COHOMOLOGY OF SCHEMES

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1. Introduction

In this chapter we first prove a number of results on the cohomology of quasi-coherent sheaves. A fundamental reference is [DG67]. Having done this we will elaborate on cohomology of coherent sheaves in the Noetherian setting. See [Ser55].

2. Cech cohomology of quasi-coherent sheaves

Let X be a scheme. Let $U \subset X$ be an affine open. Recall that a *standard open covering* of U is a covering of the form $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ where $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O}_X)$ generate the unit ideal, see Schemes, Definition 5.2.

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Lemma 2.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ be a standard open covering of an affine open of X. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. Write $U = \text{Spec}(A)$ for some ring A. In other words, $f_1, \ldots, f_n$ are elements of A which generate the unit ideal of A. Write $\mathcal{F}|_U = \widetilde{M}$ for some A-module M. Clearly the Cech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is identified with the complex

$$\prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod\nolimits_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \ldots$$

We are asked to show that the extended complex

$$0 \to M \to \prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod\nolimits_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \ldots \tag{2.1.1}$$

(whose truncation we have studied in Algebra, Lemma 22.2) is exact. It suffices to show that (2.1.1) is exact after localizing at a prime $\mathfrak{p}$, see Algebra, Lemma 23.1. In fact we will show that the extended complex localized at $\mathfrak{p}$ is homotopic to zero.

There exists an index i such that $f_i \notin \mathfrak{p}$. Choose and fix such an element i_{fix}. Note that $M_{f_{i_{\text{fix}}}, \mathfrak{p}} = M_{\mathfrak{p}}$. Similarly for a localization at a product $f_{i_0} \ldots f_{i_p}$ and $\mathfrak{p}$ we can drop any f_{i_j} for which $i_j = i_{\text{fix}}$. Let us define a homotopy

$$h : \prod\nolimits_{i_0 \ldots i_{p+1}} M_{f_{i_0} \ldots f_{i_{p+1}}, \mathfrak{p}} \longrightarrow \prod\nolimits_{i_0 \ldots i_p} M_{f_{i_0} \ldots f_{i_p}, \mathfrak{p}}$$

by the rule

$$h(s)_{i_0 \ldots i_p} = s_{i_{\text{fix}} i_0 \ldots i_p}$$

(This is "dual" to the homotopy in the proof of Cohomology, Lemma 11.4.) In other words, $h : \prod_{i_0} M_{f_{i_0}, \mathfrak{p}} \to M$ is projection onto the factor $M_{f_{i_{\text{fix}}}, \mathfrak{p}} = M_{\mathfrak{p}}$ and in general the map h equal projection onto the factors $M_{f_{i_{\text{fix}}} f_{i_1} \ldots f_{i_{p+1}}, \mathfrak{p}} = M_{f_{i_1} \ldots f_{i_{p+1}}, \mathfrak{p}}$. We compute

$$\begin{aligned}
(dh + hd)(s)_{i_0 \ldots i_p} &= \sum\nolimits_{j=0}^p (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i_{\text{fix}} i_0 \ldots i_p} \\
&= \sum\nolimits_{j=0}^p (-1)^j s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum\nolimits_{j=0}^p (-1)^{j+1} s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} \\
&= s_{i_0 \ldots i_p}
\end{aligned}$$

This proves the identity map is homotopic to zero as desired. □

The following lemma says in particular that for any affine scheme X and any quasi-coherent sheaf $\mathcal{F}$ on X we have

$$H^p(X, \mathcal{F}) = 0$$

for all $p > 0$.

Lemma 2.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. We are going to apply Cohomology, Lemma 12.8. As our basis $\mathcal{B}$ for the topology of X we are going to use the affine opens of X. As our set Cov of open coverings we are going to use the standard open coverings of affine opens of X. Next we check that conditions (1), (2) and (3) of Cohomology, Lemma 12.8 hold. Note that the intersection of standard opens in an affine is another standard open. Hence property (1) holds. The coverings form a cofinal system of open coverings

of any element of $\mathcal{B}$, see Schemes, Lemma 5.1. Hence (2) holds. Finally, condition (3) of the lemma follows from Lemma 2.1. $\square$

Here is a relative version of the vanishing of cohomology of quasi-coherent sheaves on affines.

Lemma 2.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If f is affine then $R^if_*\mathcal{F} = 0$ for all $i > 0$.*

Proof. According to Cohomology, Lemma 7.3 the sheaf $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. By assumption, whenever V is affine we have that $f^{-1}(V)$ is affine, see Morphisms, Definition 13.1. By Lemma 2.2 we conclude that $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) = 0$ whenever V is affine. Since S has a basis consisting of affine opens we win. $\square$

Lemma 2.4. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F}) = H^i(S, f_*\mathcal{F})$ for all $i \geq 0$.*

Proof. Follows from Lemma 2.3 and the Leray spectral sequence. See Cohomology, Lemma 14.6. $\square$

Lemma 2.5. *Let X be a scheme. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering such that $U_{i_0 \ldots i_p}$ is affine open for all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. In this case for any quasi-coherent sheaf $\mathcal{F}$ we have*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$$

as $\Gamma(X, \mathcal{O}_X)$-modules for all p.

Proof. In view of Lemma 2.2 this is a special case of Cohomology, Lemma 12.5. $\square$

3. Vanishing of cohomology

We have seen that on an affine scheme the higher cohomology groups of any quasi-coherent sheaf vanish (Lemma 2.2). It turns out that this also characterizes affine schemes. We give two versions.

Lemma 3.1. *Let X be a scheme. Assume that*

1. *X is quasi-compact,*
2. *for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we have $H^1(X, \mathcal{I}) = 0$.*

Then X is affine.

Proof. Let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighbourhood of x. Write $U = \text{Spec}(A)$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x. Set $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. By Schemes, Lemma 12.4 there are quasi-coherent sheaves of ideals $\mathcal{I}$, resp. $\mathcal{I}'$ cutting out the reduced closed subschemes Z, resp. Z'. Consider the short exact sequence

$$0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}/\mathcal{I}' \to 0.$$

Since x is a closed point of X and $x \notin Z$ we see that $\mathcal{I}/\mathcal{I}'$ is supported at x. In fact, the restriction of $\mathcal{I}/\mathcal{I}'$ to U corresponds to the A-module $A/\mathfrak{m}$. Hence we see that $\Gamma(X, \mathcal{I}/\mathcal{I}') = A/\mathfrak{m}$. Since by assumption $H^1(X, \mathcal{I}') = 0$ we see there exists a global section $f \in \Gamma(X, \mathcal{I})$ which maps to the element $1 \in A/\mathfrak{m}$ as a section of $\mathcal{I}/\mathcal{I}'$. Clearly we have $x \in X_f \subset U$. This implies that $X_f = D(f_A)$ where f_A is the image of f in $A = \Gamma(U, \mathcal{O}_X)$. In particular X_f is affine.

Consider the union $W = \bigcup X_f$ over all $f \in \Gamma(X, \mathcal{O}_X)$ such that X_f is affine. Obviously W is open in X. By the arguments above every closed point of X is contained in W. The closed subset $X \setminus W$ of X is also quasi-compact (see Topology, Lemma 11.3). Hence it has a closed point if it is nonempty (see Topology, Lemma 11.8). This would contradict the fact that all closed points are in W. Hence we conclude $X = W$.

Choose finitely many $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that $X = X_{f_1} \cup \ldots \cup X_{f_n}$ and such that each X_{f_i} is affine. This is possible as we've seen above. By Properties, Lemma 25.2 to finish the proof it suffices to show that $f_1, \ldots, f_n$ generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus n} \xrightarrow{f_1, \ldots, f_n} \mathcal{O}_X \longrightarrow 0$$

The arrow defined by $f_1, \ldots, f_n$ is surjective since the opens X_{f_i} cover X. We let $\mathcal{F}$ be the kernel of this surjective map. Observe that $\mathcal{F}$ has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

so that each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to a quasi-coherent sheaf of ideals. Namely we can take $\mathcal{F}_i$ to be the intersection of $\mathcal{F}$ with the first i direct summands of $\mathcal{O}_X^{\oplus n}$. The assumption of the lemma implies that $H^1(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ for all i. This implies that $H^1(X, \mathcal{F}_2) = 0$ because it is sandwiched between $H^1(X, \mathcal{F}_1)$ and $H^1(X, \mathcal{F}_2/\mathcal{F}_1)$. Continuing like this we deduce that $H^1(X, \mathcal{F}) = 0$. Therefore we conclude that the map

$$\bigoplus_{i=1,\ldots,n} \Gamma(X, \mathcal{O}_X) \xrightarrow{f_1, \ldots, f_n} \Gamma(X, \mathcal{O}_X)$$

is surjective as desired. □

Note that if X is a Noetherian scheme then every quasi-coherent sheaf of ideals is automatically a coherent sheaf of ideals and a finite type quasi-coherent sheaf of ideals. Hence the preceding lemma and the next lemma both apply in this case.

Lemma 3.2. *Let X be a scheme. Assume that*

1. *X is quasi-compact,*
2. *X is quasi-separated, and*
3. *$H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals $\mathcal{I}$ of finite type.*

Then X is affine.

Proof. By Properties, Lemma 20.3 every quasi-coherent sheaf of ideals is a directed colimit of quasi-coherent sheaves of ideals of finite type. By Cohomology, Lemma 20.1 taking cohomology on X commutes with directed colimits. Hence we see that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals on X. In other words we see that Lemma 3.1 applies. □

4. Quasi-coherence of higher direct images

We have seen that the higher cohomology groups of a quasi-coherent module on an affine is zero. For (quasi-)separated quasi-compact schemes X this implies vanishing of cohomology groups of quasi-coherent sheaves beyond a certain degree. However, it may not be the case that X has finite cohomological dimension, because that is defined in terms of vanishing of cohomology of *all* $\mathcal{O}_X$-modules.

Lemma 4.1. *Let X be a quasi-compact and quasi-separated scheme. Let P be a property of the quasi-compact opens of X. Assume that*

1. *P holds for every affine open of X,*
2. *if U is quasi-compact open, V affine open, P holds for U, V, and $U \cap V$, then P holds for $U \cup V$.*

Then P holds for every quasi-compact open of X and in particular for X.

Proof. First we argue by induction that P holds for *separated* quasi-compact opens $W \subset X$. Namely, such an open can be written as $W = U_1 \cup \ldots \cup U_n$ and we can do induction on n using property (2) with $U = U_1 \cup \ldots \cup U_{n-1}$ and $V = U_n$. This is allowed because $U \cap V = (U_1 \cap U_n) \cup \ldots \cup (U_{n-1} \cap U_n)$ is also a union of $n-1$ affine open subschemes by Schemes, Lemma 21.8 applied to the affine opens U_i and U_n of W. Having said this, for any quasi-compact open $W \subset X$ we can do induction on the number of affine opens needed to cover W using the same trick as before and using that the quasi-compact open $U_i \cap U_n$ is separated as an open subscheme of the affine scheme U_n. □

Lemma 4.2. *Let X be a quasi-compact separated scheme. Let $t = t(X)$ be the minimal number of affine opens needed to cover X. Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq t$ and all quasi-coherent sheaves $\mathcal{F}$.*

Proof. First proof. By induction on t. If $t = 1$ the result follows from Lemma 2.2. If $t > 1$ write $X = U \cup V$ with V affine open and $U = U_1 \cup \ldots \cup U_{t-1}$ a union of $t-1$ open affines. Note that in this case $U \cap V = (U_1 \cap V) \cup \ldots (U_{t-1} \cap V)$ is also a union of $t-1$ affine open subschemes, see Schemes, Lemma 21.8. We apply the Mayer-Vietoris long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

see Cohomology, Lemma 9.2. By induction we see that the groups $H^i(U, \mathcal{F})$, $H^i(V, \mathcal{F})$, $H^i(U \cap V, \mathcal{F})$ are zero for $i \geq t-1$. It follows immediately that $H^i(X, \mathcal{F})$ is zero for $i \geq t$.

Second proof. Let $\mathcal{U} : X = \bigcup_{i=1}^t U_i$ be a finite affine open covering. Since X is separated the multiple intersections $U_{i_0 \ldots i_p}$ are all affine, see Schemes, Lemma 21.8. By Lemma 2.5 the Cech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ agree with the cohomology groups. By Cohomology, Lemma 24.6 the Cech cohomology groups may be computed using the alternating Cech complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$. As the covering consists of t elements we see immediately that $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = 0$ for all $p \geq t$. Hence the result follows. □

Lemma 4.3. *Let X be a quasi-compact quasi-separated scheme. Let $X = U_1 \cup \ldots \cup U_t$ be an affine open covering. Set*

$$d = \max_{I \subset \{1, \ldots, t\}} |I| + t(\bigcap\nolimits_{i \in I} U_i)$$

where $t(U)$ is the minimal number of affines needed to cover the scheme U. Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq d$ and all quasi-coherent sheaves $\mathcal{F}$.

Proof. Note that since X is quasi-separated the numbers $t(\bigcap_{i \in I} U_i)$ are finite. Let $\mathcal{U} : X = \bigcup_{i=1}^t U_i$. By Cohomology, Lemma 12.4 there is a spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. By Cohomology, Lemma 24.6 we have

$$E_2^{p,q} = H^p(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

The alternating Čech complex with values in the presheaf $\underline{H}^q(\mathcal{F})$ vanishes in high degrees by Lemma 4.2, more precisely $E_2^{p,q} = 0$ for $p + q \geq d$. Hence the result follows. □

Lemma 4.4. *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.*

1. *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the higher direct images $R^pf_*\mathcal{F}$ are quasi-coherent on S.*
2. *If S is quasi-compact, there exists an integer $n = n(X, S, f)$ such that $R^pf_*\mathcal{F} = 0$ for all $p \geq n$ and any quasi-coherent sheaf $\mathcal{F}$ on X.*
3. *In fact, if S is quasi-compact we can find $n = n(X, S, f)$ such that for every morphism of schemes $S' \to S$ we have $R^p(f')_*\mathcal{F}' = 0$ for $p \geq n$ and any quasi-coherent sheaf $\mathcal{F}'$ on X'. Here $f' : X' = S' \times_S X \to S'$ is the base change of f.*

Proof. We first prove (1). Note that under the hypotheses of the lemma the sheaf $R^0f_*\mathcal{F} = f_*\mathcal{F}$ is quasi-coherent by Schemes, Lemma 24.1. Using Cohomology, Lemma 7.4 we see that forming higher direct images commutes with restriction to open subschemes. Since being quasi-coherent is local on S we may assume S is affine.

Assume S is affine and f quasi-compact and separated. Let $t \geq 1$ be the minimal number of affine opens needed to cover X. We will prove this case of (1) by induction on t. If $t = 1$ then the morphism f is affine by Morphisms, Lemma 13.12 and (1) follows from Lemma 2.3. If $t > 1$ write $X = U \cup V$ with V affine open and $U = U_1 \cup \ldots \cup U_{t-1}$ a union of $t - 1$ open affines. Note that in this case $U \cap V = (U_1 \cap V) \cup \ldots (U_{t-1} \cap V)$ is also a union of $t - 1$ affine open subschemes, see Schemes, Lemma 21.8. We will apply the relative Mayer-Vietoris sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U\cap V}) \to R^1f_*\mathcal{F} \to \ldots$$

see Cohomology, Lemma 9.3. By induction we see that $R^pa_*\mathcal{F}$, $R^pb_*\mathcal{F}$ and $R^pc_*\mathcal{F}$ are all quasi-coherent. This implies that each of the sheaves $R^pf_*\mathcal{F}$ is quasi-coherent since it sits in the middle of a short exact sequence with a cokernel of a map between quasi-coherent sheaves on the left and a kernel of a map between quasi-coherent sheaves on the right. Using the results on quasi-coherent sheaves in Schemes, Section 24 we see conclude $R^pf_*\mathcal{F}$ is quasi-coherent.

Assume S is affine and f quasi-compact and quasi-separated. Let $t \geq 1$ be the minimal number of affine opens needed to cover X. We will prove (1) by induction on t. In case $t = 1$ the morphism f is separated and we are back in the previous case (see previous paragraph). If $t > 1$ write $X = U \cup V$ with V affine open and U a union of $t - 1$ open affines. Note that in this case $U \cap V$ is an open subscheme of an affine scheme and hence separated (see Schemes, Lemma 21.6). We will apply the relative Mayer-Vietoris sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U\cap V}) \to R^1f_*\mathcal{F} \to \ldots$$

see Cohomology, Lemma 9.3. By induction and the result of the previous paragraph we see that $R^pa_*\mathcal{F}$, $R^pb_*\mathcal{F}$ and $R^pc_*\mathcal{F}$ are quasi-coherent. As in the previous paragraph this implies each of sheaves $R^pf_*\mathcal{F}$ is quasi-coherent.

Next, we prove (3) and a fortiori (2). Choose a finite affine open covering $S = \bigcup_{j=1,\ldots m} S_j$. For each i choose a finite affine open covering $f^{-1}(S_j) = \bigcup_{i=1,\ldots t_j} U_{ji}$. Let

$$d_j = \max_{I \subset \{1,\ldots,t_j\}} |I| + t(\bigcap\nolimits_{i \in I} U_{ji})$$

be the integer found in Lemma 4.3. We claim that $n(X, S, f) = \max d_j$ works.

Namely, let $S' \to S$ be a morphism of schemes and let $\mathcal{F}'$ be a quasi-coherent sheaf on $X' = S' \times_S X$. We want to show that $R^pf'_*\mathcal{F}' = 0$ for $p \geq n(X, S, f)$. Since this question is local on S' we may assume that S' is affine and maps into S_j for some j. Then $X' = S' \times_{S_j} f^{-1}(S_j)$ is covered by the open affines $S' \times_{S_j} U_{ji}$, $i = 1, \ldots t_j$ and the intersections

$$\bigcap\nolimits_{i \in I} S' \times_{S_j} U_{ji} = S' \times_{S_j} \bigcap\nolimits_{i \in I} U_{ji}$$

are covered by the same number of affines as before the base change. Applying Lemma 4.3 we get $H^p(X', \mathcal{F}') = 0$. By the first part of the proof we already know that each $R^qf'_*\mathcal{F}'$ is quasi-coherent hence has vanishing higher cohomology groups on our affine scheme S', thus we see that $H^0(S', R^pf'_*\mathcal{F}') = H^p(X', \mathcal{F}') = 0$ by Cohomology, Lemma 14.6. Since $R^pf'_*\mathcal{F}'$ is quasi-coherent we conclude that $R^pf'_*\mathcal{F}' = 0$. □

Lemma 4.5. *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact. Assume S is affine. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have*

$$H^q(X, \mathcal{F}) = H^0(S, R^qf_*\mathcal{F})$$

for all $q \in \mathbf{Z}$.

Proof. Consider the Leray spectral sequence $E_2^{p,q} = H^p(S, R^qf_*\mathcal{F})$ converging to $H^{p+q}(X, \mathcal{F})$, see Cohomology, Lemma 14.4. By Lemma 4.4 we see that the sheaves $R^qf_*\mathcal{F}$ are quasi-coherent. By Lemma 2.2 we see that $E_2^{p,q} = 0$ when $p > 0$. Hence the spectral sequence degenerates at E_2 and we win. See also Cohomology, Lemma 14.6 (2) for the general principle. □

5. Cohomology and base change, I

Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Suppose further that $g : S' \to S$ is any morphism of schemes. Denote $X' = X_{S'} = S' \times_S X$ the base change of X and denote $f' : X' \to S'$ the base change of f. Also write $g' : X' \to X$ the projection, and set $\mathcal{F}' = (g')^*\mathcal{F}$. Here is a diagram representing the situation:

$$\begin{array}{ccccc} \mathcal{F}' = (g')^*\mathcal{F} & & X' & \xrightarrow{g'} & X & & \mathcal{F} \\ & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \\ Rf'_*\mathcal{F}' & & S' & \xrightarrow{g} & S & & Rf_*\mathcal{F} \end{array} \tag{5.0.1}$$

Here is the simplest case of the base change property we have in mind.

Lemma 5.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is affine. In this case $f_*\mathcal{F} \cong Rf_*\mathcal{F}$ is a quasi-coherent sheaf, and for every base change diagram (5.0.1) we have*

$$g^*f_*\mathcal{F} = f'_*(g')^*\mathcal{F}.$$

Proof. The vanishing of higher direct images is Lemma 2.3. The statement is local on S and S'. Hence we may assume $X = \text{Spec}(A)$, $S = \text{Spec}(R)$, $S' = \text{Spec}(R')$ and $\mathcal{F} = \widetilde{M}$ for some A-module M. We use Schemes, Lemma 7.3 to describe pullbacks and pushforwards of $\mathcal{F}$. Namely, $X' = \text{Spec}(R' \otimes_R A)$ and $\mathcal{F}'$ is the quasi-coherent sheaf associated to $(R' \otimes_R A) \otimes_A M$. Thus we see that the lemma boils down to the equality

$$(R' \otimes_R A) \otimes_A M = R' \otimes_R M$$

as R'-modules. □

In many situations it is sufficient to know about the following special case of cohomology and base change. It follows immediately from the stronger results in Section 7, but since it is so important it deserves its own proof.

Lemma 5.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g : S' \to S$ be a morphism of schemes. Assume that g is flat and that f is quasi-compact and quasi-separated. Then for any $i \geq 0$ we have*

$$R^if'_*\mathcal{F}' = g^*R^if_*\mathcal{F}$$

with notation as in (5.0.1). Moreover, the induced isomorphism is the map given by the base change map of Cohomology, Lemma 18.1.

Proof. The statement is local on S' and hence we may assume S and S' are affine. Say $S = \text{Spec}(A)$ and $S' = \text{Spec}(B)$. In this case we are really trying to show that the map

$$H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i(X_B, \mathcal{F}_B)$$

(given by the reference in the statement of the lemma) is an isomorphism where $X_B = \text{Spec}(B) \times_{\text{Spec}(A)} X$ and $\mathcal{F}_B$ is the pullback of $\mathcal{F}$ to X_B.

In case X is separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \ldots \cup U_t$ and recall that

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}),$$

see Lemma 2.5. If $\mathcal{U}_B : X_B = (U_1)_B \cup \ldots \cup (U_t)_B$ we obtain by base change, then it is still the case that each $(U_i)_B$ is affine and that X_B is separated. Thus we obtain

$$\check{H}^p(\mathcal{U}_B, \mathcal{F}_B) = H^p(X_B, \mathcal{F}_B).$$

We have the following relation between the Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}_B, \mathcal{F}_B) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B$$

as follows from Lemma 5.1. Since $A \to B$ is flat, the same thing remains true on taking cohomology.

In case X is quasi-separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \ldots \cup U_t$. We will use the Čech-to-cohomology spectral sequence Cohomology, Lemma 12.4. The reader who wishes to avoid this spectral sequence can use Majer-Vietoris and induction on t as in the proof of Lemma 4.4. The spectral sequence has E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ and converges to $H^{p+q}(X, \mathcal{F})$. Similarly, we have a spectral sequence with E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B))$ which converges to $H^{p+q}(X_B, \mathcal{F}_B)$.

Since the intersections $U_{i_0 \ldots i_p}$ are quasi-compact and separated, the result of the second paragraph of the proof gives $\check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B)) = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \otimes_A B$. Using that $A \to B$ is flat we conclude that $H^i(X, \mathcal{F}) \otimes_A B \to H^i(X_B, \mathcal{F}_B)$ is an isomorphism for all i and we win. $\square$

6. Colimits and higher direct images

General results of this nature can be found in Cohomology, Section 20, Sheaves, Lemma 29.1, and Modules, Lemma 11.6.

Lemma 6.1. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ be a filtered colimit of quasi-coherent sheaves on X. Then for any $p \geq 0$ we have*

$$R^p f_* \mathcal{F} = \operatorname{colim} R^p f_* \mathcal{F}_i.$$

Proof. Recall that $R^p f_* \mathcal{F}$ is the sheaf associated to $U \mapsto H^p(f^{-1}U, \mathcal{F})$, see Cohomology, Lemma 7.3. Recall that the colimit is the sheaf associated to the presheaf colimit (taking colimits over opens). Hence we can apply Cohomology, Lemma 20.1 to $H^p(f^{-1}U, -)$ where U is affine to conclude. (Because the basis of affine opens in $f^{-1}U$ satisfies the assumptions of that lemma.) $\square$

7. Cohomology and base change, II

Let $f : X \to S$ be a morphism of schemes and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If f is quasi-compact and quasi-separated we would like to represent $Rf_*\mathcal{F}$ by a complex of quasi-coherent sheaves on S. This follows from the fact that the sheaves $R^i f_* \mathcal{F}$ are quasi-coherent if S is quasi-compact and has affine diagonal, using that $D_{QCoh}(S)$ is equivalent to $D(QCoh(\mathcal{O}_S))$, see Derived Categories of Schemes, Proposition 6.5.

In this section we will use a different approach which produces an explicit complex having a good base change property. The construction is particularly easy if f and S are separated, or more generally have affine diagonal. Since this is the case which by far the most often used we treat it separately.

Lemma 7.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume X is quasi-compact and X and S have affine diagonal (e.g., if X and S are separated). In this case we can compute $Rf_*\mathcal{F}$ as follows:*

1. *Choose a finite affine open covering $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$.*
2. *For $i_0, \ldots, i_p \in \{1, \ldots, n\}$ denote $f_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to S$ the restriction of f to the intersection $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$.*
3. *Set $\mathcal{F}_{i_0 \ldots i_p}$ equal to the restriction of $\mathcal{F}$ to $U_{i_0 \ldots i_p}$.*
4. *Set*
$$\check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) = \bigoplus\nolimits_{i_0 \ldots i_p} f_{i_0 \ldots i_p *} \mathcal{F}_{i_0 \ldots i_p}$$
and define differentials $d : \check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathcal{U}, f, \mathcal{F})$ as in Cohomology, Equation (10.0.1).

Then the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ is a complex of quasi-coherent sheaves on S which comes equipped with an isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf_*\mathcal{F}$$

in $D^+(S)$. This isomorphism is functorial in the quasi-coherent sheaf $\mathcal{F}$.

Proof. Consider the resolution $\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of Cohomology, Lemma 25.1. We have an equality of complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of quasi-coherent $\mathcal{O}_S$-modules. The morphisms $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to X$ and the morphisms $f_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to S$ are affine by Morphisms, Lemma 13.11. Hence $R^q j_{i_0 \ldots i_p *}\mathcal{F}_{i_0 \ldots i_p}$ as well as $R^q f_{i_0 \ldots i_p *}\mathcal{F}_{i_0 \ldots i_p}$ are zero for $q > 0$ (Lemma 2.3). Using $f \circ j_{i_0 \ldots i_p} = f_{i_0 \ldots i_p}$ and the spectral sequence of Cohomology, Lemma 14.8 we conclude that $R^q f_*(j_{i_0 \ldots i_p *}\mathcal{F}_{i_0 \ldots i_p}) = 0$ for $q > 0$. Since the terms of the complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ are finite direct sums of the sheaves $j_{i_0 \ldots i_p *}\mathcal{F}_{i_0 \ldots i_p}$ we conclude using Leray's acyclicity lemma (Derived Categories, Lemma 17.7) that

$$Rf_*\mathcal{F} = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$$

as desired. □

Next, we are going to consider what happens if we do a base change.

Lemma 7.2. *With notation as in diagram (5.0.1). Assume $f : X \to S$ and $\mathcal{F}$ satisfy the hypotheses of Lemma 7.1. Choose a finite affine open covering $\mathcal{U} : X = \bigcup U_i$ of X. There is a canonical isomorphism*

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$. Moreover, if $S' \to S$ is affine, then in fact

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\mathcal{U}' : X' = \bigcup U'_i$ where $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ is also affine.

Proof. In fact we may define $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ no matter whether S' is affine over S or not. Let $\mathcal{U}' : X' = \bigcup U'_i$ be the induced covering of X'. In this case we claim that

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$ defined in exactly the same manner as in Lemma 7.1. This is clear from the case of affine morphisms (Lemma 5.1) by working locally on S'. Moreover, exactly as in the proof of Lemma 7.1 one sees that there is an isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}') \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$ since the morphisms $U'_i \to X'$ and $U'_i \to S'$ are still affine (being base changes of affine morphisms). Details omitted. □

The lemma above says that the complex

$$\mathcal{K}^\bullet = \check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$$

is a bounded below complex of quasi-coherent sheaves on S which *universally* computes the higher direct images of $f : X \to S$. This is something about this particular complex and it is not preserved by replacing $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ by a quasi-isomorphic complex in general! In other words, this is not a statement that makes sense in the derived category. The reason is that the pullback $g^*\mathcal{K}^\bullet$ is *not* equal to the derived pullback $Lg^*\mathcal{K}^\bullet$ of $\mathcal{K}^\bullet$ in general!

Here is a more general case where we can prove this statement. We remark that the condition of S being separated is harmless in most applications, since this is usually used to prove some local property of the total derived image. The proof is significantly more involved and uses hypercoverings; it is a nice example of how you can use them sometimes.

Lemma 7.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume that f is quasi-compact and quasi-separated and that S is quasi-compact and separated. There exists a bounded below complex $\mathcal{K}^\bullet$ of quasi-coherent $\mathcal{O}_S$-modules with the following property: For every morphism $g : S' \to S$ the complex $g^*\mathcal{K}^\bullet$ is a representative for $Rf'_*\mathcal{F}'$ with notation as in diagram (5.0.1).*

Proof. (If f is separated as well, please see Lemma 7.2.) The assumptions imply in particular that X is quasi-compact and quasi-separated as a scheme. Let $\mathcal{B}$ be the set of affine opens of X. By Hypercoverings, Lemma 10.4 we can find a hypercovering $K = (I, \{U_i\})$ such that each I_n is finite and each U_i is an affine open of X. By Hypercoverings, Lemma 4.3 there is a spectral sequence with E_2-page

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. Note that $\check{H}^p(K, \underline{H}^q(\mathcal{F}))$ is the pth cohomology group of the complex

$$\prod\nolimits_{i \in I_0} H^q(U_i, \mathcal{F}) \to \prod\nolimits_{i \in I_1} H^q(U_i, \mathcal{F}) \to \prod\nolimits_{i \in I_2} H^q(U_i, \mathcal{F}) \to \ldots$$

Since each U_i is affine we see that this is zero unless $q = 0$ in which case we obtain

$$\prod\nolimits_{i \in I_0} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_1} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_2} \mathcal{F}(U_i) \to \ldots$$

Thus we conclude that $R\Gamma(X, \mathcal{F})$ is computed by this complex.

For any n and $i \in I_n$ denote $f_i : U_i \to S$ the restriction of f to U_i. As S is separated and U_i is affine this morphism is affine. Consider the complex of quasi-coherent sheaves

$$\mathcal{K}^\bullet = (\prod\nolimits_{i \in I_0} f_{i,*}\mathcal{F}|_{U_i} \to \prod\nolimits_{i \in I_1} f_{i,*}\mathcal{F}|_{U_i} \to \prod\nolimits_{i \in I_2} f_{i,*}\mathcal{F}|_{U_i} \to \ldots)$$

on S. As in Hypercoverings, Lemma 4.3 we obtain a map $\mathcal{K}^\bullet \to Rf_*\mathcal{F}$ in $D(\mathcal{O}_S)$ by choosing an injective resolution of $\mathcal{F}$ (details omitted). Consider any affine scheme V and a morphism $g : V \to S$. Then the base change X_V has a hypercovering $K_V = (I, \{U_{i,V}\})$ obtained by base change. Moreover, $g^*f_{i,*}\mathcal{F} = f_{i,V,*}(g')^*\mathcal{F}|_{U_{i,V}}$. Thus the arguments above prove that $\Gamma(V, g^*\mathcal{K}^\bullet)$ computes $R\Gamma(X_V, (g')^*\mathcal{F})$. This finishes the proof of the lemma as it suffices to prove the equality of complexes Zariski locally on S'. $\square$

8. Cohomology of projective space

In this section we compute the cohomology of the twists of the structure sheaf on $\mathbf{P}^n_S$ over a scheme S. Recall that $\mathbf{P}^n_S$ was defined as the fibre product $\mathbf{P}^n_S = S \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{P}^n_{\mathbf{Z}}$ in Constructions, Definition 13.2. It was shown to be equal to

$$\mathbf{P}^n_S = \underline{\mathrm{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n])$$

in Constructions, Lemma 21.4. In particular, projective space is a particular case of a projective bundle. If $S = \mathrm{Spec}(R)$ is affine then we have

$$\mathbf{P}^n_S = \mathbf{P}^n_R = \mathrm{Proj}(R[T_0, \ldots, T_n]).$$

All these identifications are compatible and compatible with the constructions of the twisted structure sheaves $\mathcal{O}_{\mathbf{P}^n_S}(d)$.

Before we state the result we need some notation. Let R be a ring. Recall that $R[T_0, \ldots, T_n]$ is a graded R-algebra where each T_i is homogeneous of degree 1.

Denote $(R[T_0, \ldots, T_n])_d$ the degree d summand. It is a finite free R-module of rank $\binom{n+d}{d}$ when $d \geq 0$ and zero else. It has a basis consisting of monomials $T_0^{e_0} \ldots T_n^{e_n}$ with $\sum e_i = d$. We will also use the following notation: $R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$ denotes the $\mathbf{Z}$-graded ring with $\frac{1}{T_i}$ in degree -1. In particular the $\mathbf{Z}$-graded $R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$ module

$$\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$$

which shows up in the statement below is zero in degrees $\geq -n$, is free on the generator $\frac{1}{T_0 \ldots T_n}$ in degree $-n-1$ and is free of rank $(-1)^n \binom{n+d}{d}$ for $d \leq -n-1$.

Lemma 8.1. *Let R be a ring. Let $n \geq 0$ be an integer. We have*

$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \begin{cases} (R[T_0, \ldots, T_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d & \text{if} \quad q = n \end{cases}$$

as R-modules.

Proof. We will use the standard affine open covering

$$\mathcal{U} : \mathbf{P}^n_R = \bigcup\nolimits_{i=0}^n D_+(T_i)$$

to compute the cohomology using the Cech complex. This is permissible by Lemma 2.5 since any intersection of finitely many affine $D_+(T_i)$ is also a standard affine open (see Constructions, Section 8). In fact, we can use the alternating or ordered Cech complex according to Cohomology, Lemmas 24.3 and 24.6.

The ordering we will use on $\{0, \ldots, n\}$ is the usual one. Hence the complex we are looking at has terms

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{i_0 < \ldots < i_p} (R[T_0, \ldots, T_n, \frac{1}{T_{i_0} \ldots T_{i_p}}])_d$$

Moreover, the maps are given by the usual formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$$

see Cohomology, Section 24. Note that each term of this complex has a natural $\mathbf{Z}^{n+1}$-grading. Namely, we get this by declaring a monomial $T_0^{e_0} \ldots T_n^{e_n}$ to be homogeneous with weight $(e_0, \ldots, e_n) \in \mathbf{Z}^{n+1}$. It is clear that the differential given above respects the grading. In a formula we have

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1}} \check{\mathcal{C}}^\bullet(\vec{e})$$

where not all summand on the right hand side occur (see below). Hence in order to compute the cohomology modules of the complex it suffices to compute the cohomology of the graded pieces and take the direct sum at the end.

Fix $\vec{e} = (e_0, \ldots, e_n) \in \mathbf{Z}^{n+1}$. In order for this weight to occur in the complex above we need to assume $e_0 + \ldots + e_n = d$ (if not then it occurs for a different twist of the structure sheaf of course). Assuming this set

$$NEG(\vec{e}) = \{i \in \{0, \ldots, n\} \mid e_i < 0\}.$$

With this notation the weight $\vec{e}$ summand $\check{\mathcal{C}}^\bullet(\vec{e})$ of the Cech complex above has the following terms

$$\check{\mathcal{C}}^p(\vec{e}) = \bigoplus\nolimits_{i_0<\ldots<i_p,\ NEG(\vec{e})\subset\{i_0,\ldots,i_p\}} R \cdot T_0^{e_0} \ldots T_n^{e_n}$$

In other words, the terms corresponding to $i_0 < \ldots < i_p$ such that $NEG(\vec{e})$ is not contained in $\{i_0 \ldots i_p\}$ are zero. The differential of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is still given by the exact same formula as above.

Suppose that $NEG(\vec{e}) = \{0, \ldots, n\}$, i.e., that all exponents e_i are negative. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has only one term, namely $\check{\mathcal{C}}^n(\vec{e}) = R \cdot \frac{1}{T^{-e_0} \ldots T^{-e_n}}$. Hence in this case

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot \frac{1}{T^{-e_0} \ldots T^{-e_n}} & \text{if} \quad q = n \\ 0 & \text{if} \quad \text{else} \end{cases}$$

The direct sum of all of these terms clearly gives the value

$$\left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d$$

in degree n as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

Assume $NEG(\vec{e}) = \emptyset$. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has a summand R corresponding to all $i_0 < \ldots < i_p$. Let us compare the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ to another complex. Namely, consider the affine open open covering

$$\mathcal{V} : \text{Spec}(R) = \bigcup\nolimits_{i \in \{0,\ldots,n\}} V_i$$

where $V_i = \text{Spec}(R)$ for all i. Consider the alternating Cech complex

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\text{Spec}(R)})$$

By the same reasoning as above this computes the cohomology of the structure sheaf on $\text{Spec}(R)$. Hence we see that $H^p(\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\text{Spec}(R)})) = R$ if $p = 0$ and is 0 whenever $p > 0$. For these facts, see Lemma 2.1 and its proof. Note that also $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\text{Spec}(R)})$ has a summand R for every $i_0 < \ldots < i_p$ and has exactly the same differential as $\check{\mathcal{C}}^\bullet(\vec{e})$. In other words these complexes are isomorphic complexes and hence have the same cohomology. We conclude that

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot T^{e_0} \ldots T^{e_n} & \text{if} \quad q = 0 \\ 0 & \text{if} \quad \text{else} \end{cases}$$

in the case that $NEG(\vec{e}) = \emptyset$. The direct sum of all of these terms clearly gives the value

$$(R[T_0, \ldots, T_n])_d$$

in degree 0 as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

To finish the proof of the lemma we have to show that the complexes $\check{\mathcal{C}}^\bullet(\vec{e})$ are acyclic when $NEG(\vec{e})$ is neither empty nor equal to $\{0, \ldots, n\}$. Pick an index $i_{\text{fix}} \notin NEG(\vec{e})$ (such an index exists). Consider the map

$$h : \check{\mathcal{C}}^{p+1}(\vec{e}) \to \check{\mathcal{C}}^p(\vec{e})$$

given by the rule

$$h(s)_{i_0 \ldots i_p} = s_{i_{\text{fix}} i_0 \ldots i_p}$$

(compare with the proof of Lemma 2.1). It is clear that this is well defined since

$$NEG(\vec{e}) \subset \{i_0, \ldots, i_p\} \Leftrightarrow NEG(\vec{e}) \subset \{i_{\text{fix}}, i_0, \ldots, i_p\}$$

Also $\check{\mathcal{C}}^0(\vec{e}) = 0$ so that this formula does work for all p including $p = -1$. The exact same (combinatorial) computation as in the proof of Lemma 2.1 shows that

$$(hd + dh)(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}$$

Hence we see that the identity map of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is homotopic to zero which implies that it is acyclic. □

In the following lemma we are going to use the pairing of free R-modules

$$R[T_0, \ldots, T_n] \times \frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}] \longrightarrow R$$

which is defined by the rule

$$(f, g) \longmapsto \text{coefficient of } \frac{1}{T_0 \ldots T_n} \text{ in } fg.$$

In other words, the basis element $T_0^{e_0} \ldots T_n^{e_n}$ pairs with the basis element $T_0^{d_0} \ldots T_n^{d_n}$ to give 1 if and only if $e_i + d_i = -1$ for all i, and pairs to zero in all other cases. Using this pairing we get an identification

$$\left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d = \text{Hom}_R((R[T_0, \ldots, T_n])_{-n-1-d}, R)$$

Thus we can reformulate the result of Lemma 8.1 as saying that

$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \begin{cases} (R[T_0, \ldots, T_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \text{Hom}_R((R[T_0, \ldots, T_n])_{-n-1-d}, R) & \text{if} \quad q = n \end{cases} \tag{8.1.1}$$

Lemma 8.2. *The identifications of Equation (8.1.1) are compatible with base change w.r.t. ring maps $R \to R'$. Moreover, for any $f \in R[T_0, \ldots, T_n]$ homogeneous of degree m the map multiplication by f*

$$\mathcal{O}_{\mathbf{P}^n_R}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(d + m)$$

induces the map on the cohomology group via the identifications of Equation (8.1.1) which is multiplication by f for H^0 and the contragredient of multiplication by f

$$(R[T_0, \ldots, T_n])_{-n-1-(d+m)} \longrightarrow (R[T_0, \ldots, T_n])_{-n-1-d}$$

on H^n.

Proof. Suppose that $R \to R'$ is a ring map. Let $\mathcal{U}$ be the standard affine open covering of $\mathbf{P}^n_R$, and let $\mathcal{U}'$ be the standard affine open covering of $\mathbf{P}^n_{R'}$. Note that $\mathcal{U}'$ is the pullback of the covering $\mathcal{U}$ under the canonical morphism $\mathbf{P}^n_{R'} \to \mathbf{P}^n_R$. Hence there is a map of Cech complexes

$$\gamma : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}', \mathcal{O}_{\mathbf{P}_{R'}}(d))$$

which is compatible with the map on cohomology by Cohomology, Lemma 16.1. It is clear from the computations in the proof of Lemma 8.1 that this map of Cech complexes is compatible with the identifications of the cohomology groups in question. (Namely the basis elements for the Cech complex over R simply map to the corresponding basis elements for the Cech complex over R'.) Whence the first statement of the lemma.

Now fix the ring R and consider two homogeneous polynomials $f, g \in R[T_0, \ldots, T_n]$ both of the same degree m. Since cohomology is an additive functor, it is clear that the map induced by multiplication by $f + g$ is the same as the sum of the maps induced by multiplication by f and the map induced by multiplication by g. Moreover, since cohomology is a functor a similar result holds for multiplication by a product fg where f, g are both homogeneous (but not necessarily of the same degree). Hence to verify the second statement of the lemma it suffices to prove this when $f = x \in R$ or when $f = T_i$. In the case of multiplication by an element $x \in R$ the result follows since every cohomology groups or complex in sight has the structure of an R-module or complex of R-modules. Finally, we consider the case of multiplication by T_i as a $\mathcal{O}_{\mathbf{P}^n_R}$-linear map

$$\mathcal{O}_{\mathbf{P}^n_R}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(d+1)$$

The statement on H^0 is clear. For the statement on H^n consider multiplication by T_i as a map on Cech complexes

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d+1))$$

We are going to use the notation introduced in the proof of Lemma 8.1. We consider the effect of multiplication by T_i in terms of the decompositions

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1},\ \sum e_i = d} \check{\mathcal{C}}^\bullet(\vec{e})$$

and

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d+1)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1},\ \sum e_i = d+1} \check{\mathcal{C}}^\bullet(\vec{e})$$

It is clear that it maps the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e})$ to the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e}+\vec{b}_i)$ where $\vec{b}_i = (0, \ldots, 0, 1, 0, \ldots, 0))$ the ith basis vector. In other words, it maps the summand of H^n corresponding to $\vec{e}$ with $e_i < 0$ and $\sum e_i = d$ to the summand of H^n corresponding to $\vec{e} + \vec{b}_i$ (which is zero if $e_i + b_i \geq 0$). It is easy to see that this corresponds exactly to the action of the contragredient of multiplication by T_i as a map

$$(R[T_0, \ldots, T_n])_{-n-1-(d+1)} \longrightarrow (R[T_0, \ldots, T_n])_{-n-1-d}$$

This proves the lemma. □

Before we state the relative version we need some notation. Namely, recall that $\mathcal{O}_S[T_0, \ldots, T_n]$ is a graded $\mathcal{O}_S$-module where each T_i is homogeneous of degree 1. Denote $(\mathcal{O}_S[T_0, \ldots, T_n])_d$ the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S.

Lemma 8.3. *Let S be a scheme. Let $n \geq 0$ be an integer. Consider the structure morphism*

$$f : \mathbf{P}^n_S \longrightarrow S.$$

We have

$$R^q f_*(\mathcal{O}_{\mathbf{P}^n_S}(d)) = \begin{cases} (\mathcal{O}_S[T_0, \ldots, T_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}((\mathcal{O}_S[T_0, \ldots, T_n])_{-n-1-d}, \mathcal{O}_S) & \text{if} \quad q = n \end{cases}$$

Proof. Omitted. Hint: This follows since the identifications in (8.1.1) are compatible with affine base change by Lemma 8.2. □

Next we state the version for projective bundles associated to finite locally free sheaves. Let S be a scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n+1$, see Modules, Section 14. In this case we think of $\text{Sym}(\mathcal{E})$ as a graded $\mathcal{O}_S$-module where $\mathcal{E}$ is the graded part of degree 1. And $\text{Sym}^d(\mathcal{E})$ is the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S. Recall that our normalization is that

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E})) \longrightarrow S$$

and that there are natural maps $\text{Sym}^d(\mathcal{E}) \to \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$.

Lemma 8.4. *Let S be a scheme. Let $n \geq 1$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n+1$. Consider the structure morphism*

$$\pi : \mathbf{P}(\mathcal{E}) \longrightarrow S.$$

We have

$$R^q\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) = \begin{cases} \mathcal{S}ym^d(\mathcal{E}) & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}(\mathcal{S}ym^{-n-1-d}(\mathcal{E}) \otimes_{\mathcal{O}_S} \wedge^{n+1}\mathcal{E}, \mathcal{O}_S) & \text{if} \quad q = n \end{cases}$$

These identifications are compatible with base change and isomorphism between locally free sheaves.

Proof. Consider the canonical map

$$\pi^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

and twist down by 1 to get

$$\pi^*(\mathcal{E})(-1) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}$$

This is a surjective map from a locally free rank $n+1$ sheaf onto the structure sheaf. Hence the corresponding Koszul complex is exact (insert future reference here). In other words there is an exact complex

$$0 \to \pi^*(\wedge^{n+1}\mathcal{E})(-n-1) \to \ldots \to \pi^*(\wedge^i\mathcal{E})(-i) \to \ldots \to \pi^*\mathcal{E}(-1) \to \mathcal{O}_{\mathbf{P}(\mathcal{E})} \to 0$$

We will think of the term $\pi^*(\wedge^i\mathcal{E})(-i)$ as being in degree $-i$. We are going to compute the higher direct images of this acyclic complex using the first spectral sequence of Derived Categories, Lemma 21.3. Namely, we see that there is a spectral sequence with terms

$$E_2^{p,q} = H^p(L^{\bullet,q}) \quad \text{with} \quad L^{-i,q} = R^q\pi_*\left(\pi^*(\wedge^i\mathcal{E})(-i)\right)$$

converging to zero! By the projection formula (Cohomology, Lemma 8.2) we have

$$L^{-i,q} = \wedge^i\mathcal{E} \otimes_{\mathcal{O}_S} R^q\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-i)\right).$$

Note that locally on S the sheaf $\mathcal{E}$ is trivial, i.e., isomorphic to $\mathcal{O}_S^{\oplus n+1}$, hence locally on S the morphism $\mathbf{P}(\mathcal{E}) \to S$ can be identified with $\mathbf{P}^n_S \to S$. Hence locally on S we can use the result of Lemmas 8.1, 8.2, or 8.3. It follows that $L^{-i,q} = 0$ unless $i = q = 0$ or $i = n+1$ and $q = n$. This in turn implies that $E_2^{p,q} = 0$ unless $(p,q) = (0,0)$ or $(p,q) = (-n-1, n)$, and

$$E_2^{0,0} = \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})} = \mathcal{O}_S$$
$$E_2^{-n-1,n} = \wedge^{n+1}\mathcal{E} \otimes_{\mathcal{O}_S} R^n\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)\right).$$

Hence there can only be one nonzero differential in the spectral sequence namely the map d_{n+1} inducing a map

$$d_{n+1}^{0,0} : \mathcal{O}_S \longrightarrow \wedge^{n+1}\mathcal{E} \otimes_{\mathcal{O}_S} R^n\pi_* \left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)\right)$$

which has to be an isomorphism (because the spectral sequence converges to the 0 sheaf). Since $\wedge^{n+1}\mathcal{E}$ is an invertible sheaf, this implies that $R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$ is invertible as well and canonically isomorphic to the inverse of $\wedge^{n+1}\mathcal{E}$. In other words we have proved the case $d = -n-1$ of the lemma.

Working locally on S we see immediately from the computation of cohomology in Lemmas 8.1, 8.2, or 8.3 the statements on vanishing of the lemma. Moreover the result on $R^0\pi_*$ is clear as well, since there are canonical maps $\mathrm{Sym}^d(\mathcal{E}) \to \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ for all d. It remains to show that the description of $R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ is correct for $d < -n-1$. In order to do this we consider the map

$$\pi^*(\mathrm{Sym}^{-d-n-1}(\mathcal{E})) \otimes_{\mathcal{O}_{\mathbf{P}(\mathcal{E})}} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(d) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$$

Applying $R^n\pi_*$ and the projection formula (see above) we get a map

$$\mathrm{Sym}^{-d-n-1}(\mathcal{E}) \otimes_{\mathcal{O}_S} R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) \longrightarrow R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1) = (\wedge^{n+1}\mathcal{E})^{\otimes -1}$$

(the last equality we have shown above). Again by the local calculations of Lemmas 8.1, 8.2, or 8.3 it follows that this map induces a perfect pairing between $R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d))$ and $\mathrm{Sym}^{-d-n-1}(\mathcal{E}) \otimes \wedge^{n+1}(\mathcal{E})$ as desired. □

9. Coherent sheaves on locally Noetherian schemes

We have defined the notion of a coherent module on any ringed space in Modules, Section 12. Although it is possible to consider coherent sheaves on non-Noetherian schemes we will always assume the base scheme is locally Noetherian when we consider coherent sheaves. Here is a characterization of coherent sheaves on locally Noetherian schemes.

Lemma 9.1. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is coherent,*
2. *$\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module,*
3. *$\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module,*
4. *for any affine open $\mathrm{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A-module, and*
5. *there exists an affine open covering $X = \bigcup U_i$, $U_i = \mathrm{Spec}(A_i)$ such that each $\mathcal{F}|_{U_i} = \widetilde{M_i}$ with M_i a finite A_i-module.*

In particular $\mathcal{O}_X$ is coherent, any invertible $\mathcal{O}_X$-module is coherent, and more generally any finite locally free $\mathcal{O}_X$-module is coherent.

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) hold in general, see Modules, Lemma 12.2. If $\mathcal{F}$ is finitely presented then $\mathcal{F}$ is quasi-coherent, see Modules, Lemma 11.2. Hence also (3) ⇒ (2).

Assume $\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module. By Properties, Lemma 16.1 we see that on any affine open $\mathrm{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A-module. Since A is Noetherian we see that M has a finite resolution

$$A^{\oplus m} \to A^{\oplus n} \to M \to 0.$$

Hence $\mathcal{F}$ is of finite presentation by Properties, Lemma 16.2. In other words (2) $\Rightarrow$ (3).

By Modules, Lemma 12.5 it suffices to show that $\mathcal{O}_X$ is coherent in order to show that (3) implies (1). Thus we have to show: given any open $U \subset X$ and any finite collection of sections $f_i \in \mathcal{O}_X(U)$, $i = 1, \ldots, n$ the kernel of the map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{O}_U$ is of finite type. Since being of finite type is a local property it suffices to check this in a neighbourhood of any $x \in U$. Thus we may assume $U = \text{Spec}(A)$ is affine. In this case $f_1, \ldots, f_n \in A$ are elements of A. Since A is Noetherian, see Properties, Lemma 5.2 the kernel K of the map $\bigoplus_{i=1,\ldots,n} A \to A$ is a finite A-module. See for example Algebra, Lemma 49.1. As the functor $\widetilde{}$ is exact, see Schemes, Lemma 5.4 we get an exact sequence

$$\widetilde{K} \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{O}_U$$

and by Properties, Lemma 16.1 again we see that $\widetilde{K}$ is of finite type. We conclude that (1), (2) and (3) are all equivalent.

It follows from Properties, Lemma 16.1 that (2) implies (4). It is trivial that (4) implies (5). The discussion in Schemes, Section 24 show that (5) implies that $\mathcal{F}$ is quasi-coherent and it is clear that (5) implies that $\mathcal{F}$ is of finite type. Hence (5) implies (2) and we win. □

Lemma 9.2. *Let X be a locally Noetherian scheme. The category of coherent $\mathcal{O}_X$-modules is abelian. More precisely, the kernel and cokernel of a map of coherent $\mathcal{O}_X$-modules are coherent. Any extension of coherent sheaves is coherent.*

Proof. This is a restatement of Modules, Lemma 12.4 in a particular case. □

The following lemma does not always hold for the category of coherent $\mathcal{O}_X$-modules on a general ringed space X.

Lemma 9.3. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Any quasi-coherent submodule of $\mathcal{F}$ is coherent. Any quasi-coherent quotient module of $\mathcal{F}$ is coherent.*

Proof. We may assume that X is affine, say $X = \text{Spec}(A)$. Properties, Lemma 5.2 implies that A is Noetherian. Lemma 9.1 turns this into algebra. The algebraic counter part of the lemma is that a quotient, or a submodule of a finite A-module is a finite A-module, see for example Algebra, Lemma 49.1. □

Lemma 9.4. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent.*

Proof. It is shown in Modules, Lemma 19.4 that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent. The result for tensor products is Modules, Lemma 15.5 □

Lemma 9.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$.*

1. *If $\mathcal{F}_x = 0$ then there exists an open neighbourhood $U \subset X$ of x such that $\mathcal{F}|_U = 0$.*
2. *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is injective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is injective.*

(3) *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is surjective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is surjective.*
(4) *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is bijective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is an isomorphism.*

Proof. See Modules, Lemmas 9.4, 9.5, and 12.6. □

Lemma 9.6. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $x \in X$. Suppose $\psi : \mathcal{G}_x \to \mathcal{F}_x$ is a map of $\mathcal{O}_{X,x}$-modules. Then there exists an open neighbourhood $U \subset X$ of x and a map $\varphi : \mathcal{G}|_U \to \mathcal{F}|_U$ such that $\varphi_x = \psi$.*

Proof. In view of Lemma 9.1 this is a reformulation of Modules, Lemma 19.3. □

Lemma 9.7. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then Supp($\mathcal{F}$) is closed, and $\mathcal{F}$ comes from a coherent sheaf on the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 5.5.*

Proof. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{F}$ and let $\mathcal{G}$ be the finite type quasi-coherent sheaf on Z such that $i_*\mathcal{G} \cong \mathcal{F}$. Since $Z = \text{Supp}(\mathcal{F})$ we see that the support is closed. The scheme Z is locally Noetherian by Morphisms, Lemmas 16.5 and 16.6. Finally, $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module by Lemma 9.1 □

Lemma 9.8. *Let $i : Z \to X$ be a closed immersion of locally Noetherian schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor i_* induces an equivalence between the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ and the category of coherent $\mathcal{O}_Z$-modules.*

Proof. The functor is fully faithful by Morphisms, Lemma 4.1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module annihilated by $\mathcal{I}$. By Morphisms, Lemma 4.1 we can write $\mathcal{F} = i_*\mathcal{G}$ for some quasi-coherent sheaf $\mathcal{G}$ on Z. By Modules, Lemma 13.3 we see that $\mathcal{G}$ is of finite type. Hence $\mathcal{G}$ is coherent by Lemma 9.1. Thus the functor is also essentially surjective as desired. □

Lemma 9.9. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and Y locally Noetherian. Then $R^pf_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F}$ is coherent if $\mathcal{F}$ is coherent.*

Proof. The higher direct images vanish by Lemma 2.3 and because a finite morphism is affine (by definition). Note that the assumptions imply that also X is locally Noetherian (see Morphisms, Lemma 16.6) and hence the statement makes sense. Let $\text{Spec}(A) = V \subset Y$ be an affine open subset. By Morphisms, Definition 44.1 we see that $f^{-1}(V) = \text{Spec}(B)$ with $A \to B$ finite. Lemma 9.1 turns the statement of the lemma into the following algebra fact: If M is a finite B-module, then M is also finite viewed as a A-module, see Algebra, Lemma 7.2. □

In the situation of the lemma also the higher direct images are coherent since they vanish. We will show that this is always the case for a proper morphism between locally Noetherian schemes (insert future reference here).

10. Coherent sheaves on Noetherian schemes

In this section we mention some properties of coherent sheaves on Noetherian schemes.

Lemma 10.1. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The ascending chain condition holds for quasi-coherent submodules of $\mathcal{F}$. In other words, given any sequence*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$$

of quasi-coherent submodules, then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$.

Proof. Choose a finite affine open covering. On each member of the covering we get stabilization by Algebra, Lemma 49.1. Hence the lemma follows. □

Lemma 10.2. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals corresponding to a closed subscheme $Z \subset X$. Then there is some $n \geq 0$ such that $\mathcal{I}^n\mathcal{F} = 0$ if and only if $Supp(\mathcal{F}) \subset Z$ (set theoretically).*

Proof. This follows immediately from Algebra, Lemma 61.4 because X has a finite covering by spectra of Noetherian rings. □

Lemma 10.3 (Artin-Rees)**.** *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}$ be a quasi-coherent subsheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then there exists a $c \geq 0$ such that for all $n \geq c$ we have*

$$\mathcal{I}^{n-c}(\mathcal{I}^c\mathcal{F} \cap \mathcal{G}) = \mathcal{I}^n\mathcal{F} \cap \mathcal{G}$$

Proof. This follows immediately from Algebra, Lemma 49.2 because X has a finite covering by spectra of Noetherian rings. □

Lemma 10.4. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be coherent $\mathcal{O}_X$-module. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Denote $Z \subset X$ the corresponding closed subscheme and set $U = X \setminus Z$. There is a canonical isomorphism*

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular we have an isomorphism

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}).$$

Proof. We first prove the second map is an isomorphism. It is injective by Properties, Lemma 23.3. Since $\mathcal{F}$ is the union of its coherent submodules, see Properties, Lemma 20.3 (and Lemma 9.1) we may and do assume that $\mathcal{F}$ is coherent to prove surjectivity. Let $\mathcal{F}_n$ denote the quasi-coherent subsheaf of $\mathcal{F}$ consisting of sections annihilated by $\mathcal{I}^n$, see Properties, Lemma 23.3. Since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ we see that $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$ by Lemma 10.1. Set $\mathcal{H} = \mathcal{F}_n$ for this n. By Artin-Rees (Lemma 10.3) there exists an $c \geq 0$ such that $\mathcal{I}^m\mathcal{F} \cap \mathcal{H} \subset \mathcal{I}^{m-c}\mathcal{H}$. Picking $m = n + c$ we get $\mathcal{I}^m\mathcal{F} \cap \mathcal{H} \subset \mathcal{I}^n\mathcal{H} = 0$. Thus if we set $\mathcal{F}' = \mathcal{I}^m\mathcal{F}$ then we see that $\mathcal{F}' \cap \mathcal{F}_n = 0$ and $\mathcal{F}'|_U = \mathcal{F}|_U$. Note in particular that the subsheaf $(\mathcal{F}')_N$ of sections annihilated by $\mathcal{I}^N$ is zero for all $N \geq 0$. Hence by Properties, Lemma 23.3 we deduce that the top horizontal arrow in the following commutative diagram is a bijection:

$$\begin{array}{ccc}
\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}') & \longrightarrow & \Gamma(U, \mathcal{F}') \\
\downarrow & & \downarrow \\
\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F})
\end{array}$$

Since also the right vertical arrow is a bijection we conclude that the bottom horizontal arrow is surjective as desired.

Next, we prove the first arrow of the lemma is a bijection. By Lemma 9.1 the sheaf $\mathcal{G}$ is of finite presentation and hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is quasi-coherent, see Schemes, Section 24. By definition we have

$$\mathcal{H}(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U)$$

Pick a ψ in the right hand side of the first arrow of the lemma, i.e., $\psi \in \mathcal{H}(U)$. The result just proved applies to $\mathcal{H}$ and hence there exists an $n \geq 0$ and an $\varphi : \mathcal{I}^n \to \mathcal{H}$ which recovers ψ on restriction to U. By Modules, Lemma 19.1 φ corresponds to a map

$$\varphi : \mathcal{I}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}.$$

This is almost what we want except that the source of the arrow is the tensor product of $\mathcal{I}^n$ and $\mathcal{G}$ and not the product. We will show that, at the cost of increasing n, the difference is irrelevant. Consider the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{I}^n\mathcal{G} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Note that $\mathcal{I}^n\mathcal{K} = 0$ (proof omitted). By Artin-Rees again we see that

$$\mathcal{K} \cap \mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) = 0$$

for some m large enough. In other words we see that

$$\mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) \longrightarrow \mathcal{I}^{n+m}\mathcal{G}$$

is an isomorphism. Let φ' be the restriction of φ to this submodule thought of as a map $\mathcal{I}^{m+n}\mathcal{G} \to \mathcal{F}$. Then φ' gives an element of the left hand side of the first arrow of the lemma which maps to ψ via the arrow. In other words we have proved surjectivity of the arrow. We omit the proof of injectivity. □

11. Depth

In this section we talk a little bit about depth and property (S_k) for coherent modules on locally Noetherian schemes. Note that we have already discussed this notion for locally Noetherian schemes in Properties, Section 12.

Definition 11.1. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $k \geq 0$ be an integer.

1. We say $\mathcal{F}$ has *depth k at a point x* of X if $\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = k$.
2. We say X has *depth k at a point x* of X if $\mathrm{depth}(\mathcal{O}_{X,x}) = k$.
3. We say $\mathcal{F}$ has property (S_k) if
$$\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq \min(k, \dim(\mathrm{Supp}(\mathcal{F}_x)))$$
for all $x \in X$.
4. We say X has property (S_k) if $\mathcal{O}_X$ has property (S_k).

Any coherent sheaf satisfies condition (S_0). Condition (S_1) is equivalent to having no embedded associated points, see Divisors, Lemma 4.3.

We have seen in Properties, Lemma 12.2 that a locally Noetherian scheme is Cohen-Macaulay if and only if (S_k) holds for all k. Thus it makes sense to introduce the following definition, which is equivalent to the condition that all stalks are Cohen-Macaulay modules.

Definition 11.2. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *Cohen-Macaulay* if and only if (S_k) holds for all $k \geq 0$.

12. Devissage of coherent sheaves

Let X be a Noetherian scheme. Consider an integral closed subscheme $i : Z \to X$. It is often convenient to consider coherent sheaves of the form $i_*\mathcal{G}$ where $\mathcal{G}$ is a coherent sheaf on Z. In particular we are interested in these sheaves when $\mathcal{G}$ is a torsion free rank 1 sheaf. For example $\mathcal{G}$ could be a nonzero sheaf of ideals on Z, or even more specifically $\mathcal{G} = \mathcal{O}_Z$.

Throughout this section we will use that a coherent sheaf is the same thing as a finite type quasi-coherent sheaf and that a quasi-coherent subquotient of a coherent sheaf is coherent, see Section 9. The support of a coherent sheaf is closed, see Modules, Lemma 9.6.

Lemma 12.1. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that Supp$(\mathcal{F}) = Z \cup Z'$ with Z, Z' closed. Then there exists a short exact sequence of coherent sheaves*

$$0 \to \mathcal{G}' \to \mathcal{F} \to \mathcal{G} \to 0$$

with Supp$(\mathcal{G}') \subset Z'$ and Supp$(\mathcal{G}) \subset Z$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals defining the reduced induced closed subscheme structure on Z, see Schemes, Lemma 12.4. Consider the subsheaves $\mathcal{G}'_n = \mathcal{I}^n\mathcal{F}$ and the quotients $\mathcal{G}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$. For each n we have a short exact sequence

$$0 \to \mathcal{G}'_n \to \mathcal{F} \to \mathcal{G}_n \to 0$$

For every point x of $Z' \setminus Z$ we have $\mathcal{I}_x = \mathcal{O}_{X,x}$ and hence $\mathcal{G}_{n,x} = 0$. Thus we see that $\text{Supp}(\mathcal{G}_n) \subset Z$. Note that $X \setminus Z'$ is a Noetherian scheme. Hence by Lemma 10.2 there exists an n such that $\mathcal{G}'_n|_{X\setminus Z'} = \mathcal{I}^n\mathcal{F}|_{X\setminus Z'} = 0$. For such an n we see that $\text{Supp}(\mathcal{G}'_n) \subset Z'$. Thus setting $\mathcal{G}' = \mathcal{G}'_n$ and $\mathcal{G} = \mathcal{G}_n$ works. $\square$

Lemma 12.2. *Let X be a Noetherian scheme. Let $i : Z \to X$ be an integral closed subscheme. Let $\xi \in Z$ be the generic point. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\mathcal{F}_\xi$ is annihilated by $\mathfrak{m}_\xi$. Then there exists an integer $r \geq 0$ and a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ and an injective map of coherent sheaves*

$$i_*\left(\mathcal{I}^{\oplus r}\right) \to \mathcal{F}$$

which is an isomorphism in a neighbourhood of ξ.

Proof. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of local sections of $\mathcal{F}$ which are annihilated by $\mathcal{J}$. It is a quasi-coherent sheaf by Properties, Lemma 22.2. Moreover, $\mathcal{F}'_\xi = \mathcal{F}_\xi$ because $\mathcal{J}_\xi = \mathfrak{m}_\xi$ and part (3) of Properties, Lemma 22.2. By Lemma 9.5 we see that $\mathcal{F}' \to \mathcal{F}$ induces an isomorphism in a neighbourhood of ξ. Hence we may replace $\mathcal{F}$ by $\mathcal{F}'$ and assume that $\mathcal{F}$ is annihilated by $\mathcal{J}$.

Assume $\mathcal{J}\mathcal{F} = 0$. By Lemma 9.8 we can write $\mathcal{F} = i_*\mathcal{G}$ for some coherent sheaf $\mathcal{G}$ on Z. Suppose we can find a morphism $\mathcal{I}^{\oplus r} \to \mathcal{G}$ which is an isomorphism in a neighbourhood of the generic point ξ of Z. Then applying i_* (which is left exact) we get the result of the lemma. Hence we have reduced to the case $X = Z$.

Suppose $Z = X$ is an integral Noetherian scheme with generic point ξ. Note that $\mathcal{O}_{X,\xi} = \kappa(\xi)$ is the function field of X in this case. Since $\mathcal{F}_\xi$ is a finite $\mathcal{O}_\xi$-module we see that $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is finite. Hence the sheaves $\mathcal{O}_X^{\oplus r}$ and $\mathcal{F}$ have isomorphic stalks at ξ. By Lemma 9.6 there exists a nonempty open $U \subset X$ and a morphism $\psi : \mathcal{O}_X^{\oplus r}|_U \to \mathcal{F}|_U$ which is an isomorphism at ξ, and hence an isomorphism in a neighbourhood of ξ by Lemma 9.5. By Schemes, Lemma 12.4 there exists a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose associated closed subscheme $Z \subset X$ is the complement of U. By Lemma 10.4 there exists an $n \geq 0$ and a morphism $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) \to \mathcal{F}$ which recovers our ψ over U. Since $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) = (\mathcal{I}^n)^{\oplus r}$ we get a map as in the lemma. It is injective because X is integral and it is injective at the generic point of X (easy proof omitted). $\square$

Lemma 12.3. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. There exists a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exists an integral closed subscheme $Z_j \subset X$ and a sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (Z_j \to X)_*\mathcal{I}_j$$

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{array}{c} Z \subset X \text{ closed such that there exists a coherent sheaf } \mathcal{F} \\ \text{with } \mathrm{Supp}(\mathcal{F}) = Z \text{ for which the lemma is wrong} \end{array} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because X is Noetherian we can choose a minimal element $Z \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is Z and for which the lemma does not hold. Clearly $Z \neq \emptyset$ since the only sheaf whose support is empty is the zero sheaf for which the lemma does hold (with $m = 0$).

If Z is not irreducible, then we can write $Z = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than Z. Then we can apply Lemma 12.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\mathrm{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of Z each of $\mathcal{G}_i$ has a filtration as in the statement of the lemma. By considering the induced filtration on $\mathcal{F}$ we arrive at a contradiction. Hence we conclude that Z is irreducible.

Suppose Z is irreducible. Let $\mathcal{J}$ be the sheaf of ideals cutting out the reduced induced closed subscheme structure of Z, see Schemes, Lemma 12.4. By Lemma 10.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{J}^n\mathcal{F} \subset \mathcal{J}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{J}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does. In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

In the case where Z is irreducible and $\mathcal{J}\mathcal{F} = 0$ we can apply Lemma 12.2. This gives a short exact sequence

$$0 \to i_*(\mathcal{I}^{\oplus r}) \to \mathcal{F} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is defined as the quotient. Since $\mathcal{Q}$ is zero in a neighbourhood of ξ by the lemma just cited we see that the support of $\mathcal{Q}$ is strictly smaller than Z. Hence we see that $\mathcal{Q}$ has a filtration of the desired type by minimality of Z. But then clearly $\mathcal{F}$ does too, which is our final contradiction. $\square$

Lemma 12.4. *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

1. *For any short exact sequence of coherent sheaves*
$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$
if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
2. *For every integral closed subscheme $Z \subset X$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $i_*\mathcal{I}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration
$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$
by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from the property (1) for $\mathcal{P}$. On the other hand, by Lemma 12.3 we can filter any $\mathcal{F}$ with successive subquotients as in (2). Hence the lemma follows. $\square$

Lemma 12.5. *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
2. *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*
3. *There exists some coherent sheaf $\mathcal{G}$ on X such that*
   (a) *$Supp(\mathcal{G}) = Z_0$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,*
   (c) *$\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and*
   (d) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration
$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$
by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. Or, if $\mathcal{F}$ has property $\mathcal{P}$ and all but one of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$ then so does the last one. This follows from assumption (1).

As a first application we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$. Namely, such a sheaf has a filtration (see Lemma 12.3) whose subquotients have property $\mathcal{P}$ according to (2).

Let $\mathcal{G}$ be as in (3). By Lemma 12.2 there exist a sheaf of ideals $\mathcal{I}$ on Z_0, an integer $r \geq 1$, and a short exact sequence
$$0 \to ((Z_0 \to X)_*\mathcal{I})^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is strictly contained in Z_0. By (3)(c) we see that $r = 1$. Since $\mathcal{Q}$ has property $\mathcal{P}$ too we conclude that $(Z_0 \to X)_*\mathcal{I}$ has property $\mathcal{P}$.

Next, suppose that $\mathcal{I}' \neq 0$ is another quasi-coherent sheaf of ideals on Z_0. Then we can consider the intersection $\mathcal{I}'' = \mathcal{I}' \cap \mathcal{I}$ and we get two short exact sequences

$$0 \to (Z_0 \to X)_*\mathcal{I}'' \to (Z_0 \to X)_*\mathcal{I} \to \mathcal{Q} \to 0$$

and

$$0 \to (Z_0 \to X)_*\mathcal{I}'' \to (Z_0 \to X)_*\mathcal{I}' \to \mathcal{Q}' \to 0.$$

Note that the support of the coherent sheaves $\mathcal{Q}$ and $\mathcal{Q}'$ are strictly contained in Z_0. Hence $\mathcal{Q}$ and $\mathcal{Q}'$ have property $\mathcal{P}$ (see above). Hence we conclude using (1) that $(Z_0 \to X)_*\mathcal{I}''$ and $(Z_0 \to X)_*\mathcal{I}'$ both have $\mathcal{P}$ as well.

The final step of the proof is to note that any coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0 has a filtration (see Lemma 12.3 again) whose subquotients all have property $\mathcal{P}$ by what we just said. □

Lemma 12.6. *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
2. *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$Supp(\mathcal{G}) = Z$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,*
   (c) *$\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and*
   (d) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. According to Lemma 12.4 it suffices to show that given any integral closed subscheme $Z \subset X$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$. If this fails, then since X is Noetherian there is a minimal integral closed subscheme $Z_0 \subset X$ such that $\mathcal{P}$ fails for $(Z_0 \to X)_*\mathcal{I}$ for some quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{Z_0}$. In other words, the result does hold for any integral closed subscheme of Z. According to Lemma 12.5 this cannot happen. □

Lemma 12.7. *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves*
$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$
   if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
2. *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
3. *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*
4. *There exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$Supp(\mathcal{G}) = Z_0$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
   (c) *for every quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that $\mathcal{P}$ holds for $\mathcal{G}'$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. Note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from assumption (1).

As a first application we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$. Namely, such a sheaf has a filtration (see Lemma 12.3) whose subquotients have property $\mathcal{P}$ according to (3).

Let us denote $i : Z_0 \to X$ the closed immersion. Consider a coherent sheaf $\mathcal{G}$ as in (4). By Lemma 12.2 there exists a sheaf of ideals $\mathcal{I}$ on Z_0 and a short exact sequence

$$0 \to i_*\mathcal{I}^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is strictly contained in Z_0. In particular $r > 0$ and $\mathcal{I}$ is nonzero because the support of $\mathcal{G}$ is equal to Z_0. Let $\mathcal{I}' \subset \mathcal{I}$ be any nonzero quasi-coherent sheaf of ideals on Z_0 contained in $\mathcal{I}$. Then we also get a short exact sequence

$$0 \to i_*(\mathcal{I}')^{\oplus r} \to \mathcal{G} \to \mathcal{Q}' \to 0$$

where $\mathcal{Q}'$ has support properly contained in Z_0. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals cutting out the support of $\mathcal{Q}'$ (for example the ideal corresponding to the reduced induced closed subscheme structure on the support of $\mathcal{Q}'$). Then $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. By Lemma 10.2 we see that $\mathcal{J}^n\mathcal{Q}' = 0$ for some n. Hence $\mathcal{J}^n\mathcal{G} \subset i_*(\mathcal{I}')^{\oplus r}$. By assumption (4)(c) of the lemma we see there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}^n\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ for which property $\mathcal{P}$ holds. Hence we get a short exact sequence

$$0 \to \mathcal{G}' \to i_*(\mathcal{I}')^{\oplus r} \to \mathcal{Q}'' \to 0$$

where $\mathcal{Q}''$ has support properly contained in Z_0. Thus by our initial remarks and property (1) of the lemma we conclude that $i_*(\mathcal{I}')^{\oplus r}$ satisfies $\mathcal{P}$. Hence we see that $i_*\mathcal{I}'$ satisfies $\mathcal{P}$ by (2). Finally, for an arbitrary quasi-coherent sheaf of ideals $\mathcal{I}'' \subset \mathcal{O}_{Z_0}$ we can set $\mathcal{I}' = \mathcal{I}'' \cap \mathcal{I}$ and we get a short exact sequence

$$0 \to i_*(\mathcal{I}') \to i_*(\mathcal{I}'') \to \mathcal{Q}''' \to 0$$

where $\mathcal{Q}'''$ has support properly contained in Z_0. Hence we conclude that property $\mathcal{P}$ holds for $i_*\mathcal{I}''$ as well.

The final step of the proof is to note that any coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0 has a filtration (see Lemma 12.3 again) whose subquotients all have property $\mathcal{P}$ by what we just said. □

Lemma 12.8. *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves*
$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$
if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
2. *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
3. *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*

(a) *$Supp(\mathcal{G}) = Z$,*
(b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
(c) *for every quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that $\mathcal{P}$ holds for $\mathcal{G}'$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. Follows from Lemma 12.7 in exactly the same way that Lemma 12.6 follows from Lemma 12.5. □

13. Finite morphisms and affines

In this section we use the results of the preceding sections to show that the image of a Noetherian affine scheme under a finite morphism is affine. We will see later that this result holds more generally (see Limits, Lemma 10.1).

Lemma 13.1. *Let $f : Y \to X$ be a morphism of schemes. Assume f is finite, surjective and X locally Noetherian. Let $Z \subset X$ be an integral closed subscheme with generic point ξ. Then there exists a coherent sheaf $\mathcal{F}$ on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$.*

Proof. Note that Y is locally Noetherian by Morphisms, Lemma 16.6. Because f is surjective the fibre Y_ξ is not empty. Pick $\xi' \in Y$ mapping to ξ. Let $Z' = \overline{\{\xi'\}}$. We may think of $Z' \subset Y$ as a reduced closed subscheme, see Schemes, Lemma 12.4. Hence the sheaf $\mathcal{F} = (Z' \to Y)_*\mathcal{O}_{Z'}$ is a coherent sheaf on Y (see Lemma 9.9). Look at the commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

We see that $f_*\mathcal{F} = i_*f'_*\mathcal{O}_{Z'}$. Hence the stalk of $f_*\mathcal{F}$ at ξ is the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ. Note that since Z' is integral with generic point ξ' we have that ξ' is the only point of Z' lying over ξ, see Algebra, Lemmas 35.3 and 35.18. Hence the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ equal $\mathcal{O}_{Z',\xi'} = \kappa(\xi')$. In particular the stalk of $f_*\mathcal{F}$ at ξ is not zero. This combined with the fact that $f_*\mathcal{F}$ is of the form $i_*f'_*$(something) implies the lemma. □

Lemma 13.2. *Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X. If the morphism f is affine then $\mathcal{I}f_*\mathcal{F} = f_*(f^{-1}\mathcal{I}\mathcal{F})$.*

Proof. The notation means the following. Since f^{-1} is an exact functor we see that $f^{-1}\mathcal{I}$ is a sheaf of ideals of $f^{-1}\mathcal{O}_X$. Via the map $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ this acts on $\mathcal{F}$. Then $f^{-1}\mathcal{I}\mathcal{F}$ is the subsheaf generated by sums of local sections of the form as where a is a local section of $f^{-1}\mathcal{I}$ and s is a local section of $\mathcal{F}$. It is a quasi-coherent $\mathcal{O}_Y$-submodule of $\mathcal{F}$ because it is also the image of a natural map $f^*\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{F}$.

Having said this the proof is straightforward. Namely, the question is local and hence we may assume X is affine. Since f is affine we see that Y is affine too. Thus

we may write $Y = \text{Spec}(B)$, $X = \text{Spec}(A)$, $\mathcal{F} = \widetilde{M}$, and $\mathcal{I} = \widetilde{I}$. The assertion of the lemma in this case boils down to the statement that

$$I(M_A) = ((IB)M)_A$$

where M_A indicates the A-module associated to the B-module M. □

Lemma 13.3. *Let $f : Y \to X$ be a morphism of schemes. Assume*

1. *f finite,*
2. *f surjective,*
3. *Y affine, and*
4. *X Noetherian.*

Then X is affine.

Proof. We will prove that under the assumptions of the lemma for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. This will in particular imply that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals of $\mathcal{O}_X$. Then it follows that X is affine from either Lemma 3.1 or Lemma 3.2.

Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow H^1(X, \mathcal{F}) = 0.$$

We are going to apply Lemma 12.8. Thus we have to verify (1), (2) and (3) of that lemma for $\mathcal{P}$. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves. Property (2) follows since $H^1(X, -)$ is an additive functor. To see (3) let $Z \subset X$ be an integral closed subscheme with generic point ξ. Let $\mathcal{F}$ be a coherent sheaf on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$, see Lemma 13.1. We claim that taking $\mathcal{G} = f_*\mathcal{F}$ works. We only have to verify part (3)(c) of Lemma 12.8. Hence assume that $\mathcal{J} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. A finite morphism is affine hence by Lemma 13.2 we see that $\mathcal{J}\mathcal{G} = f_*(f^{-1}\mathcal{J}\mathcal{F})$. Also, as pointed out in the proof of Lemma 13.2 the sheaf $f^{-1}\mathcal{J}\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Since Y is affine we see that $H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$, see Lemma 2.2. Since f is finite, hence affine, we see that

$$H^1(X, \mathcal{J}\mathcal{G}) = H^1(X, f_*(f^{-1}\mathcal{J}\mathcal{F})) = H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$$

by Lemma 2.4. Hence the quasi-coherent subsheaf $\mathcal{G}' = \mathcal{J}\mathcal{G}$ satisfies $\mathcal{P}$. This verifies property (3)(c) of Lemma 12.8 as desired. □

14. Ample invertible sheaves and cohomology

Given a ringed space X, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, a section $s \in \Gamma(X, \mathcal{L})$ and an $\mathcal{O}_X$-module $\mathcal{F}$ we get a map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$, $t \mapsto t \otimes s$ which we call multiplication by s. We usually denote it $t \mapsto st$.

Lemma 14.1. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}' \subset \mathcal{F}$ be quasi-coherent $\mathcal{O}_X$-modules. Assume that*

1. *X is quasi-compact,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}'|_{X_s} = \mathcal{F}|_{X_s}$.*

Then there exists an $n \geq 0$ such that multiplication by s^n on $\mathcal{F}$ factors through $\mathcal{F}'$.

Proof. In other words we claim that $s^n\mathcal{F} \subset \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for some $n \geq 0$. If this is true for n_0 then it is true for all $n \geq n_0$. Hence it suffices to show there is a finite open covering such that the result holds for each of the members of this open covering. Since X is quasi-compact we may therefore assume that X is affine and that $\mathcal{L} \cong \mathcal{O}_X$. Thus the lemma translates into the following algebra problem (use Properties, Lemma 16.1): Let A be a ring. Let $f \in A$. Let $M' \subset M$ be A-modules. Assume M is a finite A-module, and assume that $(M')_f = M_f$. Then there exists an $n \geq 0$ such that $f^nM \subset M'$. The proof of this is omitted. □

Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Assume X quasi-compact and quasi-separated. The following lemma says roughly that the category of finitely presented $\mathcal{O}_{X_s}$-modules is the category of finitely presented $\mathcal{O}_X$-modules where the map multiplication by s has been inverted.

Lemma 14.2. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}$, $\mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. Let $\psi : \mathcal{F}|_{X_s} \to \mathcal{F}'|_{X_s}$ be a map of $\mathcal{O}_{X_s}$-modules. Assume that*

1. *X is quasi-compact and quasi-separated, and*
2. *$\mathcal{F}$ is of finitely presented.*

Then there exists an $n \geq 0$ and a morphism $\alpha : \mathcal{F} \to \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ whose restriction to X_s equals ψ via the identification $\mathcal{L}^{\otimes n}|_{X_s} = \mathcal{O}_{X_s}$ coming from s. Moreover, given a pair of solutions (n, α) and (n', α') there exists an $m \geq \max(n, n')$ such that $s^{m-n}\alpha = s^{m-n'}\alpha'$.

Proof. If the lemma holds for n_0 with map α_0 then it holds for all $n \geq n_0$ simply by taking $\alpha = s^{n-n_0}\alpha_0$. Choose a finite affine open covering $X = \bigcup U_i$ such that $\mathcal{L}|_{U_i}$ is trivial. Choose finite affine open coverings $U_i \cap U_{i'} = \bigcup U_{ii'j}$. Suppose we can prove the lemma when X is affine and $\mathcal{L}$ is trivial. Then we can find $n_i \geq 0$ $\alpha_i : \mathcal{F}|_{U_i} \to \mathcal{F}'|_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{L}^{\otimes n_i}|_{U_i}$ satisfying the relation over U_i. By the uniqueness assertion of the lemma, and the finiteness of the number of affines $U_{ii'j}$ we can find a single large integer m such that the maps $s^{m-n_i}\alpha_i$ and $s^{m-n_{i'}}\alpha_{i'}$ agree over $U_{ii'j}$ and hence over $U_i \cap U_{i'}$. Thus the morphisms $s^{m-n_i}\alpha_i$ glue to give our global map α. Proof of the uniqueness statement is omitted.

Assume X affine and that $\mathcal{L} \cong \mathcal{O}_X$. Then the lemma translates into the following algebra problem (use Properties, Lemma 16.2): Let A be a ring. Let $f \in A$. Let $\psi : M_f \to (M')_f$ be a map of A_f-modules. Assume M is a finitely presented A-module. Then there exists an $n \geq 0$ and an A-module map $\alpha : M \to M'$ such that $\alpha \otimes 1_{A_f} = f^n\psi$. Moreover, given any second solution (n', α') there exists an $m \geq \max(n, n')$ such that $f^{m-n}\alpha = f^{m-n'}\alpha'$. The proof of this algebraic fact is omitted. □

Cohomology is functorial. In particular, given a ringed space X, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, a section $s \in \Gamma(X, \mathcal{L})$ we get maps

$$H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}), \quad \xi \longmapsto s\xi$$

induced by the map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ which is multiplication by s. We set $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ as a graded ring, see Modules, Definition 21.4. Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and an integer $p \geq 0$ we set

$$H^p_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus_{n \in \mathbf{Z}} H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

This is a graded $\Gamma_*(X, \mathcal{L})$-module by the multiplication defined above. Warning: the notation $H^p_*(X, \mathcal{L}, \mathcal{F})$ is nonstandard.

Lemma 14.3. *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is quasi-compact and quasi-separated, the canonical map*

$$H^p_*(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow H^p(X_s, \mathcal{F})$$

which maps ξ/s^n to $s^{-n}\xi$ is an isomorphism.

Proof. Note that for $p = 0$ this is Properties, Lemma 24.6. We will prove the statement using the induction principle (Lemma 4.1) where for $U \subset X$ quasi-compact open we let $P(U)$ be the property: for all $p \geq 0$ the map

$$H^p_*(U, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow H^p(U_s, \mathcal{F})$$

is an isomorphism.

If U is affine, then both sides of the arrow displayed above are zero for $p > 0$ by Lemma 2.2 and Properties, Lemma 24.4 and the statement is true. If P is true for U, V, and $U \cap V$, then we can use the Mayer-Vietoris sequences (Cohomology, Lemma 9.2) to obtain a map of long exact sequences

$$\begin{array}{ccccc}
H^{p-1}_*(U \cap V, \mathcal{L}, \mathcal{F})_{(s)} & \longrightarrow & H^p_*(U \cup V, \mathcal{L}, \mathcal{F})_{(s)} & \longrightarrow & H^p_*(U, \mathcal{L}, \mathcal{F})_{(s)} \oplus H^p_*(V, \mathcal{L}, \mathcal{F})_{(s)} \\
\downarrow & & \downarrow & & \downarrow \\
H^{p-1}(U_s \cap V_s, \mathcal{F}) & \longrightarrow & H^p(U_s \cup V_s, \mathcal{F}) & \longrightarrow & H^p(U_s, \mathcal{F}) \oplus H^p(V_s, \mathcal{F})
\end{array}$$

(only a snippet shown). Observe that $U_s \cap V_s = (U \cap V)_s$ and that $U_s \cup V_s = (U \cup V)_s$. Thus the left and right vertical maps are isomorphisms (as well as one more to the right and one more to the left which are not shown in the diagram). We conclude that $P(U \cup V)$ holds by the 5-lemma (Homology, Lemma 5.20). This finishes the proof. □

Lemma 14.4. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Assume that*

1. *X is quasi-compact and quasi-separated, and*
2. *X_s is affine.*

Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every $p > 0$ and all $\xi \in H^p(X, \mathcal{F})$ there exists an $n \geq 0$ such that $s^n\xi = 0$ in $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$.

Proof. Recall that $H^p(X_s, \mathcal{G})$ is zero for every quasi-coherent module $\mathcal{G}$ by Lemma 2.2. Hence the lemma follows from Lemma 14.3. □

For a more general version of the following lemma see Limits, Lemma 10.4.

Lemma 14.5. *Let $i : Z \to X$ be a closed immersion of Noetherian schemes inducing a homeomorphism of underlying topological spaces. Let $\mathcal{L}$ be an invertible sheaf on X. If $i^*\mathcal{L}$ is ample on Z, then $\mathcal{L}$ is ample on X.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals cutting out the closed subscheme Z. Since $i(Z) = X$ set theoretically we see that $\mathcal{I}^n = 0$ for some n by Lemma 10.2. Consider the sequence

$$X = Z_n \supset Z_{n-1} \supset Z_{n-2} \supset \ldots \supset Z_1 = Z$$

of closed subschemes cut out by $0 = \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I}$. Then each of the closed immersions $Z_i \to Z_{i-1}$ is defined by a coherent sheaf of ideals of square zero. In this way we reduce to the case that $\mathcal{I}^2 = 0$.

Consider the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

of quasi-coherent $\mathcal{O}_X$-modules. Tensoring with $\mathcal{L}^{\otimes n}$ we obtain short exact sequences

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n} \to i_*i^*\mathcal{L}^{\otimes n} \to 0$$

As $\mathcal{I}^2 = 0$, we can use Morphisms, Lemma 4.1 to think of $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_Z$-module and then $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} = \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n}$ with obvious abuse of notation. Moreover, the cohomology of this sheaf over Z is canonically the same as the cohomology of this sheaf over X (as i is a homeomorphism).

Let $x \in X$ be a point and denote $z \in Z$ the corresponding point. Because $i^*\mathcal{L}$ is ample there exists an n and a section $s \in \Gamma(Z, i^*\mathcal{L}^{\otimes n})$ with $z \in Z_s$ and with Z_s affine. The obstruction to lifting s to a section of $\mathcal{L}^{\otimes n}$ over X is an element ξ of

$$H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = H^1(Z, \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n})$$

If we replace s by s^{e+1} then ξ is replaced by $s^e\xi$ in $H^1(Z, \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes(e+1)n})$. Computation omitted. By Lemma 14.4 we see that $s^e\xi$ is zero for e large enough. Hence, after replacing s by a power, we can assume s is the image of a section $s' \in \Gamma(X, \mathcal{L}^{\otimes n})$. Then X_s is an open subscheme and $Z_s \to X_s$ is a surjective closed immersion of Noetherian schemes with Z_s affine. Hence X_s is affine by Lemma 13.3 and we conclude that $\mathcal{L}$ is ample. □

15. Coherent sheaves and projective morphisms

It seems illuminating to formulate an all-in-one result for projective space over a Noetherian ring.

Lemma 15.1. *Let R be a Noetherian ring. Let $n \geq 0$ be an integer. For every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_R$ we have the following:*

1. *There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection*
$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j) \longrightarrow \mathcal{F}.$$
2. *We have $H^i(\mathbf{P}^n_R, \mathcal{F}) = 0$ unless $0 \leq i \leq n$.*
3. *For any i the cohomology group $H^i(\mathbf{P}^n_R, \mathcal{F})$ is a finite R-module.*
4. *If $i > 0$, then $H^i(\mathbf{P}^n_R, \mathcal{F}(d)) = 0$ for all d large enough.*
5. *For any $k \in \mathbf{Z}$ the graded $R[T_0, \ldots, T_n]$-module*
$$\bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$
is a finite $R[T_0, \ldots, T_n]$-module.

Proof. We will use that $\mathcal{O}_{\mathbf{P}^n_R}(1)$ is an ample invertible sheaf on the scheme $\mathbf{P}^n_R$. This follows directly from the definition since $\mathbf{P}^n_R$ covered by the standard affine opens $D_+(T_i)$. Hence by Properties, Proposition 24.14 every finite type quasi-coherent $\mathcal{O}_{\mathbf{P}^n_R}$-module is a quotient of a finite direct sum of tensor powers of $\mathcal{O}_{\mathbf{P}^n_R}(1)$. On the other hand a coherent sheaves and finite type quasi-coherent sheaves are the same thing on projective space over R by Lemma 9.1. Thus we see (1).

Projective n-space $\mathbf{P}^n_R$ is covered by $n+1$ affines, namely the standard opens $D_+(T_i)$, $i = 0, \ldots, n$, see Constructions, Lemma 13.3. Hence we see that for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_R$ we have $H^i(\mathbf{P}^n_R, \mathcal{F}) = 0$ for $i \geq n+1$, see Lemma 4.2. Hence (2) holds.

Let us prove (3) and (4) simultaneously for all coherent sheaves on $\mathbf{P}^n_R$ by descending induction on i. Clearly the result holds for $i \geq n+1$ by (2). Suppose we know the result for $i+1$ and we want to show the result for i. (If $i = 0$, then part (4) is vacuous.) Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_R$. Choose a surjection as in (1) and denote $\mathcal{G}$ the kernel so that we have a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j) \to \mathcal{F} \to 0$$

By Lemma 9.2 we see that $\mathcal{G}$ is coherent. The long exact cohomology sequence gives an exact sequence

$$H^i(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j)) \to H^i(\mathbf{P}^n_R, \mathcal{F}) \to H^{i+1}(\mathbf{P}^n_R, \mathcal{G}).$$

By induction assumption the right R-module is finite and by Lemma 8.1 the left R-module is finite. Since R is Noetherian it follows immediately that $H^i(\mathbf{P}^n_R, \mathcal{F})$ is a finite R-module. This proves the induction step for assertion (3). Since $\mathcal{O}_{\mathbf{P}^n_R}(d)$ is invertible we see that twisting on $\mathbf{P}^n_R$ is an exact functor (since you get it by tensoring with an invertible sheaf, see Constructions, Definition 10.1). This means that for all $d \in \mathbf{Z}$ the sequence

$$0 \to \mathcal{G}(d) \to \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d) \to \mathcal{F}(d) \to 0$$

is short exact. The resulting cohomology sequence is

$$H^i(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d)) \to H^i(\mathbf{P}^n_R, \mathcal{F}(d)) \to H^{i+1}(\mathbf{P}^n_R, \mathcal{G}(d)).$$

By induction assumption we see the module on the right is zero for $d \gg 0$ and by the computation in Lemma 8.1 the module on the left is zero as soon as $d \geq -\min\{d_j\}$ and $i \geq 1$. Hence the induction step for assertion (4). This concludes the proof of (3) and (4).

In order to prove (5) note that for all sufficiently large d the map

$$H^0(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d)) \to H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$

is surjective by the vanishing of $H^1(\mathbf{P}^n_R, \mathcal{G}(d))$ we just proved. In other words, the module

$$M_k = \bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$

is for k large enough a quotient of the corresponding module

$$N_k = \bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d))$$

When k is sufficiently small (e.g. $k < -d_j$ for all j) then

$$N_k = \bigoplus\nolimits_{j=1,\ldots,r} R[T_0, \ldots, T_n](d_j)$$

by our computations in Section 8. In particular it is finitely generated. Suppose $k \in \mathbf{Z}$ is arbitrary. Choose $k_- \ll k \ll k_+$. Consider the diagram

$$\begin{tikzcd} N_{k_-} & N_{k_+} \arrow[l] \arrow[d] \\ M_k & M_{k_+} \arrow[l] \end{tikzcd}$$

where the vertical arrow is the surjective map above and the horizontal arrows are the obvious inclusion maps. By what was said above we see that N_{k_-} is a finitely generated $R[T_0, \ldots, T_n]$-module. Hence N_{k_+} is a finitely generated $R[T_0, \ldots, T_n]$-module because it is a submodule of a finitely generated module and the ring $R[T_0, \ldots, T_n]$ is Noetherian. Since the vertical arrow is surjective we conclude that M_{k_+} is a finitely generated $R[T_0, \ldots, T_n]$-module. The quotient M_k/M_{k_+} is finite as an R-module since it is a finite direct sum of the finite R-modules $H^0(\mathbf{P}^n_R, \mathcal{F}(d))$ for $k \leq d < k_+$. Note that we use part (3) for $i = 0$ here. Hence M_k/M_{k_+} is a fortiori a finite $R[T_0, \ldots, T_n]$-module. In other words, we have sandwiched M_k between two finite $R[T_0, \ldots, T_n]$-modules and we win. □

Lemma 15.2. *Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Set $X = Proj(A)$. Then X is a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection*
$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_X(d_j) \longrightarrow \mathcal{F}.$$
(2) *For any i the cohomology group $H^i(X, \mathcal{F})$ is a finite A_0-module.*
(3) *If $i > 0$, then $H^i(X, \mathcal{F}(d)) = 0$ for all d large enough.*
(4) *For any $k \in \mathbf{Z}$ the graded A-module*
$$\bigoplus\nolimits_{d \geq k} H^0(X, \mathcal{F}(d))$$
is a finite A-module.

Proof. By assumption there exists a surjection of graded A_0-algebras
$$A_0[T_0, \ldots, T_n] \longrightarrow A$$
where $\deg(T_j) = 1$ for $j = 0, \ldots, n$. By Constructions, Lemma 11.5 this defines a closed immersion $r : X \to \mathbf{P}^n_{A_0}$ such that $r^*\mathcal{O}_{\mathbf{P}^n_{A_0}}(1) = \mathcal{O}_X(1)$. In particular, X is Noetherian as a closed subscheme of the Noetherian scheme $\mathbf{P}^n_{A_0}$. We claim that the results of the lemma for $\mathcal{F}$ follow from the corresponding results of Lemma 15.1 for the coherent sheaf $r_*\mathcal{F}$ (Lemma 9.8) on $\mathbf{P}^n_{A_0}$. For example, by this lemma there exists a surjection
$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_{A_0}}(d_j) \longrightarrow r_*\mathcal{F}.$$
Pulling back and using that $r^*r_*\mathcal{F} = \mathcal{F}$ (see lemma cited above) we obtain a corresponding surjection over X. The statements on cohomology follow from the fact that $H^i(X, \mathcal{F}(d)) = H^i(\mathbf{P}^n_{A_0}, r_*\mathcal{F}(d))$ for example by Lemma 2.4. Some details omitted. □

Lemma 15.3. *Let A be a graded ring. Let M be a graded A-module. Assume*

(1) *A_0 is Noetherian,*

(2) *A is generated by finitely many elements of A_1 over A_0, and*
(3) *M is a finitely generated A-module.*

Set $X = Proj(A)$ and let $\widetilde{M}$ be the quasi-coherent $\mathcal{O}_X$-module on X associated to M. The maps

$$M_n \longrightarrow \Gamma(X, \widetilde{M}(n))$$

from Constructions, Lemma 10.3 are isomorphisms for all sufficiently large n.

Proof. Because M is a finite A-module we see that $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module, i.e., a coherent $\mathcal{O}_X$-module. Set $N = \bigoplus_{n \geq 0} \Gamma(X, \widetilde{M}(n))$. We have to show that the map $M \to N$ of graded A-modules is an isomorphism in all sufficiently large degrees. By Properties, Lemma 26.5 we have a canonical isomorphism $\widetilde{N} \to \widetilde{M}$ such that $M_n \to N_n = \Gamma(X, \widetilde{M}(n))$ is the canonical map. Let $K = \text{Ker}(M \to N)$ and $Q = \text{Coker}(M \to N)$. Recall that the functor $M \mapsto \widetilde{M}$ is exact, see Constructions, Lemma 8.4. Hence we see that $\widetilde{K} = 0$ and $\widetilde{Q} = 0$. On the other hand, A is a Noetherian ring and M and N are finitely generated A-modules (for N this follows from the last part of Lemma 15.2). Hence K and Q are finite A-modules. Thus it suffices to show that a finite A-module K with $\widetilde{K} = 0$ has only finitely many nonzero homogeneous parts K_d. To do this, let $x_1, \ldots, x_r \in K$ be homogeneous generators say sitting in degrees $d_1, \ldots, d_r$. Let $f_1, \ldots, f_n \in A_1$ be elements generating A over A_0. For each i and j there exists an $n_{ij} \geq 0$ such that $f_i^{n_{ij}} x_j = 0$ in $K_{d_j + n_{ij}}$: if not then $K_{(f_i)}$ would not be zero, i.e., $\widetilde{K}$ would not be zero. Then we see that K_d is zero for $d > \max_j(d_j + \sum_i n_{ij})$ as every element of K_d is a sum of terms where each term is a monomials in the f_i times one of the x_j of total degree d. □

Lemma 15.4. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

(1) *S is Noetherian,*
(2) *f is proper,*
(3) *$\mathcal{F}$ is coherent, and*
(4) *$\mathcal{L}$ is relatively ample on X/S.*

Then there exists an n_0 such that for all $n \geq n_0$ we have

$$R^p f_* \left(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \right) = 0$$

for all $p > 0$.

Proof. A proper morphism is of finite type by definition. By Morphisms, Lemma 40.7 there exists an open covering $S = \bigcup V_j$ and immersions $i_j : X_j \to \mathbf{P}^{n_j}_{V_j}$, where $X_j = f^{-1}(V_j)$ such that $i_j^* \mathcal{O}(1)$ is a power of $\mathcal{L}$. Since S is quasi-compact we may assume the covering is finite. Clearly, if we solve the question for each of the finitely many systems $(X_j \to V_j, \mathcal{L}|_{X_j}, \mathcal{F}|_{V_j})$ then the result follows. Hence we may assume there exists an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} = i^* \mathcal{O}(1)$ for some $d \geq 1$.

Repeating the argument above with a finite affine open covering of S we see that we may also assume that S is affine. In this case the vanishing of $R^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is equivalent to the vanishing of $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$, see Lemma 4.5.

Since f is proper we see that i is a closed immersion (Morphisms, Lemma 42.7). Hence we see that $R^p i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$ for all $p \geq 1$ (see Lemma 9.9 for example).

This implies that

$$H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^p(\mathbf{P}^n_S, i_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}))$$

by the Leray spectral sequence (Cohomology, Lemma 14.4). Moreover, by the projection formula (Cohomology, Lemma 8.2) we have

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \langle n \rangle_d}) \otimes_{\mathcal{O}_{\mathbf{P}^n_S}} \mathcal{O}(\lfloor n/d \rfloor)$$

for all $n \in \mathbf{Z}$ where $\langle n \rangle_d \in \{0, 1, \ldots, d-1\}$ is the unique element congruent to n module d. The sheaves $\mathcal{F}_j = i_*(\mathcal{F} \otimes \mathcal{L}^{\otimes j})$, $j \in \{0, 1, \ldots, d-1\}$ are coherent by Lemma 9.9. Thus we see that for all n large enough the cohomology groups $H^p(\mathbf{P}^n_S, \mathcal{F}_j(n))$ vanish by Lemma 15.1. Putting everything together this implies the lemma. □

16. Chow's Lemma

In this section we prove Chow's lemma in the Noetherian case (Lemma 16.1). In Limits, Section 11 we prove some variants for the non-Noetherian case.

Lemma 16.1. *Let S be a Noetherian scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exists an $n \geq 0$ and a diagram*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, we may arrange it such that there exists a dense open subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism.

Proof. All of the schemes we will encounter during the rest of the proof are going to be of finite type over the Noetherian scheme S and hence Noetherian (see Morphisms, Lemma 16.6). All morphisms between them will automatically be quasi-compact, locally of finite type and quasi-separated, see Morphisms, Lemma 16.8 and Properties, Lemmas 5.4 and 5.6.

The underlying topological space of X is Noetherian (see Properties, Lemma 5.5) and we conclude that X has only finitely many irreducible components (see Topology, Lemma 8.2). Say $X = X_1 \cup \ldots \cup X_r$ is the decomposition of X into irreducible components. Let $\eta_i \in X_i$ be the generic point. For every point $x \in X$ there exists an affine open $U_x \subset X$ which contains x and each of the generic points η_i. See Properties, Lemma 27.4. Since X is quasi-compact, we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that each U_i contains $\eta_1, \ldots, \eta_r$. In particular we conclude that the open $U = U_1 \cap \ldots \cap U_m \subset X$ is a dense open. This and the fact that the U_i are affine opens covering X is all that we will use below.

Let $X^* \subset X$ be the scheme theoretic closure of $U \to X$, see Morphisms, Definition 6.2. Let $U_i^* = X^* \cap U_i$. Note that U_i^* is a closed subscheme of U_i. Hence U_i^* is affine. Since U is dense in X the morphism $X^* \to X$ is a surjective closed immersion. It is an isomorphism over U. Hence we may replace X by X^* and U_i by U_i^* and assume that U is scheme theoretically dense in X, see Morphisms, Definition 7.1.

By Morphisms, Lemma 40.3 we can find an immersion $j_i : U_i \to \mathbf{P}^{n_i}_S$ for each i. By Morphisms, Lemma 7.7 we can find closed subschemes $Z_i \subset \mathbf{P}^{n_i}_S$ such that

$j_i : U_i \to Z_i$ is a scheme theoretically dense open immersion. Note that $Z_i \to S$ is proper, see Morphisms, Lemma 43.5. Consider the morphism

$$j = (j_1|_U, \ldots, j_n|_U) : U \longrightarrow \mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_n}_S.$$

By the lemma cited above we can find a closed subscheme Z of $\mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_n}_S$ such that $j : U \to Z$ is an open immersion and such that U is scheme theoretically dense in Z. The morphism $Z \to S$ is proper. Consider the ith projection

$$\mathrm{pr}_i|_Z : Z \longrightarrow \mathbf{P}^{n_i}_S.$$

This morphism factors through Z_i (see Morphisms, Lemma 6.6). Denote $p_i : Z \to Z_i$ the induced morphism. This is a proper morphism, see Morphisms, Lemma 42.7 for example. At this point we have that $U \subset U_i \subset Z_i$ are scheme theoretically dense open immersions. Moreover, we can think of Z as the scheme theoretic image of the "diagonal" morphism $U \to Z_1 \times_S \ldots \times_S Z_n$.

Set $V_i = p_i^{-1}(U_i)$. Note that $p_i|_{V_i} : V_i \to U_i$ is proper. Set $X' = V_1 \cup \ldots \cup V_n$. By construction X' has an immersion into the scheme $\mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_n}_S$. Thus by the Segre embedding (see Morphisms, Lemma 43.6) we see that X' has an immersion into a projective space over S.

We claim that the morphisms $p_i|_{V_i} : V_i \to U_i$ glue to a morphism $X' \to X$. Namely, it is clear that $p_i|_U$ is the identity map from U to U. Since $U \subset X'$ is scheme theoretically dense by construction, it is also scheme theoretically dense in the open subscheme $V_i \cap V_j$. Thus we see that $p_i|_{V_i \cap V_j} = p_j|_{V_i \cap V_j}$ as morphisms into the separated S-scheme X, see Morphisms, Lemma 7.10. We denote the resulting morphism $\pi : X' \to X$.

We claim that $\pi^{-1}(U_i) = V_i$. Since $\pi|_{V_i} = p_i|_{V_i}$ it follows that $V_i \subset \pi^{-1}(U_i)$. Consider the diagram

$$\begin{tikzcd} V_i \arrow[r] \arrow[dr, "p_i|_{V_i}"'] & \pi^{-1}(U_i) \arrow[d] \\ & U_i \end{tikzcd}$$

Since $V_i \to U_i$ is proper we see that the image of the horizontal arrow is closed, see Morphisms, Lemma 42.7. Since $V_i \subset \pi^{-1}(U_i)$ is scheme theoretically dense (as it contains U) we conclude that $V_i = \pi^{-1}(U_i)$ as claimed.

This shows that $\pi^{-1}(U_i) \to U_i$ is identified with the proper morphism $p_i|_{V_i} : V_i \to U_i$. Hence we see that X has a finite affine covering $X = \bigcup U_i$ such that the restriction of π is proper on each member of the covering. Thus by Morphisms, Lemma 42.3 we see that π is proper.

Finally we have to show that $\pi^{-1}(U) = U$. To see this we argue in the same way as above using the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & \pi^{-1}(U) \arrow[d] \\ & U \end{tikzcd}$$

and using that $\mathrm{id}_U : U \to U$ is proper and that U is scheme theoretically dense in $\pi^{-1}(U)$. $\square$

Remark 16.2. In the situation of Chow's Lemma 16.1:

(1) The morphism π is actually H-projective (hence projective, see Morphisms, Lemma 43.3) since the morphism $X' \to \mathbf{P}^n_S \times_S X = \mathbf{P}^n_X$ is a closed immersion (use the fact that π is proper, see Morphisms, Lemma 42.7).
(2) We may assume that $\pi^{-1}(U)$ is scheme theoretically dense in X'. Namely, we can simply replace X' by the scheme theoretic closure of $\pi^{-1}(U)$. In this case we can think of U as a scheme theoretically dense open subscheme of X'. See Morphisms, Section 6.
(3) If X is reduced then we may choose X' reduced. This is clear from (2).

17. Higher direct images of coherent sheaves

In this section we prove the fundamental fact that the higher direct images of a coherent sheaf under a proper morphism are coherent.

Lemma 17.1. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally projective morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^if_*\mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$.*

Proof. We first remark that a locally projective morphism is proper (Morphisms, Lemma 43.5) and hence of finite type. In particular X is locally Noetherian (Morphisms, Lemma 16.6) and hence the statement makes sense. Moreover, by Lemma 4.4 the sheaves $R^pf_*\mathcal{F}$ are quasi-coherent.

Having said this the statement is local on S (for example by Cohomology, Lemma 7.4). Hence we may assume $S = \text{Spec}(R)$ is the spectrum of a Noetherian ring, and X is a closed subscheme of $\mathbf{P}^n_R$ for some n, see Morphisms, Lemma 43.4. In this case, the sheaves $R^pf_*\mathcal{F}$ are the quasi-coherent sheaves associated to the R-modules $H^p(X, \mathcal{F})$, see Lemma 4.5. Hence it suffices to show that R-modules $H^p(X, \mathcal{F})$ are finite R-modules (Lemma 9.1). Denote $i : X \to \mathbf{P}^n_R$ the closed immersion. Note that $R^pi_*\mathcal{F} = 0$ by Lemma 9.9. Hence the Leray spectral sequence (Cohomology, Lemma 14.4) for $i : X \to \mathbf{P}^n_R$ degenerates, and we see that $H^p(X, \mathcal{F}) = H^p(\mathbf{P}^n_R, i_*\mathcal{F})$. Since the sheaf $i_*\mathcal{F}$ is coherent by Lemma 9.9 we see that the lemma follows from Lemma 15.1. □

Here is the general statement.

Proposition 17.2. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^if_*\mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$.*

Proof. Since the problem is local on S we may assume that S is a Noetherian scheme. Since a proper morphism is of finite type we see that in this case X is a Noetherian scheme also. Consider the property $\mathcal{P}$ of coherent sheaves on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow R^pf_*\mathcal{F} \text{ is coherent for all } p \geq 0$$

We are going to use the result of Lemma 12.6 to prove that $\mathcal{P}$ holds for every coherent sheaf on X.

Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of higher direct images

$$R^{p-1}f_*\mathcal{F}_3 \to R^pf_*\mathcal{F}_1 \to R^pf_*\mathcal{F}_2 \to R^pf_*\mathcal{F}_3 \to R^{p+1}f_*\mathcal{F}_1$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then the higher direct images of the third are sandwiched in this exact complex between two coherent sheaves. Hence these higher direct images are also coherent by Lemma 9.2 and 9.3. Hence property $\mathcal{P}$ holds for the third as well.

Let $Z \subset X$ be an integral closed subscheme. We have to find a coherent sheaf $\mathcal{F}$ on X whose support is contained in Z, whose stalk at the generic point ξ of Z is a 1-dimensional vector space over $\kappa(\xi)$ such that $\mathcal{P}$ holds for $\mathcal{F}$. Denote $g = f|_Z : Z \to S$ the restriction of f. Suppose we can find a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{G}_\xi$ is a 1-dimensional vector space over $\kappa(\xi)$, (b) $R^pg_*\mathcal{G} = 0$ for $p > 0$, and (c) $g_*\mathcal{G}$ is coherent. Then we can consider $\mathcal{F} = (Z \to X)_*\mathcal{G}$. As $Z \to X$ is a closed immersion we see that $(Z \to X)_*\mathcal{G}$ is coherent on X and $R^p(Z \to X)_*\mathcal{G} = 0$ for $p > 0$ (Lemma 9.9). Hence by the relative Leray spectral sequence (Cohomology, Lemma 14.8) we will have $R^pf_*\mathcal{F} = R^pg_*\mathcal{G} = 0$ for $p > 0$ and $f_*\mathcal{F} = g_*\mathcal{G}$ is coherent. Finally $\mathcal{F}_\xi = ((Z \to X)_*\mathcal{G})_\xi = \mathcal{G}_\xi$ which verifies the condition on the stalk at ξ. Hence everything depends on finding a coherent sheaf $\mathcal{G}$ on Z which has properties (a), (b), and (c).

We can apply Chow's Lemma 16.1 to the morphism $Z \to S$. Thus we get a diagram

$$\begin{tikzcd} Z \arrow[dr, "g"'] & Z' \arrow[l, "\pi"'] \arrow[r, "i"] \arrow[d, "g'"] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

as in the statement of Chow's lemma. Also, let $U \subset Z$ be the dense open subscheme such that $\pi^{-1}(U) \to U$ is an isomorphism. By the discussion in Remark 16.2 we see that $i' = (i, \pi) : Z' \to \mathbf{P}^n_Z$ is a closed immersion. Hence

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_S}(1) \cong (i')^*\mathcal{O}_{\mathbf{P}^n_Z}(1)$$

is g'-relatively ample and π-relatively ample (for example by Morphisms, Lemma 40.7). Hence by Lemma 15.4 there exists an $n \geq 0$ such that both $R^p\pi_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$ and $R^p(g')_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$. Property (a) holds because $\pi_*\mathcal{L}^{\otimes}|_U$ is an invertible sheaf (as $\pi^{-1}(U) \to U$ is an isomorphism). Properties (b) and (c) hold because by the relative Leray spectral sequence (Cohomology, Lemma 14.8) we have

$$E_2^{p,q} = R^pg_*R^q\pi_*\mathcal{L}^{\otimes n} \Rightarrow R^{p+q}(g')_*\mathcal{L}^{\otimes n}$$

and by choice of n the only nonzero terms in $E_2^{p,q}$ are those with $q = 0$ and the only nonzero terms of $R^{p+q}(g')_*\mathcal{L}^{\otimes n}$ are those with $p = q = 0$. This implies that $R^pg_*\mathcal{G} = 0$ for $p > 0$ and that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$. Finally, applying the previous Lemma 17.1 we see that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$ is coherent as desired. □

Remark 17.3. Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Then X is locally Noetherian (Morphisms, Lemma 16.6). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume the scheme theoretic support Z of $\mathcal{F}$ is proper over S. we claim $R^pf_*\mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $p \geq 0$. Namely, Let $i : Z \to X$ be the closed immersion and write $\mathcal{F} = i_*\mathcal{G}$ for some coherent module $\mathcal{G}$ on Z (Lemma 9.7). Denoting $g : Z \to S$ the composition $f \circ i$ we see that $R^pg_*\mathcal{G}$ is coherent on S by Proposition 17.2. On the other hand, $R^qi_*\mathcal{G} = 0$ for $q > 0$ (Lemma 9.9). By Cohomology, Lemma 14.8 we get $R^pf_*\mathcal{F} = R^pg_*\mathcal{G}$ and the claim.

Lemma 17.4. *Let $S = \mathrm{Spec}(A)$ with A a Noetherian ring. Let $f : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is finite A-module for all $i \geq 0$.*

Proof. This is just the affine case of Proposition 17.2. Namely, by Lemmas 4.4 and 4.5 we know that $R^if_*\mathcal{F}$ is the quasi-coherent sheaf associated to the A-module $H^i(X, \mathcal{F})$ and by Lemma 9.1 this is a coherent sheaf if and only if $H^i(X, \mathcal{F})$ is an A-module of finite type. □

Lemma 17.5. *Let A be a Noetherian ring. Let B be a finitely generated graded A-algebra. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Set $\mathcal{B} = f^*\widetilde{B}$. Let $\mathcal{F}$ be a quasi-coherent graded $\mathcal{B}$-module of finite type.*

1. *For every $p \geq 0$ the graded B-module $H^p(X, \mathcal{F})$ is a finite B-module.*
2. *If $\mathcal{L}$ is an ample invertible $\mathcal{O}_X$-module, then there exists an integer d_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$ and $d \geq d_0$.*

Proof. To prove this we consider the fibre product diagram

$$\begin{tikzcd} X' = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X \arrow[r, "\pi"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

Note that f' is a proper morphism, see Morphisms, Lemma 42.5. Also, B is a finitely generated A-algebra, and hence Noetherian (Algebra, Lemma 30.1). This implies that X' is a Noetherian scheme (Morphisms, Lemma 16.6). Note that X' is the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ by Constructions, Lemma 4.6. Since $\mathcal{F}$ is a quasi-coherent $\mathcal{B}$-module we see that there is a unique quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$ such that $\pi_*\mathcal{F}' = \mathcal{F}$, see Morphisms, Lemma 13.6 Since $\mathcal{F}$ is finite type as a $\mathcal{B}$-module we conclude that $\mathcal{F}'$ is a finite type $\mathcal{O}_{X'}$-module (details omitted). In other words, $\mathcal{F}'$ is a coherent $\mathcal{O}_{X'}$-module (Lemma 9.1). Since the morphism $\pi : X' \to X$ is affine we have

$$H^p(X, \mathcal{F}) = H^p(X', \mathcal{F}')$$

by Lemma 2.4. Thus (1) follows from Lemma 17.4. Given $\mathcal{L}$ as in (2) we set $\mathcal{L}' = \pi^*\mathcal{L}$. Note that $\mathcal{L}'$ is ample on X' by Morphisms, Lemma 38.7. By the projection formula (Cohomology, Lemma 8.2) we have $\pi_*(\mathcal{F}' \otimes \mathcal{L}') = \mathcal{F} \otimes \mathcal{L}$. Thus part (2) follows by the same reasoning as above from Lemma 15.4. □

18. The theorem on formal functions

In this section we study the behaviour of cohomology of sequences of sheaves either of the form $\{I^n\mathcal{F}\}_{n\geq 0}$ or of the form $\{\mathcal{F}/I^n\mathcal{F}\}_{n\geq 0}$ as n varies.

Here and below we use the following notation. Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$ we denote $\mathcal{I}^n\mathcal{F}$ the quasi-coherent subsheaf generated by products of local sections of $f^{-1}(\mathcal{I}^n)$ and $\mathcal{F}$. In a formula

$$\mathcal{I}^n\mathcal{F} = \mathrm{Im}\left(f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}\right).$$

Note that there are natural maps

$$f^{-1}(\mathcal{I}^n) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{I}^m\mathcal{F} \longrightarrow f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{I}^m\mathcal{F} \longrightarrow \mathcal{I}^{n+m}\mathcal{F}$$

Hence a section of $\mathcal{I}^n$ will give rise to a map $R^pf_*(\mathcal{I}^m\mathcal{F}) \to R^pf_*(\mathcal{I}^{n+m}\mathcal{F})$ by functoriality of higher direct images. Localizing and then sheafifying we see that there are $\mathcal{O}_Y$-module maps

$$\mathcal{I}^n \otimes_{\mathcal{O}_Y} R^pf_*(\mathcal{I}^m\mathcal{F}) \longrightarrow R^pf_*(\mathcal{I}^{n+m}\mathcal{F}).$$

In other words we see that $\bigoplus_{n\geq 0} R^pf_*(\mathcal{I}^n\mathcal{F})$ is a graded $\bigoplus_{n\geq 0}\mathcal{I}^n$-module.

If $Y = \text{Spec}(A)$ and $\mathcal{I} = \widetilde{I}$ we denote $\mathcal{I}^n\mathcal{F}$ simply $I^n\mathcal{F}$. The maps introduced above give $M = \bigoplus H^p(X, I^n\mathcal{F})$ the structure of a graded $S = \bigoplus I^n$-module. If f is proper, A is Noetherian and $\mathcal{F}$ is coherent, then this turns out to be a module of finite type.

Lemma 18.1. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Set $B = \bigoplus_{n\geq 0} I^n$. Let $f : X \to \text{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then for every $p \geq 0$ the graded B-module $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is a finite B-module.*

Proof. Let $\mathcal{B} = \bigoplus I^n\mathcal{O}_X = f^*\widetilde{B}$. Then $\bigoplus I^n\mathcal{F}$ is a finite type graded $\mathcal{B}$-module. Hence the result follows from Lemma 17.5 part (1). □

Lemma 18.2. *Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$. Assume Y locally Noetherian, f proper, and $\mathcal{F}$ coherent. Then*

$$\mathcal{M} = \bigoplus\nolimits_{n\geq 0} R^pf_*(\mathcal{I}^n\mathcal{F})$$

is a graded $\mathcal{A} = \bigoplus_{n\geq 0}\mathcal{I}^n$-module which is quasi-coherent and of finite type.

Proof. The statement is local on Y, hence this reduces to the case where Y is affine. In the affine case the result follows from Lemma 18.1. Details omitted. □

Lemma 18.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \text{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then for every $p \geq 0$ there exists an integer $c \geq 0$ such that*

1. *the multiplication map $I^{n-c} \otimes H^p(X, I^c\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is surjective for all $n \geq c$, and*
2. *the image of $H^p(X, I^{n+m}\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is contained in the submodule $I^{m-c}H^p(X, I^n\mathcal{F})$ for all $n \geq 0$, $m \geq c$.*

Proof. By Lemma 18.1 we can find $d_1, \ldots, d_t \geq 0$, and $x_i \in H^p(X, I^{d_i}\mathcal{F})$ such that $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is generated by $x_1, \ldots, x_t$ over $S = \bigoplus_{n\geq 0} I^n$. Take $c = \max\{d_i\}$. It is clear that (1) holds. For (2) let $b = \max(0, n - c)$. Consider the commutative diagram of A-modules

$$\begin{tikzcd}
I^{n+m-c-b} \otimes I^b \otimes H^p(X, I^c\mathcal{F}) \arrow[r] \arrow[d] & I^{n+m-c} \otimes H^p(X, I^c\mathcal{F}) \arrow[r] & H^p(X, I^{n+m}\mathcal{F}) \arrow[d] \\
I^{n+m-c-b} \otimes H^p(X, I^n\mathcal{F}) \arrow[rr] & & H^p(X, I^n\mathcal{F})
\end{tikzcd}$$

By part (1) of the lemma the composition of the horizontal arrows is surjective if $n + m \geq c$. On the other hand, it is clear that $n + m - c - b \geq m - c$. Hence part (2). □

In the situation of Lemmas 18.1 and 18.3 consider the inverse system

$$\mathcal{F}/I\mathcal{F} \leftarrow \mathcal{F}/I^2\mathcal{F} \leftarrow \mathcal{F}/I^3\mathcal{F} \leftarrow \ldots$$

We would like to know what happens to the cohomology groups. Here is a first result.

Lemma 18.4. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix $p \geq 0$.*

(1) *There exists a $c_1 \geq 0$ such that for all $n \geq c_1$ we have*

$$Ker(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) \subset I^{n-c_1}H^p(X, \mathcal{F}).$$

(2) *The inverse system*

$$(H^p(X, \mathcal{F}/I^n\mathcal{F}))_{n \in \mathbf{N}}$$

satisfies the Mittag-Leffler condition (see Homology, Definition 27.2).

(3) *In fact for any p and n there exists a $c_2(n) \geq n$ such that*

$$Im(H^p(X, \mathcal{F}/I^k\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = Im(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$

for all $k \geq c_2(n)$.

Proof. Let $c_1 = \max\{c_p, c_{p+1}\}$, where c_p, c_{p+1} are the integers found in Lemma 18.3 for H^p and H^{p+1}. We will use this constant in the proofs of (1), (2) and (3).

Let us prove part (1). Consider the short exact sequence

$$0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$

From the long exact cohomology sequence we see that

$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, I^n\mathcal{F}) \to H^p(X, \mathcal{F}))$$

Hence by our choice of c_1 we see that this is contained in $I^{n-c_1}H^p(X, \mathcal{F})$ for $n \geq c_1$.

Note that part (3) implies part (2) by definition of the Mittag-Leffler condition.

Let us prove part (3). Fix an n throughout the rest of the proof. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I^n\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}/I^n\mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & I^{n+m}\mathcal{F} \arrow[r] \arrow[u] & \mathcal{F} \arrow[r] \arrow[u] & \mathcal{F}/I^{n+m}\mathcal{F} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

This gives rise to the following commutative diagram

$$\begin{tikzcd}
H^p(X, I^n\mathcal{F}) \arrow[r] & H^p(X, \mathcal{F}) \arrow[r] & H^p(X, \mathcal{F}/I^n\mathcal{F}) \arrow[r, "\delta"'] & H^{p+1}(X, I^n\mathcal{F}) \\
H^p(X, I^{n+m}\mathcal{F}) \arrow[r] \arrow[u] & H^p(X, \mathcal{F}) \arrow[r] \arrow[u, "1"] & H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) \arrow[r] \arrow[u] & H^{p+1}(X, I^{n+m}\mathcal{F}) \arrow[u, "a"]
\end{tikzcd}$$

If $m \geq c_1$ we see that the image of a is contained in $I^{m-c_1}H^{p+1}(X, I^n\mathcal{F})$. By the Artin-Rees lemma (see Algebra, Lemma 49.3) there exists an integer $c_3(n)$ such that

$$I^N H^{p+1}(X, I^n\mathcal{F}) \cap \mathrm{Im}(\delta) \subset \delta\left(I^{N-c_3(n)}H^p(X, \mathcal{F}/I^n\mathcal{F})\right)$$

for all $N \geq c_3(n)$. As $H^p(X, \mathcal{F}/I^n\mathcal{F})$ is annihilated by I^n, we see that if $m \geq c_3(n) + c_1 + n$, then

$$\mathrm{Im}(H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$

In other words, part (3) holds with $c_2(n) = c_3(n) + c_1 + n$. □

Theorem 18.5 (Theorem on formal functions)**.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix $p \geq 0$. The system of maps*

$$H^p(X, \mathcal{F})/I^nH^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}/I^n\mathcal{F})$$

define an isomorphism of limits

$$H^p(X, \mathcal{F})^\wedge \longrightarrow \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

where the left hand side is the completion of the A-module $H^p(X, \mathcal{F})$ with respect to the ideal I, see Algebra, Section 93. Moreover, this is in fact a homeomorphism for the limit topologies.

Proof. In fact, this follows immediately from Lemma 18.4. We spell out the details. Set $M = H^p(X, \mathcal{F})$ and $M_n = H^p(X, \mathcal{F}/I^n\mathcal{F})$. Denote $N_n = \mathrm{Im}(M \to M_n)$. By the description of the limit in Homology, Section 27 we have

$$\lim_n M_n = \{(x_n) \in \prod M_n \mid \varphi_i(x_n) = x_{n-1},\ n = 2, 3, \ldots\}$$

Pick an element $x = (x_n) \in \lim_n M_n$. By Lemma 18.4 part (3) we have $x_n \in N_n$ for all n since by definition x_n is the image of some $x_{n+m} \in M_{n+m}$ for all m. By Lemma 18.4 part (1) we see that there exists a factorization

$$M \to N_n \to M/I^{n-c_1}M$$

of the reduction map. Denote $y_n \in M/I^{n-c_1}M$ the image of x_n for $n \geq c_1$. Since for $n' \geq n$ the composition $M \to M_{n'} \to M_n$ is the given map $M \to M_n$ we see that $y_{n'}$ maps to y_n under the canonical map $M/I^{n'-c_1}M \to M/I^{n-c_1}M$. Hence $y = (y_{n+c_1})$ defines an element of $\lim_n M/I^nM$. We omit the verification that y maps to x under the map

$$M^\wedge = \lim_n M/I^nM \longrightarrow \lim_n M_n$$

of the lemma. We also omit the verification on topologies. □

Lemma 18.6. *Let A be a ring. Let $I \subset A$ be an ideal. Assume A is Noetherian and complete with respect to I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then*

$$H^p(X, \mathcal{F}) = \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

for all $p \geq 0$.

Proof. This is a reformulation of the theorem on formal functions (Theorem 18.5) in the case of a complete Noetherian base ring. Namely, in this case the A-module $H^p(X, \mathcal{F})$ is finite (Lemma 17.4) hence I-adically complete (Algebra, Lemma 93.2) and we see that completion on the left hand side is not necessary. □

Lemma 18.7. *Given a morphism of schemes $f : X \to Y$ and a quasi-coherent sheaf $\mathcal{F}$ on X. Assume*

(1) *Y locally Noetherian,*
(2) *f proper, and*

(3) *$\mathcal{F}$ coherent.*

Let $y \in Y$ be a point. Consider the infinitesimal neighbourhoods

$$\begin{tikzcd} X_n = \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \times_Y X \arrow[r, "i_n"'] \arrow[d, "f_n"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \arrow[r, "c_n"] & Y \end{tikzcd}$$

of the fibre $X_1 = X_y$ and set $\mathcal{F}_n = i_n^*\mathcal{F}$. Then we have

$$(R^pf_*\mathcal{F})_y^\wedge \cong \lim_n H^p(X_n, \mathcal{F}_n)$$

as $\mathcal{O}_{Y,y}^\wedge$-modules.

Proof. This is just a reformulation of a special case of the theorem on formal functions, Theorem 18.5. Let us spell it out. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$, see Schemes, Equation (13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^*R^pf_*\mathcal{F} = R^pf'_*\mathcal{F}'$ by Lemma 5.2. Moreover, the infinitesimal neighbourhoods of the fibre X_y and X'_y are identified (verification omitted; hint: the morphisms c_n factor through c).

Hence we may assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a Noetherian local ring A with maximal ideal $\mathfrak{m}$ and that $y \in Y$ corresponds to the closed point (i.e., to $\mathfrak{m}$). In particular it follows that

$$(R^pf_*\mathcal{F})_y = \Gamma(Y, R^pf_*\mathcal{F}) = H^p(X, \mathcal{F}).$$

In this case also, the morphisms c_n are each closed immersions. Hence their base changes i_n are closed immersions as well. Note that $i_{n,*}\mathcal{F}_n = i_{n,*}i_n^*\mathcal{F} = \mathcal{F}/\mathfrak{m}^n\mathcal{F}$. By the Leray spectral sequence for i_n, and Lemma 9.9 we see that

$$H^p(X_n, \mathcal{F}_n) = H^p(X, i_{n,*}\mathcal{F}) = H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$$

Hence we may indeed apply the theorem on formal functions to compute the limit in the statement of the lemma and we win. □

Here is a lemma which we will generalize later to fibres of dimension > 0, namely the next lemma.

Lemma 18.8. *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume*

(1) *Y locally Noetherian,*
(2) *f is proper, and*
(3) *$f^{-1}(\{y\})$ is finite.*

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^pf_*\mathcal{F})_y = 0$ for all $p > 0$.

Proof. The fibre X_y is finite, and by Morphisms, Lemma 21.7 it is a finite discrete space. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same. Hence each of the schemes X_n is affine according to Schemes, Lemma 11.7. Hence it follows that $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > 0$. Hence we see that $(R^pf_*\mathcal{F})_y^\wedge = 0$ by Lemma 18.7. Note that $R^pf_*\mathcal{F}$ is coherent by Proposition 17.2 and hence $R^pf_*\mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$-module. By Algebra, Lemma 93.2 this implies that $(R^pf_*\mathcal{F})_y = 0$. □

Lemma 18.9. *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume*

1. *Y locally Noetherian,*
2. *f is proper, and*
3. *$\dim(X_y) = d$.*

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^pf_*\mathcal{F})_y = 0$ for all $p > d$.

Proof. The fibre X_y is of finite type over $\mathrm{Spec}(\kappa(y))$. Hence X_y is a Noetherian scheme by Morphisms, Lemma 16.6. Hence the underlying topological space of X_y is Noetherian, see Properties, Lemma 5.5. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same as that of X_y. Hence $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > d$ by Cohomology, Proposition 21.6. Hence we see that $(R^pf_*\mathcal{F})^\wedge_y = 0$ by Lemma 18.7 for $p > d$. Note that $R^pf_*\mathcal{F}$ is coherent by Proposition 17.2 and hence $R^pf_*\mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$-module. By Algebra, Lemma 93.2 this implies that $(R^pf_*\mathcal{F})_y = 0$. □

19. Applications of the theorem on formal functions

We will add more here as needed. For the moment we need the following characterization of finite morphisms (in the Noetherian case – for a more general version see the chapter More on Morphisms, Section 31).

Lemma 19.1. *(For a more general version see More on Morphisms, Lemma 31.5). Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian. The following are equivalent*

1. *f is finite, and*
2. *f is proper with finite fibres.*

Proof. A finite morphism is proper according to Morphisms, Lemma 44.10. A finite morphism is quasi-finite according to Morphisms, Lemma 44.9. A quasi-finite morphism has finite fibres, see Morphisms, Lemma 21.10. Hence a finite morphism is proper and has finite fibres.

Assume f is proper with finite fibres. We want to show f is finite. In fact it suffices to prove f is affine. Namely, if f is affine, then it follows that f is integral by Morphisms, Lemma 44.7 whereupon it follows from Morphisms, Lemma 44.4 that f is finite.

To show that f is affine we may assume that S is affine, and our goal is to show that X is affine too. Since f is proper we see that X is separated and quasi-compact. Hence we may use the criterion of Lemma 3.2 to prove that X is affine. To see this let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type ideal sheaf. In particular $\mathcal{I}$ is a coherent sheaf on X. By Lemma 18.8 we conclude that $R^1f_*\mathcal{I}_s = 0$ for all $s \in S$. In other words, $R^1f_*\mathcal{I} = 0$. Hence we see from the Leray Spectral Sequence for f that $H^1(X, \mathcal{I}) = H^1(S, f_*\mathcal{I})$. Since S is affine, and $f_*\mathcal{I}$ is quasi-coherent (Schemes, Lemma 24.1) we conclude $H^1(S, f_*\mathcal{I}) = 0$ from Lemma 2.2 as desired. Hence $H^1(X, \mathcal{I}) = 0$ as desired. □

As a consequence we have the following useful result.

Lemma 19.2. *(For a more general version see More on Morphisms, Lemma 31.6). Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume*

1. *S is locally Noetherian,*
2. *f is proper, and*

(3) $f^{-1}(\{s\})$ *is a finite set.*

Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 21.7. By Morphisms, Lemma 49.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 21.9), hence has finite fibres (Morphisms, Lemma 21.10), hence is finite by Lemma 19.1. □

20. Cohomology and base change, III

In this section we prove the simplest case of a very general phenomenon that will be discussed in Derived Categories of Schemes, Section 16. Please see Remark 20.2 for a translation of the following lemma into algebra.

Lemma 20.1. *Let A be a Noetherian ring and set $S = \mathrm{Spec}(A)$. Let $f : X \to S$ be a proper morphism of schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over S. Then*

(1) *$R\Gamma(X, \mathcal{F})$ is a perfect object of $D(A)$, and*
(2) *for any ring map $A \to A'$ the base change map*
$$R\Gamma(X, \mathcal{F}) \otimes_A^{\mathbf{L}} A' \longrightarrow R\Gamma(X_{A'}, \mathcal{F}_{A'})$$
is an isomorphism.

Proof. Choose a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. By Lemmas 7.1 and 7.2 the Čech complex $K^\bullet = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$ satisfies
$$K^\bullet \otimes_A A' = R\Gamma(X_{A'}, \mathcal{F}_{A'})$$
for all ring maps $A \to A'$. Let $K^\bullet_{alt} = \check{C}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ be the alternating Čech complex. By Cohomology, Lemma 24.6 there is a homotopy equivalence $K^\bullet_{alt} \to K^\bullet$ of A-modules. In particular, we have
$$K^\bullet_{alt} \otimes_A A' = R\Gamma(X_{A'}, \mathcal{F}_{A'})$$
as well. Since $\mathcal{F}$ is flat over A we see that each K^n_{alt} is flat over A (see Morphisms, Lemma 26.2). Since moreover $K^\bullet_{alt}$ is bounded above (this is why we switched to the alternating Čech complex) $K^\bullet_{alt} \otimes_A A' = K^\bullet_{alt} \otimes_A^{\mathbf{L}} A'$ by the definition of derived tensor products (see More on Algebra, Section 45). By Lemma 17.4 the cohomology groups $H^i(K^\bullet_{alt})$ are finite A-modules. As $K^\bullet_{alt}$ is bounded, we conclude that $K^\bullet_{alt}$ is pseudo-coherent, see More on Algebra, Lemma 50.16. Given any A-module M set $A' = A \oplus M$ where M is a square zero ideal, i.e., $(a, m) \cdot (a', m') = (aa', am' + a'm)$. By the above we see that $K^\bullet_{alt} \otimes_A^{\mathbf{L}} A'$ has cohomology in degrees $0, \ldots, n$. Hence $K^\bullet_{alt} \otimes_A^{\mathbf{L}} M$ has cohomology in degrees $0, \ldots, n$. Hence $K^\bullet_{alt}$ has finite Tor dimension, see More on Algebra, Definition 51.1. We win by More on Algebra, Lemma 56.2. □

Remark 20.2. A consequence of Lemma 20.1 is that there exists a finite complex of finite projective A-modules $M^\bullet$ such that we have
$$H^i(X_{A'}, \mathcal{F}_{A'}) = H^i(M^\bullet \otimes_A A')$$
functorially in A'. The condition that $\mathcal{F}$ is flat over A is essential, see [Har98].

21. Grothendieck's existence theorem, I

In this section we discuss Grothendieck's existence theorem for the projective case. As we do not yet have the theory of formal schemes to our disposal, we temporarily develop a bit of language that replaces the notion of a "coherent module on a Noetherian adic formal scheme". The reader who is familiar with formal schemes is encouraged to read the statement and proof of the theorem in [DG67].

Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Below we will consider inverse systems $(\mathcal{F}_n)$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_n$ is annihilated by $\mathcal{I}^n$, and
(2) the transition maps induce isomorphisms $\mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1} \to \mathcal{F}_n$.

A morphism of such inverse systems is defined as usual. Let us denote the category of these inverse systems with $Coh(X, \mathcal{I})$. We are going to proceed by proving a bunch of lemmas about objects in this category. In fact, most of the lemmas that follow are straightforward consequences of the following description of the category in the affine case.

Lemma 21.1. *If $X = \mathrm{Spec}(A)$ is the spectrum of a Noetherian ring and $\mathcal{I}$ is the quasi-coherent sheaf of ideals associated to the ideal $I \subset A$, then $Coh(X, \mathcal{I})$ is equivalent to the category of finite $A^\wedge$-modules where $A^\wedge$ is the completion of A with respect to I.*

Proof. Let $\mathrm{Mod}_{A,I}^{fg}$ be the category of inverse systems (M_n) of finite A-modules satisfying: (1) M_n is annihilated by I^n and (2) $M_{n+1}/I^nM_{n+1} = M_n$. By the correspondence between coherent sheaves on X and finite A-modules (Lemma 9.1) it suffices to show $\mathrm{Mod}_{A,I}^{fg}$ is equivalent to the category of finite $A^\wedge$-modules. To see this it suffices to prove that given an object (M_n) of $\mathrm{Mod}_{A,I}^{fg}$ the module

$$M = \lim M_n$$

is a finite $A^\wedge$-module and that $M/I^nM = M_n$. As the transition maps are surjective, we see that $M \to M_1$ is surjective. Pick $x_1, \ldots, x_t \in M$ which map to generators of M_1. This induces a map of systems $(A/I^n)^{\oplus t} \to M_n$. By Nakayama's lemma (Algebra, Lemma 19.1) these maps are surjective. Let $K_n \subset (A/I^n)^{\oplus t}$ be the kernel. Property (2) implies that $K_{n+1} \to K_n$ is surjective, in particular the system (K_n) satisfies the Mittag-Leffler condition. By Homology, Lemma 27.3 we obtain an exact sequence $0 \to K \to (A^\wedge)^{\oplus t} \to M \to 0$ with $K = \lim K_n$. Hence M is a finite $A^\wedge$-module. As $K \to K_n$ is surjective it follows that

$$M/I^nM = \mathrm{Coker}(K \to (A/I^n)^{\oplus t}) = (A/I^n)^{\oplus t}/K_n = M_n$$

as desired. □

Lemma 21.2. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.*

(1) *The category $Coh(X, \mathcal{I})$ is abelian.*
(2) *For $U \subset X$ open the restriction functor $Coh(X, \mathcal{I}) \to Coh(U, \mathcal{I}|_U)$ is exact.*
(3) *Exactness in $Coh(X, \mathcal{I})$ may be checked by restricting to the members of an open covering of X.*

Proof. Let $\alpha = (\alpha_n) : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $Coh(X, \mathcal{I})$. The cokernel of α is the inverse system $(\text{Coker}(\alpha_n))$ (details omitted). To describe the kernel let

$$\mathcal{K}'_{l,m} = \text{Im}(\text{Ker}(\alpha_l) \to \mathcal{F}_m)$$

for $l \geq m$. We claim:

(a) the inverse system $(\mathcal{K}'_{l,m})_{l \geq m}$ is eventually constant, say with value $\mathcal{K}'_m$,
(b) the system $(\mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m)_{m \geq n}$ is eventually constant, say with value $\mathcal{K}_n$,
(c) the system $(\mathcal{K}_n)$ forms an object of $Coh(X, \mathcal{I})$, and
(d) this object is the kernel of α.

To see (a), (b), and (c) we may work affine locally, say $X = \text{Spec}(A)$ and $\mathcal{I}$ corresponds to the ideal $I \subset A$. By Lemma 21.1 α corresponds to a map $f : M \to N$ of finite $A^\wedge$-modules. Denote $K = \text{Ker}(f)$. Note that $A^\wedge$ is a Noetherian ring (Algebra, Lemma 93.10). Choose an integer $c \geq 0$ such that $K \cap I^nM \subset I^{n-c}K$ for $n \geq c$ (Algebra, Lemma 49.2) and which satisfies Algebra, Lemma 49.3 for the map f and the ideal $I^\wedge = IA^\wedge$. Then $\mathcal{K}'_{l,m}$ corresponds to the A-module

$$K'_{l,m} = \frac{a^{-1}(I^lN) + I^mM}{I^mM} = \frac{K + I^{l-c}f^{-1}(I^cN) + I^mM}{I^mM} = \frac{K + I^mM}{I^mM}$$

where the last equality holds if $l \geq m + c$. So $\mathcal{K}'_m$ corresponds to the A-module $K/K \cap I^mM$ and $\mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m$ corresponds to

$$\frac{K}{K \cap I^mM + I^nK} = \frac{K}{I^nK}$$

for $m \geq n + c$ by our choice of c above. Hence $\mathcal{K}_n$ corresponds to K/I^nK.

We prove (d). It is clear from the description on affines above that the composition $(\mathcal{K}_n) \to (\mathcal{F}_n) \to (\mathcal{G}_n)$ is zero. Let $\beta : (\mathcal{H}_n) \to (\mathcal{F}_n)$ be a morphism such that $\alpha \circ \beta = 0$. Then $\mathcal{H}_l \to \mathcal{F}_l$ maps into $\text{Ker}(\alpha_l)$. Since $\mathcal{H}_m = \mathcal{H}_l/\mathcal{I}^m\mathcal{H}_l$ for $l \geq m$ we obtain a system of maps $\mathcal{H}_m \to \mathcal{K}'_{l,m}$. Thus a map $\mathcal{H}_m \to \mathcal{K}'_m$. Since $\mathcal{H}_n = \mathcal{H}_m/\mathcal{I}^n\mathcal{H}_m$ we obtain a system of maps $\mathcal{H}_n \to \mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m$ and hence a map $\mathcal{H}_n \to \mathcal{K}_n$ as desired.

To finish the proof of (1) we still have to show that $\text{Coim} = \text{Im}$ in $Coh(X, \mathcal{I})$. We have seen above that taking kernels and cokernels commutes, over affines, with the description of $Coh(X, \mathcal{I})$ as a category of modules. Since $\text{Im} = \text{Coim}$ holds in the category of modules this gives $\text{Coim} = \text{Im}$ in $Coh(X, \mathcal{I})$. Parts (2) and (3) of the lemma are immediate from our construction of kernels and cokernels. □

Lemma 21.3. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. A map $(\mathcal{F}_n) \to (\mathcal{G}_n)$ is surjective in $Coh(X, \mathcal{I})$ if and only if $\mathcal{F}_1 \to \mathcal{G}_1$ is surjective.*

Proof. Omitted. Hint: Look on affine opens, use Lemma 21.1, and use Algebra, Lemma 19.1. □

Lemma 21.4. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If $(\mathcal{F}_n)$ is an object of $Coh(X, \mathcal{I})$ then $\bigoplus Ker(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ is a finite type, graded, quasi-coherent $\bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}$-module.*

Proof. The question is local on X hence we may assume X is affine, i.e., we have a situation as in Lemma 21.1. In this case, if $(\mathcal{F}_n)$ corresponds to the finite $A^\wedge$ module M, then $\bigoplus \text{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ corresponds to $\bigoplus I^nM/I^{n+1}M$ which is clearly a finite module over $\bigoplus I^n/I^{n+1}$. □

Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There is a functor

$$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge \tag{21.4.1}$$

which associates to the coherent $\mathcal{O}_X$-module $\mathcal{F}$ the object $\mathcal{F}^\wedge = (\mathcal{F}/\mathcal{I}^n\mathcal{F})$ of $Coh(X, \mathcal{I})$.

Lemma 21.5. *The functor (21.4.1) is exact.*

Proof. It suffices to check this locally on X. Hence we may assume X is affine, i.e., we have a situation as in Lemma 21.1. The functor is the functor $\text{Mod}_A^{fg} \to \text{Mod}_{A^\wedge}^{fg}$ which associates to a finite A-module M the completion $M^\wedge$. Thus the result follows from Algebra, Lemma 93.3. □

Lemma 21.6. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$. Then*

$$\lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \text{Mor}_{Coh(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge).$$

Proof. To prove this we may work affine locally on X. Hence we may assume $X = \text{Spec}(A)$ and $\mathcal{F}$, $\mathcal{G}$ given by finite A-module M and N. Then $\mathcal{H}$ corresponds to the finite A-module $H = \text{Hom}_A(M, N)$. The statement of the lemma becomes the statement

$$H^\wedge = \text{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$$

via the equivalence of Lemma 21.1. By Algebra, Lemma 93.3 (used 3 times) we have

$$H^\wedge = \text{Hom}_A(M, N) \otimes_A A^\wedge = \text{Hom}_{A^\wedge}(M \otimes_A A^\wedge, N \otimes_A A^\wedge) = \text{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$$

where the second equality uses that $A^\wedge$ is flat over A (see More on Algebra, Remark 50.18). The lemma follows. □

Lemma 21.7. *Let A be Noetherian ring complete with respect to an ideal I. Let $f : X \to \text{Spec}(A)$ be a proper morphism. Let $\mathcal{I} = I\mathcal{O}_X$. Then the functor (21.4.1) is fully faithful.*

Proof. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Then $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is a coherent $\mathcal{O}_X$-module, see Modules, Lemma 19.4. By Lemma 21.6 the map

$$\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) \to \text{Mor}_{Coh(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective. Hence fully faithfulness of (21.4.1) follows from the theorem on formal functions (Lemma 18.6) for the coherent sheaf $\mathcal{H}$. □

Lemma 21.8. *Let A be Noetherian ring and $I \subset A$ and ideal. Let $f : X \to \text{Spec}(A)$ be a proper morphism and let $\mathcal{L}$ be an f-ample invertible sheaf. Let $\mathcal{I} = I\mathcal{O}_X$. Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. Then there exists an integer d_0 such that*

$$H^1(X, Ker(\mathcal{F}_{n+1} \to \mathcal{F}_n) \otimes \mathcal{L}^{\otimes d}) = 0$$

for all $n \geq 0$ and all $d \geq d_0$.

Proof. Set $B = \bigoplus I^n/I^{n+1}$ and $\mathcal{B} = \bigoplus \mathcal{I}^n/\mathcal{I}^{n+1} = f^*\widetilde{B}$. By Lemma 21.4 the graded quasi-coherent $\mathcal{B}$-module $\mathcal{G} = \bigoplus \text{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ is of finite type. Hence the lemma follows from Lemma 17.5 part (2). □

Lemma 21.9. *Let A be Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a projective morphism. Let $\mathcal{I} = I\mathcal{O}_X$. Then the functor (21.4.1) is an equivalence.*

Proof. We have already seen that (21.4.1) is fully faithful in Lemma 21.7. Thus it suffices to show that the functor is essentially surjective.

We first show that every object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ is the quotient of an object in the image of (21.4.1). Let $\mathcal{L}$ be an f-ample invertible sheaf on X. Choose d_0 as in Lemma 21.8. Choose a $d \geq d_0$ such that $\mathcal{F}_1 \otimes \mathcal{L}^{\otimes d}$ is globally generated by some sections $s_{1,1}, \ldots, s_{t,1}$. Since the transition maps of the system

$$H^0(X, \mathcal{F}_{n+1} \otimes \mathcal{L}^{\otimes d}) \longrightarrow H^0(X, \mathcal{F}_n \otimes \mathcal{L}^{\otimes d})$$

are surjective by the vanishing of H^1 we can lift $s_{1,1}, \ldots, s_{t,1}$ to a compatible system of global sections $s_{1,n}, \ldots, s_{t,n}$ of $\mathcal{F}_n \otimes \mathcal{L}^{\otimes d}$. These determine a compatible system of maps

$$(s_{1,n}, \ldots, s_{t,n}) : (\mathcal{L}^{\otimes -d})^{\oplus t} \longrightarrow \mathcal{F}_n$$

Using Lemma 21.3 we deduce that we have a surjective map

$$\big((\mathcal{L}^{\otimes -d})^{\oplus t}\big)^\wedge \longrightarrow (\mathcal{F}_n)$$

as desired.

The result of the previous paragraph and the fact that $Coh(X, \mathcal{I})$ is abelian (Lemma 21.2) implies that every object of $Coh(X, \mathcal{I})$ is a cokernel of a map between objects coming from $Coh(\mathcal{O}_X)$. As (21.4.1) is fully faithful and exact by Lemmas 21.7 and 21.5 we conclude. □

22. Grothendieck's existence theorem, II

In this section we discuss Grothendieck's existence theorem. Before we give the statement and proof, we need to develop a bit more theory regarding the categories $Coh(X, \mathcal{I})$ introduced in Section 21.

Lemma 22.1. *Let $f : X \to Y$ be a morphism of Noetherian schemes. Let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals and set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Then there is a right exact functor*

$$f^* : Coh(Y, \mathcal{J}) \longrightarrow Coh(X, \mathcal{I})$$

which sends $(\mathcal{G}_n)$ to $(f^*\mathcal{G}_n)$. If f is flat, then f^* is an exact functor.

Proof. Since $f^* : Coh(\mathcal{O}_Y) \to Coh(\mathcal{O}_X)$ is right exact we have

$$f^*\mathcal{G}_n = f^*(\mathcal{G}_{n+1}/\mathcal{I}^n\mathcal{G}_{n+1}) = f^*\mathcal{G}_{n+1}/f^{-1}\mathcal{I}^n f^*\mathcal{G}_{n+1} = f^*\mathcal{G}_{n+1}/\mathcal{J}^n f^*\mathcal{G}_{n+1}$$

hence the pullback of a system is a system. The construction of cokernels in the proof of Lemma 21.2 shows that $f^* : Coh(Y, \mathcal{J}) \to Coh(X, \mathcal{I})$ is always right exact. If f is flat, then $f^* : Coh(\mathcal{O}_Y) \to Coh(\mathcal{O}_X)$ is an exact functor. It follows from the construction of kernels in the proof of Lemma 21.2 that in this case $f^* : Coh(Y, \mathcal{J}) \to Coh(X, \mathcal{I})$ also transforms kernels into kernels. □

Remark 22.2. Let X be a Noetherian scheme and let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $Coh(X, \mathcal{I})$. Given an affine open $\mathrm{Spec}(A) = U \subset X$ with $\mathcal{I}|_U, \mathcal{K}|_U$ corresponding to ideals $I, K \subset A$ denote $\alpha_U : M \to N$ of finite $A^\wedge$-modules which corresponds to $\alpha|_U$ via Lemma 21.1. We claim the following are equivalent

(1) there exists an integer $t \geq 1$ such that $\mathrm{Ker}(\alpha_n)$ and $\mathrm{Coker}(\alpha_n)$ are annihilated by $\mathcal{K}^t$ for all $n \geq 1$,
(2) for any affine open $\mathrm{Spec}(A) = U \subset X$ as above the modules $\mathrm{Ker}(\alpha_U)$ and $\mathrm{Coker}(\alpha_U)$ are annihilated by K^t for some integer $t \geq 1$, and
(3) there exists a finite affine open covering $X = \bigcup U_i$ such that the conclusion of (2) holds for α_{U_i}.

If these equivalent conditions hold we will say that α is a *map whose kernel and cokernel are annihilated by a power of* $\mathcal{K}$. To see the equivalence we use the following commutative algebra fact: suppose given an exact sequence

$$0 \to T \to M \to N \to Q \to 0$$

of A-modules with T and Q annihilated by K^t for some ideal $K \subset A$. Then for every $f, g \in K^t$ there exists a canonical map "fg" $: N \to M$ such that $M \to N \to M$ is equal to multiplication by fg. Namely, for $y \in N$ we can pick $x \in M$ mapping to fy in N and then we can set "fg"$(y) = gx$. Thus it is clear that $\mathrm{Ker}(M/JM \to N/JN)$ and $\mathrm{Coker}(M/JM \to N/JN)$ are annihilated by K^{2t} for any ideal $J \subset A$.

Applying the commutative algebra fact to α_{U_i} and $J = I^n$ we see that (3) implies (1). Conversely, suppose (1) holds and $M \to N$ is equal to α_U. Then there is a $t \geq 1$ such that $\mathrm{Ker}(M/I^nM \to N/I^nN)$ and $\mathrm{Coker}(M/I^nM \to N/I^nN)$ are annihilated by K^t for all n. We obtain maps "fg" $: N/I^nN \to M/I^nM$ which in the limit induce a map $N \to M$ as N and M are I-adically complete. Since the composition with $N \to M \to N$ is multiplication by fg we conclude that fg annihilates T and Q. In other words T and Q are annihilated by K^{2t} as desired.

Lemma 22.3. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module, $(\mathcal{F}_n)$ an object of $Coh(X, \mathcal{I})$, and $\alpha : (\mathcal{F}_n) \to \mathcal{G}^\wedge$ a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$. Then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where*

(1) *$\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,*
(2) *$a : \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,*
(3) *$\beta : (\mathcal{F}_n) \to \mathcal{F}^\wedge$ is an isomorphism, and*
(4) *$\alpha = a^\wedge \circ \beta$.*

Proof. The uniqueness implies it suffices to construct $(\mathcal{F}, a, \beta)$ Zariski locally on X. Thus we may assume $X = \mathrm{Spec}(A)$ and $\mathcal{I}$ corresponds to the ideal $I \subset A$. In this situation Lemma 21.1 applies. Let M' be the finite $A^\wedge$-module corresponding to $(\mathcal{F}_n)$. Let N be the finite A-module corresponding to $\mathcal{G}$. Then α corresponds to a map

$$\varphi : M' \longrightarrow N^\wedge$$

whose kernel and cokernel are annihilated by I^t for some t. Recall that $N^\wedge = N \otimes_A A^\wedge$ (Algebra, Lemma 93.2). By More on Algebra, Lemma 63.16 there is an A-module map $\psi : M \to N$ whose kernel and cokernel are I-power torsion and an isomorphism $M \otimes_A A^\wedge = M'$ compatible with φ. As N and M' are finite modules, we conclude that M is a finite A-module, see More on Algebra, Remark 63.19. Hence $M \otimes_A A^\wedge = M^\wedge$. We omit the verification that the triple $(M, N \to M, M^\wedge \to M')$ so obtained is unique up to unique isomorphism. $\square$

Lemma 22.4. *Let X be a Noetherian scheme. Let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $X_e \subset X$ be the closed subscheme cut out by $\mathcal{K}^e$. Let $\mathcal{I}_e = \mathcal{I}\mathcal{O}_{X_e}$. Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. Assume*

1. *the functor $Coh(\mathcal{O}_{X_e}) \to Coh(X_e, \mathcal{I}_e)$ is an equivalence for all $e \geq 1$, and*
2. *there exists a coherent sheaf $\mathcal{H}$ on X and a map $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.*

Then $(\mathcal{F}_n)$ is in the essential image of (21.4.1).

Proof. During this proof we will use without further mention that for a closed immersion $i : Z \to X$ the functor i_* gives an equivalence between the category of coherent modules on Z and coherent modules on X annihilated by the ideal sheaf of Z, see Lemma 9.8. In particular we may identify $Coh(\mathcal{O}_{X_e})$ with the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{K}^e$ and $Coh(X_e, \mathcal{I}_e)$ as the full subcategory of $Coh(X, \mathcal{I})$ of objects annihilated by $\mathcal{K}^e$. Moreover (1) tells us these two categories are equivalent under the completion functor (21.4.1).

Applying this equivalence we get a coherent $\mathcal{O}_X$-module $\mathcal{G}_e$ annihilated by $\mathcal{K}^e$ corresponding to the system $(\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$. The maps $\mathcal{F}_n/\mathcal{K}^{e+1}\mathcal{F}_n \to \mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n$ correspond to canonical maps $\mathcal{G}_{e+1} \to \mathcal{G}_e$ which induce isomorphisms $\mathcal{G}_{e+1}/\mathcal{K}^e\mathcal{G}_{e+1} \to \mathcal{G}_e$. Hence $(\mathcal{G}_e)$ is an object of $Coh(X, \mathcal{K})$. The map α induces a system of maps

$$\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \longrightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$$

whence maps $\mathcal{G}_e \to \mathcal{H}/\mathcal{K}^e\mathcal{H}$ (by the equivalence of categories again). Let $t \geq 1$ be an integer, which exists by assumption (2), such that $\mathcal{K}^t$ annihilates the kernel and cokernel of all the maps $\mathcal{F}_n \to \mathcal{H}/\mathcal{I}^n\mathcal{H}$. Then $\mathcal{K}^{2t}$ annihilates the kernel and cokernel of the maps $\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \to \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$, see Remark 22.2. Whereupon we conclude that $\mathcal{K}^{4t}$ annihilates the kernel and the cokernel of the maps

$$\mathcal{G}_e \longrightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H},$$

see Remark 22.2. We apply Lemma 22.3 to obtain a coherent $\mathcal{O}_X$-module $\mathcal{F}$, a map $a : \mathcal{F} \to \mathcal{H}$ and an isomorphism $\beta : (\mathcal{G}_e) \to (\mathcal{F}/\mathcal{K}^e\mathcal{F})$ in $Coh(X, \mathcal{K})$. Working backwards, for a given n the triple $(\mathcal{F}/\mathcal{I}^n\mathcal{F}, a \bmod \mathcal{I}^n, \beta \bmod \mathcal{I}^n)$ is a triple as in the lemma for the morphism $\alpha_n \bmod \mathcal{K}^e : (\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n) \to (\mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H})$ of $Coh(X, \mathcal{K})$. Thus the uniqueness in Lemma 22.3 gives a canonical isomorphism $\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$ compatible with all the morphisms in sight. This finishes the proof of the lemma. □

Lemma 22.5. *Let Y be a Noetherian scheme. Let $\mathcal{J}, \mathcal{K} \subset \mathcal{O}_Y$ be quasi-coherent sheaves of ideals. Let $f : X \to Y$ be a proper morphism which is an isomorphism over $V = Y \setminus V(\mathcal{K})$. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Let $(\mathcal{G}_n)$ be an object of $Coh(Y, \mathcal{J})$, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and let $\beta : (f^*\mathcal{G}_n) \to \mathcal{F}^\wedge$ be an isomorphism in $Coh(X, \mathcal{I})$. Then there exists a map*

$$\alpha : (\mathcal{G}_n) \longrightarrow (f_*\mathcal{F})^\wedge$$

in $Coh(Y, \mathcal{J})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Proof. Since f is a proper morphism we see that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module (Proposition 17.2). Thus the statement of the lemma makes sense. Consider the compositions

$$\gamma_n : \mathcal{G}_n \to f_*f^*\mathcal{G}_n \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F}).$$

Here the first map is the adjunction map and the second is $f_*\beta_n$. We claim that there exists a unique α as in the lemma such that the compositions

$$\mathcal{G}_n \xrightarrow{\alpha_n} f_*\mathcal{F}/\mathcal{J}^n f_*\mathcal{F} \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F})$$

equal γ_n for all n. Because of the uniqueness we may assume that $Y = \text{Spec}(B)$ is affine. Let $J \subset B$ corresponds to the ideal $\mathcal{J}$. Set

$$M_n = H^0(X, \mathcal{F}/\mathcal{I}^n\mathcal{F}) \quad \text{and} \quad M = H^0(X, \mathcal{F})$$

By Lemma 18.4 and Theorem 18.5 the inverse limit of the modules M_n equals the completion $M^\wedge = \lim M/J^nM$. Set $N_n = H^0(Y, \mathcal{G}_n)$ and $N = \lim N_n$. Via the equivalence of categories of Lemma 21.1 the finite $B^\wedge$ modules N and $M^\wedge$ correspond to $(\mathcal{G}_n)$ and $f_*\mathcal{F}^\wedge$. It follows from this that α has to be the morphism of $Coh(Y, \mathcal{J})$ corresponding to the homomorphism

$$\lim \gamma_n : N = \lim_n N_n \longrightarrow \lim M_n = M^\wedge$$

of finite $B^\wedge$-modules.

We still have to show that the kernel and cokernel of α are annihilated by a power of $\mathcal{K}$. Set $Y' = \text{Spec}(B^\wedge)$ and $X' = Y' \times_Y X$. Let $\mathcal{K}'$, $\mathcal{J}'$, $\mathcal{G}'_n$ and $\mathcal{I}'$, $\mathcal{F}'$ be the pullback of $\mathcal{K}$, $\mathcal{J}$, $\mathcal{G}_n$ and $\mathcal{I}$, $\mathcal{F}$, to Y' and X'. The projection morphism $f' : X' \to Y'$ is the base change of f by $Y' \to Y$. Note that $Y' \to Y$ is a flat morphism of schemes as $B \to B^\wedge$ is flat by Algebra, Lemma 93.3. Hence $f'_*\mathcal{F}'$, resp. $f'_*(f')^*\mathcal{G}'_n$ is the pullback of $f_*\mathcal{F}$, resp. $f_*f^*\mathcal{G}_n$ to Y' by Lemma 5.2. The uniqueness of our construction shows the pullback of α to Y' is the corresponding map α' constructed for the situation on Y'. Moreover, to check that the kernel and cokernel of α are annihilated by $\mathcal{K}^t$ it suffices to check that the kernel and cokernel of α' are annihilated by $(\mathcal{K}')^t$. Namely, to see this we need to check this for kernels and cokernels of the maps α_n and α'_n (see Remark 22.2) and the ring map $B \to B^\wedge$ induces an equivalence of categories between modules annihilated by J^n and $(J')^n$, see More on Algebra, Lemma 63.3. Thus we may assume B is complete with respect to J.

Assume $Y = \text{Spec}(B)$ is affine, $\mathcal{J}$ corresponds to the ideal $J \subset B$, and B is complete with respect to J. In this case $(\mathcal{G}_n)$ is in the essential image of the functor $Coh(\mathcal{O}_Y) \to Coh(Y, \mathcal{J})$. Say $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module such that $(\mathcal{G}_n) = \mathcal{G}^\wedge$. Note that $f^*(\mathcal{G}^\wedge) = (f^*\mathcal{G})^\wedge$. Hence Lemma 21.7 tells us that β comes from an isomorphism $b : f^*\mathcal{G} \to \mathcal{F}$ and α is the completion functor applied to

$$\mathcal{G} \to f_*f^*\mathcal{G} \cong f_*\mathcal{F}$$

Hence we are trying to verify that the kernel and cokernel of the adjunction map $c : \mathcal{G} \to f_*f^*\mathcal{G}$ are annihilated by a power of $\mathcal{K}$. However, since the restriction $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an isomorphism we see that $c|_V$ is an isomorphism. Thus the coherent sheaves $\text{Ker}(c)$ and $\text{Coker}(c)$ are supported on $V(\mathcal{K})$ hence are annihilated by a power of $\mathcal{K}$ (Lemma 10.2) as desired. □

The following proposition is the form of Grothendieck's existence theorem which is most often used in practice.

Proposition 22.6. *Let A be a Noetherian ring complete with respect to an ideal I. Let $f : X \to \text{Spec}(A)$ be a proper morphism of schemes. Set $\mathcal{I} = I\mathcal{O}_X$. Then the functor (21.4.1) is an equivalence.*

Proof. We have already seen that (21.4.1) is fully faithful in Lemma 21.7. Thus it suffices to show that the functor is essentially surjective.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that every object $(\mathcal{F}_n)$ annihilated by $\mathcal{K}$ is in the essential image. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Lemma 10.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. (This uses the correspondence by coherent modules annihilated by $\mathcal{K}$ and coherent modules on the closed subscheme corresponding to $\mathcal{K}$, see Lemma 9.8.) Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. We will show that this object is in the essential image of the functor (21.4.1), thereby completion the proof of the proposition.

Apply Chow's lemma (Lemma 16.1) to find a proper surjective morphism $f : X' \to X$ which is an isomorphism over a dense open $U \subset X$ such that X' is projective over A. Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By the projective case of Grothendieck's existence theorem (Lemma 21.9) there exists a coherent module $\mathcal{F}'$ on X' such that $(\mathcal{F}')^\wedge \cong (f^*\mathcal{F}_n)$. By Proposition 17.2 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 22.5 there exists a morphism $(\mathcal{F}_n) \to \mathcal{H}^\wedge$ of $Coh(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. The powers $\mathcal{K}^e$ are all in Ξ so that (21.4.1) is an equivalence for the closed subschemes $X_e = V(\mathcal{K}^e)$. We conclude by Lemma 22.4. $\square$

To state the general version of Grothendieck's existence theorem we introduce a bit more notation. Let A be a Noetherian ring complete with respect to an ideal I. Let $f : X \to \text{Spec}(A)$ be a separated finite type morphism of schemes. Set $\mathcal{I} = I\mathcal{O}_X$. In this situation we let

$$Coh_{\text{support proper over } A}(\mathcal{O}_X)$$

be the full subcategory of $Coh(\mathcal{O}_X)$ consisting of those coherent $\mathcal{O}_X$-modules whose scheme theoretic support is proper over $\text{Spec}(A)$. Similarly, we let

$$Coh_{\text{support proper over } A}(X, \mathcal{I})$$

be the full subcategory of $Coh(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the scheme theoretic support of $\mathcal{F}_1$ is proper over $\text{Spec}(A)$. Since the support of a quotient module is contained in the support of the module, it follows that (21.4.1) induces a functor

$$Coh_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow Coh_{\text{support proper over } A}(X, \mathcal{I}) \tag{22.6.1}$$

We are now ready to state the main theorem of this section.

Theorem 22.7 (Grothendieck's existence theorem)**.** *In the situation described above the functor (22.6.1) is an equivalence.*

Proof. We will use the equivalence of categories of Lemma 9.8 without further mention in the proof of the theorem. Let $Z \subset X$ be a closed subscheme proper over A. By Proposition 22.6 we know that the result is true for the functor between coherent modules and systems of coherent modules supported on Z. Hence it suffices to show that every object of $Coh_{\text{support proper over } A}(\mathcal{O}_X)$ and every object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ is supported on such a closed subscheme $Z \subset X$ proper over A. This holds by definition for objects of $Coh_{\text{support proper over } A}(\mathcal{O}_X)$. We will prove this statement for objects of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ using the method of proof of Proposition 22.6. We urge the reader to read that proof first.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that the statement holds for every object $(\mathcal{F}_n)$ of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by $\mathcal{K}$. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Lemma 10.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. Let $(\mathcal{F}_n)$ be an object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. We will show that this object is supported on a closed subscheme $Z \subset X$ proper over A, thereby completing the proof of the theorem.

Apply Chow's lemma (Lemma 16.1) to find a proper surjective morphism $f : Y \to X$ which is an isomorphism over a dense open $U \subset X$ such that Y is H-quasi-projective over A. Choose an open immersion $j : Y \to Y'$ with Y' projective over A, see Morphisms, Lemma 43.11. Let T_n be the scheme theoretic support of $\mathcal{F}_n$. Note that $T_n = T_1$ set-theoretically, hence T_n is proper over A for all n (Morphisms, Lemma 42.8). Then $f^*\mathcal{F}_n$ is supported on the closed subscheme $f^{-1}T_n$ which is proper over A (by Morphisms, Lemma 42.4 and properness of f). In particular, the composition $f^{-1}T_n \to Y \to Y'$ is closed (Morphisms, Lemma 42.7). Let $T'_n \subset Y'$ be the corresponding closed subscheme; it is contained in the open subscheme Y and equal to $f^{-1}T_n$ as a closed subscheme of Y. Let $\mathcal{F}'_n$ be the coherent $\mathcal{O}_{Y'}$-module corresponding to $f^*\mathcal{F}_n$ viewed as a coherent module on Y' via the closed immersion $f^{-1}T_n = T'_n \subset Y'$. Then $(\mathcal{F}'_n)$ is an object of $Coh(Y', I\mathcal{O}_{Y'})$. By the projective case of Grothendieck's existence theorem (Lemma 21.9) there exists a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}'$ and an isomorphism $(\mathcal{F}')^\wedge \cong (\mathcal{F}'_n)$ in $Coh(Y', I\mathcal{O}_{Y'})$. Let $Z' \subset Y'$ be the scheme theoretic support of $\mathcal{F}'$. Since $\mathcal{F}'/I\mathcal{F}' = \mathcal{F}'_1$ we see that $Z' \cap V(I\mathcal{O}_{Y'}) = T'_1$ set-theoretically. The structure morphism $p' : Y' \to \text{Spec}(A)$ is proper, hence $p'(Z' \cap (Y' \setminus Y))$ is closed in $\text{Spec}(A)$. If nonempty, then it would contain a point of $V(I)$ as I is contained in the radical of A (Algebra, Lemma 93.11). But we've seen above that $Z' \cap (p')^{-1}V(I) = T'_1 \subset Y$ hence we conclude that $Z' \subset Y$. Thus $\mathcal{F}'|_Y$ is supported on a closed subscheme of Y proper over A.

Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By Proposition 17.2 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 22.5 there exists a morphism $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ of $Coh(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. Let $Z_0 \subset X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that $Z_0 \subset f(Z')$ set-theoretically. Hence $Z_0 \to \text{Spec}(A)$ is proper (Morphisms, Lemma 42.7). The kernel of α is an object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by a power of $\mathcal{K}^t$ which is in Ξ. Hence the kernel of α are supported on closed subschemes $Z_1 \subset X$ proper over A. Let $\mathcal{K}_i \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z_i for $i = 0, 1$. Set $\mathcal{K} = \mathcal{K}_0\mathcal{K}_1$ and let $Z = V(\mathcal{K}) \subset X$. Then $(\mathcal{F}_n)$ is supported on Z (details omitted). Finally, $Z_0 \amalg Z_1 \to Z$ is surjective, whence Z is proper over A by Morphisms, Lemma 42.7. This finishes the proof of the theorem. $\square$

Remark 22.8 (Unwinding Grothendieck's existence theorem)**.** Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set

$X_n = X \times_S S_n$. Picture:

$$\begin{tikzcd} X_1 \arrow[r, "i_1"'] \arrow[d] & X_2 \arrow[r, "i_2"'] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots & X \arrow[d] \\ S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \ldots & S \end{tikzcd}$$

In this situation we consider systems $(\mathcal{F}_n, \varphi_n)$ where

(1) $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module,
(2) $\varphi_n : i_n^* \mathcal{F}_{n+1} \to \mathcal{F}_n$ is an isomorphism, and
(3) $\text{Supp}(\mathcal{F}_1)$ is proper over S_1.

Theorem 22.7 says that the completion functor

$$\begin{array}{c} \text{coherent } \mathcal{O}_X\text{-modules } \mathcal{F} \\ \text{with support proper over } A \end{array} \longrightarrow \begin{array}{c} \text{systems } (\mathcal{F}_n) \\ \text{as above} \end{array}$$

is an equivalence of categories. In the special case that X is proper over A we can omit the conditions on the supports.

23. Grothendieck's algebraization theorem

Our first result is a translation of Grothendieck's existence theorem in terms of closed subschemes and finite morphisms.

Lemma 23.1. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{tikzcd} Z_1 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & Z_3 \arrow[r] \arrow[d] & \ldots \\ X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \ldots \end{tikzcd}$$

of schemes with cartesian squares. Assume that

(1) *$Z_1 \to X_1$ is a closed immersion, and*
(2) *$Z_1 \to S_1$ is proper.*

Then there exists a closed immersion of schemes $Z \to X$ such that $Z_n = Z \times_S S_n$. Moreover, Z is proper over S.

Proof. Let's write $j_n : Z_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of j_n to X_1 is j_1. Thus Morphisms, Lemma 45.5 shows that j_n is a closed immersion. Set $\mathcal{F}_n = j_{n,*}\mathcal{O}_{Z_n}$, so that $j_n^\sharp$ is a surjection $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Again using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 22.8, tells us there exists a map $\mathcal{O}_X \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to X_n recovers $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is exact (Lemma 21.5) we see that $\mathcal{O}_X \to \mathcal{F}$ is surjective. Thus $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$ for some quasi-coherent sheaf of ideals $\mathcal{J}$. Setting $Z = V(\mathcal{J})$ finishes the proof. $\square$

In the following lemma it is actually enough to assume that $Y_1 \to X_1$ is finite as it will imply that $Y_n \to X_n$ is finite too (see More on Morphisms, Lemma 2.6).

Lemma 23.2. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & \ldots \\
X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \ldots
\end{tikzcd}$$

of schemes with cartesian squares. Assume that

1. *$Y_n \to X_n$ is a finite morphism, and*
2. *$Y_1 \to S_1$ is proper.*

Then there exists a finite morphism of schemes $Y \to X$ such that $Y_n = Y \times_S S_n$. Moreover, Y is proper over S.

Proof. Let's write $f_n : Y_n \to X_n$ for the vertical morphisms. Set $\mathcal{F}_n = f_{n,*}\mathcal{O}_{Y_n}$. This is a coherent $\mathcal{O}_{X_n}$-module as f_n is finite (Lemma 9.9). Using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 22.8, tells us there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose restriction to X_n recovers $\mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is fuly faithful (Theorem 22.7) we see that the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{F}_n \to \mathcal{F}_n$ fit together to give an algebra structure on $\mathcal{F}$. Setting $Y = \underline{\mathrm{Spec}}_X(\mathcal{F})$ finishes the proof. $\square$

Lemma 23.3. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let X, Y be schemes over S. For $n \geq 1$ we set $X_n = X \times_S S_n$ and $Y_n = Y \times_S S_n$. Suppose given a compatible system of commutative diagrams*

$$\begin{tikzcd}
 & & X_{n+1} \arrow[rr, "g_{n+1}"'] \arrow[dr] & & Y_{n+1} \arrow[dl] \\
X_n \arrow[urr] \arrow[rr, "g_n"'] \arrow[dr] & & Y_n \arrow[urr] \arrow[dl] & S_{n+1} & \\
 & S_n \arrow[urr] & & &
\end{tikzcd}$$

Assume that

1. *$X \to S$ is proper, and*
2. *$Y \to S$ is separated of finite type.*

Then there exists a unique morphism of schemes $g : X \to Y$ over S such that g_n is the base change of g to S_n.

Proof. The morphisms $(1, g_n) : X_n \to X_n \times_S Y_n$ are closed immersions because $Y_n \to S_n$ is separated (Schemes, Lemma 21.12). Thus by Lemma 23.1 there exists a closed subscheme $Z \subset X \times_S Y$ proper over S whose base change to S_n recovers $X_n \subset X_n \times_S Y_n$. The first projection $p : Z \to X$ is a proper morphism (as Z is proper over S, see Morphisms, Lemma 42.7) whose base change to S_n is an isomorphism for all n. In particular, $p : Z \to X$ is finite over an open neighbourhood of X_0 by Lemma 19.2. As X is proper over S this open neighbourhood is all of X and we conclude $p : Z \to X$ is finite. Applying the equivalence of Proposition

22.6 we see that $p_*\mathcal{O}_Z = \mathcal{O}_X$ as this is true modulo I^n for all n. Hence p is an isomorphism and we obtain the morphism g as the composition $X \cong Z \to Y$. We omit the proof of uniqueness. $\square$

In order to prove an "abstract" algebraization theorem we need to assume we have an ample invertible sheaf, as the result is false without such an assumption.

Theorem 23.4 (Grothendieck's algebraization theorem)**.** *Let A be a Noetherian ring complete with respect to an ideal I. Set $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Consider a commutative diagram*

$$\begin{tikzcd} X_1 \arrow[r, "i_1"'] \arrow[d] & X_2 \arrow[r, "i_2"'] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots \\ S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \ldots \end{tikzcd}$$

of schemes with cartesian squares. Suppose given $(\mathcal{L}_n, \varphi_n)$ where each $\mathcal{L}_n$ is an invertible sheaf on X_n and $\varphi_n : i_n^*\mathcal{L}_{n+1} \to \mathcal{L}_n$ is an isomorphism. If

1. *$X_1 \to S_1$ is proper, and*
2. *$\mathcal{L}_1$ is ample on X_1*

then there exists a proper morphism of schemes $X \to S$ and an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ and isomorphisms $X_n \cong X \times_S S_n$ and $\mathcal{L}_n \cong \mathcal{L}|_{X_n}$ compatible with the morphisms i_n and φ_n.

Proof. Since the squares in the diagram are cartesian and since the morphisms $S_n \to S_{n+1}$ are closed immersions, we see that the morphisms i_n are closed immersions too. In particular we may think of X_m as a closed subscheme of X_n for $m < n$. In fact X_m is the closed subscheme cut out by the quasi-coherent sheaf of ideals $I^m\mathcal{O}_{X_n}$. Moreover, the underlying topological spaces of the schemes $X_1, X_2, X_3, \ldots$ are all identified, hence we may (and do) think of sheaves $\mathcal{O}_{X_n}$ as living on the same underlying topological space; similarly for coherent $\mathcal{O}_{X_n}$-modules. Set

$$\mathcal{F}_n = \mathrm{Ker}(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n})$$

so that we obtain short exact sequences

$$0 \to \mathcal{F}_n \to \mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n} \to 0$$

By the above we have $\mathcal{F}_n = I^n\mathcal{O}_{X_{n+1}}$. It follows $\mathcal{F}_n$ is a coherent sheaf on X_{n+1} annihilated by I, hence we may (and do) think of it as a coherent module $\mathcal{O}_{X_1}$-module. Observe that for $m > n$ the sheaf

$$I^n\mathcal{O}_{X_m}/I^{n+1}\mathcal{O}_{X_m}$$

maps isomorphically to $\mathcal{F}_n$ under the map $\mathcal{O}_{X_m} \to \mathcal{O}_{X_{n+1}}$. Hence given $n_1, n_2 \geq 0$ we can pick an $m > n_1 + n_2$ and consider the multiplication map

$$I^{n_1}\mathcal{O}_{X_m} \times I^{n_2}\mathcal{O}_{X_m} \longrightarrow I^{n_1+n_2}\mathcal{O}_{X_m} \to \mathcal{F}_{n_1+n_2}$$

This induces an $\mathcal{O}_{X_1}$-bilinear map

$$\mathcal{F}_{n_1} \times \mathcal{F}_{n_2} \longrightarrow \mathcal{F}_{n_1+n_2}$$

which in turn defines the structure of a graded $\mathcal{O}_{X_1}$-algebra on $\mathcal{F} = \bigoplus_{n\geq 0}\mathcal{F}_n$.

Set $B = \bigoplus I^n/I^{n+1}$; this is a finitely generated graded A/I-algebra. Set $\mathcal{B} = (X_1 \to S_1)^* \widetilde{B}$. The discussion above provides us with a canonical surjection

$$\mathcal{B} \longrightarrow \mathcal{F}$$

of graded $\mathcal{O}_{X_1}$-algebras. In particular we see that $\mathcal{F}$ is a finite type quasi-coherent graded $\mathcal{B}$-module. By Lemma 17.5 we can find an integer d_0 such that $H^1(X_1, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$ for all $d \geq d_0$. Pick a $d \geq d_0$ such that there exist sections $s_{0,1}, \ldots, s_{N,1} \in \Gamma(X_1, \mathcal{L}_1^{\otimes d})$ which induce an immersion

$$\psi_1 : X_1 \to \mathbf{P}^N_{S_1}$$

over S_1, see Morphisms, Lemma 40.4. As X_1 is proper over S_1 we see that ψ_1 is a closed immersion, see Morphisms, Lemma 42.7 and Schemes, Lemma 10.4. We are going to "lift" ψ_1 to a compatible system of closed immersions of X_n into $\mathbf{P}^N$.

Upon tensoring the short exact sequences of the first paragraph of the proof by $\mathcal{L}_{n+1}^{\otimes d}$ we obtain short exact sequences

$$0 \to \mathcal{F}_n \otimes \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to 0$$

Using the isomorphisms φ_n we obtain isomorphisms $\mathcal{L}_{n+1} \otimes \mathcal{O}_{X_l} = \mathcal{L}_l$ for $l \leq n$. Whence the sequence above becomes

$$0 \to \mathcal{F}_n \otimes \mathcal{L}_1^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_n^{\otimes d} \to 0$$

The vanishing of $H^1(X, \mathcal{F}_n \otimes \mathcal{L}_1^{\otimes d})$ implies we can inductively lift $s_{0,1}, \ldots, s_{N,1} \in \Gamma(X_1, \mathcal{L}_1^{\otimes d})$ to sections $s_{0,n}, \ldots, s_{N,n} \in \Gamma(X_n, \mathcal{L}_n^{\otimes d})$. Thus we obtain a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"'] \arrow[d, "\psi_1"'] & X_2 \arrow[r, "i_2"'] \arrow[d, "\psi_2"'] & X_3 \arrow[r] \arrow[d, "\psi_3"'] & \ldots \\
\mathbf{P}^N_{S_1} \arrow[r] & \mathbf{P}^N_{S_2} \arrow[r] & \mathbf{P}^N_{S_3} \arrow[r] & \ldots
\end{tikzcd}$$

where $\psi_n = \varphi_{(\mathcal{L}_n, (s_{0,n}, \ldots, s_{N,n}))}$ in the notation of Constructions, Section 13. As the squares in the statement of the theorem are cartesian we see that the squares in the above diagram are cartesian. We win by applying Lemma 23.1. $\square$

24. Other chapters

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