DIVISORS ON ALGEBRAIC SPACES

Contents

1. Introduction

In this chapter we study divisors on algebraic spaces and related topics. A basic reference for algebraic spaces is [Knu71].

2. Effective Cartier divisors

For some reason it seem convenient to define the notion of an effective Cartier divisor before anything else. Note that in Morphisms of Spaces, Section 13 we discussed the correspondence between closed subspaces and quasi-coherent sheaves of ideals. Moreover, in Properties of Spaces, Section 28, we discussed properties of quasi-coherent modules, in particular "locally generated by 1 element". These references show that the following definition is compatible with the definition for schemes.

Definition 2.1. Let S be a scheme. Let X be an algebraic space over S.
  (1) A *locally principal closed subspace* of X is a closed subspace whose sheaf of ideals is locally generated by 1 element.
  (2) An *effective Cartier divisor* on X is a closed subspace $D \subset X$ such that the ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

Thus an effective Cartier divisor is a locally principal closed subspace, but the converse is not always true. Effective Cartier divisors are closed subspaces of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is not a zerodivisor. In particular they are nowhere dense.

Lemma 2.2. *Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be a closed subspace. The following are equivalent:*
  (1) *The subspace D is an effective Cartier divisor on X.*

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(2) *For some scheme U and surjective étale morphism $U \to X$ the inverse image $D \times_X U$ is an effective Cartier divisor on U.*
(3) *For every scheme U and every étale morphism $U \to X$ the inverse image $D \times_X U$ is an effective Cartier divisor on U.*
(4) *For every $x \in |D|$ there exists an étale morphism $(U, u) \to (X, x)$ of pointed algebraic spaces such that $U = \mathrm{Spec}(A)$ and $D \times_X U = \mathrm{Spec}(A/(f))$ with $f \in A$ not a zerodivisor.*

Proof. The equivalence of (1) – (3) follows from Definition 2.1 and the references preceding it. Assume (1) and let $x \in |D|$. Choose a scheme W and a surjective étale morphism $W \to X$. Choose $w \in D \times_X W$ mapping to x. By (3) $D \times_X W$ is an effective Cartier divisor on W. Hence we can find affine étale neighbourhood U by choosing an affine open neighbourhood of w in W as in Divisors, Lemma 9.2.

Assume (2). Then we see that $\mathcal{I}_D|_U$ is invertible by Divisors, Lemma 9.2. Since we can find an étale covering of X by the collection of all such U and $X \setminus D$, we conclude that $\mathcal{I}_D$ is an invertible $\mathcal{O}_X$-module. $\square$

Lemma 2.3. *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a locally principal closed subspace. Let $U = X \setminus Z$. Then $U \to X$ is an affine morphism.*

Proof. The question is étale local on X, see Morphisms of Spaces, Lemmas 20.3 and Lemma 2.2. Thus this follows from the case of schemes which is Divisors, Lemma 9.3. $\square$

Lemma 2.4. *Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Let $U = X \setminus D$. Then $U \to X$ is an affine morphism and U is scheme theoretically dense in X.*

Proof. Affineness is Lemma 2.3. The density question is étale local on X by Morphisms of Spaces, Definition 17.3. Thus this follows from the case of schemes which is Divisors, Lemma 9.4. $\square$

Lemma 2.5. *Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Let $x \in |D|$. If $\dim_x(X) < \infty$, then $\dim_x(D) < \dim_x(X)$.*

Proof. Both the definition of an effective Cartier divisor and of the dimension of an an algebraic space at a point (Properties of Spaces, Definition 8.1) are étale local. Hence this lemma follows from the case of schemes which is Divisors, Lemma 9.5. $\square$

Definition 2.6. Let S be a scheme. Let X be an algebraic space over S. Given effective Cartier divisors D_1, D_2 on X we set $D = D_1 + D_2$ equal to the closed subspace of X corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the *sum of the effective Cartier divisors D_1 and D_2.*

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i.

Lemma 2.7. *The sum of two effective Cartier divisors is an effective Cartier divisor.*

Proof. Omitted. Étale locally this reduces to the following simple algebra fact: if $f_1, f_2 \in A$ are nonzerodivisors of a ring A, then $f_1 f_2 \in A$ is a nonzerodivisor. $\square$

Lemma 2.8. *Let S be a scheme. Let X be an algebraic space over S. Let Z, Y be two closed subspaces of X with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$. If $\mathcal{I}\mathcal{J}$ defines an effective Cartier divisor $D \subset X$, then Z and Y are effective Cartier divisors and $D = Z + Y$.*

Proof. By Lemma 2.2 this reduces to the case of schemes which is Divisors, Lemma 9.9. $\square$

Recall that we have defined the inverse image of a closed subspace under any morphism of algebraic spaces in Morphisms of Spaces, Definition 13.2.

Lemma 2.9. *Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a locally principal closed subspace. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subspace of X'.*

Proof. Omitted. $\square$

Definition 2.10. Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $D \subset X$ be an effective Cartier divisor. We say the *pullback of D by f is defined* if the closed subspace $f^{-1}(D) \subset X'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the *pullback of the effective Cartier divisor.*

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice.

Lemma 2.11. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:*
- (1) *f is flat, and*
- (2) *add more here as needed.*

Proof. Omitted. $\square$

Lemma 2.12. *Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let D_1, D_2 be effective Cartier divisors on X. If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.*

Proof. Omitted. $\square$

Definition 2.13. Let S be a scheme. Let X be an algebraic space over S and let $D \subset X$ be an effective Cartier divisor. The *invertible sheaf $\mathcal{O}_X(D)$ associated to D* is given by
$$\mathcal{O}_X(D) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X) = \mathcal{I}_D^{\otimes -1}.$$
The canonical section, usually denoted 1 or 1_D, is the global section of $\mathcal{O}_X(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \to \mathcal{O}_X$.

Lemma 2.14. *Let S be a scheme. Let X be an algebraic space over S. Let D_1, D_2 be effective Cartier divisors on X. Let $D = D_1 + D_2$. Then there is a unique isomorphism*
$$\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_X(D)$$
which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D.

Proof. Omitted. $\qquad\square$

Definition 2.15. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a *regular section* if the map $\mathcal{O}_X \to \mathcal{L}$, $f \mapsto fs$ is injective.

Lemma 2.16. *Let S be a scheme. Let X be an algebraic space over S. Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:*

(1) *f is a regular section, and*
(2) *for any $x \in X$ the image $f \in \mathcal{O}_{X,\overline{x}}$ is not a zerodivisor.*
(3) *for any affine $U = \mathrm{Spec}(A)$ étale over X the restriction $f|_U$ is a nonzero-divisor of A, and*
(4) *there exists a scheme U and a surjective étale morphism $U \to X$ such that $f|_U$ is a regular section of $\mathcal{O}_U$.*

Proof. Omitted. $\qquad\square$

Note that a global section s of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ may be seen as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$. (See Modules on Sites, Lemma 31.2 for the dual invertible sheaf.)

Definition 2.17. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$. The *zero scheme* of s is the closed subspace $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

Lemma 2.18. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$.*

(1) *Consider closed immersions $i : Z \to X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L}))$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this ordered set.*
(2) *For any morphism of algebraic spaces $f : Y \to X$ over S we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.*
(3) *The zero scheme $Z(s)$ is a locally principal closed subspace of X.*
(4) *The zero scheme $Z(s)$ is an effective Cartier divisor on X if and only if s is a regular section of $\mathcal{L}$.*

Proof. Omitted. $\qquad\square$

Lemma 2.19. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If $D \subset X$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.*
(2) *Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$, then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ which maps 1_D to s.*

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\{ \textit{effective Cartier divisors on } X \} \leftrightarrow \left\{ \begin{array}{c} \textit{pairs } (\mathcal{L}, s) \textit{ consisting of an invertible} \\ \mathcal{O}_X\textit{-module and a regular global section} \end{array} \right\}$$

Proof. Omitted. $\qquad\square$

3. Relative Proj

This section revisits the construction of the relative proj in the setting of algebraic spaces. The material in this section corresponds to the material in Constructions, Section 16 and Divisors, Section 16 in the case of schemes.

Situation 3.1. Here S is a scheme, X is an algebraic space over S, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_X$-algebra.

In Situation 3.1 we are going to define a functor $F : (Sch/S)_{fppf}^{opp} \to Sets$ which will turn out to be an algebraic space. We will follow (mutatis mutandis) the procedure of Constructions, Section 16. First, given a scheme T over S we define a *quadruple over T* to be a system $(d, f : T \to S, \mathcal{L}, \psi)$

(1) $d \geq 1$ is an integer,
(2) $f : T \to X$ is a morphism over S,
(3) $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and
(4) $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a homomorphism of graded $\mathcal{O}_T$-algebras such that $f^*\mathcal{A}_d \to \mathcal{L}$ is surjective.

We say two quadruples $(d, f, \mathcal{L}, \psi)$ and $(d', f', \mathcal{L}', \psi')$ are *equivalent*[1] if and only if we have $f = f'$ and for some positive integer $m = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes mn}$. Given a quadruple $(d, f, \mathcal{L}, \psi)$ and a morphism $h : T' \to T$ we have the pullback $(d, f \circ h, h^*\mathcal{L}, h^*\psi)$. Pullback preserves the equivalence relation. Finally, for a *quasi-compact* scheme T over S we set

$$F(T) = \text{the set of equivalence classes of quadruples over } T$$

and for an arbitrary scheme T over S we set

$$F(T) = \lim_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. Thus we have defined our functor

(3.1.1) $$F : Sch^{opp} \longrightarrow Sets$$

There is a morphism $F \to X$ of functors sending the quadruple $(d, f, \mathcal{L}, \psi)$ to f.

Lemma 3.2. *In Situation 3.1. The functor F above is an algebraic space. For any morphism $g : Z \to X$ where Z is a scheme there is a canonical isomorphism $\underline{Proj}_Z(g^*\mathcal{A}) = Z \times_X F$ compatible with further base change.*

Proof. It suffices to prove the second assertion, see Spaces, Lemma 11.1. Let $g : Z \to X$ be a morphism where Z is a scheme. Let F' be the functor of quadruples associated to the graded quasi-coherent $\mathcal{O}_Z$-algebra $g^*\mathcal{A}$. Then there is a canonical isomorphism $F' = Z \times_X F$, sending a quadruple $(d, f : T \to Z, \mathcal{L}, \psi)$ for F' to $(d, g \circ f, \mathcal{L}, \psi)$ (details omitted, see proof of Constructions, Lemma 16.1). By Constructions, Lemmas 16.4, 16.5, and 16.6 and Definition 16.7 we see that F' is representable by $\underline{Proj}_Z(g^*\mathcal{A})$. $\square$

The lemma above tells us the following definition makes sense.

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[1]This definition is motivated by Constructions, Lemma 16.4. The advantage of choosing this one is that it clearly defines an equivalence relation.

Definition 3.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. The *relative homogeneous spectrum of $\mathcal{A}$ over X*, or the *homogeneous spectrum of $\mathcal{A}$ over X*, or the *relative Proj of $\mathcal{A}$ over X* is the algebraic space F over X of Lemma 3.2. We denote it $\pi : \underline{\mathrm{Proj}}_X(\mathcal{A}) \to X$.

In particular the structure morphism of the relative Proj is representable by construction. We can also think about the relative Proj via glueing. Let $\varphi : U \to X$ be a surjective étale morphism, where U is a scheme. Set $R = U \times_X U$ with projection morphisms $s, t : R \to U$. By Lemma 3.2 there exists a canonical isomorphism

$$\gamma : \underline{\mathrm{Proj}}_U(\varphi^* \mathcal{A}) \longrightarrow \underline{\mathrm{Proj}}_X(\mathcal{A}) \times_X U$$

over U. Let $\alpha : t^* \varphi^* \mathcal{A} \to s^* \varphi^* \mathcal{A}$ be the canonical isomorphism of Properties of Spaces, Proposition 30.1. Then the diagram

$$
\begin{array}{ccc}
\underline{\mathrm{Proj}}_U(\varphi^*\mathcal{A}) \times_{U,s} R & = & \underline{\mathrm{Proj}}_R(s^*\varphi^*\mathcal{A}) \\
& {}^{s^*\gamma}\nearrow & \\
\underline{\mathrm{Proj}}_X(\mathcal{A}) \times_X R & & \text{induced by } \alpha \\
& {}_{t^*\gamma}\searrow & \\
\underline{\mathrm{Proj}}_U(\varphi^*\mathcal{A}) \times_{U,t} R & = & \underline{\mathrm{Proj}}_R(t^*\varphi^*\mathcal{A})
\end{array}
$$

is commutative (the equal signs come from Constructions, Lemma 16.10). Thus, if we denote $\mathcal{A}_U$, $\mathcal{A}_R$ the pullback of $\mathcal{A}$ to U, R, then $P = \underline{\mathrm{Proj}}_X(\mathcal{A})$ has an étale covering by the scheme $P_U = \underline{\mathrm{Proj}}_U(\mathcal{A}_U)$ and $P_U \times_P P_U$ is equal to $P_R = \underline{\mathrm{Proj}}_R(\mathcal{A}_R)$. Using these remarks we can argue in the usual fashion using étale localization to transfer results on the relative proj from the case of schemes to the case of algebraic spaces.

Lemma 3.4. *In Situation 3.1. The relative Proj comes equipped with a quasi-coherent sheaf of $\mathbf{Z}$-graded algebras $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_{\underline{Proj}_X(\mathcal{A})}(n)$ and a canonical homomorphism of graded algebras*

$$\psi : \pi^* \mathcal{A} \longrightarrow \bigoplus_{n \geq 0} \mathcal{O}_{\underline{Proj}_X(\mathcal{A})}(n)$$

whose base change to any scheme over X agrees with Constructions, Lemma 15.5.

Proof. As in the discussion following Definition 3.3 choose a scheme U and a surjective étale morphism $U \to X$, set $R = U \times_X U$ with projections $s, t : R \to U$, $\mathcal{A}_U = \mathcal{A}|_U$, $\mathcal{A}_R = \mathcal{A}|_R$, and $\pi : P = \underline{\mathrm{Proj}}_X(\mathcal{A}) \to X$, $\pi_U : P_U = \underline{\mathrm{Proj}}_U(\mathcal{A}_U)$ and $\pi_R : P_R = \underline{\mathrm{Proj}}_U(\mathcal{A}_R)$. By the Constructions, Lemma 15.5 we have a quasi-coherent sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{P_U}$-algebras $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_{P_U}(n)$ and a canonical map $\psi_U : \pi_U^* \mathcal{A}_U \to \bigoplus_{n \geq 0} \mathcal{O}_{P_U}(n)$ and similarly for P_R. By Constructions, Lemma 16.10 the pullback of $\mathcal{O}_{P_U}(n)$ and ψ_U by either projection $P_R \to P_U$ is equal to $\mathcal{O}_{P_R}(n)$ and ψ_R. By Properties of Spaces, Proposition 30.1 we obtain $\mathcal{O}_P(n)$ and ψ. We omit the verification of compatibility with pullback to arbitrary schemes over X. $\square$

Having constructed the relative Proj we turn to some basic properties.

Lemma 3.5. *Let S be a scheme. Let $g : X' \to X$ be a morphism of algebraic spaces over S and let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. Then there is a*

canonical isomorphism

$$r : \underline{Proj}_{X'}(g^*\mathcal{A}) \longrightarrow X' \times_X \underline{Proj}_X(\mathcal{A})$$

as well as a corresponding isomorphism

$$\theta : r^* pr_2^* \left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{\underline{Proj}_X(\mathcal{A})}(d) \right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{\underline{Proj}_{X'}(g^*\mathcal{A})}(d)$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{Proj}_{X'}(g^*\mathcal{A})}$-algebras.

Proof. Let F be the functor (3.1.1) and let F' be the corresponding functor defined using $g^*\mathcal{A}$ on X'. We claim there is a canonical isomorphism $r : F' \to X' \times_X F$ of functors (and of course r is the isomorphism of the lemma). It suffices to construct the bijection $r : F'(T) \to X'(T) \times_{X(T)} F(T)$ for quasi-compact schemes T over S. First, if $\xi = (d', f', \mathcal{L}', \psi')$ is a quadruple over T for F', then we can set $r(\xi) = (f', (d', g \circ f', \mathcal{L}', \psi'))$. This makes sense as $(g \circ f')^*\mathcal{A}^{(d)} = (f')^*(g^*\mathcal{A})^{(d)}$. The inverse map sends the pair $(f', (d, f, \mathcal{L}, \psi))$ to the quadruple $(d, f', \mathcal{L}, \psi)$. We omit the proof of the final assertion (hint: reduce to the case of schemes by étale localization and apply Constructions, Lemma 16.10). $\square$

Lemma 3.6. *In Situation 3.1 the morphism $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is separated.*

Proof. By Morphisms of Spaces, Lemma 4.12 and the construction of the relative Proj this follows from the case of schemes which is Constructions, Lemma 16.9. $\square$

Lemma 3.7. *In Situation 3.1. If one of the following holds*
  (1) *$\mathcal{A}$ is of finite type as a sheaf of $\mathcal{A}_0$-algebras,*
  (2) *$\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra and $\mathcal{A}_1$ is a finite type $\mathcal{A}_0$-module,*
  (3) *there exists a finite type quasi-coherent $\mathcal{A}_0$-submodule $\mathcal{F} \subset \mathcal{A}_+$ such that $\mathcal{A}_+/\mathcal{F}\mathcal{A}$ is a locally nilpotent sheaf of ideals of $\mathcal{A}/\mathcal{F}\mathcal{A}$,*

then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is quasi-compact.

Proof. By Morphisms of Spaces, Lemma 8.7 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 16.1. $\square$

Lemma 3.8. *In Situation 3.1. If $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{O}_X$-algebras, then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is of finite type.*

Proof. By Morphisms of Spaces, Lemma 23.4 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 16.2. $\square$

Lemma 3.9. *In Situation 3.1. If $\mathcal{O}_X \to \mathcal{A}_0$ is an integral algebra map[2] and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra, then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is universally closed.*

Proof. By Morphisms of Spaces, Lemma 9.5 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 16.3. $\square$

Lemma 3.10. *In Situation 3.1. The following conditions are equivalent*
  (1) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_X$-module and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra,*
  (2) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_X$-module and $\mathcal{A}$ is of finite type as an $\mathcal{O}_X$-algebra.*
If these conditions hold, then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is proper.

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[2]In other words, the integral closure of $\mathcal{O}_X$ in $\mathcal{A}_0$, see Morphisms of Spaces, Definition 43.2, equals $\mathcal{A}_0$.

Proof. By Morphisms of Spaces, Lemma 37.2 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 16.3. □

Lemma 3.11. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-modules generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. With $P = \underline{Proj}_X(\mathcal{A})$ we have*

(1) *P represents the functor F_1 which associates to T over S the set of iso-morphism classes of triples $(f, \mathcal{L}, \psi)$, where $f : T \to X$ is a morphism over S, $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and $\psi : f^*\mathcal{A} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a map of graded $\mathcal{O}_T$-algebras inducing a surjection $f^*\mathcal{A}_1 \to \mathcal{L}$,*

(2) *the canonical map $\pi^*\mathcal{A}_1 \to \mathcal{O}_P(1)$ is surjective, and*

(3) *each $\mathcal{O}_P(n)$ is invertible and the multiplication maps induce isomorphisms $\mathcal{O}_P(n) \otimes_{\mathcal{O}_P} \mathcal{O}_P(m) = \mathcal{O}_P(n + m)$.*

Proof. Omitted. See Constructions, Lemma 16.11 for the case of schemes. □

4. Functoriality of relative proj

This section is the analogue of Constructions, Section 18.

Lemma 4.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\psi : \mathcal{A} \to \mathcal{B}$ be a map of quasi-coherent graded $\mathcal{O}_X$-algebras. Set $P = \underline{Proj}_X(\mathcal{A}) \to X$ and $Q = \underline{Proj}_X(\mathcal{B}) \to X$. There is a canonical open subspace $U(\psi) \subset Q$ and a canonical morphism of algebraic spaces*

$$r_\psi : U(\psi) \longrightarrow P$$

over X and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^* \left(\bigoplus_{d \in \mathbf{Z}} \mathcal{O}_P(d) \right) \longrightarrow \bigoplus_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the property that for any scheme W étale over X the triple

$$(U(\psi) \times_X W, \quad r_\psi|_{U(\psi) \times_X W} : U(\psi) \times_X W \to P \times_X W, \quad \theta|_{U(\psi) \times_X W})$$

is equal to the triple associated to $\psi : \mathcal{A}|_W \to \mathcal{B}|_W$ of Constructions, Lemma 18.1.

Proof. This lemma follows from étale localization and the case of schemes, see discussion following Definition 3.3. Details omitted. □

Lemma 4.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be quasi-coherent graded $\mathcal{O}_X$-algebras. Set $P = \underline{Proj}_X(\mathcal{A})$, $Q = \underline{Proj}_X(\mathcal{B})$ and $R = \underline{Proj}_X(\mathcal{C})$. Let $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{C}$ be graded $\mathcal{O}_X$-algebra maps. Then we have*

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad and \quad r_{\psi \circ \varphi} = r_\varphi \circ r_\psi|_{U(\psi \circ \varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi \circ \varphi}$$

with obvious notation.

Proof. Omitted. □

Lemma 4.3. *With hypotheses and notation as in Lemma 4.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$. Then*

(1) *$U(\psi) = Q$,*

(2) $r_\psi : Q \to R$ is a closed immersion, and
(3) the maps $\theta : r_\psi^* \mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are surjective but not isomorphisms in general (even if $\mathcal{A} \to \mathcal{B}$ is surjective).

Proof. Follows from the case of schemes (Constructions, Lemma 18.3) by étale localization. $\square$

Lemma 4.4. *With hypotheses and notation as in Lemma 4.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is an isomorphism for all $d \gg 0$. Then*

(1) $U(\psi) = Q$,
(2) $r_\psi : Q \to P$ is an isomorphism, and
(3) the maps $\theta : r_\psi^* \mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are isomorphisms.

Proof. Follows from the case of schemes (Constructions, Lemma 18.4) by étale localization. $\square$

Lemma 4.5. *With hypotheses and notation as in Lemma 4.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$ and that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Then*

(1) $U(\psi) = Q$,
(2) $r_\psi : Q \to P$ is a closed immersion, and
(3) the maps $\theta : r_\psi^* \mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are isomorphisms.

Proof. Follows from the case of schemes (Constructions, Lemma 18.5) by étale localization. $\square$

5. Closed subspaces of relative proj

Some auxiliary lemmas about closed subspaces of relative proj. This section is the analogue of Divisors, Section 17.

Lemma 5.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{Proj}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to P$ be a closed subspace. Denote $\mathcal{I} \subset \mathcal{A}$ the kernel of the canonical map*

$$\mathcal{A} \longrightarrow \bigoplus_{d \geq 0} \pi_* \left((i_* \mathcal{O}_Z)(d) \right)$$

If π is quasi-compact, then there is an isomorphism $Z = \underline{Proj}_X(\mathcal{A}/\mathcal{I})$.

Proof. The morphism π is separated by Lemma 3.6. As π is quasi-compact, π_* transforms quasi-coherent modules into quasi-coherent modules, see Morphisms of Spaces, Lemma 11.2. Hence $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module. In particular, $\mathcal{B} = \mathcal{A}/\mathcal{I}$ is a quasi-coherent graded $\mathcal{O}_X$-algebra. The functoriality morphism $Z' = \underline{Proj}_X(\mathcal{B}) \to \underline{Proj}_X(\mathcal{A})$ is everywhere defined and a closed immersion, see Lemma 4.3. Hence it suffices to prove $Z = Z'$ as closed subspaces of P.

Having said this, the question is étale local on the base and we reduce to the case of schemes (Divisors, Lemma 17.1) by étale localization. $\square$

In case the closed subspace is locally cut out by finitely many equations we can define it by a finite type ideal sheaf of $\mathcal{A}$.

Lemma 5.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{Proj}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to P$ be a closed subscheme.*

If π is quasi-compact and i of finite presentation, then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{A}_d$ such that $Z = \underline{Proj}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$.

Proof. The reader can redo the arguments used in the case of schemes. However, we will show the lemma follows from the case of schemes by a trick. Let $\mathcal{I} \subset \mathcal{A}$ be the quasi-coherent graded ideal cutting out Z of Lemma 5.1. Choose an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 6.3. By the case of schemes (Divisors, Lemma 17.2) there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_U$-submodule $\mathcal{F}' \subset \mathcal{I}_d|_U \subset \mathcal{A}_d|_U$ such that $Z \times_X U$ is equal to $\underline{Proj}_U(\mathcal{A}|_U/\mathcal{F}'\mathcal{A}|_U)$. By Limits of Spaces, Lemma 9.2 we can find a finite type quasi-coherent submodule $\mathcal{F} \subset \mathcal{I}_d$ such that $\mathcal{F}' \subset \mathcal{F}|_U$. Let $Z' = \underline{Proj}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$. Then $Z' \to P$ is a closed immersion (Lemma 4.5) and $Z \subset Z'$ as $\mathcal{F}\mathcal{A} \subset \mathcal{I}$. On the other hand, $Z' \times_X U \subset Z \times_X U$ by our choice of $\mathcal{F}$. Thus $Z = Z'$ as desired. $\square$

Lemma 5.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{Proj}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subspace. Let $U \subset X$ be an open. Assume that*

(1) *π is quasi-compact,*
(2) *i of finite presentation,*
(3) *$|U| \cap |\pi|(|i|(|Z|)) = \emptyset$,*
(4) *U is quasi-compact,*
(5) *$\mathcal{A}_n$ is a finite type $\mathcal{O}_X$-module for all n.*

Then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{A}_d$ with (a) $Z = \underline{Proj}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$ and (b) the support of $\mathcal{A}_d/\mathcal{F}$ is disjoint from U.

Proof. We use the same trick as in the proof of Lemma 5.2 to reduce to the case of schemes. Let $\mathcal{I} \subset \mathcal{A}$ be the quasi-coherent graded ideal cutting out Z of Lemma 5.1. Choose an affine scheme W and a surjective étale morphism $W \to X$, see Properties of Spaces, Lemma 6.3. By the case of schemes (Divisors, Lemma 17.3) there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_W$-submodule $\mathcal{F}' \subset \mathcal{I}_d|_W \subset \mathcal{A}_d|_W$ such that (a) $Z \times_X W$ is equal to $\underline{Proj}_W(\mathcal{A}|_W/\mathcal{F}'\mathcal{A}|_W)$ and (b) the support of $\mathcal{A}_d|_W/\mathcal{F}'$ is disjoint from $U \times_X W$. By Limits of Spaces, Lemma 9.2 we can find a finite type quasi-coherent submodule $\mathcal{F} \subset \mathcal{I}_d$ such that $\mathcal{F}' \subset \mathcal{F}|_W$. Let $Z' = \underline{Proj}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$. Then $Z' \to P$ is a closed immersion (Lemma 4.5) and $Z \subset Z'$ as $\mathcal{F}\mathcal{A} \subset \mathcal{I}$. On the other hand, $Z' \times_X W \subset Z \times_X W$ by our choice of $\mathcal{F}$. Thus $Z = Z'$. Finally, we see that $\mathcal{A}_d/\mathcal{F}$ is supported on $X \setminus U$ as $\mathcal{A}_d|_W/\mathcal{F}|_W$ is a quotient of $\mathcal{A}_d|_W/\mathcal{F}'$ which is supported on $W \setminus U \times_X W$. Thus the lemma follows. $\square$

6. Blowing up

Blowing up is an important tool in algebraic geometry.

Definition 6.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subscheme corresponding to $\mathcal{I}$ (Morphisms of Spaces, Lemma 13.1). The *blowing up of X along Z*, or the *blowing up of X in the ideal sheaf $\mathcal{I}$* is the morphism

$$b : \underline{Proj}_X\left(\bigoplus_{n \geq 0} \mathcal{I}^n\right) \longrightarrow X$$

The *exceptional divisor* of the blow up is the inverse image $b^{-1}(Z)$. Sometimes Z is called the *center* of the blowup.

We will see later that the exceptional divisor is an effective Cartier divisor. More-over, the blowing up is characterized as the "smallest" algebraic space over X such that the inverse image of Z is an effective Cartier divisor.

If $b : X' \to X$ is the blow up of X in Z, then we often denote $\mathcal{O}_{X'}(n)$ the twists of the structure sheaf. Note that these are invertible $\mathcal{O}_{X'}$-modules and that $\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(1)^{\otimes n}$ because X' is the relative Proj of a quasi-coherent graded $\mathcal{O}_X$-algebra which is generated in degree 1, see Lemma 3.11.

Lemma 6.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $U = \mathrm{Spec}(A)$ be an affine scheme étale over X and let $I \subset A$ be the ideal corresponding to $\mathcal{I}|_U$. If $X' \to X$ is the blow up of X in $\mathcal{I}$, then there is a canonical isomorphism*

$$U \times_X X' = Proj(\bigoplus\nolimits_{d \geq 0} I^d)$$

of schemes over U, where the right hand side is the homogeneous spectrum of the Rees algebra of I in A. Moreover, $U \times_X X'$ has an affine open covering by spectra of the affine blowup algebras $A[\frac{I}{a}]$.

Proof. Note that the restriction $\mathcal{I}|_U$ is equal to the pullback of $\mathcal{I}$ via the morphism $U \to X$, see Properties of Spaces, Section 24. Thus the lemma follows on combining Lemma 3.2 with Divisors, Lemma 18.2. $\square$

Lemma 6.3. *Let S be a scheme. Let $X_1 \to X_2$ be a flat morphism of algebraic spaces over S. Let $Z_2 \subset X_2$ be a closed subspace. Let Z_1 be the inverse image of Z_2 in X_1. Let X'_i be the blow up of Z_i in X_i. Then there exists a cartesian diagram*

$$\begin{array}{ccc} X'_1 & \longrightarrow & X'_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

of algebraic spaces over S.

Proof. Let $\mathcal{I}_2$ be the ideal sheaf of Z_2 in X_2. Denote $g : X_1 \to X_2$ the given mor-phism. Then the ideal sheaf $\mathcal{I}_1$ of Z_1 is the image of $g^* \mathcal{I}_2 \to \mathcal{O}_{X_1}$ (see Morphisms of Spaces, Definition 13.2 and discussion following the definition). By Lemma 3.5 we see that $X_1 \times_{X_2} X'_2$ is the relative Proj of $\bigoplus_{n \geq 0} g^* \mathcal{I}_2^n$. Because g is flat the map $g^* \mathcal{I}_2^n \to \mathcal{O}_{X_1}$ is injective with image $\mathcal{I}_1^n$. Thus we see that $X_1 \times_{X_2} X'_2 = X'_1$. $\square$

Lemma 6.4. *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. The blowing up $b : X' \to X$ of Z in X has the following properties:*

(1) *$b|_{b^{-1}(X \setminus Z)} : b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism,*
(2) *the exceptional divisor $E = b^{-1}(Z)$ is an effective Cartier divisor on X',*
(3) *there is a canonical isomorphism $\mathcal{O}_{X'}(-1) = \mathcal{O}_{X'}(E)$*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 6.3) we can prove each of these statements after base change to U. This reduces us to the case of schemes. In this case the result is Divisors, Lemma 18.4. $\square$

Lemma 6.5 (Universal property blowing up)**.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. Let $\mathcal{C}$ be the full subcategory of $(Spaces/X)$ consisting of $Y \to X$ such that the inverse image of Z is an effective Cartier divisor on Y. Then the blowing up $b : X' \to X$ of Z in X is a final object of $\mathcal{C}$.*

Proof. We see that $b : X' \to X$ is an object of $\mathcal{C}$ according to Lemma 6.4. Let $f : Y \to X$ be an object of $\mathcal{C}$. We have to show there exists a unique morphism $Y \to X'$ over X. Let $D = f^{-1}(Z)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z and let $\mathcal{I}_D$ be the ideal sheaf of D. Then $f^*\mathcal{I} \to \mathcal{I}_D$ is a surjection to an invertible $\mathcal{O}_Y$-module. This extends to a map $\psi : \bigoplus f^*\mathcal{I}^d \to \bigoplus \mathcal{I}_D^d$ of graded $\mathcal{O}_Y$-algebras. (We observe that $\mathcal{I}_D^d = \mathcal{I}_D^{\otimes d}$ as D is an effective Cartier divisor.) By Lemma 3.11. the triple $(f : Y \to X, \mathcal{I}_D, \psi)$ defines a morphism $Y \to X'$ over X. The restriction

$$Y \setminus D \longrightarrow X' \setminus b^{-1}(Z) = X \setminus Z$$

is unique. The open $Y \setminus D$ is scheme theoretically dense in Y according to Lemma 2.4. Thus the morphism $Y \to X'$ is unique by Morphisms of Spaces, Lemma 17.8 (also b is separated by Lemma 3.6). $\square$

Lemma 6.6. *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be an effective Cartier divisor. The blowup of X in Z is the identity morphism of X.*

Proof. Immediate from the universal property of blowups (Lemma 6.5). $\square$

Lemma 6.7. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blow up X' of X in $\mathcal{I}$ is reduced.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 6.3) we can prove each of these statements after base change to U. This reduces us to the case of schemes. In this case the result is Divisors, Lemma 18.8. $\square$

Lemma 6.8. *Let S be a scheme. Let X be an algebraic space over S. Let $b : X' \to X$ be a blow up of X in a closed subspace. For any effective Cartier divisor D on X the pullback $b^{-1}D$ is defined (see Definition 2.10).*

Proof. By Lemmas 6.2 and 2.2 this reduces to the following algebra fact: Let A be a ring, $I \subset A$ an ideal, $a \in I$, and $x \in A$ a nonzerodivisor. Then the image of x in $A[\frac{I}{a}]$ is a nonzerodivisor. Namely, suppose that $x(y/a^n) = 0$ in $A[\frac{I}{a}]$. Then $a^m x y = 0$ in A for some m. Hence $a^m y = 0$ as x is a nonzerodivisor. Whence y/a^n is zero in $A[\frac{I}{a}]$ as desired. $\square$

Lemma 6.9. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ and $\mathcal{J}$ be quasi-coherent sheaves of ideals. Let $b : X' \to X$ be the blowing up of X in $\mathcal{I}$. Let $b' : X'' \to X'$ be the blowing up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$. Then $X'' \to X$ is canonically isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$.*

Proof. Let $E \subset X'$ be the exceptional divisor of b which is an effective Cartier divisor by Lemma 6.4. Then $(b')^{-1}E$ is an effective Cartier divisor on X'' by Lemma 6.8. Let $E' \subset X''$ be the exceptional divisor of b' (also an effective Cartier divisor). Consider the effective Cartier divisor $E'' = E' + (b')^{-1}E$. By construction the

ideal of E'' is $(b \circ b')^{-1} \mathcal{I} (b \circ b')^{-1} \mathcal{J} \mathcal{O}_{X''}$. Hence according to Lemma 6.5 there is a canonical morphism from X'' to the blowup $c : Y \to X$ of X in $\mathcal{I}\mathcal{J}$. Conversely, as $\mathcal{I}\mathcal{J}$ pulls back to an invertible ideal we see that $c^{-1} \mathcal{I} \mathcal{O}_Y$ defines an effective Cartier divisor, see Lemma 2.8. Thus a morphism $c' : Y \to X'$ over X by Lemma 6.5. Then $(c')^{-1} b^{-1} \mathcal{J} \mathcal{O}_Y = c^{-1} \mathcal{J} \mathcal{O}_Y$ which also defines an effective Cartier divisor. Thus a morphism $c'' : Y \to X''$ over X'. We omit the verification that this morphism is inverse to the morphism $X'' \to Y$ constructed earlier. $\square$

Lemma 6.10. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $b : X' \to X$ be the blowing up of X in the ideal sheaf $\mathcal{I}$. If $\mathcal{I}$ is of finite type, then $b : X' \to X$ is a proper morphism.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 6.3) we can prove each of these statements after base change to U (see Morphisms of Spaces, Lemma 37.2). This reduces us to the case of schemes. In this case the morphism b is projective by Divisors, Lemma 18.11 hence proper by Morphisms, Lemma 43.5. $\square$

Lemma 6.11. *Let S be a scheme and let X be an algebraic space over S. Assume X is quasi-compact and quasi-separated. Let $Z \subset X$ be a closed subspace of finite presentation. Let $b : X' \to X$ be the blowing up with center Z. Let $Z' \subset X'$ be a closed subspace of finite presentation. Let $X'' \to X'$ be the blowing up with center Z'. There exists a closed subspace $Y \subset X$ of finite presentation, such that*

(1) $|Y| = |Z| \cup |b|(|Z'|)$, and
(2) *the composition $X'' \to X$ is isomorphic to the blowing up of X in Y.*

Proof. The condition that $Z \to X$ is of finite presentation means that Z is cut out by a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, see Morphisms of Spaces, Lemma 27.12. Write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ so that $X' = \underline{\mathrm{Proj}}(\mathcal{A})$. Note that $X \setminus Z$ is a quasi-compact open subspace of X by Limits of Spaces, Lemma 14.1. Since $b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism (Lemma 6.4) the same result shows that $b^{-1}(X \setminus Z) \setminus Z'$ is quasi-compact open subspace in X'. Hence $U = X \setminus (Z \cup b(Z'))$ is quasi-compact open subspace in X. By Lemma 5.3 there exist a $d > 0$ and a finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{I}^d$ such that $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ and such that the support of $\mathcal{I}^d/\mathcal{F}$ is contained in $X \setminus U$.

Since $\mathcal{F} \subset \mathcal{I}^d$ is an $\mathcal{O}_X$-submodule we may think of $\mathcal{F} \subset \mathcal{I}^d \subset \mathcal{O}_X$ as a finite type quasi-coherent sheaf of ideals on X. Let's denote this $\mathcal{J} \subset \mathcal{O}_X$ to prevent confusion. Since $\mathcal{I}^d/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}^d$ are supported on $|X| \setminus |U|$ we see that $|V(\mathcal{J})|$ is contained in $|X| \setminus |U|$. Conversely, as $\mathcal{J} \subset \mathcal{I}^d$ we see that $|Z| \subset |V(\mathcal{J})|$. Over $X \setminus Z \cong X' \setminus b^{-1}(Z)$ the sheaf of ideals $\mathcal{J}$ cuts out Z' (see displayed formula below). Hence $|V(\mathcal{J})|$ equals $|Z| \cup |b|(|Z'|)$. It follows that also $|V(\mathcal{I}\mathcal{J})| = |Z| \cup |b|(|Z'|)$. Moreover, $\mathcal{I}\mathcal{J}$ is an ideal of finite type as a product of two such. We claim that $X'' \to X$ is isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$ which finishes the proof of the lemma by setting $Y = V(\mathcal{I}\mathcal{J})$.

First, recall that the blow up of X in $\mathcal{I}\mathcal{J}$ is the same as the blow up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$, see Lemma 6.9. Hence it suffices to show that the blow up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$ agrees with the blow up of X' in Z'. We will show that

$$b^{-1}\mathcal{J}\mathcal{O}_{X'} = \mathcal{I}_E^d \mathcal{I}_{Z'}$$

as ideal sheaves on X''. This will prove what we want as $\mathcal{I}_E^d$ cuts out the effective Cartier divisor dE and we can use Lemmas 6.6 and 6.9.

To see the displayed equality of the ideals we may work locally. With notation A, I, $a \in I$ as in Lemma 6.2 we see that $\mathcal{F}$ corresponds to an R-submodule $M \subset I^d$ mapping isomorphically to an ideal $J \subset R$. The condition $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ means that $Z' \cap \mathrm{Spec}(A[\frac{I}{a}])$ is cut out by the ideal generated by the elements m/a^d, $m \in M$. Say the element $m \in M$ corresponds to the function $f \in J$. Then in the affine blowup algebra $A' = A[\frac{I}{a}]$ we see that $f = (a^d m)/a^d = a^d (m/a^d)$. Thus the equality holds. $\qquad\square$

7. Strict transform

This section is the analogue of Divisors, Section 19. Let S be a scheme, let B be an algebraic space over S, and let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up of B in Z and denote $E \subset B'$ the exceptional divisor $E = b^{-1}Z$. In the following we will often consider an algebraic space X over B and form the cartesian diagram

$$
\begin{array}{ccccc}
\mathrm{pr}_{B'}^{-1}E & \longrightarrow & X \times_B B' & \xrightarrow{\ \mathrm{pr}_X\ } & X \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{B'}} & & \downarrow{\scriptstyle f} \\
E & \longrightarrow & B' & \longrightarrow & B
\end{array}
$$

Since E is an effective Cartier divisor (Lemma 6.4) we see that $\mathrm{pr}_{B'}^{-1}E \subset X \times_B B'$ is locally principal (Lemma 2.9). Thus the inclusion morphism of the complement of $\mathrm{pr}_{B'}^{-1}E$ in $X \times_B B'$ is affine and in particular quasi-compact (Lemma 2.3). Consequently, for a quasi-coherent $\mathcal{O}_{X \times_B B'}$-module $\mathcal{G}$ the subsheaf of sections supported on $|\mathrm{pr}_{B'}^{-1}E|$ is a quasi-coherent submodule, see Limits of Spaces, Lemma 14.5. If $\mathcal{G}$ is a quasi-coherent sheaf of algebras, e.g., $\mathcal{G} = \mathcal{O}_{X \times_B B'}$, then this subsheaf is an ideal of $\mathcal{G}$.

Definition 7.1. With $Z \subset B$ and $f : X \to B$ as above.
   (1) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the *strict transform* of $\mathcal{F}$ with respect to the blowup of B in Z is the quotient $\mathcal{F}'$ of $\mathrm{pr}_X^*\mathcal{F}$ by the submodule of sections supported on $|\mathrm{pr}_{B'}^{-1}E|$.
   (2) The *strict transform* of X is the closed subscheme $X' \subset X \times_B B'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X \times_B B'}$ supported on $|\mathrm{pr}_{B'}^{-1}E|$.

Note that taking the strict transform along a blowup depends on the closed subspace used for the blowup (and not just on the morphism $B' \to B$).

Lemma 7.2 (Étale localization and strict transform)**.** *In the situation of Definition 7.1. Let*

$$
\begin{array}{ccc}
U & \longrightarrow & X \\
\downarrow & & \downarrow \\
V & \longrightarrow & B
\end{array}
$$

be a commutative diagram of morphisms with U and V schemes and étale horizontal arrows. Let $V' \to V$ be the blowup of V in $Z \times_B V$. Then
   (1) $V' = V \times_B B'$ *and the maps $V' \to B'$ and $U \times_V V' \to X \times_B B'$ are étale,*

(2) *the strict transform U' of U relative to $V' \to V$ is equal to $X' \times_X U$ where X' is the strict transform of X relative to $B' \to B$, and*

(3) *for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the restriction of the strict transform $\mathcal{F}'$ to $U \times_V V'$ is the strict transform of $\mathcal{F}|_U$ relative to $V' \to V$.*

Proof. Part (1) follows from the fact that blowup commutes with flat base change (Lemma 6.3), the fact that étale morphisms are flat, and that the base change of an étale morphism is étale. Part (3) then follows from the fact that taking the sheaf of sections supported on a closed commutes with pullback by étale morphisms, see Limits of Spaces, Lemma 14.5. Part (2) follows from (3) applied to $\mathcal{F} = \mathcal{O}_X$. $\square$

Lemma 7.3. *In the situation of Definition 7.1.*

(1) *The strict transform X' of X is the blowup of X in the closed subspace $f^{-1}Z$ of X.*

(2) *For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform $\mathcal{F}'$ is canonically isomorphic to the pushforward along $X' \to X \times_B B'$ of the strict transform of $\mathcal{F}$ relative to the blowing up $X' \to X$.*

Proof. Let $X'' \to X$ be the blowup of X in $f^{-1}Z$. By the universal property of blowing up (Lemma 6.5) there exists a commutative diagram

$$
\begin{array}{ccc}
X'' & \longrightarrow & X \\
\downarrow & & \downarrow \\
B' & \longrightarrow & B
\end{array}
$$

whence a morphism $i : X'' \to X \times_B B'$. The first assertion of the lemma is that i is a closed immersion with image X'. The second assertion of the lemma is that $\mathcal{F}' = i_*\mathcal{F}''$ where $\mathcal{F}''$ is the strict transform of $\mathcal{F}$ with respect to the blowing up $X'' \to X$. We can check these assertions étale locally on X, hence we reduce to the case of schemes (Divisors, Lemma 19.2). Some details omitted. $\square$

Lemma 7.4. *In the situation of Definition 7.1.*

(1) *If X is flat over B at all points lying over Z, then the strict transform of X is equal to the base change $X \times_B B'$.*

(2) *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over B at all points lying over Z, then the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is equal to the pullback $pr_X^*\mathcal{F}$.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 19.3) by étale localization (Lemma 7.2). $\square$

Lemma 7.5. *Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up of Z in B. Let $g : X \to Y$ be an affine morphism of spaces over B. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g' : X \times_B B' \to Y \times_B B'$ be the base change of g. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ relative to b. Then $g'_*\mathcal{F}'$ is the strict transform of $g_*\mathcal{F}$.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 19.4) by étale localization (Lemma 7.2). $\square$

Lemma 7.6. *Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $D \subset B$ be an effective Cartier divisor. Let $Z' \subset B$ be the closed subspace cut out by the product of the ideal sheaves of Z and D. Let $B' \to B$ be the blowup of B in Z.*

(1) *The blowup of B in Z' is isomorphic to $B' \to B$.*
(2) *Let $f : X \to B$ be a morphism of algebraic spaces and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If the subsheaf of $\mathcal{F}$ of sections supported on $|f^{-1}D|$ is zero, then the strict transform of $\mathcal{F}$ relative to the blowing up in Z agrees with the strict transform of $\mathcal{F}$ relative to the blowing up of B in Z'.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 19.5) by étale localization (Lemma 7.2). $\square$

Lemma 7.7. *Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up with center Z. Let $Z' \subset B'$ be a closed subspace. Let $B'' \to B'$ be the blowing up with center Z'. Let $Y \subset B$ be a closed subscheme such that $|Y| = |Z| \cup |b|(|Z'|)$ and the composition $B'' \to B$ is isomorphic to the blowing up of B in Y. In this situation, given any scheme X over B and $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have*

(1) *the strict transform of $\mathcal{F}$ with respect to the blowing up of B in Y is equal to the strict transform with respect to the blowup $B'' \to B'$ in Z' of the strict transform of $\mathcal{F}$ with respect to the blowup $B' \to B$ of B in Z, and*
(2) *the strict transform of X with respect to the blowing up of B in Y is equal to the strict transform with respect to the blowup $B'' \to B'$ in Z' of the strict transform of X with respect to the blowup $B' \to B$ of B in Z.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 19.6) by étale localization (Lemma 7.2). $\square$

Lemma 7.8. *In the situation of Definition 7.1. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of quasi-coherent sheaves on X which remains exact after any base change $T \to B$. Then the strict transforms of $\mathcal{F}_i'$ relative to any blowup $B' \to B$ form a short exact sequence $0 \to \mathcal{F}_1' \to \mathcal{F}_2' \to \mathcal{F}_3' \to 0$ too.

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 19.7) by étale localization (Lemma 7.2). $\square$

8. Admissible blowups

To have a bit more control over our blowups we introduce the following standard terminology.

Definition 8.1. Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. A morphism $X' \to X$ is called a U-*admissible blowup* if there exists a closed immersion $Z \to X$ of finite presentation with Z disjoint from U such that X' is isomorphic to the blow up of X in Z.

We recall that $Z \to X$ is of finite presentation if and only if the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is of finite type, see Morphisms of Spaces, Lemma 27.12. In particular, a U-admissible blowup is a proper morphism, see Lemma 6.10. Note that there can be multiple centers which give rise to the same morphism. Hence the requirement is just the existence of some center disjoint from U which produces X'. Finally, as the morphism $b : X' \to X$ is an isomorphism over U (see Lemma 6.4) we will often abuse notation and think of U as an open subspace of X' as well.

Lemma 8.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open subspace. Let $b : X' \to X$ be a U-admissible blowup. Let $X'' \to X'$ be a U-admissible blowup. Then the composition $X'' \to X$ is a U-admissible blowup.*

Proof. Immediate from the more precise Lemma 6.11. $\square$

Lemma 8.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U, V \subset X$ be quasi-compact open subspaces. Let $b : V' \to V$ be a $U \cap V$-admissible blowup. Then there exists a U-admissible blowup $X' \to X$ whose restriction to V is V'.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_V$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I})$ is disjoint from $U \cap V$ and such that V' is isomorphic to the blow up of V in $\mathcal{I}$. Let $\mathcal{I}' \subset \mathcal{O}_{U \cup V}$ be the quasi-coherent sheaf of ideals whose restriction to U is $\mathcal{O}_U$ and whose restriction to V is $\mathcal{I}$. By Limits of Spaces, Lemma 9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ whose restriction to $U \cup V$ is $\mathcal{I}'$. The lemma follows. $\square$

Lemma 8.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open subspace. Let $b_i : X_i \to X$, $i = 1, \ldots, n$ be U-admissible blowups. There exists a U-admissible blowup $b : X' \to X$ such that (a) b factors as $X' \to X_i \to X$ for $i = 1, \ldots, n$ and (b) each of the morphisms $X' \to X_i$ is a U-admissible blowup.*

Proof. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_i)$ is disjoint from U and such that X_i is isomorphic to the blow up of X in $\mathcal{I}_i$. Set $\mathcal{I} = \mathcal{I}_1 \cdot \ldots \cdot \mathcal{I}_n$ and let X' be the blowup of X in $\mathcal{I}$. Then $X' \to X$ factors through b_i by Lemma 6.9. $\square$

Lemma 8.5. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let U, V be quasi-compact disjoint open subspaces of X. Then there exist a $U \cup V$-admissible blowup $b : X' \to X$ such that X' is a disjoint union of open subspaces $X' = X_1' \amalg X_2'$ with $b^{-1}(U) \subset X_1'$ and $b^{-1}(V) \subset X_2'$.*

Proof. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}$, resp. $\mathcal{J}$ such that $X \setminus U = V(\mathcal{I})$, resp. $X \setminus V = V(\mathcal{J})$, see Limits of Spaces, Lemma 14.1. Then $|V(\mathcal{I}\mathcal{J})| = |X|$. Hence $\mathcal{I}\mathcal{J}$ is a locally nilpotent sheaf of ideals. Since $\mathcal{I}$ and $\mathcal{J}$ are of finite type and X is quasi-compact there exists an $n > 0$ such that $\mathcal{I}^n \mathcal{J}^n = 0$. We may and do replace $\mathcal{I}$ by $\mathcal{I}^n$ and $\mathcal{J}$ by $\mathcal{J}^n$. Whence $\mathcal{I}\mathcal{J} = 0$. Let $b : X' \to X$ be the blowing up in $\mathcal{I} + \mathcal{J}$. This is $U \cup V$-admissible as $|V(\mathcal{I} + \mathcal{J})| = |X| \setminus |U| \cup |V|$. We will show that X' is a disjoint union of open subspaces $X' = X_1' \amalg X_2'$ as in the statement of the lemma.

Since $|V(\mathcal{I} + \mathcal{J})|$ is the complement of $|U \cup V|$ we conclude that $V \cup U$ is scheme theoretically dense in X', see Lemmas 6.4 and 2.4. Thus if such a decomposition $X' = X_1' \amalg X_2'$ into open and closed subspaces exists, then X_1' is the scheme theoretic closure of U in X' and similarly X_2' is the scheme theoretic closure of V in X'. Since $U \to X'$ and $V \to X'$ are quasi-compact taking scheme theoretic closures commutes with étale localization (Morphisms of Spaces, Lemma 16.3). Hence to verify the existence of X_1' and X_2' we may work étale locally on X. This reduces us to the case of schemes which is treated in the proof of Divisors, Lemma 20.5. $\square$

9. Other chapters

References

[Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, vol. 203, Springer-Verlag, 1971.