

MORE ON FLATNESS

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1. Introduction

In this chapter, we discuss some advanced results on flat modules and flat morphisms of schemes. Most of these results can be found in the paper [GR71] by Raynaud and Gruson.

Before reading this chapter we advise the reader to take a look at the following results (this list also serves as a pointer to previous results):

- (1) General discussion on flat modules in Algebra, Section 38.
- (2) The relationship between Tor-groups and flatness, see Algebra, Section 72.
- (3) Criteria for flatness, see Algebra, Section 95 (Noetherian case), Algebra, Section 97 (Artinian case), Algebra, Section 124 (non-Noetherian case), and finally More on Morphisms, Section 13.
- (4) Generic flatness, see Algebra, Section 114 and Morphisms, Section 28.
- (5) Openness of the flat locus, see Algebra, Section 125 and More on Morphisms, Section 12.
- (6) Flattening, see More on Algebra, Sections 9, 10, 11, 12, and 13.
- (7) Additional results in More on Algebra, Sections 14, 15, 17, and 18.

2. Lemmas on étale localization

In this section we list some lemmas on étale localization which will be useful later in this chapter. Please skip this section on a first reading.

Lemma 2.1. *Let $i : Z \rightarrow X$ be a closed immersion of affine schemes. Let $Z' \rightarrow Z$ be an étale morphism with Z' affine. Then there exists an étale morphism $X' \rightarrow X$ with X' affine such that $Z' \cong Z \times_X X'$ as schemes over Z .*

Proof. See Algebra, Lemma 138.11. □

Lemma 2.2. *Let*

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

be a commutative diagram of schemes with $X' \rightarrow X$ and $S' \rightarrow S$ étale. Let $s' \in S'$ be a point. Then

$$X' \times_{S'} \operatorname{Spec}(\mathcal{O}_{S', s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'})$$

is étale.

Proof. This is true because $X' \rightarrow X_{S'}$ is étale as a morphism of schemes étale over X , see Morphisms, Lemma 37.18 and the base change of an étale morphism is étale, see Morphisms, Lemma 37.4. □

Lemma 2.3. *Let $X \rightarrow T \rightarrow S$ be morphisms of schemes with $T \rightarrow S$ étale. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $x \in X$ be a point. Then*

$$\mathcal{F} \text{ flat over } S \text{ at } x \Leftrightarrow \mathcal{F} \text{ flat over } T \text{ at } x$$

In particular \mathcal{F} is flat over S if and only if \mathcal{F} is flat over T .

Proof. As an étale morphism is a flat morphism (see Morphisms, Lemma 37.12) the implication “ \Leftarrow ” follows from Algebra, Lemma 38.3. For the converse assume that \mathcal{F} is flat at x over S . Denote $\tilde{x} \in X \times_S T$ the point lying over x in X and over the image of x in T in T . Then $(X \times_S T \rightarrow X)^* \mathcal{F}$ is flat at \tilde{x} over T via $\operatorname{pr}_2 : X \times_S T \rightarrow T$, see Morphisms, Lemma 26.6. The diagonal $\Delta_{T/S} : T \rightarrow T \times_S T$ is an open immersion; combine Morphisms, Lemmas 36.13 and 37.5. So X is identified with open subscheme of $X \times_S T$, the restriction of pr_2 to this open is the

given morphism $X \rightarrow T$, the point \tilde{x} corresponds to the point x in this open, and $(X \times_S T \rightarrow X)^*\mathcal{F}$ restricted to this open is \mathcal{F} . Whence we see that \mathcal{F} is flat at x over T . \square

Lemma 2.4. *Let $T \rightarrow S$ be an étale morphism. Let $t \in T$ with image $s \in S$. Let M be a $\mathcal{O}_{T,t}$ -module. Then*

$$M \text{ flat over } \mathcal{O}_{S,s} \Leftrightarrow M \text{ flat over } \mathcal{O}_{T,t}.$$

Proof. We may replace S by an affine neighbourhood of s and after that T by an affine neighbourhood of t . Set $\mathcal{F} = (\mathrm{Spec}(\mathcal{O}_{T,t}) \rightarrow T)_* \widetilde{M}$. This is a quasi-coherent sheaf (see Schemes, Lemma 24.1 or argue directly) on T whose stalk at t is M (details omitted). Apply Lemma 2.3. \square

Lemma 2.5. *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}_{S,s}^h$ (resp. $\mathcal{O}_{S,s}^{sh}$) the henselization (resp. strict henselization), see Algebra, Definition 145.18. Let M^{sh} be a $\mathcal{O}_{S,s}^{sh}$ -module. The following are equivalent*

- (1) M^{sh} is flat over $\mathcal{O}_{S,s}$,
- (2) M^{sh} is flat over $\mathcal{O}_{S,s}^h$, and
- (3) M^{sh} is flat over $\mathcal{O}_{S,s}^{sh}$.

If $M^{sh} = M^h \otimes_{\mathcal{O}_{S,s}^h} \mathcal{O}_{S,s}^{sh}$ this is also equivalent to

- (4) M^h is flat over $\mathcal{O}_{S,s}$, and
- (5) M^h is flat over $\mathcal{O}_{S,s}^h$.

If $M^h = M \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}^h$ this is also equivalent to

- (6) M is flat over $\mathcal{O}_{S,s}$.

Proof. We may assume that S is an affine scheme. It is shown in Algebra, Lemmas 145.21 and 145.27 that $\mathcal{O}_{S,s}^h$ and $\mathcal{O}_{S,s}^{sh}$ are filtered colimits of the rings $\mathcal{O}_{T,t}$ where $T \rightarrow S$ is étale and affine. Hence the local ring maps $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{S,s}^h \rightarrow \mathcal{O}_{S,s}^{sh}$ are flat as directed colimits of étale ring maps, see Algebra, Lemma 38.2. Hence (3) \Rightarrow (2) \Rightarrow (1) and (5) \Rightarrow (4) follow from Algebra, Lemma 38.3. Of course these maps are faithfully flat, see Algebra, Lemma 38.16. Hence the equivalences (6) \Leftrightarrow (5) and (5) \Leftrightarrow (3) follow from Algebra, Lemma 38.7. Thus it suffices to show that (1) \Rightarrow (2) \Rightarrow (3) and (4) \Rightarrow (5).

Assume (1). By Lemma 2.4 we see that M^{sh} is flat over $\mathcal{O}_{T,t}$ for any étale neighbourhood $(T, t) \rightarrow (S, s)$. Since $\mathcal{O}_{S,s}^h$ and $\mathcal{O}_{S,s}^{sh}$ are directed colimits of local rings of the form $\mathcal{O}_{T,t}$ (see above) we conclude that M^{sh} is flat over $\mathcal{O}_{S,s}^h$ and $\mathcal{O}_{S,s}^{sh}$ by Algebra, Lemma 38.5. Thus (1) implies (2) and (3). Of course this implies also (2) \Rightarrow (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}_{S,s}^h$. The same argument applies to prove (4) \Rightarrow (5). \square

Lemma 2.6. *Let $g : T \rightarrow S$ be a finite flat morphism of schemes. Let \mathcal{G} be a quasi-coherent \mathcal{O}_S -module. Let $t \in T$ be a point with image $s \in S$. Then*

$$t \in \mathrm{WeakAss}(g^*\mathcal{G}) \Leftrightarrow s \in \mathrm{WeakAss}(\mathcal{G})$$

Proof. The implication “ \Leftarrow ” follows immediately from Divisors, Lemma 6.4. Assume $t \in \mathrm{WeakAss}(g^*\mathcal{G})$. Let $\mathrm{Spec}(A) \subset S$ be an affine open neighbourhood of s . Let \mathcal{G} be the quasi-coherent sheaf associated to the A -module M . Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s . As g is finite flat we have $g^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$

for some finite flat A -algebra B . Note that $g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\mathrm{Spec}(B)}$ -module associated to the B -module $M \otimes_A B$ and $g_*g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\mathrm{Spec}(A)}$ -module associated to the A -module $M \otimes_A B$. By Algebra, Lemma 75.4 we have $B_{\mathfrak{p}} \cong A_{\mathfrak{p}}^{\oplus n}$ for some integer $n \geq 0$. Note that $n \geq 1$ as we assumed there exists at least one point of T lying over s . Hence we see by looking at stalks that

$$s \in \mathrm{WeakAss}(\mathcal{G}) \Leftrightarrow s \in \mathrm{WeakAss}(g_*g^*\mathcal{G})$$

Now the assumption that $t \in \mathrm{WeakAss}(g^*\mathcal{G})$ implies that $s \in \mathrm{WeakAss}(g_*g^*\mathcal{G})$ by Divisors, Lemma 6.3 and hence by the above $s \in \mathrm{WeakAss}(\mathcal{G})$. \square

Lemma 2.7. *Let $h : U \rightarrow S$ be an étale morphism of schemes. Let \mathcal{G} be a quasi-coherent \mathcal{O}_S -module. Let $u \in U$ be a point with image $s \in S$. Then*

$$u \in \mathrm{WeakAss}(h^*\mathcal{G}) \Leftrightarrow s \in \mathrm{WeakAss}(\mathcal{G})$$

Proof. After replacing S and U by affine neighbourhoods of s and u we may assume that g is a standard étale morphism of affines, see Morphisms, Lemma 37.14. Thus we may assume $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A[x, 1/g]/(f))$, where f is monic and f' is invertible in $A[x, 1/g]$. Note that $A[x, 1/g]/(f) = (A[x]/(f))_g$ is also the localization of the finite free A -algebra $A[x]/(f)$. Hence we may think of U as an open subscheme of the scheme $T = \mathrm{Spec}(A[x]/(f))$ which is finite locally free over S . This reduces us to Lemma 2.6 above. \square

3. The local structure of a finite type module

The key technical lemma that makes a lot of the arguments in this chapter work is the geometric Lemma 3.2.

Lemma 3.1. *Let $f : X \rightarrow S$ be a finite type morphism of affine schemes. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Let $x \in X$ with image $s = f(x)$ in S . Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$. Then there exist a closed immersion $i : Z \rightarrow X$ of finite presentation, and a quasi-coherent finite type \mathcal{O}_Z -module \mathcal{G} such that $i_*\mathcal{G} = \mathcal{F}$ and $Z_s = \mathrm{Supp}(\mathcal{F}_s)$.*

Proof. Say the morphism $f : X \rightarrow S$ is given by the ring map $A \rightarrow B$ and that \mathcal{F} is the quasi-coherent sheaf associated to the B -module M . By Morphisms, Lemma 16.2 we know that $A \rightarrow B$ is a finite type ring map, and by Properties, Lemma 16.1 we know that M is a finite B -module. In particular the support of \mathcal{F} is the closed subscheme of $\mathrm{Spec}(B)$ cut out by the annihilator $I = \{x \in B \mid xm = 0 \ \forall m \in M\}$ of M , see Algebra, Lemma 39.5. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s . Note that $X_s = \mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ and that \mathcal{F}_s is the quasi-coherent sheaf associated to the $B \otimes_A \kappa(\mathfrak{p})$ module $M \otimes_A \kappa(\mathfrak{p})$. By Morphisms, Lemma 5.3 the support of \mathcal{F}_s is equal to $V(I(B \otimes_A \kappa(\mathfrak{p})))$. Since $B \otimes_A \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$ there exist finitely many elements $f_1, \dots, f_m \in I$ such that

$$I(B \otimes_A \kappa(\mathfrak{p})) = (f_1, \dots, f_m)(B \otimes_A \kappa(\mathfrak{p})).$$

Denote $i : Z \rightarrow X$ the closed subscheme cut out by (f_1, \dots, f_m) , in a formula $Z = \mathrm{Spec}(B/(f_1, \dots, f_m))$. Since M is annihilated by I we can think of M as an $B/(f_1, \dots, f_m)$ -module. In other words, \mathcal{F} is the pushforward of a finite type module on Z . As $Z_s = \mathrm{Supp}(\mathcal{F}_s)$ by construction, this proves the lemma. \square

Lemma 3.2. *Let $f : X \rightarrow S$ be morphism of schemes which is locally of finite type. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Let $x \in X$ with image $s = f(x)$ in S . Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$ and $n = \dim_x(\text{Supp}(\mathcal{F}_s))$. Then we can construct*

- (1) *elementary étale neighbourhoods $g : (X', x') \rightarrow (X, x)$, $e : (S', s') \rightarrow (S, s)$,*
- (2) *a commutative diagram*

$$\begin{array}{ccccc}
 X & \xleftarrow{g} & X' & \xleftarrow{i} & Z' \\
 \downarrow f & & \downarrow & & \downarrow \pi \\
 & & & & Y' \\
 & & \downarrow & & \downarrow h \\
 S & \xleftarrow{e} & S' & \xlongequal{\quad} & S'
 \end{array}$$

- (3) *a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,*
- (4) *a finite type quasi-coherent $\mathcal{O}_{Z'}$ -module \mathcal{G} ,*

such that the following properties hold

- (1) *X' , Z' , Y' , S' are affine schemes,*
- (2) *i is a closed immersion of finite presentation,*
- (3) *$i_*(\mathcal{G}) \cong g^*\mathcal{F}$,*
- (4) *π is finite and $\pi^{-1}(\{y'\}) = \{z'\}$,*
- (5) *the extension $\kappa(s') \subset \kappa(y')$ is purely transcendental,*
- (6) *h is smooth of relative dimension n with geometrically integral fibres.*

Proof. Let $V \subset S$ be an affine neighbourhood of s . Let $U \subset f^{-1}(V)$ be an affine neighbourhood of x . Then it suffices to prove the lemma for $f|_U : U \rightarrow V$ and $\mathcal{F}|_U$. Hence in the rest of the proof we assume that X and S are affine.

First, suppose that $X_s = \text{Supp}(\mathcal{F}_s)$, in particular $n = \dim_x(X_s)$. Apply More on Morphisms, Lemmas 33.2 and 33.3. This gives us a commutative diagram

$$\begin{array}{ccc}
 X & \xleftarrow{g} & X' \\
 \downarrow & & \downarrow \pi \\
 & & Y' \\
 \downarrow & & \downarrow h \\
 S & \xleftarrow{e} & S'
 \end{array}$$

and point $x' \in X'$. We set $Z' = X'$, $i = \text{id}$, and $\mathcal{G} = g^*\mathcal{F}$ to obtain a solution in this case.

In general choose a closed immersion $Z \rightarrow X$ and a sheaf \mathcal{G} on Z as in Lemma 3.1. Applying the result of the previous paragraph to $Z \rightarrow S$ and \mathcal{G} we obtain a

diagram

$$\begin{array}{ccccc}
 X & \longleftarrow & Z & \xleftarrow{g} & Z' \\
 \downarrow f & & \downarrow f|_Z & & \downarrow \pi \\
 & & & & Y' \\
 & & & & \downarrow h \\
 S & \xlongequal{\quad} & S & \xleftarrow{e} & S'
 \end{array}$$

and point $z' \in Z'$ satisfying all the required properties. We will use Lemma 2.1 to embed Z' into a scheme étale over X . We cannot apply the lemma directly as we want X' to be a scheme over S' . Instead we consider the morphisms

$$Z' \longrightarrow Z \times_S S' \longrightarrow X \times_S S'$$

The first morphism is étale by Morphisms, Lemma 37.18. The second is a closed immersion as a base change of a closed immersion. Finally, as X, S, S', Z, Z' are all affine we may apply Lemma 2.1 to get an étale morphism of affine schemes $X' \rightarrow X \times_S S'$ such that

$$Z' = (Z \times_S S') \times_{(X \times_S S')} X' = Z \times_X X'.$$

As $Z \rightarrow X$ is a closed immersion of finite presentation, so is $Z' \rightarrow X'$. Let $x' \in X'$ be the point corresponding to $z' \in Z'$. Then the completed diagram

$$\begin{array}{ccccc}
 X & \longleftarrow & X' & \xleftarrow{i} & Z' \\
 \downarrow & & \downarrow & & \downarrow \pi \\
 & & & & Y' \\
 \downarrow & & \downarrow & & \downarrow h \\
 S & \xleftarrow{e} & S' & \xlongequal{\quad} & S'
 \end{array}$$

is a solution of the original problem. \square

Lemma 3.3. *Assumptions and notation as in Lemma 3.2. If f is locally of finite presentation then π is of finite presentation. In this case the following are equivalent*

- (1) \mathcal{F} is an \mathcal{O}_X -module of finite presentation in a neighbourhood of x ,
- (2) \mathcal{G} is an $\mathcal{O}_{Z'}$ -module of finite presentation in a neighbourhood of z' , and
- (3) $\pi_* \mathcal{G}$ is an $\mathcal{O}_{Y'}$ -module of finite presentation in a neighbourhood of y' .

Still assuming f locally of finite presentation the following are equivalent to each other

- (a) \mathcal{F}_x is an $\mathcal{O}_{X,x}$ -module of finite presentation,
- (b) $\mathcal{G}_{z'}$ is an $\mathcal{O}_{Z',z'}$ -module of finite presentation, and
- (c) $(\pi_* \mathcal{G})_{y'}$ is an $\mathcal{O}_{Y',y'}$ -module of finite presentation.

Proof. Assume f locally of finite presentation. Then $Z' \rightarrow S$ is locally of finite presentation as a composition of such, see Morphisms, Lemma 22.3. Note that $Y' \rightarrow S$ is also locally of finite presentation as a composition of a smooth and an étale morphism. Hence Morphisms, Lemma 22.11 implies π is locally of finite presentation. Since π is finite we conclude that it is also separated and quasi-compact, hence π is actually of finite presentation.

To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is a $\mathcal{O}_{X'}$ -module of finite presentation in a neighbourhood of x' . The pullback of a module of finite presentation is of finite presentation, see Modules, Lemma 11.4. Hence (1) \Rightarrow (4). The étale morphism g is open, see Morphisms, Lemma 37.13. Hence for any open neighbourhood $U' \subset X'$ of x' , the image $g(U')$ is an open neighbourhood of x and the map $\{U' \rightarrow g(U')\}$ is an étale covering. Thus (4) \Rightarrow (1) by Descent, Lemma 6.3. Using Descent, Lemma 6.10 and some easy topological arguments (see More on Morphisms, Lemma 33.4) we see that (4) \Leftrightarrow (2) \Leftrightarrow (3).

To prove the equivalence of (a), (b), (c) consider the ring maps

$$\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X',x'} \rightarrow \mathcal{O}_{Z',z'} \leftarrow \mathcal{O}_{Y',y'}$$

The first ring map is faithfully flat. Hence \mathcal{F}_x is of finite presentation over $\mathcal{O}_{X,x}$ if and only if $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$, see Algebra, Lemma 80.2. The second ring map is surjective (hence finite) and finitely presented by assumption, hence $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$ if and only if $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$, see Algebra, Lemma 7.4. Because π is finite, of finite presentation, and $\pi^{-1}(\{y'\}) = \{x'\}$ the ring homomorphism $\mathcal{O}_{Y',y'} \leftarrow \mathcal{O}_{Z',z'}$ is finite and of finite presentation, see More on Morphisms, Lemma 33.4. Hence $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$ if and only if $\pi_*\mathcal{G}_{y'}$ is of finite presentation over $\mathcal{O}_{Y',y'}$, see Algebra, Lemma 7.4. \square

Lemma 3.4. *Assumptions and notation as in Lemma 3.2. The following are equivalent*

- (1) \mathcal{F} is flat over S in a neighbourhood of x ,
- (2) \mathcal{G} is flat over S' in a neighbourhood of z' , and
- (3) $\pi_*\mathcal{G}$ is flat over S' in a neighbourhood of y' .

The following are equivalent also

- (a) \mathcal{F}_x is flat over $\mathcal{O}_{S,s}$,
- (b) $\mathcal{G}_{z'}$ is flat over $\mathcal{O}_{S',s'}$, and
- (c) $(\pi_*\mathcal{G})_{y'}$ is flat over $\mathcal{O}_{S',s'}$.

Proof. To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is flat over S in a neighbourhood of x' . We will use Lemma 2.3 to equate flatness over S and S' without further mention. The étale morphism g is flat and open, see Morphisms, Lemma 37.13. Hence for any open neighbourhood $U' \subset X'$ of x' , the image $g(U')$ is an open neighbourhood of x and the map $U' \rightarrow g(U')$ is surjective and flat. Thus (4) \Leftrightarrow (1) by Morphisms, Lemma 26.11. Note that

$$\Gamma(X', g^*\mathcal{F}) = \Gamma(Z', \mathcal{G}) = \Gamma(Y', \pi_*\mathcal{G})$$

Hence the flatness of $g^*\mathcal{F}$, \mathcal{G} and $\pi_*\mathcal{G}$ over S' are all equivalent (this uses that X' , Z' , Y' , and S' are all affine). Some omitted topological arguments (compare More on Morphisms, Lemma 33.4) regarding affine neighbourhoods now show that (4) \Leftrightarrow (2) \Leftrightarrow (3).

To prove the equivalence of (a), (b), (c) consider the commutative diagram of local ring maps

$$\begin{array}{ccccccc}
 \mathcal{O}_{X',x'} & \xrightarrow{\iota} & \mathcal{O}_{Z',z'} & \xleftarrow{\alpha} & \mathcal{O}_{Y',y'} & \xleftarrow{\beta} & \mathcal{O}_{S',s'} \\
 \uparrow \gamma & & & & & & \uparrow \epsilon \\
 \mathcal{O}_{X,x} & \xleftarrow{\varphi} & & & & & \mathcal{O}_{S,s}
 \end{array}$$

We will use Lemma 2.4 to equate flatness over $\mathcal{O}_{S,s}$ and $\mathcal{O}_{S',s'}$ without further mention. The map γ is faithfully flat. Hence \mathcal{F}_x is flat over $\mathcal{O}_{S,s}$ if and only if $g^*\mathcal{F}_{x'}$ is flat over $\mathcal{O}_{S',s'}$, see Algebra, Lemma 38.8. As $\mathcal{O}_{S',s'}$ -modules the modules $g^*\mathcal{F}_{x'}$, $\mathcal{G}_{z'}$, and $\pi_*\mathcal{G}_{y'}$ are all isomorphic, see More on Morphisms, Lemma 33.4. This finishes the proof. \square

4. One step dévissage

In this section we explain what is a one step dévissage of a module. A one step dévissage exist étale locally on base and target. We discuss base change, Zariski shrinking and étale localization of a one step dévissage.

Definition 4.1. Let S be a scheme. Let X be locally of finite type over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Let $s \in S$ be a point. A *one step dévissage of $\mathcal{F}/X/S$ over s* is given by morphisms of schemes over S

$$X \xleftarrow{i} Z \xrightarrow{\pi} Y$$

and a quasi-coherent \mathcal{O}_Z -module \mathcal{G} of finite type such that

- (1) X, S, Z and Y are affine,
- (2) i is a closed immersion of finite presentation,
- (3) $\mathcal{F} \cong i_*\mathcal{G}$,
- (4) π is finite, and
- (5) the structure morphism $Y \rightarrow S$ is smooth with geometrically irreducible fibres of dimension $\dim(\text{Supp}(\mathcal{F}_s))$.

In this case we say $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s .

Note that such a one step dévissage can only exist if X and S are affine. In the definition above we only require X to be (locally) of finite type over S and we continue working in this setting below. In [GR71] the authors use consistently the setup where $X \rightarrow S$ is locally of finite presentation and \mathcal{F} quasi-coherent \mathcal{O}_X -module of finite type. The advantage of this choice is that it “makes sense” to ask for \mathcal{F} to be of finite presentation as an \mathcal{O}_X -module, whereas in our setting it “does not make sense”. Please see More on Morphisms, Section 38 for a discussion; the observations made there show that in our setup we may consider the condition of \mathcal{F} being “locally of finite presentation relative to S ”, and we could work consistently with this notion. Instead however, we will rely on the results of Lemma 3.3 and the observations in Remark 6.3 to deal with this issue in an ad hoc fashion whenever it comes up.

Definition 4.2. Let S be a scheme. Let X be locally of finite type over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Let $x \in X$ be a point with image s in S . A *one step dévissage of $\mathcal{F}/X/S$ at x* is a system $(Z, Y, i, \pi, \mathcal{G}, z, y)$, where $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s and

- (1) $\dim_x(\text{Supp}(\mathcal{F}_s)) = \dim(\text{Supp}(\mathcal{F}_s))$,

- (2) $z \in Z$ is a point with $i(z) = x$ and $\pi(z) = y$,
- (3) we have $\pi^{-1}(\{y\}) = \{z\}$,
- (4) the extension $\kappa(s) \subset \kappa(y)$ is purely transcendental.

A one step dévissage of $\mathcal{F}/X/S$ at x can only exist if X and S are affine. Condition (1) assures us that $Y \rightarrow S$ has relative dimension equal to $\dim_x(\text{Supp}(\mathcal{F}_s))$ via condition (5) of Definition 4.1.

Lemma 4.3. *Let $f : X \rightarrow S$ be morphism of schemes which is locally of finite type. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Let $x \in X$ with image $s = f(x)$ in S . Then there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ f \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (S', s') \end{array}$$

such that $(S', s') \rightarrow (S, s)$ and $(X', x') \rightarrow (X, x)$ are elementary étale neighbourhoods, and such that $g^\mathcal{F}/X'/S'$ has a one step dévissage at x' .*

Proof. This is immediate from Definition 4.2 and Lemma 3.2. \square

Lemma 4.4. *Let S, X, \mathcal{F}, s be as in Definition 4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s . Let $(S', s') \rightarrow (S, s)$ be any morphism of pointed schemes. Given this data let X', Z', Y', i', π' be the base changes of X, Z, Y, i, π via $S' \rightarrow S$. Let \mathcal{F}' be the pullback of \mathcal{F} to X' and let \mathcal{G}' be the pullback of \mathcal{G} to Z' . If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s' .*

Proof. Fibre products of affines are affine, see Schemes, Lemma 17.2. Base change preserves closed immersions, morphisms of finite presentation, finite morphisms, smooth morphisms, morphisms with geometrically irreducible fibres, and morphisms of relative dimension n , see Morphisms, Lemmas 2.4, 22.4, 44.6, 35.5, 30.2, and More on Morphisms, Lemma 22.2. We have $i'_*\mathcal{G}' \cong \mathcal{F}'$ because push-forward along the finite morphism i commutes with base change, see Cohomology of Schemes, Lemma 5.1. We have $\dim(\text{Supp}(\mathcal{F}_s)) = \dim(\text{Supp}(\mathcal{F}'_{s'}))$ by Morphisms, Lemma 29.3 because

$$\text{Supp}(\mathcal{F}_s) \times_s s' = \text{Supp}(\mathcal{F}'_{s'}).$$

This proves the lemma. \square

Lemma 4.5. *Let S, X, \mathcal{F}, x, s be as in Definition 4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x . Let $(S', s') \rightarrow (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z', Y', i', \pi', \mathcal{G}')$ be as constructed in Lemma 4.4 and let $x' \in X'$ (resp. $z' \in Z', y' \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z \in Z, y \in Y$) and $s' \in S'$. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}', z', y')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ at x' .*

Proof. By Lemma 4.4 $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s' . Properties (1) – (4) of Definition 4.2 hold for $(Z', Y', i', \pi', \mathcal{G}', z', y')$ as the assumption that $\kappa(s) = \kappa(s')$ insures that the fibres $X'_{s'}$, $Z'_{s'}$, and $Y'_{s'}$ are isomorphic to X_s , Z_s , and Y_s . \square

Definition 4.6. Let S, X, \mathcal{F}, x, s be as in Definition 4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x . Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S, X' \subset X, Z' \subset Z$, and $Y' \subset Y$ such that $s \in S', x \in X', z \in Z'$, and $y \in Y'$ and such that

$$(Z', Y', i|_{Z'}, \pi|_{Z'}, \mathcal{G}|_{Z'}, z, y)$$

is a one step dévissage of $\mathcal{F}|_{X'}/X'/S'$ at x .

Lemma 4.7. *With assumption and notation as in Definition 4.6 we have:*

- (1) *If $S' \subset S$ is a standard open neighbourhood of s , then setting $X' = X_{S'}$, $Z' = Z_{S'}$ and $Y' = Y_{S'}$ we obtain a standard shrinking.*
- (2) *Let $W \subset Y$ be a standard open neighbourhood of y . Then there exists a standard shrinking with $Y' = W \times_S S'$.*
- (3) *Let $U \subset X$ be an open neighbourhood of x . Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemma 4.5 and the fact that the inverse image of a standard open under a morphism of affine schemes is a standard open, see Algebra, Lemma 16.4.

Let $W \subset Y$ as in (2). Because $Y \rightarrow S$ is smooth it is open, see Morphisms, Lemma 35.10. Hence we can find a standard open neighbourhood S' of s contained in the image of W . Then the fibres of $W_{S'} \rightarrow S'$ are nonempty open subschemes of the fibres of $Y \rightarrow S$ over S' and hence geometrically irreducible too. Setting $Y' = W_{S'}$ and $Z' = \pi^{-1}(Y')$ we see that $Z' \subset Z$ is a standard open neighbourhood of z . Let $\bar{h} \in \Gamma(Z, \mathcal{O}_Z)$ be a function such that $Z' = D(\bar{h})$. As $i : Z \rightarrow X$ is a closed immersion, we can find a function $h \in \Gamma(X, \mathcal{O}_X)$ such that $i^\#(h) = \bar{h}$. Take $X' = D(h) \subset X$. In this way we obtain a standard shrinking as in (2).

Let $U \subset X$ be as in (3). We may after shrinking U assume that U is a standard open. By More on Morphisms, Lemma 33.4 there exists a standard open $W \subset Y$ neighbourhood of y such that $\pi^{-1}(W) \subset i^{-1}(U)$. Apply (2) to get a standard shrinking X', S', Z', Y' with $Y' = W_{S'}$. Since $Z' \subset \pi^{-1}(W) \subset i^{-1}(U)$ we may replace X' by $X' \cap U$ (still a standard open as U is also standard open) without violating any of the conditions defining a standard shrinking. Hence we win. \square

Lemma 4.8. *Let S, X, \mathcal{F}, x, s be as in Definition 4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x . Let*

$$\begin{array}{ccc} (Y, y) & \longleftarrow & (Y', y') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

be a commutative diagram of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods. Then there exists a commutative diagram

$$\begin{array}{ccccc}
 & & (X'', x'') & \longleftarrow & (Z'', z'') \\
 & \swarrow & \downarrow & \swarrow & \downarrow \\
 (X, x) & \longleftarrow & (Z, z) & & (S'', s'') \longleftarrow (Y'', y'') \\
 \downarrow & & \downarrow & \swarrow & \downarrow \\
 (S, s) & \longleftarrow & (Y, y) & &
 \end{array}$$

of pointed schemes with the following properties:

- (1) $(S'', s'') \rightarrow (S', s')$ is an elementary étale neighbourhood and the morphism $S'' \rightarrow S$ is the composition $S'' \rightarrow S' \rightarrow S$,
- (2) Y'' is an open subscheme of $Y' \times_{S'} S''$,
- (3) $Z'' = Z \times_Y Y''$,
- (4) $(X'', x'') \rightarrow (X, x)$ is an elementary étale neighbourhood, and
- (5) $(Z'', Y'', i'', \pi'', \mathcal{G}'', z'', y'')$ is a one step dévissage at x'' of the sheaf \mathcal{F}'' .

Here \mathcal{F}'' (resp. \mathcal{G}'') is the pullback of \mathcal{F} (resp. \mathcal{G}) via the morphism $X'' \rightarrow X$ (resp. $Z'' \rightarrow Z$) and $i'' : Z'' \rightarrow X''$ and $\pi'' : Z'' \rightarrow Y''$ are as in the diagram.

Proof. Let $(S'', s'') \rightarrow (S', s')$ be any elementary étale neighbourhood with S'' affine. Let $Y'' \subset Y' \times_{S'} S''$ be any affine open neighbourhood containing the point $y'' = (y', s'')$. Then we obtain an affine (Z'', z'') by (3). Moreover $Z_{S''} \rightarrow X_{S''}$ is a closed immersion and $Z'' \rightarrow Z_{S''}$ is an étale morphism. Hence Lemma 2.1 applies and we can find an étale morphism $X'' \rightarrow X_{S'}$ of affines such that $Z'' \cong X'' \times_{X_{S'}} Z_{S'}$. Denote $i'' : Z'' \rightarrow X''$ the corresponding closed immersion. Setting $x'' = i''(z'')$ we obtain a commutative diagram as in the lemma. Properties (1), (2), (3), and (4) hold by construction. Thus it suffices to show that (5) holds for a suitable choice of $(S'', s'') \rightarrow (S', s')$ and Y'' .

We first list those properties which hold for any choice of $(S'', s'') \rightarrow (S', s')$ and Y'' as in the first paragraph. As we have $Z'' = X'' \times_X Z$ by construction we see that $i''_* \mathcal{G}'' = \mathcal{F}''$ (with notation as in the statement of the lemma), see Cohomology of Schemes, Lemma 5.1. Set $n = \dim(\text{Supp}(\mathcal{F}_s)) = \dim_x(\text{Supp}(\mathcal{F}_s))$. The morphism $Y'' \rightarrow S''$ is smooth of relative dimension n (because $Y' \rightarrow S'$ is smooth of relative dimension n as the composition $Y' \rightarrow Y_{S'} \rightarrow S'$ of an étale and smooth morphism of relative dimension n and because base change preserves smooth morphisms of relative dimension n). We have $\kappa(y'') = \kappa(y)$ and $\kappa(s) = \kappa(s'')$ hence $\kappa(y'')$ is a purely transcendental extension of $\kappa(s'')$. The morphism of fibres $X''_{s''} \rightarrow X_s$ is an étale morphism of affine schemes over $\kappa(s) = \kappa(s'')$ mapping the point x'' to the point x and pulling back \mathcal{F}_s to $\mathcal{F}''_{s''}$. Hence

$$\dim(\text{Supp}(\mathcal{F}''_{s''})) = \dim(\text{Supp}(\mathcal{F}_s)) = n = \dim_x(\text{Supp}(\mathcal{F}_s)) = \dim_{x''}(\text{Supp}(\mathcal{F}''_{s''}))$$

because dimension is invariant under étale localization, see Descent, Lemma 17.2. As $\pi'' : Z'' \rightarrow Y''$ is the base change of π we see that π'' is finite and as $\kappa(y) = \kappa(y'')$ we see that $\pi''^{-1}(\{y''\}) = \{z''\}$.

At this point we have verified all the conditions of Definition 4.1 except we have not verified that $Y'' \rightarrow S''$ has geometrically irreducible fibres. Of course in general this is not going to be true, and it is at this point that we will use that $\kappa(s) \subset \kappa(y)$

is purely transcendental. Namely, let $T \subset Y'_s$ be the irreducible component of Y'_s containing $y' = (y, s')$. Note that T is an open subscheme of Y'_s as this is a smooth scheme over $\kappa(s')$. By Varieties, Lemma 5.14 we see that T is geometrically connected because $\kappa(s') = \kappa(s)$ is algebraically closed in $\kappa(y') = \kappa(y)$. As T is smooth we see that T is geometrically irreducible. Hence More on Morphisms, Lemma 32.3 applies and we can find an elementary étale morphism $(S'', s'') \rightarrow (S', s')$ and an affine open $Y'' \subset Y'_{S''}$ such that all fibres of $Y'' \rightarrow S''$ are geometrically irreducible and such that $T = Y''_{s''}$. After shrinking (first Y'' and then S'') we may assume that both Y'' and S'' are affine. This finishes the proof of the lemma. \square

Lemma 4.9. *Let S, X, \mathcal{F}, s be as in Definition 4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s . Let $\xi \in Y_s$ be the (unique) generic point. Then there exists an integer $r > 0$ and an \mathcal{O}_Y -module map*

$$\alpha : \mathcal{O}_Y^{\oplus r} \longrightarrow \pi_* \mathcal{G}$$

such that

$$\alpha : \kappa(\xi)^{\oplus r} \longrightarrow (\pi_* \mathcal{G})_\xi \otimes_{\mathcal{O}_{Y, \xi}} \kappa(\xi)$$

is an isomorphism. Moreover, in this case we have

$$\dim(\text{Supp}(\text{Coker}(\alpha)_s)) < \dim(\text{Supp}(\mathcal{F}_s)).$$

Proof. By assumption the schemes S and Y are affine. Write $S = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. As π is finite the \mathcal{O}_Y -module $\pi_* \mathcal{G}$ is a finite type quasi-coherent \mathcal{O}_Y -module. Hence $\pi_* \mathcal{G} = \tilde{N}$ for some finite B -module N . Let $\mathfrak{p} \subset B$ be the prime ideal corresponding to ξ . To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_B \kappa(\mathfrak{p})$ and pick $x_1, \dots, x_r \in N$ which form a basis of $N \otimes_B \kappa(\mathfrak{p})$. Take $\alpha : B^{\oplus r} \rightarrow N$ to be the map given by the formula $\alpha(b_1, \dots, b_r) = \sum b_i x_i$. It is clear that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \rightarrow N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism as desired. Finally, suppose α is any map with this property. Then $N' = \text{Coker}(\alpha)$ is a finite B -module such that $N' \otimes_B \kappa(\mathfrak{p}) = 0$. By Nakayama's lemma (Algebra, Lemma 19.1) we see that $N'_\mathfrak{p} = 0$. Since the fibre Y_s is geometrically irreducible of dimension n with generic point ξ and since we have just seen that ξ is not in the support of $\text{Coker}(\alpha)$ the last assertion of the lemma holds. \square

5. Complete dévissage

In this section we explain what is a complete dévissage of a module and prove that such exist. The material in this section is mainly bookkeeping.

Definition 5.1. Let S be a scheme. Let X be locally of finite type over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Let $s \in S$ be a point. A *complete*

déviissage of $\mathcal{F}/X/S$ over s is given by a diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{i_1} & Z_1 & & \\
 & & \downarrow \pi_1 & & \\
 Y_1 & \xleftarrow{i_2} & Z_2 & & \\
 & & \downarrow \pi_2 & & \\
 & & Y_2 & \xleftarrow{\quad} & Z_3 \\
 & & & & \downarrow \\
 & & & & \dots \xleftarrow{\quad} \dots \\
 & & & & \downarrow \\
 & & & & Y_n
 \end{array}$$

of schemes over S , finite type quasi-coherent \mathcal{O}_{Z_k} -modules \mathcal{G}_k , and \mathcal{O}_{Y_k} -module maps

$$\alpha_k : \mathcal{O}_{Y_k}^{\oplus r_k} \longrightarrow \pi_{k,*} \mathcal{G}_k, \quad k = 1, \dots, n$$

satisfying the following properties:

- (1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ is a one step déviissage of $\mathcal{F}/X/S$ over s ,
- (2) the map α_k induces an isomorphism

$$\kappa(\xi_k)^{\oplus r_k} \longrightarrow (\pi_{k,*} \mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k, \xi_k}} \kappa(\xi_k)$$

where $\xi_k \in (Y_k)_s$ is the unique generic point,

- (3) for $k = 2, \dots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k)$ is a one step déviissage of $\text{Coker}(\alpha_{k-1})/Y_{k-1}/S$ over s ,
- (4) $\text{Coker}(\alpha_n) = 0$.

In this case we say that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1, \dots, n}$ is a complete déviissage of $\mathcal{F}/X/S$ over s .

Definition 5.2. Let S be a scheme. Let X be locally of finite type over S . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Let $x \in X$ be a point with image $s \in S$. A *complete déviissage of $\mathcal{F}/X/S$ at x* is given by a system

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1, \dots, n}$$

such that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)$ is a complete déviissage of $\mathcal{F}/X/S$ over s , and such that

- (1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1, z_1, y_1)$ is a one step déviissage of $\mathcal{F}/X/S$ at x ,
- (2) for $k = 2, \dots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, z_k, y_k)$ is a one step déviissage of $\text{Coker}(\alpha_{k-1})/Y_{k-1}/S$ at y_{k-1} .

Again we remark that a complete déviissage can only exist if X and S are affine.

Lemma 5.3. Let S, X, \mathcal{F}, s be as in Definition 5.1. Let $(S', s') \rightarrow (S, s)$ be any morphism of pointed schemes. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1, \dots, n}$ be a complete déviissage of $\mathcal{F}/X/S$ over s . Given this data let $X', Z'_k, Y'_k, i'_k, \pi'_k$ be the base changes of X, Z_k, Y_k, i_k, π_k via $S' \rightarrow S$. Let \mathcal{F}' be the pullback of \mathcal{F} to X' and let \mathcal{G}'_k be the pullback of \mathcal{G}_k to Z'_k . Let α'_k be the pullback of α_k to Y'_k . If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1, \dots, n}$ is a complete déviissage of $\mathcal{F}'/X'/S'$ over s' .

Proof. By Lemma 4.4 we know that the base change of a one step dévissage is a one step dévissage. Hence it suffices to prove that formation of $\text{Coker}(\alpha_k)$ commutes with base change and that condition (2) of Definition 5.1 is preserved by base change. The first is true as $\pi'_{k,*}\mathcal{G}'_k$ is the pullback of $\pi_{k,*}\mathcal{G}_k$ (by Cohomology of Schemes, Lemma 5.1) and because \otimes is right exact. The second because by the same token we have

$$(\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k, \xi_k}} \kappa(\xi_k) \otimes_{\kappa(\xi_k)} \kappa(\xi'_k) \cong (\pi'_{k,*}\mathcal{G}'_k)_{\xi'_k} \otimes_{\mathcal{O}_{Y'_k, \xi'_k}} \kappa(\xi'_k)$$

with obvious notation. \square

Lemma 5.4. *Let S, X, \mathcal{F}, x, s be as in Definition 5.2. Let $(S', s') \rightarrow (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1, \dots, n}$ be a complete dévissage of $\mathcal{F}/X/S$ at x . Let $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1, \dots, n}$ be as constructed in Lemma 5.3 and let $x' \in X'$ (resp. $z'_k \in Z'_k, y'_k \in Y'_k$) be the unique point mapping to both $x \in X$ (resp. $z_k \in Z_k, y_k \in Y_k$) and $s' \in S'$. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z'_k, y'_k)_{k=1, \dots, n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ at x' .*

Proof. Combine Lemma 5.3 and Lemma 4.5. \square

Definition 5.5. Let S, X, \mathcal{F}, x, s be as in Definition 5.2. Consider a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1, \dots, n}$ of $\mathcal{F}/X/S$ at x . Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S, X' \subset X, Z'_k \subset Z_k$, and $Y'_k \subset Y_k$ such that $s_k \in S', x_k \in X', z_k \in Z'_k$, and $y_k \in Y'_k$ and such that

$$(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z_k, y_k)_{k=1, \dots, n}$$

is a one step dévissage of $\mathcal{F}'/X'/S'$ at x where $\mathcal{G}'_k = \mathcal{G}_k|_{Z'_k}$ and $\mathcal{F}' = \mathcal{F}|_{X'}$.

Lemma 5.6. *With assumption and notation as in Definition 5.5 we have:*

- (1) *If $S' \subset S$ is a standard open neighbourhood of s , then setting $X' = X_{S'}$, $Z'_k = Z_{S'}$ and $Y'_k = Y_{S'}$ we obtain a standard shrinking.*
- (2) *Let $W \subset Y_n$ be a standard open neighbourhood of y . Then there exists a standard shrinking with $Y'_n = W \times_S S'$.*
- (3) *Let $U \subset X$ be an open neighbourhood of x . Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemmas 5.4 and 4.7.

Proof of (2). For convenience denote $X = Y_0$. We apply Lemma 4.7 (2) to find a standard shrinking S', Y'_{n-1}, Z'_n, Y'_n of the one step dévissage of $\text{Coker}(\alpha_{n-1})/Y_{n-1}/S$ at y_{n-1} with $Y'_n = W \times_S S'$. We may repeat this procedure and find a standard shrinking $S'', Y''_{n-2}, Z''_{n-1}, Y''_{n-1}$ of the one step dévissage of $\text{Coker}(\alpha_{n-2})/Y_{n-2}/S$ at y_{n-2} with $Y''_{n-1} = Y'_{n-1} \times_S S''$. We may continue in this manner until we obtain $S^{(n)}, Y_0^{(n)}, Z_1^{(n)}, Y_1^{(n)}$. At this point it is clear that we obtain our desired standard shrinking by taking $S^{(n)}, X^{(n)}, Z_k^{(n-k)} \times_S S^{(n)}$, and $Y_k^{(n-k)} \times_S S^{(n)}$ with the desired property.

Proof of (3). We use induction on the length of the complete dévissage. First we apply Lemma 4.7 (3) to find a standard shrinking S', X', Z'_1, Y'_1 of the one step dévissage of $\mathcal{F}/X/S$ at x with $X' \subset U$. If $n = 1$, then we are done. If $n > 1$, then by induction we can find a standard shrinking S'', Y''_1, Z''_k , and Y''_k of the complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2, \dots, n}$ of $\text{Coker}(\alpha_1)/Y_1/S$ at

x such that $Y_1'' \subset Y_1'$. Using Lemma 4.7 (2) we can find $S''' \subset S'$, $X''' \subset X'$, Z_1''' and $Y_1''' = Y_1'' \times_S S'''$ which is a standard shrinking. The solution to our problem is to take

$$S''', X''', Z_1''', Y_1''', Z_2'' \times_S S''', Y_2'' \times_S S''', \dots, Z_n'' \times_S S''', Y_n'' \times_S S'''$$

This ends the proof of the lemma. \square

Proposition 5.7. *Let S be a scheme. Let X be locally of finite type over S . Let $x \in X$ be a point with image $s \in S$. There exists a commutative diagram*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x .

Proof. We prove this by induction on the integer $d = \dim_x(\text{Supp}(\mathcal{F}_s))$. By Lemma 4.3 there exists a diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x' . The local nature of the problem implies that we may replace $(X, x) \rightarrow (S, s)$ by $(X', x') \rightarrow (S', s')$. Thus after doing so we may assume that there exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x .

We apply Lemma 4.9 to find a map

$$\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \longrightarrow \pi_{1,*}\mathcal{G}_1$$

which induces an isomorphism of vector spaces over $\kappa(\xi_1)$ where $\xi_1 \in Y_1$ is the unique generic point of the fibre of Y_1 over s . Moreover $\dim_{y_1}(\text{Supp}(\text{Coker}(\alpha_1)_s)) < d$. It may happen that the stalk of $\text{Coker}(\alpha_1)_s$ at y_1 is zero. In this case we may shrink Y_1 by Lemma 4.7 (2) and assume that $\text{Coker}(\alpha_1) = 0$ so we obtain a complete dévissage of length zero.

Assume now that the stalk of $\text{Coker}(\alpha_1)_s$ at y_1 is not zero. In this case, by induction, there exists a commutative diagram

$$(5.7.1) \quad \begin{array}{ccc} (Y_1, y_1) & \xleftarrow{h} & (Y'_1, y'_1) \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $h^*\text{Coker}(\alpha_1)/Y'_1/S'$ has a complete dévissage

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2, \dots, n}$$

at y'_1 . (In particular $i_2 : Z_2 \rightarrow Y'_1$ is a closed immersion into Y'_2 .) At this point we apply Lemma 4.8 to S, X, \mathcal{F}, x, s , the system $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ and diagram (5.7.1). We obtain a diagram

$$\begin{array}{ccccc}
 & & (X'', x'') & \longleftarrow & (Z''_1, z''_1) \\
 & \swarrow & \downarrow & \swarrow & \downarrow \\
 (X, x) & \longleftarrow & (Z_1, z_1) & \longleftarrow & (S'', s'') & \longleftarrow & (Y''_1, y''_1) \\
 \downarrow & & \downarrow & \swarrow & \downarrow & \swarrow & \\
 (S, s) & \longleftarrow & (Y_1, y_1) & \longleftarrow & & &
 \end{array}$$

with all the properties as listed in the referenced lemma. In particular $Y''_1 \subset Y'_1 \times_{S'} S''$. Set $X_1 = Y'_1 \times_{S'} S''$ and let \mathcal{F}_1 denote the pullback of $\text{Coker}(\alpha_1)$. By Lemma 5.4 the system

$$(5.7.2) \quad (Z_k \times_{S'} S'', Y_k \times_{S'} S'', i''_k, \pi''_k, \mathcal{G}''_k, \alpha''_k, z''_k, y''_k)_{k=2, \dots, n}$$

is a complete dévissage of \mathcal{F}_1 to X_1 . Again, the nature of the problem allows us to replace $(X, x) \rightarrow (S, s)$ by $(X'', x'') \rightarrow (S'', s'')$. In this we see that we may assume:

- (a) There exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x ,
- (b) there exists an $\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \rightarrow \pi_{1,*} \mathcal{G}_1$ such that $\alpha \otimes \kappa(\xi_1)$ is an isomorphism,
- (c) $Y_1 \subset X_1$ is open, $y_1 = x_1$, and $\mathcal{F}_1|_{Y_1} \cong \text{Coker}(\alpha_1)$, and
- (d) there exists a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2, \dots, n}$ of $\mathcal{F}_1/X_1/S$ at x_1 .

To finish the proof all we have to do is shrink the one step dévissage and the complete dévissage such that they fit together to a complete dévissage. (We suggest the reader do this on their own using Lemmas 4.7 and 5.6 instead of reading the proof that follows.) Since $Y_1 \subset X_1$ is an open neighbourhood of x_1 we may apply Lemma 5.6 (3) to find a standard shrinking $S', X'_1, Z'_2, Y'_2, \dots, Y'_n$ of the datum (d) so that $X'_1 \subset Y_1$. Note that X'_1 is also a standard open of the affine scheme Y_1 . Next, we shrink the datum (a) as follows: first we shrink the base S to S' , see Lemma 4.7 (1) and then we shrink the result to S'', X'', Z''_1, Y''_1 using Lemma 4.7 (2) such that eventually $Y''_1 = X'_1 \times_S S''$ and $S'' \subset S'$. Then we see that

$$Z''_1, Y''_1, Z'_2 \times_{S'} S'', Y'_2 \times_{S'} S'', \dots, Y'_n \times_{S'} S''$$

gives the complete dévissage we were looking for. \square

Some more bookkeeping gives the following consequence.

Lemma 5.8. *Let $X \rightarrow S$ be a finite type morphism of schemes. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Let $s \in S$ be a point. There exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and étale morphisms $h_i : Y_i \rightarrow X_{S'}$, $i = 1, \dots, n$ such that for each i there exists a complete dévissage of $\mathcal{F}_i/Y_i/S'$ over s' , where \mathcal{F}_i is the pullback of \mathcal{F} to Y_i and such that $X_s = (X_{S'})_{s'} \subset \bigcup h_i(Y_i)$.*

Proof. For every point $x \in X_s$ we can find a diagram

$$\begin{array}{ccc}
 (X, x) & \xleftarrow{g} & (X', x') \\
 \downarrow & & \downarrow \\
 (S, s) & \longleftarrow & (S', s')
 \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x' . As $X \rightarrow S$ is of finite type the fibre X_s is quasi-compact, and since each $g : X' \rightarrow X$ as above is open we can cover X_s by a finite union of $g(X'_s)$. Thus we can find a finite family of such diagrams

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_i} & (X'_i, x'_i) \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (S'_i, s'_i) \end{array} \quad i = 1, \dots, n$$

such that $X_s = \bigcup g_i(X'_i)$. Set $S' = S'_1 \times_S \dots \times_S S'_n$ and let $Y_i = X_i \times_{S'_i} S'$ be the base change of X'_i to S' . By Lemma 5.3 we see that the pullback of \mathcal{F} to Y_i has a complete dévissage over s and we win. \square

6. Translation into algebra

It may be useful to spell out algebraically what it means to have a complete dévissage. We introduce the following notion (which is not that useful so we give it an impossibly long name).

Definition 6.1. Let $R \rightarrow S$ be a ring map. Let \mathfrak{q} be a prime of S lying over the prime \mathfrak{p} of R . A *elementary étale localization of the ring map $R \rightarrow S$ at \mathfrak{q}* is given by a commutative diagram of rings and accompanying primes

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array} \quad \begin{array}{ccc} \mathfrak{q} & \longrightarrow & \mathfrak{q}' \\ \downarrow & & \downarrow \\ \mathfrak{p} & \longrightarrow & \mathfrak{p}' \end{array}$$

such that $R \rightarrow R'$ and $S \rightarrow S'$ are étale ring maps and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$ and $\kappa(\mathfrak{q}) = \kappa(\mathfrak{q}')$.

Definition 6.2. Let $R \rightarrow S$ be a finite type ring map. Let \mathfrak{r} be a prime of R . Let N be a finite S -module. A *complete dévissage of $N/S/R$ over \mathfrak{r}* is given by R -algebra maps

$$\begin{array}{ccccccc} & & A_1 & & A_2 & & \dots & & A_n \\ & \nearrow & & \nwarrow & \nearrow & & \nwarrow & & \nearrow \\ S & & & & B_1 & & \dots & & B_n \end{array}$$

finite A_i -modules M_i and B_i -module maps $\alpha_i : B_i^{\oplus r_i} \rightarrow M_i$ such that

- (1) $S \rightarrow A_1$ is surjective and of finite presentation,
- (2) $B_i \rightarrow A_{i+1}$ is surjective and of finite presentation,
- (3) $B_i \rightarrow A_i$ is finite,
- (4) $R \rightarrow B_i$ is smooth with geometrically irreducible fibres,
- (5) $N \cong M_1$ as S -modules,
- (6) $\text{Coker}(\alpha_i) \cong M_{i+1}$ as B_i -modules,
- (7) $\alpha_i : \kappa(\mathfrak{p}_i)^{\oplus r_i} \rightarrow M_i \otimes_{B_i} \kappa(\mathfrak{p}_i)$ is an isomorphism where $\mathfrak{p}_i = \mathfrak{r}B_i$, and
- (8) $\text{Coker}(\alpha_n) = 0$.

In this situation we say that $(A_i, B_i, M_i, \alpha_i)_{i=1, \dots, n}$ is a complete dévissage of $N/S/R$ over \mathfrak{r} .

Remark 6.3. Note that the R -algebras B_i for all i and A_i for $i \geq 2$ are of finite presentation over R . If S is of finite presentation over R , then it is also the case that A_1 is of finite presentation over R . In this case all the ring maps in the complete dévissage are of finite presentation. See Algebra, Lemma 6.2. Still assuming S of finite presentation over R the following are equivalent

- (1) M is of finite presentation over S ,
- (2) M_1 is of finite presentation over A_1 ,
- (3) M_1 is of finite presentation over B_1 ,
- (4) each M_i is of finite presentation both as an A_i -module and as a B_i -module.

The equivalences (1) \Leftrightarrow (2) and (2) \Leftrightarrow (3) follow from Algebra, Lemma 7.4. If M_1 is finitely presented, so is $\text{Coker}(\alpha_1)$ (see Algebra, Lemma 5.3) and hence M_2 , etc.

Definition 6.4. Let $R \rightarrow S$ be a finite type ring map. Let \mathfrak{q} be a prime of S lying over the prime \mathfrak{r} of R . Let N be a finite S -module. A *complete dévissage of $N/S/R$ at \mathfrak{q}* is given by a complete dévissage $(A_i, B_i, M_i, \alpha_i)_{i=1, \dots, n}$ of $N/S/R$ over \mathfrak{r} and prime ideals $\mathfrak{q}_i \subset B_i$ lying over \mathfrak{r} such that

- (1) $\kappa(\mathfrak{r}) \subset \kappa(\mathfrak{q}_i)$ is purely transcendental,
- (2) there is a unique prime $\mathfrak{q}'_i \subset A_i$ lying over $\mathfrak{q}_i \subset B_i$,
- (3) $\mathfrak{q} = \mathfrak{q}'_1 \cap S$ and $\mathfrak{q}_i = \mathfrak{q}'_{i+1} \cap A_i$,
- (4) $R \rightarrow B_i$ has relative dimension $\dim_{\mathfrak{q}_i}(\text{Supp}(M_i \otimes_R \kappa(\mathfrak{r})))$.

Remark 6.5. Let $A \rightarrow B$ be a finite type ring map and let N be a finite B -module. Let \mathfrak{q} be a prime of B lying over the prime \mathfrak{r} of A . Set $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and $\mathcal{F} = \tilde{N}$ on X . Let x be the point corresponding to \mathfrak{q} and let $s \in S$ be the point corresponding to \mathfrak{p} . Then

- (1) if there exists a complete dévissage of $\mathcal{F}/X/S$ over s then there exists a complete dévissage of $N/B/A$ over \mathfrak{p} , and
- (2) there exists a complete dévissage of $\mathcal{F}/X/S$ at x if and only if there exists a complete dévissage of $N/B/A$ at \mathfrak{q} .

There is just a small twist in that we omitted the condition on the relative dimension in the formulation of “a complete dévissage of $N/B/A$ over \mathfrak{p} ” which is why the implication in (1) only goes in one direction. The notion of a complete dévissage at \mathfrak{q} does have this condition built in. In any case we will only use that existence for $\mathcal{F}/X/S$ implies the existence for $N/B/A$.

Lemma 6.6. *Let $R \rightarrow S$ be a finite type ring map. Let M be a finite S -module. Let \mathfrak{q} be a prime ideal of S . There exists an elementary étale localization $R' \rightarrow S', \mathfrak{q}', \mathfrak{p}'$ of the ring map $R \rightarrow S$ at \mathfrak{q} such that there exists a complete dévissage of $(M \otimes_S S')/S'/R'$ at \mathfrak{q}' .*

Proof. This is a reformulation of Proposition 5.7 via Remark 6.5 □

7. Localization and universally injective maps

Lemma 7.1. *Let $R \rightarrow S$ be a ring map. Let N be a S -module. Assume*

- (1) R is a local ring with maximal ideal \mathfrak{m} ,
- (2) $\overline{S} = S/\mathfrak{m}S$ is Noetherian, and
- (3) $\overline{N} = N/\mathfrak{m}_R N$ is a finite \overline{S} -module.

Let $\Sigma \subset S$ be the multiplicative subset of elements which are not a zerodivisor on \overline{N} . Then $\Sigma^{-1}S$ is a semi-local ring whose spectrum consists of primes $\mathfrak{q} \subset S$ contained

in an element of $\text{Ass}_S(\overline{N})$. Moreover, any maximal ideal of $\Sigma^{-1}S$ corresponds to an associated prime of \overline{N} over \overline{S} .

Proof. Note that $\text{Ass}_S(\overline{N}) = \text{Ass}_{\overline{S}}(\overline{N})$, see Algebra, Lemma 62.13. This is a finite set by Algebra, Lemma 62.5. Say $\{\mathfrak{q}_1, \dots, \mathfrak{q}_r\} = \text{Ass}_S(\overline{N})$. We have $\Sigma = S \setminus (\bigcup \mathfrak{q}_i)$ by Algebra, Lemma 62.9. By the description of $\text{Spec}(\Sigma^{-1}S)$ in Algebra, Lemma 16.5 and by Algebra, Lemma 14.2 we see that the primes of $\Sigma^{-1}S$ correspond to the primes of S contained in one of the \mathfrak{q}_i . Hence the maximal ideals of $\Sigma^{-1}S$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$. This proves the lemma. \square

Lemma 7.2. *Assumption and notation as in Lemma 7.1. Assume moreover that*

- (1) *S is local and $R \rightarrow S$ is a local homomorphism,*
- (2) *S is essentially of finite presentation over R ,*
- (3) *N is finitely presented over S , and*
- (4) *N is flat over R .*

Then each $s \in \Sigma$ defines a universally injective R -module map $s : N \rightarrow N$, and the map $N \rightarrow \Sigma^{-1}N$ is R -universally injective.

Proof. By Algebra, Lemma 124.4 the sequence $0 \rightarrow N \rightarrow N \rightarrow N/sN \rightarrow 0$ is exact and N/sN is flat over R . This implies that $s : N \rightarrow N$ is universally injective, see Algebra, Lemma 38.11. The map $N \rightarrow \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \rightarrow N$. \square

Lemma 7.3. *Let $R \rightarrow S$ be a ring map. Let N be an S -module. Let $S \rightarrow S'$ be a ring map. Assume*

- (1) *$R \rightarrow S$ is a local homomorphism of local rings*
- (2) *S is essentially of finite presentation over R ,*
- (3) *N is of finite presentation over S ,*
- (4) *N is flat over R ,*
- (5) *$S \rightarrow S'$ is flat, and*
- (6) *the image of $\text{Spec}(S') \rightarrow \text{Spec}(S)$ contains all primes \mathfrak{q} of S lying over \mathfrak{m}_R such that \mathfrak{q} is an associated prime of $N/\mathfrak{m}_R N$.*

Then $N \rightarrow N \otimes_S S'$ is R -universally injective.

Proof. Set $N' = N \otimes_S S'$. Consider the commutative diagram

$$\begin{array}{ccc} N & \longrightarrow & N' \\ \downarrow & & \downarrow \\ \Sigma^{-1}N & \longrightarrow & \Sigma^{-1}N' \end{array}$$

where $\Sigma \subset S$ is the set of elements which are not a zerodivisor on $N/\mathfrak{m}_R N$. If we can show that the map $N \rightarrow \Sigma^{-1}N'$ is universally injective, then $N \rightarrow N'$ is too (see Algebra, Lemma 79.10).

By Lemma 7.1 the ring $\Sigma^{-1}S$ is a semi-local ring whose maximal ideals correspond to associated primes of $N/\mathfrak{m}_R N$. Hence the image of $\text{Spec}(\Sigma^{-1}S') \rightarrow \text{Spec}(\Sigma^{-1}S)$ contains all these maximal ideals by assumption. By Algebra, Lemma 38.15 the ring map $\Sigma^{-1}S \rightarrow \Sigma^{-1}S'$ is faithfully flat. Hence $\Sigma^{-1}N \rightarrow \Sigma^{-1}N'$, which is the map

$$N \otimes_S \Sigma^{-1}S \longrightarrow N \otimes_S \Sigma^{-1}S'$$

is universally injective, see Algebra, Lemmas 79.11 and 79.8. Finally, we apply Lemma 7.2 to see that $N \rightarrow \Sigma^{-1}N$ is universally injective. As the composition of universally injective module maps is universally injective (see Algebra, Lemma 79.9) we conclude that $N \rightarrow \Sigma^{-1}N'$ is universally injective and we win. \square

Lemma 7.4. *Let $R \rightarrow S$ be a ring map. Let N be an S -module. Let $S \rightarrow S'$ be a ring map. Assume*

- (1) $R \rightarrow S$ is of finite presentation and N is of finite presentation over S ,
- (2) N is flat over R ,
- (3) $S \rightarrow S'$ is flat, and
- (4) the image of $\text{Spec}(S') \rightarrow \text{Spec}(S)$ contains all primes \mathfrak{q} such that \mathfrak{q} is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where \mathfrak{p} is the inverse image of \mathfrak{q} in R .

Then $N \rightarrow N \otimes_S S'$ is R -universally injective.

Proof. By Algebra, Lemma 79.12 it suffices to show that $N_{\mathfrak{q}} \rightarrow (N \otimes_S S')_{\mathfrak{q}}$ is a $R_{\mathfrak{p}}$ -universally injective for any prime \mathfrak{q} of S lying over \mathfrak{p} in R . Thus we may apply Lemma 7.3 to the ring maps $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}} \rightarrow S'_{\mathfrak{q}}$ and the module $N_{\mathfrak{q}}$. \square

The reader may want to compare the following lemma to Algebra, Lemmas 95.1 and 124.4. In each case the conclusion is that the map $u : M \rightarrow N$ is universally injective with flat cokernel.

Lemma 7.5. *Let (R, \mathfrak{m}) be a local ring. Let $u : M \rightarrow N$ be an R -module map. If M is a projective R -module, N is a flat R -module, and $\bar{u} : M/\mathfrak{m}M \rightarrow N/\mathfrak{m}N$ is injective then u is universally injective.*

Proof. By Algebra, Theorem 82.4 the module M is free. If we show the result holds for every finitely generated direct summand of M , then the lemma follows. Hence we may assume that M is finite free. Write $N = \text{colim}_i N_i$ as a directed colimit of finite free modules, see Algebra, Theorem 78.4. Note that $u : M \rightarrow N$ factors through N_i for some i (as M is finite free). Denote $u_i : M \rightarrow N_i$ the corresponding R -module map. As \bar{u} is injective we see that $\bar{u}_i : M/\mathfrak{m}M \rightarrow N_i/\mathfrak{m}N_i$ is injective and remains injective on composing with the maps $N_i/\mathfrak{m}N_i \rightarrow N_{i'}/\mathfrak{m}N_{i'}$ for all $i' \geq i$. As M and $N_{i'}$ are finite free over the local ring R this implies that $M \rightarrow N_{i'}$ is a split injection for all $i' \geq i$. Hence for any R -module Q we see that $M \otimes_R Q \rightarrow N_{i'} \otimes_R Q$ is injective for all $i' \geq i$. As $- \otimes_R Q$ commutes with colimits we conclude that $M \otimes_R Q \rightarrow N \otimes_R Q$ is injective as desired. \square

Lemma 7.6. *Assumption and notation as in Lemma 7.1. Assume moreover that N is projective as an R -module. Then each $s \in \Sigma$ defines a universally injective R -module map $s : N \rightarrow N$, and the map $N \rightarrow \Sigma^{-1}N$ is R -universally injective.*

Proof. Pick $s \in \Sigma$. By Lemma 7.5 the map $s : N \rightarrow N$ is universally injective. The map $N \rightarrow \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \rightarrow N$. \square

8. Completion and Mittag-Leffler modules

Lemma 8.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I . The completion $(\bigoplus_{\alpha \in A} R)^{\wedge}$ is flat and Mittag-Leffler.*

Proof. By More on Algebra, Lemma 19.1 the map $(\bigoplus_{\alpha \in A} R)^\wedge \rightarrow \prod_{\alpha \in A} R$ is universally injective. Thus, by Algebra, Lemmas 79.7 and 86.7 it suffices to show that $\prod_{\alpha \in A} R$ is flat and Mittag-Leffler. By Algebra, Proposition 87.5 (and Algebra, Lemma 87.4) we see that $\prod_{\alpha \in A} R$ is flat. Thus we conclude because a product of copies of R is Mittag-Leffler, see Algebra, Lemma 88.3. \square

Lemma 8.2. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R -module. Assume*

- (1) *R is Noetherian and I -adically complete,*
- (2) *M is flat over R , and*
- (3) *M/IM is a projective R/I -module.*

Then the I -adic completion M^\wedge is a flat Mittag-Leffler R -module.

Proof. Choose a surjection $F \rightarrow M$ where F is a free R -module. By Algebra, Lemma 93.20 the module M^\wedge is a direct summand of the module F^\wedge . Hence it suffices to prove the lemma for F . In this case the lemma follows from Lemma 8.1. \square

In Lemmas 8.3 and 8.4 the assumption that S be Noetherian holds if $R \rightarrow S$ is of finite type, see Algebra, Lemma 30.1.

Lemma 8.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \rightarrow S$ be a ring map, and N an S -module. Assume*

- (1) *R is a Noetherian ring,*
- (2) *S is a Noetherian ring,*
- (3) *N is a finite S -module, and*
- (4) *for any finite R -module Q , any $\mathfrak{q} \in \text{Ass}_S(Q \otimes_R N)$ satisfies $IS + \mathfrak{q} \neq S$.*

Then the map $N \rightarrow N^\wedge$ of N into the I -adic completion of N is universally injective as a map of R -modules.

Proof. We have to show that for any finite R -module Q the map $Q \otimes_R N \rightarrow Q \otimes_R N^\wedge$ is injective, see Algebra, Theorem 79.3. As there is a canonical map $Q \otimes_R N^\wedge \rightarrow (Q \otimes_R N)^\wedge$ it suffices to prove that the canonical map $Q \otimes_R N \rightarrow (Q \otimes_R N)^\wedge$ is injective. Hence we may replace N by $Q \otimes_R N$ and it suffices to prove the injectivity for the map $N \rightarrow N^\wedge$.

Let $K = \text{Ker}(N \rightarrow N^\wedge)$. It suffices to show that $K_{\mathfrak{q}} = 0$ for $\mathfrak{q} \in \text{Ass}(N)$ as N is a submodule of $\prod_{\mathfrak{q} \in \text{Ass}(N)} N_{\mathfrak{q}}$, see Algebra, Lemma 62.18. Pick $\mathfrak{q} \in \text{Ass}(N)$. By the last assumption we see that there exists a prime $\mathfrak{q}' \supset IS + \mathfrak{q}$. Since $K_{\mathfrak{q}}$ is a localization of $K_{\mathfrak{q}'}$ it suffices to prove the vanishing of $K_{\mathfrak{q}'}$. Note that $K = \bigcap I^n N$, hence $K_{\mathfrak{q}'} \subset \bigcap I^n N_{\mathfrak{q}'}$. Hence $K_{\mathfrak{q}'} = 0$ by Algebra, Lemma 49.4. \square

Lemma 8.4. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \rightarrow S$ be a ring map, and N an S -module. Assume*

- (1) *R is a Noetherian ring,*
- (2) *S is a Noetherian ring,*
- (3) *N is a finite S -module,*
- (4) *N is flat over R , and*
- (5) *for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.*

Then the map $N \rightarrow N^\wedge$ of N into the I -adic completion of N is universally injective as a map of R -modules.

Proof. This follows from Lemma 8.3 because Algebra, Lemma 64.5 and Remark 64.6 guarantee that the set of associated primes of tensor products $N \otimes_R Q$ are contained in the set of associated primes of the modules $N \otimes_R \kappa(\mathfrak{p})$. \square

9. Projective modules

The following lemma can be used to prove projectivity by Noetherian induction on the base, see Lemma 9.2.

Lemma 9.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \rightarrow S$ be a ring map, and N an S -module. Assume*

- (1) *R is Noetherian and I -adically complete,*
- (2) *$R \rightarrow S$ is of finite type,*
- (3) *N is a finite S -module,*
- (4) *N is flat over R ,*
- (5) *N/IN is projective as a R/I -module, and*
- (6) *for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.*

Then N is projective as an R -module.

Proof. By Lemma 8.4 the map $N \rightarrow N^\wedge$ is universally injective. By Lemma 8.2 the module N^\wedge is Mittag-Leffler. By Algebra, Lemma 86.7 we conclude that N is Mittag-Leffler. Hence N is countably generated, flat and Mittag-Leffler as an R -module, whence projective by Algebra, Lemma 90.1. \square

Lemma 9.2. *Let R be a ring. Let $R \rightarrow S$ be a ring map. Assume*

- (1) *R is Noetherian,*
- (2) *$R \rightarrow S$ is of finite type and flat, and*
- (3) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R -module.

Proof. Consider the set

$$\{I \subset R \mid S/IS \text{ not projective as } R/I\text{-module}\}$$

We have to show this set is empty. To get a contradiction assume it is nonempty. Then it contains a maximal element I . Let $J = \sqrt{I}$ be its radical. If $I \neq J$, then S/JS is projective as a R/J -module, and S/IS is flat over R/I and J/I is a nilpotent ideal in R/I . Applying Algebra, Lemma 74.5 we see that S/IS is a projective R/I -module, which is a contradiction. Hence we may assume that I is a radical ideal. In other words we are reduced to proving the lemma in case R is a reduced ring and S/IS is a projective R/I -module for every nonzero ideal I of R .

Assume R is a reduced ring and S/IS is a projective R/I -module for every nonzero ideal I of R . By generic flatness, Algebra, Lemma 114.1 (applied to a localization R_g which is a domain) or the more general Algebra, Lemma 114.7 there exists a nonzero $f \in R$ such that S_f is free as an R_f -module. Denote $R^\wedge = \lim R/(f^n)$ the (f) -adic completion of R . Note that the ring map

$$R \longrightarrow R_f \times R^\wedge$$

is a faithfully flat ring map, see Algebra, Lemma 93.3. Hence by faithfully flat descent of projectivity, see Algebra, Theorem 92.5 it suffices to prove that $S \otimes_R R^\wedge$ is a projective R^\wedge -module. To see this we will use the criterion of Lemma 9.1. First of all, note that $S/fS = (S \otimes_R R^\wedge)/f(S \otimes_R R^\wedge)$ is a projective $R/(f)$ -module and that $S \otimes_R R^\wedge$ is flat and of finite type over R^\wedge as a base change of such. Next, suppose that \mathfrak{p}^\wedge is a prime ideal of R^\wedge . Let $\mathfrak{p} \subset R$ be the corresponding prime of R . As $R \rightarrow S$ has geometrically integral fibre rings, the same is true for the fibre rings of any base change. Hence $\mathfrak{q}^\wedge = \mathfrak{p}^\wedge(S \otimes_R R^\wedge)$, is a prime ideal lying over \mathfrak{p}^\wedge and it is the unique associated prime of $S \otimes_R \kappa(\mathfrak{p}^\wedge)$. Thus we win if $f(S \otimes_R R^\wedge) + \mathfrak{q}^\wedge \neq S \otimes_R R^\wedge$. This is true because $\mathfrak{p}^\wedge + fR^\wedge \neq R^\wedge$ as f lies in the radical of the f -adically complete ring R^\wedge and because $R^\wedge \rightarrow S \otimes_R R^\wedge$ is surjective on spectra as its fibres are nonempty (irreducible spaces are nonempty). \square

Lemma 9.3. *Let R be a ring. Let $R \rightarrow S$ be a ring map. Assume*

- (1) *$R \rightarrow S$ is of finite presentation and flat, and*
- (2) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R -module.

Proof. We can find a cocartesian diagram of rings

$$\begin{array}{ccc} S_0 & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_0 & \longrightarrow & R \end{array}$$

such that R_0 is of finite type over \mathbf{Z} , the map $R_0 \rightarrow S_0$ is of finite type and flat with geometrically integral fibres, see More on Morphisms, Lemmas 26.4, 26.6, 26.7, and 26.11. By Lemma 9.2 we see that S_0 is a projective R_0 -module. Hence $S = S_0 \otimes_{R_0} R$ is a projective R -module, see Algebra, Lemma 91.1. \square

Remark 9.4. Lemma 9.3 is a key step in the development of results in this chapter. The analogue of this lemma in [GR71] is [GR71, I Proposition 3.3.1]: If $R \rightarrow S$ is smooth with geometrically integral fibres, then S is projective as an R -module. This is a special case of Lemma 9.3, but as we will later improve on this lemma anyway, we do not gain much from having a stronger result at this point. We briefly sketch the proof of this as it is given in [GR71].

- (1) First reduce to the case where R is Noetherian as above.
- (2) Since projectivity descends through faithfully flat ring maps, see Algebra, Theorem 92.5 we may work locally in the fppf topology on R , hence we may assume that $R \rightarrow S$ has a section $\sigma : S \rightarrow R$. (Just by the usual trick of base changing to S .) Set $I = \text{Ker}(S \rightarrow R)$.
- (3) Localizing a bit more on R we may assume that I/I^2 is a free R -module and that the completion S^\wedge of S with respect to I is isomorphic to $R[[t_1, \dots, t_n]]$, see Morphisms, Lemma 35.20. Here we are using that $R \rightarrow S$ is smooth.
- (4) To prove that S is projective as an R -module, it suffices to prove that S is flat, countably generated and Mittag-Leffler as an R -module, see Algebra, Lemma 90.1. The first two properties are evident. Thus it suffices to prove that S is Mittag-Leffler as an R -module. By Algebra, Lemma 88.4 the module $R[[t_1, \dots, t_n]]$ is Mittag-Leffler over R . Hence Algebra, Lemma 86.7 shows that it suffices to show that the $S \rightarrow S^\wedge$ is universally injective as a map of R -modules.

- (5) Apply Lemma 7.4 to see that $S \rightarrow S^\wedge$ is R -universally injective. Namely, as $R \rightarrow S$ has geometrically integral fibres, any associated point of any fibre ring is just the generic point of the fibre ring which is in the image of $\text{Spec}(S^\wedge) \rightarrow \text{Spec}(S)$.

There is an analogy between the proof as sketched just now, and the development of the arguments leading to the proof of Lemma 9.3. In both a completion plays an essential role, and both times the assumption of having geometrically integral fibres assures one that the map from S to the completion of S is R -universally injective.

10. Flat finite type modules, Part I

In some cases given a ring map $R \rightarrow S$ of finite presentation and a finite S -module N the flatness of N over R implies that N is of finite presentation. In this section we prove this is true “pointwise”. We remark that the first proof of Proposition 10.3 uses the geometric results of Section 3 but not the existence of a complete dévissage.

Lemma 10.1. *Let (R, \mathfrak{m}) be a local ring. Let $R \rightarrow S$ be a finitely presented flat ring map with geometrically integral fibres. Write $\mathfrak{p} = \mathfrak{m}S$. Let $\mathfrak{q} \subset S$ be a prime ideal lying over \mathfrak{m} . Let N be a finite S -module. There exist $r \geq 0$ and an S -module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \rightarrow N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

- (1) $N_{\mathfrak{q}}$ is R -flat,
- (2) α is R -universally injective and $\text{Coker}(\alpha)_{\mathfrak{q}}$ is R -flat,
- (3) α is injective and $\text{Coker}(\alpha)_{\mathfrak{q}}$ is R -flat,
- (4) $\alpha_{\mathfrak{p}}$ is an isomorphism and $\text{Coker}(\alpha)_{\mathfrak{q}}$ is R -flat, and
- (5) $\alpha_{\mathfrak{q}}$ is injective and $\text{Coker}(\alpha)_{\mathfrak{q}}$ is R -flat.

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \dots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \dots, s_r) = \sum s_i x_i$. This proves the existence.

Fix an α . The most interesting implication is (1) \Rightarrow (2) which we prove first. Assume (1). Because $S/\mathfrak{m}S$ is a domain with fraction field $\kappa(\mathfrak{p})$ we see that $(S/\mathfrak{m}S)^{\oplus r} \rightarrow N_{\mathfrak{p}}/\mathfrak{m}N_{\mathfrak{p}} = N \otimes_S \kappa(\mathfrak{p})$ is injective. Hence by Lemmas 7.5 and 9.3. the map $S^{\oplus r} \rightarrow N_{\mathfrak{p}}$ is R -universally injective. It follows that $S^{\oplus r} \rightarrow N$ is R -universally injective, see Algebra, Lemma 79.10. Then also the localization $\alpha_{\mathfrak{q}}$ is R -universally injective, see Algebra, Lemma 79.13. We conclude that $\text{Coker}(\alpha)_{\mathfrak{q}}$ is R -flat by Algebra, Lemma 79.7.

The implication (2) \Rightarrow (3) is immediate. If (3) holds, then $\alpha_{\mathfrak{p}}$ is injective as a localization of an injective module map. By Nakayama’s lemma (Algebra, Lemma 19.1) $\alpha_{\mathfrak{p}}$ is surjective too. Hence (3) \Rightarrow (4). If (4) holds, then $\alpha_{\mathfrak{p}}$ is an isomorphism, so α is injective as $S_{\mathfrak{q}} \rightarrow S_{\mathfrak{p}}$ is injective. Namely, elements of $S \setminus \mathfrak{p}$ are nonzerodivisors on S by a combination of Lemmas 7.6 and 9.3. Hence (4) \Rightarrow (5). Finally, if (5) holds, then $N_{\mathfrak{q}}$ is R -flat as an extension of flat modules, see Algebra, Lemma 38.12. Hence (5) \Rightarrow (1) and the proof is finished. \square

Lemma 10.2. *Let (R, \mathfrak{m}) be a local ring. Let $R \rightarrow S$ be a ring map of finite presentation. Let N be a finite S -module. Let \mathfrak{q} be a prime of S lying over \mathfrak{m} . Assume that $N_{\mathfrak{q}}$ is flat over R , and assume there exists a complete dévissage of*

$N/S/R$ at \mathfrak{q} . Then N is a finitely presented S -module, free as an R -module, and there exists an isomorphism

$$N \cong B_1^{\oplus r_1} \oplus \dots \oplus B_n^{\oplus r_n}$$

as R -modules where each B_i is a smooth R -algebra with geometrically irreducible fibres.

Proof. Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1, \dots, n}$ be the given complete dévissage. We prove the lemma by induction on n . Note that N is finitely presented as an S -module if and only if M_1 is finitely presented as a B_1 -module, see Remark 6.3. Note that $N_{\mathfrak{q}} \cong (M_1)_{\mathfrak{q}_1}$ as R -modules because (a) $N_{\mathfrak{q}} \cong (M_1)_{\mathfrak{q}'_1}$ where \mathfrak{q}'_1 is the unique prime in A_1 lying over \mathfrak{q}_1 and (b) $(A_1)_{\mathfrak{q}'_1} = (A_1)_{\mathfrak{q}_1}$ by Algebra, Lemma 40.11, so (c) $(M_1)_{\mathfrak{q}'_1} \cong (M_1)_{\mathfrak{q}_1}$. Hence $(M_1)_{\mathfrak{q}_1}$ is a flat R -module. Thus we may replace (S, N) by (B_1, M_1) in order to prove the lemma. By Lemma 10.1 the map $\alpha_1 : B_1^{\oplus r_1} \rightarrow M_1$ is R -universally injective and $\text{Coker}(\alpha_1)_{\mathfrak{q}}$ is R -flat. Note that $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=2, \dots, n}$ is a complete dévissage of $\text{Coker}(\alpha_1)/B_1/R$ at \mathfrak{q}_1 . Hence the induction hypothesis implies that $\text{Coker}(\alpha_1)$ is finitely presented as a B_1 -module, free as an R -module, and has a decomposition as in the lemma. This implies that M_1 is finitely presented as a B_1 -module, see Algebra, Lemma 5.3. It further implies that $M_1 \cong B_1^{\oplus r_1} \oplus \text{Coker}(\alpha_1)$ as R -modules, hence a decomposition as in the lemma. Finally, B_1 is projective as an R -module by Lemma 9.3 hence free as an R -module by Algebra, Theorem 82.4. This finishes the proof. \square

Proposition 10.3. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $x \in X$ with image $s \in S$. Assume that*

- (1) *f is locally of finite presentation,*
- (2) *\mathcal{F} is of finite type, and*
- (3) *\mathcal{F} is flat at x over S .*

Then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and an open subscheme

$$V \subset X \times_S \text{Spec}(\mathcal{O}_{S', s'})$$

which contains the unique point of $X \times_S \text{Spec}(\mathcal{O}_{S', s'})$ mapping to x such that the pullback of \mathcal{F} to V is an \mathcal{O}_V -module of finite presentation and flat over $\mathcal{O}_{S', s'}$.

First proof. This proof is longer but does not use the existence of a complete dévissage. The problem is local around x and s , hence we may assume that X and S are affine. During the proof we will finitely many times replace S by an elementary étale neighbourhood of (S, s) . The goal is then to find (after such a replacement) an open $V \subset X \times_S \text{Spec}(\mathcal{O}_{S, s})$ containing x such that $\mathcal{F}|_V$ is flat over S and finitely presented. Of course we may also replace S by $\text{Spec}(\mathcal{O}_{S, s})$ at any point of the proof, i.e., we may assume S is a local scheme. We will prove the lemma by induction on the integer $n = \dim_x(\text{Supp}(\mathcal{F}_s))$.

We can choose

- (1) elementary étale neighbourhoods $g : (X', x') \rightarrow (X, x)$, $e : (S', s') \rightarrow (S, s)$,

(2) a commutative diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{g} & X' & \xleftarrow{i} & Z' \\
 \downarrow f & & \downarrow & & \downarrow \pi \\
 & & & & Y' \\
 & & & & \downarrow h \\
 S & \xleftarrow{e} & S' & \xlongequal{\quad} & S'
 \end{array}$$

(3) a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,

(4) a finite type quasi-coherent $\mathcal{O}_{Z'}$ -module \mathcal{G} ,

as in Lemma 3.2. We are going to replace S by $\text{Spec}(\mathcal{O}_{S',s'})$, see remarks in first paragraph of the proof. Consider the diagram

$$\begin{array}{ccccc}
 X_{\mathcal{O}_{S',s'}} & \xleftarrow{g} & X'_{\mathcal{O}_{S',s'}} & \xleftarrow{i} & Z'_{\mathcal{O}_{S',s'}} \\
 & \searrow f & \downarrow & & \downarrow \pi \\
 & & & & Y'_{\mathcal{O}_{S',s'}} \\
 & & & \nearrow h & \\
 & & \text{Spec}(\mathcal{O}_{S',s'}) & &
 \end{array}$$

Here we have base changed the schemes X', Z', Y' over S' via $\text{Spec}(\mathcal{O}_{S',s'}) \rightarrow S'$ and the scheme X over S via $\text{Spec}(\mathcal{O}_{S',s'}) \rightarrow S$. It is still the case that g is étale, see Lemma 2.2. After replacing X by $X_{\mathcal{O}_{S',s'}}$, X' by $X'_{\mathcal{O}_{S',s'}}$, Z' by $Z'_{\mathcal{O}_{S',s'}}$, and Y' by $Y'_{\mathcal{O}_{S',s'}}$, we may assume we have a diagram as Lemma 3.2 where in addition $S = S'$ is a local scheme with closed point s . By Lemmas 3.3 and 3.4 the result for $Y' \rightarrow S$, the sheaf $\pi_*\mathcal{G}$, and the point y' implies the result for $X \rightarrow S$, \mathcal{F} and x . Hence we may assume that S is local and $X \rightarrow S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension n .

The base case of the induction: $n = 0$. As $X \rightarrow S$ is smooth with geometrically irreducible fibres of dimension 0 we see that $X \rightarrow S$ is an open immersion, see Descent, Lemma 21.2. As S is local and the closed point is in the image of $X \rightarrow S$ we conclude that $X = S$. Thus we see that \mathcal{F} corresponds to a finite flat $\mathcal{O}_{S,s}$ module. In this case the result follows from Algebra, Lemma 75.4 which tells us that \mathcal{F} is in fact finite free.

The induction step. Assume the result holds whenever the dimension of the support in the closed fibre is $< n$. Write $S = \text{Spec}(A)$, $X = \text{Spec}(B)$ and $\mathcal{F} = \tilde{N}$ for some B -module N . Note that A is a local ring; denote its maximal ideal \mathfrak{m} . Then $\mathfrak{p} = \mathfrak{m}B$ is the unique minimal prime lying over \mathfrak{m} as $X \rightarrow S$ has geometrically irreducible fibres. Finally, let $\mathfrak{q} \subset B$ be the prime corresponding to x . By Lemma 10.1 we can choose a map

$$\alpha : B^{\oplus r} \rightarrow N$$

such that $\kappa(\mathfrak{p})^{\oplus r} \rightarrow N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism. Moreover, as $N_{\mathfrak{q}}$ is A -flat the lemma also shows that α is injective and that $\text{Coker}(\alpha)_{\mathfrak{q}}$ is A -flat. Set $Q = \text{Coker}(\alpha)$. Note that the support of $Q/\mathfrak{m}Q$ does not contain \mathfrak{p} . Hence it is certainly the case that $\dim_{\mathfrak{q}}(\text{Supp}(Q/\mathfrak{m}Q)) < n$. Combining everything we know about Q

we see that the induction hypothesis applies to Q . It follows that there exists an elementary étale morphism $(S', s) \rightarrow (S, s)$ such that the conclusion holds for $Q \otimes_A A'$ over $B \otimes_A A'$ where $A' = \mathcal{O}_{S', s'}$. After replacing A by A' we have an exact sequence

$$0 \rightarrow B^{\oplus r} \rightarrow N \rightarrow Q \rightarrow 0$$

(here we use that α is injective as mentioned above) of finite B -modules and we also get an element $g \in B$, $g \notin \mathfrak{q}$ such that Q_g is finitely presented over B_g and flat over A . Since localization is exact we see that

$$0 \rightarrow B_g^{\oplus r} \rightarrow N_g \rightarrow Q_g \rightarrow 0$$

is still exact. As B_g and Q_g are flat over A we conclude that N_g is flat over A , see Algebra, Lemma 38.12, and as B_g and Q_g are finitely presented over B_g the same holds for N_g , see Algebra, Lemma 5.3. \square

Second proof. We apply Proposition 5.7 to find a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x . (In particular S' and X' are affine.) By Morphisms, Lemma 26.11 we see that $g^*\mathcal{F}$ is flat at x' over S and by Lemma 2.3 we see that it is flat at x' over S' . Via Remark 6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage at the prime of $\Gamma(X', \mathcal{O}_{X'})$ corresponding to x' . We may base change this complete dévissage to the local ring $\mathcal{O}_{S', s'}$ of $\Gamma(S', \mathcal{O}_{S'})$ at the prime corresponding to s' . Thus Lemma 10.2 implies that

$$\Gamma(X', \mathcal{F}) \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S', s'}$$

is flat over $\mathcal{O}_{S', s'}$ and of finite presentation over $\Gamma(X', \mathcal{O}_{X'}) \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S', s'}$. In other words, the restriction of \mathcal{F} to $X' \times_{S'} \text{Spec}(\mathcal{O}_{S', s'})$ is of finite presentation and flat over $\mathcal{O}_{S', s'}$. Since the morphism $X' \times_{S'} \text{Spec}(\mathcal{O}_{S', s'}) \rightarrow X \times_S \text{Spec}(\mathcal{O}_{S', s'})$ is étale (Lemma 2.2) its image $V \subset X \times_S \text{Spec}(\mathcal{O}_{S', s'})$ is an open subscheme, and by étale descent the restriction of \mathcal{F} to V is of finite presentation and flat over $\mathcal{O}_{S', s'}$. (Results used: Morphisms, Lemma 37.13, Descent, Lemma 6.3, and Morphisms, Lemma 26.11.) \square

Lemma 10.4. *Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Let $s \in S$. Then the set*

$$\{x \in X_s \mid \mathcal{F} \text{ flat over } S \text{ at } x\}$$

is open in the fibre X_s .

Proof. Suppose $x \in U$. Choose an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and open $V \subset X \times_S \text{Spec}(\mathcal{O}_{S', s'})$ as in Proposition 10.3. Note that $X_{s'} = X_s$ as $\kappa(s) = \kappa(s')$. If $x' \in V \cap X_{s'}$, then the pullback of \mathcal{F} to $X \times_S S'$ is flat over S' at x' . Hence \mathcal{F} is flat at x' over S , see Morphisms, Lemma 26.11. In other words $X_s \cap V \subset U$ is an open neighbourhood of x in U . \square

Lemma 10.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $x \in X$ with image $s \in S$. Assume that*

- (1) *f is locally of finite type,*
- (2) *\mathcal{F} is of finite type, and*
- (3) *\mathcal{F} is flat at x over S .*

Then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'})$$

which contains the unique point of $X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'})$ mapping to x such that the pullback of \mathcal{F} to V is flat over $\mathcal{O}_{S', s'}$.

Proof. (The only difference between this and Proposition 10.3 is that we do not assume f is of finite presentation.) The question is local on X and S , hence we may assume X and S are affine. Write $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$ and write $B = A[x_1, \dots, x_n]/I$. In other words we obtain a closed immersion $i : X \rightarrow \mathbf{A}_S^n$. Denote $t = i(x) \in \mathbf{A}_S^n$. We may apply Proposition 10.3 to $\mathbf{A}_S^n \rightarrow S$, the sheaf $i_*\mathcal{F}$ and the point t . We obtain an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and an open subscheme

$$W \subset \mathbf{A}_{\mathcal{O}_{S', s'}}^n$$

such that the pullback of $i_*\mathcal{F}$ to W is flat over $\mathcal{O}_{S', s'}$. This means that $V := W \cap (X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'}))$ is the desired open subscheme. \square

Lemma 10.6. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $s \in S$. Assume that*

- (1) *f is of finite presentation,*
- (2) *\mathcal{F} is of finite type, and*
- (3) *\mathcal{F} is flat over S at every point of the fibre X_s .*

Then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of \mathcal{F} to V is an \mathcal{O}_V -module of finite presentation and flat over $\mathcal{O}_{S', s'}$.

Proof. For every point $x \in X_s$ we can use Proposition 10.3 to find an elementary étale neighbourhood $(S_x, s_x) \rightarrow (S, s)$ and an open $V_x \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S_x, s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of \mathcal{F} to V_x is an \mathcal{O}_{V_x} -module of finite presentation and flat over \mathcal{O}_{S_x, s_x} . In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s . Because X_s is quasi-compact we can find a finite number of points $x_1, \dots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}) , see More on Morphisms, Lemma 27.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S_{x_i}, s_{x_i}})$$

By construction V contains X_s and by construction the pullback of \mathcal{F} to V is an \mathcal{O}_V -module of finite presentation and flat over $\mathcal{O}_{S', s'}$. \square

Lemma 10.7. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $s \in S$. Assume that*

- (1) f is of finite type,
- (2) \mathcal{F} is of finite type, and
- (3) \mathcal{F} is flat over S at every point of the fibre X_s .

Then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of \mathcal{F} to V is flat over $\mathcal{O}_{S', s'}$.

Proof. (The only difference between this and Lemma 10.6 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 10.5 to find an elementary étale neighbourhood $(S_x, s_x) \rightarrow (S, s)$ and an open $V_x \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S_x, s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of \mathcal{F} to V_x is flat over \mathcal{O}_{S_x, s_x} . In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s . Because X_s is quasi-compact we can find a finite number of points $x_1, \dots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}) , see More on Morphisms, Lemma 27.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S_{x_i}, s_{x_i}})$$

By construction V contains X_s and by construction the pullback of \mathcal{F} to V is flat over $\mathcal{O}_{S', s'}$. \square

Lemma 10.8. *Let S be a scheme. Let X be locally of finite type over S . Let $x \in X$ with image $s \in S$. If X is flat at x over S , then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and an open subscheme*

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'})$$

which contains the unique point of $X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'})$ mapping to x such that $V \rightarrow \operatorname{Spec}(\mathcal{O}_{S', s'})$ is flat and of finite presentation.

Proof. The question is local on X and S , hence we may assume X and S are affine. Write $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$ and write $B = A[x_1, \dots, x_n]/I$. In other words we obtain a closed immersion $i : X \rightarrow \mathbf{A}_S^n$. Denote $t = i(x) \in \mathbf{A}_S^n$. We may apply Proposition 10.3 to $\mathbf{A}_S^n \rightarrow S$, the sheaf $\mathcal{F} = i_* \mathcal{O}_X$ and the point t . We obtain an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and an open subscheme

$$W \subset \mathbf{A}_{\mathcal{O}_{S', s'}}^n$$

such that the pullback of $i_* \mathcal{O}_X$ is flat and of finite presentation. This means that $V := W \cap (X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'}))$ is the desired open subscheme. \square

Lemma 10.9. *Let $f : X \rightarrow S$ be a morphism which is locally of finite presentation. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. If $x \in X$ and \mathcal{F} is flat at x over S , then \mathcal{F}_x is an $\mathcal{O}_{X, x}$ -module of finite presentation.*

Proof. Let $s = f(x)$. By Proposition 10.3 there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ such that the pullback of \mathcal{F} to $X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'})$ is of finite presentation in a neighbourhood of the point $x' \in X_{s'} = X_s$ corresponding to x . The ring map

$$\mathcal{O}_{X, x} \longrightarrow \mathcal{O}_{X \times_S \operatorname{Spec}(\mathcal{O}_{S', s'}), x'} = \mathcal{O}_{X \times_S S', x'}$$

is flat and local as a localization of an étale ring map. Hence \mathcal{F}_x is of finite presentation over $\mathcal{O}_{X,x}$ by descent, see Algebra, Lemma 80.2 (and also that a flat local ring map is faithfully flat, see Algebra, Lemma 38.16). \square

Lemma 10.10. *Let $f : X \rightarrow S$ be a morphism which is locally of finite type. Let $x \in X$ with image $s \in S$. If f is flat at x over S , then $\mathcal{O}_{X,x}$ is essentially of finite presentation over $\mathcal{O}_{S,s}$.*

Proof. We may assume X and S affine. Write $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and write $B = A[x_1, \dots, x_n]/I$. In other words we obtain a closed immersion $i : X \rightarrow \mathbf{A}_S^n$. Denote $t = i(x) \in \mathbf{A}_S^n$. We may apply Lemma 10.9 to $\mathbf{A}_S^n \rightarrow S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t . We conclude that $\mathcal{O}_{X,x}$ is of finite presentation over $\mathcal{O}_{\mathbf{A}_S^n, t}$ which implies what we want. \square

Lemma 10.11. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $U \subset S$ be open. Assume*

- (1) *f is locally of finite presentation,*
- (2) *\mathcal{F} is of finite type and flat over S ,*
- (3) *$U \subset S$ is retrocompact and scheme theoretically dense,*
- (4) *$\mathcal{F}|_{f^{-1}U}$ is of finite presentation.*

Then \mathcal{F} is of finite presentation.

Proof. The problem is local on X and S , hence we may assume X and S affine. Write $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$. Let N be a finite B -module such that \mathcal{F} is the quasi-coherent sheaf associated to N . We have $U = D(f_1) \cup \dots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 28.1. As U is schematically dense the map $A \rightarrow A_{f_1} \times \dots \times A_{f_n}$ is injective. Pick a prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ corresponding to $x \in X$ mapping to $s \in S$. By Lemma 10.9 the module $N_{\mathfrak{q}}$ is of finite presentation over $B_{\mathfrak{q}}$. Choose a surjection $\varphi : B^{\oplus m} \rightarrow N$ of B -modules. Choose $k_1, \dots, k_t \in \text{Ker}(\varphi)$ and set $N' = B^{\oplus m} / \sum Bk_j$. There is a canonical surjection $N' \rightarrow N$ and N is the filtered colimit of the B -modules N' constructed in this manner. Thus we see that we can choose k_1, \dots, k_t such that (a) $N'_{f_i} \cong N_{f_i}$, $i = 1, \dots, n$ and (b) $N'_{\mathfrak{q}} \cong N_{\mathfrak{q}}$. This in particular implies that $N'_{\mathfrak{q}}$ is flat over A . By openness of flatness, see Algebra, Theorem 125.4 we conclude that there exists a $g \in B$, $g \notin \mathfrak{q}$ such that N'_g is flat over A . Consider the commutative diagram

$$\begin{array}{ccc} N'_g & \longrightarrow & N_g \\ \downarrow & & \downarrow \\ \prod N'_{gf_i} & \longrightarrow & \prod N_{gf_i} \end{array}$$

The bottom arrow is an isomorphism by choice of k_1, \dots, k_t . The left vertical arrow is an injective map as $A \rightarrow \prod A_{f_i}$ is injective and N'_g is flat over A . Hence the top horizontal arrow is injective, hence an isomorphism. This proves that N_g is of finite presentation over B_g . We conclude by applying Algebra, Lemma 23.2. \square

Lemma 10.12. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $U \subset S$ be open. Assume*

- (1) *f is locally of finite type and flat,*
- (2) *$U \subset S$ is retrocompact and scheme theoretically dense,*
- (3) *$f|_{f^{-1}U} : f^{-1}U \rightarrow U$ is locally of finite presentation.*

Then f is of locally of finite presentation.

Proof. The question is local on X and S , hence we may assume X and S affine. Choose a closed immersion $i : X \rightarrow \mathbf{A}_S^n$ and apply Lemma 10.11 to $i_*\mathcal{O}_X$. Some details omitted. \square

11. Flat finitely presented modules

In some cases given a ring map $R \rightarrow S$ of finite presentation and a finitely presented S -module N the flatness of N over R implies that N is projective as an R -module, at least after replacing S by an étale extension. In this section we collect a some results of this nature.

Lemma 11.1. *Let R be a ring. Let $R \rightarrow S$ be a finitely presented flat ring map with geometrically integral fibres. Let $\mathfrak{q} \subset S$ be a prime ideal lying over the prime $\mathfrak{r} \subset R$. Set $\mathfrak{p} = \mathfrak{r}S$. Let N be a finitely presented S -module. There exists $r \geq 0$ and an S -module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \rightarrow N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

- (1) $N_{\mathfrak{q}}$ is R -flat,
- (2) *there exists an $f \in R$, $f \notin \mathfrak{r}$ such that $\alpha_f : S_f^{\oplus r} \rightarrow N_f$ is R_f -universally injective and a $g \in S$, $g \notin \mathfrak{q}$ such that $\text{Coker}(\alpha)_g$ is R -flat,*
- (3) $\alpha_{\mathfrak{r}}$ is $R_{\mathfrak{r}}$ -universally injective and $\text{Coker}(\alpha)_{\mathfrak{q}}$ is R -flat
- (4) $\alpha_{\mathfrak{r}}$ is injective and $\text{Coker}(\alpha)_{\mathfrak{q}}$ is R -flat,
- (5) $\alpha_{\mathfrak{p}}$ is an isomorphism and $\text{Coker}(\alpha)_{\mathfrak{q}}$ is R -flat, and
- (6) $\alpha_{\mathfrak{q}}$ is injective and $\text{Coker}(\alpha)_{\mathfrak{q}}$ is R -flat.

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \dots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \dots, s_r) = \sum s_i x_i$. This proves the existence.

Fix a choice of α . We may apply Lemma 10.1 to the map $\alpha_{\mathfrak{r}} : S_{\mathfrak{r}}^{\oplus r} \rightarrow N_{\mathfrak{r}}$. Hence we see that (1), (3), (4), (5), and (6) are all equivalent. Since it is also clear that (2) implies (3) we see that all we have to do is show that (1) implies (2).

Assume (1). By openness of flatness, see Algebra, Theorem 125.4, the set

$$U_1 = \{\mathfrak{q}' \subset S \mid N_{\mathfrak{q}'} \text{ is flat over } R\}$$

is open in $\text{Spec}(S)$. It contains \mathfrak{q} by assumption and hence \mathfrak{p} . Because $S^{\oplus r}$ and N are finitely presented S -modules the set

$$U_2 = \{\mathfrak{q}' \subset S \mid \alpha_{\mathfrak{q}'} \text{ is an isomorphism}\}$$

is open in $\text{Spec}(S)$, see Algebra, Lemma 76.2. It contains \mathfrak{p} by (5). As $R \rightarrow S$ is finitely presented and flat the map $\Phi : \text{Spec}(S) \rightarrow \text{Spec}(R)$ is open, see Algebra, Proposition 40.8. For any prime $\mathfrak{r}' \in \Phi(U_1 \cap U_2)$ we see that there exists a prime \mathfrak{q}' lying over \mathfrak{r}' such that $N_{\mathfrak{q}'}$ is flat and such that $\alpha_{\mathfrak{q}'}$ is an isomorphism, which implies that $\alpha \otimes \kappa(\mathfrak{p}')$ is an isomorphism where $\mathfrak{p}' = \mathfrak{r}'S$. Thus $\alpha_{\mathfrak{r}'}$ is $R_{\mathfrak{r}'}$ -universally injective by the implication (1) \Rightarrow (3). Hence if we pick $f \in R$, $f \notin \mathfrak{r}$ such that $D(f) \subset \Phi(U_1 \cap U_2)$ then we conclude that α_f is R_f -universally injective, see Algebra, Lemma 79.12. The same reasoning also shows that for any $\mathfrak{q}' \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ the module $\text{Coker}(\alpha)_{\mathfrak{q}'}$ is R -flat. Note that $\mathfrak{q} \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$. Hence we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g) \subset U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ and we win. \square

Lemma 11.2. *Let $R \rightarrow S$ be a ring map of finite presentation. Let N be a finitely presented S -module flat over R . Let $\mathfrak{r} \subset R$ be a prime ideal. Assume there exists a complete dévissage of $N/S/R$ over \mathfrak{r} . Then there exists an $f \in R$, $f \notin \mathfrak{r}$ such that*

$$N_f \cong B_1^{\oplus r_1} \oplus \dots \oplus B_n^{\oplus r_n}$$

as R -modules where each B_i is a smooth R_f -algebra with geometrically irreducible fibres. Moreover, N_f is projective as an R_f -module.

Proof. Let $(A_i, B_i, M_i, \alpha_i)_{i=1, \dots, n}$ be the given complete dévissage. We prove the lemma by induction on n . Note that the assertions of the lemma are entirely about the structure of N as an R -module. Hence we may replace N by M_1 , and we may think of M_1 as a B_1 -module. See Remark 6.3 in order to see why M_1 is of finite presentation as a B_1 -module. By Lemma 11.1 we may, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{r}$, assume the map $\alpha_1 : B_1^{\oplus r_1} \rightarrow M_1$ is R -universally injective. Since M_1 and $B_1^{\oplus r_1}$ are R -flat and finitely presented as B_1 -modules we see that $\text{Coker}(\alpha_1)$ is R -flat (Algebra, Lemma 79.7) and finitely presented as a B_1 -module. Note that $(A_i, B_i, M_i, \alpha_i)_{i=2, \dots, n}$ is a complete dévissage of $\text{Coker}(\alpha_1)$. Hence the induction hypothesis implies that, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{r}$, we may assume that $\text{Coker}(\alpha_1)$ has a decomposition as in the lemma and is projective. In particular $M_1 = B_1^{\oplus r_1} \oplus \text{Coker}(\alpha_1)$. This proves the statement regarding the decomposition. The statement on projectivity follows as B_1 is projective as an R -module by Lemma 9.3. \square

Remark 11.3. There is a variant of Lemma 11.2 where we weaken the flatness condition by assuming only that N is flat at some given prime \mathfrak{q} lying over \mathfrak{r} but where we strengthen the dévissage condition by assuming the existence of a complete dévissage *at* \mathfrak{q} . Compare with Lemma 10.2.

The following is the main result of this section.

Proposition 11.4. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $x \in X$ with image $s \in S$. Assume that*

- (1) *f is locally of finite presentation,*
- (2) *\mathcal{F} is of finite presentation, and*
- (3) *\mathcal{F} is flat at x over S .*

Then there exists a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (S', s') \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that X' , S' are affine and such that $\Gamma(X', g^\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$ -module.*

Proof. By openness of flatness, see More on Morphisms, Theorem 12.1 we may replace X by an open neighbourhood of x and assume that \mathcal{F} is flat over S . Next, we apply Proposition 5.7 to find a diagram as in the statement of the proposition such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage over s' . (In particular S' and X' are affine.) By Morphisms, Lemma 26.11 we see that $g^*\mathcal{F}$ is flat over S and by Lemma 2.3 we see that it is flat over S' . Via Remark 6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage over the prime of $\Gamma(S', \mathcal{O}_{S'})$ corresponding to s' . Thus Lemma 11.2 implies that the result of the proposition holds after replacing S' by a standard open neighbourhood of s' . \square

In the rest of this section we prove a number of variants on this result. The first is a “global” version.

Lemma 11.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $s \in S$. Assume that*

- (1) *f is of finite presentation,*
- (2) *\mathcal{F} is of finite presentation, and*
- (3) *\mathcal{F} is flat over S at every point of the fibre X_s .*

Then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \xleftarrow{\quad} & S' \end{array}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$ -module.*

Proof. For every point $x \in X_s$ we can use Proposition 11.4 to find a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (S_x, s_x) \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that Y_x, S_x are affine and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a projective $\Gamma(S_x, \mathcal{O}_{S_x})$ -module. In particular $g_x(Y_x) \cap X_s$ is an open neighbourhood of x in X_s . Because X_s is quasi-compact we can find a finite number of points $x_1, \dots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}(Y_{x_i}) \cap X_s$. Choose an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}) , see More on Morphisms, Lemma 27.4. We may also assume that S' is affine. Set $X' = \coprod Y_{x_i} \times_{S_{x_i}} S'$ and endow it with the obvious morphism $g : X' \rightarrow X$. By construction $g(X')$ contains X_s and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\Gamma(S_{x_i}, \mathcal{O}_{S_{x_i}})} \Gamma(S', \mathcal{O}_{S'}).$$

This is a projective $\Gamma(S', \mathcal{O}_{S'})$ -module, see Algebra, Lemma 91.1. \square

The following two lemmas are reformulations of the results above in case $\mathcal{F} = \mathcal{O}_X$.

Lemma 11.6. *Let $f : X \rightarrow S$ be locally of finite presentation. Let $x \in X$ with image $s \in S$. If f is flat at x over S , then there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (S', s') \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$ -module.

Proof. This is a special case of Proposition 11.4. \square

Lemma 11.7. *Let $f : X \rightarrow S$ be of finite presentation. Let $s \in S$. If X is flat over S at all points of X_s , then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \xleftarrow{\quad} & S' \end{array}$$

with g étale, $X_s \subset g(X')$, such that X', S' are affine, and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$ -module.

Proof. This is a special case of Lemma 11.5. \square

The following lemmas explain consequences of Proposition 11.4 in case we only assume the morphism and the sheaf are of finite type (and not necessarily of finite presentation).

Lemma 11.8. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $x \in X$ with image $s \in S$. Assume that*

- (1) f is locally of finite presentation,
- (2) \mathcal{F} is of finite type, and
- (3) \mathcal{F} is flat at x over S .

Then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (\mathrm{Spec}(\mathcal{O}_{S', s'}), s') \end{array}$$

such that $X' \rightarrow X \times_S \mathrm{Spec}(\mathcal{O}_{S', s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S', s'}$, the sheaf $g^\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S', s'}$ -module.*

Proof. To prove the lemma we may replace (S, s) by any elementary étale neighbourhood, and we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S, s})$. Hence by Proposition 10.3 we may assume that \mathcal{F} is finitely presented and flat over S in a neighbourhood of x . In this case the result follows from Proposition 11.4 because Algebra, Theorem 82.4 assures us that projective = free over a local ring. \square

Lemma 11.9. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $x \in X$ with image $s \in S$. Assume that*

- (1) f is locally of finite type,
- (2) \mathcal{F} is of finite type, and
- (3) \mathcal{F} is flat at x over S .

Then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (\mathrm{Spec}(\mathcal{O}_{S', s'}), s') \end{array}$$

such that $X' \rightarrow X \times_S \mathrm{Spec}(\mathcal{O}_{S', s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine, and such that $\Gamma(X', g^* \mathcal{F})$ is a free $\mathcal{O}_{S', s'}$ -module.

Proof. (The only difference with Lemma 11.8 is that we do not assume f is of finite presentation.) The problem is local on X and S . Hence we may assume X and S are affine, say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$. Since B is a finite type A -algebra we can find a surjection $A[x_1, \dots, x_n] \rightarrow B$. In other words, we can choose a closed immersion $i : X \rightarrow \mathbf{A}_S^n$. Set $t = i(x)$ and $\mathcal{G} = i_* \mathcal{F}$. Note that $\mathcal{G}_t \cong \mathcal{F}_x$ are $\mathcal{O}_{S, s}$ -modules. Hence \mathcal{G} is flat over S at t . We apply Lemma 11.8 to the morphism $\mathbf{A}_S^n \rightarrow S$, the point t , and the sheaf \mathcal{G} . Thus we can find an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and a commutative diagram of pointed schemes

$$\begin{array}{ccc} (\mathbf{A}_S^n, t) & \xleftarrow{h} & (Y, y) \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (\mathrm{Spec}(\mathcal{O}_{S', s'}), s') \end{array}$$

such that $Y \rightarrow \mathbf{A}_{\mathcal{O}_{S', s'}}^n$ is étale, $\kappa(t) = \kappa(y)$, the scheme Y is affine, and such that $\Gamma(Y, h^* \mathcal{G})$ is a projective $\mathcal{O}_{S', s'}$ -module. Then a solution to the original problem is given by the closed subscheme $X' = Y \times_{\mathbf{A}_S^n} X$ of Y . \square

Lemma 11.10. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $s \in S$. Assume that*

- (1) f is of finite presentation,
- (2) \mathcal{F} is of finite type, and
- (3) \mathcal{F} is flat over S at all points of X_s .

Then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \xleftarrow{\quad} & \mathrm{Spec}(\mathcal{O}_{S', s'}) \end{array}$$

such that $X' \rightarrow X \times_S \mathrm{Spec}(\mathcal{O}_{S', s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S', s'}$, the sheaf $g^* \mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^* \mathcal{F})$ is a free $\mathcal{O}_{S', s'}$ -module.

Proof. For every point $x \in X_s$ we can use Lemma 11.8 to find an elementary étale neighbourhood $(S_x, s_x) \rightarrow (S, s)$ and a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (\text{Spec}(\mathcal{O}_{S_x, s_x}), s_x) \end{array}$$

such that $Y_x \rightarrow X \times_S \text{Spec}(\mathcal{O}_{S_x, s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine of finite presentation over \mathcal{O}_{S_x, s_x} , the sheaf $g_x^* \mathcal{F}$ is of finite presentation over \mathcal{O}_{Y_x} , and such that $\Gamma(Y_x, g_x^* \mathcal{F})$ is a free \mathcal{O}_{S_x, s_x} -module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s . Because X_s is quasi-compact we can find a finite number of points $x_1, \dots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}) , see More on Morphisms, Lemma 27.4. Set

$$X' = \coprod Y_{x_i} \times_{\text{Spec}(\mathcal{O}_{S_{x_i}, s_{x_i}})} \text{Spec}(\mathcal{O}_{S', s'})$$

and endow it with the obvious morphism $g : X' \rightarrow X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^* \mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^* \mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i}, s_{x_i}}} \mathcal{O}_{S', s'}.$$

This is a free $\mathcal{O}_{S', s'}$ -module as a direct sum of base changes of free modules. Some minor details omitted. \square

Lemma 11.11. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $s \in S$. Assume that*

- (1) *f is of finite type,*
- (2) *\mathcal{F} is of finite type, and*
- (3) *\mathcal{F} is flat over S at all points of X_s .*

Then there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \xleftarrow{\quad} & \text{Spec}(\mathcal{O}_{S', s'}) \end{array}$$

such that $X' \rightarrow X \times_S \text{Spec}(\mathcal{O}_{S', s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^ \mathcal{F})$ is a free $\mathcal{O}_{S', s'}$ -module.*

Proof. (The only difference with Lemma 11.10 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 11.9 to find an elementary étale neighbourhood $(S_x, s_x) \rightarrow (S, s)$ and a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{\quad} & (\text{Spec}(\mathcal{O}_{S_x, s_x}), s_x) \end{array}$$

such that $Y_x \rightarrow X \times_S \text{Spec}(\mathcal{O}_{S_x, s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine, and such that $\Gamma(Y_x, g_x^* \mathcal{F})$ is a free \mathcal{O}_{S_x, s_x} -module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s . Because X_s is quasi-compact we can find a finite

number of points $x_1, \dots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}) , see More on Morphisms, Lemma 27.4. Set

$$X' = \coprod Y_{x_i} \times_{\text{Spec}(\mathcal{O}_{S_{x_i}, s_{x_i}})} \text{Spec}(\mathcal{O}_{S', s'})$$

and endow it with the obvious morphism $g : X' \rightarrow X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^* \mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^* \mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i}, s_{x_i}}} \mathcal{O}_{S', s'}.$$

This is a free $\mathcal{O}_{S', s'}$ -module as a direct sum of base changes of free modules. \square

12. Flat finite type modules, Part II

The following lemma will be superseded by the stronger Lemma 12.3 below.

Lemma 12.1. *Let (R, \mathfrak{m}) be a local ring. Let $R \rightarrow S$ be of finite presentation. Let N be a finitely presented S -module which is free as an R -module. Let M be an R -module. Let \mathfrak{q} be a prime of S lying over \mathfrak{m} . Then*

- (1) *if $\mathfrak{q} \in \text{WeakAss}_S(M \otimes_R N)$ then $\mathfrak{m} \in \text{WeakAss}_R(M)$ and $\bar{\mathfrak{q}} \in \text{Ass}_{\bar{S}}(\bar{N})$,*
- (2) *if $\mathfrak{m} \in \text{WeakAss}_R(M)$ and $\bar{\mathfrak{q}} \in \text{Ass}_{\bar{S}}(\bar{N})$ is a maximal element then $\mathfrak{q} \in \text{WeakAss}_S(M \otimes_R N)$.*

Here $\bar{S} = S/\mathfrak{m}S$, $\bar{\mathfrak{q}} = \mathfrak{q}\bar{S}$, and $\bar{N} = N/\mathfrak{m}N$.

Proof. Suppose that $\bar{\mathfrak{q}} \notin \text{Ass}_{\bar{S}}(\bar{N})$. By Algebra, Lemmas 62.9, 62.5, and 14.2 there exists an element $\bar{g} \in \bar{\mathfrak{q}}$ which is not a zerodivisor on \bar{N} . Let $g \in \mathfrak{q}$ be an element which maps to \bar{g} in $\bar{\mathfrak{q}}$. By Lemma 7.6 the map $g : N \rightarrow N$ is R -universally injective. In particular we see that $g : M \otimes_R N \rightarrow M \otimes_R N$ is injective. Clearly this implies that $\mathfrak{q} \notin \text{WeakAss}_S(M \otimes_R N)$. We conclude that $\mathfrak{q} \in \text{WeakAss}_S(M \otimes_R N)$ implies $\bar{\mathfrak{q}} \in \text{Ass}_{\bar{S}}(\bar{N})$.

Assume $\mathfrak{q} \in \text{WeakAss}_S(M \otimes_R N)$. Let $z \in M \otimes_R N$ be an element whose annihilator in S has radical \mathfrak{q} . As N is a free R -module, we can find a finite free direct summand $F \subset N$ such that $z \in M \otimes_R F$. The radical of the annihilator of $z \in M \otimes_R F$ in R is \mathfrak{m} (by our assumption on z and because \mathfrak{q} lies over \mathfrak{m}). Hence we see that $\mathfrak{m} \in \text{WeakAss}_R(M \otimes_R F)$ which implies that $\mathfrak{m} \in \text{WeakAss}_R(M)$ by Algebra, Lemma 65.3. This finishes the proof of (1).

Assume that $\mathfrak{m} \in \text{WeakAss}_R(M)$ and $\bar{\mathfrak{q}} \in \text{Ass}_{\bar{S}}(\bar{N})$ is a maximal element. Let $y \in M$ be an element whose annihilator $I = \text{Ann}_R(y)$ has radical \mathfrak{m} . Then $R/I \subset M$ and by flatness of N over R we get $N/IN = R/I \otimes_R N \subset M \otimes_R N$. Hence it is enough to show that $\mathfrak{q} \in \text{WeakAss}(N/IN)$. Write $\bar{\mathfrak{q}} = (\bar{g}_1, \dots, \bar{g}_n)$ for some $\bar{g}_i \in \bar{S}$. Choose lifts $g_i \in \mathfrak{q}$. Consider the map

$$\Psi : N/IN \longrightarrow N/IN^{\oplus n}, \quad z \longmapsto (g_1 z, \dots, g_n z).$$

We may think of this as a map of free R/I -modules. As the ring R/I is auto-associated (since \mathfrak{m}/I is locally nilpotent) and since $\Psi \otimes R/\mathfrak{m}$ isn't injective (since $\bar{\mathfrak{q}} \in \text{Ass}(\bar{N})$) we see by More on Algebra, Lemma 8.4 that Ψ isn't injective. Pick $z \in N/IN$ nonzero in the kernel of Ψ . The annihilator of z contains I and g_i , whence its radical $J = \sqrt{\text{Ann}_S(z)}$ contains \mathfrak{q} . Let $\mathfrak{q}' \supset J$ be a minimal prime over J . Then $\mathfrak{q}' \in \text{WeakAss}(M \otimes_R N)$ (by definition) and by (1) we see that $\bar{\mathfrak{q}}' \in \text{Ass}(\bar{N})$. Then since $\mathfrak{q} \subset \mathfrak{q}'$ by construction the maximality of $\bar{\mathfrak{q}}$ implies $\mathfrak{q} = \mathfrak{q}'$ whence $\mathfrak{q} \in \text{WeakAss}(M \otimes_R N)$. This proves part (2) of the lemma. \square

Lemma 12.2. *Let S be a scheme. Let $f : X \rightarrow S$ be locally of finite type. Let $x \in X$ with image $s \in S$. Let \mathcal{F} be a finite type quasi-coherent sheaf on X . Let \mathcal{G} be a quasi-coherent sheaf on S . If \mathcal{F} is flat at x over S , then*

$$x \in \text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G}) \Leftrightarrow s \in \text{WeakAss}_S(\mathcal{G}) \text{ and } x \in \text{Ass}_{X_s}(\mathcal{F}_s).$$

Proof. The question is local on X and S , hence we may assume X and S are affine. Write $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and write $B = A[x_1, \dots, x_n]/I$. In other words we obtain a closed immersion $i : X \rightarrow \mathbf{A}_S^n$ over S . Denote $t = i(x) \in \mathbf{A}_S^n$. Note that $i_* \mathcal{F}$ is a finite type quasi-coherent sheaf on \mathbf{A}_S^n which is flat at t over S and note that

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G}) = i_* \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{A}_S^n}} p^* \mathcal{G}$$

where $p : \mathbf{A}_S^n \rightarrow S$ is the projection. Note that t is a weakly associated point of $i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G})$ if and only if x is a weakly associated point of $\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G}$, see Divisors, Lemma 6.3. Similarly $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $t \in \text{Ass}_{\mathbf{A}_S^n}((i_* \mathcal{F})_s)$ (see Algebra, Lemma 62.13). Hence it suffices to prove the lemma in case $X = \mathbf{A}_S^n$. In particular we may assume that $X \rightarrow S$ is of finite presentation.

Recall that $\text{Ass}_{X_s}(\mathcal{F}_s)$ is a locally finite subset of the locally Noetherian scheme X_s , see Divisors, Lemma 2.5. After replacing X by a suitable affine neighbourhood of x we may assume that

$$(*) \text{ if } x' \in \text{Ass}_{X_s}(\mathcal{F}_s) \text{ and } x \rightsquigarrow x' \text{ then } x = x'.$$

(Proof omitted. Hint: using Algebra, Lemma 14.2 invert a function which does not vanish at x but does vanish in all the finitely many points of $\text{Ass}_{X_s}(\mathcal{F}_s)$ which are specializations of x but not equal to x .) In words, no point of $\text{Ass}_{X_s}(\mathcal{F}_s)$ is a proper specialization of x .

Suppose given a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{e} & (S', s') \end{array}$$

of pointed schemes whose horizontal arrows are elementary étale neighbourhoods. Then it suffices to prove the statement for x' , s' , $g^* \mathcal{F}$ and $e^* \mathcal{G}$, see Lemma 2.7. Note that property $(*)$ is preserved by such an étale localization by the same lemma (if there is a proper specialization $x' \rightsquigarrow x''$ on X'_s , then this maps to a proper specialization on X_s because the fibres of an étale morphism are discrete). We may also replace S by the spectrum of its local ring as the condition of being an associated point of a quasi-coherent sheaf depends only on the stalk of the sheaf. Again property $(*)$ is preserved by this as well. Thus we may first apply Proposition 10.3 to reduce to the case where \mathcal{F} is of finite presentation and flat over S , whereupon we may use Proposition 11.4 to reduce to the case that $X \rightarrow S$ is a morphism of affines and $\Gamma(X, \mathcal{F})$ is a finitely presented $\Gamma(X, \mathcal{O}_X)$ -module which is projective as a $\Gamma(S, \mathcal{O}_S)$ -module. Localizing S once more we may assume that $\Gamma(S, \mathcal{O}_S)$ is a local ring such that s corresponds to the maximal ideal. In this case Algebra, Theorem 82.4 guarantees that $\Gamma(X, \mathcal{F})$ is free as an $\Gamma(S, \mathcal{O}_S)$ -module. The implication $x \in \text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G}) \Rightarrow s \in \text{WeakAss}_S(\mathcal{G})$ and $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ follows from part (1) of Lemma 12.1. The converse implication follows from part (2) of Lemma 12.1 as property $(*)$ insures that the prime corresponding to x gives

rise to a maximal element of $\text{Ass}_{\overline{S}}(\overline{N})$ exactly as in the statement of part (2) of Lemma 12.1. \square

Lemma 12.3. *Let $R \rightarrow S$ be a ring map which is essentially of finite type. Let N be a localization of a finite S -module flat over R . Let M be an R -module. Then*

$$\text{WeakAss}_S(M \otimes_R N) = \bigcup_{\mathfrak{p} \in \text{WeakAss}_R(M)} \text{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

Proof. This lemma is a translation of Lemma 12.2 into algebra. Details of translation omitted. \square

Lemma 12.4. *Let $f : X \rightarrow S$ be a morphism which is locally of finite type. Let \mathcal{F} be a finite type quasi-coherent sheaf on X which is flat over S . Let \mathcal{G} be a quasi-coherent sheaf on S . Then we have*

$$\text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G}) = \bigcup_{s \in \text{WeakAss}_S(\mathcal{G})} \text{Ass}_{X_s}(\mathcal{F}_s)$$

Proof. Immediate consequence of Lemma 12.2. \square

Theorem 12.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Assume*

- (1) $X \rightarrow S$ is locally of finite presentation,
- (2) \mathcal{F} is an \mathcal{O}_X -module of finite type, and
- (3) the set of weakly associated points of S is locally finite in S .

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is an \mathcal{O}_U -module of finite presentation and flat over S .

Proof. Let $x \in X$ be such that \mathcal{F} is flat at x over S . We have to find an open neighbourhood of x such that \mathcal{F} restricts to a S -flat finitely presented module on this neighbourhood. The problem is local on X and S , hence we may assume that X and S are affine. As \mathcal{F}_x is a finitely presented $\mathcal{O}_{X,x}$ -module by Lemma 10.9 we conclude from Algebra, Lemma 12.5 there exists a finitely presented \mathcal{O}_X -module \mathcal{F}' and a map $\varphi : \mathcal{F}' \rightarrow \mathcal{F}$ which induces an isomorphism $\varphi_x : \mathcal{F}'_x \rightarrow \mathcal{F}_x$. In particular we see that \mathcal{F}' is flat over S at x , hence by openness of flatness More on Morphisms, Theorem 12.1 we see that after shrinking X we may assume that \mathcal{F}' is flat over S . As \mathcal{F} is of finite type after shrinking X we may assume that φ is surjective, see Modules, Lemma 9.4 or alternatively use Nakayama's lemma (Algebra, Lemma 19.1). By Lemma 12.4 we have

$$\text{WeakAss}_X(\mathcal{F}') \subset \bigcup_{s \in \text{WeakAss}(S)} \text{Ass}_{X_s}(\mathcal{F}'_s)$$

As $\text{WeakAss}(S)$ is finite by assumption and since $\text{Ass}_{X_s}(\mathcal{F}'_s)$ is finite by Divisors, Lemma 2.5 we conclude that $\text{WeakAss}_X(\mathcal{F}')$ is finite. Using Algebra, Lemma 14.2 we may, after shrinking X once more, assume that $\text{WeakAss}_X(\mathcal{F}')$ is contained in the generalization of x . Now consider $\mathcal{K} = \text{Ker}(\varphi)$. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F}')$ (by Divisors, Lemma 5.4) but on the other hand, φ_x is an isomorphism, also $\varphi_{x'}$ is an isomorphism for all $x' \rightsquigarrow x$. We conclude that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$ whence $\mathcal{K} = 0$ by Divisors, Lemma 5.5. \square

Lemma 12.6. *Let $R \rightarrow S$ be a ring map of finite presentation. Let M be a finite S -module. Assume $\text{WeakAss}_S(S)$ is finite. Then*

$$U = \{\mathfrak{q} \subset S \mid M_{\mathfrak{q}} \text{ flat over } R\}$$

is open in $\mathrm{Spec}(S)$ and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is a finitely presented S_g -module flat over R .

Proof. Follows immediately from Theorem 12.5. \square

Lemma 12.7. *Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Assume the set of weakly associated points of S is locally finite in S . Then the set of points $x \in X$ where f is flat is an open subscheme $U \subset X$ and $U \rightarrow S$ is flat and locally of finite presentation.*

Proof. The problem is local on X and S , hence we may assume that X and S are affine. Then $X \rightarrow S$ corresponds to a finite type ring map $A \rightarrow B$. Choose a surjection $A[x_1, \dots, x_n] \rightarrow B$ and consider B as an $A[x_1, \dots, x_n]$ -module. An application of Lemma 12.6 finishes the proof. \square

Lemma 12.8. *Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type and flat. If S is integral, then f is locally of finite presentation.*

Proof. Special case of Lemma 12.7. \square

Proposition 12.9. *Let R be a domain. Let $R \rightarrow S$ be a ring map of finite type. Let M be a finite S -module.*

- (1) *If S is flat over R , then S is a finitely presented R -algebra.*
- (2) *If M is flat as an R -module, then M is finitely presented as an S -module.*

Proof. Part (1) is a special case of Lemma 12.8. For Part (2) choose a surjection $R[x_1, \dots, x_n] \rightarrow S$. By Lemma 12.6 we find that M is finitely presented as an $R[x_1, \dots, x_n]$ -module. We conclude by Algebra, Lemma 6.4. \square

Remark 12.10 (Finite type version of Theorem 12.5). Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Assume

- (1) $X \rightarrow S$ is locally of finite type,
- (2) \mathcal{F} is an \mathcal{O}_X -module of finite type, and
- (3) the set of weakly associated points of S is locally finite in S .

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is flat over S and locally finitely presented relative to S (see More on Morphisms, Definition 38.1). If we ever need this result in the stacks project we will convert this remark into a lemma with a proof.

Remark 12.11 (Algebra version of Remark 12.10). Let $R \rightarrow S$ be a ring map of finite type. Let M be a finite S -module. Assume $\mathrm{WeakAss}_S(S)$ is finite. Then

$$U = \{\mathfrak{q} \subset S \mid M_{\mathfrak{q}} \text{ flat over } R\}$$

is open in $\mathrm{Spec}(S)$ and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is flat over R and an S_g -module finitely presented relative to R (see More on Algebra, Definition 58.2). If we ever need this result in the stacks project we will convert this remark into a lemma with a proof.

13. Examples of relatively pure modules

In the short section we discuss some examples of results that will serve as motivation for the notion of a *relatively pure module* and the concept of an *impurity* which we will introduce later. Each of the examples is stated as a lemma. Note the similarity with the condition on associated primes to the conditions appearing in Lemmas 7.4, 8.3, 8.4, and 9.1. See also Algebra, Lemma 64.1 for a discussion.

Lemma 13.1. *Let R be a local ring with maximal ideal \mathfrak{m} . Let $R \rightarrow S$ be a ring map. Let N be an S -module. Assume*

- (1) *N is projective as an R -module, and*
- (2) *$S/\mathfrak{m}S$ is Noetherian and $N/\mathfrak{m}N$ is a finite $S/\mathfrak{m}S$ -module.*

Then for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.

Proof. Note that the hypotheses of Lemmas 7.1 and 7.6 are satisfied. We will use the conclusions of these lemmas without further mention. Let $\Sigma \subset S$ be the multiplicative set of elements which are not zerodivisors on $N/\mathfrak{m}N$. The map $N \rightarrow \Sigma^{-1}N$ is R -universally injective. Hence we see that any $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ is also an associated prime of $\Sigma^{-1}N \otimes_R \kappa(\mathfrak{p})$. Clearly this implies that \mathfrak{q} corresponds to a prime of $\Sigma^{-1}S$. Thus $\mathfrak{q} \subset \mathfrak{q}'$ where \mathfrak{q}' corresponds to an associated prime of $N/\mathfrak{m}N$ and we win. \square

The following lemma gives another (slightly silly) example of this phenomenon.

Lemma 13.2. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \rightarrow S$ be a ring map. Let N be an S -module. If N is I -adically complete, then for any R -module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + IS \neq S$.*

Proof. Let S^\wedge denote the I -adic completion of S . Note that N is an S^\wedge -module, hence also $N \otimes_R M$ is an S^\wedge -module. Let $z \in N \otimes_R M$ be an element such that $\mathfrak{q} = \text{Ann}_S(z)$. Since $z \neq 0$ we see that $\text{Ann}_{S^\wedge}(z) \neq S^\wedge$. Hence $\mathfrak{q}S^\wedge \neq S^\wedge$. Hence there exists a maximal ideal $\mathfrak{m} \subset S^\wedge$ with $\mathfrak{q}S^\wedge \subset \mathfrak{m}$. Since $IS^\wedge \subset \mathfrak{m}$ by Algebra, Lemma 93.11 we win. \square

Note that the following lemma gives an alternative proof of Lemma 13.1 as a projective module over a local ring is free, see Algebra, Theorem 82.4.

Lemma 13.3. *Let R be a local ring with maximal ideal \mathfrak{m} . Let $R \rightarrow S$ be a ring map. Let N be an S -module. Assume N is isomorphic as an R -module to a direct sum of finite R -modules. Then for any R -module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.*

Proof. Write $N = \bigoplus_{i \in I} M_i$ with each M_i a finite R -module. Let M be an R -module and let $\mathfrak{q} \subset S$ be an associated prime of $N \otimes_R M$ such that $\mathfrak{q} + \mathfrak{m}S = S$. Let $z \in N \otimes_R M$ be an element with $\mathfrak{q} = \text{Ann}_S(z)$. After modifying the direct sum decomposition a little bit we may assume that $z \in M_1 \otimes_R M$ for some element $1 \in I$. Write $1 = f + \sum x_j g_j$ for some $f \in \mathfrak{q}$, $x_j \in \mathfrak{m}$, and $g_j \in S$. For any $g \in S$ denote g' the R -linear map

$$M_1 \rightarrow N \xrightarrow{g} N \rightarrow M_1$$

where the first arrow is the inclusion map, the second arrow is multiplication by g and the third arrow is the projection map. Because each $x_j \in R$ we obtain the

equality

$$f' + \sum x_j g'_j = \text{id}_{M_1} \in \text{End}_R(M_1)$$

By Nakayama's lemma (Algebra, Lemma 19.1) we see that f' is surjective, hence by Algebra, Lemma 15.4 we see that f' is an isomorphism. In particular the map

$$M_1 \otimes_R M \rightarrow N \otimes_R M \xrightarrow{f} N \otimes_R M \rightarrow M_1 \otimes_R M$$

is an isomorphism. This contradicts the assumption that $fz = 0$. \square

Lemma 13.4. *Let R be a henselian local ring with maximal ideal \mathfrak{m} . Let $R \rightarrow S$ be a ring map. Let N be an S -module. Assume N is countably generated and Mittag-Leffler as an R -module. Then for any R -module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.*

Proof. This lemma reduces to Lemma 13.3 by Algebra, Lemma 145.32. \square

Suppose $f : X \rightarrow S$ is a morphism of schemes and \mathcal{F} is a quasi-coherent module on X . Let $\xi \in \text{Ass}_{X/S}(\mathcal{F})$ and let $Z = \overline{\{\xi\}}$. Picture

$$\begin{array}{ccc} \xi & & Z \longrightarrow X \\ \downarrow & & \searrow \downarrow f \\ f(\xi) & & S \end{array}$$

Note that $f(Z) \subset \overline{\{f(\xi)\}}$ and that $f(Z)$ is closed if and only if equality holds, i.e., $f(Z) = \overline{\{f(\xi)\}}$. It follows from Lemma 13.1 that if S, X are affine, the fibres X_s are Noetherian, \mathcal{F} is of finite type, and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$ -module, then $f(Z) = \overline{\{f(\xi)\}}$ is a closed subset. Slightly different analogous statements holds for the cases described in Lemmas 13.2, 13.3, and 13.4.

14. Impurities

We want to formalize the phenomenon of which we gave examples in Section 13 in terms of specializations of points of $\text{Ass}_{X/S}(\mathcal{F})$. We also want to work locally around a point $s \in S$. In order to do so we make the following definitions.

Situation 14.1. Here S, X are schemes and $f : X \rightarrow S$ is a finite type morphism. Also, \mathcal{F} is a finite type quasi-coherent \mathcal{O}_X -module. Finally s is a point of S .

In this situation consider a morphism $g : T \rightarrow S$, a point $t \in T$ with $g(t) = s$, a specialization $t' \rightsquigarrow t$, and a point $\xi \in X_{T'}$ in the base change of X lying over t' . Picture

$$(14.1.1) \quad \begin{array}{ccc} \xi & & X_{T'} \longrightarrow X \\ \downarrow & & \downarrow \quad \downarrow \\ t' & \rightsquigarrow & t \\ & \downarrow & \\ & s & \end{array} \quad \begin{array}{ccc} X_{T'} & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \xrightarrow{g} & S \\ \downarrow g & & \\ S & & \end{array}$$

Moreover, denote $\mathcal{F}_{T'}$ the pullback of \mathcal{F} to $X_{T'}$.

Definition 14.2. In Situation 14.1 we say a diagram (14.1.1) defines an *impurity* of \mathcal{F} above s if $\xi \in \text{Ass}_{X_T/T}(\mathcal{F}_T)$ and $\overline{\{\xi\}} \cap X_t = \emptyset$. We will indicate this by saying “let $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ be an impurity of \mathcal{F} above s ”.

Lemma 14.3. *In Situation 14.1. If there exists an impurity of \mathcal{F} above s , then there exists an impurity $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ of \mathcal{F} above s such that g is locally of finite presentation and t a closed point of the fibre of g above s .*

Proof. Let $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ be any impurity of \mathcal{F} above s . We apply Limits, Lemma 13.1 to $t \in T$ and $Z = \overline{\{\xi\}}$ to obtain an open neighbourhood $V \subset T$ of t , a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

- (1) the morphism $b : T' \rightarrow S$ is locally of finite presentation,
- (2) we have $Z' \cap X_{a(t)} = \emptyset$, and
- (3) $Z \cap X_V$ maps into Z' via the morphism $X_V \rightarrow X_{T'}$.

As t' specializes to t we may replace T by the open neighbourhood V of t . Thus we have a commutative diagram

$$\begin{array}{ccccc} X_T & \longrightarrow & X_{T'} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T & \xrightarrow{a} & T' & \xrightarrow{b} & S \end{array}$$

where $b \circ a = g$. Let $\xi' \in X_{T'}$ denote the image of ξ . By Divisors, Lemma 7.2 we see that $\xi' \in \text{Ass}_{X_{T'}/T'}(\mathcal{F}_{T'})$. Moreover, by construction the closure of $\overline{\{\xi'\}}$ is contained in the closed subset Z' which avoids the fibre $X_{a(t)}$. In this way we see that $(T' \rightarrow S, a(t') \rightsquigarrow a(t), \xi')$ is an impurity of \mathcal{F} above s .

Thus we may assume that $g : T \rightarrow S$ is locally of finite presentation. Let $Z = \overline{\{\xi\}}$. By assumption $Z_t = \emptyset$. By More on Morphisms, Lemma 19.1 this means that $Z_{t''} = \emptyset$ for t'' in an open subset of $\overline{\{t\}}$. Since the fibre of $T \rightarrow S$ over s is a Jacobson scheme, see Morphisms, Lemma 17.10 we find that there exist a closed point $t'' \in \overline{\{t\}}$ such that $Z_{t''} = \emptyset$. Then $(g : T \rightarrow S, t' \rightsquigarrow t'', \xi)$ is the desired impurity. \square

Lemma 14.4. *In Situation 14.1. Let $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ be an impurity of \mathcal{F} above s . Assume S is affine and that T is written $T = \lim_{i \in I} T_i$ as a directed colimit of affine schemes over S . Then for some i the triple $(T_i \rightarrow S, t'_i \rightsquigarrow t_i, \xi_i)$ is an impurity of \mathcal{F} above s .*

Proof. The notation in the statement means this: Let $f_i : T \rightarrow T_i$ be the projection morphisms, let $t_i = f_i(t)$ and $t'_i = f_i(t')$. Finally $\xi_i \in X_{T_i}$ is the image of ξ . By Divisors, Lemma 7.2 it is true that ξ_i is a point of the relative assassin of \mathcal{F}_{T_i} over T_i . Thus the only point is to show that $\overline{\{\xi_i\}} \cap X_{t_i} = \emptyset$ for some i . Set $Z = \overline{\{\xi\}}$. Apply Limits, Lemma 13.1 to this situation to obtain an open neighbourhood $V \subset T$ of

t , a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

- (1) the morphism $b : T' \rightarrow S$ is locally of finite presentation,
- (2) we have $Z' \cap X_{a(t)} = \emptyset$, and
- (3) $Z \cap X_V$ maps into Z' via the morphism $X_V \rightarrow X_{T'}$.

We may assume V is an affine open of T , hence by Limits, Lemmas 3.8 and 3.10 we can find an i and an affine open $V_i \subset T_i$ with $V = f_i^{-1}(V_i)$. By Limits, Proposition 5.1 after possibly increasing i a bit we can find a morphism $a_i : V_i \rightarrow T'$ such that $a = a_i \circ f_i|_V$. The induced morphism $X_{T_i} \rightarrow X_{T'}$ maps ξ_i into Z' . As $Z' \cap X_{a(t)} = \emptyset$ we conclude that $(T_i \rightarrow S, t'_i \rightsquigarrow t_i, \xi_i)$ is an impurity of \mathcal{F} above s . \square

Lemma 14.5. *In Situation 14.1. If there exists an impurity $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ of \mathcal{F} above s with g quasi-finite at t , then there exists an impurity $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ such that $(T, t) \rightarrow (S, s)$ is an elementary étale neighbourhood.*

Proof. Let $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ be an impurity of \mathcal{F} above s such that g is quasi-finite at t . After shrinking T we may assume that g is locally of finite type. Apply More on Morphisms, Lemma 30.1 to $T \rightarrow S$ and $t \mapsto s$. This gives us a diagram

$$\begin{array}{ccccc} T & \longleftarrow & T \times_S U & \longleftarrow & V \\ \downarrow & & \downarrow & \swarrow & \\ S & \longleftarrow & U & & \end{array}$$

where $(U, u) \rightarrow (S, s)$ is an elementary étale neighbourhood and $V \subset T \times_S U$ is an open neighbourhood of $v = (t, u)$ such that $V \rightarrow U$ is finite and such that v is the unique point of V lying over u . Since the morphism $V \rightarrow T$ is étale hence flat we see that there exists a specialization $v' \rightsquigarrow v$ such that $v' \mapsto t'$. Note that $\kappa(t') \subset \kappa(v')$ is finite separable. Pick any point $\zeta \in X_{v'}$ mapping to $\xi \in X_{t'}$. By Divisors, Lemma 7.2 we see that $\zeta \in \text{Ass}_{X_V/V}(\mathcal{F}_V)$. Moreover, the closure $\overline{\{\zeta\}}$ does not meet the fibre X_v as by assumption the closure $\overline{\{\xi\}}$ does not meet X_t . In other words $(V \rightarrow S, v' \rightsquigarrow v, \zeta)$ is an impurity of \mathcal{F} above S .

Next, let $u' \in U'$ be the image of v' and let $\theta \in X_U$ be the image of ζ . Then $\theta \mapsto u'$ and $u' \rightsquigarrow u$. By Divisors, Lemma 7.2 we see that $\theta \in \text{Ass}_{X_U/U}(\mathcal{F})$. Moreover, as $\pi : X_V \rightarrow X_U$ is finite we see that $\pi(\overline{\{\zeta\}}) = \overline{\{\pi(\zeta)\}}$. Since v is the unique point of V lying over u we see that $X_u \cap \overline{\{\pi(\zeta)\}} = \emptyset$ because $X_v \cap \overline{\{\zeta\}} = \emptyset$. In this way we conclude that $(U \rightarrow S, u' \rightsquigarrow u, \theta)$ is an impurity of \mathcal{F} above s and we win. \square

Lemma 14.6. *In Situation 14.1. Assume that S is locally Noetherian. If there exists an impurity of \mathcal{F} above s , then there exists an impurity $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ of \mathcal{F} above s such that g is quasi-finite at t .*

Proof. We may replace S by an affine neighbourhood of s . By Lemma 14.3 we may assume that we have an impurity $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ of such that g is locally of finite type and t a closed point of the fibre of g above s . We may replace T by

the reduced induced scheme structure on $\overline{\{t'\}}$. Let $Z = \overline{\{\xi\}} \subset X_T$. By assumption $Z_t = \emptyset$ and the image of $Z \rightarrow T$ contains t' . By More on Morphisms, Lemma 20.1 there exists a nonempty open $V \subset Z$ such that for any $w \in f(V)$ any generic point ξ' of V_w is in $\text{Ass}_{X_T/T}(\mathcal{F}_T)$. By More on Morphisms, Lemma 19.2 there exists a nonempty open $W \subset T$ with $W \subset f(V)$. By More on Morphisms, Lemma 35.7 there exists a closed subscheme $T' \subset T$ such that $t \in T'$, $T' \rightarrow S$ is quasi-finite at t , and there exists a point $z \in T' \cap W$, $z \rightsquigarrow t$ which does not map to s . Choose any generic point ξ' of the nonempty scheme V_z . Then $(T' \rightarrow S, z \rightsquigarrow t, \xi')$ is the desired impurity. \square

In the following we will use the henselization $S^h = \text{Spec}(\mathcal{O}_{S,s}^h)$ of S at s , see Étale Cohomology, Definition 33.2. Since $S^h \rightarrow S$ maps to closed point of S^h to s and induces an isomorphism of residue fields, we will indicate $s \in S^h$ this closed point also. Thus $(S^h, s) \rightarrow (S, s)$ is a morphism of pointed schemes.

Lemma 14.7. *In Situation 14.1. If there exists an impurity $(S^h \rightarrow S, s' \rightsquigarrow s, \xi)$ of \mathcal{F} above s then there exists an impurity $(T \rightarrow S, t' \rightsquigarrow t, \xi)$ of \mathcal{F} above s where $(T, t) \rightarrow (S, s)$ is an elementary étale neighbourhood.*

Proof. We may replace S by an affine neighbourhood of s . Say $S = \text{Spec}(A)$ and s corresponds to the prime $\mathfrak{p} \subset A$. Then $\mathcal{O}_{S,s}^h = \text{colim}_{(T,t)} \Gamma(T, \mathcal{O}_T)$ where the limit is over the opposite of the cofiltered category of affine elementary étale neighbourhoods (T, t) of (S, s) , see More on Morphisms, Lemma 27.5 and its proof. Hence $S^h = \lim_i T_i$ and we win by Lemma 14.4. \square

Lemma 14.8. *In Situation 14.1 the following are equivalent*

- (1) *there exists an impurity $(S^h \rightarrow S, s' \rightsquigarrow s, \xi)$ of \mathcal{F} above s where S^h is the henselization of S at s ,*
- (2) *there exists an impurity $(T \rightarrow S, t' \rightsquigarrow t, \xi)$ of \mathcal{F} above s such that $(T, t) \rightarrow (S, s)$ is an elementary étale neighbourhood, and*
- (3) *there exists an impurity $(T \rightarrow S, t' \rightsquigarrow t, \xi)$ of \mathcal{F} above s such that $T \rightarrow S$ is quasi-finite at t .*

Proof. As an étale morphism is locally quasi-finite it is clear that (2) implies (3). We have seen that (3) implies (2) in Lemma 14.5. We have seen that (1) implies (2) in Lemma 14.7. Finally, if $(T \rightarrow S, t' \rightsquigarrow t, \xi)$ is an impurity of \mathcal{F} above s such that $(T, t) \rightarrow (S, s)$ is an elementary étale neighbourhood, then we can choose a factorization $S^h \rightarrow T \rightarrow S$ of the structure morphism $S^h \rightarrow S$. Choose any point $s' \in S^h$ mapping to t' and choose any $\xi' \in X_{s'}$ mapping to $\xi \in X_{t'}$. Then $(S^h \rightarrow S, s' \rightsquigarrow s, \xi')$ is an impurity of \mathcal{F} above s . We omit the details. \square

15. Relatively pure modules

The notion of a module pure relative to a base was introduced in [GR71].

Definition 15.1. Let $f : X \rightarrow S$ be a morphism of schemes which is of finite type. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module.

- (1) Let $s \in S$. We say \mathcal{F} is *pure along X_s* if there is no impurity $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ of \mathcal{F} above s with $(T, t) \rightarrow (S, s)$ an elementary étale neighbourhood.
- (2) We say \mathcal{F} is *universally pure along X_s* if there does not exist any impurity of \mathcal{F} above s .
- (3) We say that X is *pure along X_s* if \mathcal{O}_X is pure along X_s .

- (4) We say \mathcal{F} is *universally S -pure*, or *universally pure relative to S* if \mathcal{F} is universally pure along X_s for every $s \in S$.
- (5) We say \mathcal{F} is *S -pure*, or *pure relative to S* if \mathcal{F} is pure along X_s for every $s \in S$.
- (6) We say that X is *S -pure* or *pure relative to S* if \mathcal{O}_X is pure relative to S .

We intentionally restrict ourselves here to morphisms which are of finite type and not just morphisms which are locally of finite type, see Remark 15.2 for a discussion. In the situation of the definition Lemma 14.8 tells us that the following are equivalent

- (1) \mathcal{F} is pure along X_s ,
- (2) there is no impurity $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ with g quasi-finite at t ,
- (3) there does not exist any impurity of the form $(S^h \rightarrow S, s' \rightsquigarrow s, \xi)$, where S^h is the henselization of S at s .

If we denote $X^h = X \times_S S^h$ and \mathcal{F}^h the pullback of \mathcal{F} to X^h , then we can formulate the last condition in the following more positive way:

- (4) All points of $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$ specialize to points of X_s .

In particular, it is clear that \mathcal{F} is pure along X_s if and only if the pullback of \mathcal{F} to $X \times_S \text{Spec}(\mathcal{O}_{S,s})$ is pure along X_s .

Remark 15.2. Let $f : X \rightarrow S$ be a morphism which is locally of finite type and \mathcal{F} a quasi-coherent finite type \mathcal{O}_X -module. In this case it is still true that (1) and (2) above are equivalent because the proof of Lemma 14.5 does not use that f is quasi-compact. It is also clear that (3) and (4) are equivalent. However, we don't know if (1) and (3) are equivalent. In this case it may sometimes be more convenient to define purity using the equivalent conditions (3) and (4) as is done in [GR71]. On the other hand, for many applications it seems that the correct notion is really that of being universally pure.

A natural question to ask is if the property of being pure relative to the base is preserved by base change, i.e., if being pure is the same thing as being universally pure. It turns out that this is true over Noetherian base schemes (see Lemma 15.5), or if the sheaf is flat (see Lemmas 17.3 and 17.4). It is not true in general, even if the morphism and the sheaf are of finite presentation, see Examples, Section 32 for a counter example. First we match our usage of “universally” to the usual notion.

Lemma 15.3. *Let $f : X \rightarrow S$ be a morphism of schemes which is of finite type. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Let $s \in S$. The following are equivalent*

- (1) \mathcal{F} is *universally pure* along X_s , and
- (2) *for every morphism of pointed schemes $(S', s') \rightarrow (S, s)$ the pullback $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

In particular, \mathcal{F} is universally pure relative to S if and only if every base change $\mathcal{F}_{S'}$ of \mathcal{F} is pure relative to S' .

Proof. This is formal. □

Lemma 15.4. *Let $f : X \rightarrow S$ be a morphism of schemes which is of finite type. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Let $s \in S$. Let $(S', s') \rightarrow (S, s)$ be a morphism of pointed schemes. If $S' \rightarrow S$ is quasi-finite at s' and \mathcal{F} is pure along X_s , then $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

Proof. It $(T \rightarrow S', t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}_{S'}$ above s' with $T \rightarrow S'$ quasi-finite at t , then $(T \rightarrow S, t' \rightarrow t, \xi)$ is an impurity of \mathcal{F} above s with $T \rightarrow S$ quasi-finite at t , see Morphisms, Lemma 21.12. Hence the lemma follows immediately from the characterization (2) of purity given following Definition 15.1. \square

Lemma 15.5. *Let $f : X \rightarrow S$ be a morphism of schemes which is of finite type. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Let $s \in S$. If $\mathcal{O}_{S,s}$ is Noetherian then \mathcal{F} is pure along X_s if and only if \mathcal{F} is universally pure along X_s .*

Proof. First we may replace S by $\text{Spec}(\mathcal{O}_{S,s})$, i.e., we may assume that S is Noetherian. Next, use Lemma 14.6 and characterization (2) of purity given in discussion following Definition 15.1 to conclude. \square

Purity satisfies flat descent.

Lemma 15.6. *Let $f : X \rightarrow S$ be a morphism of schemes which is of finite type. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Let $s \in S$. Let $(S', s') \rightarrow (S, s)$ be a morphism of pointed schemes. Assume $S' \rightarrow S$ is flat at s' .*

- (1) *If $\mathcal{F}_{S'}$ is pure along $X_{s'}$, then \mathcal{F} is pure along X_s .*
- (2) *If $\mathcal{F}_{S'}$ is universally pure along $X_{s'}$, then \mathcal{F} is universally pure along X_s .*

Proof. Let $(T \rightarrow S, t' \rightsquigarrow t, \xi)$ be an impurity of \mathcal{F} above s . Set $T_1 = T \times_S S'$, and let t_1 be the unique point of T_1 mapping to t and s' . Since $T_1 \rightarrow T$ is flat at t_1 , see Morphisms, Lemma 26.7, there exists a specialization $t'_1 \rightsquigarrow t_1$ lying over $t' \rightsquigarrow t$, see Algebra, Section 40. Choose a point $\xi_1 \in X_{t'_1}$ which corresponds to a generic point of $\text{Spec}(\kappa(t'_1) \otimes_{\kappa(t')} \kappa(\xi))$, see Schemes, Lemma 17.5. By Divisors, Lemma 7.2 we see that $\xi_1 \in \text{Ass}_{X_{T_1}/T_1}(\mathcal{F}_{T_1})$. As the Zariski closure of $\{\xi_1\}$ in X_{T_1} maps into the Zariski closure of $\{\xi\}$ in X_T we conclude that this closure is disjoint from X_{t_1} . Hence $(T_1 \rightarrow S', t'_1 \rightsquigarrow t_1, \xi_1)$ is an impurity of $\mathcal{F}_{S'}$ above s' . In other words we have proved the contrapositive to part (2) of the lemma. Finally, if $(T, t) \rightarrow (S, s)$ is an elementary étale neighbourhood, then $(T_1, t_1) \rightarrow (S', s')$ is an elementary étale neighbourhood too, and in this way we see that (1) holds. \square

Lemma 15.7. *Let $i : Z \rightarrow X$ be a closed immersion of schemes of finite type over a scheme S . Let $s \in S$. Let \mathcal{F} be a finite type, quasi-coherent sheaf on Z . Then \mathcal{F} is (universally) pure along Z_s if and only if $i_*\mathcal{F}$ is (universally) pure along X_s .*

Proof. Omitted. \square

16. Examples of relatively pure sheaves

Here are some example cases where it is possible to see what purity means.

Lemma 16.1. *Let $f : X \rightarrow S$ be a proper morphism of schemes. Then every finite type, quasi-coherent \mathcal{O}_X -module \mathcal{F} is universally pure relative to S . In particular X is universally pure relative to S .*

Proof. Let $(g : T \rightarrow S, t' \rightsquigarrow t, \xi)$ be an impurity of \mathcal{F} above $s \in S$. Since f is proper, it is universally closed. Hence $f_T : X_T \rightarrow T$ is closed. Since $f_T(\xi) = t'$ this implies that $t \in f(\overline{\{\xi\}})$ which is a contradiction. \square

Lemma 16.2. *Let $f : X \rightarrow S$ be a separated, finite type morphism of schemes. Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_X -module. Assume that $\text{Supp}(\mathcal{F}_s)$ is finite for every $s \in S$. Then the following are equivalent*

- (1) \mathcal{F} is pure relative to S ,
- (2) the scheme theoretic support of \mathcal{F} is finite over S , and
- (3) \mathcal{F} is universally pure relative to S .

In particular, given a quasi-finite separated morphism $X \rightarrow S$ we see that X is pure relative to S if and only if $X \rightarrow S$ is finite.

Proof. Let $Z \subset X$ be the scheme theoretic support of \mathcal{F} , see Morphisms, Definition 5.5. Then $Z \rightarrow S$ is a separated, finite type morphism of schemes with finite fibres. Hence it is separated and quasi-finite, see Morphisms, Lemma 21.10. By Lemma 15.7 it suffices to prove the lemma for $Z \rightarrow S$ and the sheaf \mathcal{F} viewed as a finite type quasi-coherent module on Z . Hence we may assume that $X \rightarrow S$ is separated and quasi-finite and that $\text{Supp}(\mathcal{F}) = X$.

It follows from Lemma 16.1 and Morphisms, Lemma 44.10 that (2) implies (3). Trivially (3) implies (1). Assume (1) holds. We will prove that (2) holds. It is clear that we may assume S is affine. By More on Morphisms, Lemma 31.3 we can find a diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & T \\ & \searrow f & \swarrow \pi \\ & S & \end{array}$$

with π finite and j a quasi-compact open immersion. If we show that j is closed, then j is a closed immersion and we conclude that $f = \pi \circ j$ is finite. To show that j is closed it suffices to show that specializations lift along j , see Schemes, Lemma 19.8. Let $x \in X$, set $t' = j(x)$ and let $t' \rightsquigarrow t$ be a specialization. We have to show $t \in j(X)$. Set $s' = f(x)$ and $s = \pi(t)$ so $s' \rightsquigarrow s$. By More on Morphisms, Lemma 30.4 we can find an elementary étale neighbourhood $(U, u) \rightarrow (S, s)$ and a decomposition

$$T_U = T \times_S U = V \amalg W$$

into open and closed subschemes, such that $V \rightarrow U$ is finite and there exists a unique point v of V mapping to u , and such that v maps to t in T . As $V \rightarrow T$ is étale, we can lift generalizations, see Morphisms, Lemmas 26.8 and 37.12. Hence there exists a specialization $v' \rightsquigarrow v$ such that v' maps to $t' \in T$. In particular we see that $v' \in X_U \subset T_U$. Denote $u' \in U$ the image of t' . Note that $v' \in \text{Ass}_{X_U/U}(\mathcal{F})$ because $X_{u'}$ is a finite discrete set and $X_{u'} = \text{Supp}(\mathcal{F}_{u'})$. As \mathcal{F} is pure relative to S we see that v' must specialize to a point in X_u . Since v is the only point of V lying over u (and since no point of W can be a specialization of v') we see that $v \in X_u$. Hence $t \in X$. \square

Lemma 16.3. *Let $f : X \rightarrow S$ be a finite type, flat morphism of schemes with geometrically integral fibres. Then X is universally pure over S .*

Proof. Let $\xi \in X$ with $s' = f(\xi)$ and $s' \rightsquigarrow s$ a specialization of S . If ξ is an associated point of $X_{s'}$, then ξ is the unique generic point because $X_{s'}$ is an integral scheme. Let ξ_0 be the unique generic point of X_s . As $X \rightarrow S$ is flat we can lift $s' \rightsquigarrow s$ to a specialization $\xi' \rightsquigarrow \xi_0$ in X , see Morphisms, Lemma 26.8. The $\xi \rightsquigarrow \xi'$ because ξ is the generic point of $X_{s'}$ hence $\xi \rightsquigarrow \xi_0$. This means that $(\text{id}_S, s' \rightarrow s, \xi)$ is not an impurity of \mathcal{O}_X above s . Since the assumption that f is finite type, flat with geometrically integral fibres is preserved under base change,

we see that there doesn't exist an impurity after any base change. In this way we see that X is universally S -pure. \square

Lemma 16.4. *Let $f : X \rightarrow S$ be a finite type, affine morphism of schemes. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module such that $f_*\mathcal{F}$ is locally projective on S , see Properties, Definition 19.1. Then \mathcal{F} is universally pure over S .*

Proof. After reducing to the case where S is the spectrum of a henselian local ring this follows from Lemma 13.1. \square

17. A criterion for purity

We first prove that given a flat family of finite type quasi-coherent sheaves the points in the relative assassin specialize to points in the relative assassins of nearby fibres (if they specialize at all).

Lemma 17.1. *Let $f : X \rightarrow S$ be a morphism of schemes of finite type. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Let $s \in S$. Assume that \mathcal{F} is flat over S at all points of X_s . Let $x' \in \text{Ass}_{X/S}(\mathcal{F})$ with $f(x') = s'$ such that $s' \rightsquigarrow s$ is a specialization in S . If x' specializes to a point of X_s , then $x' \rightsquigarrow x$ with $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$.*

Proof. Let $x' \rightsquigarrow t$ be a specialization with $t \in X_s$. We may replace X by an affine neighbourhood of t and S by an affine neighbourhood of s . Choose a closed immersion $i : X \rightarrow \mathbb{A}_S^n$. Then it suffices to prove the lemma for the module $i_*\mathcal{F}$ on \mathbb{A}_S^n and the point $i(x')$. Hence we may assume $X \rightarrow S$ is of finite presentation.

Let $x' \rightsquigarrow t$ be a specialization with $t \in X_s$. Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,t}$, and $N = \mathcal{F}_t$. Note that B is essentially of finite presentation over A and that N is a finite B -module flat over A . Also N is a finitely presented B -module by Lemma 10.9. Let $\mathfrak{q}' \subset B$ be the prime ideal corresponding to x' and let $\mathfrak{p}' \subset A$ be the prime ideal corresponding to s' . The assumption $x' \in \text{Ass}_{X/S}(\mathcal{F})$ means that \mathfrak{q}' is an associated prime of $N \otimes_A \kappa(\mathfrak{p}')$. Let $\Sigma \subset B$ be the multiplicative subset of elements which are not zerodivisors on $N/\mathfrak{m}_A N$. By Lemma 7.2 the map $N \rightarrow \Sigma^{-1}N$ is universally injective. In particular, we see that $N \otimes_A \kappa(\mathfrak{p}') \rightarrow \Sigma^{-1}N \otimes_A \kappa(\mathfrak{p}')$ is injective which implies that \mathfrak{q}' is an associated prime of $\Sigma^{-1}N \otimes_A \kappa(\mathfrak{p}')$ and hence \mathfrak{q}' is in the image of $\text{Spec}(\Sigma^{-1}B) \rightarrow \text{Spec}(B)$. Thus Lemma 7.1 implies that $\mathfrak{q}' \subset \mathfrak{q}$ for some prime $\mathfrak{q} \in \text{Ass}_B(N/\mathfrak{m}_A N)$ (which in particular implies that $\mathfrak{m}_A = A \cap \mathfrak{q}$). If $x \in X_s$ denotes the point corresponding to \mathfrak{q} , then $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ and $x' \rightsquigarrow x$ as desired. \square

Lemma 17.2. *Let $f : X \rightarrow S$ be a morphism of schemes of finite type. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Let $s \in S$. Let $(S', s') \rightarrow (S, s)$ be an elementary étale neighbourhood and let*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \xleftarrow{\quad} & S' \end{array}$$

be a commutative diagram of morphisms of schemes. Assume

- (1) \mathcal{F} is flat over S at all points of X_s ,
- (2) $X' \rightarrow S'$ is of finite type,

- (3) $g^*\mathcal{F}$ is pure along $X'_{s'}$,
- (4) $g : X' \rightarrow X$ is étale, and
- (5) $g(X')$ contains $\text{Ass}_{X_s}(\mathcal{F}_s)$.

In this situation \mathcal{F} is pure along X_s if and only if the image of $X' \rightarrow X \times_S S'$ contains the points of $\text{Ass}_{X \times_S S'/S'}(\mathcal{F} \times_S S')$ lying over points in S' which specialize to s' .

Proof. Since the morphism $S' \rightarrow S$ is étale, we see that if \mathcal{F} is pure along X_s , then $\mathcal{F} \times_S S'$ is pure along X_s , see Lemma 15.4. Since purity satisfies flat descent, see Lemma 15.6, we see that if $\mathcal{F} \times_S S'$ is pure along $X_{s'}$, then \mathcal{F} is pure along X_s . Hence we may replace S by S' and assume that $S = S'$ so that $g : X' \rightarrow X$ is an étale morphism between schemes of finite type over S . Moreover, we may replace S by $\text{Spec}(\mathcal{O}_{S,s})$ and assume that S is local.

First, assume that \mathcal{F} is pure along X_s . In this case every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of X_s by purity. Hence by Lemma 17.1 we see that every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of $\text{Ass}_{X_s}(\mathcal{F}_s)$. Thus every point of $\text{Ass}_{X/S}(\mathcal{F})$ is in the image of g (as the image is open and contains $\text{Ass}_{X_s}(\mathcal{F}_s)$).

Conversely, assume that $g(X')$ contains $\text{Ass}_{X/S}(\mathcal{F})$. Let $S^h = \text{Spec}(\mathcal{O}_{S,s}^h)$ be the henselization of S at s . Denote $g^h : (X')^h \rightarrow X^h$ the base change of g by $S^h \rightarrow S$, and denote \mathcal{F}^h the pullback of \mathcal{F} to X^h . By Divisors, Lemma 7.2 and Remark 7.3 the relative assassin $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$ is the inverse image of $\text{Ass}_{X/S}(\mathcal{F})$ via the projection $X^h \rightarrow X$. As we have assumed that $g(X')$ contains $\text{Ass}_{X/S}(\mathcal{F})$ we conclude that the base change $g^h((X')^h) = g(X') \times_S S^h$ contains $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$. In this way we reduce to the case where S is the spectrum of a henselian local ring. Let $x \in \text{Ass}_{X/S}(\mathcal{F})$. To finish the proof of the lemma we have to show that x specializes to a point of X_s , see criterion (4) for purity in discussion following Definition 15.1. By assumption there exists a $x' \in X'$ such that $g(x') = x$. As $g : X' \rightarrow X$ is étale, we see that $x' \in \text{Ass}_{X'/S}(g^*\mathcal{F})$, see Lemma 2.7 (applied to the morphism of fibres $X'_w \rightarrow X_w$ where $w \in S$ is the image of x'). Since $g^*\mathcal{F}$ is pure along X'_s we see that $x' \rightsquigarrow y$ for some $y \in X'_s$. Hence $x = g(x') \rightsquigarrow g(y)$ and $g(y) \in X_s$ as desired. \square

Lemma 17.3. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $s \in S$. Assume*

- (1) f is of finite type,
- (2) \mathcal{F} is of finite type,
- (3) \mathcal{F} is flat over S at all points of X_s , and
- (4) \mathcal{F} is pure along X_s .

Then \mathcal{F} is universally pure along X_s .

Proof. We first make a preliminary remark. Suppose that $(S', s') \rightarrow (S, s)$ is an elementary étale neighbourhood. Denote \mathcal{F}' the pullback of \mathcal{F} to $X' = X \times_S S'$. By the discussion following Definition 15.1 we see that \mathcal{F}' is pure along $X'_{s'}$. Moreover, \mathcal{F}' is flat over S' along $X'_{s'}$. Then it suffices to prove that \mathcal{F}' is universally pure along $X'_{s'}$. Namely, given any morphism $(T, t) \rightarrow (S, s)$ of pointed schemes the fibre product $(T', t') = (T \times_S S', (t, s'))$ is flat over (T, t) and hence if $\mathcal{F}_{T'}$ is pure along $X_{t'}$ then \mathcal{F}_T is pure along X_t by Lemma 15.6. Thus during the proof we may always replace (s, S) by an elementary étale neighbourhood. We may also replace S by $\text{Spec}(\mathcal{O}_{S,s})$ due to the local nature of the problem.

Choose an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \xleftarrow{\quad} & \mathrm{Spec}(\mathcal{O}_{S', s'}) \end{array}$$

such that $X' \rightarrow X \times_S \mathrm{Spec}(\mathcal{O}_{S', s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^* \mathcal{F})$ is a free $\mathcal{O}_{S', s'}$ -module, see Lemma 11.11. Note that $X' \rightarrow \mathrm{Spec}(\mathcal{O}_{S', s'})$ is of finite type (as a quasi-compact morphism which is the composition of an étale morphism and the base change of a finite type morphism). By our preliminary remarks in the first paragraph of the proof we may replace S by $\mathrm{Spec}(\mathcal{O}_{S', s'})$. Hence we may assume there exists a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow & \swarrow \\ & S & \end{array}$$

of schemes of finite type over S , where g is étale, $X_s \subset g(X')$, with S local with closed point s , with X' affine, and with $\Gamma(X', g^* \mathcal{F})$ a free $\Gamma(S, \mathcal{O}_S)$ -module. Note that in this case $g^* \mathcal{F}$ is universally pure over S , see Lemma 16.4.

In this situation we apply Lemma 17.2 to deduce that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ from our assumption that \mathcal{F} is pure along X_s and flat over S along X_s . By Divisors, Lemma 7.2 and Remark 7.3 we see that for any morphism of pointed schemes $(T, t) \rightarrow (S, s)$ we have

$$\mathrm{Ass}_{X_T/T}(\mathcal{F}_T) \subset (X_T \rightarrow X)^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) \subset g(X') \times_S T = g_T(X'_T).$$

Hence by Lemma 17.2 applied to the base change of our displayed diagram to (T, t) we conclude that \mathcal{F}_T is pure along X_t as desired. \square

Lemma 17.4. *Let $f : X \rightarrow S$ be a finite type morphism of schemes. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Assume \mathcal{F} is flat over S . In this case \mathcal{F} is pure relative to S if and only if \mathcal{F} is universally pure relative to S .*

Proof. Immediate consequence of Lemma 17.3 and the definitions. \square

Lemma 17.5. *Let I be a directed partially ordered set. Let $(S_i, g_{ii'})$ be an inverse system of affine schemes over I . Set $S = \lim_i S_i$ and $s \in S$. Denote $g_i : S \rightarrow S_i$ the projections and set $s_i = g_i(s)$. Suppose that $f : X \rightarrow S$ is a morphism of finite presentation, \mathcal{F} a quasi-coherent \mathcal{O}_X -module of finite presentation which is pure along X_s and flat over S at all points of X_s . Then there exists an $i \in I$, a morphism of finite presentation $X_i \rightarrow S_i$, a quasi-coherent \mathcal{O}_{X_i} -module \mathcal{F}_i of finite presentation which is pure along $(X_i)_{s_i}$ and flat over S_i at all points of $(X_i)_{s_i}$ such that $X \cong X_i \times_{S_i} S$ and such that the pullback of \mathcal{F}_i to X is isomorphic to \mathcal{F} .*

Proof. Let $U \subset X$ be the set of points where \mathcal{F} is flat over S . By More on Morphisms, Theorem 12.1 this is an open subscheme of X . By assumption $X_s \subset U$. As X_s is quasi-compact, we can find a quasi-compact open $U' \subset U$ with $X_s \subset U'$. By Limits, Lemma 9.1 we can find an $i \in I$ and a morphism of finite presentation $f_i : X_i \rightarrow S_i$ whose base change to S is isomorphic to f . Fix such a choice and set $X_{i'} = X_i \times_{S_i} S_{i'}$. Then $X = \lim_{i'} X_{i'}$ with affine transition morphisms. By Limits,

Lemma 9.2 we can, after possibly increasing i assume there exists a quasi-coherent \mathcal{O}_{X_i} -module \mathcal{F}_i of finite presentation whose base change to S is isomorphic to \mathcal{F} . By Limits, Lemma 3.8 after possibly increasing i we may assume there exists an open $U'_i \subset X_i$ whose inverse image in X is U' . Note that in particular $(X_i)_{s_i} \subset U'_i$. By Limits, Lemma 9.3 (after increasing i once more) we may assume that \mathcal{F}_i is flat on U'_i . In particular we see that \mathcal{F}_i is flat along $(X_i)_{s_i}$.

Next, we use Lemma 11.5 to choose an elementary étale neighbourhood $(S'_i, s'_i) \rightarrow (S_i, s_i)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X_i & \xleftarrow{g_i} & X'_i \\ \downarrow & & \downarrow \\ S_i & \xleftarrow{\quad} & S'_i \end{array}$$

such that g_i is étale, $(X_i)_{s_i} \subset g_i(X'_i)$, the schemes X'_i, S'_i are affine, and such that $\Gamma(X'_i, g_i^* \mathcal{F}_i)$ is a projective $\Gamma(S'_i, \mathcal{O}_{S'_i})$ -module. Note that $g_i^* \mathcal{F}_i$ is universally pure over S'_i , see Lemma 16.4. We may base change the diagram above to a diagram with morphisms $(S'_{i'}, s'_{i'}) \rightarrow (S_{i'}, s_{i'})$ and $g_{i'} : X'_{i'} \rightarrow X_{i'}$ over $S_{i'}$ for any $i' \geq i$ and we may base change the diagram to a diagram with morphisms $(S', s') \rightarrow (S, s)$ and $g : X' \rightarrow X$ over S .

At this point we can use our criterion for purity. Set $W'_i \subset X_i \times_{S_i} S'_i$ equal to the image of the étale morphism $X'_i \rightarrow X_i \times_{S_i} S'_i$. For every $i' \geq i$ we have similarly the image $W'_{i'} \subset X_{i'} \times_{S_{i'}} S'_{i'}$ and we have the image $W' \subset X \times_S S'$. Taking images commutes with base change, hence $W'_{i'} = W'_i \times_{S'_i} S'_{i'}$ and $W' = W_i \times_{S_i} S'$. Because \mathcal{F} is pure along X_s the Lemma 17.2 implies that

$$(17.5.1) \quad f^{-1}(\text{Spec}(\mathcal{O}_{S', s'})) \cap \text{Ass}_{X \times_S S'/S'}(\mathcal{F} \times_S S') \subset W'$$

By More on Morphisms, Lemma 20.5 we see that

$$E = \{t \in S' \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset W'\} \quad \text{and} \quad E_{i'} = \{t \in S'_{i'} \mid \text{Ass}_{X_t}(\mathcal{F}_{i', t}) \subset W'_{i'}\}$$

are locally constructible subsets of S' and $S'_{i'}$. By More on Morphisms, Lemma 20.4 we see that $E_{i'}$ is the inverse image of E_i under the morphism $S'_{i'} \rightarrow S'_i$ and that E is the inverse image of E_i under the morphism $S' \rightarrow S'_i$. Thus Equation (17.5.1) is equivalent to the assertion that $\text{Spec}(\mathcal{O}_{S', s'})$ maps into E_i . As $\mathcal{O}_{S', s'} = \text{colim}_{i' \geq i} \mathcal{O}_{S'_{i'}, s'_{i'}}$ we see that $\text{Spec}(\mathcal{O}_{S'_{i'}, s'_{i'}})$ maps into E_i for some $i' \geq i$, see Limits, Lemma 3.7. Then, applying Lemma 17.2 to the situation over $S_{i'}$, we conclude that $\mathcal{F}_{i'}$ is pure along $(X_{i'})_{s_{i'}}$. \square

Lemma 17.6. *Let $f : X \rightarrow S$ be a morphism of finite presentation. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite presentation flat over S . Then the set*

$$U = \{s \in S \mid \mathcal{F} \text{ is pure along } X_s\}$$

is open in S .

Proof. Let $s \in U$. Using Lemma 11.5 we can find an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \xleftarrow{\quad} & S' \end{array}$$

such that g is étale, $X_s \subset g(X')$, the schemes X' , S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$ -module. Note that $g^*\mathcal{F}$ is universally pure over S' , see Lemma 16.4. Set $W' \subset X \times_S S'$ equal to the image of the étale morphism $X' \rightarrow X \times_S S'$. Note that W is open and quasi-compact over S' . Set

$$E = \{t \in S' \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset W'\}.$$

By More on Morphisms, Lemma 20.5 E is a constructible subset of S' . By Lemma 17.2 we see that $\text{Spec}(\mathcal{O}_{S', s'}) \subset E$. By Morphisms, Lemma 23.4 we see that E contains an open neighbourhood V' of s' . Applying Lemma 17.2 once more we see that for any point s_1 in the image of V' in S the sheaf \mathcal{F} is pure along X_{s_1} . Since $S' \rightarrow S$ is étale the image of V' in S is open and we win. \square

18. How purity is used

Here are some examples of how purity can be used. The first lemma actually uses a slightly weaker form of purity.

Lemma 18.1. *Let $f : X \rightarrow S$ be a morphism of finite type. Let \mathcal{F} be a quasi-coherent sheaf of finite type on X . Assume S is local with closed point s . Assume \mathcal{F} is pure along X_s and that \mathcal{F} is flat over S . Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ of quasi-coherent \mathcal{O}_X -modules. Then the following are equivalent*

- (1) *the map on stalks φ_x is injective for all $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$, and*
- (2) *φ is injective.*

Proof. Let $\mathcal{K} = \text{Ker}(\varphi)$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 5.5. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 5.4. As \mathcal{F} is flat we see from Lemma 12.4 that $\text{WeakAss}_X(\mathcal{F}) \subset \text{Ass}_{X/S}(\mathcal{F})$. By purity any point x' of $\text{Ass}_{X/S}(\mathcal{F})$ is a generalization of a point of X_s , and hence is the specialization of a point $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$, by Lemma 17.1. Hence the injectivity of φ_x implies the injectivity of $\varphi_{x'}$, whence $\mathcal{K}_{x'} = 0$. \square

Proposition 18.2. *Let $f : X \rightarrow S$ be an affine, finitely presented morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite presentation, flat over S . Then the following are equivalent*

- (1) *$f_*\mathcal{F}$ is locally projective on S , and*
- (2) *\mathcal{F} is pure relative to S .*

In particular, given a ring map $A \rightarrow B$ of finite presentation and a finitely presented B -module N flat over A we have: N is projective as an A -module if and only if \tilde{N} on $\text{Spec}(B)$ is pure relative to $\text{Spec}(A)$.

Proof. The implication (1) \Rightarrow (2) is Lemma 16.4. Assume \mathcal{F} is pure relative to S . Note that by Lemma 17.3 this implies \mathcal{F} remains pure after any base change. By Descent, Lemma 6.7 it suffices to prove $f_*\mathcal{F}$ is fpqc locally projective on S . Pick $s \in S$. We will prove that the restriction of $f_*\mathcal{F}$ to an étale neighbourhood of s is locally projective. Namely, by Lemma 11.5, after replacing S by an affine elementary étale neighbourhood of s , we may assume there exists a diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow & \swarrow \\ & S & \end{array}$$

of schemes affine and of finite presentation over S , where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$ -module. Note that in this case $g^*\mathcal{F}$ is universally pure over S , see Lemma 16.4. Hence by Lemma 17.2 we see that the open $g(X')$ contains the points of $\text{Ass}_{X/S}(\mathcal{F})$ lying over $\text{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 20.5 E is a constructible subset of S . We have seen that $\text{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 23.4 we see that E contains an open neighbourhood of s . Hence after replacing S by an affine neighbourhood of s we may assume that $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$. By Lemma 7.4 this means that

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X', g^*\mathcal{F})$$

is $\Gamma(S, \mathcal{O}_S)$ -universally injective. By Algebra, Lemma 86.7 we conclude that $\Gamma(X, \mathcal{F})$ is Mittag-Leffler as an $\Gamma(S, \mathcal{O}_S)$ -module. Since $\Gamma(X, \mathcal{F})$ is countably generated and flat as a $\Gamma(S, \mathcal{O}_S)$ -module, we conclude it is projective by Algebra, Lemma 90.1. \square

We can use the proposition to improve some of our earlier results. The following lemma is an improvement of Proposition 11.4.

Lemma 18.3. *Let $f : X \rightarrow S$ be a morphism which is locally of finite presentation. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module which is of finite presentation. Let $x \in X$ with $s = f(x) \in S$. If \mathcal{F} is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and an affine open $U' \subset X \times_S S'$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$ -module.*

Proof. During the proof we may replace X by an open neighbourhood of x and we may replace S by an elementary étale neighbourhood of s . Hence, by openness of flatness (see More on Morphisms, Theorem 12.1) we may assume that \mathcal{F} is flat over S . We may assume S and X are affine. After shrinking X some more we may assume that any point of $\text{Ass}_{X_s}(\mathcal{F}_s)$ is a generalization of x . This property is preserved on replacing (S, s) by an elementary étale neighbourhood. Hence we may apply Lemma 11.5 to arrive at the situation where there exists a diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow & \swarrow \\ & S & \end{array}$$

of schemes affine and of finite presentation over S , where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$ -module. Note that in this case $g^*\mathcal{F}$ is universally pure over S , see Lemma 16.4.

Let $U \subset g(X')$ be an affine open neighbourhood of x . We claim that $\mathcal{F}|_U$ is pure along U_s . If we prove this, then the lemma follows because $\mathcal{F}|_U$ will be pure relative to S after shrinking S , see Lemma 17.6, whereupon the projectivity follows from Proposition 18.2. To prove the claim we have to show, after replacing (S, s) by an arbitrary elementary étale neighbourhood, that any point ξ of $\text{Ass}_{U/S}(\mathcal{F}|_U)$ lying over some $s' \in S$, $s' \rightsquigarrow s$ specializes to a point of U_s . Since $U \subset g(X')$ we can find a $\xi' \in X'$ with $g(\xi') = \xi$. Because $g^*\mathcal{F}$ is pure over S , using Lemma 17.1, we see there exists a specialization $\xi' \rightsquigarrow x'$ with $x' \in \text{Ass}_{X'_s}(g^*\mathcal{F}_s)$. Then $g(x') \in \text{Ass}_{X_s}(\mathcal{F}_s)$ (see for example Lemma 2.7 applied to the étale morphism $X'_s \rightarrow X_s$ of Noetherian

schemes) and hence $g(x') \rightsquigarrow x$ by our choice of X above! Since $x \in U$ we conclude that $g(x') \in U$. Thus $\xi = g(\xi') \rightsquigarrow g(x') \in U_s$ as desired. \square

The following lemma is an improvement of Lemma 11.9.

Lemma 18.4. *Let $f : X \rightarrow S$ be a morphism which is locally of finite type. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module which is of finite type. Let $x \in X$ with $s = f(x) \in S$. If \mathcal{F} is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ and an affine open $U' \subset X \times_S \text{Spec}(\mathcal{O}_{S', s'})$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a free $\mathcal{O}_{S', s'}$ -module.*

Proof. The question is Zariski local on X and S . Hence we may assume that X and S are affine. Then we can find a closed immersion $i : X \rightarrow \mathbf{A}_S^n$ over S . It is clear that it suffices to prove the lemma for the sheaf $i_*\mathcal{F}$ on \mathbf{A}_S^n and the point $i(x)$. In this way we reduce to the case where $X \rightarrow S$ is of finite presentation. After replacing S by $\text{Spec}(\mathcal{O}_{S', s'})$ and X by an open of $X \times_S \text{Spec}(\mathcal{O}_{S', s'})$ we may assume that \mathcal{F} is of finite presentation, see Proposition 10.3. In this case we may appeal to Lemma 18.3 and Algebra, Theorem 82.4 to conclude. \square

Lemma 18.5. *Let $A \rightarrow B$ be a local ring map of local rings which is essentially of finite type. Let N be a finite B -module which is flat as an A -module. If A is henselian, then N is a filtered colimit*

$$N = \text{colim}_i F_i$$

of free A -modules F_i such that all transition maps $u_i : F_i \rightarrow F_{i'}$ of the system induce injective maps $\bar{u}_i : F_i/\mathfrak{m}_A F_i \rightarrow F_{i'}/\mathfrak{m}_A F_{i'}$. Also, N is a Mittag-Leffler A -module.

Proof. We can find a morphism of finite type $X \rightarrow S = \text{Spec}(A)$ and a point $x \in X$ lying over the closed point s of S and a finite type quasi-coherent \mathcal{O}_X -module \mathcal{F} such that $\mathcal{F}_x \cong N$ as an A -module. After shrinking X we may assume that each point of $\text{Ass}_{X_s}(\mathcal{F}_s)$ specializes to x . By Lemma 18.4 we see that there exists a fundamental system of affine open neighbourhoods $U_i \subset X$ of x such that $\Gamma(U_i, \mathcal{F})$ is a free A -module F_i . Note that if $U_{i'} \subset U_i$, then

$$F_i/\mathfrak{m}_A F_i = \Gamma(U_{i,s}, \mathcal{F}_s) \longrightarrow \Gamma(U_{i',s}, \mathcal{F}_s) = F_{i'}/\mathfrak{m}_A F_{i'}$$

is injective because a section of the kernel would be supported at a closed subset of X_s not meeting x which is a contradiction to our choice of X above. Since the maps $F_i \rightarrow F_{i'}$ are A -universally injective (Lemma 7.5) it follows that N is Mittag-Leffler by Algebra, Lemma 86.8. \square

19. Flattening functors

Let S be a scheme. Recall that a functor $F : (\text{Sch}/S)^{\text{opp}} \rightarrow \text{Sets}$ is called limit preserving if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes with limit T we have $F(T) = \text{colim}_i F(T_i)$.

Situation 19.1. Let $f : X \rightarrow S$ be a morphism of schemes. Let $u : \mathcal{F} \rightarrow \mathcal{G}$ be a homomorphism of quasi-coherent \mathcal{O}_X -modules. For any scheme T over S we will denote $u_T : \mathcal{F}_T \rightarrow \mathcal{G}_T$ the base change of u to T , in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_S T \rightarrow X$. In this situation we can consider the functor

$$(19.1.1) \quad F_{\text{iso}} : (\text{Sch}/S)^{\text{opp}} \longrightarrow \text{Sets}, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants F_{inj} , F_{surj} , F_{zero} where we ask that u_T is injective, surjective, or zero.

Lemma 19.2. *In Situation 19.1.*

- (1) *Each of the functors F_{iso} , F_{inj} , F_{surj} , F_{zero} satisfies the sheaf property for the fpqc topology.*
- (2) *If f is quasi-compact and \mathcal{G} is of finite type, then F_{surj} is limit preserving.*
- (3) *If f is quasi-compact and \mathcal{F} of finite type, then F_{zero} is limit preserving.*
- (4) *If f is quasi-compact, \mathcal{F} is of finite type, and \mathcal{G} is of finite presentation, then F_{iso} is limit preserving.*

Proof. Let $\{T_i \rightarrow T\}_{i \in I}$ be an fpqc covering of schemes over S . Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \rightarrow X_T\}_{i \in I}$ is an fpqc covering of X_T , see Topologies, Lemma 8.7. In particular, for every $x \in X_T$ there exists an $i \in I$ and an $x_i \in X_i$ mapping to x . Since $\mathcal{O}_{X_T, x} \rightarrow \mathcal{O}_{X_i, x_i}$ is flat, hence faithfully flat (see Algebra, Lemma 38.16) we conclude that $(u_i)_{x_i}$ is injective, surjective, bijective, or zero if and only if $(u_T)_x$ is injective, surjective, bijective, or zero. Whence part (1) of the lemma.

Proof of (2). Assume f quasi-compact and \mathcal{G} of finite type. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S -schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \rightarrow \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove part (2) we have to show that u_i is surjective for some i . Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since f is quasi-compact the scheme X_{i_0} is quasi-compact. Hence we may choose affine opens $W_1, \dots, W_m \subset X$ and an affine open covering $X_{i_0} = U_{1, i_0} \cup \dots \cup U_{m, i_0}$ such that U_{j, i_0} maps into W_j under the projection morphism $X_{i_0} \rightarrow X$. For any $i \in I$ let $U_{j, i}$ be the inverse image of U_{j, i_0} . Setting $U_j = \lim_{i \in I} U_{j, i}$ we see that $X_T = U_1 \cup \dots \cup U_m$ is an affine open covering of X_T . Now it suffices to show, for a given $j \in \{1, \dots, m\}$ that $u_i|_{U_{j, i}}$ is surjective for some $i = i(j) \in I$. Using Properties, Lemma 16.1 this translates into the following algebra problem: Let A be a ring and let $u : M \rightarrow N$ be an A -module map. Suppose that $R = \text{colim}_{i \in I} R_i$ is a directed colimit of A -algebras. If N is a finite A -module and if $u \otimes 1 : M \otimes_A R \rightarrow N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \rightarrow N \otimes_A R_i$ is surjective. This is Algebra, Lemma 123.3 part (2).

Proof of (3). Exactly the same arguments as given in the proof of (2) reduces this to the following algebra problem: Let A be a ring and let $u : M \rightarrow N$ be an A -module map. Suppose that $R = \text{colim}_{i \in I} R_i$ is a directed colimit of A -algebras. If M is a finite A -module and if $u \otimes 1 : M \otimes_A R \rightarrow N \otimes_A R$ is zero, then for some i the map $u \otimes 1 : M \otimes_A R_i \rightarrow N \otimes_A R_i$ is zero. This is Algebra, Lemma 123.3 part (1).

Proof of (4). Assume f quasi-compact and \mathcal{F}, \mathcal{G} of finite presentation. Arguing in exactly the same manner as in the previous paragraph (using in addition also Properties, Lemma 16.2) part (3) translates into the following algebra statement: Let A be a ring and let $u : M \rightarrow N$ be an A -module map. Suppose that $R = \text{colim}_{i \in I} R_i$ is a directed colimit of A -algebras. Assume M is a finite A -module, N is a finitely presented A -module, and $u \otimes 1 : M \otimes_A R \rightarrow N \otimes_A R$ is an isomorphism. Then for some i the map $u \otimes 1 : M \otimes_A R_i \rightarrow N \otimes_A R_i$ is an isomorphism. This is Algebra, Lemma 123.3 part (3). \square

Situation 19.3. Let (A, \mathfrak{m}_A) be a local ring. Denote \mathcal{C} the category whose objects are A -algebras A' which are local rings such that the algebra structure $A \rightarrow A'$ is

a local homomorphism of local rings. A morphism between objects A', A'' of \mathcal{C} is a local homomorphism $A' \rightarrow A''$ of A -algebras. Let $A \rightarrow B$ be a local ring map of local rings and let M be a B -module. If A' is an object of \mathcal{C} we set $B' = B \otimes_A A'$ and we set $M' = M \otimes_A A'$ as a B' -module. Given $A' \in \text{Ob}(\mathcal{C})$, consider the condition

$$(19.3.1) \quad \forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset \text{Spec}(B') : M'_\mathfrak{q} \text{ is flat over } A'.$$

Note the similarity with More on Algebra, Equation (12.1.1). In particular, if $A' \rightarrow A''$ is a morphism of \mathcal{C} and (19.3.1) holds for A' , then it holds for A'' , see More on Algebra, Lemma 12.2. Hence we obtain a functor

$$(19.3.2) \quad F_{lf} : \mathcal{C} \longrightarrow \text{Sets}, \quad A' \longrightarrow \begin{cases} \{*\} & \text{if (19.3.1) holds,} \\ \emptyset & \text{else.} \end{cases}$$

Lemma 19.4. *In Situation 19.3.*

- (1) *If $A' \rightarrow A''$ is a flat morphism in \mathcal{C} then $F_{fl}(A') = F_{lf}(A'')$.*
- (2) *If $A \rightarrow B$ is essentially of finite presentation and M is a B -module of finite presentation, then F_{fl} is limit preserving: If $\{A_i\}_{i \in I}$ is a directed system of objects of \mathcal{C} , then $F_{fl}(\text{colim}_i A_i) = \text{colim}_i F_{fl}(A_i)$.*

Proof. Part (1) is a special case of More on Algebra, Lemma 12.3. Part (2) is a special case of More on Algebra, Lemma 12.4. \square

Lemma 19.5. *In Situation 19.3 suppose that $B \rightarrow C$ is a local map of local A -algebras and that $M \cong N$ as B -modules. Denote $F'_{lf} : \mathcal{C} \rightarrow \text{Sets}$ the functor associated to the pair (C, N) . If $B \rightarrow C$ is finite, then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of \mathcal{C} . Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A'$ similarly to the definitions of B', M' in Situation 19.3. Note that $M' \cong N'$ as B' -modules. The assumption that $B \rightarrow C$ is finite has two consequences: (a) $\mathfrak{m}_C = \sqrt{\mathfrak{m}_B C}$ and (b) $B' \rightarrow C'$ is finite. Consequence (a) implies that

$$V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C') = (\text{Spec}(C') \rightarrow \text{Spec}(B'))^{-1} V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B').$$

Suppose $\mathfrak{q} \subset V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$. Then $M'_\mathfrak{q}$ is flat over A' if and only if the $C'_\mathfrak{q}$ -module $N'_\mathfrak{q}$ is flat over A' (because these are isomorphic as A' -modules) if and only if for every maximal ideal \mathfrak{r} of $C'_\mathfrak{q}$ the module $N'_\mathfrak{r}$ is flat over A' (see Algebra, Lemma 38.19). As $B'_\mathfrak{q} \rightarrow C'_\mathfrak{q}$ is finite by (b), the maximal ideals of $C'_\mathfrak{q}$ correspond exactly to the primes of C' lying over \mathfrak{q} (see Algebra, Lemma 35.20) and these primes are all contained in $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ by the displayed equation above. Thus the result of the lemma holds. \square

Lemma 19.6. *In Situation 19.3 suppose that $B \rightarrow C$ is a flat local homomorphism of local rings. Set $N = M \otimes_B C$. Denote $F'_{lf} : \mathcal{C} \rightarrow \text{Sets}$ the functor associated to the pair (C, N) . Then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of \mathcal{C} . Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A' = M' \otimes_{B'} C'$ similarly to the definitions of B', M' in Situation 19.3. Note that

$$V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') = \text{Spec}(\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}))$$

and similarly for $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$. The ring map

$$\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}) \longrightarrow \kappa(\mathfrak{m}_C) \otimes_A \kappa(\mathfrak{m}_{A'})$$

is faithfully flat, hence $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C') \rightarrow V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$ is surjective. Finally, if $\mathfrak{r} \in V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ maps to $\mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$, then $M'_\mathfrak{q}$ is flat over A'

if and only if N'_t is flat over A' because $B' \rightarrow C'$ is flat, see Algebra, Lemma 38.8. The lemma follows formally from these remarks. \square

Situation 19.7. Let $f : X \rightarrow S$ be a smooth morphism with geometrically irreducible fibres. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. For any scheme T over S we will denote \mathcal{F}_T the base change of \mathcal{F} to T , in other words, \mathcal{F}_T is the pullback of \mathcal{F} via the projection morphism $X_T = X \times_S T \rightarrow X$. Note that $X_T \rightarrow T$ is smooth with geometrically irreducible fibres, see Morphisms, Lemma 35.5 and More on Morphisms, Lemma 22.2. Let $p \geq 0$ be an integer. Given a point $t \in T$ consider the condition

$$(19.7.1) \quad \mathcal{F}_T \text{ is free of rank } p \text{ in a neighbourhood of } \xi_t$$

where ξ_t is the generic point of the fibre X_t . This condition for all $t \in T$ is stable under base change, and hence we obtain a functor

$$(19.7.2) \quad H_p : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ satisfies (19.7.1) } \forall t \in T, \\ \emptyset & \text{else.} \end{cases}$$

Lemma 19.8. *In Situation 19.7.*

- (1) *The functor H_p satisfies the sheaf property for the fpqc topology.*
- (2) *If \mathcal{F} is of finite presentation, then functor H_p is limit preserving.*

Proof. Let $\{T_i \rightarrow T\}_{i \in I}$ be an fpqc¹ covering of schemes over S . Set $X_i = X_{T_i} = X \times_S T_i$ and denote \mathcal{F}_i the pullback of \mathcal{F} to X_i . Assume that \mathcal{F}_i satisfies (19.7.1) for all i . Pick $t \in T$ and let $\xi_t \in X_T$ denote the generic point of X_t . We have to show that \mathcal{F} is free in a neighbourhood of ξ_t . For some $i \in I$ we can find a $t_i \in T_i$ mapping to t . Let $\xi_i \in X_i$ denote the generic point of X_{t_i} , so that ξ_i maps to ξ_t . The fact that \mathcal{F}_i is free of rank p in a neighbourhood of ξ_i implies that $(\mathcal{F}_i)_{x_i} \cong \mathcal{O}_{X_i, x_i}^{\oplus p}$ which implies that $\mathcal{F}_{T, \xi_t} \cong \mathcal{O}_{X_T, \xi_t}^{\oplus p}$ as $\mathcal{O}_{X_T, \xi_t} \rightarrow \mathcal{O}_{X_i, x_i}$ is flat, see for example Algebra, Lemma 75.5. Thus there exists an affine neighbourhood U of ξ_t in X_T and a surjection $\mathcal{O}_U^{\oplus p} \rightarrow \mathcal{F}_U = \mathcal{F}_T|_U$, see Modules, Lemma 9.4. After shrinking T we may assume that $U \rightarrow T$ is surjective. Hence $U \rightarrow T$ is a smooth morphism of affines with geometrically irreducible fibres. Moreover, for every $t' \in T$ we see that the induced map

$$\alpha : \mathcal{O}_{U, \xi_{t'}}^{\oplus p} \longrightarrow \mathcal{F}_{U, \xi_{t'}}$$

is an isomorphism (since by the same argument as before the module on the right is free of rank p). It follows from Lemma 10.1 that

$$\Gamma(U, \mathcal{O}_U^{\oplus p}) \otimes_{\Gamma(T, \mathcal{O}_T)} \mathcal{O}_{T, t'} \longrightarrow \Gamma(U, \mathcal{F}_U) \otimes_{\Gamma(T, \mathcal{O}_T)} \mathcal{O}_{T, t'}$$

is injective for every $t' \in T$. Hence we see the surjection α is an isomorphism. This finishes the proof of (1).

Assume that \mathcal{F} is of finite presentation. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S -schemes and assume that \mathcal{F}_T satisfies (19.7.1). Set $X_i = X_{T_i} = X \times_S T_i$ and denote \mathcal{F}_i the pullback of \mathcal{F} to X_i . Let $U \subset X_T$ denote the open subscheme of points where \mathcal{F}_T is flat over T , see More on Morphisms, Theorem 12.1. By assumption every generic point of every fibre is a point of U , i.e., $U \rightarrow T$ is a smooth surjective morphism with geometrically irreducible fibres. We may shrink

¹It is quite easy to show that H_p is a sheaf for the fppf topology using that flat morphisms of finite presentation are open. This is all we really need later on. But it is kind of fun to prove directly that it also satisfies the sheaf condition for the fpqc topology.

U a bit and assume that U is quasi-compact. Using Limits, Lemma 3.8 we can find an $i \in I$ and a quasi-compact open $U_i \subset X_i$ whose inverse image in X_T is U . After increasing i we may assume that $\mathcal{F}_i|_{U_i}$ is flat over T_i , see Limits, Lemma 9.3. In particular, $\mathcal{F}_i|_{U_i}$ is finite locally free hence defines a locally constant rank function $\rho : U_i \rightarrow \{0, 1, 2, \dots\}$. Let $(U_i)_p \subset U_i$ denote the open and closed subset where ρ has value p . Let $V_i \subset T_i$ be the image of $(U_i)_p$; note that V_i is open and quasi-compact. By assumption the image of $T \rightarrow T_i$ is contained in V_i . Hence there exists an $i' \geq i$ such that $T_{i'} \rightarrow T_i$ factors through V_i by Limits, Lemma 3.8. Then $\mathcal{F}_{i'}$ satisfies (19.7.1) as desired. Some details omitted. \square

Situation 19.9. Let $f : X \rightarrow S$ be a morphism of schemes which is of finite type. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. For any scheme T over S we will denote \mathcal{F}_T the base change of \mathcal{F} to T , in other words, \mathcal{F}_T is the pullback of \mathcal{F} via the projection morphism $X_T = X \times_S T \rightarrow X$. Note that $X_T \rightarrow T$ is of finite type and that \mathcal{F}_T is an \mathcal{O}_{X_T} -module of finite type, see Morphisms, Lemma 16.4 and Modules, Lemma 9.2. Let $n \geq 0$. We say that \mathcal{F}_T is flat over T in dimensions $\geq n$ if for every $t \in T$ the closed subset $Z \subset X_t$ of points where \mathcal{F}_T is not flat over T (see Lemma 10.4) satisfies $\dim(Z) < n$ for all $t \in T$. Note that if this is the case, and if $T' \rightarrow T$ is a morphism, then $\mathcal{F}_{T'}$ is also flat in dimensions $\geq n$ over T' , see Morphisms, Lemmas 26.6 and 29.3. Hence we obtain a functor

$$F_n : (\text{Sch}/S)^{\text{opp}} \longrightarrow \text{Sets}, \quad T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T \text{ in } \dim \geq n, \\ \emptyset & \text{else.} \end{cases}$$

Lemma 19.10. *In Situation 19.9.*

- (1) *The functor F_n satisfies the sheaf property for the fpqc topology.*
- (2) *If f is quasi-compact and locally of finite presentation and \mathcal{F} is of finite presentation, then the functor F_n is limit preserving.*

Proof. Let $\{T_i \rightarrow T\}_{i \in I}$ be an fpqc covering of schemes over S . Set $X_i = X_{T_i} = X \times_S T_i$ and denote \mathcal{F}_i the pullback of \mathcal{F} to X_i . Assume that \mathcal{F}_i is flat over T_i in dimensions $\geq n$ for all i . Let $t \in T$. Choose an index i and a point $t_i \in T_i$ mapping to t . Consider the cartesian diagram

$$\begin{array}{ccc} X_{\text{Spec}(\mathcal{O}_{T,t})} & \longleftarrow & X_{\text{Spec}(\mathcal{O}_{T_i,t_i})} \\ \downarrow & & \downarrow \\ \text{Spec}(\mathcal{O}_{T,t}) & \longleftarrow & \text{Spec}(\mathcal{O}_{T_i,t_i}) \end{array}$$

As the lower horizontal morphism is flat we see from More on Morphisms, Lemma 12.2 that the set $Z_i \subset X_{t_i}$ where \mathcal{F}_i is not flat over T_i and the set $Z \subset X_t$ where \mathcal{F}_T is not flat over T are related by the rule $Z_i = Z_{\kappa(t_i)}$. Hence we see that \mathcal{F}_T is flat over T in dimensions $\geq n$ by Morphisms, Lemma 29.3.

Assume that f is quasi-compact and locally of finite presentation and that \mathcal{F} is of finite presentation. In this paragraph we first reduce the proof of (2) to the case where f is of finite presentation. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S -schemes and assume that \mathcal{F}_T is flat in dimensions $\geq n$. Set $X_i = X_{T_i} = X \times_S T_i$ and denote \mathcal{F}_i the pullback of \mathcal{F} to X_i . We have to show that \mathcal{F}_i is flat in dimensions $\geq n$ for some i . Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since T_{i_0} is affine (hence quasi-compact) there exist finitely many affine opens $W_j \subset S$, $j = 1, \dots, m$ and an

affine open overing $T_{i_0} = \bigcup_{j=1, \dots, m} V_{j, i_0}$ such that $T_{i_0} \rightarrow S$ maps V_{j, i_0} into W_j . For $i \geq i_0$ denote $V_{j, i}$ the inverse image of V_{j, i_0} in T_i . If we can show, for each j , that there exists an i such that $\mathcal{F}_{V_{j, i_0}}$ is flat in dimensions $\geq n$, then we win. In this way we reduce to the case that S is affine. In this case X is quasi-compact and we can choose a finite affine open covering $X = W_1 \cup \dots \cup W_m$. In this case the result for $(X \rightarrow S, \mathcal{F})$ is equivalent to the result for $(\coprod W_j, \coprod \mathcal{F}|_{W_j})$. Hence we may assume that f is of finite presentation.

Assume f is of finite presentation and \mathcal{F} is of finite presentation. Let $U \subset X_T$ denote the open subscheme of points where \mathcal{F}_T is flat over T , see More on Morphisms, Theorem 12.1. By assumption the dimension of every fibre of $Z = X_T \setminus U$ over T has dimension $\leq n$. By Limits, Lemma 14.3 we can find a closed subscheme $Z \subset Z' \subset X_T$ such that $\dim(Z'_t) < n$ for all $t \in T$ and such that $Z' \rightarrow X_T$ is of finite presentation. By Limits, Lemmas 9.1 and 7.4 there exists an $i \in I$ and a closed subscheme $Z'_i \subset X_i$ of finite presentation whose base change to T is Z' . By Limits, Lemma 14.1 we may assume all fibres of $Z'_i \rightarrow T_i$ have dimension $< n$. By Limits, Lemma 9.3 we may assume that $\mathcal{F}_i|_{X_i \setminus T'_i}$ is flat over T_i . This implies that \mathcal{F}_i is flat in dimensions $\geq n$; here we use that $Z' \rightarrow X_T$ is of finite presentation, and hence the complement $X_T \setminus Z'$ is quasi-compact! Thus part (2) is proved and the proof of the lemma is complete. \square

Situation 19.11. Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. For any scheme T over S we will denote \mathcal{F}_T the base change of \mathcal{F} to T , in other words, \mathcal{F}_T is the pullback of \mathcal{F} via the projection morphism $X_T = X \times_S T \rightarrow X$. Since the base change of a flat module is flat we obtain a functor

$$(19.11.1) \quad F_{flat} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T, \\ \emptyset & \text{else.} \end{cases}$$

Lemma 19.12. *In Situation 19.11.*

- (1) *The functor F_{flat} satisfies the sheaf property for the fpqc topology.*
- (2) *If f is quasi-compact and locally of finite presentation and \mathcal{F} is of finite presentation, then the functor F_{flat} is limit preserving.*

Proof. Part (1) follows from the following statement: If $T' \rightarrow T$ is a surjective flat morphism of schemes over S , then $\mathcal{F}_{T'}$ is flat over T' if and only if \mathcal{F}_T is flat over T , see More on Morphisms, Lemma 12.2. Part (2) follows from Limits, Lemma 9.3 after reducing to the case where X and S are affine (compare with the proof of Lemma 19.10). \square

20. Flattening stratifications

Just the definitions and an important baby case.

Definition 20.1. Let $X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. We say that the *universal flattening of \mathcal{F} exists* if the functor F_{flat} defined in Situation 19.11 is representable by a scheme S' over S . We say that the *universal flattening of X exists* if the universal flattening of \mathcal{O}_X exists.

Note that if the universal flattening S'^2 of \mathcal{F} exists, then the morphism $S' \rightarrow S$ is a monomorphism of schemes such that $\mathcal{F}_{S'}$ is flat over S' and such that a morphism $T \rightarrow S$ factors through S' if and only if \mathcal{F}_T is flat over T .

We define (compare with Topology, Remark 27.4) a (locally finite, scheme theoretic) *stratification* of a scheme S to be given by closed subschemes $Z_i \subset S$ indexed by a partially ordered set I such that $S = \bigcup Z_i$ (set theoretically), such that every point of S has a neighbourhood meeting only a finite number of Z_i , and such that

$$Z_i \cap Z_j = \bigcup_{k \leq i, j} Z_k.$$

Setting $S_i = Z_i \setminus \bigcup_{j < i} Z_j$ the actual stratification is the decomposition $S = \coprod S_i$ into locally closed subschemes. We often only indicate the strata S_i and leave the construction of the closed subschemes Z_i to the reader. Given a stratification we obtain a monomorphism

$$S' = \coprod_{i \in I} S_i \longrightarrow S.$$

We will call this the *monomorphism associated to the stratification*. With this terminology we can define what it means to have a flattening stratification.

Definition 20.2. Let $X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. We say that \mathcal{F} has a *flattening stratification* if the functor F_{flat} defined in Situation 19.11 is representable by a monomorphism $S' \rightarrow S$ associated to a stratification of S by locally closed subschemes. We say that X has a *flattening stratification* if \mathcal{O}_X has a flattening stratification.

When a flattening stratification exists, it is often important to understand the index set labeling the strata and its partial ordering. This often has to do with ranks of modules, as in the baby case below.

Lemma 20.3. *Let S be a scheme. Let \mathcal{F} be a finite type, quasi-coherent \mathcal{O}_S -module. The closed subschemes*

$$S = Z_{-1} \supset Z_0 \supset Z_1 \supset Z_2 \dots$$

defined by the fitting ideals of \mathcal{F} have the following properties

- (1) *The intersection $\bigcap Z_r$ is empty.*
- (2) *The functor $(\text{Sch}/S)^{\text{opp}} \rightarrow \text{Sets}$ defined by the rule*

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is locally generated by } \leq r \text{ sections} \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the open subscheme $S \setminus Z_r$.

- (3) *The functor $F_r : (\text{Sch}/S)^{\text{opp}} \rightarrow \text{Sets}$ defined by the rule*

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ locally free rank } r \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the locally closed subscheme $Z_{r-1} \setminus Z_r$ of S .

If \mathcal{F} is of finite presentation, then $Z_r \rightarrow S$, $S \setminus Z_r \rightarrow S$, and $Z_{r-1} \setminus Z_r \rightarrow S$ are of finite presentation.

²The scheme S' is sometimes called the *universal flatifier*. In [GR71] it is called the *platificateur universel*. Existence of the universal flattening should not be confused with the type of results discussed in More on Algebra, Section 18.

Proof. We refer to More on Algebra, Section 5 for the construction of the *fitting ideals* in the algebraic setting. Here we will construct the sequence

$$0 = \mathcal{I}_{-1} \subset \mathcal{I}_0 \subset \mathcal{I}_1 \subset \dots \subset \mathcal{O}_S$$

of fitting ideals of \mathcal{F} as an \mathcal{O}_S -module. Namely, if $U \subset X$ is open, and

$$\bigoplus_{i \in I} \mathcal{O}_U \rightarrow \mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F}|_U \rightarrow 0$$

is a presentation of \mathcal{F} over U , then $\mathcal{I}_r|_U$ is generated by the $(n-r) \times (n-r)$ -minors of the matrix defining the first arrow of the presentation. In particular, \mathcal{I}_r is locally generated by sections, whence quasi-coherent. If $U = \operatorname{Spec}(A)$ and $\mathcal{F}|_U = \widetilde{M}$, then $\mathcal{I}_r|_U$ is the ideal sheaf associated to the fitting ideal $\operatorname{Fit}_r(M)$ as in More on Algebra, Definition 5.3. Let $Z_r \subset S$ be the closed subscheme corresponding to \mathcal{I}_r .

For any morphism $g : T \rightarrow S$ we see from More on Algebra, Lemma 5.6 that \mathcal{F}_T is locally generated by $\leq r$ sections if and only if $\mathcal{I}_r \cdot \mathcal{O}_T = \mathcal{O}_T$. This proves (2).

For any morphism $g : T \rightarrow S$ we see from More on Algebra, Lemma 5.7 that \mathcal{F}_T is free of rank r if and only if $\mathcal{I}_r \cdot \mathcal{O}_T = \mathcal{O}_T$ and $\mathcal{I}_{r-1} \cdot \mathcal{O}_T = 0$. This proves (3).

The final statement of the lemma follows from the fact that if \mathcal{F} is of finite presentation, then each of the morphisms $Z_r \rightarrow S$ is of finite presentation as \mathcal{I}_r is locally generated by finitely many minors. This implies that $Z_{r-1} \setminus Z_r$ is a retrocompact open in Z_r and hence the morphism $Z_{r-1} \setminus Z_r \rightarrow Z_r$ is of finite presentation as well. \square

Lemma 20.3 notwithstanding the following lemma does not hold if \mathcal{F} is a finite type quasi-coherent module. Namely, the stratification still exists but it isn't true that it represents the functor F_{flat} in general.

Lemma 20.4. *Let S be a scheme. Let \mathcal{F} be a quasi-coherent \mathcal{O}_S -module of finite presentation. There exists a flattening stratification $S' = \coprod_{r \geq 0} S_r$ for \mathcal{F} (relative to $\operatorname{id}_S : S \rightarrow S$) such that $\mathcal{F}|_{S_r}$ is locally free of rank r . Moreover, each $S_r \rightarrow S$ is of finite presentation.*

Proof. Suppose that $g : T \rightarrow S$ is a morphism of schemes such that the pullback $\mathcal{F}_T = g^* \mathcal{F}$ is flat. Then \mathcal{F}_T is a flat \mathcal{O}_T -module of finite presentation. Hence \mathcal{F}_T is finite locally free, see Properties, Lemma 18.2. Thus $T = \coprod_{r \geq 0} T_r$, where $\mathcal{F}_T|_{T_r}$ is locally free of rank r . This implies that

$$F_{flat} = \coprod_{r \geq 0} F_r$$

in the category of Zariski sheaves on Sch/S where F_r is as in Lemma 20.3. It follows that F_{flat} is represented by $\coprod_{r \geq 0} (Z_{r-1} \setminus Z_r)$ where Z_r is as in Lemma 20.3. \square

21. Flattening stratification over an Artinian ring

A flattening stratification exists when the base scheme is the spectrum of an Artinian ring.

Lemma 21.1. *Let S be the spectrum of an Artinian ring. For any scheme X over S , and any quasi-coherent \mathcal{O}_X -module there exists a universal flattening. In fact the universal flattening is given by a closed immersion $S' \rightarrow S$, and hence is a flattening stratification for \mathcal{F} as well.*

Proof. Choose an affine open covering $X = \bigcup U_i$. Then F_{flat} is the product of the functors associated to each of the pairs $(U_i, \mathcal{F}|_{U_i})$. Hence it suffices to prove the result for each $(U_i, \mathcal{F}|_{U_i})$. In the affine case the lemma follows immediately from More on Algebra, Lemma 10.2. \square

22. Flattening a map

Theorem 22.3 is the key to further flattening statements.

Lemma 22.1. *Let S be a scheme. Let $g : X' \rightarrow X$ be a flat morphism of schemes over S with X locally of finite type over S . Let \mathcal{F} be a finite type \mathcal{O}_X -module which is flat over S . If $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ then the canonical map*

$$\mathcal{F} \longrightarrow g_* g^* \mathcal{F}$$

is injective, and remains injective after any base change.

Proof. The final assertion means that $\mathcal{F}_T \rightarrow (g_T)_* g_T^* \mathcal{F}_T$ is injective for any morphism $T \rightarrow S$. The assumption $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ is preserved by base change, see Divisors, Lemma 7.2 and Remark 7.3. The same holds for the assumption of flatness and finite type. Hence it suffices to prove the injectivity of the displayed arrow. Let $\mathcal{K} = \text{Ker}(\mathcal{F} \rightarrow g_* g^* \mathcal{F})$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 5.5. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 5.4. As \mathcal{F} is flat we see from Lemma 12.4 that $\text{WeakAss}_X(\mathcal{F}) \subset \text{Ass}_{X/S}(\mathcal{F})$. By assumption any point x of $\text{Ass}_{X/S}(\mathcal{F})$ is the image of some $x' \in X'$. Since g is flat the local ring map $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X',x'}$ is faithfully flat, hence the map

$$\mathcal{F}_x \longrightarrow g^* \mathcal{F}_{x'} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

is injective (see Algebra, Lemma 79.11). This implies that $\mathcal{K}_x = 0$ as desired. \square

Lemma 22.2. *Let A be a ring. Let $u : M \rightarrow N$ be a surjective map of A -modules. If M is projective as an A -module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \rightarrow B$ the following are equivalent*

- (1) $u \otimes 1 : M \otimes_A B \rightarrow N \otimes_A B$ is an isomorphism, and
- (2) $\varphi(I) = 0$.

Proof. As M is projective we can find a projective A -module C such that $F = M \oplus C$ is a free R -module. By replacing u by $u \oplus 1 : F = M \oplus C \rightarrow N \oplus C$ we see that we may assume M is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\text{Ker}(u)$ with respect to some (fixed) basis of M . The reason this works is that, since u is surjective and $\otimes_A B$ is right exact, $\text{Ker}(u \otimes 1)$ is the image of $\text{Ker}(u) \otimes_A B$ in $M \otimes_A B$. \square

Theorem 22.3. *In Situation 19.1 assume*

- (1) f is of finite presentation,
- (2) \mathcal{F} is of finite presentation, flat over S , and pure relative to S , and
- (3) u is surjective.

Then F_{iso} is representable by a closed immersion $Z \rightarrow S$. Moreover $Z \rightarrow S$ is of finite presentation if \mathcal{G} is of finite presentation.

Proof. We will use without further mention that \mathcal{F} is universally pure over S , see Lemma 17.3. By Lemma 19.2 and Descent, Lemmas 33.2 and 35.1 the question is local for the étale topology on S . Hence it suffices to prove, given $s \in S$, that there exists an étale neighbourhood of (S, s) so that the theorem holds.

Using Lemma 11.5 and after replacing S by an elementary étale neighbourhood of s we may assume there exists a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow & \swarrow \\ & S & \end{array}$$

of schemes of finite presentation over S , where g is étale, $X_s \subset g(X')$, the schemes X' and S are affine, $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$ -module. Note that $g^*\mathcal{F}$ is universally pure over S , see Lemma 16.4. Hence by Lemma 17.2 we see that the open $g(X')$ contains the points of $\text{Ass}_{X/S}(\mathcal{F})$ lying over $\text{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 20.5 E is a constructible subset of S . We have seen that $\text{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 23.4 we see that E contains an open neighbourhood of s . Hence after replacing S by a smaller affine neighbourhood of s we may assume that $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$.

Since we have assumed that u is surjective we have $F_{iso} = F_{inj}$. From Lemma 22.1 it follows that $u : \mathcal{F} \rightarrow \mathcal{G}$ is injective if and only if $g^*u : g^*\mathcal{F} \rightarrow g^*\mathcal{G}$ is injective, and the same remains true after any base change. Hence we have reduced to the case where, in addition to the assumptions in the theorem, $X \rightarrow S$ is a morphism of affine schemes and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$ -module. This case follows immediately from Lemma 22.2.

To see that Z is of finite presentation if \mathcal{G} is of finite presentation, combine Lemma 19.2 part (4) with Limits, Remark 5.2. \square

Lemma 22.4. *Let $f : X \rightarrow S$ be a morphism of schemes which is of finite presentation, flat, and pure. Let Y be a closed subscheme of X . Let $F = f_*Y$ be the Weil restriction functor of Y along f , defined by*

$$F : (\text{Sch}/S)^{opp} \rightarrow \text{Sets}, \quad T \mapsto \begin{cases} \{*\} & \text{if } Y_T \rightarrow X_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

Then F is representable by a closed immersion $Z \rightarrow S$. Moreover $Z \rightarrow S$ is of finite presentation if $Y \rightarrow S$ is.

Proof. Let \mathcal{I} be the ideal sheaf defining Y in X and let $u : \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{I}$ be the surjection. Then for an S -scheme T , the closed immersion $Y_T \rightarrow X_T$ is an isomorphism if and only if u_T is an isomorphism. Hence the result follows from Theorem 22.3. \square

23. Flattening in the local case

In this section we start applying the earlier material to obtain a shadow of the flattening stratification.

Theorem 23.1. *In Situation 19.3 assume A is henselian, B is essentially of finite type over A , and M is a finite B -module. Then there exists an ideal $I \subset A$ such that A/I corepresents the functor F_{lf} on the category \mathcal{C} . In other words given a local homomorphism of local rings $\varphi : A \rightarrow A'$ with $B' = B \otimes_A A'$ and $M' = M \otimes_A A'$ the following are equivalent:*

- (1) $\forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset \text{Spec}(B') : M'_{\mathfrak{q}}$ is flat over A' , and
- (2) $\varphi(I) = 0$.

If B is essentially of finite presentation over A and M of finite presentation over B , then I is a finitely generated ideal.

Proof. Choose a finite type ring map $A \rightarrow C$ and a finite C -module N and a prime \mathfrak{q} of C such that $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. In the following, when we say “the theorem holds for $(N/C/A, \mathfrak{q})$ ” we mean that it holds for $(A \rightarrow B, M)$ where $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. By Lemma 19.6 the functor F_{lf} is unchanged if we replace B by a local ring flat over B . Hence, since A is henselian, we may apply Lemma 6.6 and assume that there exists a complete dévissage of $N/C/A$ at \mathfrak{q} .

Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1, \dots, n}$ be such a complete dévissage of $N/C/A$ at \mathfrak{q} . Let $\mathfrak{q}'_i \subset A_i$ be the unique prime lying over $\mathfrak{q}_i \subset B_i$ as in Definition 6.4. Since $C \rightarrow A_1$ is surjective and $N \cong M_1$ as C -modules, we see by Lemma 19.5 it suffices to prove the theorem holds for $(M_1/A_1/A, \mathfrak{q}'_1)$. Since $B_1 \rightarrow A_1$ is finite and \mathfrak{q}_1 is the only prime of B_1 over \mathfrak{q}'_1 we see that $(A_1)_{\mathfrak{q}'_1} \rightarrow (B_1)_{\mathfrak{q}_1}$ is finite (see Algebra, Lemma 40.11 or More on Morphisms, Lemma 33.4). Hence by Lemma 19.5 it suffices to prove the theorem holds for $(M_1/B_1/A, \mathfrak{q}_1)$.

At this point we may assume, by induction on the length n of the dévissage, that the theorem holds for $(M_2/B_2/A, \mathfrak{q}_2)$. (If $n = 1$, then $M_2 = 0$ which is flat over A .) Reversing the last couple of steps of the previous paragraph, using that $M_2 \cong \text{Coker}(\alpha_2)$ as B_1 -modules, we see that the theorem holds for $(\text{Coker}(\alpha_1)/B_1/A, \mathfrak{q}_1)$.

Let A' be an object of \mathcal{C} . At this point we use Lemma 10.1 to see that if $(M_1 \otimes_A A')_{\mathfrak{q}'}$ is flat over A' for a prime \mathfrak{q}' of $B_1 \otimes_A A'$ lying over $\mathfrak{m}_{A'}$, then $(\text{Coker}(\alpha_1) \otimes_A A')_{\mathfrak{q}'}$ is flat over A' . Hence we conclude that F_{lf} is a subfunctor of the functor F'_{lf} associated to the module $\text{Coker}(\alpha_1)_{\mathfrak{q}_1}$ over $(B_1)_{\mathfrak{q}_1}$. By the previous paragraph we know F'_{lf} is corepresented by A/J for some ideal $J \subset A$. Hence we may replace A by A/J and assume that $\text{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A .

Since $\text{Coker}(\alpha_1)$ is a B_1 -module for which there exist a complete dévissage of $N_1/B_1/A$ at \mathfrak{q}_1 and since $\text{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A by Lemma 10.2 we see that $\text{Coker}(\alpha_1)$ is free as an A -module, in particular flat as an A -module. Hence Lemma 10.1 implies $F_{lf}(A')$ is nonempty if and only if $\alpha \otimes 1_{A'}$ is injective. Let $N_1 = \text{Im}(\alpha_1) \subset M_1$ so that we have exact sequences

$$0 \rightarrow N_1 \rightarrow M_1 \rightarrow \text{Coker}(\alpha_1) \rightarrow 0 \quad \text{and} \quad B_1^{\oplus r_1} \rightarrow N_1 \rightarrow 0$$

The flatness of $\text{Coker}(\alpha_1)$ implies the first sequence is universally exact (see Algebra, Lemma 79.5). Hence $\alpha \otimes 1_{A'}$ is injective if and only if $B_1^{\oplus r_1} \otimes_A A' \rightarrow N_1 \otimes_A A'$ is an isomorphism. Finally, Theorem 22.3 applies to show this functor is corepresentable by A/I for some ideal I and we conclude F_{lf} is corepresentable by A/I also.

To prove the final statement, suppose that $A \rightarrow B$ is essentially of finite presentation and M of finite presentation over B . Let $I \subset A$ be the ideal such that F_{lf} is corepresented by A/I . Write $I = \bigcup I_{\lambda}$ where I_{λ} ranges over the finitely generated

ideals contained in I . Then, since $F_{lf}(A/I) = \{*\}$ we see that $F_{lf}(A/I_\lambda) = \{*\}$ for some λ , see Lemma 19.4 part (2). Clearly this implies that $I = I_\lambda$. \square

Remark 23.2. Here is a scheme theoretic reformulation of Theorem 23.1. Let $(X, x) \rightarrow (S, s)$ be a morphism of pointed schemes which is locally of finite type. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Assume S henselian local with closed point s . There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \rightarrow (S, s)$ the following are equivalent

- (1) \mathcal{F}_T is flat over T at all points of the fibre X_t which map to $x \in X_s$, and
- (2) $\text{Spec}(\mathcal{O}_{T,t}) \rightarrow S$ factors through Z .

Moreover, if $X \rightarrow S$ is of finite presentation at x and \mathcal{F}_x of finite presentation over $\mathcal{O}_{X,x}$, then $Z \rightarrow S$ is of finite presentation.

At this point we can obtain some very general results completely for free from the result above. Note that perhaps the most interesting case is when $E = X_s$!

Lemma 23.3. *Let S be the spectrum of a henselian local ring with closed point s . Let $X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module. Let $E \subset X_s$ be a subset. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \rightarrow (S, s)$ the following are equivalent*

- (1) \mathcal{F}_T is flat over T at all points of the fibre X_t which map to a point of $E \subset X_s$, and
- (2) $\text{Spec}(\mathcal{O}_{T,t}) \rightarrow S$ factors through Z .

Moreover, if $X \rightarrow S$ is locally of finite presentation, \mathcal{F} is of finite presentation, and $E \subset X_s$ is closed and quasi-compact, then $Z \rightarrow S$ is of finite presentation.

Proof. For $x \in X_s$ denote $Z_x \subset S$ the closed subscheme we found in Remark 23.2. Then it is clear that $Z = \bigcap_{x \in E} Z_x$ works!

To prove the final statement assume X locally of finite presentation, \mathcal{F} of finite presentation and Z closed and quasi-compact. First, choose finitely many affine opens $W_j \subset X$ such that $E \subset \bigcup W_j$. It clearly suffices to prove the result for each morphism $W_j \rightarrow S$ with sheaf $\mathcal{F}|_{W_j}$ and closed subset $E \cap W_j$. Hence we may assume X is affine. In this case, More on Algebra, Lemma 12.4 shows that the functor defined by (1) is “limit preserving”. Hence we can show that $Z \rightarrow S$ is of finite presentation exactly as in the last part of the proof of Theorem 23.1. \square

Remark 23.4. Tracing the proof of Lemma 23.3 to its origins we find a long and winding road. But if we assume that

- (1) f is of finite type,
- (2) \mathcal{F} is a finite type \mathcal{O}_X -module,
- (3) $E = X_s$, and
- (4) S is the spectrum of a Noetherian complete local ring.

then there is a proof relying completely on more elementary algebra as follows: first we reduce to the case where X is affine by taking a finite affine open cover. In this case Z exists by More on Algebra, Lemma 13.3. The key step in this proof is constructing the closed subscheme Z step by step inside the truncations $\text{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^n)$. This relies on the fact that flattening stratifications always exist when the base is Artinian, and the fact that $\mathcal{O}_{S,s} = \lim \mathcal{O}_{S,s}/\mathfrak{m}_s^n$.

24. Variants of a lemma

In this section we discuss variants of Algebra, Lemma 124.4.

Situation 24.1. Let $\varphi : A \rightarrow B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A -module, N a finite B -module and $u : N \rightarrow M$ an A -module map such that $\bar{u} : N/\mathfrak{m}_A N \rightarrow M/\mathfrak{m}_A M$ is injective.

In this situation it is our goal to show (Lemma 24.6) that u is A -universally injective, N is of finite presentation over B , and N is flat as an A -module. If this is true, we will say *the lemma holds* in the given situation.

Lemma 24.2. *If in Situation 24.1 the ring A is Noetherian then the lemma holds.*

Proof. Applying Algebra, Lemma 95.1 we see that u is injective and that $N/u(M)$ is flat over A . Then u is A -universally injective (Algebra, Lemma 38.11) and N is A -flat (Algebra, Lemma 38.12). Since B is Noetherian in this case we see that N is of finite presentation. \square

Lemma 24.3. *Let A_0 be a local ring. If the lemma holds for every Situation 24.1 with $A = A_0$, with B a localization of a polynomial algebra over A , and N of finite presentation over B , then the lemma holds for every Situation 24.1 with $A = A_0$.*

Proof. Let $A \rightarrow B$, $u : N \rightarrow M$ be as in Situation 24.1. Write $B = C/I$ where C is the localization of a polynomial algebra over A at a prime. If we can show that N is finitely presented as a C -module, then a fortiori this shows that N is finitely presented as a B -module (see Algebra, Lemma 6.4). Hence we may assume that B is the localization of a polynomial algebra. Next, write $N = B^{\oplus n}/K$ for some submodule $K \subset B^{\oplus n}$. Since $B/\mathfrak{m}_A B$ is Noetherian (as it is essentially of finite type over a field), there exist finitely many elements $k_1, \dots, k_s \in K$ such that for $K' = \sum Bk_i$ and $N' = B^{\oplus n}/K'$ the canonical surjection $N' \rightarrow N$ induces an isomorphism $N'/\mathfrak{m}_A N' \cong N/\mathfrak{m}_A N$. Now, if the lemma holds for the composition $u' : N' \rightarrow M$, then u' is injective, hence $N' = N$ and $u' = u$. Thus the lemma holds for the original situation. \square

Lemma 24.4. *If in Situation 24.1 the ring A is henselian then the lemma holds.*

Proof. It suffices to prove this when B is essentially of finite presentation over A and N is of finite presentation over B , see Lemma 24.3. Let us temporarily make the additional assumption that N is flat over A . Then N is a filtered colimit $N = \text{colim}_i F_i$ of free A -modules F_i such that the transition maps $u_{ii'} : F_i \rightarrow F_{i'}$ are injective modulo \mathfrak{m}_A , see Lemma 18.5. Each of the compositions $u_i : F_i \rightarrow M$ is A -universally injective by Lemma 7.5 wherefore $u = \text{colim } u_i$ is A -universally injective as desired.

Assume A is a henselian local ring, B is essentially of finite presentation over A , N of finite presentation over B . By Theorem 23.1 there exists a finitely generated ideal $I \subset A$ such that N/IN is flat over A/I and such that $N/I^2 N$ is not flat over A/I^2 unless $I = 0$. The result of the previous paragraph shows that the lemma holds for u mod $I : N/IN \rightarrow M/IM$ over A/I . Consider the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M \otimes_A I/I^2 & \longrightarrow & M/I^2 M & \longrightarrow & M/IM \longrightarrow 0 \\
 & & \uparrow u & & \uparrow u & & \uparrow u \\
 & & N \otimes_A I/I^2 & \longrightarrow & N/I^2 N & \longrightarrow & N/IN \longrightarrow 0
 \end{array}$$

whose rows are exact by right exactness of \otimes and the fact that M is flat over A . Note that the left vertical arrow is the map $N/IN \otimes_{A/I} I/I^2 \rightarrow M/IM \otimes_{A/I} I/I^2$, hence is injective. A diagram chase shows that the lower left arrow is injective, i.e., $\text{Tor}_{A/I^2}^1(I/I^2, M/I^2) = 0$ see Algebra, Remark 72.8. Hence N/I^2N is flat over A/I^2 by Algebra, Lemma 95.8 a contradiction unless $I = 0$. \square

The following lemma discusses the special case of Situation 24.1 where M has a B -module structure and u is B -linear. This is the case most often used in practice and it is significantly easier to prove than the general case.

Lemma 24.5. *Let $A \rightarrow B$ be a local ring homomorphism of local rings which is essentially of finite type. Let $u : N \rightarrow M$ be a B -module map. If N is a finite B -module, M is flat over A , and $\bar{u} : N/\mathfrak{m}_A N \rightarrow M/\mathfrak{m}_A M$ is injective, then u is A -universally injective, N is of finite presentation over B , and N is flat over A .*

Proof. Let $A \rightarrow A^h$ be the henselization of A . Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Since $B \rightarrow B'$ is flat (hence faithfully flat, see Algebra, Lemma 38.16), we may replace $A \rightarrow B$ with $A^h \rightarrow B'$, the module M by $M \otimes_B B'$, the module N by $N \otimes_B B'$, and u by $u \otimes \text{id}_{B'}$, see Algebra, Lemmas 80.2 and 38.8. Thus we may assume that A is a henselian local ring. In this case our lemma follows from the more general Lemma 24.4. \square

Lemma 24.6. *Let $\varphi : A \rightarrow B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A -module, N a finite B -module and $u : N \rightarrow M$ an A -module map such that $\bar{u} : N/\mathfrak{m}_A N \rightarrow M/\mathfrak{m}_A M$ is injective. Then u is A -universally injective, N is of finite presentation over B , and N is flat over A .*

Proof. Omitted. Proof will be added soon. \square

25. Flat finite type modules, Part III

The following result is one of the main results of this chapter.

Theorem 25.1. *Let $f : X \rightarrow S$ be locally of finite type. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Let $x \in X$ with image $s \in S$. The following are equivalent*

- (1) \mathcal{F} is flat at x over S , and
- (2) for every $x' \in \text{Ass}_{X_s}(\mathcal{F}_s)$ which specializes to x we have that \mathcal{F} is flat at x' over S .

Proof. It is clear that (1) implies (2) as $\mathcal{F}_{x'}$ is a localization of \mathcal{F}_x for every point which specializes to x . Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$ and $N = \mathcal{F}_x$. Let $\Sigma \subset B$ be the multiplicative subset of B of elements which act as nonzerodivisors on $N/\mathfrak{m}_A N$. Assumption (2) implies that $\Sigma^{-1}N$ is A -flat by the description of $\text{Spec}(\Sigma^{-1}N)$ in Lemma 7.1. On the other hand, the map $N \rightarrow \Sigma^{-1}N$ is injective modulo \mathfrak{m}_A by construction. Hence applying Lemma 24.5 we win. \square

Now we apply this directly to obtain the following useful results.

Lemma 25.2. *Let S be a local scheme with closed point s . Let $f : X \rightarrow S$ be locally of finite type. Let \mathcal{F} be a finite type \mathcal{O}_X -module. Assume that*

- (1) every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of the closed fibre X_s^3 ,

³For example this holds if f is finite type and \mathcal{F} is pure along X_s , or if f is proper.

(2) \mathcal{F} is flat over S at every point of X_s .

Then \mathcal{F} is flat over S .

Proof. This is immediate from the fact that it suffices to check for flatness at points of the relative assassin of \mathcal{F} over S by Theorem 25.1. \square

26. Universal flattening

If $f : X \rightarrow S$ is a proper, finitely presented morphism of schemes then one can find a universal flattening of f . In this section we discuss this and some of its variants.

Lemma 26.1. *In Situation 19.7. For each $p \geq 0$ the functor H_p (19.7.2) is representable by a locally closed immersion $S_p \rightarrow S$. If \mathcal{F} is of finite presentation, then $S_p \rightarrow S$ is of finite presentation.*

Proof. For each S we will prove the statement for all $p \geq 0$ concurrently. The functor H_p is a sheaf for the fppf topology by Lemma 19.8. Hence combining Descent, Lemma 35.1, More on Morphisms, Lemma 37.1, and Descent, Lemma 20.1 we see that the question is local for the étale topology on S . In particular, the question is Zariski local on S .

For $s \in S$ denote ξ_s the unique generic point of the fibre X_s . Note that for every $s \in S$ the restriction \mathcal{F}_s of \mathcal{F} is locally free of some rank $p(s) \geq 0$ in some neighbourhood of ξ_s . (As X_s is irreducible and smooth this follows from generic flatness for \mathcal{F}_s over X_s , see Algebra, Lemma 114.1 although this is overkill.) For future reference we note that

$$p(s) = \dim_{\kappa(\xi_s)}(\mathcal{F}_{\xi_s} \otimes_{\mathcal{O}_{X, \xi_s}} \kappa(\xi_s)).$$

In particular $H_{p(s)}(s)$ is nonempty and $H_q(s)$ is empty if $q \neq p(s)$.

Let $U \subset X$ be an open subscheme. As $f : X \rightarrow S$ is smooth, it is open. It is immediate from (19.7.2) that the functor H_p for the pair $(f|_U : U \rightarrow f(U), \mathcal{F}|_U)$ and the functor H_p for the pair $(f|_{f^{-1}(f(U))}, \mathcal{F}|_{f^{-1}(f(U))})$ are the same. Hence to prove the existence of S_p over $f(U)$ we may always replace X by U .

Pick $s \in S$. There exists an affine open neighbourhood U of ξ_s such that $\mathcal{F}|_U$ can be generated by at most $p(s)$ elements. By the arguments above we see that in order to prove the statement for $H_{p(s)}$ in an neighbourhood of s we may assume that \mathcal{F} is generated by $p(s)$ elements, i.e., that there exists a surjection

$$u : \mathcal{O}_X^{\oplus p(s)} \longrightarrow \mathcal{F}$$

In this case it is clear that $H_{p(s)}$ is equal to F_{iso} (19.1.1) for the map u (this follows immediately from Lemma 18.1 but also from Lemma 11.1 after shrinking a bit more so that both S and X are affine.) Thus we may apply Theorem 22.3 to see that $H_{p(s)}$ is representable by a closed immersion in a neighbourhood of s .

The result follows formally from the above. Namely, the arguments above show that locally on S the function $s \mapsto p(s)$ is bounded. Hence we may use induction on $p = \max_{s \in S} p(s)$. The functor H_p is representable by a closed immersion $S_p \rightarrow S$ by the above. Replace S by $S \setminus S_p$ which drops the maximum by at least one and we win by induction hypothesis.

To see that $S_p \rightarrow S$ is of finite presentation if \mathcal{F} is of finite presentation combine Lemma 19.8 part (2) with Limits, Remark 5.2. \square

Lemma 26.2. *In Situation 19.9. Let $h : X' \rightarrow X$ be an étale morphism. Set $\mathcal{F}' = h^*\mathcal{F}$ and $f' = f \circ h$. Let F'_n be (19.9.1) associated to $(f' : X' \rightarrow S, \mathcal{F}')$. Then F_n is a subfunctor of F'_n and if $h(X') \supset \text{Ass}_{X/S}(\mathcal{F})$, then $F_n = F'_n$.*

Proof. Let $T \rightarrow S$ be any morphism. Then $h_T : X'_T \rightarrow X_T$ is étale as a base change of the étale morphism g . For $t \in T$ denote $Z \subset X_t$ the set of points where \mathcal{F}_T is not flat over T , and similarly denote $Z' \subset X'_t$ the set of points where \mathcal{F}'_T is not flat over T . As $\mathcal{F}'_T = h_T^*\mathcal{F}_T$ we see that $Z' = h_t^{-1}(Z)$, see Morphisms, Lemma 26.11. Hence $Z' \rightarrow Z$ is an étale morphism, so $\dim(Z') \leq \dim(Z)$ (for example by Descent, Lemma 17.2 or just because an étale morphism is smooth of relative dimension 0). This implies that $F_n \subset F'_n$.

Finally, suppose that $h(X') \supset \text{Ass}_{X/S}(\mathcal{F})$ and that $T \rightarrow S$ is a morphism such that $F'_n(T)$ is nonempty, i.e., such that \mathcal{F}'_T is flat in dimensions $\geq n$ over T . Pick a point $t \in T$ and let $Z \subset X_t$ and $Z' \subset X'_t$ be as above. To get a contradiction assume that $\dim(Z) \geq n$. Pick a generic point $\xi \in Z$ corresponding to a component of dimension $\geq n$. Let $x \in \text{Ass}_{X_t}(\mathcal{F}_t)$ be a generalization of ξ . Then x maps to a point of $\text{Ass}_{X/S}(\mathcal{F})$ by Divisors, Lemma 7.2 and Remark 7.3. Thus we see that x is in the image of h_T , say $x = h_T(x')$ for some $x' \in X'_T$. But $x' \notin Z'$ as $x \rightsquigarrow \xi$ and $\dim(Z') < n$. Hence \mathcal{F}'_T is flat over T at x' which implies that \mathcal{F}_T is flat at x over T (by Morphisms, Lemma 26.11). Since this holds for every such x we conclude that \mathcal{F}_T is flat over T at ξ by Theorem 25.1 which is the desired contradiction. \square

Lemma 26.3. *Assume that $X \rightarrow S$ is a smooth morphism of affine schemes with geometrically irreducible fibres of dimension d and that \mathcal{F} is a quasi-coherent \mathcal{O}_X -module of finite presentation. Then $F_d = \coprod_{p=0, \dots, c} H_p$ for some $c \geq 0$ with F_d as in (19.9.1) and H_p as in (19.7.2).*

Proof. As X is affine and \mathcal{F} is quasi-coherent of finite presentation we know that \mathcal{F} can be generated by $c \geq 0$ elements. Then $\dim_{\kappa(x)}(\mathcal{F}_x \otimes \kappa(x))$ in any point $x \in X$ never exceeds c . In particular $H_p = \emptyset$ for $p > c$. Moreover, note that there certainly is an inclusion $\coprod H_p \rightarrow F_d$. Having said this the content of the lemma is that, if a base change \mathcal{F}_T is flat in dimensions $\geq d$ over T and if $t \in T$, then \mathcal{F}_T is free of some rank r in an open neighbourhood $U \subset X_T$ of the unique generic point ξ of X_t . Namely, then H_r contains the image of U which is an open neighbourhood of t . The existence of U follows from More on Morphisms, Lemma 13.7. \square

Lemma 26.4. *In Situation 19.9. Let $s \in S$ let $d \geq 0$. Assume*

- (1) *there exists a complete dévissage of $\mathcal{F}/X/S$ over some point $s \in S$,*
- (2) *X is of finite presentation over S ,*
- (3) *\mathcal{F} is an \mathcal{O}_X -module of finite presentation, and*
- (4) *\mathcal{F} is flat in dimensions $\geq d+1$ over S .*

Then after possibly replacing S by an open neighbourhood of s the functor F_d (19.9.1) is representable by a monomorphism $Z_d \rightarrow S$ of finite presentation.

Proof. A preliminary remark is that X, S are affine schemes and that it suffices to prove F_d is representable by a closed subscheme on the category of affine schemes over S . Hence throughout the proof of the lemma we work in the category of affine schemes over S .

Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1, \dots, n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s , see Definition 5.1. We will use induction on the length n of the dévissage. Recall that

$Y_k \rightarrow S$ is smooth with geometrically irreducible fibres, see Definition 4.1. Let d_k be the relative dimension of Y_k over S . Recall that $i_{k,*}\mathcal{G}_k = \text{Coker}(\alpha_k)$ and that i_k is a closed immersion. By the definitions referenced above we have $d_1 = \dim(\text{Supp}(\mathcal{F}_s))$ and

$$d_k = \dim(\text{Supp}(\text{Coker}(\alpha_{k-1})_s)) = \dim(\text{Supp}(\mathcal{G}_{k,s}))$$

for $k = 2, \dots, n$. It follows that $d_1 > d_2 > \dots > d_n \geq 0$ because α_k is an isomorphism in the generic point of $(Y_k)_s$.

Note that i_1 is a closed immersion and $\mathcal{F} = i_{1,*}\mathcal{G}_1$. Hence for any morphism of schemes $T \rightarrow S$ with T affine, we have $\mathcal{F}_T = i_{1,T,*}\mathcal{G}_{1,T}$ and $i_{1,T}$ is still a closed immersion of schemes over T . Thus \mathcal{F}_T is flat in dimensions $\geq d$ over T if and only if $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T . Because $\pi_1 : Z_1 \rightarrow Y_1$ is finite we see in the same manner that $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T if and only if $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T . The same arguments work for “flat in dimensions $\geq d+1$ ” and we conclude in particular that $\pi_{1,*}\mathcal{G}_1$ is flat over S in dimensions $\geq d+1$ by our assumption on \mathcal{F} .

Suppose that $d_1 > d$. It follows from the discussion above that in particular $\pi_{1,*}\mathcal{G}_1$ is flat over S at the generic point of $(Y_1)_s$. By Lemma 11.1 we may replace S by an affine neighbourhood of s and assume that α_1 is S -universally injective. Because α_1 is S -universally injective, for any morphism $T \rightarrow S$ with T affine, we have a short exact sequence

$$0 \rightarrow \mathcal{O}_{Y_{1,T}}^{\oplus r_1} \rightarrow \pi_{1,T,*}\mathcal{G}_{1,T} \rightarrow \text{Coker}(\alpha_1)_T \rightarrow 0$$

and still the first arrow is T -universally injective. Hence the set of points of $(Y_1)_T$ where $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat over T is the same as the set of points of $(Y_1)_T$ where $\text{Coker}(\alpha_1)_T$ is flat over S . In this way the question reduces to the sheaf $\text{Coker}(\alpha_1)$ which has a complete dévissage of length $n-1$ and we win by induction.

If $d_1 < d$ then F_d is represented by S and we win.

The last case is the case $d_1 = d$. This case follows from a combination of Lemma 26.3 and Lemma 26.1. \square

Theorem 26.5. *In Situation 19.9. Assume moreover that f is of finite presentation, that \mathcal{F} is an \mathcal{O}_X -module of finite presentation, and that \mathcal{F} is pure relative to S . Then F_n is representable by a monomorphism $Z_n \rightarrow S$ of finite presentation.*

Proof. The functor F_n is a sheaf for the fppf topology by Lemma 19.10. Hence combining Descent, Lemma 35.1, More on Morphisms, Lemma 37.1, and Descent, Lemmas 19.29 and 19.11 we see that the question is local for the étale topology on S .

In particular the situation is local for the Zariski topology on S and we may assume that S is affine. In this case the dimension of the fibres of f is bounded above, hence we see that F_n is representable for n large enough. Thus we may use descending induction on n . Suppose that we know F_{n+1} is representable by a monomorphism $Z_{n+1} \rightarrow S$ of finite presentation. Consider the base change $X_{n+1} = Z_{n+1} \times_S X$ and the pullback \mathcal{F}_{n+1} of \mathcal{F} to X_{n+1} . The morphism $Z_{n+1} \rightarrow S$ is quasi-finite as it is a monomorphism of finite presentation, hence Lemma 15.4 implies that \mathcal{F}_{n+1} is pure relative to Z_{n+1} . Since F_n is a subfunctor of F_{n+1} we conclude that in order to prove the result for F_n it suffices to prove the result for the corresponding functor for the situation $\mathcal{F}_{n+1}/X_{n+1}/Z_{n+1}$. In this way we reduce to proving the result for

F_n in case $S_{n+1} = S$, i.e., we may assume that \mathcal{F} is flat in dimensions $\geq n+1$ over S .

Fix n and assume \mathcal{F} is flat in dimensions $\geq n+1$ over S . To finish the proof we have to show that F_n is representable by a monomorphism $Z_n \rightarrow S$ of finite presentation. Since the question is local in the étale topology on S it suffices to show that for every $s \in S$ there exists an elementary étale neighbourhood $(S', s') \rightarrow (S, s)$ such that the result holds after base change to S' . Thus by Lemma 5.8 we may assume there exist étale morphisms $h_j : Y_j \rightarrow X$, $j = 1, \dots, m$ such that for each i there exists a complete dévissage of $\mathcal{F}_j/Y_j/S$ over s , where \mathcal{F}_j is the pullback of \mathcal{F} to Y_j and such that $X_s \subset \bigcup h_j(Y_j)$. Note that by Lemma 26.2 the sheaves \mathcal{F}_j are still flat over in dimensions $\geq n+1$ over S . Set $W = \bigcup h_j(Y_j)$, which is a quasi-compact open of X . As \mathcal{F} is pure along X_s we see that

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset W\}.$$

contains all generalizations of s . By More on Morphisms, Lemma 20.5 E is a constructible subset of S . We have seen that $\text{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 23.4 we see that E contains an open neighbourhood of s . Hence after shrinking S we may assume that $E = S$. It follows from Lemma 26.2 that it suffices to prove the lemma for the functor F_n associated to $X = \coprod Y_j$ and $\mathcal{F} = \coprod \mathcal{F}_j$. If $F_{j,n}$ denotes the functor for $Y_j \rightarrow S$ and the sheaf \mathcal{F}_j we see that $F_n = \prod F_{j,n}$. Hence it suffices to prove each $F_{j,n}$ is representable by some monomorphism $Z_{j,n} \rightarrow S$ of finite presentation, since then

$$Z_n = Z_{1,n} \times_S \dots \times_S Z_{m,n}$$

Thus we have reduced the theorem to the special case handled in Lemma 26.4. \square

We make explicit what the theorem means in terms of universal flattenings in the following lemma.

Lemma 26.6. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module.*

- (1) *If f is of finite presentation, \mathcal{F} is an \mathcal{O}_X -module of finite presentation, and \mathcal{F} is pure relative to S , then there exists a universal flattening $S' \rightarrow S$ of \mathcal{F} . Moreover $S' \rightarrow S$ is a monomorphism of finite presentation.*
- (2) *If f is of finite presentation and X is pure relative to S , then there exists a universal flattening $S' \rightarrow S$ of X . Moreover $S' \rightarrow S$ is a monomorphism of finite presentation.*
- (3) *If f is proper and of finite presentation and \mathcal{F} is an \mathcal{O}_X -module of finite presentation, then there exists a universal flattening $S' \rightarrow S$ of \mathcal{F} . Moreover $S' \rightarrow S$ is a monomorphism of finite presentation.*
- (4) *If f is proper and of finite presentation then there exists a universal flattening $S' \rightarrow S$ of X .*

Proof. These statements follow immediately from Theorem 26.5 applied to $F_0 = F_{\text{flat}}$ and the fact that if f is proper then \mathcal{F} is automatically pure over the base, see Lemma 16.1. \square

27. Blowing up and flatness

In this section we begin our discussion of results of the form: “After a blowup the strict transform becomes flat”. We will use the following (more or less standard)

notation in this section. If $X \rightarrow S$ is a morphism of schemes, \mathcal{F} is a quasi-coherent module on X , and $T \rightarrow S$ is a morphism of schemes, then we denote \mathcal{F}_T the pullback of \mathcal{F} to the base change $X_T = X \times_S T$.

Remark 27.1. Let S be a quasi-compact and quasi-separated scheme. Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent module on X . Let $U \subset S$ be a quasi-compact open subscheme. Given a U -admissible blowup $S' \rightarrow S$ we denote X' the strict transform of X and \mathcal{F}' the strict transform of \mathcal{F} which we think of as a quasi-coherent module on X' (via Divisors, Lemma 19.2). Let P be a property of $\mathcal{F}/X/S$ which is stable under strict transform (as above) for U -admissible blowups. The general problem in this section is: Show (under auxiliary conditions on $\mathcal{F}/X/S$) there exists a U -admissible blowup $S' \rightarrow S$ such that the strict transform $\mathcal{F}'/X'/S'$ has P .

The general strategy will be to use that a composition of U -admissible blowups is a U -admissible blowup, see Divisors, Lemma 20.2. In fact, we will make use of the more precise Divisors, Lemma 18.12 and combine it with Divisors, Lemma 19.6. The result is that it suffices to find a sequence of U -admissible blowups

$$S = S_0 \leftarrow S_1 \leftarrow \dots \leftarrow S_n$$

such that, setting $\mathcal{F}_0 = \mathcal{F}$ and $X_0 = X$ and setting \mathcal{F}_i/X_i equal to the strict transform of $\mathcal{F}_{i-1}/X_{i-1}$, we arrive at $\mathcal{F}_n/X_n/S_n$ with property P .

In particular, choose a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$ such that $V(\mathcal{I}) = S \setminus U$, see Properties, Lemma 22.1. Let $S' \rightarrow S$ be the blowup in \mathcal{I} and let $E \subset S'$ be the exceptional divisor (Divisors, Lemma 18.4). Then we see that we've reduced the problem to the case where there exists an effective Cartier divisor $D \subset S$ whose support is $X \setminus U$. In particular we may assume U is scheme theoretically dense in S (Divisors, Lemma 9.4).

Suppose that P is local on S : If $S = \bigcup S_i$ is a finite open covering by quasi-compact opens and P holds for $\mathcal{F}_{S_i}/X_{S_i}/S_i$ then P holds for $\mathcal{F}/X/S$. In this case the general problem above is local on S as well, i.e., if given $s \in S$ we can find a quasi-compact open neighbourhood W of s such that the problem for $\mathcal{F}_W/X_W/W$ is solvable, then the problem is solvable for $\mathcal{F}/X/S$. This follows from Divisors, Lemmas 20.3 and 20.4.

Lemma 27.2. *Let R be a local ring. Let M be a finite R -module. Let $k \geq 0$. Assume that $\text{Fit}_k(M) = (f)$ for some $f \in R$. Let M' be the quotient of M by $\{x \in M \mid fx = 0\}$. Then M' can be generated by k elements.*

Proof. Choose generators $x_1, \dots, x_n \in M$ corresponding to the surjection $R^{\oplus n} \rightarrow M$. Since R is local if a set of elements $E \subset (f)$ generates (f) , then some $e \in E$ generates (f) , see Algebra, Lemma 19.1. Hence we may pick z_1, \dots, z_{n-k} in the kernel of $R^{\oplus n} \rightarrow M$ such that some $(n-k) \times (n-k)$ minor of the $n \times (n-k)$ matrix $A = (z_{ij})$ is (f) . After renumbering the x_i we may assume the first minor $\det(z_{ij})_{1 \leq i, j \leq n-k}$ generates (f) , i.e., $\det(z_{ij})_{1 \leq i, j \leq n-k} = uf$ for some unit $u \in R$. Every other minor is a multiple of f . By Algebra, Lemma 14.5 there exists a $(n-k) \times (n-k)$ matrix B such that

$$AB = f \begin{pmatrix} u1_{n-k \times n-k} \\ C \end{pmatrix}$$

for some matrix C with coefficients in R . This implies that for every $i \leq n - k$ the element $y_i = ux_i + \sum_j c_{ji}x_j$ is annihilated by f . Since $M/\sum Ry_i$ is generated by the images of x_{n-k+1}, \dots, x_n we win. \square

Lemma 27.3. *Let R be a ring and let $f \in R$. Let $r, d \geq 0$ be integers. Let $R \rightarrow S$ be a ring map and let M be an S -module. Assume*

- (1) $R \rightarrow S$ is of finite presentation and flat,
- (2) every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over R ,
- (3) M is a finite S -module,
- (4) M_f is a finitely presented S_f -module,
- (5) for all $\mathfrak{p} \in R$, $f \notin \mathfrak{p}$ with $\mathfrak{q} = \mathfrak{p}S$ the module $M_{\mathfrak{q}}$ is free of rank r over $S_{\mathfrak{q}}$.

Then there exists a finitely generated ideal $I \subset R$ with $V(f) = V(I)$ such that for all $a \in I$ with $R' = R[\frac{I}{a}]$ the quotient

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

over $S' = S \otimes_R R'$ satisfies the following: for every prime $\mathfrak{p}' \subset R'$ there exists a $g \in S'$, $g \notin \mathfrak{p}'S'$ such that M'_g is a free S'_g -module of rank r .

Proof. Choose a surjection $S^{\oplus n} \rightarrow M$, which is possible by (1). Choose a finite submodule $K \subset \text{Ker}(S^{\oplus n} \rightarrow M)$ such that $S^{\oplus n}/K \rightarrow M$ becomes an isomorphism after inverting f . This is possible by (2). Set $M_1 = S^{\oplus n}/K$ and suppose we can prove the lemma for M_1 . Say $I \subset R$ is the corresponding ideal. Then for $a \in I$ the map

$$M'_1 = (M_1 \otimes_R R')/a\text{-power torsion} \longrightarrow M' = (M \otimes_R R')/a\text{-power torsion}$$

is surjective. It is also an isomorphism after inverting a in R' as $R'_a = R_f$, see Algebra, Lemma 56.3. But a is a nonzerodivisor on M'_1 , whence the displayed map is an isomorphism. Thus it suffices to prove the lemma in case M is a finitely presented S -module.

Assume M is a finitely presented S -module satisfying (3). Then $J = \text{Fit}_r(M) \subset S$ is a finitely generated ideal. By Lemma 9.3 we can write S as a direct summand of a free R -module: $\bigoplus_{\alpha \in A} R = S \oplus C$. For any element $h \in S$ writing $h = \sum a_{\alpha}$ in the decomposition above, we say that the a_{α} are the coefficients of h . Let $I' \subset R$ be the ideal generated by the coefficients of the elements of J . Multiplication by an element of S defines an R -linear map $S \rightarrow S$, hence I' is generated by the coefficients of the generators of J , i.e., I' is a finitely generated ideal. We claim that $I = fI'$ works.

We first check that $V(f) = V(I)$. The inclusion $V(f) \subset V(I)$ is clear. Conversely, if $f \notin \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}S$ is not an element of $V(J)$ by property (3) and the fact that formation of fitting ideals commute with base change (More on Algebra, Lemma 5.4). Hence there is an element of J which does not map to zero in $S \otimes_R \kappa(\mathfrak{p})$. Thus there exists an element of I' which is not contained in \mathfrak{p} , so $\mathfrak{p} \notin V(fI') = V(I)$.

Let $a \in I$ and let $\mathfrak{p}' \subset R' = R[\frac{I}{a}]$ be a prime ideal. Set $S' = S \otimes_S R'$ and $\mathfrak{q}' = \mathfrak{p}'S'$. Every element g of $JS' = \text{Fit}_r(M \otimes_S S')$ can be written as $g = \sum_{\alpha} c_{\alpha}$ for some $c_{\alpha} \in IS'$. Since $IR' = aR'$ we can write $c_{\alpha} = ac'_{\alpha}$ for some $c'_{\alpha} \in R'$ and $g = (\sum c'_{\alpha})a = g'a$ in S' . Moreover, we can find some $g_0 \in JS'$ such that $a = c_{\alpha}$ for some α . For this element $g_0 = g'_0a$ where g'_0 is a unit in $S'_{\mathfrak{q}'}$. Thus we see that $JS'_{\mathfrak{q}'}$ is the principal ideal generated by the nonzerodivisor a . It follows

from Lemma 27.2 that $M'_{\mathfrak{q}'}$ can be generated by r elements. Since M' is finite, there exist $m_1, \dots, m_r \in M'$ and $g \in S'$, $g \notin \mathfrak{q}'$ such that the corresponding map $(S')^{\oplus r} \rightarrow M'$ becomes surjective after inverting g .

Finally, consider the finitely generated ideal $J' = \text{Fit}_{k-1}(M')$. Note that $J'S'_g$ is generated by the coefficients of relations between m_1, \dots, m_r (compatibility of fitting ideal with base change). Thus it suffices to show that $J' = 0$, see More on Algebra, Lemma 5.7. Since $R'_a = R_f$ (see above) and $M'_a = M_f$ we see from (3) that J'_a maps to zero in $S_{\mathfrak{q}''}$ for any prime $\mathfrak{q}'' \subset S'$ of the form $\mathfrak{q}'' = \mathfrak{p}''S'$ where $\mathfrak{p}'' \subset R'_a$. Since $S'_a \subset \prod_{\mathfrak{q}'' \text{ as above}} S'_{\mathfrak{q}''}$ (as $(S'_a)_{\mathfrak{p}''} \subset S'_{\mathfrak{q}''}$ by Lemma 7.4) we see that $J'R'_a = 0$. Since a is a nonzerodivisor in R' we conclude that $J' = 0$ and we win. \square

Lemma 27.4. *Let S be a quasi-compact and quasi-separated scheme. Let $X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent module on X . Let $U \subset S$ be a quasi-compact open. Assume*

- (1) $X \rightarrow S$ is affine, of finite presentation, flat, geometrically integral fibres,
- (2) \mathcal{F} is a module of finite type,
- (3) \mathcal{F}_U is of finite presentation,
- (4) \mathcal{F} is flat over S at all generic points of fibres lying over points of U .

Then there exists a U -admissible blowup $S' \rightarrow S$ and an open subscheme $V \subset X_{S'}$ such that (a) the strict transform \mathcal{F}' of \mathcal{F} restricts to a finitely locally free \mathcal{O}_V -module and (b) $V \rightarrow S'$ is surjective.

Proof. Given $\mathcal{F}/X/S$ and $U \subset S$ with hypotheses as in the lemma, denote P the property “ \mathcal{F} is flat over S at all generic points of fibres”. It is clear that P is preserved under strict transform, see Divisors, Lemma 19.3 and Morphisms, Lemma 26.6. It is also clear that P is local on S . Hence any and all observations of Remark 27.1 apply to the problem posed by the lemma.

Consider the function $r : U \rightarrow \mathbf{Z}_{\geq 0}$ which assigns to $u \in U$ the integer

$$r(u) = \dim_{\kappa(\xi_u)}(\mathcal{F}_{\xi_u} \otimes \kappa(\xi_u))$$

where ξ_u is the generic point of the fibre X_u . By More on Morphisms, Lemma 13.7 and the fact that the image of an open in X_S in S is open, we see that $r(u)$ is locally constant. Accordingly $U = U_0 \amalg U_1 \amalg \dots \amalg U_c$ is a finite disjoint union of open and closed subschemes where r is constant with value i on U_i . By Divisors, Lemma 20.5 we can find a U -admissible blowup to decompose S into the disjoint union of two schemes, the first containing U_0 and the second $U_1 \cup \dots \cup U_c$. Repeating this $c-1$ more times we may assume that S is a disjoint union $S = S_0 \amalg S_1 \amalg \dots \amalg S_c$ with $U_i \subset S_i$. Thus we may assume the function r defined above is constant, say with value r .

By Remark 27.1 we see that we may assume that we have an effective Cartier divisor $D \subset S$ whose support is $S \setminus U$. Another application of Remark 27.1 combined with Divisors, Lemma 9.2 tells us we may assume that $S = \text{Spec}(R)$ and $D = \text{Spec}(R/(f))$ for some nonzerodivisor $f \in R$. This case is handled by Lemma 27.3. \square

Lemma 27.5. *Let $A \rightarrow C$ be a finite locally free ring map of rank d . Let $h \in C$ be an element such that C_h is étale over A . Let $J \subset C$ be an ideal. Set $I = \text{Fit}_0(C/J)$*

where we think of C/J as a finite A -module. Then $IC_h = JJ'$ for some ideal $J' \subset C_h$. If J is finitely generated so are I and J' .

Proof. We will use basic properties of fitting ideals, see More on Algebra, Lemma 5.4. Then IC is the fitting ideal of $C/J \otimes_A C$. Note that $C \rightarrow C \otimes_A C$, $c \mapsto 1 \otimes c$ has a section (the multiplication map). By assumption $C \rightarrow C \otimes_A C$ is étale at every prime in the image of $\text{Spec}(C_h)$ under this section. Hence the multiplication map $C \otimes_A C_h \rightarrow C_h$ is étale in particular flat, see Algebra, Lemma 138.9. Hence there exists a C_h -algebra such that $C \otimes_A C_h \cong C_h \oplus C'$ as C_h -algebras, see Algebra, Lemma 138.10. Thus $(C/J) \otimes_A C_h \cong (C_h/J_h) \oplus C'/I'$ as C_h -modules for some ideal $I' \subset C'$. Hence $IC_h = JJ'$ with $J' = \text{Fit}_0(C'/I')$ where we view C'/J' as a C_h -module. \square

Lemma 27.6. *Let $A \rightarrow B$ be an étale ring map. Let $a \in A$ be a nonzerodivisor. Let $J \subset B$ be a finite type ideal with $V(J) \subset V(aB)$. For every $\mathfrak{q} \subset B$ there exists a finite type ideal $I \subset A$ with $V(I) \subset V(a)$ and $g \in B$, $g \notin \mathfrak{q}$ such that $IB_g = JJ'$ for some finite type ideal $J' \subset B_g$.*

Proof. We may replace B by a principal localization at an element $g \in B$, $g \notin \mathfrak{q}$. Thus we may assume that B is standard étale, see Algebra, Proposition 138.17. Thus we may assume B is a localization of $C = A[x]/(f)$ for some monic $f \in A[x]$ of some degree d . Say $B = C_h$ for some $h \in C$. Choose elements $h_1, \dots, h_n \in C$ which generate J over B . The condition $V(J) \subset V(aB)$ signifies that $a^m = \sum b_i h_i$ in B for some large m . Set $h_{n+1} = a^m$. As in Lemma 27.5 we take $I = \text{Fit}_0(C/(h_1, \dots, h_{r+1}))$. Since the module $C/(h_1, \dots, h_{r+1})$ is annihilated by a^m we see that $a^{dm} \in I$ which implies that $V(I) \subset V(a)$. \square

Lemma 27.7. *Let S be a quasi-compact and quasi-separated scheme. Let $X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent module on X . Let $U \subset S$ be a quasi-compact open. Assume there exist finitely many commutative diagrams*

$$\begin{array}{ccccc} & X_i & \xrightarrow{\quad} & X & \\ & \downarrow & & \downarrow j_i & \\ S_i^* & \longrightarrow & S_i & \xrightarrow{e_i} & S \end{array}$$

where

- (1) $e_i : S_i \rightarrow S$ are quasi-compact étale morphisms and $S = \bigcup e_i(S_i)$,
- (2) $j_i : X_i \rightarrow X$ are étale morphisms and $X = \bigcup j_i(X_i)$,
- (3) $S_i^* \rightarrow S_i$ is an $e_i^{-1}(U)$ -admissible blowup such that the strict transform \mathcal{F}_i^* of $j_i^* \mathcal{F}$ is flat over S_i^* .

Then there exists a U -admissible blowup $S' \rightarrow S$ such that the strict transform of \mathcal{F} is flat over S' .

Proof. We claim that the hypotheses of the lemma are preserved under U -admissible blowups. Namely, suppose $b : S' \rightarrow S$ is a U -admissible blowup in the quasi-coherent sheaf of ideals \mathcal{I} . Moreover, let $S'_i \rightarrow S_i$ be the blowup in the quasi-coherent sheaf of ideals \mathcal{J}_i . Then the collection of morphisms $e'_i : S'_i \times_S S' \rightarrow S'$ and $j'_i : X'_i = X_i \times_S S' \rightarrow X \times_S S'$ satisfy conditions (1), (2), (3) for the strict transform \mathcal{F}' of \mathcal{F} relative to the blowup $S' \rightarrow S$. First, observe that S'_i is the blowup of S_i in the pullback of \mathcal{I} , see Divisors, Lemma 18.3. Second, consider the

blowup $S_i'^* \rightarrow S_i'$ of S_i' in the pullback of the ideal \mathcal{J}_i . By Divisors, Lemma 18.10 we get a commutative diagram

$$\begin{array}{ccc} S_i'^* & \longrightarrow & S_i' \\ \downarrow & \searrow & \downarrow \\ S_i^* & \longrightarrow & S_i \end{array}$$

and all the morphisms in the diagram above are blowups. Hence by Divisors, Lemmas 19.3 and 19.6 we see

$$\begin{aligned} & \text{the strict transform of } (j_i')^* \mathcal{F}' \text{ under } S_i'^* \rightarrow S_i' \\ &= \text{the strict transform of } j_i^* \mathcal{F} \text{ under } S_i'^* \rightarrow S_i \\ &= \text{the strict transform of } \mathcal{F}' \text{ under } S_i'^* \rightarrow S_i' \\ &= \text{the pullback of } \mathcal{F}_i^* \text{ via } X_i \times_{S_i} S_i'^* \rightarrow X_i \end{aligned}$$

which is therefore flat over $S_i'^*$ (Morphisms, Lemma 26.6). Having said this, we see that all observations of Remark 27.1 apply to the problem of finding a U -admissible blowup such that the strict transform of \mathcal{F} becomes flat over the base under assumptions as in the lemma. In particular, we may assume that $S \setminus U$ is the support of an effective Cartier divisor $D \subset S$. Another application of Remark 27.1 combined with Divisors, Lemma 9.2 shows we may assume that $S = \operatorname{Spec}(A)$ and $D = \operatorname{Spec}(A/(a))$ for some nonzerodivisor $a \in A$.

Pick an i and $s \in S_i$. Lemma 27.6 implies we can find an open neighbourhood $s \in W_i \subset S_i$ and a finite type quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_S$ such that $\mathcal{I} \cdot \mathcal{O}_{W_i} = \mathcal{J}_i \mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{W_i}$ and such that $V(\mathcal{I}) \subset V(a) = S \setminus U$. Since S_i is quasi-compact we can replace S_i by a finite collection W_1, \dots, W_n of these opens and assume that for each i there exists a quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_S$ such that $\mathcal{I}_i \cdot \mathcal{O}_{S_i} = \mathcal{J}_i \mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{S_i}$. As in the discussion of the first paragraph of the proof, consider the blowup S' of S in the product $\mathcal{I}_1 \dots \mathcal{I}_n$ (this blowup is U -admissible by construction). The base change of $S' \rightarrow S$ to S_i is the blowup in

$$\mathcal{J}_i \cdot \mathcal{J}_i' \mathcal{I}_1 \dots \hat{\mathcal{I}}_i \dots \mathcal{I}_n$$

which factors through the given blowup $S_i^* \rightarrow S_i$ (Divisors, Lemma 18.10). In the notation of the diagram above this means that $S_i'^* = S_i'$. Hence after replacing S by S' we arrive in the situation that $j_i^* \mathcal{F}$ is flat over S_i . Hence $j_i^* \mathcal{F}$ is flat over S , see Lemma 2.3. By Morphisms, Lemma 26.11 we see that \mathcal{F} is flat over S . \square

Theorem 27.8. *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S . Let \mathcal{F} be a quasi-coherent module on X . Let $U \subset S$ be a quasi-compact open. Assume*

- (1) X is quasi-compact,
- (2) X is locally of finite presentation over S ,
- (3) \mathcal{F} is a module of finite type,
- (4) \mathcal{F}_U is of finite presentation, and
- (5) \mathcal{F}_U is flat over U .

Then there exists a U -admissible blowup $S' \rightarrow S$ such that the strict transform \mathcal{F}' of \mathcal{F} is an $\mathcal{O}_{X \times_S S'}$ -module of finite presentation and flat over S' .

Proof. We first prove that we can find a U -admissible blowup such that the strict transform is flat. The question is étale local on the source and the target, see Lemma 27.7 for a precise statement. In particular, we may assume that $S = \operatorname{Spec}(R)$ and $X = \operatorname{Spec}(A)$ are affine. For $s \in S$ write $\mathcal{F}_s = \mathcal{F}|_{X_s}$ (pullback of \mathcal{F} to the fibre). As $X \rightarrow S$ is of finite type $d = \max_{s \in S} \dim(\operatorname{Supp}(\mathcal{F}_s))$ is an integer. We will do induction on d .

Let $x \in X$ be a point of X lying over $s \in S$ with $\dim_x(\operatorname{Supp}(\mathcal{F}_s)) = d$. Apply Lemma 3.2 to get $g : X' \rightarrow X$, $e : S' \rightarrow S$, $i : Z' \rightarrow X'$, and $\pi : Z' \rightarrow Y'$. Observe that $Y' \rightarrow S'$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d . Because the problem is étale local it suffices to prove the theorem for $g^*\mathcal{F}/X'/S'$. Because $i : Z' \rightarrow X'$ is a closed immersion of finite presentation (and since strict transform commutes with affine pushforward, see Divisors, Lemma 19.4) it suffices to prove the flattening result for \mathcal{G} . Since π is finite (hence also affine) it suffices to prove the flattening result for $\pi_*\mathcal{G}/Y'/S'$. Thus we may assume that $X \rightarrow S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d .

Next, we apply a blow up as in Lemma 27.4. Doing so we reach the situation where there exists an open $V \subset X$ surjecting onto S such that $\mathcal{F}|_V$ is finite locally free. Let $\xi \in X$ be the generic point of X_s . Let $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi \otimes \kappa(\xi)$. Choose a map $\alpha : \mathcal{O}_X^{\oplus r} \rightarrow \mathcal{F}$ which induces an isomorphism $\kappa(\xi)^{\oplus r} \rightarrow \mathcal{F}_\xi \otimes \kappa(\xi)$. Because \mathcal{F} is locally free over V we find an open neighbourhood W of ξ where α is an isomorphism. Shrink S to an affine open neighbourhood of s such that $W \rightarrow S$ is surjective. Say \mathcal{F} is the quasi-coherent module associated to the A -module N . Since \mathcal{F} is flat over S at all generic points of fibres (in fact at all points of W), we see that

$$\alpha_{\mathfrak{p}} : A_{\mathfrak{p}}^{\oplus r} \rightarrow N_{\mathfrak{p}}$$

is universally injective for all primes \mathfrak{p} of R , see Lemma 10.1. Hence α is universally injective, see Algebra, Lemma 79.12. Set $\mathcal{H} = \operatorname{Coker}(\alpha)$. By Divisors, Lemma 19.7 we see that, given a U -admissible blowup $S' \rightarrow S$ the strict transforms of \mathcal{F}' and \mathcal{H}' fit into an exact sequence

$$0 \rightarrow \mathcal{O}_{X \times_S S'}^{\oplus r} \rightarrow \mathcal{F}' \rightarrow \mathcal{H}' \rightarrow 0$$

Hence Lemma 10.1 also shows that \mathcal{F}' is flat at a point x' if and only if \mathcal{H}' is flat at that point. In particular \mathcal{H}_U is flat over U and \mathcal{H}_U is a module of finite presentation. We may apply the induction hypothesis to \mathcal{H} to see that there exists a U -admissible blowup such that the strict transform \mathcal{H}' is flat as desired.

To finish the proof of the theorem we still have to show that \mathcal{F}' is a module of finite presentation (after possibly another U -admissible blowup). This follows from Lemma 10.11 as we can assume $U \subset S$ is scheme theoretically dense (see third paragraph of Remark 27.1). This finishes the proof of the theorem. \square

28. Applications

In this section we apply some of the results above.

Lemma 28.1. *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S . Let $U \subset S$ be a quasi-compact open. Assume*

- (1) $X \rightarrow S$ is of finite type and quasi-separated, and

(2) $X_U \rightarrow U$ is flat and locally of finite presentation.

Then there exists a U -admissible blowup $S' \rightarrow S$ such that the strict transform of X is flat and of finite presentation over S' .

Proof. Since $X \rightarrow S$ is quasi-compact and quasi-separated by assumption, the strict transform of X with respect to a blowing up $S' \rightarrow S$ is also quasi-compact and quasi-separated. Hence to prove the lemma it suffices to find a U -admissible blowup such that the strict transform is flat and locally of finite presentation. Let $X = W_1 \cup \dots \cup W_n$ be a finite affine open covering. If we can find a U -admissible blowup $S_i \rightarrow S$ such that the strict transform of W_i is flat and locally of finite presentation, then there exists a U -admissible blowing up $S' \rightarrow S$ dominating all $S_i \rightarrow S$ which does the job (see Divisors, Lemma 20.4; see also Remark 27.1). Hence we may assume X is affine.

Assume X is affine. By Morphisms, Lemma 40.2 we can choose an immersion $j : X \rightarrow \mathbf{A}_S^n$ over S . Let $V \subset \mathbf{A}_S^n$ be a quasi-compact open subscheme such that j induces a closed immersion $i : X \rightarrow V$ over S . Apply Theorem 27.8 to $V \rightarrow S$ and the quasi-coherent module $i_* \mathcal{O}_X$ to obtain a U -admissible blowup $S' \rightarrow S$ such that the strict transform of $i_* \mathcal{O}_X$ is flat over S' and of finite presentation over $\mathcal{O}_{V \times_S S'}$. Let X' be the strict transform of X with respect to $S' \rightarrow S$. Let $i' : X' \rightarrow V \times_S S'$ be the induced morphism. Since taking strict transform commutes with pushforward along affine morphisms (Divisors, Lemma 19.4), we see that $i'_* \mathcal{O}_{X'}$ is flat over S and of finite presentation as a $\mathcal{O}_{V \times_S S'}$ -module. This implies the lemma. \square

Lemma 28.2. *Let $\varphi : X \rightarrow S$ be a separated morphism of finite type with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \rightarrow U$ is an isomorphism. Then there exists a U -admissible blowup $S' \rightarrow S$ such that the strict transform X' of X is isomorphic to an open subscheme of S' .*

Proof. The discussion in Remark 27.1 applies. Thus we may do a first U -admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D . In particular U is scheme theoretically dense in S . Next, we do another U -admissible blowup to get to the situation where $X \rightarrow S$ is flat and of finite presentation, see Lemma 28.1. In this case the result follows from More on Morphisms, Lemma 31.4. \square

The following lemma says that a proper modification can be dominated by a blowup.

Lemma 28.3. *Let $\varphi : X \rightarrow S$ be a proper morphism with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \rightarrow U$ is an isomorphism. Then there exists a U -admissible blowup $S' \rightarrow S$ which dominates X , i.e., such that there exists a factorization $S' \rightarrow X \rightarrow S$ of the blowup morphism.*

Proof. The discussion in Remark 27.1 applies. Thus we may do a first U -admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D . In particular U is scheme theoretically dense in S . Choose another U -admissible blowup $S' \rightarrow S$ such that the strict transform X' of X is an open subscheme of S' , see Lemma 28.2. Since $X' \rightarrow S'$ is proper, and $U \subset S'$ is dense, we see that $X' = S'$. Some details omitted. \square

29. Other chapters

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References

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