|X \times_k Y| \to |Y|$ over $y$ is the image of $|Y_K| \to |X \times_k Y|$ by Properties of Spaces, Lemma 4.3. Thus these fibres are connected by our assumption that $Y$ is geometrically connected. By Morphisms of Spaces, Lemma 6.6 the map $|p|$ is open. Thus we may apply Topology, Lemma 6.5 to conclude. $\square$

**Lemma 8.4.** *Let $k \subset k'$ be an extension of fields. Let $X$ be an algebraic space over $k$. Assume $k$ separably algebraically closed. Then the morphism $X_{k'} \to X$ induces a bijection of connected components. In particular, $X$ is geometrically connected over $k$ if and only if $X$ is connected.*

**Proof.** Since $k$ is separably algebraically closed we see that $k'$ is geometrically connected over $k$, see Algebra, Lemma 46.4. Hence $Z = \mathrm{Spec}(k')$ is geometrically connected over $k$ by Varieties, Lemma 5.5. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 8.3. $\square$

**Lemma 8.5.** *Let $k$ be a field. Let $X$ be an algebraic space over $k$. Let $\bar{k}$ be a separable algebraic closure of $k$. Then $X$ is geometrically connected if and only if the base change $X_{\bar{k}}$ is connected.*

**Proof.** Assume $X_{\bar{k}}$ is connected. Let $k \subset k'$ be a field extension. There exists a field extension $\bar{k} \subset \bar{k}'$ such that $k'$ embeds into $\bar{k}'$ as an extension of $k$. By Lemma

8.4 we see that $X_{\overline{k}'}$ is connected. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. $\square$

Let $k$ be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. For example $\overline{k}$ could be the separable algebraic closure of $k$. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we get a corresponding automorphism $\mathrm{Spec}(\sigma) : \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$. Note that $\mathrm{Spec}(\sigma) \circ \mathrm{Spec}(\tau) = \mathrm{Spec}(\tau \circ \sigma)$. Hence we get an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$$

of the opposite group on the scheme $\mathrm{Spec}(\overline{k})$. Let $X$ be an algebraic space over $k$. Since $X_{\overline{k}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} X$ by definition we see that the action above induces a canonical action

(8.5.1) $$\mathrm{Gal}(\overline{k}/k)^{opp} \times X_{\overline{k}} \longrightarrow X_{\overline{k}}.$$

**Lemma 8.6.** *Let $k$ be a field. Let $X$ be an algebraic space over $k$. Let $\overline{k}$ be a (possibly infinite) Galois extension of $k$. Let $V \subset X_{\overline{k}}$ be a quasi-compact open. Then*

(1) *there exists a finite subextension $k \subset k' \subset \overline{k}$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\overline{k}}$,*
(2) *there exists an open subgroup $H \subset \mathrm{Gal}(\overline{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.*

**Proof.** Choose a scheme $U$ and a surjective étale morphism $U \to X$. Choose a quasi-compact open $W \subset U_{\overline{k}}$ whose image in $X_{\overline{k}}$ is $V$. This is possible because $|U_{\overline{k}}| \to |X_{\overline{k}}|$ is continuous and because $|U_{\overline{k}}|$ has a basis of quasi-compact opens. We can apply Varieties, Lemma 5.9 to $W \subset U_{\overline{k}}$ to obtain the lemma. $\square$

**Lemma 8.7.** *Let $k$ be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. Let $X$ be an algebraic space over $k$. Let $\overline{T} \subset |X_{\overline{k}}|$ have the following properties*

(1) *$\overline{T}$ is a closed subset of $|X_{\overline{k}}|$,*
(2) *for every $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we have $\sigma(\overline{T}) = \overline{T}$.*

*Then there exists a closed subset $T \subset |X|$ whose inverse image in $|X_{k'}|$ is $\overline{T}$.*

**Proof.** Let $T \subset |X|$ be the image of $\overline{T}$. Since $|X_{\overline{k}}| \to |X|$ is surjective, the statement means that $T$ is closed and that its inverse image is $\overline{T}$. Choose a scheme $U$ and a surjective étale morphism $U \to X$. By the case of schemes (see Varieties, Lemma 5.10) there exists a closed subset $T' \subset |U|$ whose inverse image in $|U_{\overline{k}}|$ is the inverse image of $\overline{T}$. Since $|U_{\overline{k}}| \to |X_{\overline{k}}|$ is surjective, we see that $T'$ is the inverse image of $T$ via $|U| \to |X|$. By our construction of the topology on $|X|$ this means that $T$ is closed. In the same manner one sees that $\overline{T}$ is the inverse image of $T$. $\square$

**Lemma 8.8.** *Let $k$ be a field. Let $X$ be an algebraic space over $k$. The following are equivalent*

(1) *$X$ is geometrically connected,*
(2) *for every finite separable field extension $k \subset k'$ the scheme $X_{k'}$ is connected.*

**Proof.** This proof is identical to the proof of Varieties, Lemma 5.11 except that we replace Varieties, Lemma 5.7 by Lemma 8.5, we replace Varieties, Lemma 5.9 by Lemma 8.6, and we replace Varieties, Lemma 5.10 by Lemma 8.7. We urge the reader to read that proof in stead of this one.

It follows immediately from the definition that (1) implies (2). Assume that $X$ is not geometrically connected. Let $k \subset \bar{k}$ be a separable algebraic closure of $k$. By Lemma 8.5 it follows that $X_{\bar{k}}$ is disconnected. Say $X_{\bar{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open, closed, and nonempty algebraic subspaces of $X_{\bar{k}}$.

Suppose that $W \subset X$ is any quasi-compact open subspace. Then $W_{\bar{k}} \cap \overline{U}$ and $W_{\bar{k}} \cap \overline{V}$ are open and closed subspaces of $W_{\bar{k}}$. In particular $W_{\bar{k}} \cap \overline{U}$ and $W_{\bar{k}} \cap \overline{V}$ are quasi-compact, and by Lemma 8.6 both $W_{\bar{k}} \cap \overline{U}$ and $W_{\bar{k}} \cap \overline{V}$ are defined over a finite subextension and invariant under an open subgroup of $\mathrm{Gal}(\bar{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open subspace such that both $W_{0,\bar{k}} \cap \overline{U}$ and $W_{0,\bar{k}} \cap \overline{V}$ are nonempty. Choose a finite subextension $k \subset k' \subset \bar{k}$ and a decomposition $W_{0,k'} = U_0' \amalg V_0'$ into open and closed subsets such that $W_{0,\bar{k}} \cap \overline{U} = (U_0')_{\bar{k}}$ and $W_{0,\bar{k}} \cap \overline{V} = (V_0')_{\bar{k}}$. Let $H = \mathrm{Gal}(\bar{k}/k') \subset \mathrm{Gal}(\bar{k}/k)$. In particular $\sigma(W_{0,\bar{k}} \cap \overline{U}) = W_{0,\bar{k}} \cap \overline{U}$ and similarly for $\overline{V}$.

Having chosen $W_0$, $k'$ as above, for every quasi-compact open subspace $W \subset X$ we set
$$U_W = \bigcap\nolimits_{\sigma \in H} \sigma(W_{\bar{k}} \cap \overline{U}), \quad V_W = \bigcup\nolimits_{\sigma \in H} \sigma(W_{\bar{k}} \cap \overline{V}).$$
Now, since $W_{\bar{k}} \cap \overline{U}$ and $W_{\bar{k}} \cap \overline{V}$ are fixed by an open subgroup of $\mathrm{Gal}(\bar{k}/k)$ we see that the union and intersection above are finite. Hence $U_W$ and $V_W$ are both open and closed subspaces. Also, by construction $W_{\bar{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open subspaces, then $W_{\bar{k}} \cap U_{W'} = U_W$ and $W_{\bar{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\bar{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that $V$ is nonempty as it is constructed by taking unions (locally). On the other hand, $U$ is nonempty since it contains $W_0 \cap \overline{U}$ by construction. Finally, $U, V \subset X_{\bar{k}}$ are closed and $H$-invariant by construction. Hence by Lemma 8.7 we have $U = (U')_{\bar{k}}$, and $V = (V')_{\bar{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired.                    $\square$

## 9. Spaces smooth over fields

**Lemma 9.1.** *Let $k$ be a field. Let $X$ be an algebraic space smooth over $k$. Then $X$ is a regular algebraic space.*

**Proof.** Choose a scheme $U$ and a surjective étale morphism $U \to X$. The morphism $U \to \mathrm{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 36.6 and 34.2). Hence $U$ is regular by Varieties, Lemma 15.3. By Properties of Spaces, Definition 7.2 this means that $X$ is regular.                    $\square$

**Lemma 9.2.** *Let $k$ be a field. Let $X$ be an algebraic space smooth over $\mathrm{Spec}(k)$. The set of $x \in |X|$ which are image of morphisms $\mathrm{Spec}(k') \to X$ with $k' \supset k$ finite separable is dense in $|X|$.*

**Proof.** Choose a scheme $U$ and a surjective étale morphism $U \to X$. The morphism $U \to \mathrm{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 36.6 and 34.2). Hence we can apply Varieties, Lemma 15.6 to see that the closed points of $U$ whose

residue fields are finite separable over $k$ are dense. This implies the lemma by our definition of the topology on $|X|$. $\square$

## 10. Other chapters

## References

[Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, vol. 203, Springer-Verlag, 1971.