

TOPOLOGIES ON ALGEBRAIC SPACES

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1. Introduction

In this chapter we introduce some topologies on the category of algebraic spaces. Compare with the material in [Gro71], [BLR90], [LMB00] and [Knu71]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

2. The general procedure

In this section we explain a general procedure for producing the sites we will be working with. This discussion will make little or no sense unless the reader has read Topologies, Section 2.

Let S be a base scheme. Take any category Sch_α constructed as in Sets, Lemma 9.2 starting with S and any set of schemes over S you want to be included. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained, and let $(Sch/S)_{fppf}$ denote the corresponding big fppf site of S . (The above is entirely as prescribed in Topologies, Section 7.)

Given choices as above the category of algebraic spaces over S has a set of isomorphism classes. One way to see this is to use the fact that any algebraic space over S is of the form U/R for some étale equivalence relation $j : R \rightarrow U \times_S U$ with $U, R \in \text{Ob}((Sch/S)_{fppf})$, see Spaces, Lemma 9.1. Hence we can find a full subcategory $Spaces/S$ of the category of algebraic spaces over S which has a set of objects such that each algebraic space is isomorphic to an object of $Spaces/S$. We fix a choice of such a category.

In the sections below, given a topology τ , the big site $(\text{Spaces}/S)_\tau$ (resp. the big site $(\text{Spaces}/X)_\tau$ of an algebraic space X over S) has as underlying category the category Spaces/S (resp. the subcategory Spaces/X of Spaces/S , see Categories, Example 2.13). The procedure for turning this into a site is as usual by defining a class of τ -coverings and using Sets, Lemma 11.1 to choose a sufficiently large set of coverings which defines the topology.

We point out that the *small étale site* $X_{\text{étale}}$ of an algebraic space X has already been defined in Properties of Spaces, Definition 15.1. Its objects are schemes étale over X , of which there are plenty by definition of an algebraic spaces. However, a more natural site, from the perspective of this chapter (compare Topologies, Definition 4.8) is the site $X_{\text{spaces, étale}}$ of Properties of Spaces, Definition 15.2. These two sites define the same topos, see Properties of Spaces, Lemma 15.3. We will not redefine these in this chapter; instead we will simply use them.

Finally, we intend not to define the Zariski sites, since these do not seem particularly useful (although the Zariski topology is occasionally useful).

3. Fpqc topology

We briefly discuss the notion of an fpqc covering of algebraic spaces. Please compare with Topologies, Section 8. We will show in Descent on Spaces, Proposition 4.1 that quasi-coherent sheaves descent along these.

Definition 3.1. Let S be a scheme, and let X be an algebraic space over S . An *fpqc covering* of X is a family of morphisms $\{f_i : X_i \rightarrow X\}_{i \in I}$ of algebraic spaces such that each f_i is flat and such that for every affine scheme Z and morphism $h : Z \rightarrow X$ there exists a standard fpqc covering $\{g_j : Z_j \rightarrow Z\}_{j=1, \dots, n}$ which refines the family $\{X_i \times_X Z \rightarrow Z\}_{i \in I}$.

In other words, there exists indices $i_1, \dots, i_n \in I$ and morphisms $h_j : U_j \rightarrow X_{i_j}$ such that $f_{i_j} \circ h_j = h \circ g_j$. Note that if X and all X_i are representable, this is the same as a fpqc covering of schemes by Topologies, Lemma 8.11.

Lemma 3.2. *Let S be a scheme. Let X be an algebraic space over S .*

- (1) *If $X' \rightarrow X$ is an isomorphism then $\{X' \rightarrow X\}$ is an fpqc covering of X .*
- (2) *If $\{X_i \rightarrow X\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$, then $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$ is an fpqc covering.*
- (3) *If $\{X_i \rightarrow X\}_{i \in I}$ is an fpqc covering and $X' \rightarrow X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$ is an fpqc covering.*

Proof. Part (1) is clear. Consider $g : X' \rightarrow X$ and $\{X_i \rightarrow X\}_{i \in I}$ an fpqc covering as in (3). By Morphisms of Spaces, Lemma 28.4 the morphisms $X' \times_X X_i \rightarrow X'$ are flat. If $h' : Z \rightarrow X'$ is a morphism from an affine scheme towards X' , then set $h = g \circ h' : Z \rightarrow X$. The assumption on $\{X_i \rightarrow X\}_{i \in I}$ means there exists a standard fpqc covering $\{Z_j \rightarrow Z\}_{j=1, \dots, n}$ and morphisms $Z_j \rightarrow X_{i(j)}$ covering h for certain $i(j) \in I$. By the universal property of the fibre product we obtain morphisms $Z_j \rightarrow X' \times_X X_{i(j)}$ over h' also. Hence $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$ is an fpqc covering. This proves (3).

Let $\{X_i \rightarrow X\}_{i \in I}$ and $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$ be as in (2). Let $h : Z \rightarrow X$ be a morphism from an affine scheme towards X . By assumption there exists a standard fpqc covering $\{Z_j \rightarrow Z\}_{j=1, \dots, n}$ and morphisms $h_j : Z_j \rightarrow X_{i(j)}$ covering h for

some indices $i(j) \in I$. By assumption there exist standard fpqc coverings $\{Z_{j,l} \rightarrow Z_j\}_{l=1,\dots,n(j)}$ and morphisms $Z_{j,l} \rightarrow X_{i(j)j(l)}$ covering h_j for some indices $j(l) \in J_{i(j)}$. By Topologies, Lemma 8.10 the family $\{Z_{j,l} \rightarrow Z\}$ is a standard fpqc covering. Hence we conclude that $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$ is an fpqc covering. \square

Lemma 3.3. *Let S be a scheme, and let X be an algebraic space over S . Suppose that $\{f_i : X_i \rightarrow X\}_{i \in I}$ is a family of morphisms of algebraic spaces with target X . Let $U \rightarrow X$ be a surjective étale morphism from a scheme towards X . Then $\{f_i : X_i \rightarrow X\}_{i \in I}$ is an fpqc covering of X if and only if $\{U \times_X X_i \rightarrow U\}_{i \in I}$ is an fpqc covering of U .*

Proof. If $\{X_i \rightarrow X\}_{i \in I}$ is an fpqc covering, then so is $\{U \times_X X_i \rightarrow U\}_{i \in I}$ by Lemma 3.2. Assume that $\{U \times_X X_i \rightarrow U\}_{i \in I}$ is an fpqc covering. Let $h : Z \rightarrow X$ be a morphism from an affine scheme towards X . Then we see that $U \times_X Z \rightarrow Z$ is a surjective étale morphism of schemes, in particular open. Hence we can find finitely many affine opens W_1, \dots, W_t of $U \times_X Z$ whose images cover Z . For each j we may apply the condition that $\{U \times_X X_i \rightarrow U\}_{i \in I}$ is an fpqc covering to the morphism $W_j \rightarrow U$, and obtain a standard fpqc covering $\{W_{jl} \rightarrow W_j\}$ which refines $\{W_j \times_X X_i \rightarrow W_j\}_{i \in I}$. Hence $\{W_{jl} \rightarrow Z\}$ is a standard fpqc covering of Z (see Topologies, Lemma 8.10) which refines $\{Z \times_X X_i \rightarrow Z\}$ and we win. \square

Lemma 3.4. *Let S be a scheme, and let X be an algebraic space over S . Suppose that $\mathcal{U} = \{f_i : X_i \rightarrow X\}_{i \in I}$ is an fpqc covering of X . Then there exists a refinement $\mathcal{V} = \{g_i : T_i \rightarrow X\}$ of \mathcal{U} which is an fpqc covering such that each T_i is a scheme.*

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \rightarrow X_i$. Then check that $\{T_i \rightarrow X\}$ is an fpqc covering. \square

To be continued...

4. Fppf topology

In this section we discuss the notion of an fppf covering of algebraic spaces, and we define the big fppf site of an algebraic space. Please compare with Topologies, Section 7.

Definition 4.1. Let S be a scheme, and let X be an algebraic space over S . An *fppf covering* of X is a family of morphisms $\{f_i : X_i \rightarrow X\}_{i \in I}$ of algebraic spaces over S such that each f_i is flat and locally of finite presentation and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 7.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of an fppf covering of schemes.

Lemma 4.2. *Let S be a scheme. Let X be an algebraic space over S .*

- (1) *If $X' \rightarrow X$ is an isomorphism then $\{X' \rightarrow X\}$ is an fppf covering of X .*
- (2) *If $\{X_i \rightarrow X\}_{i \in I}$ is an fppf covering and for each i we have an fppf covering $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$, then $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$ is an fppf covering.*
- (3) *If $\{X_i \rightarrow X\}_{i \in I}$ is an fppf covering and $X' \rightarrow X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$ is an fppf covering.*

Proof. Omitted. \square

Lemma 4.3. *Let S be a scheme, and let X be an algebraic space over S . Suppose that $\mathcal{U} = \{f_i : X_i \rightarrow X\}_{i \in I}$ is an fppf covering of X . Then there exists a refinement $\mathcal{V} = \{g_i : T_i \rightarrow X\}$ of \mathcal{U} which is an fppf covering such that each T_i is a scheme.*

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \rightarrow X_i$. Then check that $\{T_i \rightarrow X\}$ is an fppf covering. \square

Lemma 4.4. *Let S be a scheme. Let $\{f_i : X_i \rightarrow X\}_{i \in I}$ be an fppf covering of algebraic spaces over S . Then the map of sheaves*

$$\coprod X_i \longrightarrow X$$

is surjective.

Proof. This follows from Spaces, Lemma 5.9. See also Spaces, Remark 5.2 in case you are confused about the meaning of this lemma. \square

To be continued...

5. Syntomic topology

In this section we discuss the notion of a syntomic covering of algebraic spaces, and we define the big syntomic site of an algebraic space. Please compare with Topologies, Section 6.

Definition 5.1. Let S be a scheme, and let X be an algebraic space over S . A *syntomic covering* of X is a family of morphisms $\{f_i : X_i \rightarrow X\}_{i \in I}$ of algebraic spaces over S such that each f_i is syntomic and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 6.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a syntomic covering of schemes.

Lemma 5.2. *Let S be a scheme. Let X be an algebraic space over S .*

- (1) *If $X' \rightarrow X$ is an isomorphism then $\{X' \rightarrow X\}$ is a syntomic covering of X .*
- (2) *If $\{X_i \rightarrow X\}_{i \in I}$ is a syntomic covering and for each i we have a syntomic covering $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$, then $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$ is a syntomic covering.*
- (3) *If $\{X_i \rightarrow X\}_{i \in I}$ is a syntomic covering and $X' \rightarrow X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$ is a syntomic covering.*

Proof. Omitted. \square

To be continued...

6. Smooth topology

In this section we discuss the notion of a smooth covering of algebraic spaces, and we define the big smooth site of an algebraic space. Please compare with Topologies, Section 5.

Definition 6.1. Let S be a scheme, and let X be an algebraic space over S . A *smooth covering* of X is a family of morphisms $\{f_i : X_i \rightarrow X\}_{i \in I}$ of algebraic spaces over S such that each f_i is smooth and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 5.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a smooth covering of schemes.

Lemma 6.2. *Let S be a scheme. Let X be an algebraic space over S .*

- (1) *If $X' \rightarrow X$ is an isomorphism then $\{X' \rightarrow X\}$ is a smooth covering of X .*
- (2) *If $\{X_i \rightarrow X\}_{i \in I}$ is a smooth covering and for each i we have a smooth covering $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$, then $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$ is a smooth covering.*
- (3) *If $\{X_i \rightarrow X\}_{i \in I}$ is a smooth covering and $X' \rightarrow X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$ is a smooth covering.*

Proof. Omitted. □

To be continued...

7. Étale topology

In this section we discuss the notion of a étale covering of algebraic spaces, and we define the big étale site of an algebraic space. Please compare with Topologies, Section 4.

Definition 7.1. Let S be a scheme, and let X be an algebraic space over S . A *étale covering* of X is a family of morphisms $\{f_i : X_i \rightarrow X\}_{i \in I}$ of algebraic spaces over S such that each f_i is étale and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 4.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a étale covering of schemes.

Lemma 7.2. *Let S be a scheme. Let X be an algebraic space over S .*

- (1) *If $X' \rightarrow X$ is an isomorphism then $\{X' \rightarrow X\}$ is a étale covering of X .*
- (2) *If $\{X_i \rightarrow X\}_{i \in I}$ is a étale covering and for each i we have a étale covering $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$, then $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$ is a étale covering.*
- (3) *If $\{X_i \rightarrow X\}_{i \in I}$ is a étale covering and $X' \rightarrow X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$ is a étale covering.*

Proof. Omitted. □

To be continued...

8. Zariski topology

In Spaces, Section 12 we introduced the notion of a Zariski covering of an algebraic space by open subspaces. Here is the corresponding notion with open subspaces replaces by open immersions.

Definition 8.1. Let S be a scheme, and let X be an algebraic space over S . A *Zariski covering* of X is a family of morphisms $\{f_i : X_i \rightarrow X\}_{i \in I}$ of algebraic spaces over S such that each f_i is an open immersion and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

Although Zariski coverings are occasionally useful the corresponding topology on the category of algebraic spaces is really too coarse, and not particularly useful. Still, it does define a site.

Lemma 8.2. *Let S be a scheme. Let X be an algebraic space over S .*

- (1) *If $X' \rightarrow X$ is an isomorphism then $\{X' \rightarrow X\}$ is a Zariski covering of X .*
- (2) *If $\{X_i \rightarrow X\}_{i \in I}$ is a Zariski covering and for each i we have a Zariski covering $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$, then $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$ is a Zariski covering.*
- (3) *If $\{X_i \rightarrow X\}_{i \in I}$ is a Zariski covering and $X' \rightarrow X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$ is a Zariski covering.*

Proof. Omitted. □

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References

- [BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 21, Springer-Verlag, 1990.
- [Gro71] Alexander Grothendieck, *Revêtements étales et groupe fondamental (sga 1)*, Lecture notes in mathematics, vol. 224, Springer-Verlag, 1971.
- [Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, vol. 203, Springer-Verlag, 1971.
- [LMB00] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 39, Springer-Verlag, 2000.